Supersymmetric Large Extra Dimensions

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DEDICATION

To Cole,

The world is a strange place; take pride in what you are able to understand,
but delight in the mystery that remains.
There are a number of people to whom I am indebted, since without their help this thesis would not have been completed. Foremost in this list is my thesis supervisor, Cliff Burgess, who is one of the smartest people I have had the pleasure of meeting. Ask him a question on any topic in physics, and besides giving you the correct answer, you can expect at least one excellent physical reason why it has to be so. I have also had the pleasure of working with some truly wonderful collaborators, including Andrew Tolley, Claudia de Rham, and Gianmassimo Tasinato; I can only hope that they gained as much from our collaborations as I did. I’d also like to thank my fellow graduate student, Aristide Baratin, for his excellent translation of the thesis abstract. I’d like to acknowledge Paula Domingues, the graduate secretary at McGill, for her tremendous help in dealing with the administrative side of my degree. Of course, this research was only possible because of generous financial support from NSERC, McGill University, and Perimeter Institute. Much of the research for this thesis was carried out at Perimeter Institute, and so I am grateful for their hospitality during this time. On a more personal level, I would like to thank my parents for their steadfast support and encouragement. They gave me the tools and the confidence to succeed, and for that I am eternally grateful. Finally, I thank my wife Nicole, for believing in me and for never once asking why in the world I work in six dimensions.
CONTRIBUTION OF AUTHORS

This thesis contains portions of text from papers where the thesis author is a co-author. Chapter 3 is an amalgamation of the papers [1] and [2]. Portions of these papers to which the thesis author did not contribute have been removed, while the remaining sections represent work to which the thesis author made significant contributions. Chapter 4 contains work from [3], with contributions from the thesis author in the majority of this work. Chapter 5 is taken virtually unaltered from the paper [4], whose primary author is the thesis author. Chapter 6 is also an amalgamation of results for two papers, [5] and [6]. Again, the thesis author is the primary author of these papers.
CONVENTIONS

Throughout this thesis, except where explicitly stated, we use the following convention for the indices. Spacetime coordinates which run over the full dimensionality of space are labeled generically by $x^M$, with $M = 0, 1, 2, 3, 5, 6, \ldots$ (N.B. there is no coordinate $x^4$ in our convention), while coordinates referring to the four large dimensions are labelled by greek indices, $\mu = 0, 1, 2, 3$. As is customary, the coordinate $x$ with no index serves double-duty; at times, it refers to only the four large dimensions, $x^\mu$, while at other times it refers to all spacetime coordinates, $x^M$ (if it is unclear from the context, we write $x^\mu$ or $x^M$ explicitly). To single out the coordinates of the extra dimensions, we use $y^m$ with $m = 5, 6, \ldots$. Tangent space indices will be denoted by capital latin indices from the beginning of the alphabet, $A, B, \ldots$.

Our metric and curvature conventions are the same as in Weinberg [7]. Specifically, our metric signature is \textit{mostly-plus}, while the Riemann tensor is defined with a relative minus sign compared with MTW [8], implying $R^M_{NPQ} = -\partial_P \Gamma^M_{NQ} + \ldots$. We work exclusively in units where $\hbar = c = 1$, while at times we may also set either $\kappa^2 = 8\pi G_N$ or the Boltzmann constant $k$ to unity. At various times, we’ll find it convenient to write the metric as $\hat{g}_{MN} dx^M dx^N = g_{\mu\nu} dx^\mu dx^\nu + \tilde{g}_{mn} dy^m dy^n$. Thus, when necessary to distinguish between quantities built out of $\hat{g}_{MN}$ from those built out of $g_{\mu\nu}$, we use a caret. Similarly, tildes imply the object is built from the internal metric, $\tilde{g}_{mn}$. Finally, we reserve the coordinate $r$ for Gaussian-Normal (GN) gauge, defined such that the ‘internal’ metric is given by $\bar{g}_{mn} = dr^2 + h_{ab} dz^a dz^b$. 

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ABSTRACT

In this thesis we examine the viability of a recent proposal, known as Supersymmetric Large Extra Dimensions (SLED), for solving both the cosmological constant and the hierarchy problems. Central to this proposal is the requirement of two large extra dimensions of size $r_c \sim 10 \mu m$ together with a low value for the higher-dimensional scale of gravity, $M_\ast \sim 10$ TeV. In order not to run into immediate conflict with experiment, it is presumed that all fields of the Standard Model are confined to a four-dimensional domain wall (brane). A realization of the SLED idea is achieved by relying on the 6D supergravity of Nishino and Sezgin (NS), which is known to have 4D-flat compactifications.

When work on this thesis first began, there were many open questions which are now answered either partially or completely. In particular, we expand on the known solutions of NS supergravity, which now include: warped compactifications having either 4D de Sitter or 4D anti-de Sitter symmetry, static solutions with broken 4D Lorentz invariance, and time-dependent “scaling” solutions. We elucidate the connection between brane properties and the asymptotic form of bulk fields as they approach the brane. Marginal stability of the 4D-flat solutions is demonstrated for a broad range of boundary conditions. Given that the warped solutions of NS supergravity which we consider are singular at the brane locations, we present an explicit regularization procedure for dealing with these singularities. Finally, we derive general formulae for the one-loop quantum corrections for both massless and
massive field in arbitrary dimensions, with an eye towards applying these results to NS supergravity.
Cette thèse examine la viabilité d’une approche récente, dite des Dimensions Supplémentaires Larges Supersymétriques (Supersymmetric Large Extra Dimensions, or SLED), qui propose une solution au problème de la constante cosmologique et à celui de la hiérarchie. Un aspect central de cette approche est l’existence de deux dimensions supplémentaires de grande taille $r_c \sim 10 \mu m$, et la faible valeur de l’échelle de gravité, $M_* \sim 10$ TeV. Afin d’éviter un conflit immédiat avec l’expérience, tous les champs du Modèle Standard sont supposés être confinés dans les quatre dimensions observées (i.e. sur une brane). Une implémentation de cette idée de SLED est réalisée par le biais de la supergravité $6D$ de Nishino et Sezgin (NS), dont on sait qu’elle a des compactifications $4D$-plates.

Un certain nombre de questions, laissées ouvertes lorsque cette thèse à débutée, sont à présent partiellement ou complètement résolues. En particulier, nous étendons les solutions connues de la supergravité NS; elle incluent à présent: compactifications déformées ayant la symétrie de Sitter ou anti-de Sitter $4D$, solutions statiques avec invariance de Lorentz $4D$ brisée, et solutions d’échelle (“scaling”) dépendentes du temps. La relation entre les propriétés des branes et la forme asymptotique des champs de bulk lorsqu’ils approchent la brane est mise en lumière et expliquée. La stabilité marginale des solutions $4D$-plate est démontrée pour une large classe de conditions de bord. Etant donné que les solutions déformées de la supergravité NS que l’on considère sont singulières à l’emplacement de la brane, une procédure explicite de régularisation qui traite ces singularités est présentée. Enfin, des formules
générales sont mises en place pour calculer les corrections quantiques à une boucle pour des champs sans et avec masse en toute dimension, avec pour objectif à terme d’appliquer ces résultats à la supergravité NS.
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The unprecedented success of both the particle physics Standard Model (SM) and Einstein’s theory of General Relativity (GR) is both a blessing and a curse. On the one hand, we have now confirmed aspects of the Standard Model to an accuracy its inventors could scarcely have imagined forty years ago. One of its best measured parameters, the fine structure constant $\alpha = 1/137.035\,999\,11(46)$ [10], has been determined from a myriad of diverse experiments, including: precision measurements of the anomalous magnetic moment of the electron, the quantum hall effect, hyperfine splitting, low-energy atomic recoil measurements, high-energy collider experiments, as well as many others [11]. Likewise, GR has withstood intense experimental scrutiny, ranging from solar system tests, where it correctly predicts the precession of Mercury’s orbit [12], to more exotic systems such as binary pulsars [13], where the predicted rate of energy loss through gravitational waves is in excellent agreement with observation [12].

On the other hand, there are compelling reasons to believe that the final theory of everything is not so simple as SM + GR. Physicists thus find themselves in a quandary, knowing current theories are inadequate yet without clear guidance from experiment as to the nature of this final theory. There are hints, however, and these come in the form of “naturalness” problems. As we describe in greater detail later, one such naturalness problem suggests that General Relativity should be modified
at distances smaller than approximately 10 µm, while another advocates for a scale of gravity much lower than the Planck mass. It is the position of this thesis that both of these suggestions should be taken seriously. These ideas are embodied in the *Supersymmetric Large Extra Dimension* (SLED) scenario, where it is assumed that gravity is fundamentally six-dimensional with scale $M_* \sim 10\text{ TeV}$, only appearing four-dimensional at distances much greater than 10 µm. Along with such a dramatic modification comes a large number of experimental signatures, many of which will be tested in upcoming experiments. In fact, it is a great virtue of this proposal that it faces so many nontrivial tests in the near future. If it proves incorrect the damning evidence will come swiftly and from many fronts, but should it succeed it will do so spectacularly.

The remainder of this first chapter is organized as follows. In §1.1.1 we introduce effective field theory and in §1.1.2 define what is meant by a naturalness problem. In §1.2.1 and §1.2.2 we discuss two such naturalness problems: the hierarchy problem and the cosmological constant problem. We then go on to argue in §1.3 that extra spacetime dimensions can help with both of these problems. As in any theory with extra dimensions, we must eventually be concerned with how the theory can *appear* four-dimensional; this is the subject of domain walls, §1.3.1, and Kaluza-Klein reduction, §1.3.2. In §1.4 we introduce in earnest the SLED proposal, highlighting some of its key features such as scale invariance, §1.4.1, large extra dimensions, §1.4.2, and supersymmetry, §1.4.3. Finally, §1.5 provides a brief overview of the subsequent chapters.
1.1 Effective Field Theory

Much of the work in this thesis relies, in some way or another, on concepts from Effective Field Theory (EFT), and so in this section we provide a cursory overview of the subject. For a more thorough treatment, the reader is instead referred to some of the many excellent reviews, e.g. [14, 15].

1.1.1 EFT: An Overview

The credo of effective field theory is that to understand physics at one particular scale, \( m \), we’re not required to understand physics at the scales \( M \gg m \); indeed, if this were not the case it’s hard to imagine how any progress in physics could be made at all! Note that this does not imply that the high-energy physics is completely irrelevant, only that its effect on the low-energy theory is generically manifested through a finite number of coupling constants. This phenomenon is known as decoupling.

The canonical example of an effective field theory is Fermi theory, which describes physics well below the mass the of the \( W \) boson, \( M_W \). Here, the effects of \( W \) and \( Z \) bosons in fermion scattering \( ff \rightarrow ff \) are accounted for by a four-fermion interaction term in the Lagrangian, having coupling constant \( G_F \), and so these bosons themselves no longer appear in the theory. We should expect that this effective theory will be accurate up to corrections of order \( (E_{cm}/M_W)^n \), where \( E_{cm} \) is the centre-of-mass energy for the scattering process, and \( n \) is some integer.\(^1\)

\(^1\) Should we desire a higher accuracy, corresponding to a larger \( n \), we simply include higher and higher dimension operators (e.g. a six-fermion interaction term) together with their associated coupling constants.
If we happen to know the high-energy theory, we can calculate all the low-energy coupling constants in terms of known parameters of the high-energy theory. On the other hand, if we’re ignorant of the high-energy theory, then we must appeal to experiment to determine these couplings.

Continuing in this manner, if we’re interested only in processes with an energy much less than the muon mass, $m_\mu$, then there’s nothing to stop us from repeating the above procedure. In this case, we remove from the Lagrangian all fields with mass greater than or equal to $m_\mu$. As before, in order to account for the effects of the fields we remove, we must add to our Lagrangian possible higher dimension operators involving only low-energy fields.\(^2\) In this way, we arrive at the effective theory of quantum electrodynamics (QED). Schematically, we have the following progression:

\[
\text{Standard Model} \rightarrow \text{Fermi Theory} \rightarrow \text{QED}.
\]

Of course, we could also go directly from the Standard Model to QED. This process of successively removing heavy fields from our theory is referred to as \textit{integrating out} heavy degrees of freedom.

1.1.2 Technical Naturalness

We are now in a position to discuss the concept of \textit{technical naturalness} \cite{16}. Imagine for this purpose that we have a high-energy theory, valid at scale $\Lambda_{\text{HE}}$, which contains a light field $\ell$ having mass $m \ll \Lambda_{\text{HE}}$. Note that in this section, we are careful

\(^2\) As it turns out, QED is a renormalizable EFT, so there are no such higher-dimension operators in this case. This effective theory breaks down at energy $2m_\mu$. 

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to distinguish between the bare mass, $\mu$, which appears in the Lagrangian as $\mu^2 \ell^* \ell$, and the physical mass, $m$, as measured by an experimentalist. At the classical level, these two quantities are the same; however, taking into account radiative correction from quantum mechanics, $\delta \mu^2$, we instead find $m^2 = \mu^2 + \delta \mu^2$.

Should we so desire, we can work with a low-energy theory valid up to an energy scale $\Lambda_{\text{LE}} \gtrsim m$. As explained above, we obtain this effective theory by integrating out heavy particles with mass greater than $\Lambda_{\text{LE}}$. This situation is shown pictorially in fig. 1–1, where heavy particles are represented by large circles, and light particles by small circles. The bare mass of the light field is given by $\mu_1$ in the low-energy theory (EFT$_1$) and by $\mu_2$ in the high-energy theory (EFT$_2$). In the low-energy theory, the physical mass of $\ell$ is calculated to be $m^2 = \mu_1^2 + \delta_1 \mu^2$, while in the high-energy theory one has instead $m^2 = \mu_2^2 + \delta_1 \mu^2 + \delta_2 \mu^2$. Here, $\delta_1 \mu^2$ are radiative corrections from loops of light particles only, while $\delta_2 \mu^2$ are the radiative corrections from loops containing at least one heavy particle.

In EFT$_1$ we say the bare mass $\mu_1$ is natural if it is close to $\Lambda_{\text{LE}}$. The radiative corrections to $\mu_1$ are also expected to be close to the cut-off, $\Lambda_{\text{LE}}$, since generically we have $\delta \mu^2 \sim \mathcal{O}(M^2)$, where $M$ is the mass of the heaviest particle in the theory. We thus expect the physical mass of $\ell$ to be $m \sim \Lambda_{\text{LE}}$. On the other hand, in the high-energy theory, EFT$_2$, we expect that $\delta_2 \mu^2 \sim \Lambda_{\text{HE}}^2$ since $\ell$ will
generically couple to particles with mass of order $\Lambda_{\text{HE}}$. Thus, in order to maintain a small physical mass, we must choose $\mu_2^2$ to be very nearly equal to $-\delta_2^2\mu_2^2$ in order that their sum be $\mathcal{O}(m^2) \ll \Lambda_{\text{HE}}^2$. This is an example of fine-tuning. Alternatively, it could be that there is a symmetry which prevents radiative corrections from being as large as $\mathcal{O}(M^2)$ in which case no fine-tuning is required to maintain the hierarchy $m \ll M$: we say that the hierarchy is technically natural. What’s more, we demand a technically natural understanding of the given hierarchy at any energy scale (e.g. EFT$_1$ or EFT$_2$).

As an example of a technically natural hierarchy, consider the electron in the Standard Model, with mass $m_e \ll M_W$. This hierarchy is technically natural because radiative corrections to the electron’s mass are suppressed by an approximate chiral symmetry,

$$
\psi_e \rightarrow e^{i\alpha\gamma_5}\psi_e,
$$

which is broken by the term $\mu \bar{\psi}_e\psi_e$, where for our purposes here $\mu$ is just a constant (analogous to bare mass discussed earlier). Here, $\alpha = \alpha(x)$ is an arbitrary function and $\gamma_5$ is the usual chirality matrix. Chiral symmetry implies that corrections to the electron mass must vanish as $\mu \rightarrow 0$, from which we conclude that these corrections are proportional to $\mu$. In fact, detailed calculation shows that at one-loop $\delta\mu \propto \mu \ln(M/\mu)$, where $M$ is the mass of some heavy particle. Given that $\mu \ll M$, we see that the radiative corrections also satisfy $\delta\mu \ll M$ and so this hierarchy is technically natural, as claimed.

While the preceding discussion focused on naturalness as it pertains to mass, this can be easily generalized to other couplings. Given an EFT with a cut-off $\Lambda_{\text{UV}},$
any bare coupling can be written as $g_0 \cdot \Lambda_{UV}^n$, where $g_0$ is dimensionless and $n$ is an integer chosen to ensure correct dimensions. If the corresponding physical coupling $g$ is measured and found to satisfy $g \ll 1$, then naturalness requires two conditions: (i) that the smallness of $g_0$ be understood within the microscopic theory, and (ii) that radiative corrections to $g_0$ also be small.

1.2 Two Troubling Naturalness Problems

In this section, we introduce both the hierarchy problem and the cosmological constant problem. Roughly speaking, the hierarchy problem is the question of how the hierarchy $M_{EW} \ll M_{Pl}$ can be technically natural given that the scale of electroweak symmetry breaking, $M_{EW}$, is assumed to be set by the mass of a scalar field, the Higgs, whose radiative corrections should demand that $M_{EW} \sim M_{Pl}$. The cosmological constant problem is the disparity between vacuum energy we expect from calculating loops of heavy particles, and the vacuum energy we actually observe. Both are examples of fine-tuning problems, as we now discuss.

1.2.1 The Hierarchy Problem

The Higgs field is an as-yet-unobserved scalar doublet $\phi$ which is the postulated source of mass within the Standard Model. Its potential is given by

$$V(\phi) = \lambda (\phi^\dagger \phi - \mu^2 / 2\lambda)^2,$$

implying that it has a bare mass $m_{0H}^2 = 2\mu^2$. In the case of the Higgs field, we note that there is no approximate symmetry (analogous to chiral symmetry for fermions) which can protect its mass from receiving large corrections.
The hierarchy problem now stems from two basic facts. First, there is a general consensus among theorists that the Standard Model must be treated as an effective field theory. That the Standard Model is not a complete theory is itself not a surprise, as it already contains a glaring omission — gravity! However, we might assume that it is at least valid up to some intermediate ‘Grand Unified’ (GUT) scale, \( M_{\text{GUT}} \sim 10^{16} \text{GeV} \). Second, in the Standard Model, electroweak symmetry breaking is assumed to occur through a process whereby a scalar field — the Higgs — obtains a vacuum expectation value. As it stands, there is now overwhelming evidence that the electroweak symmetry breaking should occur at the scale \( M_{\text{EW}} \sim 1 \text{TeV} \), and thus the Higgs mass also needs to be \( m_H \lesssim 1 \text{TeV} \) \cite{10}. To see how this leads to a naturalness problem, we imagine calculating the Higgs mass within the high-energy GUT theory. This is the same situation described in the previous section, and so we expect

\[
m_H^2 = m_{0H}^2 + \delta m_H^2 \tag{1.3}
\]

where the radiative corrections \( \delta m_H^2 \) are expected to be of order \( M_{\text{GUT}}^2 \). Given that \( m_H \) is order TeV, we see that the bare mass \( m_{0H}^2 \) must be fine-tuned to order 1 part in \( 10^{26} \).

Notice that the existence of this fine-tuning presupposes that the Higgs description of electroweak symmetry breaking is correct. Since the Higgs field has not yet been observed, it is reasonable to question whether this simple description is really correct. For example, it could be that the Higgs is not a fundamental scalar, as postulated by Technicolor theories \cite{17, 18}. Alternatively, even if the Standard Model Higgs is the correct low-energy description, it could happen that new particles
emerge at the TeV scale which work to suppress the radiative corrections to $m_H^2$, as happens in the Minimal Supersymmetric Standard Model [19].

1.2.2 The Cosmological Constant Problem

We now discuss the cosmological constant which, as we show, also flouts technical naturalness. It is well-known that we are free to include in the classical gravitational action, $S_g$, a constant term, $\Lambda_0$, so that

$$S_g = -\frac{1}{2\kappa^2} \int d^4x \sqrt{g} \left( R + 2\Lambda_0 \right).$$

(1.4)

Of course, $\Lambda_0$ is the (bare) cosmological constant, which all the fuss is about. Here, we include a subscript zero on $\Lambda$ in order to emphasize the analogy with bare mass, discussed in the previous section.

There is nothing particularly threatening about the way $\Lambda_0$ appears in the above equation, and indeed if eq. (1.4) were the whole story, we would not be led to a problem. However, we know that the subatomic world is well-described by Quantum Field Theory, and in particular the Standard Model with a constant spacetime metric, $\eta_{\mu\nu}$. Among other things, this predicts that the zero-point (Casimir) energy density of a Standard Model field is

$$\rho_{\text{SM}} = \langle T_{00} \rangle \sim \Lambda_{\text{UV}}^4,$$

(1.5)

As mentioned already, we consistently follow the conventions in Weinberg [7] and define the Ricci scalar, $R$, such that the curvature of a 2-sphere is negative. Note that this equation also defines $\kappa^2 = 8\pi G_N$, where $G_N$ is Newton’s gravitational constant.

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where \( \Lambda_{UV} \) is the ultraviolet (UV) cut-off where the theory is expected to break down, and \( T_{\mu\nu} \) is the Energy-Momentum tensor for the field in question. Unbroken Lorentz invariance necessarily implies that \( \langle T_{\mu\nu} \rangle \) is proportional to \( \eta_{\mu\nu} \), and so we have \( \langle T_{\mu\nu} \rangle = -\rho_{\text{SM}} \eta_{\mu\nu} \). Coupling this theory to gravity in the usual way, one obtains that the vacuum field equations obey:

\[
G_{\mu\nu} - \Lambda_0 g_{\mu\nu} = -\kappa^2 \langle T_{\mu\nu} \rangle = \kappa^2 \rho_{\text{SM}} g_{\mu\nu}.
\]

Notice that we can define the \textit{physical} cosmological constant,

\[
\Lambda = \Lambda_0 + \kappa^2 \rho_{\text{SM}}
\]

in which case the Einstein equation becomes

\[
G_{\mu\nu} - \Lambda g_{\mu\nu} = 0.
\]

Just as it is physical mass and not the bare mass which is measured in particle physics experiments, it is this physical cosmological constant which is directly measurable from cosmological observations. Evidence from such observations, including the cosmic microwave background (CMB) [20, 21] and Type 1a supernovae [22, 23, 24], implies that \( \Lambda \) is given by

\[
\kappa^{-2}\Lambda \approx (10^{-12} \text{ GeV})^4.
\]

For later discussions, it is useful to define what we call the \textit{vacuum energy density} — also known as \textit{dark energy} — which is simply \( \rho_{\text{vac}} = \kappa^{-2} \Lambda \). Notice that the vacuum energy, \( \rho_{\text{vac}} \), includes contributions from the bare cosmological constant, \( \Lambda_0 \) as well as radiative corrections from all matter fields, \( \rho_{\text{SM}} \).
To understand why this leads to a problem, we note that any reasonable estimate for $\rho_{SM}$ gives an answer many orders of magnitude larger than this observed value for $\Lambda$. For example, an upper bound on this quantity is obtained by taking the cut-off to be the Planck scale, $M_{Pl}$, giving $\rho_{SM} \sim M_{Pl}^4 = (10^{18} \text{ GeV})^4$. A more conservative estimate takes the cut-off to be the scale of electroweak symmetry breaking, $M_{EW}$, in which case we find $\rho_{SM} \sim M_{EW}^4 = (10^3 \text{ GeV})^4$ — still many orders of magnitude larger than the observational bound. In fact, any particle physics scale (except perhaps neutrino masses) inserted into eq. (1.5) gives a contribution which is much larger than the observed vacuum energy, $\rho_{vac}$. Similarly to the hierarchy problem, we must fine-tune parameters in order to enforce a small physical cosmological constant. For example, taking the cut-off as $M_{EW}$, we see that the cancellation between $\Lambda_0$ and $\rho_{SM}$ in eq. (1.6) must be exact to $1 \text{ part in } 10^{60}$! Certainly, it is very unnatural to expect that these two quantities, which have nothing apparently to do with one another, should be so very nearly equal.

Despite the similarities with the hierarchy problem, the cosmological constant problem is worse by far. While the existence of the hierarchy problem relies on what we expect physics to be above the TeV scale, the cosmological constant problem can be phrased in terms of physics we think we understand. For example, we can choose to calculate the cosmological constant within the effective theory of QED. There, we find that electron loops contribute to the vacuum energy an amount of order $m_e^4 \gg \Lambda$. Thus, even in a theory such as QED which we understand well, it is necessary to fine-tune parameters.
1.2.3 Naturalness: A Beacon in the Fog

Our discussion thus far belies the fact that technical naturalness is ubiquitous in nature. In all but a handful of cases, *e.g.* the hierarchy and cosmological constant problem, a large hierarchy of scales can be understood in a technically natural way [25]. (Indeed, if that were not the case, few physicists would lose sleep over fine-tuning.) Even so, the egregiousness of the cosmological constant problem has led some to abandon technical naturalness as a useful principle [26, 27]. This attitude, which also goes under the name of the *Anthropic Principle*, was first legitimized through Weinberg’s famous prediction for the value of the cosmological constant [28]. The anthropic principle has also undergone a recent revival thanks to ideas from string theory [27].

In this thesis we do not take such an attitude; in stark contrast, we hold technical naturalness as our guiding light. The benefit of this approach is immediate, since it allows us to infer that something peculiar must happen at around the dark energy scale, $\rho_{\text{vac}} \sim (10^{-3}\text{ eV})^4$. Treating this scale as a physical cut-off, we see that below $\rho_{\text{vac}}$ the cosmological constant in the corresponding low-energy EFT is naturally the correct order of magnitude. However, if the theory above this scale is the Standard Model together with General Relativity, then we have already seen that fine-tuning problems arise. Given that the Standard Model is tightly constraint below 1 TeV, we conclude that it is General Relativity that must be modified within this range. As we discuss shortly, SLED is perhaps the simplest scenario for how gravity is modified at the dark energy scale. There are other proposals which take a similar tack, for example postulating that gravity is a composite particle [29]. However, we argue that
SLED is the most promising such proposal as it can be realized within a fully-fledged microscopic model which is valid all the way up to the weak scale.

1.3 Extra Dimensions

As mentioned at the start of this chapter, the modification to gravity which SLED proposes is the addition of two large extra dimensions, with Standard Model fields trapped on a four-dimensional surface. Before getting into the details, however, we first pause to motivate why extra dimensions might be useful for solving the two aforementioned problems. Because in SLED gravity is not fundamentally 4D, this in turn implies that neither is the 4D Planck mass, $M_{Pl}$, fundamental. If, as proposed by SLED, the fundamental scale of gravity is instead $M_\ast \sim 10$ TeV, then new physics could begin to appear at around the TeV scale. Accordingly, the physical cut-off for the Standard Model would also be around 1 TeV and so the Higgs mass would be technically natural.

With regard to the cosmological constant problem, recall we showed that 4D Lorentz invariance implies that gravity responds to the zero point energy of a quantum field as if it were a cosmological constant, cf. eq. (1.6). Extra dimensions change this story in two fundamental ways: first, there is obviously no requirement for 6D Lorentz invariance; and second, because Standard Model fields are assumed to be trapped on a lower-dimensional surface, their Casimir energy acts not as a 6D cosmological constant but rather as an energy density which is localized in the extra dimensions. In chapter 2 we show explicitly how these two loopholes are exploited.

Having established the case for extra dimensions, we now discuss some of the more technical details. In §1.3.1 we argue that mechanisms exist to trap the Standard
Model fields on a sub-surface (brane) within a higher-dimensional spacetime, which is an essential component of the SLED proposal. Given that gravity cannot similarly be trapped, we discuss in §1.3.2 how gravity appears to a low-energy observer on a brane and how it can be that this situation is not already ruled out experimentally.

1.3.1 Domain Walls & Branes

The idea that the universe may have more than four dimensions is an old one, attributed to Kaluza [30] and Klein [31]. While this may seem contradictory to our everyday experience, there are in fact two basic mechanisms one can employ in order not to run into conflict with experiment (in general, a given model will employ some combination of these two mechanisms). The simplest is to assume that the extra dimensions are very small, having “compactification radius” $r_c$, in which case the effect of the extra dimensions is negligible at energies much less than $r_c^{-1}$ (this statement is made more precise in §1.3.2). Here, we are assuming that the extra dimensions are universal, meaning fields propagate freely in all dimensions. Within this picture, we must have $r_c^{-1} \gtrsim \text{TeV}$ which is above the energy scale currently probed by colliders.

Alternatively, one can assume that fields are trapped on a lower-dimensional surface (domain wall) while only gravity is free to propagate within the full dimensionality of spacetime. This is the picture suggested by string theory, where these lower-dimensional surfaces are generically referred to a $p$-branes, with $p$ being the spatial dimension of the brane. Notice that because fields can be localized to 3 spatial dimensions, the extra dimensions can be taken to be large (or infinite). We point out that it’s really only necessary to trap the Standard Model fields onto a 3-brane;
“bulk” fields can freely propagate in the large extra dimensions without immediate problems.

**String Theory Branes**

In string theory, Dirichlet $p$-branes are extended objects onto which open strings can end (see [32] for a pedagogical introduction). Thus, string theory provides a natural mechanism with which to localize objects onto a lower-dimensional manifold. It is also true that gravitons in string theory are described by closed strings, and as such they cannot be localized in the same way that open strings can. This should be expected since gravity can be interpreted as the curvature of space, and so intuitively gravitons must be free to propagate in all dimensions.

For the most part, however, we are not concerned with the string theory origin of branes. The main reason for this is that it much more convenient to be able to choose for ourselves brane properties, such as their couplings to the dilaton, gauge fields, etc., as opposed to using those dictated by string theory. Of course, it would be desirable to eventually have a string theory derivation, but for now we content ourselves with this more pragmatic approach.\(^4\)

**Field Theory Domain Walls**

Even within humble field theory it is possible to trap fields on domain walls, which we also refer to as branes although they need not have any assumed connection to string theory (thus, unless otherwise indicated, “domain wall” and “brane” should

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\(^4\) A string theory derivation of the $6D$ supergravity which we focus on in this thesis is given in [33].
be taken as synonymous). For example, we can consider the action

\[ S = \int d^{p+1}x \, d^y \mathcal{L}_{\text{bulk}}(\Phi(x, y)) + \int d^{p+1}x \mathcal{L}_{\text{brane}}(\Phi(x, 0), \phi(x)), \]  

(1.9)

where \( \Phi(x, y) \) and \( \phi(x) \) symbolize any number of bulk fields and brane fields, respectively, and we have used our coordinate freedom to place the \( p \)-brane at the bulk location \( y = 0 \). The simplest action for the brane is one due to pure tension, corresponding to a Lagrangian \( \mathcal{L}_{\text{brane}} = -\sqrt{h} \, T \) with \( h_{ab} \) the induced metric on the brane and \( T \) its tension.

Starting from this action, one can derive equations of motion from the usual variational principle, although slightly more care must be taken because of the explicit boundary term. For the codimension-1 case \( (d = 1) \) in the above action), this procedure leads to solutions which are generically well-defined at \( y = 0 \). However, as we discuss in greater detail in subsequent chapters, the case \( d \geq 2 \) can be problematic.

1.3.2 Kaluza-Klein Theory

Since gravity must propagate in all dimensions, one might wonder how it can be that extra dimensions (of any size and number) are not already ruled out by gravitational experiments. To be more precise, we should ask why low-energy experiments have not ruled out this situation. To answer this question we construct the effective 4D theory relevant to the low-energy observer, which we do in two steps. First, we show that the 6D theory is exactly equivalent to a 4D theory, albeit with an infinite number of particles in the 4D version. Second, we integrate out all but the massless modes of the exact 4D theory to obtain the effective 4D theory. We go through this
calculation in some detail, as the lessons learned will be important for SLED (and, indeed, any other theory with extra dimensions).

To see how this works, consider the $D$-dimensional action

$$S = -\frac{1}{2\kappa^2} \int d^D x \sqrt{\hat{g}} \hat{R}. \tag{1.10}$$

We assume that one of the spatial dimensions, $z$, is compactified to a circle, while indices $a, b, \ldots$ label the other $(D - 1)$ coordinates. Without loss of generality, we may parameterize the metric $\hat{g}_{MN}$ according to

$$\hat{g}_{ab} = e^{2\alpha\hat{\kappa}\phi} g_{ab} + e^{2\beta\hat{\kappa}\phi} \hat{\kappa}^2 A_a A_b, \quad \hat{g}_{az} = e^{2\beta\hat{\kappa}\phi} \hat{\kappa} A_a, \quad \hat{g}_{zz} = e^{2\beta\hat{\kappa}\phi}, \tag{1.11}$$

where $\alpha$ and $\beta (\neq 0)$ are freely chosen constants, while $\phi = \phi(x^M)$ and $A_a = A_a(x^M)$ are unspecified functions. Making the convenient choices $\alpha^{-2} = (D - 2)(D - 3)$ and $\beta = -(D - 3)\alpha$, and ignoring total derivatives, the above action becomes

$$S = -\int d^D x \sqrt{g} \left( \frac{1}{2\kappa^2} R + \frac{1}{4} e^{-2(D-2)\alpha\hat{\kappa}\phi} F_{ab} F^{ab} + \frac{1}{2} g^{ab} \partial_a \phi \partial_b \phi + \cdots \right), \tag{1.12}$$

where $F_{ab} = \partial_a A_b - \partial_b A_a$ is the usual field strength tensor and $R$ is the Ricci scalar constructed from $g_{ab}$. The ellipsis denotes terms which depend on $z$-derivatives.

To obtain the equivalent $(D - 1)$-dimensional version of this theory, we write the $D$-dimensional fields as a Fourier sum of $(D - 1)$-dimensional fields. For example, in the case of the scalar $\phi$ we write

$$\phi(x^a, z) = \sum_{n=-\infty}^{\infty} \phi(n)(x^a) e^{inz/r_c}, \tag{1.13}$$
where $r_c$ is the radius of the circle, so that $0 \leq z < 2\pi r_c$. Our assertion is that fields with $n \neq 0$ represent massive fields in the lower-dimensional theory, and so can be integrated out from the low-energy theory. To justify this claim, consider the simpler toy model of a single massless scalar field which we Fourier decompose as above. When we integrate over the extra dimension, we obtain the result

$$S = -\int d^D x \partial_M \phi \partial^M \phi = -(2\pi r_c) \sum_n \int d^{D-1} x \left[ \partial_a \phi(n) \partial^a \phi(n) + \left( \frac{n}{r_c} \right)^2 \phi^2(n) \right]. \quad (1.14)$$

Indeed, we see that the fields $\phi(n)$ have mass $m(n) = (n/r_c)$ in the lower-dimensional theory.

Performing the analogous Fourier decomposition and integration in the action (1.12), we obtain a $(D-1)$-dimensional theory involving both massive and massless fields, with complicated interactions. Integrating out these massive fields at the classical level amounts to using the equations of motion to eliminate these fields from the action. In the limit where the Kaluza-Klein mass goes to infinity, this amounts to a truncation to the massless sector, in which case we may substitute in the solutions $\phi(n \neq 0) = 0$, $g^{ab}(n \neq 0) = 0$, and $A^a(n \neq 0) = 0$. The result of this procedure is the $(D-1)$-dimensional action

$$S = -\int d^{D-1} x \sqrt{g} \left( \frac{1}{2\kappa^2} R(0) + \frac{1}{4} e^{-4\alpha \kappa \phi(0)} F_{(0)ab} F_{(0)^{ab}} + \frac{1}{2} \partial_a \phi(0) \partial^a \phi(0) \right), \quad (1.15)$$

where the length, $L = 2\pi r_c$, of the extra dimension has been absorbed by a constant field redefinition of $\phi(0)$ and $A^a(0)$, and we include the zero subscript to emphasize that these are massless modes. Notice that we have made the definition $\kappa^2 = \hat{\kappa}^2 / L$, and that the fields depend only on $x^a$. 

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**Take-home Messages**

There are several noteworthy features of the above *Kaluza-Klein reduction*. First, we see that eq. (1.15) must be viewed as an effective action whose physical cut-off is given by the lowest non-zero Kaluza-Klein (KK) mass, $m_{(1)} = 1/r_c$. Above this scale, no effective field theory description exists for the simple reason that there is no large hierarchy of scales (the ratio between two successive KK levels is always $\mathcal{O}(1)$, except between $n = 0$ and $n = 1$). Of course, in more complicated situations the KK spectrum will not be so simple, although the moral will almost always be the same: above the KK scale we forced to work with the higher dimensional theory.

We also note here that it is a typical feature of dimensional reduction that massless fields (besides the graviton) appear in the lower-dimensional theory. Since massless fields can have potentially interesting cosmological implications, a careful treatment of their effects is usually warranted.

We also learn how the fundamental scale of gravity is related to the lower-dimensional scale measured in the effective field theory. For example, specializing to the case $D = 5$, the above analysis shows that the $5D$ Planck mass, $M_5^3 = 1/\kappa^2$, is related to the usual $4D$ Planck mass by $M_5^3 L = M_{Pl}^2$. It is straightforward to generalize this result to the case where there are $d$ extra dimensions. To do so, we use the metric ansatz

$$\hat{g}_{MN} dx^M dx^N = g_{\mu\nu}(x) dx^\mu dx^\nu + \tilde{g}_{mn}(y) dy^m dy^n. \quad (1.16)$$
From this, it follows that the gravitational action reduces to

\[ S_g = -\frac{1}{2} M^2_{*+d} \int d^d y \sqrt{\tilde{g}} \int d^4 x \sqrt{\tilde{g}} R(x) + \cdots = -\frac{1}{2} (M^2_{*+d} V_d) \int d^4 x \sqrt{g} R(x) + \cdots \]

where the ellipses denote \( y \)-dependent quantities. To obtain the second equality we have integrated over the extra dimensions, yielding the \( d \)-dimensional volume factor, \( V_d \). We thus find that the 4D Planck mass is related to \( M_* \) via

\[ M^2_{*+d} V_d = M^2_{Pl}. \]  

(1.17)

In §1.4, we show how this result can be exploited in order to solve — or, at least, rephrase — the hierarchy problem.

1.4 The SLED Proposal

Having covered the necessary background, we are now in a position to introduce the SLED proposal. To start, we discuss scale invariance and the role it plays in ensuring that the 4D effective vacuum energy vanishes classically. We next introduce the Large Extra Dimension (LED) scenario of Arkani-Hamed, et al. [34, 35] which addresses the hierarchy problem. Next, we introduce supersymmetry, motivated by the fact that this symmetry promotes the LED scenario to a possible solution to both the cosmological constant problem and the hierarchy problem.

1.4.1 Scale Invariance in SLED

In this section, we define scale invariance and show why it is important for SLED. A scale transformation is nothing but a rescaling of the metric, so its action on the metric can be represented as

\[ g_{MN} \rightarrow e^\sigma g_{MN} \]  

(1.18)
where $\sigma$ is a constant. In general, other fields may also transform; for example, there is often a scalar field — the dilaton — which shifts by a constant which is proportional to $\sigma$, e.g.

$$\varphi \rightarrow \varphi - \sigma.$$  \hfill (1.19)

Given an action,  

$$S = \int d^Dx \mathcal{L}(g_{MN}, \phi, \ldots),$$

where the ellipsis denotes the other fields in the theory (which may or may not transform under the above scale transformation), we say that the theory is **scale invariant** if the action transforms as $S \rightarrow \text{const.} \times S$. Since equations of motion are unchanged by a constant rescaling of the action, we see that given any solution to the equations of motion a one-parameter family of new solutions can be generated by application of a scale transformation.

In the context of SLED, we now ask what scale invariance implies for the vacuum energy in the 4D effective theory, whose calculation was considered in §1.3.2. To proceed, we need to evaluate the on-shell action, $S^{\text{cl}}$, obtained by evaluating the action assuming all fields obey their classical equations of motion. Scale invariance allows $S^{\text{cl}}$ to be written in the following equivalent ways:

$$S^{\text{cl}} = \int d^Dx \mathcal{L}(g^{\text{cl}}_{MN}, \phi^{\text{cl}}, \ldots) = e^{-\omega_B \sigma} \int d^Dx \mathcal{L}(e^{\sigma} g^{\text{cl}}_{MN}, \phi^{\text{cl}} - \sigma, \ldots),$$  \hfill (1.20)

where the last equality assumes $S$ transforms to $e^{\omega_B \sigma} S$ under a scale transformation. Here, $\omega_B$ is a constant which is easily determined once the functional form of the action is given (the actions we encounter in later chapters have $\omega_B = 2$). Notice that we consider here the case where spacetime has no boundaries; the general result including boundaries is a straightforward generalization [36]. Noting that the middle equality in eq. (1.20) is independent of $\sigma$, we differentiate this equation with respect
to $\sigma$ and then set $\sigma = 0$ to obtain the result

$$0 = -\omega_B S^{\text{cl}} + \int d^D x \left[ \frac{\delta S^{\text{cl}}}{\delta g^e_{MN}} g^{e}_{MN} - \frac{\delta S^{\text{cl}}}{\delta \phi^e} + \cdots \right].$$

(1.21)

Notice that each term in the above integral individually vanishes as the fields are assumed to satisfy the classical equations of motion. Thus, we obtain the important result $\omega_B S^{\text{cl}} = 0$, which for $\omega_B \neq 0$ implies that the on-shell action, $S^{\text{cl}}$, vanishes. In the case where there are boundary branes $b_i$ — whose actions transform under a scale transformation as $S_{b_i} \rightarrow e^{\omega_{b_i} \sigma} S_{b_i}$ — we again find $S^{\text{cl}} = 0$ provided $\omega_B = \omega_{b_i} \neq 0$.

To see what this result implies for the vacuum energy in the 4D effective theory, we consider integrating out all massive modes. As described earlier, at the classical level this is accomplished by using the classical equations of motion to eliminate these modes from the action. For the purpose of calculating the 4D vacuum energy, it suffices to consider the case where all fields are independent of $x^\mu$, in which case we find

$$\rho_{\text{eff}} = \int d^d y L^{\text{cl}} \propto S^{\text{cl}} = 0.$$  

(1.22)

Thus, we see that unbroken scale invariance ensures that the 4D vacuum energy vanishes. One might nonetheless worry that since scale invariance is a symmetry of the equations of motion and not of the action, we should expect this “symmetry” to be broken by quantization. Indeed, it is the task of supersymmetry (discussed shortly) to ensure that the scale of breaking is given by the supersymmetry breaking scale, which is typically small in the cases of interest.
1.4.2 Why Large Extra Dimensions?

We have already seen that a natural solution to the cosmological constant problem suggests that gravity is fundamentally modified at the scale $\rho_{\text{vac}}$ — a fact that can be easily accommodated by having extra dimensions with size $\rho_{\text{vac}}^{-1}$. We now show that the hierarchy problem can also be solved using large extra dimensions. Remarkably, for the case of two extra dimensions, the size of the dimensions required to solve the hierarchy problem is also $\rho_{\text{vac}}^{-1}$.

The LED scenario relies on the observation made in §1.3.2 that the higher-dimensional scale of gravity is related to the 4D Planck mass by the equation\(^5\)

$$M^{2+d}V_d = M_{\text{Pl}}^2. \quad (1.23)$$

This equation suggests a novel solution to the hierarchy problem: assume $M_* \sim$ TeV! Given that the $M_{\text{Pl}} \sim 10^{18}$ GeV, eq. (1.17) implies that the size of the extra dimensions is very large relative to $M_*^{-1}$. We see that the LED scenario solves the hierarchy problem in a trivial way, since it assume that there is only one fundamental scale, $M_{\text{EW}}$. To be fair, the hierarchy problem has not really disappeared, but rather it has been transmuted to the question of whether large extra dimensions, *viz.* $M_*^d V_d \gg 1$, can be technically natural.

Substituting in numbers, one finds that for one extra dimension its size must be $r_c \sim 10^{13}$ cm $\sim$ 1 AU in order to have TeV-scale gravity. Not surprisingly, this

\(^5\) Among other things, this result assumes that our four large dimensions are not strongly “warped”, a concept introduced in §2.3.
situation is empirically ruled out by solar system tests. On the other hand, for $d = 2$ one finds that the choice $M_* \sim 10 \text{ TeV}$ leads to extra dimensions of size $r_c \sim 10 \mu m \sim (10^{-2} \text{ eV})^{-1}$. Again, one can ask whether this situation is observationally excluded, although here the answer is less obvious.

We know that Standard Model processes are sensitive to the dimensionality of space, and since the ($4D$) Standard Model has been confirmed down to a distance of roughly TeV$^{-1}$ its fields must be confined to a $4D$ brane. Even though Standard Model fields are confined to a brane, there can still be observable signatures in collider experiments since these fields must interact with gravity, which is not similarly confined. Thus, there will be “missing energy” signatures coming from decays of Standard Model particles into unobserved Kaluza-Klein modes of the graviton, for example

$$e^+ + e^- \rightarrow \gamma + G_{KK}.$$ 

The low scale of quantum gravity, $M_*$, also implies that virtual graviton exchange could lead to dangerous corrections to well-measured Standard Model processes. Considerations such as these lead to a collider physics bound $M_* \gtrsim 1 \text{ TeV}$ [37, 38, 39].

Because gravity is not trapped on a brane, we must ask whether precision tests of gravity have already ruled out large extra dimensions. As shown in appendix A, for $d$ extra dimensions of size $r_c$ the gravitational force between objects separated by a distance $r \gg r_c$ obeys the usual inverse square law (ISL), varying as $1/r^2$; at distances $r \ll r_c$, this relationship changes to $1/r^{2+d}$. Current tests of the ISL imply the extra dimensions must have a size $r_c \lesssim 100 \mu m$ [40].
More thorough treatments of other experimental signatures can be found elsewhere, for example [35] in the case of LED and [41] in the case of SLED. These references also include cosmological and astrophysical bounds (one of the strongest, coming from SN1987a, implies $r_c \lesssim 10 \, \mu m$ [42, 43]). Recalling that SLED posits two extra dimensions of size $r_c \sim 10 \, \mu m$ and a fundamental scale of gravity $M_* \sim 10 \, \text{TeV}$ we see that, while still within experimental bounds, the SLED proposal teeters dangerously close to the boundary; whether it falls into the realm of fact or fiction will soon be determined by upcoming experiments.

### 1.4.3 Why Supersymmetry?

Supersymmetry is a spacetime symmetry relating bosons and fermions, and although it has not yet been observed to occur in nature, there are good reasons for believing it soon will be. As we now argue, one of its main virtues is that it allows for a naturally small vacuum energy. To see why this might be so, we recall that the generator $Q$ of a global supersymmetry transformation obeys

$$\{Q, Q^\dagger\} = \sigma_\mu P^\mu,$$

where $P^\mu$ is the energy-momentum operator and $\sigma_i$ are the Pauli matrices (with $\sigma_0 \equiv 1$). It is then observed that for a state, $|\psi\rangle$, which preserves supersymmetry — implying $Q|\psi\rangle = Q^\dagger|\psi\rangle = 0$ — we have

$$\langle \psi | P^\mu | \psi \rangle = 0,$$

and so $|\psi\rangle$ is necessarily a state with zero energy and momentum. In particular, if the vacuum is supersymmetric then it also has zero energy.
Unfortunately, the lack of observational evidence for superpartners in collider experiments suggests that supersymmetry, if it exists, is broken at the TeV scale or higher. In this case, while the vacuum energy is not zero, it is suppressed by factors of \((m_{sb}/M)^2\), where \(m_{sb}\) is the supersymmetry breaking scale and \(M\) is some appropriate heavy mass scale. This suppression is easily understood since the vacuum energy must vanish in the limit \(m_{sb} \rightarrow 0\). In the usual 4D picture, we thus find that radiative corrections to the cosmological constant are \(M^2 m_{sb}^2\) instead of \(M^4\). Unfortunately, given that we must take \(m_{sb} \gtrsim \text{TeV}\), this contribution is still much too large to allow for a technically natural solution to the cosmological constant problem.

Crucially, SLED does not rely on supersymmetry on the brane to suppress corrections to the localized energy density on the brane (a.k.a. brane tension). Instead, the suppression mechanism occurs through the bulk’s classical response to such a localized energy density, which is to curve the two extra dimensions as opposed to the four large dimensions we observe.\(^6\) As argued in §1.4.1, the fact that the 4D space remains flat is a consequence of the classical scale invariance of SLED. We now see how quantum effects alter this picture.

\(^6\) Obviously, this ‘off-loading of curvature’ to the bulk is only possible within extra dimensional theories.
In SLED, the quantum corrected 6D action can be written as an expansion in powers of the small 6D curvature [9]. Schematically, we have

\[ \mathcal{L}_{\text{eff}} = -\sqrt{g} \left[ c_0 M_6^6 + c_1 M_4^4 \mathcal{R} + c_2 M_6^2 \mathcal{R}^2 + c_3 \mathcal{R}^3 + \cdots \right] \]  

(1.26)

where the \( c_i \) are dimensionless parameters which can depend at most logarithmically on \( M_6 \), and the expression \( \mathcal{R}^n \) stands for all possible Lorentz invariant contractions of terms containing \( 2n \) derivatives; for example, \( \mathcal{R}^2 \) denotes terms such as \( R^2, R_{MN}R^{MN} \), etc. Given this quantum effective action, we ask how the classical result \( \rho_{\text{eff}} = 0 \) is modified. With this goal in mind, notice that the \( c_0 \) term corresponds to a renormalization of the 6D cosmological constant while the \( c_1 \) term renormalizes \( M_6 \); the remaining terms represent new interactions. Ignoring for now these higher-derivative interactions, we see that the quantum corrected action differs from the bare classical action only by a renormalization of its couplings. To the extent that the vanishing of \( \rho_{\text{eff}} \) is not dependent on these couplings taking specific values, then we should also find in the quantum-corrected action that \( \rho_{\text{eff}} = 0 \).

Fortunately, we don’t expect that these higher order terms should all vanish, though the success of SLED hinges on having \( c_2 = 0 \). To see why, we estimate \( \rho_{\text{eff}} \) by using the explicit solution of chapter 2 — where we find \( \sqrt{g} \sim \mathcal{R}^{-1} \sim r_c^2 \) — implying that the leading contribution to 4D vacuum energy coming from eq. (1.26) is

\[ \rho_{\text{eff}} \sim c_2 M_6^2 r_c^{-2} \sim (10 \text{ KeV})^4. \]

While this is an enormous improvement over the usual cosmological constant problem, it still necessitates a fine-tuning of 1 part in \( 10^{28} \). As it turns out, in cases where explicit calculations are known [6] supersymmetry can often ensure the desired result \( c_2 = 0 \). In this case, we see that the leading order
contribution to the 4D vacuum energy is given by \( r_c^{-4} \sim (10^{-2}\text{eV})^4 \), which is the right order of magnitude to account for the observed dark energy.

1.5 Thesis Summary

Before closing this chapter, we summarize the direction we take in this thesis towards determining the viability of the SLED proposal. As we have argued already, scale invariance places a crucial role in SLED since it ensures that the effective 4D vacuum energy vanishes at the classical level. It is therefore appropriate to ask what brane properties are required in order to maintain this classical scale invariance. As it turns out, for the case of a 3-brane we should demand that the brane not couple to a particular bulk field, the dilaton. More generally, we see that it’s important to identify, first of all, what choices we must make for the brane couplings in order to have 4D-flat solutions, and secondly, whether these choices are UV stable. Part of the goal of this thesis is therefore to identify how changing various brane couplings affects both the bulk spacetime properties and the properties intrinsic to the brane. Work towards this goal is begun in chapter 2, and then continued at various points in later chapters. Most of the next chapter is devoted to a review of 6D supergravity.

Since one of the eventual goals of SLED is to provide a realistic cosmology, it is important to expand upon the catalogue of known solutions to 6D supergravity to include solutions with 4D de Sitter geometry as well as to include time-dependent solutions. Thus, we devote ourselves in chapter 3 to finding new solutions to the 6D supergravity field equations. A related issue which we study in chapter 4 is the stability of a very general class of 4D-flat solutions. It is not surprising that stability turns out to rely crucially on the brane properties. Given that so much
depends on the brane properties, in chapter 5 we attempt to make more precise this
dependence. However, as already alluded to, there are difficulties when we work with
codimension-2 branes as bulk fields generally tend to diverge at the brane locations.
To make progress, in this chapter we choose to regulate these infinities by modeling
the codimension-2 brane as a small codimension-1 brane. Finally, since the success or
failure of SLED depends crucially on the bulk radiative corrections are small enough,
we study this question in chapter 6. Chapter 7 lists our conclusions as well as possible
areas of future research.
CHAPTER 2

6D Nishino-Sezgin Supergravity

We have seen in the previous chapter that the SLED proposal relies critically on supersymmetry, for otherwise we could not expect the vacuum energy induced from bulk loops to be small enough to account for the dark energy observed today. Accordingly, any concrete model espousing the principles of SLED must itself be rooted within a 6D supergravity framework. While there are many such supergravities to choose from, we adopt the one written down by Nishino and Sezgin [44].

Before getting lost in calculation, we draw attention to a few details which might otherwise lead to confusion. The gauge group for this supergravity minimally contains a $\text{U}(1)_R$ symmetry (i.e. $\mathcal{G} = \text{U}(1)_R \times \cdots$) whose gauge coupling $g_1$ appears explicitly in the potential for a set of charged hyperscalars. Given a generic group the theory is anomalous, however there are a number of anomaly free choices [45, 46, 47, 48]. For the most part, until we study the ultraviolet behaviour of this supergravity in chapter 6, we do not need to be explicit about which anomaly-free group we choose. Furthermore, there are a number of known 4D compactifications of this theory where the 2 compact dimensions are supported by a background monopole configuration; however, we do not necessarily assume that the gauge field whose condensate acquires the monopole configuration corresponds to the $\text{U}(1)_R$ symmetry. The gauge coupling for this background monopole field will generically be denoted by $e$. 

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2.1 Nishino-Sezgin Supergravity

We now introduce the $\mathcal{N} = 1$ supergravity of Nishino and Sezgin [44]. The bosonic field content of this model consists of a vielbein, $e_M^A$; an antisymmetric Kalb-Ramond tensor, $B_{MN}$; gauge fields, $A^a_M$, falling in the adjoint representation a group $\mathcal{G}$; hyperscalars, $\Phi^\alpha$, where $\alpha = 1 \ldots 4n_H$; and a dilaton, $\varphi$. The fermionic fields are the gravitino, $\psi^i_M$, the gaugini, $\lambda^i$, the dilatino, $\chi^i$ — all $Sp(1)$ Majorana-Weyl spinors with $i = 1 \ldots 2$ — and the hyperini, $\psi^{\hat{\alpha}}$, which are $Sp(n_H)$ Majorana-Weyl with $\hat{\alpha} = 1 \ldots 2n_H$. The gravitino and gaugini are right-handed spinors, satisfying $\Gamma_7 \psi_M = \Psi_M$ and $\Gamma_7 \lambda = \lambda$, while the dilatino and hyperini are left-handed, $\Gamma_7 \chi = -\chi$ and $\Gamma_7 \psi = -\psi$.

These fields are grouped into supersymmetric multiplets, according to

- gravity: $(e_M^A, \psi^i_M, B^+_{MN})$
- tensor: $(B^-_{MN}, \chi^i, \varphi)$
- vector: $(A^a_M, \lambda^{ai})$
- hypermatter: $(\Phi^\alpha, \psi^{\hat{\alpha}})$

where the superscripts on $B^+_{MN}$ and $B^-_{MN}$ indicate whether the field strengths they give rise to are self-dual or anti-self-dual, respectively. In the model we consider, these fields only occur in the combination $B^+_{MN} + B^-_{MN}$.

The gauge group $\mathcal{G}$ is a product of simple groups that includes a $U(1)_R$ gauged R-symmetry. Because the fermions are chiral this gauge group is restricted by anomaly considerations; however, these anomalies can be cancelled via the Green-Schwarz

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mechanism [49]. In the case where there are \( n_V \) vector multiplets and \( n_H \) hypermultiplets, anomaly cancellation requires that \( n_H = n_V + 244 \). For example, one such anomaly-free model [45] corresponds to the choice \( \mathcal{G} = E_7 \times E_6 \times U(1)_R \), and so the gauge fields transform in the representation \((78, 133, 1)\). The anomaly constraint then tells us that we must have \( n_H = 212 + 244 = 456 \) hypermultiplets, which can be assigned to the \( 912 \) of \( E_7 \subset Sp(456) \). There are now many other consistent models which have also been discovered [46, 48, 47].

The supergravity action we work with has a bosonic action given by

\[
e^{-1} \mathcal{L}_B = - \frac{1}{2\kappa^2} R - \frac{1}{2\kappa^2} \partial_M \varphi \partial^M \varphi - \frac{1}{2} G_{\alpha\beta}(\Phi) D_M \Phi^\alpha D^M \Phi^\beta \\
- \frac{1}{12} e^{-2\varphi} G_{MNP} G^{MNP} - \frac{1}{4} e^{-\varphi} F^a_{MN} F^a_{MN} - e^\varphi v(\Phi).
\]  

Here, \( G_{\alpha\beta}(\Phi) \) is the metric on the target manifold of the hyperscalars, \( F^a_{MN} \) is the usual field strength for the gauge fields, while \( G_{MNP} \) is the field strength of the Kalb-Ramond field, including the usual Chern-Simons terms. The detailed functional form of the hyperscalar potential, \( v(\Phi) \), is not important for our purposes; it is enough to know that it’s minimized for \( \Phi = 0 \) with value \( v(0) = 2\kappa^{-4}g_1^2 \), where \( g_1 \) is the gauge coupling corresponding to the \( U(1)_R \) symmetry.
The portion of the Lagrangian bilinear in fermions is given by

$$\begin{align*}
e^{-1} \mathcal{L}_F &= -\bar{\psi}_M \Gamma^{MNP} D_N \psi_P - \bar{\chi} \Gamma^M D_M \chi - \bar{\lambda} \Gamma^M D_M \lambda \\
&\quad - \frac{1}{2} \partial_M \varphi \left( \bar{\chi} \Gamma^N \Gamma^M \psi_N + \bar{\psi}_N \Gamma^M \Gamma^N \chi \right) \\
&\quad + \frac{\kappa e^{-\varphi}}{12 \sqrt{2}} G_{MNP} \left( -\bar{\psi}_R \Gamma_{[R} \Gamma^{MNP} \Gamma_{S]} \psi^S + \bar{\psi}_R \Gamma^{MNP} \Gamma^R \chi \right) \\
&\quad - \bar{\chi} \Gamma^R \Gamma^{MNP} \psi_R + \bar{\chi} \Gamma^{MNP} \chi - \bar{\lambda} \Gamma^{MNP} \lambda \\
&\quad - \frac{\kappa e^{-\varphi/2}}{4} F_{MN} \left( \bar{\psi}_Q \Gamma^M \Gamma^N \lambda + \bar{\lambda} \Gamma^Q \Gamma^M \psi_Q - \bar{\chi} \Gamma^M \psi_Q + \bar{\lambda} \Gamma^M \lambda \right) \\
&\quad + \frac{i g_1 e^{-\varphi/2}}{\kappa} \left( \bar{\psi}_M \Gamma^M \lambda + \bar{\lambda} \Gamma^M \psi_M + \bar{\chi} \lambda - \bar{\lambda} \chi \right). \quad (2.2)
\end{align*}$$

The supersymmetry transformations for this Lagrangian are

$$\begin{align*}
\delta e^A_M &= \frac{\kappa}{\sqrt{2}} \left( \bar{\epsilon} \Gamma^A \psi_M - \bar{\psi}_M \Gamma^A \epsilon \right) \\
\delta \varphi &= - \frac{\kappa}{\sqrt{2}} \left( \bar{\epsilon} \chi + \bar{\chi} \epsilon \right) \\
\delta A^a_M &= \frac{e^{\varphi/2}}{\sqrt{2}} \left( \bar{\epsilon} \Gamma^a M \lambda^a - \bar{\lambda}^a \Gamma^a_M \epsilon \right) \\
\delta B_{MN} &= \sqrt{2} \kappa A^a_{[m} \delta A^a_{n]} + \frac{e^{\varphi}}{2} \left( \bar{\epsilon} \Gamma^a M \psi_N - \bar{\psi}_N \Gamma^a M \epsilon \right) \\
&\quad \quad \quad - \bar{\epsilon} \Gamma^a_N \psi_M + \bar{\psi}_M \Gamma^a_N \epsilon - \bar{\epsilon} \Gamma^a_{MN} \chi + \bar{\chi} \Gamma^a_{MN} \epsilon \right) \quad (2.3) \\
\delta \chi &= \frac{1}{\sqrt{2} \kappa} \partial_M \varphi \Gamma^M \epsilon + \frac{e^{-\varphi}}{12} G_{MNP} \Gamma^{MNP} \epsilon \\
\delta \lambda &= \frac{e^{-\varphi/2}}{2 \sqrt{2}} F_{MN} \Gamma^M \epsilon - \sqrt{2} i \frac{g_1}{\kappa^2} \frac{e^{\varphi/2}}{\epsilon} \\
\delta \psi_M &= \sqrt{2} D_M \epsilon + \frac{e^{-\varphi}}{24} G_{PQR} \Gamma^{PQR} \Gamma_M \epsilon ,
\end{align*}$$

1 We ignore the hypermatter from now on.
where we have set to zero the hypermultiplets.\textsuperscript{2} The supersymmetry transformation to focus on here is the one for the dilatino, $\chi$. Throughout this thesis, we focus on the case where the antisymmetric tensor vanishes, and so the dilatino transformation tells us that supersymmetry is completely broken whenever we consider a background where the dilaton is not constant.

For the most part, we will be interested in the bosonic part of the action only. However, when discussing the UV properties of this model in chapter 6 it will be necessary to know the full Lagrangian.

\subsection*{2.2 Unwarped Solutions}

Many explicit compactifications of the above field equations to four dimensions have been constructed over the years, starting 20 years ago with the Salam-Sezgin spherical solution \cite{50}. These now include compactifications to flat 4\textit{D} space on unwarped, rugby-ball solutions \cite{9}, as well as warped axially-symmetric internal dimensions having conical \cite{51} and more general \cite{52, 53, 54} singularities at the positions of two source branes. More recent generalizations have also found configurations for which the hyperscalars and 3-form fluxes are nontrivial \cite{55}.

We now go on to discuss some of the various solutions to Nishino-Sezgin supergravity of the previous section. We will be concerned only with the case where the hyperscalars vanish, in which case the field equations following from the bosonic

\footnote{We correct for a factor of $\kappa$ in the transformation law for $\lambda$ which was missing in \cite{50}.}
An important feature of these equations is their classical scaling property. This property states that given any solution to these equations another can be obtained by making the replacement

\[ e^\varphi \rightarrow e^{\varphi - \sigma} \quad \text{and} \quad g_{MN} \rightarrow e^\sigma g_{MN}, \]

with all other fields held fixed. This property follows quite generally from the fact that the action, eq. (2.1), scales under this transformation according to \( S \rightarrow e^{2\sigma} S \) [56].

### 2.2.1 Spherical Compactification

One of the simplest solutions to these equations was written down by Salam and Sezgin [50], and corresponds to the gauge group \( G = U(1)_R \). While this choice of gauge group leads to anomalies, this can easily be corrected by considering a larger group. In any case, none of the details of this section depend on this subtlety. The
metric takes a direct product form, where the internal two dimensions are compactified to a sphere, with metric \( r_c^2(d\theta^2 + \sin^2 \theta d\phi^2) \), while the four-dimensional metric is flat Minkowski space. There are several noteworthy features of this solution. First of all, as we will see, the dimensionally reduced theory preserves \( \mathcal{N} = 1 \) supersymmetry, and contains chiral fermions. The internal space is supported by a monopole gauge field configuration, whose flux is quantized.

This solution is obtained by taking the dilaton to be constant, and setting to zero the Kalb-Ramond field. Maximal symmetry of the noncompact dimensions ensures that \( F_{\mu M} = 0 \), and so we can immediately see that the Maxwell equation is solved for \( F_{mn} = f \epsilon_{mn} \), where \( \epsilon_{mn} \) is the volume form for the sphere and \( f \) is a constant. Inspection of the dilaton equation then shows that the dilaton is related to the flux \( f \) via

\[
f^2 = \frac{4g_1^2}{\kappa^4} e^{2\varphi}
\]  

while the Einstein equation shows that the radius of the sphere, \( r_c \), and the dilaton are related by

\[
r_c^2 e^\varphi = \frac{\kappa^2}{4g_1^2},
\]  

from which one can also derive that \( f = n/2g_1r_c^2 \), with \( n = \pm 1 \) the monopole number.

**Flux Quantization**

The flux quantization condition is a consequence of the nontrivial topology of the compact space. To see how it arises, note that the solution to the Maxwell equation can be written

\[
\partial_\theta A_\phi = f(r_c^2 \sin \theta).
\]
For positive (negative) $f$, $\partial_\theta A_\phi$ is therefore a strictly increasing (decreasing) function on $S^2$, and so in particular $A_\phi$ cannot be zero at both poles of the sphere. This is problematic as the field strength is then singular.

The cure is well-known, and involves defining the gauge field on two separate coordinate patches such that the field strength is non-singular in each patch, and such that in the overlap the two definitions differ by at most a gauge transformation. Explicitly, we define

$$A_\phi^{(N)} = fr_c^2(1 - \cos \theta) \quad \text{and} \quad A_\phi^{(S)} = fr_c^2(-1 - \cos \theta) \quad (2.9)$$

in the ‘north’ and ‘south’ hemisphere, respectively, while in the overlap region one finds $A_\phi^{(N)} - A_\phi^{(S)} = 2fr_c^2$. This difference indeed corresponds to the gauge transformation $\delta A_\phi = e^{-1}\partial_\phi \Lambda$ if we take $\Lambda = (2efr_c^2)\phi$, where $e$ is the gauge field coupling constant. To see now how flux quantization appears, assume there exists a field $\psi$ with charge $e$ under this gauge transformation, so that $\psi \rightarrow e^{i\Lambda}\psi$. Requiring $e^{i\Lambda}\psi$ to be single-valued under $\phi \rightarrow \phi + 2\pi$ immediately gives the quantization condition

$$2efr_c^2 = N \quad (2.10)$$

with $N$ an integer. In the case at hand, we see that this equation is trivially satisfied for $N = 1$ as we have already found that $f = 1/2g_1r_c^2$ and $e = g_1$ (since $\mathcal{G} = U(1)_R$ in the Salam-Sezgin solution). However, when we consider larger gauge groups, it becomes possible for the background monopole to lie in a gauge group other than the $U(1)_R$, and so in general $e \neq g_1$. This equation will be generalized further when we consider the ‘rugby-ball’ solutions.
Supersymmetry of the Solution

To see whether the spherical solution leaves unbroken any supersymmetry, it is enough to check that the supersymmetry variations, eq. (2.3), of the fermionic fields vanish. We see immediately that the variation of dilatino, $\chi$, is trivially zero, as we have assumed constant dilaton and vanishing Kalb-Ramond field. Requiring that the variation of gaugino vanishes gives the constraint

$$ (\Gamma^{56} - i\eta) \epsilon = 0. \quad (2.11) $$

To proceed, we adopt a specific representation of the Dirac gamma matrices given by

$$ \Gamma_\mu = \gamma_\mu \times \sigma_1, \quad \Gamma_5 = \gamma_5 \times \sigma_1, \quad \Gamma_6 = 1 \times \sigma_2, \quad \gamma_5^2 = 1, \quad (2.12) $$

where these matrices satisfy the Clifford algebra, $\{\Gamma_M, \Gamma_N\} = 2\eta_{MN}$, and $\sigma^i$ are the Pauli matrices. Note that we use $\Gamma_M$ to denote the eight-dimensional representation of the Clifford algebra, and $\gamma_\mu$ to denote the usual four-dimensional representation. From the above equations, we see that

$$ \Gamma_{56} = \gamma_5 \times i\sigma_3, \quad \text{and} \quad \Gamma_7 = \Gamma_0 \Gamma_1 \cdots \Gamma_6 = 1 \times \sigma_3. \quad (2.13) $$

For reference, the Pauli matrices are

$$ \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. $$
From the given chirality of the fermions and the supersymmetry transformations, we have that \( \Gamma_7 \epsilon = \epsilon \), or equivalently\(^4\) from eq. (2.13), \( \sigma_3 \epsilon = \epsilon \). Thus, we find that the constraint (2.11) becomes

\[
(\gamma_5 - n) \epsilon = 0. \tag{2.14}
\]

Performing the decomposition \( \epsilon = \epsilon_R + \epsilon_L \), with \( \gamma_5 \epsilon_R = \epsilon_R \) and \( \gamma_5 \epsilon_L = -\epsilon_L \), we see that depending on the sign of \( n \), we can satisfy the above equation either for \( \epsilon_R \) or \( \epsilon_L \), but not both. Note also that had \( |n| \) not been unity, then supersymmetry would necessarily have been broken.

The final supersymmetry transformation to check is the one for the gravitino, which tells us that \( \epsilon \) must be covariantly constant,

\[
D_M \epsilon = (\partial_M + \frac{1}{2} \omega_{M56} \Gamma^{56} - ig_1 A_M) \epsilon = 0. \tag{2.15}
\]

Writing \( A_M = \frac{n}{2g_1} e_M \), we have \([57]\) \( \omega_{M56} = \epsilon_M \), and so the above equation becomes

\[
\left[ \partial_M + \frac{1}{2} \left( \Gamma^{56} - i n \right) \right] \epsilon = 0. \tag{2.16}
\]

Comparing with eq. (2.11), we immediately find the condition \( \partial_M \epsilon = 0 \). This completes the proof that the spherical compactification preserves one supersymmetry, with constant supersymmetry parameter \( \epsilon_R \) or \( \epsilon_L \).

\(^4\) In the following, we no longer write explicitly the appropriate identity matrices.
Low Energy Spectrum

We go on to briefly describe the spectrum of this theory, well below the Kaluza-Klein compactification scale $m_{KK} \sim 1/r_c$. The dilaton, $\varphi$, and the metric radion field, $r_c$, combine to form a massless field, $s = r_c^2 e^{-\varphi}$, and a massive field, $t = r_c^2 e^{\varphi}$. The mass of $t$ is inversely proportion to $s$, and so can be parametrically smaller than the Kaluza-Klein scale [58]. There is a massless field corresponding to the unbroken Kalb-Ramond symmetry, $\delta B_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$, and this dualizes to a 4D scalar, $b_0$. Unbroken $SO(3)$ invariance, corresponding to the isometries of the internal metric, gives rise to three massless spin-1 fields. The field $A_\mu$ acquires a mass through its Chern-Simons coupling, although this mass is also suppressed by $1/s$ and so can appear in the low energy theory.\footnote{Performing the Kaluza-Klein expansion with $B_{mn} = b(x)\epsilon_{mn}$, one can show that $b$ is the Goldstone boson which gets eaten by the gauge field.} The graviton remains massless, as ensured by unbroken 4D Poincaré invariance. For the fermions, unbroken $\mathcal{N} = 1$ supersymmetry implies that the gravitino, $\psi_\mu$, is massless. There are similarly four massless spin-1/2 fields which correspond to $\psi_m$. The dilatino, $\chi$, and the gaugino, $\lambda$, both remain massless.

2.2.2 Rugby Ball

An interesting generalization of the spherical compactification can be obtained by including codimension-2 sources at the two poles of the sphere. This corresponds to supplementing the Nishino-Sezgin action with the following (non-supersymmetric)
boundary action

\[ S_b = \sum_{i=1}^{2} \int_{y_i} d^4x \, e_4 \, e^{\lambda_i \varphi} T_i, \]  

(2.17)

where \( e_4 = \sqrt{-h}, \) \( h \) being the determinant of the induced metric at \( y_i \), \( T_i \) are the constant brane tensions, and \( \lambda_i \) are coupling constants.

This boundary action modifies the dilaton and Einstein equations by including localized source terms, which become

\[ e_6 \left[ \kappa^{-2} \Box \varphi + \frac{1}{6} e^{-2\varphi} G^2 + \frac{1}{4} e^{-\varphi} F^2 - 2\kappa^{-4} g_1^2 e^{\varphi} \right] = \sum_i e_4 \lambda_i \, e^{\lambda_i \varphi} T_i \delta^2(y - y_i), \]  

(2.18)

and

\[ e_6 \left[ \kappa^{-2} (R_{MN} + \partial_M \varphi \partial_N \varphi) + G_{\alpha\beta}(\Phi) D_M \Phi^\alpha D_N \Phi^\beta + \frac{1}{2} e^{-2\varphi} G_{MPQ} G_N^{PQ} \right. \]
\[ + \left. e^{-\varphi} F_{MP}^a F_N^{a\rho} - \left( \frac{1}{12} e^{-2\varphi} G^2 + \frac{1}{8} e^{-\varphi} F^2 - \kappa^{-4} g_1^2 e^{\varphi} \right) g_{MN} \right] \]
\[ = \sum_i e_4 \left( g_{\mu\nu} \delta_M^\mu \delta_N^\nu - g_{MN} \right) e^{\lambda_i \varphi} T_i \delta^2(y - y_i). \]  

(2.19)

In the rugby-ball solution, the dilaton’s coupling to the brane is turned off, so that \( \lambda_i = 0 \) in the above formulae. This choice simplifies the analysis considerably since the fields now have smooth limits at the poles of the sphere. In this case the dilaton equation is the unchanged, since the right-hand side of eq. (2.18) is zero as before.

Decomposing the two-dimensional internal Ricci tensor into a smooth and a singular

\[ \int d^2y \delta^2(y) = 1 \] for any region of integration containing the point \( y = 0 \). Otherwise, the integral vanishes.  

\[ \text{41} \]
part according to \( R_{mn} = R_{mn}^{(sm)} + R_{mn}^{(\infty)} \), we find the boundary conditions

\[
e_2 R_{mn}^{(\infty)} \bigg|_{y_i} = -g_{mn} \kappa^2 T_i \delta^2 (y - y_i). \tag{2.20}
\]

at the poles, while our ansatz is that all other fields remain finite there. In order to understand what this boundary condition implies, we digress briefly to discuss conical spaces.

**Conical Spaces**

Since many of the solutions we encounter have metrics which are approximately conical in certain regimes, we now discuss the properties of such conical spaces. Consider the 2D metric \( ds^2 = dr^2 + (1 - \epsilon)^2 r^2 d\phi^2 \), where \( \phi \) has period \( 2\pi \). For \( \epsilon = 0 \) this is simply flat space in polar coordinates, but otherwise there is a conical singularity at the origin. To see this, note that a circle centered at the origin has circumference \( 2\pi r (1 - \epsilon) \), which is the same answer one obtains for a cone with deficit angle \( \delta = 2\pi \epsilon \). It is also easy to check that the above metric has vanishing Riemann tensor everywhere, except possibly at \( r = 0 \) where the coordinate system breaks down. In fact, the Ricci scalar is proportional to a delta function, and is given by

\[
e_2 R = -4\pi \epsilon \delta^2 (y). \tag{2.21}
\]

Furthermore, in two dimensions we know that the Ricci tensor is proportional to the metric, from which we obtain

\[
e_2 R_{mn} = -2\pi \epsilon g_{mn} \delta^2 (y). \tag{2.22}
\]
The Explicit Solution

The rugby-ball solution is obtained from the spherical one by removing a ‘wedge’ of constant deficit angle, $\delta$, and then identifying the edges, as shown in figure 2–1. In the bulk, the only effect of this modification is that the coordinate $\phi$ now has period $(2\pi - \delta)$. Thus, the spherical solution found earlier, eqs. (2.6)–(2.8), still applies in the bulk region while at the poles we must ensure that the boundary condition (2.20) is properly accounted for. Comparing this boundary condition with the result (2.22), we find that the brane tension at the poles is related to the deficit angle by

$$\delta = \kappa^2 T.$$  \hspace{1cm} (2.23)

Note that because the deficit angle is necessarily the same at both poles, the brane tensions must also be identical.

With this last result in hand, we can now calculate the 4$D$ effective vacuum energy which, at the classical level, is obtained by substituting the classical equations...
of motion into the action. Doing so, it is a straightforward exercise to show that

$$\rho_{\text{eff}} = -\frac{1}{2} \int d^2 y e_2 \Box \phi.$$  \hspace{1cm} (2.24)

Thus, we find that in the rugby-ball solution where the dilaton is constant, a low-energy observer would measure zero vacuum energy. We point out that the brane tensions are automatically cancelled by the singular parts of the Ricci scalar, which is why they don’t appear in the above equation (this cancellation must carry through regardless of the brane tension).

While we have managed to solve all equations of motion, including boundary conditions, there is one extra topological constraint — the flux quantization condition derived in §2.2.1 — which must be satisfied. It is trivial to generalize this condition to the case where \(\phi\) has period \(2\pi(1-\epsilon)\). In this case, one finds

$$2e_f r_c^2 = \frac{N}{1 - \epsilon}.$$  \hspace{1cm} (2.25)

This immediately leads to a problem in the Salam-Sezgin compactification where \(G = U(1)_R\) and so the background monopole necessarily lies in the \(U(1)_R\) gauge group. This implies that \(e = g_1\), and so the left-hand side of the above equation is unity, while the right-hand is necessarily greater than one. The solution to this problem is to consider a larger gauge group, with the monopole lying in some other factor of the algebra. In fact, as already discussed, we are forced to consider a larger gauge group in order to cancel anomalies.
Anomaly-Free Version

As a specific example, assume we have $G = E_6 \times E_7 \times U(1)_R$, for which the resulting theory is free of anomalies. If the monopole lies in the algebra of $E_6$, then we have $e = g_6$ and the above equation implies

$$\frac{g_6}{g_1} = \frac{N}{1 - \epsilon}.  \quad (2.26)$$

This can be rewritten as a condition on the allowed tension, giving

$$\left(1 - \frac{\kappa^2 T}{2\pi}\right) = N \left(\frac{g_1}{g_6}\right).  \quad (2.27)$$

In this solution, however, it turns out that supersymmetry is broken. To see this, note that the variation $\delta \chi$ vanishes as before since the dilaton is constant and the hyperscalars vanish. Similarly, the variation $\delta \lambda$ leads to the same constraint as before, eq. (2.11). On the other hand, because the gravitino is not charged under $E_6$, the vanishing of the covariant derivative $D_M \epsilon$ leads to a new constraint

$$0 = \left[\partial_M + \frac{1}{2} \Gamma^{56} e_M\right] \epsilon = \left[\partial_M + \frac{in}{2} e_M\right] \epsilon.  \quad (2.28)$$

where the second equality follows from (2.11), and we recall $e_M \propto A_M$. Given that $A_\theta = 0$, the $\theta$-component of this equation tells us that $\epsilon$ is independent of $\theta$. Since $A_\phi = A_\phi(\theta)$, this is inconsistent with the $\phi$-component of the equation, unless $\epsilon = 0$. Hence, we find that supersymmetry is broken.
**Low Energy Spectrum**

In order to deduce the low energy spectrum coming from the above (anomaly-free) rugby-ball solution, we first need to be explicit about how the symmetry breaking occurs. For this purpose, we follow [45] and take the monopole to lie in the $U'(1)$ factor of $SO(10) \times U'(1) \subset E_6$. As before, we argue on symmetry grounds what the low energy theory should be. There are several noteworthy differences from the spherical Salam-Sezgin compactification described earlier. The most obvious is that supersymmetry is broken and so the 4D gravitino will be massive. Also broken is the $SO(3)$ isometry of the internal dimensions, which is reduced to a $U(1)$ symmetry. Thus, we find one massless and two massive gauge fields, whose mass is set by the deficit angle, $\epsilon$, and so can be parametrically smaller than the KK scale. As before, the gauge monopole will be massive, with a mass which can also be small compared with the KK scale.

Notably, the scaling symmetry is not broken, since the brane Lagrangian transforms under the scaling symmetry as $L_b \rightarrow e^{2\sigma} L_b$, which is the same as the bulk Lagrangian. The scalar sector of the low energy theory will therefore be the same as before. The 4D graviton will be massless as before, as will be the gauge fields corresponding to the unbroken $E_7 \times SO(10) \times U(1)_R$. The only massless fermions are contained among the gauginos of $E_6$, which fill out $2|N|$ families of $SO(10)$.

**2.3 Non-Supersymmetric Warped Solutions**

There have been a number of papers dealing with warped solutions to the bosonic sector of the 6D supergravities described above. We present here the most general solution found by Gibbons, Güven, and Pope (henceforth known as GGP), subject
to the ansätze that the noncompact four dimensions are maximally symmetric, and that the the internal dimensions are axially-symmetric [52]. In fact, without much more work, we can generalize the arguments of GGP to the case where their are \( n \) maximally symmetric large dimensions, while the internal space is \( d \)-dimensional [1]. The total dimension of space is then \( D = n + d \).

The \( D \)-dimensional bosonic action we use in this section is an obvious generalization of eq. (2.1). The only nontrivial additional feature is that the coupling of the dilaton to the \( p_r \)-form field strength is chosen such that term proportional to \( g_{MN} \) in the Einstein equation is \( \Box \phi \). Our starting action is then

\[
S = -\int d^D x \sqrt{-g} \left[ \frac{1}{2\kappa^2} g^{MN} \left( R_{MN} + \partial_M \varphi \partial_N \varphi \right) + \frac{1}{2} \sum_r \frac{1}{(p_r + 1)!} e^{-p_r \varphi} F^2_r + v e^\varphi \right],
\]

(2.29)

where \( v \) is a dimensional constant. The fields \( F_r \) are the \((p_r + 1)\)-form field strengths for a collection of \( p_r \)-form gauge potentials, \( A_r \), and \( F^2_r = F_{M_1 \cdots M_{p_r+1}} F^{M_1 \cdots M_{p_r+1}} \). When \( v = 0 \) this is sufficiently general to encompass the bosonic parts of a variety of higher-dimensional, ungauged supergravity Lagrangians (for example, see [59]). Of course, for \( v \neq 0 \) the dilaton potential has the form found in Nishino-Sezgin supergravity.

The field equations obtained from this action are:

\[
\Box \varphi - \kappa^2 v e^\varphi + \kappa^2 \sum_r \frac{p_r}{2(p_r + 1)!} e^{-p_r \varphi} F^2_r = 0 \quad \text{(dilaton)}
\]

\[
\nabla_M \left( e^{-p_r \varphi} F^M_{rN\cdots Q} \right) + \text{(CS terms)} = 0 \quad \text{\((p_r\)-form\)}
\]

\[
R_{MN} + \partial_M \varphi \partial_N \varphi + \kappa^2 \sum_r \frac{1}{p_r!} e^{-p_r \varphi} \left[ F^2_r \right]_{MN} + \frac{2}{D-2} \left( \Box \varphi \right) g_{MN} = 0 \quad \text{(Einstein)},
\]

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where ‘(CS terms)’ denotes terms arising from any Chern-Simons terms within the definition of $F(r)$, and we define

$$[F^2]_{MN} = F_M^{\rho_1...R}F_{NP...R}. \quad (2.31)$$

We now follow essentially the same steps as GGP, and in so doing we arrive at the same conclusion as they (albeit, generalized to arbitrary dimensions). In particular, it’s shown that the assumptions of maximally symmetry of the non-compact dimensions and smooth internal manifold is enough to ensure the non-compact manifold is flat. To proceed, we take our metric ansatz to be the one consistent with the above assumptions. Namely,

$$d s^2 = \delta_{MN} d x^M d x^N = W(y)^2 g_{\mu \nu}(x) d x^\mu d x^\nu + \delta_{mn}(y) d y^m d y^n, \quad (2.32)$$

where $g_{\mu \nu}$ is an $n$-dimensional maximally-symmetric metric and $\delta_{mn}$ a generic $d$-dimensional metric. As always, we use the convention that hats denote objects constructed from the full $D$-dimensional metric $\delta_{MN}$, while tildes denote objects constructed from the metric $\delta_{mn}$. Tensors without hats or tildes are constructed from the metric $g_{\mu \nu}$.

Using the metric ansatz, eq. (2.32), we may write

$$\hat{R}_{\mu \nu} = R_{\mu \nu} + \frac{1}{n} \left[ W^{2-n} \hat{\nabla}^2 W \right] g_{\mu \nu}, \quad (2.33)$$

$$\hat{R}_{mn} = \hat{R}_{mn} + n W^{-1} \hat{\nabla}_m \partial_n W \quad (2.34)$$

$$\hat{\Box} \phi = \Box \phi + n \delta_{mn} \partial_m \phi \partial_n \ln W \quad (2.35)$$
where $\tilde{\nabla}^2 = \tilde{g}^{mn}\tilde{\nabla}_m\tilde{\nabla}_n$. Since maximal symmetry implies $R_{\mu\nu} = (R/n)g_{\mu\nu}$ and $F_{\mu M N \ldots R} = \partial_{\mu}\varphi = 0$, these equations allow the $(\mu\nu)$-Einstein equation to be simplified to

$$\frac{1}{n} W^{m-2} R = -\tilde{\nabla}_p \left[ W^n \tilde{g}^{pq} \partial_q \left( \ln W + \frac{2\varphi}{D-2} \right) \right]. \quad (2.36)$$

The significance of eq. (2.36) is most easily seen once it is integrated over the compact $d$ dimensions and Gauss’ Law is used to rewrite the right-hand side as a surface term:

$$\frac{1}{n} \int_M d^d y \sqrt{\tilde{g}} W^{m-2} R = -\sum_\alpha \int_{\Sigma_\alpha} d^{d-1} y \sqrt{h} N_p \left[ W^n \tilde{g}^{pq} \partial_q \left( \ln W + \frac{2\varphi}{D-2} \right) \right], \quad (2.37)$$

where $N_m$ is an outward-pointing normal, with $\tilde{g}^{mn}N_mN_n = 1$, and $h$ is the induced metric on the surface $\Sigma_\alpha$. (If time is one of the $d$ dimensions then the surface terms must include spacelike surfaces in the remote future and past, for which $\tilde{g}^{mn}N_mN_n = -1$.) If there are no singularities or boundaries in the dimensions being integrated then the right-hand side vanishes, leading to the conclusion that the product $W^{m-2} R$ integrates to zero. Since $R$ is constant and $W^{m-2}$ is strictly positive, this immediately implies $R = 0$, as concluded for $6D$ in ref. [52].

### 2.3.1 Uniqueness of Salam-Sezgin Vacuum

For the remainder of this chapter, we focus on $6D$ solutions where $n = 4$ and $d = 2$. Following GGP, we now show that the only singularity-free solution with maximal $4D$ symmetry and a smooth compact internal space is the Salam-Sezgin vacuum solution, $M^4 \otimes S^2$, discussed in §2.2. We have already shown that $R = 0$ in this case, and so eq. (2.36) gives

$$\tilde{\nabla}^m \left[ W^4 \partial_m (2 \ln W + \varphi) \right] = 0. \quad (2.38)$$
Multiplying by \((2 \ln W + \varphi)\) and then integrating over the internal space gives

\[
\int d^2 y \sqrt{\tilde{g}} W^4 |\nabla (2 \ln W + \varphi)|^2 = 0, \tag{2.39}
\]

where we have integrated by parts. From this, we immediately find

\[
\varphi = -2 \ln W, \tag{2.40}
\]

up to an irrelevant constant which corresponds to the classical scaling symmetry of the action, cf. eq. (2.5). The equation of motion for the gauge field can be easily integrated, giving

\[
F_{mn} = q W^{-6} \epsilon_{mn}, \tag{2.41}
\]

where \(\epsilon_{mn}\) is the volume form of the compact space and \(q\) is a constant of integration. These last two solutions, together with eqs. (2.30) and (2.34), allow us to write the \((mn)\)-Einstein equation as

\[
\tilde{R}_{mn} + 2 W^{-2} \tilde{D}_m \partial_n W^2 + \frac{3}{4} \kappa^2 q^2 W^{-10} g_{mn} + \frac{1}{2} \kappa^2 v W^{-2} \tilde{g}_{mn} = 0. \tag{2.42}
\]

Taking the trace, and integrating over the compact space gives

\[
\chi = - \frac{1}{4 \pi} \int d^2 y \sqrt{\tilde{g}} \tilde{R} = \frac{1}{4 \pi} \int d^2 y \sqrt{\tilde{g}} \left( \frac{4 |\nabla W|^2}{W^2} + \frac{3}{2} \kappa^2 q^2 W^{-10} + \kappa^2 v W^{-2} \right) \tag{2.43}
\]

where \(\chi\) is the Euler number. Noting that \(v \geq 0\) in the models we consider, we find that \(\chi\) must be positive. By the assumption that \(Y\) is compact, non-singular, and complete, we find we must take \(\chi = 2\), and so \(Y\) is topologically a sphere. Finally,
the trace-free part of the \((mn)\)-Einstein equation yields

\[
\tilde{\nabla}_m \partial_n W^2 = \frac{1}{2} (\tilde{\nabla}^2 W^2) \tilde{g}_{mn}
\] (2.44)

from which it can be easily verified that \(K^m = \epsilon^{mn} \partial_n W^2\) is a Killing vector, and moreover it is orthogonal to the level sets of \(W\) and \(\varphi\). Thus, we learn that \(Y\) must have axial symmetry.

In the next section, we discuss the most general axisymmetric solutions having flat four dimensions. There, we will see that the only non-singular solution occurs when \(Y\) is a sphere, thus completing the proof.

### 2.3.2 General Axisymmetric Solutions

We now go on to discuss the most general 4\(D\)-flat axially symmetric solutions to the supergravity equations, eqs. (2.4), with \(v = 2\kappa^{-4} g_r^2\). Given these assumptions, we take as our metric ansatz

\[
ds^2 = W^2(\eta) \eta_{\mu\nu} dx^\mu dx^\nu + A^2(\eta) W^8(\eta) d\eta^2 + A^2(\eta) d\psi^2,
\] (2.45)

where \(x^\mu\) label the four noncompact dimensions, and \(\{\eta, \psi\}\) are coordinates in the two extra dimensions. The coordinate \(\psi\) satisfies the periodicity condition \(0 \leq \psi \leq 2\pi \sqrt{\kappa}\), while we will find the range of \(\eta\) to be \(-\infty < \eta < \infty\). (The reason we include a factor of \(\sqrt{\kappa}\) in the definition of the ‘angular’ coordinate \(\psi\) is to ensure that the dimensions work out correctly.)

Note that we use our freedom to redefine the radial coordinate \(\eta\) in order to choose the function which premultiplies \(d\eta^2\) in the line element. At times, however,
we will find it convenient to choose a different radial coordinate,

$$\text{d}r^2 = A^2 \mathcal{W}^8 \text{d}\eta^2. \quad (2.46)$$

This is nothing more than Gaussian-Normal (GN) gauge, which is defined by the requirement that the radial coordinate measure proper distance to the origin. We use the convention throughout that $r$ is reserved for calculations performed in Gaussian-Normal gauge, while the use of $\eta$ implies the former gauge choice.

Given the above metric ansatz, we now write the ordinary differential equations which determine the unknown functions $\varphi$, $A$, $W$, and $F_{\eta\psi}$. Starting with the Maxwell equation, we note that maximal symmetry of the noncompact space ensures that the only nonzero component of the field strength is $F_{\eta\psi}$. This equation then reduces to $(e^{-\varphi} F_{\eta\psi}/A^2)' = 0$, where primes denote differentiation with respect to $\eta$. Integrating gives

$$F_{\eta\psi} = q A^2 e^{\varphi}, \quad (2.47)$$

where $q$ is an integration constant. So, in particular we find that $F_{MN} F^{MN} = 2q^2 e^{2\varphi}/W^8$.

Using $\Box \varphi = \varphi''/(A^2 W^8)$ the equation of motion for the dilaton becomes

$$\varphi'' + \frac{\kappa^2}{2} q^2 A^2 e^\varphi - 2\kappa^{-2} g_1^2 A^2 W^8 e^\varphi = 0. \quad (2.48)$$
Finally, the Einstein equations are obtained using the following expression for the nonzero components of the Ricci tensor:

\[
R_{\mu\nu} = \frac{1}{A^2W^8} \left[ WW'' - (W')^2 \right] \eta_{\mu\nu} \\
R_{\psi\psi} = \frac{AA'' - (A')^2}{A^2W^8} \\
R_{\eta\eta} = \frac{1}{A^2W^2} \left[ AW^2A'' + 4A^2WW'' - W^2(A')^2 - 8AWA'W' - 16A^2(W')^2 \right].
\]

Two of the corresponding Einstein equations become

\[
(\mu\nu) : \quad \frac{W''}{W} - \frac{(W')^2}{W^2} + \frac{1}{2} \varphi'' = 0 \quad (2.50)
\]

\[
(\psi\psi) : \quad \frac{A''}{A} - \frac{(A')^2}{A^2} + \kappa^2 q^2 A^2 e^\varphi + \frac{1}{2} \varphi'' = 0 \quad (2.51)
\]

while use of the \(\eta\eta\) component of the Einstein tensor

\[
G_{\eta\eta} = -\frac{2}{AW^2} \left[ 2WA'W' + 3A(W')^2 \right], \quad (2.52)
\]

allows the third to be written

\[
(\eta\eta) : \quad -\frac{4A'W'}{AW} - \frac{6(W')^2}{W^2} + \frac{1}{2} \left( \varphi' \right)^2 + \frac{\kappa^2}{2} q^2 A^2 e^\varphi - 2\kappa^{-2} g_1^2 A^2 W^8 e^\varphi = 0. \quad (2.53)
\]

Solving the field equations gives\(^7\) the following formulae for the unknown functions \(A(\eta)\) and \(W(\eta)\) [52]

---

\(^7\) Beware that ref. [52] instead uses \(\kappa^2 = \frac{1}{2}\), while their dilaton is related to ours by \(\varphi_{GGP} = -2\varphi\).
\[ W^4 = \frac{\kappa^2 q \lambda_2}{2 g_1 \lambda_1} \cosh[\lambda_1 (\eta - \xi_1)] \cosh[\lambda_2 (\eta - \xi_2)] \]
\[ A^{-4} = \frac{2 \kappa^2 g_1 q^3}{\lambda_1 \lambda_2} e^{-2 \lambda_3 \eta} \cosh[\lambda_1 (\eta - \xi_1)] \cosh[\lambda_2 (\eta - \xi_2)] \]

while \( e^{-\varphi} = W^2 e^{\lambda_3 \eta} \) and \( F_{\eta \psi} = \frac{q A^2}{W^2} e^{-\lambda_3 \eta} \). \hfill (2.54)

Here \( q, \lambda_i (i = 1, 2, 3) \) and \( \xi_a (a = 1, 2) \) are arbitrary integration constants, subject only to the constraint

\[ \lambda_2^2 = \lambda_1^2 + \lambda_3^2 \] \hfill (2.55)

which follows from eq. (2.53). Notice that the signs of both \( \lambda_1 \) and \( \lambda_2 \) are irrelevant in these solutions, and so without loss of generality we take \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \). Also, since in all subsequent equations it is only the magnitude of \( g_1 \) which appears, we simplify notation by writing \( g_1 \) in lieu of \( |g_1| \). For future reference, we note here the mass dimension of the various constants (where square brackets are defined here such that [mass] = 1)

\[ [\lambda_i] = 1 \quad [\xi_a] = -1 \quad [g_1] = -1 \quad [q] = 3 \quad [\kappa] = -2 \]. \hfill (2.56)

It is also useful to display here the form of the gauge potential, \( A_M \), whose differentiation gives the above field strength, \( F_{\eta \psi} \):

\[ A_\psi = \frac{\lambda_1}{\kappa^2 q} \left( \tanh [\lambda_1 (\eta - \xi_1)] + \alpha \right), \] \hfill (2.57)

where \( \alpha \) is an arbitrary integration constant.
Note that there are only six independent integration constants (which we can take to be $\lambda_1$, $\lambda_2$, $\xi_1$, $\xi_2$, $q$, and $\alpha$), whereas we should expect seven since we have solved four second-order differential equations subject to one constraint equation. The missing seventh integration constant corresponds to the classical scaling symmetry discussed earlier. Recall that by this symmetry we can generate a new solution to the equations of motion by shifting the dilaton, $\varphi \rightarrow \varphi + \sigma$ and similarly rescaling the metric, $g_{MN} \rightarrow e^{-\sigma} g_{MN}$. Since these solutions are in some sense trivially different, we have set $\sigma = 0$ for simplicity.\(^8\)

Before concluding this section we pause to point out that, as usual, there is a flux quantization constraint that must be satisfied. If the solution for the gauge field is to be non-singular in the patches containing $\eta = \pm \infty$, then we need to choose $\alpha = \mp 1$ respectively. The same argument as before then implies that the gauge coupling $e$ is quantized according to

$$\frac{2\lambda}{\kappa^{3/2} q} = \frac{N}{e}.$$  \hspace{1cm} (2.58)

### 2.3.3 Singularities of the GGP Solutions

From the explicit solution, eqs. (2.54), we see that there are at most two singularities—as expected since we have assumed axial symmetry—whose locations can be at $\eta = \pm \infty$. It is useful to change to Gaussian-Normal gauge, where we define $r_\pm$ to be the proper distance from the singularities at $\eta = \pm \infty$. In the limit

\(^8\) Note, however, that we will be forced to reintroduce this integration constant when we study the regularized solutions in chapter 5.
\( \eta \to \pm \infty \), eq. (2.46) can be easily integrated to give [53]

\[
    r_{\pm} = c_{\pm} \exp \left[ -\frac{1}{4} (5\lambda_2 - \lambda_1 \mp 2\lambda_3) |\eta| \right],
\]

(2.59)

where \( c_{\pm} \) are constants which don’t concern us. It is important to note that \( \Delta_{\pm} = (5\lambda_2 - \lambda_1 \mp 2\lambda_3) \) is manifestly positive,\(^9\) as required to make sense of the above definition of \( r_{\pm} \). Had this not been the case, we would have learned that the proper distance between singularities was infinite.

Rewriting the GGP solutions using the proper distance coordinate, we find the asymptotic behaviour \( W \sim r^{\omega_{\pm}} \), \( A \sim r^{\beta_{\pm}} \) and \( \varphi \sim \zeta_{\pm} \ln r \), where we now omit the subscripts on \( r \) as it is clear from the exponent to which singularity we refer. Explicitly, these coefficients are

\[
    \omega_{\pm} = \frac{\lambda_2 - \lambda_1}{5\lambda_2 - \lambda_1 \mp 2\lambda_3}, \quad \beta_{\pm} = \frac{\lambda_2 + 3\lambda_1 \mp 2\lambda_3}{5\lambda_2 - \lambda_1 \mp 2\lambda_3} \quad \text{and} \quad \zeta_{\pm} = -\frac{2(\lambda_2 - \lambda_1 \mp 2\lambda_3)}{5\lambda_2 - \lambda_1 \mp 2\lambda_3}.
\]

(2.60)

Several things are of note here. First, the warp factor \( W \) remains bounded since we have \( \omega_{\pm} > 0 \), which follows from \( \lambda_2 > \lambda_1 \). It is also immediately obvious from this expression that the metric is non-singular at \( r_{\pm} = 0 \) only if \( \lambda_1 = \lambda_2 \equiv \lambda \). In this special case we also find \( \beta_{\pm} = 1 \), and so we see that the 2-dimensional internal

\(^9\) To see this, we need only show that \( 5\lambda_2 > \lambda_1 \pm 2\lambda_3 \). If the rhs is negative, then this is satisfied as \( \lambda_2 \) is positive. If the rhs is positive, then squaring and using the constraint eq. (2.55) results in the inequality \( 20(\lambda_1^2 + \lambda_3^2) + (2\lambda_1 \mp \lambda_3)^2 > 0 \), which is necessarily true.
metric,
\[ ds^2_{(2)} \sim dr^2 + \left( 1 - \frac{\delta_{\pm}}{2\pi} \right)^2 r^2 d\theta^2 \]

is conical. Here, the angular coordinate \( \theta \) ranges from \( 0 \leq \theta < 2\pi \) and the deficit angle, \( \delta_{\pm} \), is found to be
\[
\delta_{\pm} = 2\pi \left[ 1 - \left( \frac{2\lambda g_1}{|q|\kappa^{3/2}} \right) e^{\pm\lambda(\xi_1-\xi_2)} \right].
\]

Using the result found earlier which relates the deficit angle to the brane tension, \( \delta_{\pm} = \kappa^2 T_{\pm} \), we can use the above equation to rewrite the topological constraint, eq. (2.58), in terms of a constraint among brane tensions,
\[
\left( 1 - \frac{\kappa^2 T_+}{2\pi} \right) \left( 1 - \frac{\kappa^2 T_-}{2\pi} \right) = N^2 \left( \frac{g_1}{e} \right)^2.
\]

In the special case \( T_+ = T_- \), this reduces to the formula found in the rugby-ball solution. While this constraint may appear like a fine-tuning between brane tensions, it must be kept in mind that this equation is a topological constraint since it is nothing but a rewriting of eq. (2.58). However, topological constraints are always technically natural since they are invariant under continuous processes like renormalization [25].

We now proceed to discuss the three possible cases, corresponding to the number of singularities in the solutions.

**Zero Singularities**

As already discovered, the GGP solutions can be singularity-free only if \( \lambda_1 = \lambda_2 \equiv \lambda \). Furthermore, inspection of eq. (2.62) shows that in order for there to be no
conical singularities — corresponding to $\delta_\pm = 0$ — we must have $\xi_1 = \xi_2$ and

$$q = \frac{2\lambda g_1}{\kappa^{3/2}}. \quad (2.64)$$

In this case, both $\xi_1$ and $\xi_2$ can be gauged away by a shift in $\eta$, and so they are no longer physical parameters. We see that $q$ is now related to $\lambda$, which is now the only physical integration constant remaining.\textsuperscript{10} Using eq. (2.46), it is now straightforward to put the $2D$ metric into the standard form for a 2-sphere, and we find that the sphere’s radius, $r_c$, is related to $\lambda$ via

$$r_c^2 = \frac{\kappa^{1/4} \sqrt{\lambda}}{4g^2}. \quad (2.65)$$

This completes the proof, begun in §2.3.1, of the uniqueness of the Salam-Sezgin vacuum.

**One Singularity**

We start with the observation that there can be one singularity only if $\lambda_1 = \lambda_2 = \lambda$. As already noted, any singularities in this case are conical, with deficit angle given by eq. (2.62). From this equation, we see that even if $\xi_1 \neq \xi_2$, we can still tune parameters such that one of the conical singularities disappears. For example, imposing that there be no singularity at $\eta = -\infty$ gives

$$|q| = \left( \frac{2\lambda g_1}{\kappa^{3/2}} \right) e^{\lambda(\xi_2 - \xi_1)}. \quad (2.66)$$

\textsuperscript{10} Recall, however, that we ignore the integration constant corresponding to the scaling symmetry.
For obvious reasons, this will be referred to as a ‘teardrop’ geometry. We will make use of this particular solution in chapter 5 when we study the regularization of the codimension-2 branes.

**Two Singularities**

In the case where \( \lambda_1 = \lambda_2 \) and \( \xi_1 = \xi_2 \), but \( q \) is not chosen to satisfy eq. (2.64), then we recover the rugby-ball solution introduced earlier. On the other hand, when \( \lambda_1 \neq \lambda_2 \) (or, equivalently \( \lambda_3 \neq 0 \)), we see that there are necessarily two singularities, neither of which is conical. These solutions can be characterized in two broad classes, depending on the sign of \( \beta_\pm \) [53]. In the first case, both \( \beta_+ \) and \( \beta_- \) are positive and so, given that the angular part of the metric near these singularities behaves as \( r^{2\beta_\pm} \), we see that the circumference of circles centred on these singularities vanish as \( r \to 0 \). These singularities can be naturally interpreted as being sourced by codimension-2 branes. However, we have already seen that the branes cannot be pure tension, as this would result in conical singularities. It will be the pursuit of chapter 5 to discover what type of brane energy-momentum tensor sources such geometries. Alternatively, it is possible that \( \beta_+ \) and \( \beta_- \) have opposite signs. As noted in [53], however, these cannot both be negative (this can be shown by considering the definition of \( \beta_\pm \), eq. (2.60), together with eq. (2.55)). At the singularity satisfying \( \beta < 0 \), we see that circles which surround this singularity have their circumference grow without bound as they approach it. Thus, in such solutions one dimension decompactifies within a finite proper distance. To make sense of such a geometry, it will likely be necessary to cut-off the space by surrounding the singularity with a codimension-1 brane.
CHAPTER 3
New Solutions to 6D Supergravity

In this chapter, several new solutions to the supergravity equations of motion are presented. As a warm-up, in §3.1 we first identify the general asymptotic form taken by the bulk fields in the immediate vicinity of any source branes, with an eye to its use in eq. (2.37) of the previous section. We are able to keep our analysis quite general by arguing that these asymptotic forms are given by powers of the distance from the source for codimension > 2 (or possibly logs for codimension-2) with the powers determined by explicitly solving the bulk equations. Assuming these equations are dominated near the branes by the contributions of the kinetic terms they may be integrated quite generally, leading to solutions corresponding to Kasner-like [60] near-brane geometries. Given these solutions the validity of the assumption that kinetic terms dominate can be checked \textit{a posteriori}. Our arguments closely resemble similar arguments used long ago [61, 62] to identify the time-dependence of spacetimes in the vicinity of space-like singularities.

In §3.2, we then find solutions which exhibit maximal 4D symmetry. In the case where the 4D metric is either de Sitter or anti-de Sitter, then part of the solution cannot be written in closed-form, although we show that this solution is simple to integrate numerically. The results of this integration are shown in Figures 3–1 and 3–2.
Finally, we present a number of different time-dependent scaling solutions. As we show in §3.4, by appropriately choosing how the various fields scale with time, it is possible to write the equations of motion in such a way that all time-dependence factors out. For example, the $(\mu\nu)$ Einstein equation becomes $t^p(G_{\mu\nu} + \kappa^2 T_{\mu\nu}) = 0$, for some constant $p$. Because the time dependence can be factored out in this way, the differential equations become tractable and we are thus able to find various classes of solutions.

### 3.1 Asymptotic Near-Brane Geometries

As was done in the beginning of §2.3, we keep the discussion here as general as possible by allowing the number of maximally symmetric dimensions, $n$, and the number of compact dimensions, $d$, to be arbitrary. Our starting action is then the one appropriate for such a generalization, and is given explicitly by eq. (2.29). We will later specialize to the case of most interest, with $n = 4$ and $d = 2$, when discussing the de Sitter, anti-de Sitter, and scaling solutions.

We assume here that the dilaton field, $\varphi$, and the metric near the brane have the form

\[
\varphi \approx \zeta \ln r,
\]

\[
ds^2 = \tilde{g}_{MN} dx^M dx^N \approx r^{2\omega} g_{\mu\nu}(x) dx^\mu dx^\nu + dr^2 + r^{2\beta} f_{ab}(z) dz^a dz^b, \tag{3.1}
\]

where $\omega$, $\beta$ and $\zeta$ are constants. Comparing this with the metric ansatz, eq. (2.32), we see that this corresponds to the choices

\[
\mathcal{W}(y) = r^\omega \quad \text{and} \quad \tilde{g}_{mn} dy^m dy^n = dr^2 + r^{2\beta} f_{ab} dz^a dz^b, \tag{3.2}
\]
where \( \{ y^m \} = \{ r, z^a \} \). If the supergravity of interest is regarded as describing the low-energy limit of a perturbative string theory then our conventions are such that \( e^{\varphi} \to 0 \) represents the limit of weak string coupling. We see that if \( \zeta < 0 \) then the region of small \( r \) lies beyond the domain of the weak-coupling approximation.

We imagine the brane location to be given by \( r = 0 \) and the coordinate \( r \) is then seen to represent the proper distance away from the brane. With this choice a surface having proper radius \( r \) has an area which varies with \( r \) like \( r^{\beta(d-1)} \), and so this area only grows with increasing \( r \) if \( \beta > 0 \). The geometry in general has a curvature singularity at \( r = 0 \), except for the special case \( \beta = 1 \) for which the singularity can be smooth (or purely conical).

Finally, we specialize for simplicity to the case where there is only one non-vanishing gauge flux which we take to be for a \( p \)-form potential whose field strength is \( F \). With a Freund-Rubin ansatz [63] in mind we also specialize to \( p = d - 1 \) and take \( F \) proportional to the volume form of the \( d \)-dimensional metric \( \tilde{g}_{mn} \). Near \( r = 0 \), we assume

\[
F^{r a_1 \ldots a_p} \sim r^\gamma. 
\]  

(3.3)

With these assumptions, we now determine the powers \( \beta, \omega, \zeta \) and \( \gamma \) by solving the field equations in the region \( r \approx 0 \). We do so by neglecting the contributions of fluxes or the dilaton potential in the dilaton and Einstein equations, and by neglecting any Chern-Simons contributions to the equations for the background \( p \)-form gauge potential. Once we find the solutions we return to verify that the neglect of these terms is indeed justified.
The \( p \)-form equation gives the condition

\[
0 = \partial_r \left( \sqrt{g} e^{-p \phi} F^{rz_1 \ldots z_p} \right) \sim \partial_r \left[ r^{\omega n + \beta(d-1) - p \zeta + \gamma} \right]
\]  

(3.4)

which leads (when \( p = d - 1 \)) to the condition \( \gamma = (\zeta - \beta)(d - 1) - \omega n \), and so

\[
F^2 \sim r^{2\zeta(d-1)-2\omega n}.
\]

(3.5)

Consider next the dilaton equation. We first note that

\[
\hat{\Box} \phi = \frac{1}{\sqrt{\hat{g}}} \partial_M \left( \sqrt{\hat{g}} \hat{g}^{MN} \partial_N \phi \right) \sim \zeta \left[ n \omega + \beta(d-1) - 1 \right] r^{-2}.
\]

(3.6)

For comparison, the other terms in the dilaton equation of motion depend on \( r \) as follows:

\[
e^{-p \phi} F^2 \sim r^{\zeta(d-1)-2\omega n} \quad \text{and} \quad e^\phi \sim r^\zeta.
\]

(3.7)

Thus, provided \( \zeta > -2 \) and \( \zeta(d - 1) - 2\omega n > -2 \) (whose domains of validity we explore below) all of the terms in the dilaton equation are subdominant to \( \hat{\Box} \phi \), and so may be neglected. The dilaton therefore effectively satisfies \( \hat{\Box} \phi = 0 \) near \( r = 0 \), and so from eq. (3.6) we see that this requires

\[
n \omega + \beta(d - 1) = 1.
\]

(3.8)

Next consider the \( rr \)-component of the Einstein equation. Given the assumed asymptotic form for the metric, we calculate

\[
\hat{R}_{rr} = [-\omega n + n \omega^2 + (\beta^2 - \beta)(d - 1)] r^{-2} = [n \omega^2 + \beta^2(d - 1) - 1] r^{-2}.
\]

(3.9)
As before, we find that the $F^2$ term is subdominant if $\zeta(d - 1) - 2\omega n > -2$. The \textit{rr}-Einstein equation therefore gives the additional constraint

$$n\omega^2 + \beta^2(d - 1) + \zeta^2 = 1.$$ \hfill (3.10)

Notice that this equation restricts the ranges of $\omega$, $\beta$ and $\zeta$ to be

$$-\frac{1}{\sqrt{n}} \leq \omega \leq \frac{1}{\sqrt{n}}, \quad -\frac{1}{\sqrt{d-1}} \leq \beta \leq \frac{1}{\sqrt{d-1}} \quad \text{and} \quad -1 \leq \zeta \leq 1.$$ \hfill (3.11)

In particular it allows a regular solution (or one having a conical singularity) — \textit{i.e.} one having $\beta = 1$ — only if $d = 2$ and $\zeta = \omega = 0$.

The Einstein equations in the maximally symmetric dimensions can be similarly evaluated using the assumed asymptotic form for the metric. The contribution of the induced $n$-dimensional curvature tensor contributes to this equation subdominantly in $r$, and so is not constrained to leading order. The leading term vanishes as a consequence of eq. (3.8), and so does not impose any new conditions. Neither do the Einstein equations in the $z^a$ directions.

The net summary of the bulk field equations on the parameters $\beta$, $\omega$ and $\zeta$ is therefore given by the two Kasner-like conditions (3.8) and (3.10). These two conditions therefore allow a one-parameter family (parameterized, say, by $\zeta$) of solutions in the vicinity of any given singularity. Notice that the symmetry of these conditions under $\zeta \rightarrow -\zeta$ implies that for any given asymptotic solution there is a new one which can be obtained from the first through the weak-to-strong-coupling replacement $e^\varphi \rightarrow e^{-\varphi}$.
Regarding these singularities as brane sources, the one-parameter set of asymptotic bulk configurations presumably corresponds to a one-parameter choice which is possible for the couplings of the brane to bulk fields. For instance, at the lowest-derivative level considered here this is plausibly related to the choice of dilaton coupling, such as if the brane action were to take the $D$-brane form

$$
S_b = -T \int d^n \xi \sqrt{-h} e^{\lambda \varphi},
$$

where $\xi^\mu$ represent coordinates on the brane world-volume, $T$ is the brane tension, $h_{\mu\nu}$ is the induced metric on the brane. Here the choice for $\lambda$ (which is a known function of brane dimension for $D$-branes) plausibly determines the value of $\zeta$, and so the value of this parameter is not determined purely from the bulk equations of motion.

We must now go back to ask whether the Kasner-like conditions (3.8) and (3.10) are consistent with the requirements $\zeta > -2$ and $\zeta(d - 1) - 2\omega n > -2$. The first inequality clearly follows from the last of eqs. (3.11), and so is automatic for the solutions of interest. By contrast, constraints (3.8) and (3.10) are not sufficient to ensure that the second inequality is satisfied, as is seen by using eq. (3.8) to rewrite it as $\zeta + 2 \beta \geq 0$. This is clearly not satisfied by the choices $\beta = 0$, $\omega = 1/n$ and $\zeta = -\sqrt{1-1/n}$. Since its violation requires either $\beta$ or $\zeta$ to be negative, it necessarily involves either surfaces, $\Sigma_\beta$, whose area does not grow with their radius ($\beta < 0$) or the breakdown of the perturbative supergravity approximation ($\zeta < 0$). We exclude such solutions in what follows.
3.1.1 Asymptotics and Curvature

We now use the above expressions to evaluate the combination of bulk fields which appears on the right-hand side of eq. (2.37). The surface quantity which appears there is

\[ f_\alpha \equiv - \int_{\Sigma_\alpha} d^{d-1} y \sqrt{h} N_i \left[ W^m \tilde{g}^{ij} \partial_j \left( \ln W + \frac{2 \varphi}{D - 2} \right) \right]_{\Sigma_\alpha}, \tag{3.13} \]

and so evaluating this using \( N_i \, dy^i = -dr \) (since the outward-pointing normal points towards the brane at \( r = 0 \)) and the asymptotic forms given above we find

\[ \sum_\alpha f_\alpha \sim \sum_\alpha \lim_{r \to 0} \left( \omega_\alpha + \frac{2 \zeta_\alpha}{D - 2} \right) c_\alpha r^{\beta_\alpha(d-1)+n\omega_\alpha-1} = \sum_\alpha c_\alpha \left( w_\alpha + \frac{2 \zeta_\alpha}{D - 2} \right), \tag{3.14} \]

where the last equality uses eq. (3.8). The positive constants \( c_\alpha \) are defined by the condition \( \int_{\Sigma_\alpha} d^{d-1} y \sqrt{h} W^m \sim c_\alpha r^{\beta_\alpha(d-1)+n\omega_\alpha}. \)

It is the sign (or vanishing) of the sum in eq. (3.14) which governs the sign (or vanishing) of the maximally-symmetric \( n \)-dimensional curvature. Several points here are noteworthy.

- \( f_\alpha \) always vanishes for any source at which the bulk equations are non-singular (or only has a conical singularity), because \( \omega_\alpha = \zeta_\alpha = 0 \) at any such point. Consequently the maximally-symmetric large \( n \) dimensions must be flat in the absence of any extra-dimensional brane sources at whose positions the bulk fields are singular.

- The \( n \)-dimensional curvature can vanish even if \( f_\alpha \neq 0 \) provided that the sum of the \( f_\alpha \)'s over all of the sources vanishes. For example, specializing to the GGP solutions, eq. (3.13) relating brane asymptotics to the curvature of the
4D space in this case becomes

\[ f_\alpha = \pm 2\pi \left( \ln \mathcal{W} + \frac{\varphi}{2} \right)' \]  

(3.15)

for branes at \( \eta \to \pm \infty \). Simplifying the right-hand side using the relation \( e^{\varphi} = \mathcal{W}^{-2} e^{-\lambda_3 \eta} \) implies \( \left( \ln \mathcal{W} + \frac{1}{2} \varphi \right)' = -\frac{1}{2} \lambda_3 \), which vanishes only for the conical-singularity case. However we see that the sum \( \sum_\alpha f_\alpha \) nevertheless vanishes once summed over the two singularities, consistent with the flatness of the 4D geometries.

### 3.2 6D de Sitter Solutions

In this section, we construct new solutions to 6D supergravity which go beyond previously known solutions [52, 55, 64, 65, 66, 67, 68, 69] by having 4 maximally-symmetric dimensions which are not flat (see [70] for a recent discussion of similar solutions in the non-supersymmetric context).

#### 3.2.1 Equations of Motion

In order to find solutions with constant 4D curvature, we take as our metric ansatz the minimal extension of the one used by GGP. In particular, we assume

\[ ds^2 = \hat{g}_{MN} dx^M dx^N = \mathcal{W}^2 q_{\mu\nu} dx^\mu dx^\nu + A^2 d\theta^2 + A^2 \mathcal{W}^8 d\eta^2, \]  

(3.16)

In what follows we take \( q_{\mu\nu} \) to be the 4D de Sitter metric having Hubble constant \( H \). The anti-de Sitter case can be obtained from the final results by taking \( H^2 \to -H^2 \). Inspection of eq. (2.33) shows that the only change in the equations of motions comes because the 4D curvature tensor gets an added contribution, \( \hat{R}_{\mu\nu} = \hat{R}_{\mu\nu}|_{H=0} - 3H^2 q_{\mu\nu} \). We see then that the only equations of motion which are affected by this
change are the \((\mu\nu)\) and \((\eta\eta)\) Einstein equations, which become respectively

\[
\frac{W''}{W} - \frac{(W')^2}{W^2} - 3 H^2 A^2 W^6 + \frac{1}{2} \varphi'' = 0
\]  
(3.17)

\[
6 H^2 A^2 W^6 - \frac{4 A W'}{A W} - \frac{6(W')^2}{W^2} + \frac{1}{2} (\varphi')^2 + \frac{\kappa^2}{2} q^2 A^2 e^\varphi - \frac{2 g_1^2}{\kappa^2} A^2 W^8 e^\varphi = 0.
\]  
(3.18)

Here, we have used the other equations of motion in order to write the \((\eta\eta)\) equation in a form involving only first derivatives. The equation of motion for the gauge field and the dilaton have been given already by eqs. (2.47) and (2.48), respectively.

For numerical purposes we use eqs. (2.47), (2.48) and (3.17) to determine \(\varphi''\), \(a''\) and \(W''\) as a function of \(\varphi\), \(a\), \(W\), \(\varphi'\), \(a'\) and \(W'\), and by stepping forward in \(\eta\) generate a solution as a function of \(\eta\). By contrast, eq. (3.18) must be read as a constraint rather than an evolution equation because it contains no second derivatives. The consistency of this constraint with the evolution equations is guaranteed (as usual) by general covariance and the Bianchi identities. Evaluating this constraint at the ‘initial’ point, \(\eta = \eta_0\), gives \(H\) in terms of the assumed initial conditions.

### 3.2.2 Solutions

Recall that the special case of \(H = 0\) corresponds precisely to the solutions found by GGP. Our goal now is to construct more general solutions to the same field equations, but with \(H\) nonzero, and so for which the maximally-symmetric 4D geometries are not flat. Although we could do so by directly integrating the field equations as given above, we instead follow ref. [52] and regard these equations as
coming from the following equivalent Lagrangian

\[
L = \left[ (\varphi')^2 - 8(\ln \mathcal{W})(\ln \mathcal{A})' - 12[(\ln \mathcal{W})']^2 \right] N^{-1} \\
- \mathcal{N} \mathcal{A}^2 e^\varphi \left( \kappa^2 q^2 - \frac{4g_1^2}{\kappa^2} \mathcal{W}^8 + 12H^2 \mathcal{W}^6 e^{-\varphi} \right).
\]  

(3.19)

This agrees with the form used in [52] when \( H = 0 \). We temporarily re-introduce here the ‘lapse’ function, \( g_{\eta\eta} = N^2 \mathcal{A}^2 \mathcal{W}^8 \), which we choose to reset to unity after it has been varied in the action. Varying with respect to \( N \) gives the constraint equation (3.18) where we set \( N = 1 \) after variation.

The equivalent Lagrangian simplifies if we diagonalize the ‘kinetic’ terms, by defining the new variables \( \mathcal{X}, \mathcal{Y} \) and \( \mathcal{Z} \) using\(^1\)

\[
\varphi = \frac{1}{2}(\mathcal{X} - \mathcal{Y} - 2\mathcal{Z}), \quad \ln \mathcal{W} = \frac{1}{4}(\mathcal{Y} - \mathcal{X}) \quad \text{and} \quad \ln \mathcal{A} = \frac{1}{4}(3\mathcal{X} + \mathcal{Y} + 2\mathcal{Z}).
\]  

(3.20)

In terms of these variables the Lagrangian becomes

\[
L = (\mathcal{X}')^2 - (\mathcal{Y}')^2 + (\mathcal{Z}')^2 - \kappa^2 q^2 e^{2\mathcal{X}} + \frac{4g_1^2}{\kappa^2} e^{2\mathcal{Y}} - 12H^2 e^{2\mathcal{Y} + \mathcal{Z}}.
\]  

(3.21)

We have set \( N = 1 \) but continue to keep in mind its role in determining the constraint. At this point, it pays to make a further redefinition to absorb all the constants appearing in the above Lagrangian. The main reason for doing this is that we will eventually integrate numerically some of the resulting equations of motion. With this

\(^1\) In fact, this field redefinition is often used in this thesis as it simplifies the equations of motion considerably.
goal in mind, we see that the ‘potential’ terms simplify further if we also redefine

\[
\begin{align*}
X &= \frac{1}{2} \ln(\kappa^2 q^2) + X' \\
Y &= \frac{1}{2} \ln(4g_1^2 / \kappa^2) + Y' \\
Z &= \ln |3\kappa^2 H^2 / g_1^2| + Z
\end{align*}
\]

and so

\[
L = (X')^2 + (Y')^2 + (Z')^2 - e^{2X} + e^{2Y} - \epsilon e^{2Y+Z}.
\]

where \(\epsilon = +1\) for de Sitter and \(-1\) for anti-de Sitter solutions. We now integrate the equations of motion obtained from this Lagrangian to obtain explicit solutions for the extra-dimensional geometries. Since \(X\) has the equation of motion

\[
X'' + e^{2X} = 0
\]

it decouples from the other variables. Its equation can be directly integrated to give

\[
(X')^2 + e^{2X} = \lambda_1^2,
\]

and so \(e^{-X} = \lambda_1^{-1} \cosh[\lambda_1 (\eta - \xi_1)]\). As before, we have chosen to take \(\lambda_1 > 0\). The remaining two nontrivial equations of motion become in these variables

\[
\begin{align*}
Y'' + e^{2Y} - \epsilon e^{2Y+Z} &= 0 \\
Z'' + \frac{\epsilon}{2} e^{2Y+Z} &= 0,
\end{align*}
\]

along with the constraint \(\lambda_1^2 - (Y')^2 + (Z')^2 - e^{2Y} + \epsilon e^{2Y+Z} = 0\), whose solutions we obtain numerically below.
In terms of these variables the asymptotic behaviour of the solutions assumed in previous sections near the singularities is linear in \( \eta \). For example, using eqs. (3.20) and (3.22) to write \( X \) in terms of \( \varphi \) and \( \mathcal{W} \), and then using the asymptotic forms given by eqs. (3.1) and (3.2), we see

\[
2X = \varphi + 2 \ln A + \ln (\kappa^2 q^2) \approx (\zeta_\pm + 2 \beta_\pm) \ln r_\pm \approx \mp (\zeta_\pm + 2 \beta_\pm) \eta, \tag{3.27}
\]

where in the last step we have used that \( \eta \approx \mp \ln r_\pm \) in the asymptotic region \( \eta \to \pm \infty \). Alternatively, from the exact solution for \( X \) it is clear that

\[
\lim_{\eta \to \pm \infty} X \to \mp \lambda_1 \eta. \tag{3.28}
\]

For the other dependent variables we may similarly write

\[
\lim_{\eta \to \pm \infty} Y \to \mp \lambda_2^\pm \eta, \\
\lim_{\eta \to \pm \infty} Z \to \mp \lambda_3^\pm \eta, \tag{3.29}
\]

with independent constants \( \lambda_i^\pm \) at \( \eta \to \pm \infty \). By substituting these asymptotic forms into the differential equations, eqs. (3.26), we immediately obtain the two constraints \( \lambda_2^\pm > 0 \) and \( (2 \lambda_2^\pm + \lambda_3^\pm) > 0 \). Note that there is no restriction on the sign of \( \lambda_3^\pm \).

Finally, the Kasner-like condition in the asymptotic region also imposes the following constraint on these constants: \( (\lambda_3^\pm)^2 = \lambda_1^2 + (\lambda_3^\pm)^2 \).

The solutions of ref. [52] discussed above satisfy these condition in the special case where \( \lambda_3^\pm = \pm |\lambda_3| \), and in this case we know the 4D geometries are flat. In general, however, both the parameters \( \lambda_3^\pm \) are not determined by the one constant \( \lambda_3 \). Using that \( Z = -\varphi - 2 \ln \mathcal{W} \to \mp \lambda_3^\pm \eta \) and comparing with eq. (3.13), we see
Figure 3–1: Typical behaviour of $Y$ as a function of $\eta$ for de Sitter solutions ($\epsilon = +1$). The function interpolates between two asymptotically linear regimes. The gradient is always positive as $\eta \to -\infty$ and negative as $\eta \to +\infty$.

Figure 3–2: Typical behaviour of $Z$ as a function of $\eta$ for de Sitter solutions ($\epsilon = +1$). The solutions are asymptotically linear with different gradients. For a suitable choice of initial data the gradient can change sign as in Fig. 3–1.
in this case that the sum $\sum_{\pm} f_{\pm}$ does not vanish, leading to the conclusion that the corresponding $4D$ geometries cannot be flat. We have been unable to obtain analytic solutions to these equations, but there is no obstruction to their integration. They can be solved numerically leading to numerical profiles such as those given in Figures 3–1 and 3–2.

3.3 Static Solutions with Broken 4D Lorentz Symmetry

In this section we describe a broad class of static compactifications which are more general than previous solutions in that they allow $4D$ Lorentz symmetry to be broken. When the parameter which controls this breaking is turned off, these solutions reduce to the GGP solutions discussed in §2.3.2.

3.3.1 Equations of Motion

We start by writing out the field equations for configurations which are (i) time-independent; (ii) translation and rotation invariant in the 3 noncompact spatial dimensions; and (iii) are axially symmetric in the extra dimensions. That is, we take

$$\begin{align*}
    ds^2 &= -e^{2w(\eta)} \, dt^2 + e^{2a(\eta)} \, \delta_{ij} \, dx^i dx^j + e^{2v(\eta)} \, d\eta^2 + e^{2b(\eta)} \, d\theta^2 \\
    A_\theta &= a_\theta(\eta) \quad \text{and} \quad e^\varphi = e^{\varphi(\eta)},
\end{align*}$$

(3.30)
leading to the following system of coupled ordinary differential equations:

\[ a''_\theta + \left( w' + 3a' - v' - b' - \varphi' \right) a'_\theta = 0 \] (Maxwell)

\[ \varphi'' + \left( w' + 3a' - v' + b' \right) \varphi' + \frac{\kappa^2}{2} e^{-2b-\varphi}(a'_\theta)^2 - \frac{2g_1^2}{\kappa^2} e^{2v+\varphi} = 0 \] (Dilaton)

\[ w'' + \left( w' + 3a' - v' + b' \right) w' - \frac{\kappa^2}{4} e^{-2b-\varphi}(a'_\theta)^2 + \frac{g_1^2}{\kappa^2} e^{2v+\varphi} = 0 \] (tt Einstein)

\[ a'' + \left( w' + 3a' - v' + b' \right) a' - \frac{\kappa^2}{4} e^{-2b-\varphi}(a'_\theta)^2 + \frac{g_1^2}{\kappa^2} e^{2v+\varphi} = 0 \] (ii Einstein)

\[ b'' + \left( w' + 3a' - v' + b' \right) b' + \frac{3\kappa^2}{4} e^{-2b-\varphi}(a'_\theta)^2 + \frac{g_1^2}{\kappa^2} e^{2v+\varphi} = 0 \] (θθ Einstein)

\[ w'' + 3a'' + b'' + (w')^2 + 3(a')^2 + (b')^2 + (\varphi')^2 - \left( w' + 3a' + b' \right) v' + \frac{3\kappa^2}{4} e^{-2b-\varphi}(a'_\theta)^2 + \frac{g_1^2}{\kappa^2} e^{2v+\varphi} = 0 \] (ηη Einstein)

Here ' denotes a derivative with respect to η. These equations must be supplemented with boundary conditions at the locations of the two branes.

Although this appears to provide 6 equations for the 6 unknown functions \( w, a, v, b, \varphi \) and \( a_\theta \), this is deceptive because we can ensure that one of these functions (say \( v \)) takes any particular form simply by appropriately changing the coordinate η. Also, one combination of these equations, found by taking the combination (ηη) − (tt) − 3 (ii) − (θθ), can be thought of as a ‘constraint’ on the evolution of the fields into the η direction because all second derivatives, \( (d/d\eta)^2 \), drop out:

\[ (\varphi')^2 - 6(w' + a' + b')a' - 2b'w' + \kappa^2 e^{-2b-\varphi}(a'_\theta)^2 - \frac{4g_1^2}{\kappa^2} e^{2v+\varphi} = 0 \] (3.32)

As a consequence of the Bianchi identities, one of the remaining equations — which for definiteness we take to be the (ηη) Einstein equation — is then not independent and can be derived from the derivative of the constraint and the other equations.
3.3.2 Solutions

Before continuing, we pause to point out that one useful trick for solving these complicated equations of motion is making a smart choice for the radial coordinate. Naively, one might think that the Gaussian-Normal gauge choice, $v = 1$, is best since that leads to one fewer equation of motion. However, it pays to not commit to a specific gauge until after examining the equations of motion. In this case, we see that 4 of the 5 equations of motion can be simplified immensely by the choice

$$v = w + 3a + b + \ln N. \quad (3.33)$$

We will eventually choose the gauge $N = 1$, but until then the above equation should be treated simply as a field redefinition, trading $v$ for $N$. The remaining Maxwell equation can be integrated exactly, giving

$$a'_{\theta} = qe^{-(w+3a-v-b-\varphi)} = qNe^{2b+\varphi} \quad (3.34)$$

where in the second equality we have used the field redefinition for $v$. Plugging this solution into the equations of motion, we note that the arguments of the exponentials in these equations are functions of either $(2b + \varphi)$, or $(2v + \varphi) = 2(w + 3a + b + \varphi/2)$. Thus, it will pay to make the redefinitions

$$\mathcal{X} = b + \frac{1}{2} \varphi \quad (3.35)$$
$$\mathcal{Y} = w + 3a + b + \frac{1}{2} \varphi. \quad (3.36)$$
Inspection of the equations shows that two other useful choices for redefinitions are

\[
Z = -\frac{1}{2}(w + 3a + 2\varphi) \quad (3.37)
\]
\[
\xi = \frac{1}{4}(w - a) \quad (3.38)
\]

At this point, it is straightforward to integrate the resulting equations.

Before proceeding, however, we pause to comment that these equations of motion as well as the constraint equation can be derived from an equivalent Lagrangian, just as in the case of the de Sitter solutions in §3.2. This Lagrangian is given by

\[
S = \int d\eta \left[ N^{-1} \left[ (\mathcal{X}')^2 - (\mathcal{Y}')^2 + (\mathcal{Z}')^2 + 2(\xi')^2 \right] - N \left[ \kappa^2 q^2 e^{2\mathcal{X}} - \frac{4g_1^2}{\kappa^2} e^{2\mathcal{Y}} \right] \right],
\]

where we see now that \( N \) plays the role of a Lagrange multiplier and so can be set to unity after variation. Notice that for \( \xi' = 0 \), this is the same as the equivalent Lagrangian in §3.2 with \( H = 0 \). This should be expected, since from eqs. (3.30) and (3.38) we see that \( \xi' \) is the quantity which measures 4D Lorentz breaking. Thus, for \( \xi' = 0 \) these solutions must reduce to the GGP solutions of §2.3.2.

This system can be solved exactly, giving (with \( N = 1 \)):

\[
e^{-\mathcal{X}} = \frac{\kappa q}{\lambda_1} \cosh[\lambda_1 (\eta - \xi_1)] \quad (3.40)
\]
\[
e^{-\mathcal{Y}} = \frac{2g_1}{\kappa \lambda_2} \cosh[\lambda_2 (\eta - \xi_2)] \quad (3.41)
\]
\[
\mathcal{Z} = z_0 + \lambda_3 \eta \quad (3.42)
\]
\[
\xi = \xi_0 + \lambda_4 \eta, \quad (3.43)
\]
while integrating the Maxwell equation (3.34) gives

\[ a_{\theta} = \frac{\lambda_1}{q} (\tanh[\lambda_1(\eta - \xi_1)] + \alpha) \]  

(3.44)

The constraint equation amounts to the condition \( \lambda_2^2 = \lambda_1^2 + \lambda_3^2 + 12 \lambda_4^2 \). Note that it is the integration constant \( \lambda_4 \) which controls 4D Lorentz breaking, since \( \lambda_4 = 0 \) implies \( \xi' = 0 \). As discussed already, these solutions then reduce to the GGP ones.

Finally, note that we can set two of the parameters \( \xi_1, \xi_2, z_0, \xi_0 \) to zero by coordinate rescalings without loss of generality.

### 3.3.3 Asymptotic Forms

These solutions describe geometries which become singular at \( \eta = \pm \infty \), which are interpreted as being the positions of the 3-branes which source this configuration. Since it is ultimately the internal structure of the brane (if any) which is responsible for resolving these singularities, one expects this structure to be related to the asymptotic limit of the above solutions in the near-brane limit. We therefore pause here to outline what this asymptotic near-brane behaviour is.

As before, it is useful to adopt Gaussian-Normal coordinates near the brane for which \( ds^2 = g_{ab} dx^a dx^b + dr^2 \), where \( x^a \) denotes the 5 other coordinates, \( \{x^a\} = \{x^\mu, \theta\} = \{t, x^i, \theta\} \), with the brane position being described by \( r = 0 \). We take the asymptotic form of the bulk fields in the near-brane limit \( (r \gtrsim \ell) \) to be generically given by a power law [1]

\[
\begin{align*}
    ds^2 &\sim -[c_\omega(H_1 r)^\omega']^2 dt^2 + [c_\alpha(H_1 r)^\alpha']^2 \delta_{ij} dx^i dx^j + dr^2 + [c_\theta(H_1 r)^\theta']^2 r^2 d\theta^2 \\
    e^\varphi &\sim c_\varphi(H_1 r)^\zeta \quad \text{and} \quad F^{\gamma \theta} \sim c_f(H_1 r)^\gamma, \\
\end{align*}
\]

(3.45)
where $\omega$, $\beta$, $\alpha$, $\zeta$, $\gamma$, $c_\omega$, $c_\beta$, $c_\theta$, $c_\phi$ and $c_f$ are constants, and $H_1$ is an arbitrary scale.\footnote{Take care not to confuse the constant $\omega$ with the function $w$ used in the previous section.}

Only two of the five powers $\beta$, $\alpha$, $\omega$, $\gamma$ and $\zeta$ defined above are independent, since the bulk field equations impose the following three conditions amongst them

$$\omega^2 + 3\alpha^2 + \beta^2 + \zeta^2 = \omega + 3\alpha + \beta = 1 \quad \text{and} \quad \gamma = \zeta - 1.$$

(3.46)

For example, explicit calculation with the general static solutions given above gives

$$\omega_{\pm} = \frac{\lambda_2 - \lambda_1 \mp 12\lambda_4}{5\lambda_2 - \lambda_1 \mp 2\lambda_3}, \quad \alpha_{\pm} = \frac{\lambda_2 - \lambda_1 \pm 4\lambda_4}{5\lambda_2 - \lambda_1 \mp 2\lambda_3},$$

$$\beta_{\pm} = \frac{\lambda_2 + 3\lambda_1 \mp 2\lambda_3}{5\lambda_2 - \lambda_1 \mp 2\lambda_3}, \quad \text{and} \quad \zeta_{\pm} = -\frac{2(\lambda_2 - \lambda_1 \pm 2\lambda_3)}{5\lambda_2 - \lambda_1 \mp 2\lambda_3},$$

(3.47)

at the brane positions $\eta \to \pm \infty$. It is easy to verify that these satisfy the expressions (3.46) above. As expected, in the special case where $\lambda_4 = 0$, these reduce to eq. (2.60) found for the GGP solutions.

### 3.4 Time-Dependent Scaling Solutions

We now generalize the previous discussion to a new class of time-dependent scaling configurations which provide exact solutions to the same 6D field equations. The idea behind the construction is to choose the time dependence of the fields in such a way that it factors out of all equations of motion, leaving behind a set of coupled ordinary differential equations. As we will see, these equations are very similar to the ones studied in the general static case, §3.3.2.
Before going on to solve the time-dependent solutions, we first pause to demonstrate how the scaling symmetry discussed earlier can be exploited. Recall that this symmetry tells us that given any solution to the equations of motion, a new solution can be obtained through the transformations \( \bar{g}_{MN} = e^\sigma g_{MN} \) and \( \bar{\varphi} = \varphi - \sigma \), where \( \sigma \) is a constant. The idea now will be to take \( \sigma = \sigma(x) \). We know that under this conformal transformation, the Ricci tensor transforms as

\[
\bar{R}_{MN} = R_{MN} - \frac{1}{2}(D - 2)\sigma;_{MN} - \frac{1}{2}g_{MN}\sigma;_{P}^{P} + \frac{1}{4}(D - 2)(\sigma;_{M}\sigma;_{N} - g_{MN}\sigma;_{P}\sigma;^{P})
\]

where a semicolon denotes covariant differentiation with respect to the original metric, \( g_{MN} \), and we have left arbitrary the dimension of spacetime, \( D \). It can also be shown that

\[
\bar{\square}\bar{\varphi} = e^{-\sigma}\left[\square\varphi - \square\sigma + \frac{1}{2}(D - 2)(\varphi;_{M} - \sigma;_{M})\sigma;^{M}\right].
\]

Notice that each of these expressions scales by a specific overall factor of \( e^{\sigma} \). Scale invariance now assures us that this will factor out of all equations of motion. For example, the dilaton equation will be \( e^{-\sigma}(\square\varphi + \ldots) = 0 \), and so we can multiply both sides by \( e^{\sigma} \) to remove this overall factor.

Now, we specialize to the case \( \sigma = \sigma(t) \), and take the fields \( \varphi \) and \( g_{MN} \) to be independent of time. Assuming also that the metric is diagonal, we have

\[
\bar{\square}\bar{\varphi} = e^{-\alpha}\left[\square\varphi - g^{tt}(\ddot{\sigma} + 2\dot{\sigma}^2)\right],
\]

\[\text{See, for example, appendix F in [72]}\]
where we now specialize to the case $D = 6$, and we use a dot to denote differentiation with respect to time. Using this result in the dilaton equation of motion gives

$$\square \varphi + \frac{\kappa^2}{4} e^{-\varphi} F_{MN} F^{MN} - \frac{2g_{tt}^2}{\kappa^2} e^\varphi - g^{tt}(\ddot{\sigma} + 2\dot{\sigma}^2) = 0. \quad (3.52)$$

Since all fields except $\sigma(t)$ are assumed to be independent of time, we see that this is only consistent if $(\ddot{\sigma} + 2\dot{\sigma}^2)$ is a constant. The simplest way to satisfy this constraint is to take $\sigma = Ht$. It is easy to check now that with this choice, $\ddot{R}_{tt} = R_{tt}$, $\dddot{R}_{ij} = R_{ij}$, and $\dddot{R}_{ti} = R_{ti} + \frac{1}{2}H g^{tt} \partial_i g_{tt}$, where $i$ and $j$ are spatial indices. We can again use these results in the equations of motion and we find that all time-dependence disappears. This special choice of time-dependence therefore results in a considerable simplification.

In the following sections, we focus on the case where all fields depend on time as well as a single spatial coordinate, $\eta$. Choosing the time-dependence of the fields in a manner similar to above, we see that the equations of motion reduce to a set of ordinary — as opposed to partial — differential equations.

### 3.4.1 Equations of Motion

Our ansatz for the metric is a generalization of the static solution which allows for power law time dependence. In particular, we take

$$ds^2 = (H_0 t)^c \left[-e^{2w(\eta)} dt^2 + e^{2a(\eta)} \delta_{ij} dx^i dx^j\right] + (H_0 t)^{c+2} \left[e^{2v(\eta)} d\eta^2 + e^{2b(\eta)} d\theta^2\right], \quad (3.53)$$

where $c$ is an arbitrary dimensionless constant and $H_0$ is a constant having dimension of inverse time. In some cases it will be natural to take $H_0 < 0$ so that the direction of increasing time corresponds to $t \to 0^-$. Notice that this is a more general time
dependence than discussed in the previous section because the large and compact
dimensions scale differently with time. We should expect, however, that this will
reduce to the simpler case in the limit of large \(c\).

The key point to recognize is that with this choice the components of the Ricci
tensor also scale as a simple power of \(t\): \(R_{\mu\nu} \propto t^{-2}\), \(R_{\eta\eta} \propto t^{-1}\) and \(R_{mn} \propto t^0\). Because
of this, and of the scale-invariance of the supergravity equations, it is also possible to
scale the other fields in the problem with \(t\) in such a way as to ensure that the field
equations are also proportional to specific powers of \(t\). From the above comments,
we see that we require the following time dependence for the gauge field and the
dilaton:

\[
A_\theta = a_\theta(\eta) \quad \text{and} \quad e^\phi = \frac{e^{\phi(\eta)}}{(H_0 t)^{c+2}}. \tag{3.54}
\]

By virtue of the way the Ricci tensor scales with \(t\), with the above ansatz all
of the field equations reduce to the following set of coupled ordinary differential
equations which govern the \(\eta\)-dependence of the various undetermined functions.
The Maxwell equation is

\[
a''_\theta + \left(w' + 3a' - b' - v' - \varphi'\right)a'_\theta = 0, \tag{3.55}
\]

while the Dilaton equation similarly becomes

\[
\varphi'' + \left(w' + 3a' - v' + b'\right)\varphi' + (c + 2)(2c + 1) e^{-2w+2v} H_0^2 \frac{\kappa^2}{2} e^{-2b-\varphi} (a'_\theta)^2 - \frac{2d_1^2}{\kappa^2} e^{2v+\varphi} = 0. \tag{3.56}
\]
The Ricci tensor for this class of metrics is easily computed and leads to the following components for the Einstein equations. The $(t\eta)$ component is

\[(2c + 1) w' + 3 a' + (c + 2) \varphi' = 0, \tag{3.57}\]

while the $(tt)$ equation is

\[w'' + \left(w' + 3a' - v' + b'\right) w' - \frac{\kappa^2}{4} e^{-2b-\varphi} (a')^2 + \frac{g_1^2}{\kappa^2} e^{2v+\varphi} - \left(c^2 + \frac{5}{2} c + 4\right) e^{-2w+2v} H_0^2 = 0. \tag{3.58}\]

The $(\theta\theta)$ equation is

\[b'' + \left(w' + 3a' - v' + b'\right) b' + \frac{3\kappa^2}{4} e^{-2b-\varphi} (a')^2 + \frac{g_1^2}{\kappa^2} e^{2v+\varphi} - \frac{1}{2} (c + 2)(2c + 1) e^{-2w+2v} H_0^2 = 0, \tag{3.59}\]

the $(ii)$ Einstein equation becomes

\[a'' + \left(w' + 3a' - v' + b'\right) a' - \frac{\kappa^2}{4} e^{-2b-\varphi} (a')^2 + \frac{g_1^2}{\kappa^2} e^{2v+\varphi} - \frac{1}{2} (c + 2)(2c + 1) e^{-2w+2v} H_0^2 = 0, \tag{3.60}\]

and finally the $(\eta\eta)$ equation is

\[w'' + 3a'' + b'' + (w')^2 + 3(a')^2 + (b')^2 + (\varphi')^2 - \left(w' + 3a' + b'\right) v' - \frac{3\kappa^2}{4} e^{-2b-\varphi} (a')^2 + \frac{g_1^2}{\kappa^2} e^{2v+\varphi} - \frac{1}{2} (c + 2)(2c + 1) e^{-2w+2v} H_0^2 = 0, \tag{3.61}\]

where $' = d/d\eta$.

Note that because of the inclusion of time dependence, two components of the Bianchi identity are nontrivial rather than one. This implies that now two of the field equations are not independent of the others. Related to this is the existence in this case of two constraint equations which do not involve $d^2/d\eta^2$, which the
Bianchi identities ensure are preserved when evolved using the field equations in the \(\eta\) direction. These constraints can be taken to be the \((t\eta)\) Einstein equation, eq. (3.57), and the combination \((\eta\eta) - (tt) - 3(ii) - (\theta\theta)\):

\[
(\varphi')^2 - 6(w' + a' + b')a' - 2b'w' + \kappa^2 e^{-2b - \varphi} (a'_\theta)^2 - \frac{4g_1^2}{\kappa^2} e^{2w + \varphi} + 4(c^2 + c + 1) e^{-2w + 2w} H_0^2 = 0.
\]

(3.62)

Notice that if we set \(H_0 = 0\), all equations except the \((t\eta)\) Einstein equation are the same as we found in the static case, §3.3.2. This suggests that we can again simplify these equations by using the same field redefinitions as in the previous section.

3.4.2 Solutions

Proceeding as before, then, we make the gauge choice \(v = w + 3a + b\) (i.e. we take \(N = 1\) from the start). Then the Maxwell equation integrates as before to give

\[
a'_\theta = q e^{2\lambda}.
\]

(3.63)

while the two constraint equations written in the variables of the previous section are

\[
0 = 6c \xi' - (c + 2) Z'
\]

(3.64)

\[
0 = (\lambda')^2 - (\gamma')^2 + \frac{4(c^2 + c + 1)}{3c^2} (Z')^2 + \kappa^2 q^2 e^{2\lambda} - \frac{4g_1^2}{\kappa^2} e^{2\gamma} + 4H_0^2 (c^2 + c + 1) e^{2\gamma - 2Z/c}.
\]

(3.65)
The remaining equations of motion simplify to

\[ \mathcal{X}'' + \kappa^2 q^2 e^{2\mathcal{X}} = 0 \]
\[ \mathcal{Y}'' + \frac{4g_1^2}{\kappa^2} e^{2\mathcal{Y}} - 4H_0^2 (c^2 + c + 1) e^{2\mathcal{Y} - 2Z/c} = 0 \quad (3.66) \]
\[ \mathcal{Z}'' - 3cH_0^2 e^{2\mathcal{Y} - 2Z/c} = 0. \]

We note that, as in previous sections, we can obtain these equations of motion, as well as the constraint equation (3.65), from the equivalent Lagrangian

\[ S = \int d\eta \left\{ N^{-1} \left[ (\mathcal{X}')^2 - (\mathcal{Y}')^2 + \frac{4(c^2 + c + 1)}{3c^2} (\mathcal{Z}')^2 \right] - N \left[ \kappa^2 q^2 e^{2\mathcal{X}} - \frac{4g_1^2}{\kappa^2} e^{2\mathcal{Y}} + 4H_0^2 (c^2 + c + 1)e^{2\mathcal{Y} - 2Z/c} \right] \right\}, \quad (3.67) \]

where again \( N \) plays the role of a Lagrange multiplier.

As before, both \( x \) and \( a_\theta \) decouple and have solutions given previously by eqs. (3.40) and (3.44), respectively. The remaining two equations do not appear to have closed-form solutions, although they are again straightforward to integrate numerically, having qualitatively the same form as the de Sitter solutions considered in §3.2.\(^4\) This relation with the de Sitter solutions is not surprising since these solutions may be obtained as the special case \( c = -2 \) of the above scaling solutions. The asymptotic form of these solutions as \( \eta \to \pm \infty \) is \( \mathcal{Y} \to \mp \lambda_2^\pm \eta \) and \( \mathcal{Z} \to \mp \lambda_3^\pm \eta \), with \( \lambda_2^\pm > 0 \) and \( 2\lambda_2^\pm - (\frac{2}{c}) \lambda_3^\pm > 0 \). The constraint implies \((\lambda_2^\pm)^2 = \lambda_1^2 + 4(c^2 + c + 1)(\lambda_3^\pm)^2/(3c^2)\).

\(^4\) See also [70] for de Sitter solutions in the non-supersymmetric case.
3.4.3 Useful Special Cases

There are several special cases of the previous solutions which are of particular interest.

**Connection to 4D scaling solutions**

If $c = -1$ then the metric and dilaton have the time dependence

$$ds^2 = \frac{1}{t} \hat{g}_{\mu\nu}(y) dx^\mu dx^\nu + t \hat{g}_{mn}(y) dy^m dy^n \quad \text{and} \quad e^\phi = \frac{e^{\phi'}}{t},$$

which implies in particular that $\sqrt{-g} g^{\mu\nu}$ is independent of $t$.

Such a scaling solution has a simple interpretation in the limit where the extra dimensions are large enough to justify a description in terms of an appropriate low-energy 4D effective theory [58]. In the classical limit this theory contains two massless modes, corresponding to the 4D metric and one combination of the dilaton, $\phi$, and radius, $r$, of the extra dimensions (for which $r^2 e^\phi$ is fixed) which parameterizes a flat direction of the 4D scalar potential. The above scaling solution describes a time-dependent scaling along this flat direction with a fixed metric in the 4D Einstein frame. (More general choices for $c$ also rescale the Einstein-frame 4D metric.)

Explicit scaling solutions to the field equations of the effective 4D theory describing these modes are known, many of which are attractor solutions to which a broad class of initial conditions are drawn [73]. The solutions found here show how to extend those of the effective 4D theory to see the profiles of the other nonzero KK modes. Because the 4D solutions are attractors for the 4D field equations, we might also expect that the same may be true for the higher-dimensional solutions found here.
Pure tension branes

It is known that even for the special case of pure tension branes, maximally-symmetric solutions to the 6D field equations only exist when the tensions of the two branes are adjusted relative to one another [51, 52, 53]. One might hope that the above scaling solutions describe the late time behaviour of the solutions in the case that the brane tensions are not adjusted in the appropriate way.

We now show that a subset of the solutions found above can indeed describe this situation. In order to do so we must identify when the asymptotic form of the solutions near the branes have the pure tension form where \( a \sim w \), i.e. \( \xi \to 0 \). Since the \((\eta t)\) constraint implies \( \xi' = (c + 2)Z'/(6c) \) there are only two circumstances for which \( \xi' \to 0 \) near a brane:

\[
(i) \quad Z' \to 0, \quad \text{or} \quad (ii) \quad c = -2. \tag{3.69}
\]

**Special case \( c = \infty \)**

Case (i) of (3.69) corresponds to the special case where the geometry near the brane has a conical singularity, since this always requires \( Z' \to 0 \) in the near-brane limit. While this is always possible for one brane, in general it is not possible for both branes since the equation of motion for \( Z \) is

\[
Z'' - 3cH_0^2 e^{2Y - 2Z/c} = 0, \tag{3.70}
\]

and so \(|Z''| > 0\). There is however, one special case for which this can be achieved. Defining \( \bar{H}_0 = c H_0 \) such that \( \bar{H}_0 \) remains constant (i.e. we take \( H_0 \to 0 \) as \( c \to \infty \)) and taking the limit \( c \to \infty \) we recognize that the originally coupled system for \( Y, Z \)
decouples,
\[ Y'' + \frac{4g_1^2}{\kappa^2} e^{2Y} - 4\tilde{H}_0^2 e^{2Y} = Z'' = 0. \] (3.71)

For which the conical solution is
\[ e^{-Y} = \frac{2}{\kappa \lambda_2} \sqrt{(g_1^2 - \kappa^2 \tilde{H}_0^2)} \cosh[\lambda_2 (\eta - \xi_2)], \quad Z = Z_0. \] (3.72)

As before we can choose \( Z_0 = 0 \) without loss of generality. Although the metric, eq. (3.53), appears to be singular in this limit, this is only a consequence of an inconvenient choice for the time coordinate. If we instead convert to ‘proper’ time, \( \tau \), defined by \( d\tau = (\tilde{H}_0 \tau)^{c/2} dt \), then \( \tau \propto t^{1+c/2} \) and the metric of eq. (3.53) has a smooth large-\( c \) limit:
\[ ds^2 = -e^{2w(\eta)} d\tau^2 + \frac{1}{4} (\tilde{H}_0 \tau)^2 \left[ e^{2a(\eta)} \delta_{ij} dx^i dx^j + e^{2v(\eta)} d\eta^2 + e^{2b(\eta)} d\theta^2 \right]. \] (3.73)

We now convert back to ‘conformal’ time, \( t \), using \( d\tau = \frac{1}{2} (\tilde{H}_0 \tau) dt \), then \( (\tilde{H}_0 \tau)^2 \propto e^{\tilde{H}_0 t} \), and so
\[ ds^2 = e^{\tilde{H}_0 t} \left[ -e^{2w(\eta)} dt^2 + e^{2a(\eta)} \delta_{ij} dx^i dx^j + e^{2v(\eta)} d\eta^2 + e^{2b(\eta)} d\theta^2 \right]. \] (3.74)

In these same coordinates, we note that the dilaton behaves as as \( \varphi(t, \eta) = \varphi(\eta) - \tilde{H}_0 t. \)

Defining \( \sigma(t) = \tilde{H}_0 t \), we can write this more suggestively as
\[ ds^2 = e^{\sigma(t)} g_{MN}(\eta) dx^M dx^N \quad \text{and} \quad \varphi(t, \eta) = \varphi(\eta) - \sigma(t), \] (3.75)

which we recognize as the solution discussed in the introduction to this section.

As before, we can now determine the singularities of this solution in the limit \( \eta \to \pm \infty \). Converting to proper radius \( dr = e^{v(\eta)} d\eta \) and identifying with the conical
deficit form $dr^2 + (1 - \delta/2\pi)^2 r^2 d\theta^2$ we infer the deficit angles

$$
\left(1 - \frac{\delta_+}{2\pi}\right) = \frac{2\lambda_2 e^{\pm \lambda_2 (\xi_1 - \xi_2)}}{|q| \kappa^{3/2}} \sqrt{g^2 - \kappa^2 \bar{H}_0^2}.
$$

(3.76)

It is clear that by choosing any two of $\lambda_2, \xi_1, \xi_2$ and $\bar{H}_0$ appropriately, we can match this solution onto two conical branes of arbitrary tensions.

**Special case $c = -2$**

Option $(ii)$ of (3.69) makes the choice $c = -2$, since in this case the $(t\eta)$ Einstein equation implies the strong statement that $\xi' = \frac{1}{4}(w - a) = 0$ everywhere throughout the bulk. For this special case the 6D geometry is everywhere maximally symmetric in the noncompact 4 dimensions, taking the form

$$
ds^2 = e^{2w(\eta)} ds_4^2 + e^{2v(\eta)} d\eta^2 + e^{2b(\eta)} d\theta^2,
$$

(3.77)

where $dS^4$ is the 4D de Sitter metric. These are just the de Sitter geometries considered previously. Notice that there are no de Sitter solutions where both branes are conical.

**3.4.4 Asymptotic Forms**

We now re-examine the near-brane behaviour of these scaling solutions in order to connect their properties to those of the source branes. If we repeat the analysis of asymptotic forms given earlier for static solutions for the scaling solutions, with near-brane asymptotic form assumed to be given by eq. (3.53), we find two changes relative to the static case. The simplest change is simply that the assumed time dependence implies that the coefficients $c_w, c_a, \text{etc.}$ now depend explicitly on $t$, with

$$
c^2_w(t) \propto c^2_a(t) \propto t^c, \quad c^2_\theta(t) \propto t^{2+c} \quad \text{and} \quad c^2_\phi(t) \propto t^{-2-c}.
$$

(3.78)
The second change is to do with the relationship amongst the powers $\alpha$, $\beta$, $\gamma$, $\omega$ and $\zeta$ which is dictated by the bulk equations. For instance, in the static case these equations required the powers to be related to one another by the Kasner-like conditions $\omega + 3\alpha + \beta = \omega^2 + 3\alpha^2 + \beta^2 + \zeta^2 = 1$. These conditions also apply for the scaling solutions, unchanged by $c$ and $H_0$ because the relevant terms in the field equations are subdominant in powers of $r$. However, in the scaling case there is also a new constraint, eq. (3.57), coming from the $(t\eta)$ Einstein equation, which implies the following for the asymptotic powers:

\[(2c + 1)\omega + 3\alpha + (2 + c)\zeta = 0.\] (3.79)

### 3.5 Discussion

Scaling solutions are often attractor solutions towards which general time-dependent configurations tend after the passage of any initial transients. If this is also so for the 6$D$ supergravity field equations, we would be led to the following attractive picture. It has long been known [51, 52, 53] that static (and maximally-symmetric, but curved [1]) solutions can only exist if the properties of the two source branes are appropriately adjusted relative to one another. But it has been unknown what happens to the bulk geometry in the generic case where such adjustments are not made, although it has been suspected that these would produce time-dependent bulk configurations. Based on the above considerations, in the generic case we expect that the bulk is indeed time-dependent, and in particular this time-dependence approaches one of the scaling solutions given here (once transients pass) at late times.
CHAPTER 4
Perturbative Stability of the GGP Solutions

In the previous chapter we presented a broad class of time-dependent scaling solutions for 4D compactifications of 6D supergravity. These solutions aid in understanding the evolution produced by generic brane properties by showing what the late-time evolution of the system is likely to be for a wide class of initial conditions. The present chapter is aimed at a complementary part of the problem of time-dependence. Instead of seeking the late-time behaviour towards which the system evolves, we here study how small perturbations can initiate the beginnings of time-dependent behaviour. (See ref. [74] for a description of some of the intermediate and transient time-dependent phenomena which lies between these two cases.) We do so by investigating the linear perturbations of a broad class of static solutions to 6D supergravity, in order to see how they respond to arbitrary small perturbations. In this analysis, we use a Kaluza Klein decomposition which includes both the modes oscillating in time as well as the zero-mass non-oscillating ones.

Our main result is that these systems are marginally stable for normalizable perturbations around conical geometries, and for non-singular perturbations around nonconical ones. (We also examine a class of physically motivated non-regular perturbations, for which our stability analysis is inconclusive.) The stability is only marginal because of the generic scaling property which the 6D supergravity solutions have. (It is the absence of an energy barrier in this marginal direction which
allows the development of the scaling solutions of the previous chapter.) We identify the set of equations which govern the time evolution of the system after all of the constraints have been used to eliminate some of the field fluctuations, and reproduce the form found by earlier workers [75].

We begin in §4.1 and §4.2 with a derivation of the equations governing linearized perturbations in comoving gauge. §4.3 then solves analytically the linearized equations in the conical case and identifies the asymptotic behaviour in the more general non-conical case. We then give in §4.4 several general arguments in favour of the stability of the modes examined and relate them with the boundary conditions in §4.3.

4.1 Equations of Motion

We take the following as our metric ansatz

\[ ds^2 = e^{2a} \eta_{\mu\nu} dx^\mu dx^\nu + e^{2v} d\eta^2 + e^{2b} d\theta^2 \]  

and consider these component functions, as well as \( \varphi \) and \( A_\theta \), to depend only on the coordinates \( \eta \) and \( t \). (In what follows we sometimes generalize this assumption to allow dependence on \( \eta \) and \( x^\mu \).) Denoting differentiation with respect to \( \eta \) and \( t \) by primes and dots, the field equations for these functions then reduce to the following set of coupled partial differential equations.

\footnote{While our results disagree with earlier versions (as found on the arxiv) of [75], the latest version has corrected an error and so there is no longer a disagreement.}
The Maxwell equation is:

\[-e^{2(v-a)} \left[ \ddot{A}_\theta + (2\dot{a} + \dot{v} - \dot{b} - \dot{\varphi})\dot{A}_\theta \right] + A''_\theta + (4a' - v' - b' - \varphi')A'_\theta = 0. \quad (4.2)\]

The dilaton equation is:

\[-e^{2(v-a)} \left[ \ddot{\varphi} + (2\dot{a} + \dot{v} + \dot{b})\dot{\varphi} \right] + \varphi'' + (4a' - v' + b')\varphi'
- \frac{1}{2} e^{-2a+2v-2b-\varphi} (\dot{A}_\theta)^2 + \frac{1}{2} e^{-2b-\varphi} (A'_\theta)^2 - \frac{2g_1^2}{\kappa^2} e^{2v+\varphi} = 0. \quad (4.3)\]

The \((\tau \eta)\) Einstein equation is

\[3 \dot{a}' + \dot{b}' - 3 \dot{v} a' + \dot{b} b' - \dot{b} a' - \dot{v} b' + \varphi' \varphi' + e^{-2b-\varphi} \dot{A}_\theta A'_\theta = 0. \quad (4.4)\]

The \((tt)\) Einstein equation is:

\[-e^{2(v-a)} \left[ 3 \ddot{a} + \ddot{b} + \ddot{v} + (\dot{v})^2 + (\dot{b})^2 + (\dot{\varphi})^2 - \ddot{a}(\dot{v} + \dot{b}) \right] + a'' + a'(4a' - v' + b')
- \frac{3\kappa^2}{4} e^{-2a+2v-2b-\varphi} (\dot{A}_\theta)^2 - \frac{\kappa^2}{4} e^{-2b-\varphi} (A'_\theta)^2 + \frac{g_1^2}{\kappa^2} e^{2v+\varphi} = 0. \quad (4.5)\]

The \((\eta \eta)\) Einstein equation is:

\[-e^{2(v-a)} \left[ \ddot{v} + \ddot{v}(2\dot{a} + \dot{v} + \dot{b}) \right] + 4a'' + b'' + 4(a')^2 + (b')^2 - v'(4a' + b') + (\varphi')^2
+ \frac{\kappa^2}{4} e^{-2a+2v-2b-\varphi} (\dot{A}_\theta)^2 + \frac{3\kappa^2}{4} e^{-2b-\varphi} (A'_\theta)^2 + \frac{g_1^2}{\kappa^2} e^{2v+\varphi} = 0. \quad (4.6)\]

The \((\theta \theta)\) Einstein equation is:

\[-e^{2(v-a)} \left[ \ddot{b} + \ddot{b}(2\dot{a} + \dot{v} + \dot{b}) \right] + b'' + b'(4a' - v' + b')
- \frac{3\kappa^2}{4} e^{-2a+2v-2b-\varphi} (\dot{A}_\theta)^2 + \frac{3\kappa^2}{4} e^{-2b-\varphi} (A'_\theta)^2 + \frac{g_1^2}{\kappa^2} e^{2v+\varphi} = 0. \quad (4.7)\]
The \((ij)\) Einstein equation is:

\[
-e^{2(\nu - a)} \left[ \ddot{a} + \dot{a}(2 \dot{a} + \dot{\nu} + \dot{b}) \right] + a'' + a'(4a' - \nu' + b')
+ \frac{\kappa^2}{4} e^{-2a+2\nu-2b-\varphi}(\dot{A}_\theta)^2 - \frac{\kappa^2}{4} e^{-2b-\varphi}(A'_\theta)^2 + \frac{g_1^2}{\kappa^2} e^{2\nu+\varphi} = 0 .
\]

4.1.1 Maximally Symmetric 4D Compactifications

Since we wish to perturb about the 4\(D\)-flat GGP solutions, we briefly recall their properties here. These solutions may be written

\[
e^a = W, \quad e^\nu = A W^4 \quad \text{and} \quad e^b = A ,
\]

and so the bulk fields become

\[
ds^2 = W^2(\eta) \eta_{\mu\nu} dx^\mu dx^\nu + A^2(\eta) \left[ W^8(\eta) d\eta^2 + d\theta^2 \right]
\]

\[
F_{\eta\theta} = \left( \frac{q A^2}{W^2} \right) e^{-\lambda_3 \eta} \quad \text{and} \quad e^{-\varphi} = W^2 e^{\lambda_3 \eta} ,
\]

where

\[
W^4 = \frac{\kappa^2 q \lambda_2 \cosh[\lambda_1(\eta - \xi_1)]}{2g_1 \lambda_1 \cosh[\lambda_2(\eta - \xi_2)]}
\]

\[
A^{-4} = \frac{2\kappa^2 g_1 q^3 \lambda_2^3}{\lambda_1^3 \lambda_2} e^{-2\lambda_3 \eta} \cosh[\lambda_1(\eta - \xi_1)] \cosh[\lambda_2(\eta - \xi_2)] ,
\]

where the field equations imply the constraint \(\lambda_1^2 = \lambda_2^2 + \lambda_3^2\).

We also recall the extremely useful change of variables introduced in §3.2,

\[
\varphi = \frac{1}{2} (\mathcal{X} - \mathcal{Y} - 2\mathcal{Z}) , \quad \ln W = \frac{1}{4} (\mathcal{Y} - \mathcal{X}) \quad \text{and} \quad \ln A = \frac{1}{4} (3\mathcal{X} + \mathcal{Y} + 2\mathcal{Z}) .
\]

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and so

\[ e^{-\mathcal{X}} = e^{-\varphi/2} \mathcal{A}^{-1} = \left| \frac{\kappa q}{\lambda_1} \right| \cosh \left[ \lambda_1 (\eta - \xi_1) \right] \quad (4.13) \]

\[ e^{-\mathcal{Y}} = e^{-\varphi/2} \mathcal{A}^{-1} \mathcal{W}^{-4} = \left| \frac{2g_1}{\kappa \lambda_2} \right| \cosh \left[ \lambda_2 (\eta - \xi_2) \right] \quad (4.14) \]

\[ e^{-Z} = e^{\varphi} \mathcal{W}^2 = e^{-\lambda_3 \eta}. \quad (4.15) \]

It is straightforward to check that these are related by the Hamiltonian constraint (for evolution in the \( \eta \) direction):

\[- \frac{4g_1^2}{\kappa^2} e^{2\mathcal{Y}} + \kappa^2 q^2 e^{2\mathcal{X}} + \mathcal{X}'^2 - \mathcal{Y}'^2 + Z'^2 = 0. \quad (4.16)\]

## 4.2 Linearization

In this section we set up the equations which govern axially-symmetric perturbations about the solutions described above which transform as scalars in 4D. The restriction to axially symmetric perturbations does not limit the ensuing stability analysis because the most unstable modes are generally the most symmetric, since any angular dependence contributes positively to the corresponding Kaluza-Klein mass.

### 4.2.1 Symmetries and Gauge Choices

The most general 4D scalar perturbations have the form

\[ ds^2 = e^{2a} \left[ \eta_{\mu \nu} + M_{i, \mu \nu} \right] dx^\mu dx^\nu + 2N_{m, \mu} dx^\mu dy^m + g_{mn} dy^m dy^n \]

\[ B = B_{\mu \nu} dx^\mu \wedge dx^\nu + 2B_{m, \mu} dx^\mu \wedge dy^m + B_{mn} dy^m \wedge dy^n \]

and

\[ A = \Omega_{i, \mu} dx^\mu + A_m dy^m, \quad (4.17) \]
where in four dimensions $B_{\mu\nu}$ dualizes to a 4D scalar $\zeta$ through a relation of the form $H_{\mu\nu\lambda} \propto \epsilon_{\mu\nu\lambda\rho} \partial^\rho \zeta$. We are free to use gauge symmetries to locally set $N_m = B_m = \Omega = 0$.

**Discrete Symmetry and Mode Mixing**

As has been pointed out by earlier analyses [75, 76], for the purposes of a stability analysis it is not necessary to keep all of the remaining perturbations. To see why, notice that since our interest is in fluctuations depending only on $t$, $x^i$, and $\eta$, the absence of a dependence on the coordinate $\theta$ implies also a symmetry under the reflection, $\theta \rightarrow -\theta$. If $A_\theta$ is nonzero in the background configuration (as is the case when a flux $F_{\eta\theta}$ is turned on in the extra dimensions), this is only a symmetry if we also independently require $A_M \rightarrow -A_M$.

This symmetry ensures that the linearized fluctuations can be divided into two classes, according to whether or not they are even or odd under these reflections. In particular we have

$$\{\delta a, \delta g_{\theta\theta}, \delta g_{\eta\eta}, \delta \varphi, \delta A_\theta, M\} \quad \text{(even)}$$

$$\{\delta g_{\eta\theta}, \delta \zeta, \delta B_{\eta\theta}, \delta A_\eta\} \quad \text{(odd)}. \quad (4.18)$$

Since the symmetry guarantees that it is consistent with all of the equations of motion to set the odd fluctuations to vanish, these two categories of fluctuations cannot mix with one another at the linearized level. We take advantage of this fact to choose $\delta g_{\eta\theta} = \delta \zeta = \delta B_{\eta\theta} = \delta A_\eta = 0$. Again, the omission of these modes does not restrict the stability analysis which follows.
We are led in this way to the following ansatz for the metric and other bulk fields
\[
\begin{align*}
\text{d}s^2 &= e^{2a} \left( \eta_{\mu\nu} + M_{\mu\nu} \right) \text{d}x^\mu \text{d}x^\nu + e^{2v} \text{d}\eta^2 + e^{2b} \text{d}\theta^2 \\
A &= A_\theta \text{d}\theta \quad \text{and} \quad B = 0. 
\end{align*}
\] (4.19)

To linearize the field equations about a static background we write
\[
\begin{align*}
a(\eta, x) &= a_0(\eta) + A(\eta, x), \quad v(\eta, x) = v_0(\eta) + V(\eta, x), \quad b(\eta, x) = b_0(\eta) + B(\eta, x), \\
\varphi(\eta, x) &= \varphi_0(\eta) + \Phi(\eta, x) \quad \text{and} \quad A_\theta(\eta, x) = a_\theta(\eta) + A_\theta(\eta, x), 
\end{align*}
\] (4.20)

where we allow the fluctuations to depend on all 4 coordinates, \( x^\mu \), and the background field configurations are given as above:
\[
\begin{align*}
e^{-\varphi_0} &= W(\eta), \quad e^{v_0} = A(\eta) W^4(\eta), \quad e^{b_0} = A(\eta), \\
e^{-\varphi_0} &= W^2(\eta) e^{\lambda_3 \eta} \quad \text{and} \quad a'_\theta = \frac{qA^2(\eta)}{W^2(\eta)} e^{-\lambda_3 \eta}. 
\end{align*}
\] (4.21)

For some of the discussion to follow it is useful to choose proper distance computed using the background metric as a coordinate within the extra dimensions. As always, we reserve the variable \( r \) for proper distance, and it is related explicitly to \( \eta \) by \( \text{d}r = e^{v_0(\eta)} \text{d}\eta = A W^4 \text{d}\eta \). Notice that this choice of coordinates can be made independently of the gauge choice we make on the linearized fluctuations, which we now describe.

**Three useful gauge choices**

Finally, it is also not necessary to keep \( M, V \) and \( A_\theta \) independent, because one of these can be set to zero using an appropriate coordinate choice. There are three gauge choices which we use in what follows.
• **Comoving** \((c)\) gauge: defined by the condition \(A^{(c)}_\theta = 0\), leading to

\[
ds^2 = e^{2a_0}e^{2A^{(c)}} \eta_{\mu\nu} dx^\mu dx^\nu + e^{2\nu_0}e^{2V^{(c)}} d\eta^2 + e^{2b_0}e^{2B^{(c)}} d\theta^2 + 2N_\mu d\eta dx^\mu,
\]

(4.22)

together with \(A = a_\theta d\theta\) and \(\varphi = \varphi_0 + \Phi^{(c)}\). Notice that this gauge does not fall completely into the ansatz of eq. (4.1), because the coordinate transformation required to get to the form (4.1) does not preserve the condition \(\delta A_\theta = 0\). This is the origin of the new metric variable, \(\delta g_{\eta\mu}\), in the above expression. Notice also that we have used the gauge freedom to set \(M = 0\). (For counting purposes, we see that we have removed one degree of freedom, \(A_\theta\), while \(N\) simply replaces the degree of freedom \(M\).

• **Gaussian Normal** \((GN)\) gauge: defined by \(V^{(GN)} = 0\), and so:

\[
ds^2 = e^{2a_0} \left( e^{2A^{(GN)}} \eta_{\mu\nu} + M^{(GN)}_{\mu\nu} \right) dx^\mu dx^\nu + e^{2\nu_0}e^{2B^{(GN)}} d\theta^2 + dr^2,
\]

(4.23)

together with \(\varphi = \varphi_0 + \Phi^{(GN)}\) and \(A_\theta = a_\theta + A^{(GN)}_\theta\). It is obviously convenient also to use background-metric proper distance, \(r\), defined via \(dr = e^{\nu_0} d\eta\).

• **Longitudinal** \((l)\) gauge: defined by \(M = 0\) and so

\[
ds^2 = e^{2a_0}e^{2A^{(l)}} \eta_{\mu\nu} dx^\mu dx^\nu + e^{2b_0}e^{2B^{(l)}} d\theta^2 + e^{2V^{(l)}} dr^2,
\]

(4.24)

together with \(\varphi = \varphi_0 + \Phi^{(l)}\) and \(A_\theta = a_\theta + A^{(l)}_\theta\). Notice that we again choose to use background-metric proper distance, \(r\), as the radial coordinate.

Which of these gauges is most convenient depends on the goal of the calculation. Comoving gauge is physically intuitive (since one follows perturbations along hypersurfaces of constant \(A_\theta\)), and has the enormous benefit that the field equations
completely decouple in this gauge, making the linearization analysis much easier. Longitudinal gauge is convenient for making contact with earlier calculations in the literature, since this is the gauge which has often been used. Finally, GN gauge is useful for analyzing the boundary conditions which fluctuations must satisfy near the source branes.

It is clearly useful to be able to transform between expressions obtained in these three gauges, so we next briefly summarize the main formulae which are required. Under the coordinate transformation $x^M \rightarrow x^M + \xi^M$, where $\xi^\mu = \partial^\mu \epsilon$ and $\xi^r = \epsilon$, we have that to linear order the line element transforms as

$$ds^2 \rightarrow e^{2a_0} \left[ e^{2(\epsilon \partial_r a_0)} \eta_{\mu\nu} + \partial_\mu \partial_\nu \left( M + 2\epsilon \right) \right] dx^\mu dx^\nu + 2\partial_\mu \left[ e^{2a_0} \partial_r \epsilon + \epsilon \right] dr dx^\mu$$

$$+ e^{2(V + \partial_r \epsilon)} dr^2 + e^{2a_0} \left[ e^{2(B + \partial_r \phi_0)} \right] d\theta^2,$$  

(4.25)

while the gauge field and dilaton transform according to

$$A_\theta \rightarrow A_\theta + \epsilon \partial_\theta a_\theta$$  

(4.26)

$$\Phi \rightarrow \Phi + \epsilon \partial_r \phi_0.$$  

(4.27)

Note that in these expressions we choose to use the radial coordinate $r$ as opposed to $\eta$. From these transformations, it is straightforward to move between the various gauges.

To change from longitudinal to comoving gauge requires the transformation $x^M_{(l)} = x^M_{(c)} + \xi^M$, with $\xi^M = \epsilon \delta^M_r$ so that $r_{(l)} = r_{(c)} + \epsilon$, with $\epsilon = -A^{(l)}_\theta / \partial_r a_\theta$ required to ensure $A^{(c)}_\theta = 0$ while $\epsilon = 0$ is chosen to ensure that $M^{(c)} = M^{(l)} = 0$. This implies that the various fields in longitudinal gauge are related to the ones in comoving gauge
by:

\[ A^{(c)} = A^{(l)} + \varepsilon \partial_r a_0, \quad B^{(c)} = B^{(l)} + \varepsilon \partial_r b_0, \quad V^{(c)} = V^{(l)} + \partial_r \varepsilon, \quad (4.28) \]
\[ N^{(c)} = e^{4a_0 + b_0} \varepsilon, \quad \Phi^{(c)} = \Phi^{(l)} + \varepsilon \partial_r \varphi_0. \]

Similarly, to go from comoving to GN gauge, we perform the transformations
\[ r^{(c)} = r^{(GN)} + \varepsilon \text{ with } \partial_r \varepsilon = -V^{(c)} \text{ chosen to ensure } V^{(GN)} = 0. \]

The further transformation, \( x^{\mu}_{(c)} = x^{\mu}_{(GN)} + \partial^{\mu} \varepsilon \), with \( \partial^{\mu} \varepsilon = -e^{-(6a_0 + b_0)} N^{(c)} - e^{-2a_0} \varepsilon \) is also required in order to ensure the vanishing of \( N^{(GN)} \). Quantities in comoving gauge are then related to those in GN gauge by

\[ A^{(GN)} = A^{(c)} + \varepsilon \partial_r a_0, \quad M^{(GN)} = 2\varepsilon, \quad B^{(GN)} = B^{(c)} + \varepsilon \partial_r b_0 \]
\[ \Phi^{(GN)} = \Phi^{(c)} + \varepsilon \partial_r \varphi_0, \quad A^{(GN)}_{\theta} = \varepsilon \partial_r a_{\theta}. \quad (4.29) \]

### 4.2.2 Linearized Equations in Comoving Gauge

We now work out the perturbed equations in comoving gauge, since in this gauge we are able to decouple the fluctuations even when perturbing about non-conical background solutions. Note that the equations of motion given in §4.1 were in longitudinal gauge, so we could proceed by linearizing these equations and then using the results of the previous section in order to transform to the desired comoving gauge. However, it turns out to be easier to derive from scratch the equations of motion using the metric ansatz appropriate for comoving gauge, eq. (4.22).

In the following, we use the coordinate \( \eta \) with the derivative \( d/d\eta \) denoted by primes, and we allow to the perturbations to depend on all \( x^{\mu} \) as well as on \( \eta \). We
adopt a notation for the perturbations where $A^{(c)} = -\Psi/2$ and $V^{(c)} = \xi/2$, and so

$$
\text{d}s^2 = e^{(x+y)/2}e^{-\Psi} \eta_{\mu\nu} \text{d}x^\mu \text{d}x^\nu + e^{(x+y+2z)/2} \left[ e^{2(x+y)}e^{\xi} \text{d}\eta^2 + e^{2B} \text{d}\theta^2 \right] + 2N_{,\mu} \text{d}\eta \text{d}x^\mu,
$$

(4.30)

while $e^\phi = e^{(x+y-2z)/2}e^\Phi$ and $F_{\eta\theta} = a_{\theta} = qe^{2x}$. In terms of these variables, and Fourier transforming with respect to time and space\(^2\) so $\Box = \eta^{\mu\nu} \partial_\mu \partial_\nu = \omega^2$, the Maxwell equation is

$$
2 \omega^2 e^{(x-y)/2}N + 2B' + 2\Phi' + \xi' + 4\Psi' = 0; \ (4.31)
$$

the dilaton equation is

$$
\Phi'' + \omega^2 e^{2y+z} \Phi - \frac{1}{2} \left( \kappa^2 \alpha^2 e^{2y} + \frac{4g^2}{\kappa^2} e^{2y} \right) \Phi = \varphi'_0 \left( \frac{1}{2} \xi' + 2\Psi' - B' + \omega^2 e^{(x-y)/2}N \right) + \frac{2g^2}{\kappa^2} e^{2y} \xi + \kappa^2 \alpha^2 e^{2x} B; \ (4.32)
$$

the $(\mu\eta)$ Einstein equation is

$$
\partial_\mu \left[ 2(2\chi' + Z')B - (2\chi' + Z')\xi + 4\varphi'_0 \Phi + 4B' - 6\Psi \right] = 0; \ (4.33)
$$

while the other off-diagonal term in the Einstein equation $(\mu \neq \nu)$ is

$$
\partial_\mu \partial_\nu \left[ e^{1/2(x-5y-2z)} \left[ 2N' + (\chi' - \chi')N \right] - 2B - \xi + 2\Psi \right] = 0. \ (4.34)
$$

\(^2\) We emphasize that $\omega$ is the mass of the corresponding Kaluza-Klein mode and should be understood as the eigenvalue of the operator $\Box$. In this analysis, we do not make the assumption that the modes have a specific time-oscillatory behavior of the form $e^{ik_0t}$. Thus this analysis includes both time-oscillating modes, and non-oscillating ones which correspond to zero-mass Kaluza Klein modes.
A combination of the \((\eta \eta)\) Einstein equation, the dilaton equation of motion and the trace of the Einstein equation which involves only first derivatives with respect to \(\eta\) is

\[
\frac{\omega^2}{2} e^{(X - Y)/2} (2Y' + Z') N - \frac{2g_1^2}{\kappa^2} e^{2Y} \xi + \frac{3\omega^2}{2} e^{2Y + Z} \Psi
\]

\[
+ (2Y' + Z') \Psi' - \frac{1}{2} \left( \kappa^2 q^2 e^{2X} + \frac{4g_1^2}{\kappa^2} e^{2Y} \right) \Phi
\]

\[
+ \varphi_0 \Phi' - \kappa^2 q^2 e^{2X} B - \omega^2 e^{2Y + Z} B + (\chi' - Y') B' = 0 .
\] (4.35)

The \((\theta \theta)\) Einstein equation is similarly

\[
(2B'' + \Phi'') + \omega^2 e^{2Y + Z} (2B + \Phi) - 2\kappa^2 q^2 e^{2X} (2B + \Phi) + 2\chi' B'
\]

\[
= \chi' \left( 4\Psi' + \xi' + 2\omega^2 e^{(X - Y)/2} N \right) ,
\] (4.36)

and finally, tracing the \((\mu \mu)\) Einstein equations gives

\[
(\Phi'' + \omega^2 e^{2Y + Z} \Phi) - (\Psi'' + \omega^2 e^{2Y + Z} \Psi - 2Z' \Psi')
\]

\[
= Z' \left( B' - \frac{1}{2} \xi' - \omega^2 e^{(X - Y)/2} N \right) .
\] (4.37)

We have here a set of seven equations for five variables \((\xi, N, \Psi, B\) and \(\Phi)\). Two of these equations appear as constraints and can be used to fix the lapse and shift functions: using the \((\mu \eta)\) equation (4.33) and the \((\eta \eta)\) equation (4.35), we can express both \(\xi\) and \(\omega^2 N\) in terms of the three remaining variables \(\Psi, B\) and \(\Phi\), leading to five remaining equations for three variables. However these remaining equations are not independent since the Bianchi identities ensure that two of them are redundant. One can be expressed as a linear combination of the others, whilst the derivative of
the Maxwell equation (4.31) is simply the derivative of (4.33) and a combination of the other equations.

We arrive in this way with three equations for three unknowns, which can be written as

\[
B'' + \omega^2 e^{2\gamma + z} B + \frac{1}{4} \left( 6\lambda' + 2\gamma' + 4Z' + \frac{16 g_1^2}{\kappa^2} \frac{e^{2\gamma}}{2\gamma' + Z'} \right) B' + \frac{1}{8} \left( -12\kappa^2 q^2 e^{2\gamma} + \frac{16 g_1^2}{\kappa^2} e^{2\gamma} \frac{2\lambda' + Z'}{2\gamma' + Z'} \right) B + \frac{1}{4} \left( 3\lambda' + \gamma' + 2Z' \right) \Phi' = 0, \tag{4.38}
\]

\[
\Psi'' + \omega^2 e^{2\gamma + z} \Psi + \frac{12 g_1^2}{\kappa^2} \frac{e^{2\gamma}}{(2\gamma' + Z')} \Psi' + \frac{1}{2} (\lambda' - \gamma') (\Phi' + 2B') - \frac{8 g_1^2}{\kappa^2} \frac{e^{2\gamma}}{2\gamma' + Z'} B' - \frac{\kappa^2 q^2}{2} e^{2\gamma} (\Phi + 2B) \]
\[
- \frac{2 g_1^2}{\kappa^2} \frac{e^{2\gamma}}{2\gamma' + Z'} [(2\lambda' - 3Z') \Phi + 2(2\lambda' + Z') B] = 0, \tag{4.39}
\]

and

\[
\Phi'' + \omega^2 e^{2\gamma + z} \Phi + \varphi'_0 (\Phi' + 2B') + \frac{4 g_1^2}{\kappa^2} \frac{e^{2\gamma}}{2\gamma' + Z'} (3\Psi' - 2B') - \frac{\kappa^2 q^2}{2} e^{2\lambda} (\Phi + 2B) - \frac{2 g_1^2}{\kappa^2} \frac{e^{2\gamma}}{2\gamma' + Z'} [(2\lambda' - 3Z') \Phi + 2(2\lambda' + Z') B] = 0. \tag{4.40}
\]
These equations dramatically simplify once the change of variables \((\Phi, B, \Psi) \rightarrow (f, \chi, \psi)\) is performed, with\(^3\)

\[
\begin{align*}
\Phi &= \frac{2U}{\kappa q} e^{-\chi} f - 2B \\
B &= \frac{\mathcal{V}}{\sqrt{3U}} \chi - \frac{1}{2} \Psi + \frac{2U^2 + \chi'\mathcal{Z}'}{2\kappa q U} e^{-\chi} f \\
\Psi &= \frac{2\mathcal{V}' + \mathcal{Z}'}{4\mathcal{V}} \psi + \frac{2U^2 - 3\mathcal{V}' \mathcal{Z}'}{4\sqrt{3U}\mathcal{V}} \chi + \frac{U^2 - \chi'\mathcal{V}'}{2\kappa q U} e^{-\chi} f ,
\end{align*}
\]

where use of the background field equations shows that the nominally field-dependent quantities \(U\) and \(V\) are really both constants:

\[
\begin{align*}
U^2 &\equiv \chi'^2 + \kappa^2 q^2 e^{2\mathcal{V}} = \lambda_1^2 \\
V^2 &\equiv \chi'^2 + \kappa^2 q^2 e^{2\mathcal{V}} + \frac{3}{4} \mathcal{Z}'^2 = \lambda_1^2 + \frac{3}{4} \lambda_3^2 .
\end{align*}
\]

Notice that for conical backgrounds \(U = V = \lambda_1\). In terms of these new variables the three field equations — eqs. (4.38), (4.39) and (4.40) — take the remarkably simple, decoupled form

\[
\begin{align*}
\chi'' + \omega^2 e^{2\mathcal{V} + \mathcal{Z}} \chi &= 0 \\
f'' + \omega^2 e^{2\mathcal{V} + \mathcal{Z}} f - U^2 f &= 0 \\
\psi'' + \omega^2 e^{2\mathcal{V} + \mathcal{Z}} \psi - \frac{32 g_1^2 \mathcal{V}^2 e^{3\mathcal{V}}}{\kappa^2 (2\mathcal{V}' + \mathcal{Z}'^2)} \psi &= 0 .
\end{align*}
\]

Because these equations decouple from one another each can be solved separately, and presents no problem for numerical analysis. Exact solutions are also

---

\(^3\) Take care not to confuse \(\chi\) and \(\mathcal{X}\)!
available in the case of backgrounds having conical singularities (including but not restricted to the rugby-ball geometries), as shall be described in detail in later sections. It is noteworthy for later purposes that the decoupled equations (4.43 – 4.45) are independent of the background field $X$ (and hence also of the parameter $\xi_1$). This implies, for example, that the linearized equations of motion for the variables $\chi, f,$ and $\psi$ are identical for a general conical background geometry (for which $Z' = 0$) and a rugby-ball geometry (for which $Z' = 0$ and $X' = Y'$). However once the solutions are inserted into the expressions for the metric fluctuations using relations (4.41), the results do depend on $X$ and so the full fluctuations ‘know’ about the complete background geometry.

4.3 Properties of Solutions

Up to this point there has been no restriction on the form of the background geometry. In this section we specialize to special conical backgrounds for which it is possible to solve the above linearized equations in closed form. We then return to the more general non-conical geometry in order to discuss the asymptotic behaviour of the solutions in the near-brane limit. The section closes with an outline of how this asymptotic form is related to the physical properties of the branes which source the bulk geometries.

4.3.1 Analytic Solutions for General Conical Backgrounds

We now concentrate on the behaviour of the perturbations to solutions having only conical singularities, which are defined by the condition $\lambda_3 = 0$ (and so $Z' = 0$). Performing the change of variable $z = \tanh [\lambda_2 (\eta - \xi_2)]$, the perturbation equations
become
\[
\frac{d^2\chi}{dz^2} - \left(\frac{2z}{1-z^2}\right)\frac{d\chi}{dz} + \left(\frac{\mu^2}{1-z^2}\right)\chi = 0 \quad (4.46)
\]
\[
\frac{d^2\psi}{dz^2} - \left(\frac{2z}{1-z^2}\right)\frac{d\psi}{dz} + \left(\frac{1}{1-z^2}\right)\left(\mu^2 - \frac{2}{z^2}\right)\psi = 0 \quad (4.47)
\]
\[
\frac{d^2f}{dz^2} - \left(\frac{2z}{1-z^2}\right)\frac{df}{dz} + \left(\frac{1}{1-z^2}\right)\left(\mu^2 - \frac{1}{1-z^2}\right)f = 0, \quad (4.48)
\]
with \(\mu^2 = \omega^2\kappa^2/4g_1^2\), and \(\omega\) is the 4D mode mass. Notice that these equations are symmetric under \(z \rightarrow -z\), which corresponds to reflections of \(\eta\) around \(\eta = \xi_2\). The reason for this symmetry is clear in the special case of the rugby-ball solutions (for which \(\xi_1 = \xi_2\)), since it then corresponds to reflections of the spherical geometry about its equator. In the more general warped conical geometries this symmetry instead follows from the above-mentioned circumstance that the equations are completely independent of \(\xi_1\), and so take the same form for general conical geometries as they do for the rugby ball.

Defining \(\nu = \frac{1}{2}\left[-1 + \sqrt{1 + 4\mu^2}\right]\) — or equivalently writing \(\mu^2 = \nu(\nu + 1)\) — the solutions to these equations become

\[
\chi = C_1 P_\nu(z) + C_2 \text{Re} Q_\nu(z) \quad (4.49)
\]
\[
= C_1 F\left[-\nu, 1 + \nu; 1; \frac{1}{2}(1-z)\right] + C_2 \sqrt{\pi} \frac{\Gamma(\nu + 1)}{\Gamma(\nu + \frac{3}{2})} \text{Re} \frac{1}{(2z)^{\nu+1}} F\left[\frac{1}{2}(1 + \nu), 1 + \nu + \frac{3}{2}; z^{-2}\right] \quad (4.50)
\]
\[
\psi = C_3 z^{-1} F\left[-\frac{1}{2}(1 + \nu), \frac{\nu}{2}; -\frac{1}{2}; z^2\right] + C_4 z^2 F\left[1 - \nu, \frac{1}{2}(3 + \nu); \frac{5}{2}; z^2\right] \quad (4.51)
\]
\[
f = \frac{1}{\sqrt{1-z^2}} \left(C_5 F\left[\frac{\nu}{2}, -\frac{1}{2}(1 + \nu); \frac{1}{2}; z^2\right] + C_6 z F\left[\frac{1}{2}(1 + \nu), -\nu, \frac{3}{2}; z^2\right]\right)
\]
where the $C_i$, $i = 1, \ldots, 6$ are integration constants and $F[a, b; c; z]$ is the hypergeometric function. Neglecting the overall normalization of all three modes, we see that the above expressions provide a 9-parameter family of solutions. Notice also that these solutions can be decomposed into symmetric and anti-symmetric combinations under the symmetry $z \to -z$, and this is already explicit for the functions $\psi$ and $f$.

We see that the mode energies satisfy

$$\omega^2 = \frac{\nu(\nu + 1)}{r_0^2},$$

with $1/r_0 = 2g_1/\kappa$, and so the spectrum depends on the values which are allowed for the parameter $\nu$. Notice that for real $\nu$ we have $\omega^2 \geq 0$ except for the interval $-1 < \nu < 0$. As usual, the values which are allowed for $\nu$ are related to the behaviour which is demanded of the solutions at the boundaries, which in the present instance corresponds to the near-brane limits $z \to \pm 1$. For this reason we next explore the asymptotic form taken by the solutions in the near-brane limit.

4.3.2 Asymptotic Forms

We next examine three aspects of the near-brane limit. In this section we first explore the asymptotic forms taken by the solutions just constructed for fluctuations about backgrounds having only conical singularities. We then see how these asymptotic forms generalize to the broader situation where the background has more generic singularities. A preliminary connection is drawn in the next section between this asymptotic behaviour and the properties of the source branes.
**Conical backgrounds**

The asymptotic form of the solutions found above near the brane positions at $z = \pm 1$ may be found by using the properties of the Hypergeometric functions given in appendix B. This shows that for generic $\nu$, $\chi$ and $\psi$ diverge logarithmically as one approaches the branes, whereas $f$ generically diverges as $(1 - z^2)^{-1/2} \sim \cosh[\lambda_2(\eta - \xi_2)] \sim r^{-1}$.

As usual, less singular asymptotic behaviour can be possible for those $\nu$ for which the hypergeometric series terminates, which for $F[a, b; c; z]$ occurs when either $a$ or $b$ is a negative integer (or zero). Inspection of the explicit solutions then shows that termination can happen when $\nu$ is quantized to be a non-negative integer: $\nu = \ell = 0, 1, 2, \ldots$. Notice that the resulting quantization of KK mass, from eq. (4.52), implies in this case $\omega^2 \geq 0$.

For $\nu = \ell$ one — but not both — of the two Hypergeometric functions appearing for each of $\chi$, $\psi$ and $f$ terminates, allowing a less singular near-brane limit to be obtained through an appropriate choice for the integration constants $C_i$. For instance, if we require — as we shall argue on the grounds of normalizability below — that all fluctuations must be less singular than $(1 - z^2)^{-1/2}$, then this is possible for $f$ only if $\nu = \ell$ and one of $C_5$ or $C_6$ vanishes (which one depends on whether $\ell$ is even or
odd).\footnote{4} Once this has been done the functions $\chi$ and $\psi$ can still diverge logarithmically as $z \to \pm 1$, leaving a 4-parameter family of potentially logarithmically singular solutions (up to overall normalization).

Completely regular solutions are also possible when $\nu = \ell$, provided that one of $C_1$ or $C_2$ and one of $C_3$ or $C_4$ also vanishes. For instance, $\chi$ is regular everywhere if $\nu = \ell$ and $C_2 = 0$ since in this case the solution $P_\ell$ degenerates to the Legendre polynomials. This leaves a 2-parameter family of non-singular solutions (up to overall normalization).

**Non-conical backgrounds**

Although it is difficult to find explicit solutions in the generic case where the background has non-conical singularities, it is possible to use the linearized equations to estimate the asymptotic behaviour of the mode functions as $\eta \to \pm \infty$ (or $z \to \pm 1$).

Repeating the steps which led to the equations of the previous section for this more general case, we instead find

\[
\frac{d^2 \chi}{dz^2} - \left( \frac{2z}{1 - z^2} \right) \frac{d \chi}{dz} + \frac{\tilde{\mu}^2}{(1 - z^2)} \left( \frac{1 + z}{1 - z} \right)^{\delta \lambda} \chi = 0 \tag{4.53}
\]

\[
\frac{d^2 \psi}{dz^2} - \left( \frac{2z}{1 - z^2} \right) \frac{d \psi}{dz} + \left( \frac{1}{1 - z^2} \right) \left[ \tilde{\mu}^2 \left( \frac{1 + z}{1 - z} \right)^{\delta \lambda} - \frac{2}{z^2} \right] \psi = 0 \tag{4.54}
\]

\[
\frac{d^2 f}{dz^2} - \left( \frac{2z}{1 - z^2} \right) \frac{d f}{dz} + \left( \frac{1}{1 - z^2} \right) \left[ \tilde{\mu}^2 \left( \frac{1 + z}{1 - z} \right)^{\delta \lambda} - \frac{1 - 4\delta \lambda^2}{1 - z^2} \right] f = 0, \tag{4.55}
\]

with $\tilde{\mu}^2 = \mu^2 e^{\lambda_3 \xi_2}$ and $\delta \lambda = \lambda_3 / 2 \lambda_2$.

\footnote{4 We give explicit expressions for some of the lowest of these non-singular modes for $\psi$ and $f$ in appendix A for longitudinal gauge.}
Since $|\delta \lambda| < 1/2$ the term in eq. (4.53) which is proportional to $\chi$ remains subdominant compared to the terms in $(1 - z^2)^{-1} \partial \chi/\partial z$ and $d^2 \chi/dz^2$, such that the asymptotic behaviour of $\chi$ still remains logarithmic near the brane. Similar arguments show that $\psi$ and $f$ also keep the same asymptotic behaviour as in the conical case.

**Normalizability**

A restriction on the asymptotic forms follows from the condition that the mode functions be normalizable, as we now explore. As we show in detail in the next section, the equations of motion for each of the decoupled modes can be derived from a reduced action, eq. (4.71). In this action all modes, $u$, have the same kinetic term, $e^{2Y+Z} u \Box u$, where $e^{2Y+Z} = A^2 \mathcal{W}^6$, and so the definition of the inner product can be taken to be the same for all three of $\chi, \psi$, and $f$, having the form:

$$
\langle u, v \rangle = \frac{i}{2} \int_{\Sigma} d^3x \, d\theta \, d\eta \, A^2 \mathcal{W}^6 (u \partial_t v^* - u^* \partial_t v),
$$

for any two modes $u$ and $v$. Here $\Sigma$ denotes a surface of constant $t$. The norm of a mode is therefore defined by

$$
\langle u, u \rangle = \omega \int_{\Sigma} d^3x \, d\theta \, d\eta \, A^2 \mathcal{W}^6 uu^* = \omega \int_{\Sigma} d^3x \, d\theta \, dr \, A \mathcal{W}^2 uu^*,
$$

(4.56)

where we use $dr/d\eta = A \mathcal{W}^4$ and $\partial_t u = -i \omega u$.

Our interest is in the conditions placed on the asymptotic behaviour of linearized modes by the convergence of these integrals. We take the usual ansatz for the
asymptotic form of the metric and fields, namely

\[ ds^2 \sim [c_\alpha (Hr)^\alpha]^2 \eta_{\mu \nu} \, dx^\mu dx^\nu + dr^2 + [c_\theta (Hr)^\beta]^2 d\theta^2 \]

\[ e^\varphi \sim c_\varphi (Hr)^\xi \quad \text{and} \quad F^{\rho \theta} \sim c_f (Hr)^\gamma. \]  

Using a subscript zero to denote background quantities, we have \( AW^2 \propto r^{\beta_0 + 2\alpha_0} \propto r^{1-2\alpha_0} \) in the near-brane limit \((r \to 0)\). In order to be normalizable, modes should therefore vary as \( u \propto r^\upsilon \) with power \( \upsilon > -1 + \alpha_0 \) as \( r \to 0 \). In particular, for a conical background with \( \alpha_0 = 0 \) and \( \beta_0 = 1 \), modes must diverge less rapidly than \( 1/r \) near both branes.

4.3.3 Boundary Conditions and Brane Properties

The above considerations show that the stability of the linearized fluctuations is related to the behaviour of the modes as they approach the branes. In this section we briefly explore to what extent the properties of the linearized solutions capture what we know about the asymptotic properties of the exact solutions. Exploring this connection also allows us to make some contact between the conditions for stability and the physical properties of the branes which source the bulk field configurations, through considerations along the lines of those given in refs. [2, 53, 71].

Asymptotics in GN gauge

Recall for these purposes that on general grounds the various bulk fields are known (in GN gauge) to approach the branes through a power-law form given by eqs. (3.1). In principle there are two cases to consider, depending on whether or not the background and perturbed fields share the same powers in these asymptotic expressions. That is, if we imagine that the background and perturbed metric
components, $e^{a_0}$ and $e^a = e^{a_0} e^A$, satisfy
\[ e^{a_0} \to c_{a_0} (H r)^{a_0} \quad \text{and} \quad e^a \to c_a (H r)^a, \tag{4.58} \]

near $r = 0$, then we see that the fluctuation field, $A$, must satisfy
\[ A \to (\alpha - \alpha_0) \ln (H r) + \ln \left( \frac{c_a}{c_{a_0}} \right) + \cdots, \tag{4.59} \]
in the same limit. Similar expressions also hold for the fields $B$, $V$, and $\Phi$.

Since the Maxwell field strength varies as $F_{r \theta} \propto r^{\gamma + 2 \beta} \propto r^{\zeta + 2 \beta - 1}$ the same argument has slightly different implications for $A_\theta$, since $a_\theta \propto r^{\zeta_0 + 2 \beta_0}$ and $A_\theta = a_\theta + A_\theta \propto r^{\zeta + 2 \beta}$. (Notice that this gives the expected results $F_{r \theta} \propto r$ and $A_\theta \propto r^2$ in the conical case, for which $\beta = 1$ and $\zeta = 0$.) In this case we have $A_\theta \propto r^{\zeta + 2 \beta}$ if $\zeta + 2 \beta < \zeta_0 + 2 \beta_0$, because the small-$r$ behaviour of $A_\theta$ dominates that of $a_\theta$. Things are different if $\zeta + 2 \beta > \zeta_0 + 2 \beta_0$, however, since in this case $a_\theta$ dominates $A_\theta$, and in order to achieve this $A_\theta$ must contain a piece which varies in the same way and cancels the contribution of $a_\theta$ as $r \to 0$. As a consequence in this case $A_\theta \propto r^{\zeta_0 + 2 \beta_0}$ for small $r$.

How singular this is depends on the allowed range of $\zeta + 2 \beta$. Notice that in general the constraints $4 \alpha + \beta = 4 \alpha^2 + \beta^2 + \zeta^2 = 1$ require $\frac{5}{4} \beta^2 - \frac{1}{2} \beta + \zeta^2 = \frac{3}{4}$. Solving this equation for $\zeta$, the sum $\zeta + 2 \beta$ is thus minimal (‘–’ sign) and maximal (‘+’ sign) for $\beta = \frac{1}{5} \pm \frac{16}{5 \sqrt{21}}$, and $\zeta = \mp \frac{2}{\sqrt{21}}$, so that the sum is in the range $-1.433 \simeq \frac{2}{5} (1 - \sqrt{21}) \leq \zeta + 2 \beta \leq \frac{2}{5} (1 + \sqrt{21}) \simeq 2.233$. We see that perturbations to $A_\theta$ can become quite singular. In the particular case of conical singularities $\zeta = 0$ and $\beta = 1$ and so $\zeta + 2 \beta = 2$, leading to smooth behaviour near the branes.
We see from the above that the perturbations need not be smooth at the brane positions, but that in GN gauge the metric perturbations can at worst diverge logarithmically in the near-brane limit. (Perturbations to the Maxwell field can be more singular than logarithmic, but only if $\zeta$ and $\beta$ are sufficiently negative.) Since the arguments of refs. [2, 53, 71] relate powers like $\alpha$ to physical properties on the branes, these logarithmically singular perturbations only arise if some of the brane properties are themselves perturbed. It is only in the limit that the asymptotic near-brane behaviour is the same before and after perturbation that the perturbed solutions are non-singular at the brane positions. This is the case in particular if it is assumed that the geometry has purely conical singularities before and after perturbation, or if the sources are represented as delta functions with the implicit associated assumption that the bulk fields are well-defined when evaluated at the brane positions — as is common in higher-codimension stability analyses, such as that of ref. [75].

**Asymptotic forms in comoving gauge**

Although we have seen that, within GN gauge, metric fluctuations diverge at most logarithmically near the source branes, we next determine what this implies for the near-brane behaviour in the (comoving) gauge we use in our analysis. In this section we examine in particular the near-brane asymptotics in comoving gauge.

Choosing background proper distance $r$ (rather than $\eta$) as the radial coordinate, we have seen that the perturbations in comoving and GN gauge are related by eqs. (4.29), which we reproduce here for ease of reference:

\[
A^{(c)} = A^{(GN)} - \varepsilon \partial_r a_0, \quad B^{(c)} = B^{(GN)} - \varepsilon \partial_r b_0
\]

\[
\Phi^{(c)} = \Phi^{(GN)} + \varepsilon \partial_r \varphi_0, \quad V^{(c)} = -\partial_r \varepsilon,
\]

(4.60)
with \( \varepsilon = A_\theta^{(GN)}/(\partial_r a_\theta) \), and \( \epsilon = M^{(GN)}/2 \) chosen to ensure that \( M^{(c)} = 0 \).

Since we know how the fluctuations vary with \( r \) as \( r \to 0 \) within GN gauge, the above equations allow this information to be carried over to comoving gauge. In particular, since \( a_\theta + A_\theta \propto r^{\zeta+2\beta} \) and \( a_\theta \propto r^{\zeta_0+2\beta_0} \), we saw \( A_\theta \propto r^{\Xi} \), where \( \Xi = \min(\zeta + 2\beta, \zeta_0 + 2\beta_0) \). We see that \( \varepsilon \propto r^{1+\Delta} \) and that there are two cases: (i) \( \Delta = (\zeta - \zeta_0) + 2(\beta - \beta_0) \) if \( \zeta + 2\beta < \zeta_0 + 2\beta_0 \), or (ii) \( \Delta = 0 \) if \( \zeta + 2\beta \geq \zeta_0 + 2\beta_0 \).

Since \( a_0, b_0, \varphi_0, A^{(GN)}, B^{(GN)} \) and \( \Phi^{(GN)} \) all vary at most logarithmically for small \( r \), we see from the above that all of the fluctuations in comoving gauge are also at worst logarithmic provided only that \( \Delta = (\zeta + 2\beta) - (\zeta_0 + 2\beta_0) \geq 0 \). (In particular, for conical backgrounds the perturbations are at most logarithmic provided \( \Delta = (\zeta + 2\beta - 2) \geq 0 \).)

These arguments show that the fluctuations \( \Phi, B \) and \( \Psi \) diverge at most logarithmically within comoving gauge for a broad range of physical situations. Inspection of the definitions, eqs. (4.43)-(4.45), shows that this implies the same conclusion for \( e^{-X} f, \chi \) and \( \psi \).

4.4 General Stability Analysis

We now give two general arguments in favour of stability for a broad class of boundary conditions. The first of these works directly with the linearized equations derived above, while the second argument is instead cast in terms of the action, and closes a loophole left by the previous equation-of-motion analysis.
4.4.1 Equations of Motion and Tachyons

We start by arguing for stability directly with the equations of motion. The goal of this argument is to relate the sign of the energy eigenvalue, \( \omega^2 \), to the boundary conditions which the fluctuations satisfy near the positions of the source branes.

The most direct way to do so is to multiply eq. (4.43) by \( \chi^* \); sum the result with its complex conjugate; and integrate the answer over the extra dimensions, to get:

\[
\omega^2 \int_{-\infty}^{\infty} d\eta e^{2\gamma^+ \bar{Z}} |\chi|^2 = -\frac{1}{2} \left[ (|\chi|^2)' \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} d\eta |\chi'|^2. \tag{4.61}
\]

This shows how the sign of \( \omega^2 \) is related to the behaviour of \( \chi \) near the two brane positions. Notice that the combination \( e^{2\gamma^+ \bar{Z}} \) appearing on the left-hand side of (4.61) is related to \( \mathcal{A} \) and \( \mathcal{W} \) by \( e^{2\gamma^+ \bar{Z}} = \mathcal{A}^2 \mathcal{W}^6 \).

Although we do not have explicit solutions to the linearized equations for general non-conical backgrounds, the asymptotic behaviour of these solutions was argued above to be the same as in the conical case. In particular, both fluctuations \( \chi \) and \( \psi \) are typically proportional to \( \log r \sim \eta \) as \( \eta \to \pm \infty \), whereas \( f \) usually diverges as \( e^{\lambda_2 |\eta|} \) in this limit. Thus the general asymptotic behaviour as \( \eta \to \pm \infty \) is

\[
\chi \sim (C_\pm \eta + D_\pm), \quad \psi \sim (E_\pm \eta + F_\pm),
\]

and

\[
f \sim G_\pm e^{\lambda_\pm |\eta|},
\]

for constants \( C_\pm, D_\pm, E_\pm, F_\pm \), and \( G_\pm \). Using this in eq. (4.61), and cutting off the integrations at \( \eta = \pm \Lambda \), gives

\[
\omega^2 \int_{-\Lambda}^{\Lambda} d\eta e^{2\gamma^+ \bar{Z}} |\chi|^2 = -\left( |C_+|^2 + |C_-|^2 \right) \Lambda - \text{Re} \left( D^*_+ C_+ - D^*_- C_- \right) + \int_{-\Lambda}^{\Lambda} d\eta |\chi'|^2. \tag{4.63}
\]
We see from this that if the fluctuation is required to remain finite on both branes — i.e. $C_+ = C_- = 0$ — the boundary term vanishes and the squared energy of any mode is necessarily positive: $\omega^2 \geq 0$. Furthermore, $\omega$ only vanishes if $\chi$ is a constant throughout the entire bulk.

Applying the same argument to the $\psi$ equation gives:

$$\omega^2 \int_{-\Lambda}^{+\Lambda} d\eta e^{2\gamma+Z} |\psi|^2 = -\frac{1}{2} \left[ (|\psi|^2)' \right]_{-\Lambda}^{+\Lambda} + \int_{-\Lambda}^{+\Lambda} d\eta \left( |\psi'|^2 + \frac{32 g_1^2 V_2 e^{2\gamma}}{k^2} \frac{V_2^2 e^{2\gamma}}{(2\gamma + Z')^2} |\psi|^2 \right)$$

$$= - (|E_+|^2 + |E_-|^2) \Lambda - \text{Re} \left( F_+^* E_+ - F_-^* E_- \right)$$

$$+ \int_{-\Lambda}^{+\Lambda} d\eta \left( |\psi'|^2 + \frac{32 g_1^2 V_2 e^{2\gamma}}{k^2} \frac{V_2^2 e^{2\gamma}}{(2\gamma + Z')^2} |\psi|^2 \right).$$

Here again, $\omega^2 \geq 0$ if $E_+ = E_- = 0$ is imposed so that the fluctuation remains finite on both branes, and can only vanish if $\psi$ vanishes identically.

Finally, the same argument applied to the mode $f$ gives

$$\omega^2 \int_{-\Lambda}^{+\Lambda} d\eta e^{2\gamma+Z} |f|^2 = -\frac{1}{2} \left[ (|f|^2)' \right]_{-\Lambda}^{+\Lambda} + \int_{-\Lambda}^{+\Lambda} d\eta \left( |f'|^2 + U^2 |f|^2 \right)$$

$$= -\lambda_2 (|G_+|^2 + |G_-|^2) e^{2\lambda_2 \Lambda} + \int_{-\Lambda}^{+\Lambda} d\eta \left( |f'|^2 + U^2 |f|^2 \right).$$

so that finiteness of the boundary terms requires $G_\pm = 0$. Thus, we see that $\omega^2 \geq 0$ and can only vanish if $f \equiv 0$ throughout the bulk.

In this way we see that perturbations about any background (be it conical or not) are marginally stable, provided that the fluctuations are restricted to be non-singular at the brane positions.
4.4.2 Action Analysis and Ghosts

The previous argument shows that the eigenmode frequency $\omega$ must be real for a broad choice of boundary conditions at the brane positions. However one might worry that the Lagrangian density for the linearized fluctuation might have the form

$$ L = \Phi^* f(\varphi)[\Box - g(\varphi)]\Phi, $$

where $\varphi$ (resp. $\Phi$) denotes a generic background (resp. fluctuation) field, and $f(\varphi)$ and $g(\varphi)$ are background-field dependent quantities. Notice that the equation of motion for $\Phi$ implies $[-\Box + g(\varphi)]\Phi = 0$, and so implies $\omega^2 \geq 0$ if $g(\varphi) \geq 0$. But this might still imply a negative contribution to the fluctuation energy if the prefactor $f(\varphi)$ should happen to be negative for some configurations $\varphi$. We next present a second argument for stability, which closes this particular loophole.

The starting point in this approach is to expand the action to second order in the perturbations. Working in comoving gauge and using the coordinate $\eta$ we get in this way the following quadratic action

$$ S^{(2)} = \int d^6 x \left( L_{\text{kin}} + L_{\text{dyn}} \right), $$

$$ L_{\text{kin}} = \frac{1}{4\kappa^2} e^\frac{2}{\kappa^2} \left[ 6\Psi' - 4B' - 4\varphi_0' \Phi - 2(2\lambda' + Z')B + (2\lambda' + Z')\xi \right] \Box N $$

$$ + \frac{3}{4\kappa^2} e^{2\lambda+Z} (2B - \Psi) \Box \Psi + \frac{1}{4\kappa^2} e^{2\lambda+Z} (3\Psi - 2B) \Box \xi + \frac{1}{2\kappa^2} e^{2\lambda+Z} \Phi \Box \Phi, $$

$$ L_{\text{dyn}} = \frac{1}{2\kappa^2} (3\Psi' - 4B') \Psi' + \frac{1}{\kappa^2} (2\lambda' + Z') \Psi B' - \frac{1}{2\kappa^2} \Phi'^2 - \frac{1}{2\kappa^2} (\lambda' - \lambda') \xi B' $$

$$ - \frac{1}{2\kappa^2} (4\Psi' - 2B' + \xi') \varphi_0' \Phi + \frac{1}{2\kappa^2} (2\lambda' + Z') \xi \Psi' - \frac{q^2}{4} e^{2\lambda} \Phi^2 $$

$$ - \frac{g_1^2}{2} e^{2\lambda} B (B + \xi + 4\Psi + 2\Phi) - \frac{g_2^2}{\kappa^4} e^{2\lambda} (\Phi + 2\xi) \Phi - \frac{g_3^2}{2\kappa^4} e^{2\lambda} \xi^2. $$

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where we allow the fluctuations to depend on all four noncompact coordinates, $x^\mu$, $\Box = \eta^{\mu\nu}\partial_\mu \partial_\nu$ denotes the flat d’Alembertian, and the split into ‘dyn’ and ‘kin’ distinguishes terms which involve $\Box$ from those which do not. We obtain the above expression by freely integrating by parts while ignoring the surface terms which this procedure introduces at the brane positions. This neglect of surface terms is justified inasmuch as our goal is to exclude the possibility of the negative (ghost-like) kinetic terms, as described above.

Differentiating this action with respect to the shift function $N$ leads to the constraint

$$\xi = \frac{2(2\mathcal{X}' + \mathcal{Z}')B + 4\varphi_0'\Phi + 2(2B' - 3\Psi')}{2\mathcal{Y}' + \mathcal{Z}'}$$

(4.69)

while differentiating with respect to the lapse function $\xi$ gives the constraint for $N$:

$$\Box N = \frac{e^{\frac{1}{2}(\mathcal{Y} - \mathcal{X})}}{2\mathcal{Y}' + \mathcal{Z}'} \left[ \kappa^2 q^2 e^{2\mathcal{X}} (2B + \Phi) + \frac{4 g_1^2}{\kappa^2} e^{2\mathcal{Y}} (\xi + \Phi) + e^{2\mathcal{Y} + \mathcal{Z}} \Box (2B - 3\Psi) \\
+ 2(\mathcal{Y}' - \mathcal{X}')B' - 2(2\mathcal{Y}' + \mathcal{Z}')\Psi' - 2\varphi_0'\Phi' \right]$$

(4.70)

These two constraints are consistent with what was obtained in eqs. (4.33) and (4.35). As before, we may use both these constraints to obtain the equations of motion for the three remaining degrees of freedom, $\Phi, B$ and $\Psi$. Substituting eqs. (4.69) and (4.70) into the action, together with the change of variable (4.41), the resulting action
takes the remarkably simple form

\[ S^{(2)} = \frac{1}{\kappa^2} \int d^6 x \left\{ \chi \left[ \partial^2 \eta + e^{2Y + Z\Box} \right] \chi + \psi \left[ \partial^2 \eta + e^{2Y + Z\Box} - \frac{32 g_1^2}{\kappa^2} \frac{V^2 e^{2Y}}{(2Y' + Z')^2} \right] \psi 
+ f \left[ \partial^2 \eta + e^{2Y + Z\Box} - U^2 \right] f \right\}. \] (4.71)

As a check, we note that varying this action gives the same equations as the ones obtained in (4.43, 4.44, 4.45). Since all of the kinetic terms in the action are positive, it follows immediately that the theory has no ghost modes (for which \( f(\varphi) < 0 \)).

4.5 Discussion

In this chapter we have studied the linearized evolution of perturbations to gauged chiral 6D supergravity compactified to 4D on a broad class of static and axially-symmetric vacuum solutions, sourced by two space-filling 3-branes. We follow previous authors in focussing on scalar modes which share the axial symmetry of the background, and which are even under a convenient parity transformation. These restrictions are consistent with the equations of motion, since they are enforced by symmetries. They do not compromise the stability conclusions because the excluded modes necessarily have higher squared-frequencies than do the lowest of the modes kept (which we find are bounded below by \( \omega^2 = 0 \)).

Although none of the backgrounds we perturb are supersymmetric, we find they are all marginally stable to perturbations which are well-behaved at the brane positions. The marginal direction is unique and is the one required on general grounds by a general scaling property of the 6D supergravity equations. Besides providing a general stability argument which works for a broad class of static vacua, we also provide analytic solutions to the fluctuation equation for the special case of conical
backgrounds, including (but not restricted to) the rugby ball in comoving gauge. These allow us to explicitly verify that the near-brane behaviour of the solutions has the properties required by general asymptotic properties of the field equations.
CHAPTER 5
Regularization of Codimension-2 Brane Worlds

One of the obstacles to the study of branes having codimension-$d$, with $d \geq 2$, is that bulk fields generically diverge at the brane location. To see why this might be so, it is enough to consider the axially symmetric solutions to the equation of motion for a scalar field in the vicinity of such an object. In the case of massless field, the equation of motion $\Box \Phi = 0$ implies in Gaussian-Normal gauge that $\sqrt{g} \partial_r \Phi = \zeta$, where $\zeta$ is an integration constant and $\Phi$ is assumed to depend only on the proper distance, $r$. Unless $\zeta = 0$, the solution

$$\partial_r \Phi = \frac{\zeta}{\sqrt{g}} \quad (5.1)$$

diverges at the brane location where $g \to 0$ by assumption.\(^1\) So long as $g$ vanishes at least as fast as $r$, we see that the field $\Phi$ must also diverge. For codimension-1 branes, such as in the familiar Randall-Sundrum models [77, 78], we note that there is no requirement that $g = 0$ at the brane location.

\(^1\) Close to the brane, the metric can be expanded as $ds^2 \approx d^2 r + c_\theta^2 r^{2\alpha} d^2 \theta + h_{ab} dx^a dx^b$, where the proper circumference $2\pi c_\theta r^\alpha$ must vanish as $r \to 0$. One could imagine exceptional cases where $g$ does not vanish at the brane location, but this would require $h$ to diverge there.
One case of particular interest to us is when the metric exhibits a conical singularity, so that the $r$-dependence of the metric goes like $\sqrt{g} \sim r$. In this case we see that $\Phi$ diverges logarithmically at the brane location. Furthermore, even if one includes a bulk potential for the scalar field having the very general form

$$V(\Phi) = \sum_{n=-\infty}^{\infty} c_n \Phi^n$$

(5.2)

the conclusion that the field diverges logarithmically will not be altered. This can be seen simply by substituting the logarithmic solution into the equation of motion $(\sqrt{g}\Phi')' = \sqrt{g} V(\Phi)$ and noting that it solves this equation in the limit $r \to 0$. In detail, this works because the factor of $\sqrt{g} \sim r$ which premultiplies $V(\Phi)$ vanishes faster than any power of $\ln r$, and so the potential is negligible near the brane.

That bulk fields generically diverge at the brane location is just the tip of the iceberg, however. When an action such as (1.9) is considered, we are forced to admit solutions which are not smooth at the brane location (generally, the first derivative is discontinuous there). In that case, we should treat fields as distributions to ensure that differentiation is a well-defined operation. In a linear theory, such as Electromagnetism, this treatment presents no difficulties. Non-linear theories such as General Relativity, on the other hand, are problematic [79] since the product of two distributions is not necessarily another distribution.

There have been various attempts to circumvent this difficulty, from the more mundane — such as “smearing out” the singularity introduced by the branes [71, 80] or explicitly replacing the codimension-2 brane by a small codimension-1 brane [81, 82, 83, 84, 85] — to the more extreme — such as modifying Einstein gravity, for
example by adding Gauss-Bonnet terms in 6D [86, 87]. In this chapter, we take
the more pedestrian route of introducing some internal structure to the brane, our
attitude being that a codimension-2 delta function source is really just an idealization
which outlives its usefulness rather abruptly when confronted with non-linear theories
such as General Relativity. Note that we shouldn’t expect this approach to have a
sensible codimension-2 limit. For that, we need the machinery of renormalization,
which in the codimension-2 case has been considered in other recent works [88, 89, 90].

We have seen already in §2.3.3 that the GGP solutions contain singularities as
the radial coordinate, \( \eta \), goes to \( \pm \infty \). Moving to Gaussian-Normal gauge, eq. (2.46),
the angular part of the metric can be written as \( r^\alpha d\theta^2 \), so we see that it is the sign
of alpha which determines whether we interpret the singularity as being sourced by
either a codimension-1 brane (\( \alpha < 0 \)), or a codimension-2 brane (\( \alpha > 0 \)). Physically,
this singular behaviour arises because of the back-reaction of these branes onto the
bulk fields. Furthermore, the precise kind of singularity is expected to be related to
the properties of these source branes [2, 53, 71], with branes that source the dilaton
field \( \phi \) typically giving rise to a bulk scalar field configuration which diverges at the
brane position, and so whose energy density can give rise to curvature singularities
there.

Our goal in this chapter is to sharpen this connection, by relating more precisely
the integration constants of the bulk solutions to the properties of the two source
branes. We do so by explicitly resolving the singularities at \( \eta \to \pm \infty \) in terms of a
model of the microscopic structure of these two codimension-2 branes. Schematically,
this is done by cutting off the GGP spacetime before it reaches the singular regions,
and then “pasting” onto this excised region a non-singular GGP solution. We do so in such a manner that the extra-dimensional space remains compact. At the location of the junction will be a 4-brane, whose properties are determined by the appropriate junction conditions.

Our main result is to provide explicit relations between the properties of the 4-branes (and their capped geometries) and those of the external bulk, a connection which pays at least two dividends.

- First, by sharpening the general relations between the brane and the bulk, our results provide the tools required to definitively explore the sensitivity of bulk properties to the UV structure on the source branes.
- Second, because the capped branes generically break the classical degeneracy between re-scaled bulk geometries, their presence lifts this degeneracy and so provides a stabilization mechanism which relates the size of the extra dimensions to properties of the source branes. This stabilization mechanism can be regarded as a particular form of the general Goldberger-Wise mechanism [91] which arises particularly naturally within 6D supergravity.

Our presentation of these results proceeds as follows. Next, in §5.1, we review the general 4D flat, cylindrically symmetric solutions of ref. [52], and use these to identify the form taken by the smooth geometries which cap the interiors of the cylindrical 4-branes. §5.2 then follows with a detailed discussion of the matching conditions which apply at the position of the 4-branes, and we use these to identify the relationships which must exist between the parameters of the bulk solutions and those which govern the capped geometries and the intervening 4-branes. §5.3 then
focusses on the implications of these relations for the parameters which govern the sizes of the bulk and capped geometries, and identify the choices which must be made on the branes in order to ensure a large hierarchy between the size of the bulk and the size of the ‘thick’ branes. Some conclusions are summarized in §5.4.

5.1 GGP Solutions Redux

We here rewrite the GGP solutions, eqs. (2.54), but in a more general form. In particular, we explicitly include the integration constant, \( \sigma \), corresponding to the classical scaling symmetry discussed in §2.1. However, the GGP solutions contain a second scaling symmetry, which arises because these solutions are independent of the large dimensions, \( x^\mu \). In fact, it is a trivial statement that given any \( x \)-independent set of solutions, we may obtain ‘new’ solutions by letting \( x \to e^p x \), for some constant \( p \). We choose to parameterize this scaling symmetry by the constant, \( p \), in the definition of the metric.

Writing the line element then as

\[
\text{d} s^2 = e^{\sigma - p} W^2(\eta) \eta_{\mu\nu} \text{d} x^\mu \text{d} x^\nu + A^2(\eta) W^8(\eta) \text{d} \eta^2 + A^2(\eta) \text{d} \psi^2, \tag{5.3}
\]

the GGP solutions are now given by

\[
W^4 = \left| \frac{g_{1} q_{2}}{\lambda_{1} \lambda_{2}} \right| \cosh[\lambda_{1}(\eta - \xi_{1})] \cosh[\lambda_{2}(\eta - \xi_{2})]
\]

\[
A^{-4} = \left| \frac{2g_{1} q_{2}^{3}}{\lambda_{1}^{3} \lambda_{2}} \right| e^{-2(\lambda_{1} q_{2} + \sigma)} \cosh^{3}[\lambda_{1}(\eta - \xi_{1})] \cosh[\lambda_{2}(\eta - \xi_{2})]
\]

while \( e^{-\phi} = W^2 e^{\lambda_{3} q_{2} + \sigma} \) and \( F_{\eta\psi} = \frac{q A^2}{W^2} e^{-\lambda_{3} q_{2} - \sigma} \). \tag{5.4}
For $\sigma = p = 0$, these solutions are exactly the same as given earlier. We also see that when $\omega$ is nonzero, its effect is to scale the metric and dilaton according to

$$g_{MN} \rightarrow e^{\sigma} g_{MN} \quad \text{and} \quad e^\phi \rightarrow e^{\phi - \sigma}, \quad (5.5)$$

while the gauge field does not transform. Comparing with eqs. (2.5), we see that $\sigma$ does indeed parameterize the scaling symmetry. While it is clear that $\sigma$ is a physical parameter,\(^2\) the same is not obviously true of $p$.

What’s more, a simple counting of integration constants now tells us that we have one too many! As discussion in section 2.3.2, we should have seven integration constants, since we have solved four second order differential equations subject to one constraint. However, because of the inclusion of the parameter $p$, we now have a total of eight integration constants, which we take to be: $\lambda_1, \lambda_2, \xi_1, \xi_2, q, \sigma, p,$ and $\alpha$ (recall $\alpha$ is the integration constant for $A_M$). It is clear, then, that we should be able to remove one of these integration constants by a redefinition of the others. Indeed, we see that the effect of shifting the radial coordinate by $\eta \rightarrow \eta + \delta$ is to generate a new solution which differs from the original one by making the changes

$$\xi_i \rightarrow \xi_i + \delta, \quad \sigma \rightarrow \sigma - \lambda_3 \delta \quad \text{and} \quad p \rightarrow p + \lambda_3 \delta. \quad (5.6)$$

\(^2\) For example, changing $\sigma$ changes the size of the compact dimensions.
This fact is important later since it tells us that one of the parameters which governs the bulk solutions, for example $p$, can be arbitrarily removed by making an appropriate choice for the origin of coordinates for $\eta$.\(^3\) The reason we choose to include one redundant parameter (e.g. $p$ or $\xi_1$) is because when we paste together various GGP solutions, we find that new physical parameters appear which correspond to the 4-brane locations. In the next sections, we see how these previously unphysical parameters take on physical significance.

**Boundary contributions**

For later purposes we also record here the additional Gibbons-Hawking term [92] with which the action (2.1) must be supplemented when the field equations are investigated in the presence of boundaries. If the 6D spacetime region of interest, $M$, has a 5D boundary, $\Sigma = \partial M$, then the full action for the bulk fields is

$$S = \int_M d^6x \mathcal{L} - \int_\Sigma \sqrt{-\gamma} K,$$

(5.7)

where $\gamma_{mn}$ is the induced metric on $\Sigma$, and $K = \gamma^{mn} K_{mn}$ is the trace of the extrinsic curvature tensor, $K_{mn}$, on $\Sigma$. Note that in this chapter only, lower case later letters $(m, n)$ run over the 4-brane directions, $0 \ldots 4$.

\(^3\) For the special case $\lambda_3 = 0$, then this is no longer true. However, in this case we could instead choose to set to zero $\xi_1$. The point remains that one constant can still be removed.
5.1.1 Capped Solutions

To proceed with the regularization, we model each of the source branes as a cylindrical codimension-1 4-brane, situated at a fixed value of $\eta$, whose interior is filled in with one of the GGP solutions that is non-singular everywhere within the interior of the cylinder.

Consider then pasting together the following two metrics, along the 4+1 dimensional surface at $\eta = \eta_a$:

$$ds^2 = e^{p_a} \hat{W}^2(\eta) \eta_{\mu\nu} dx^\mu dx^\nu + \hat{A}^2(\eta) \hat{W}^8(\eta) d\eta^2 + \hat{A}^2(\eta) d\psi^2, \quad -\infty < \eta \leq \eta_a,$$

$$ds^2 = e^{p} \mathcal{W}^2(\eta) \eta_{\mu\nu} dx^\mu dx^\nu + \mathcal{A}^2(\eta) \mathcal{W}^8(\eta) d\eta^2 + \mathcal{A}^2(\eta) d\psi^2, \quad \eta_a \leq \eta \leq \eta_b,$$

with a similar splicing being performed at $\eta = \eta_b$ onto a non-singular cap geometry which is defined for $\eta_b < \eta < \infty$. Codimension-1 4-branes will be located at the two boundaries $\eta = \eta_a$ and $\eta = \eta_b$, whose properties we determine below by using the appropriate jump conditions. Notice that we use the freedom to shift the bulk coordinate $\eta$ to set $p = 0$ in the bulk geometry (for $\eta_a < \eta < \eta_b$). However, in the caps we have already exhausted this “shift-symmetry” in order to ensure that the location of the brane in the cap coordinate location, call it $\hat{\eta}_a$, is the same as the brane’s location in the bulk coordinate system, $\eta_a$ (and similarly for $\eta_b$). Because of this choice, we can no longer set $p_a$ and $p_b$ equal to zero.

---

4 Throughout this chapter we use the convention that integration constants corresponding to the cap solutions will include a subscript $a$ or $b$. Also, whenever ambiguity could arise, we use $\hat{a}$ to indicate that the quantity in question refers to the cap.
For the cap solution which applies for $\eta < \eta_a$ we take one of the solutions of eqs. (5.4), subject to the condition that it be singularity free as $\eta \to -\infty$. Recalling the discussion in §2.3.3, the most general axisymmetric solution with only one singularity is the ‘teardrop’ geometry. This geometry is defined by the requirement that $\lambda_3 = 0$ — and so $\lambda_1 = \lambda_2 \equiv \lambda_a$ — leading to the form

$$e^{-\hat{\phi}} = \hat{W}^2 e^{\omega_a},$$

$$\hat{W}^4 = \frac{|g_a|}{2g_a} \frac{\cosh[\lambda_a(\eta - \xi_1 a)]}{\cosh[\lambda_a(\eta - \xi_2 a)]},$$

$$\hat{A}^{-4} = \frac{2g_a q_a^2}{\lambda_a^4} e^{-2\omega_a} \cosh^3[\lambda_a(\eta - \xi_1 a)] \cosh[\lambda_a(\eta - \xi_2 a)]$$

$$\hat{F}_{\eta\psi} = \frac{q_a \hat{A}^2}{\hat{W}^2} e^{-\omega_a}. \quad (5.8)$$

We reiterate here that $\lambda_a > 0$ without loss of generality, and similarly for $g_a$ which we write as a shorthand for $|g_a|$. We also recall another result for §2.3.3, where we found that requiring the teardrop geometry to be singularity-free at $\eta = -\infty$ imposes the constraint

$$|q_a| = 2\lambda_a g_a e^{\lambda_a(\xi_2 a - \xi_1 a)}. \quad (5.9)$$

We read this equation as imposing the value of $\xi_1 a$ in terms of the other cap integration constants. Thus, we are led in this way to the following six parameters describing the geometry at cap $a$: $\lambda_a$, $p_a$, $q_a$, $\sigma_a$, $\xi_2 a$ and $\eta_a$. By contrast, the constant $g_a$ is not a free parameter, but is the $U_a(1)$ gauge coupling which appears in the bulk action whose equations of motion govern the solutions of interest. Although we keep $g_a$ and $g_1$ distinct in what follows, this is not crucial for our results, and one could instead choose to use the same action for the cap regions and the bulk between
the two branes: \( g_a = g_1 \). The analogous story holds for cap \( b \), where regularity at \( \eta_b \to \infty \) requires

\[
|q_b| = 2\lambda_b g_b e^{-\lambda_b (\xi_{2b} - \xi_{1b})}. \tag{5.10}
\]

**Parameter counting**

For future convenience it is useful at this point to count the number of integration constants associated with each of the solutions.

- **The Bulk:** Using the coordinate freedom to re-scale \( g_{\mu\nu} \) and to shift \( \eta \), we may set \( p = 0 \) and fix \( \xi_2 \) to a particular value. This leaves the general bulk solutions characterized by the 5 integration constants \( \lambda_1, \lambda_2, \xi_2, q \) and \( \sigma \).

- **The Caps:** The same coordinate freedom cannot again be used to similarly simplify the teardrop cap geometries for the regions \( \eta < \eta_a \) and \( \eta > \eta_b \). Once one parameter (e.g. \( \xi_{1a} \)) is used to ensure the cap geometry is everywhere smooth — cf. eq. (5.9) — each cap is therefore described by 6 parameters. For the cap at \( \eta < \eta_a \) these are \( \lambda_a, \xi_{2a}, q_a, p_a \) and \( \sigma_a \), together with the 4-brane location, \( \eta_a \). For the cap at \( \eta > \eta_b \) we instead have \( \lambda_b, \xi_{2b}, q_b, p_b, \sigma_b \) and \( \eta_b \).

To these parameters we must also add those that characterize the 4-brane action, as is discussed in some detail in the next section.

We do not include the gauge potential integration constant, \( \alpha \), in the above counting because we handle its matching conditions separately in what follows. Besides \( \alpha \), the gauge potential also potentially hides other moduli describing how the background gauge field is embedded within the full gauge group. This can show up in the present analysis by making the gauge coupling constant, \( e \), associated with
the background gauge field potentially different from the coupling $g_1$ which appears in the supergravity action, eq. (2.1), and so also in the solutions, eqs. (5.4) [9].

5.2 Matching Conditions

We next impose the matching conditions which apply across the 4-brane position, where the cap geometry meets that of the bulk. These come in two types: continuity of the fields $g_{MN}, A_M$ and $\phi$ across $\eta = \eta_a$, and jump conditions which relate the discontinuity in the derivatives of these fields to properties of the 4-brane action.

5.2.1 Continuity Conditions

Continuity of the bulk fields at each brane position provides 4 conditions among the parameters which define the caps. For instance, continuity across the 4-brane situated at $\eta_a$ gives:

$$e^{\sigma_a - p_a} \hat{\mathcal{V}}^2(\eta_a) = e^{\sigma} \mathcal{V}^2(\eta_a), \quad \hat{A}^2(\eta_a) = A^2(\eta_a), \quad \hat{\phi}(\eta_a) = \phi(\eta_a)$$

(5.11)

and

$$\hat{A}_\psi(\eta_a) = A_\psi(\eta_a).$$

(5.12)

After some simplification, the three conditions of eqs. (5.11) reduce to the following relations amongst the parameters of the capped and bulk solutions

$$\frac{\cosh[\lambda_1 (\eta_a - \xi_1)]}{\cosh[\lambda_a (\eta_a - \xi_{1a})]} = \frac{|\lambda_1 g_a|}{\lambda_a q}$$

(5.13)

$$\frac{\cosh[\lambda_2 (\eta_a - \xi_2)]}{\cosh[\lambda_a (\eta_a - \xi_{2a})]} = \frac{|g_a \lambda_2|}{g_1 \lambda_a} e^{2(\sigma - \sigma_a + \lambda_3 \eta_a)}$$

(5.14)

$$p_a = \lambda_3 \eta_a,$$  

(5.15)
with a similar set of relations holding for brane $b$. As we see below in more detail in §5.3.1, these equations can be regarded as fixing the three parameters $p_a$, $\xi_{2a}$ and $q_a$, leaving $\lambda_a$, $\sigma_a$ and $\eta_a$ free.

**Topological constraint**

We treat the continuity condition for the gauge potential separately, because of a topological subtlety which arises in this case. Recall that the gauge potential for the bulk and capped regions can be written in the form

$$A_\psi = \frac{\lambda_1}{q} \left( \tanh [\lambda_1 (\eta - \xi_1)] + \alpha \right) \quad \eta_a < \eta < \eta_b$$

$$\hat{A}_\psi = \frac{\lambda_a}{q_a} \left( \tanh [\lambda_a (\eta - \xi_{1a})] + 1 \right) \quad -\infty < \eta < \eta_a$$

(5.16)

where the integration constant is chosen in the capped region to ensure that $A_\psi$ vanishes as $\eta \to -\infty$, as is required for a non-singular gauge potential. The same reasoning applied to the second capped region similarly gives

$$A_\psi = \frac{\lambda_1}{q} \left( \tanh [\lambda_1 (\eta - \xi_1)] + \alpha' \right) \quad \eta_a < \eta < \eta_b$$

$$\hat{A}_\psi = \frac{\lambda_b}{q_b} \left( \tanh [\lambda_b (\eta - \xi_{1b})] - 1 \right) \quad \eta_b < \eta < \infty$$

(5.17)

where the integration constant is in this case chosen in the capped region to ensure that $A_\psi$ vanishes as $\eta \to +\infty$.

Naively we would determine $\alpha$ and $\alpha'$ by working within a gauge for which $A_\psi$ is continuous for all $\eta$. However, the crucial point is that there is in general a topological obstruction to making such a choice for $A_M$ everywhere. Instead we choose a gauge for which $A_\psi(\eta_a) = \hat{A}_\psi(\eta_a)$ and $A_\psi(\eta_b) = \hat{A}_\psi(\eta_b)$, and use these conditions to determine $\alpha$ and $\alpha'$. Here, $\alpha$ and $\alpha'$ cannot be taken to be equal on the
region of overlap, $\eta_a < \eta < \eta_b$, but must differ instead by a gauge transformation. Following standard arguments, this leads to the quantization condition

$$\frac{\lambda_1}{q} (\alpha - \alpha') = \frac{N}{e}$$

(5.18)

where $N$ is an integer, and $e$ is the gauge coupling for the background gauge field (which need not equal $g_1$ if the background flux is not the one gauging the specific $U_R(1)$ symmetry).

We find in this way that eq. (5.18) implies the following quantization condition on the various parameters:

$$\frac{N}{e} = \frac{\lambda_1}{q} \left( \tanh[\lambda_1(\eta_b - \xi_1)] - \tanh[\lambda_1(\eta_a - \xi_1)] \right)$$

$$+ \frac{\lambda_a}{q_a} \left( \tanh[\lambda_a(\eta_a - \xi_{1a})] + 1 \right) - \frac{\lambda_b}{q_b} \left( \tanh[\lambda_b(\eta_b - \xi_{1b})] - 1 \right).$$

(5.19)

This generalizes to the case of thick branes the well-known Dirac quantization condition $N/e = 2\lambda_1/q$ [93], which is retrieved from eq. (5.19) in the thin-brane limit obtained by taking $\eta_a \to -\infty$ and $\eta_b \to +\infty$.

Such arguments show that in general the continuity of the gauge potential across the two 4-branes, $\eta = \eta_a$ and $\eta = \eta_b$, determines the integration constants, $\alpha$ and $\alpha'$ which are specific to the gauge potentials. But the topological constraint then implies a single additional condition, eq. (5.19), which relates the bulk parameters, $\lambda_1$, $\xi_1$ and $q$, to the undetermined brane quantities, $\eta_a$, $\xi_{1a}$, $\eta_b$, $\xi_{1b}$ and the flux integer $N$. 

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5.2.2 Jump Conditions

Having examined the continuity conditions, we next examine the relevant jump conditions which govern the discontinuity of derivatives of the bulk fields across the brane positions at $\eta = \eta_a$ and $\eta = \eta_b$. These junction conditions relate any such discontinuity to the dependence of the intervening 4-brane action, $S$, on these bulk fields, and may be derived by integrating the equations of motion across a narrow interval around the 4-brane position: $\eta_a - \epsilon < \eta < \eta_a + \epsilon$, with $\epsilon$ taken negligibly small. Specialized to the metric these conditions are known as the Israel junction conditions [94].

One finds in this way

$$[K_{mn}]_J = -T_{mn}, \quad [\sqrt{-g} e^{-\phi} F_{mn}]_J = -\frac{\delta S}{\delta A_m} \quad \text{and} \quad [\sqrt{-g} \partial^\eta \phi]_J = -\frac{\delta S}{\delta \phi}, \quad (5.20)$$

where we use the definition $[f(\eta)]_{\eta_a} \equiv f(\eta_a + \epsilon) - f(\eta_a - \epsilon)$. Here we define $K = \gamma^{mn}K_{mn}$ and $K_{mn} = K_{mn} - \gamma_{mn}K$, where $K_{mn}$ is the extrinsic curvature of the appropriate 4-brane surface.

4-Brane action

In order to proceed we require an ansatz for the 4-brane action. Consider therefore the following general choice

$$S = -\int_{\Sigma} d^5x \sqrt{-\gamma} \left[ V(\phi) + \frac{1}{2} U(\phi) (D_m\sigma D^n \sigma) \right], \quad (5.21)$$

where $\gamma_{mn}$ is the induced metric on the brane, and $V(\phi)$ and $U(\phi)$ are functions which determine the 4-brane couplings to the 6D dilaton.
Following ref. [83] we introduce a Stückelberg field, $\sigma$, living on the brane, whose gauge covariant derivative is $D_m \sigma = \partial_m \sigma - e A_m$. We imagine this to be the low-energy effective action obtained by integrating out the massive mode of some brane-localized Higgs field, $H = v e^{i \sigma}$, where $v$ is an appropriate expectation value. Physically, this field describes supercurrents whose circulation can support changes in the background flux across the position of the 4-brane. (We return to the necessity for including such a field in subsequent sections.) The equation of motion for $\sigma$, together with the periodicity requirement $\psi \simeq \psi + 2\pi$, allows us to write the background configuration for $\sigma$ as

$$\sigma = k \psi,$$

(5.22)

for some integer $k \in \mathbb{Z}$.

With these choices the jump conditions, eqs. (5.20), become

$$[K_{\mu\nu}]_J = -T_{\mu\nu},$$

(5.23)

$$[K_{\psi\psi}]_J = -T_{\psi\psi},$$

(5.24)

$$\left[ \sqrt{-g} \ e^{-\phi} F_{\eta\psi} \right]_J = -e U \sqrt{-\gamma} \ D^\psi \sigma,$$

(5.25)

$$\left[ \sqrt{-g} \ \partial^\phi \phi \right]_J = \sqrt{-\gamma} \left[ \frac{dV}{d\phi} + \frac{1}{2} (D_m \sigma D^m \sigma) \frac{d\U}{d\phi} \right],$$

(5.26)

where the energy-momentum tensor derived from the above action is

$$T_{\mu\nu} = -e^\sigma \left( \frac{W}{A} \right)^2 \left[ A^2 V + \frac{1}{2} U(k - e A_\psi)^2 \right] \eta_{\mu\nu},$$

$$T_{\psi\psi} = - \left[ A^2 V - \frac{1}{2} U(k - e A_\psi)^2 \right].$$

(5.27)
Here we see one reason for including the Stückelberg field: without the function $U$ the expressions for $T_{\mu\nu}$ and $T_{\psi\psi}$ are not independent since their ratio would be independent of parameters from the 4-brane action, leading to too restrictive a set of geometries which could be described in the bulk.

Evaluating the junction conditions

We next specialize the junction conditions to the explicit bulk fields discussed above. We first require the extrinsic curvature, $K_{mn}$, evaluated on both sides of the brane. In the bulk region, the unit normal to surfaces of constant $\eta$ is

$$n_M = \mathcal{A} \mathcal{W}^4 \delta_M^\eta$$ (5.28)

and so the extrinsic curvature is given by $K_{mn} = \nabla_m n_n = -\mathcal{A} \mathcal{W}^4 \Gamma_m^\eta$, where $\Gamma_m^\eta$ is the Christoffel symbol calculated from the full 6D metric. We find

$$K_{\mu\nu} = -\frac{e^\sigma}{\mathcal{A} \mathcal{W}^2} \left[ 3 \frac{\mathcal{W}'}{\mathcal{W}} + \frac{\mathcal{A}'}{\mathcal{A}} \right] \eta_{\mu\nu}$$
$$K_{\psi\psi} = -\frac{4 \mathcal{A} \mathcal{W}'}{\mathcal{W}^5},$$ (5.29)

where primes denote differentiation with respect to $\eta$. Similarly, in the cap regions we have

$$\dot{K}_{\mu\nu} = -\frac{e^{\sigma a - p a}}{\dot{\mathcal{A}} \dot{\mathcal{W}}^2} \left[ 3 \frac{\dot{\mathcal{W}}'}{\dot{\mathcal{W}}} + \frac{\dot{\mathcal{A}}'}{\dot{\mathcal{A}}} \right] \eta_{\mu\nu}$$
$$\dot{K}_{\psi\psi} = -\frac{4 \dot{\mathcal{A}} \dot{\mathcal{W}}'}{\dot{\mathcal{W}}^5}.$$ (5.30)
• Evaluating the \((\mu \nu)\) Israel junction condition at \(\eta = \eta_a\) then gives\(^5\)

\[
\left(\frac{\lambda_3}{2} + e^{-2(\sigma-\sigma_a+\lambda_3\eta_a)} \lambda_a \tanh[\lambda_a(\eta_a - \xi_{2a})] - \lambda_2 \tanh[\lambda_2(\eta_a - \xi_2)]\right)
\]

\[
= -W^4 \left( (A V_a) + \frac{1}{2} \left( \frac{U_a}{A} \right) (k_a - eA_\psi)^2 \right) \tag{5.31}
\]

where the subscript ‘\(a\)’ on \(V\), \(U\) and \(k\) denotes the corresponding 4-brane property specialized to the brane at \(\eta = \eta_a\).

• The \((\psi \psi)\) Israel junction condition similarly gives

\[
\left[ \lambda_1 \tanh[\lambda_1(\eta_a - \xi_1)] - \lambda_2 \tanh[\lambda_2(\eta_a - \xi_2)] - e^{-2(\sigma-\sigma_a+\lambda_3\eta_a)} (\lambda_a \tanh[\lambda_a(\eta_a - \xi_{1a})]
\]

\[
- \lambda_a \tanh[\lambda_a(\eta_a - \xi_{2a})]) \right] = -W^4 \left( (A V_a) - \frac{1}{2} \left( \frac{U_a}{A} \right) (k_a - eA_\psi)^2 \right). \tag{5.32}
\]

Taking the sum and the difference of these last two conditions allows the isolation of conditions for \(V_a\) and \(U_a\) separately. It is also easy to see that the resulting equations always admit real solutions for any value of the bulk parameters and the brane position.

• The junction condition for the gauge field similarly evaluates to

\[
q - q_a e^{-2(\sigma-\sigma_a+\lambda_3\eta_a)} = -eW^4 \left( \frac{U_a}{A} \right) (k_a - eA_\psi). \tag{5.33}
\]

\(^5\) It is understood in what follows that all functions depending on \(\eta\) are evaluated at \(\eta = \eta_a\).
Notice that we can eliminate the two brane quantities, $U_a$ and $V_a$, from the previous three jump conditions to obtain a constraint that does not depend on 4-brane parameters. Indeed, by subtracting eq. (5.31) from eq. (5.32), and then dividing the result by eq. (5.33), we obtain the expression

\[
\frac{1}{2} \lambda_3 + \lambda_a \tanh[\lambda_a (\eta_a - \xi_{1a})] - \lambda_1 \tanh[\lambda_1 (\eta_a - \xi_1)]
\]

\[
e^{-2(\sigma_a + \lambda_3 \eta_a)} q_a - q
\]

\[
= -\frac{k_a}{e} + \frac{\lambda_a}{q_a} \left( \tanh[\lambda_a (\eta_a - \xi_{1a})] + 1 \right).
\]

(5.34)

An identical argument for brane $b$ similarly gives:

\[
\frac{1}{2} \lambda_3 + \lambda_b \tanh[\lambda_b (\eta_b - \xi_{1b})] - \lambda_1 \tanh[\lambda_1 (\eta_b - \xi_1)]
\]

\[
e^{-2(\sigma_b + \lambda_3 \eta_b)} q_b - q
\]

\[
= -\frac{k_b}{e} + \frac{\lambda_b}{q_b} \left( \tanh[\lambda_b (\eta_b - \xi_{1b})] - 1 \right).
\]

(5.35)

- By contrast, the dilaton junction condition gives a condition on the $\phi$-derivatives of $U_a$ and $V_a$:

\[
2 \lambda_3 + \lambda_1 \tanh[\lambda_1 (\eta_a - \xi_1)] - \lambda_2 \tanh[\lambda_2 (\eta_a - \xi_2)]
\]

\[
- e^{-2(\sigma_a + \lambda_3 \eta_a)} \left( \lambda_a \tanh[\lambda_a (\eta_a - \xi_{1a})] - \lambda_a \tanh[\lambda_a (\eta_a - \xi_{2a})] \right)
\]

\[
= -2 \mathcal{W}^4 \left[ \mathcal{A} \frac{dV_a}{d\phi} + \frac{1}{2\mathcal{A}} \frac{dU_a}{d\phi} (k_a - eA_\psi)^2 \right],
\]

which, using the $(\psi \psi)$ Israel jump condition, simplifies to

\[
2 \lambda_3 = \mathcal{W}^4 \left[ \mathcal{A} \left( V_a - 2 \frac{dV_a}{d\phi} \right) - \frac{1}{2\mathcal{A}} \left( U_a + 2 \frac{dU_a}{d\phi} \right) (k_a - eA_\psi)^2 \right].
\]

(5.37)
Conditions for scale invariance

Before proceeding it is useful to pause at this point to record the unique choice for the functions $V_a$ and $U_a$ which preserves the classical scaling symmetry of the bulk equations of motion, corresponding to the transformation $\sigma \rightarrow \sigma + \Delta$ and $\sigma_a \rightarrow \sigma_a + \Delta$.

Inspection shows that the continuity equations remain unchanged by this transformation because $\sigma$ and $\sigma_a$ only appear there in the combination $\sigma - \sigma_a$. The left-hand-sides of the various jump conditions remain similarly unchanged. On the right-hand-side, however, we see that $A$ transforms, and so invariance requires $V_a(\phi)$ and $U_a(\phi)$ to transform in a way which cancels the transformation of $A$. Such an invariant choice for $U_a$ and $V_a$ is possible for the Israel and Maxwell jump conditions, eqs. (5.31), (5.32), and (5.33), because within these $U_a$ and $V_a$ only appear with $A$ in the combinations $Av_a$ and $U_a/A$. It follows that preservation of the classical scaling symmetry requires

\[ V_a = u_a e^{\phi/2} \quad \text{and} \quad U_a = u_a e^{-\phi/2}, \quad (5.38) \]

in agreement with the analysis of ref. [51]. Any other choices for these functions necessarily breaks the classical scale invariance of the problem.

It then remains to determine what invariance requires for the dilaton jump condition, eq. (5.37). When this is specialized to the scale invariant case, eqs. (5.38), the right-hand side degenerates to zero, giving the simple condition $\lambda_3 = 0$. Besides imposing no new conditions on $U_a$ and $V_a$, this tells us that scale-invariant brane configurations can only source bulk geometries satisfying $\lambda_3 = 0$, and hence only having
conical singularities. Since all of the geometries having two conical singularities are $4D$-flat [1], we see in detail how the jump conditions enforce the connection between scale invariance and $4D$ flatness.

5.3 Applications

Given the general bulk and cap solutions, and a complete set of matching conditions, we may now see what the solutions to these conditions tell us about bulk-brane dynamics in six dimensions. In this section we use the above formalism to address two questions. First: given a bulk geometry what kinds of caps are possible? Second: given specific brane properties, what kinds of bulk are generated? In particular, in this second case we ask how the breaking of scale invariance by the branes can lead to the stabilization of the extra-dimensional size.

Answering this last question allows us also to address an issue of potential importance for phenomenology: what conditions must the cap and bulk parameters satisfy in order to have a large hierarchy between the volumes of the caps and the volume of the bulk? This point is important when the regularizing 4-branes and caps are regarded as specifying the microscopic structure of 3-branes that sit at the singular points of the geometry.

5.3.1 Capping a Given Bulk

We begin by studying what kinds of caps can be used to smooth a generic bulk solution. In this section we therefore regard the 5 bulk parameters $\lambda_1, \lambda_2, \xi_1, q$ and $\sigma$ as given (we remove both $p$ and $\xi_2$ using appropriate coordinate conditions), and look for solutions for the kinds of branes which can smooth the singularities at $\eta = \pm\infty$. 

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We emphasize that our purpose here is simply to show that a regularization procedure exists for any choice of bulk solution, through an appropriate choice for the 4-branes and caps. We return in subsequent sections to the relations which must exist between the parameters governing the branes and caps, due to the interpolation between them of a $4D$-flat bulk.

**Parameter counting**

It is instructive to count parameters and constraints, to get a sense of whether or not the problem of capping a given bulk is over-determined. To this end it is worth distinguishing between those parameters which are integration constants in the capped region, and those which arise within the action, $S$, governing the 4-brane. We start by counting only those relations which are independent of the 4-brane action, before returning to those which are not.

*S-independent conditions:* We have seen that each cap naively involves 7 integration constants, $\lambda_a$, $\xi_{i_a}$, $\xi_{2a}$, $p_a$, $q_a$, $\sigma_a$ and $\eta_a$ that are related by the smooth-geometry condition, (5.9), at each cap. Counting the two caps this gives a total of $6 + 6 = 12$ independent cap integration constants.

At each cap these parameters are subject to 3 continuity conditions, eqs. (5.13) – (5.15), as well as the 1 jump condition, (5.34) or (5.35), constructed by eliminating $U(\phi)$ and $V(\phi)$ from eqs. (5.31) – (5.33). The topological constraint then imposes one more overall relation which relates the properties of the bulk to those of both caps, giving a grand total of $4 + 4 + 1 = 9$ conditions. Barring other obstructions we then expect to find a $12 - 9 = 3$-parameter family of capped geometries which can match properly to the given bulk.
\textbf{S-dependent conditions:} In addition to these are the parameters \( U_a(\phi) \) and \( V_a(\phi) \) governing the 4-brane action, \( S \). For each brane these two functions are related by the three remaining conditions, eqs. (5.31), (5.32) and (5.37). Solving the two linear equations, (5.31) and (5.32), immediately gives \( U_a \) and \( V_a \) as explicit functions of \( \eta_a \):

\[
U_a = U_a(\eta_a) \quad \text{and} \quad V_a = V_a(\eta_a)
\]

(where we suppress the dependence on the other cap and bulk parameters).

We are then left with one remaining relation: the dilaton jump condition, eq. (5.37). Since this requires knowing the derivatives, \( dU_a/d\phi \) and \( dV_a/d\phi \), further progress requires making some choices for the functional form of these quantities.

- If \( U_a \) and \( V_a \) are both constant, then both are fixed by eqs. (5.31) and (5.32).

  In this case the dilaton jump condition, eq. (5.37), imposes an additional 2 constraints (one at each cap) on the 3 cap integration constants which remain to this point undetermined. We are then led to expect a 1-parameter family of capped solutions.

- If \( U_a \) and \( V_a \) preserve scale invariance, then \( U_a = u_a e^{-\phi/2} \) and \( V_a = v_a e^{\phi/2} \), have 2 free parameters. In this case the counting naively goes through as above, with one change: although \( u_a \) and \( v_a \) are fixed by solving the Israel junction conditions, eqs. (5.31) and (5.32), the dilaton jump condition, eq. (5.37), degenerates to \( \lambda_3 = 0 \) and so does not further constrain any 4-brane or cap parameters. (None of these matching conditions fix the scale symmetry \( \sigma \rightarrow \sigma + \Delta \), \( \sigma_a \rightarrow \sigma_a + \Delta \) and \( \sigma_b \rightarrow \sigma_b + \Delta \). However, because we here regard the bulk
parameter $\sigma$ to have been specified this symmetry does not preclude the determination of $\sigma_a$ and $\sigma_b$ in terms of $\sigma$.) We are therefore led in this case to 3 free parameters in the capped solution.

- More general choices for $U_a$ and $V_a$ potentially involve more parameters, and so allow more freedom of choice for the capped geometry. For instance, if $U_a = u_a e^{s_a \phi}$ and $V_a = v_a e^{t_a \phi}$, then the three conditions, (5.31), (5.32) and (5.36), provide three relations amongst the four parameters $u_a$, $v_a$, $s_a$ and $t_a$, and in particular (5.36) no longer constrains the parameters of the caps. In this case we’d expect a total of 5 free parameters to describe the capped geometry.

Considerations such as these lead us to expect that capped solutions of the type we entertain should exist for any given kind of bulk geometry, barring an obstruction to solving the relevant equations. Furthermore, we expect to find at least a 1-parameter family of such solutions, and this has a simple physical interpretation: in the absence of the topological constraint the caps have 2 free parameters, corresponding to the freedom to choose the positions, $\eta_a$ and $\eta_b$, where we choose to position the two caps. The topological constraint can then impose one relation amongst these two positions, relating them to the quantum number, $N$, which governs the total amount of Maxwell flux.

Notice that our counting here regards $U$ and $V$ as parameters to be adjusted even though these arise within the brane action rather than as integration constants in the solutions to the field equations. So the existence of the caps requires these parameters in the action to be tuned relative to one in a way which depends on the properties of the given bulk solution. We also do not distinguish here whether the
solutions found give positive values for $U$ and $V$, as would normally be required by positivity of the kinetic energy associated with brane motion ($V$) and the Stückenberg field ($U$).

**Freely-floating 4-branes**

The previous section takes the point of view that the $\phi$-dependence of the 4-brane action can be arbitrarily parameterized, with the parameters required to cap the given bulk geometry being fixed in terms of the positions of the caps and other variables. Another point of view is to ask for a 4-brane action to be defined so that the same 4-brane action can be used at any 4-brane position, for a given bulk geometry. As we shall see, consistency also requires the cap geometry to be varied as a function of the brane position. This approach is similar in spirit to what is done for the actions of end-of-the-world branes which mark the boundary of bulk spaces in discussions of the AdS/CFT correspondence [95].

This amounts to asking that the $\eta_a$-dependence inferred by solving eqs. (5.31) and (5.32) for $U_a(\eta_a)$ and $V_a(\eta_a)$ is completely given by the implicit $\eta_a$-dependence which $U_a$ and $V_a$ inherit as functions of $\phi(\eta_a)$ (with $\eta_a$-independent constants). That is, we demand $U_a(\eta_a) = U_a[\phi(\eta_a)]$ and $V_a(\eta_a) = V_a[\phi(\eta_a)]$. We call such a 4-brane action the ‘floating’ action which is defined by the given bulk and capped geometries. In principle, the functional form that this requires for both $U_a(\phi)$ and $V_a(\phi)$ can be inferred in this way using the known expressions for the bulk dilaton profile, $\phi(\eta_a)$, together with the expressions for $U_a(\eta_a)$ and $V_a(\eta_a)$ obtained by solving eqs. (5.31) and (5.32).
Finally, the dilaton jump condition, (5.37), is then read as an additional constraint on the parameters which govern the capped geometry. To identify this constraint more explicitly, we notice that we could use either the bulk dilaton profile, \( \phi(\eta) \), or the profile in the cap, \( \hat{\phi}(\eta) \), to convert the \( \eta_a \) dependence of \( U_a \) and \( V_a \) into their dependence on the dilaton. In particular, we have two ways of evaluating the dilaton derivative of the 4-brane quantities like \( U_a \), which must agree with each other:

\[
\left( \frac{dU_a}{d\eta_a} \right) = \left( \frac{dU_a}{d\phi} \right) \left( \frac{d\phi}{d\eta_a} \right) = \left( \frac{dU_a}{d\phi} \right) \left( \frac{d\hat{\phi}}{d\eta_a} \right).
\]

(5.39)

Here \( d\phi/d\eta_a = (\partial\phi/\partial\eta)|_{\eta \rightarrow \eta_a} \), while \( d\hat{\phi}/d\eta_a \) also includes the implicit dependence on \( \eta_a \) that that \( \hat{\phi} \) acquires through its dependence on the \( \eta_a \)-dependent cap parameters. Collectively denoting these cap parameters by \( \{ \hat{c}_s \} = \{ \lambda_a, \xi_{1a}, \ldots \} \), we have

\[
\frac{d\hat{\phi}}{d\eta} = \left[ \left( \frac{\partial\hat{\phi}}{\partial\eta} \right) + \left( \frac{\partial\hat{\phi}}{\partial\hat{c}_s} \right) \frac{\partial\hat{c}_s}{\partial\eta_a} \right]_{\eta \rightarrow \eta_a}.
\]

(5.40)

The desired consistency condition on the cap parameters comes from equating \( (\partial\hat{\phi}/\partial\eta)_{\eta \rightarrow \eta_a} \) obtained by solving eqs. (5.39) and (5.40), with that inferred from the dilaton jump condition, eq. (5.37).

We see from this that the number of independent constraints on the cap geometry is the same as it was when we made the simpler assumption that \( U \) and \( V \) were constants. We have not yet tried to solve these constraints to determine the functional form for \( U_a(\phi) \) and \( V_a(\phi) \) which would be obtained.

**Solving the matching conditions**

In order to see in more detail if obstructions to solutions to the matching conditions might exist, we next examine some of these conditions in more detail. Recall
that if the smoothness condition is not used then each cap is described by 7 inte-
gration constants — $\lambda_a$, $q_a$, $\xi_{1a}$, $\xi_{2a}$, $\sigma_a$, $p_a$ and $\eta_a$ — for a total of 14 once both
branes are included. Smoothness of the caps and continuity at both branes, with
the topological condition cut these down by a total of 9 conditions, leaving 5 un-
determined. There is also one combination of jump conditions at each brane which
does not involve the potentials $U$ and $V$, reducing us to 3 parameters. If $U$ and $V$
are $\phi$-independent, then the dilaton jump condition for each brane removes 2 more.
This leaves 1 cap parameter undetermined. By contrast, the integers $k_a$, $k_b$ and $N$
describing the monopole flux and background configuration for the St"uckelberg field
are not solved for, but are instead regarded as choices we get to pick by hand. We
show there is a solution to the junction conditions for a range of $k_a$, $k_b$ and $N$.

In what follows it is convenient to define first the quantities

\[ \Lambda_{ia} = \lambda_a(\eta_a - \xi_{ia}), \quad \Lambda_{ib} = \lambda_b(\eta_b - \xi_{ib}), \quad (5.41) \]

and \[ \Delta_{ia} = \lambda_i(\eta_a - \xi_i), \quad \Delta_{ib} = \lambda_i(\eta_b - \xi_i) \quad (5.42) \]

where $i = 1, 2$. In our counting, the parameters $\Lambda_{ia}$ and $\Lambda_{ib}$ replace $\xi_{ia}$ and $\xi_{ib}$,
whereas $\Delta_{ia}$ and $\Delta_{ib}$ are known functions of $\eta_a$ and $\eta_b$.

Recall that there are a total of 14 cap parameters, and these are subject to a
total of 11 conditions before the three conditions (per brane) involving $U$ and $V$ are
used, leaving 3 parameters undetermined. (Depending on what we assume about
the 4-brane action — such as if $U$ and $V$ are constants — two of these can then
be fixed by the dilaton jump conditions, leaving the single undetermined parameter,
although we do not yet apply this constraint in this section.) Although other choices
are possible, we find it easiest to solve for the cap parameters as functions of the
three undetermined quantities \((\eta_a, \eta_b, \Lambda_{1b})\).

We start with the topological constraint, eq. (5.19), which we simplify by using
eq. (5.13) and its counterpart for brane \(b\) to eliminate the combinations \(\lambda_a/q_a\) and
\(\lambda_b/q_b\). Using the resulting expressions in eq. (5.19) gives
\[
\tanh \Delta_{1b} - \tanh \Delta_{1a} = \frac{qN}{e\lambda_1} - \frac{\varepsilon_a e^{\Lambda_{1a}}}{\cosh \Delta_{1a}} - \frac{\varepsilon_b e^{-\Lambda_{1b}}}{\cosh \Delta_{1b}},
\]
where we define \(\varepsilon_a = |q_a|/q_a = \text{sign} q_a\), and similarly for \(\varepsilon_b\) and \(\varepsilon\). Writing this as
\(\varepsilon_a e^{\Lambda_{1a}} = F\), with \(F = F(\eta_a, \eta_b, \Lambda_{1b})\) given by
\[
F = \cosh \Delta_{1a} \left( \frac{qN}{e\lambda_1} - \frac{\varepsilon_b e^{-\Lambda_{1b}}}{\cosh \Delta_{1b}} - \tanh \Delta_{1b} + \tanh \Delta_{1a} \right),
\]
shows that solutions exist so long as we choose \(\varepsilon_a = \text{sign} F\), and gives these solutions as
\[
\Lambda_{1a} = \ln |F|.
\]

Using the smoothness condition together with the continuity condition, eq. (5.13),
and the above solution for \(\Lambda_{1a}\), then gives
\[
\Lambda_{2a} = \ln \left| \frac{\lambda_1 g_a(1 + F^2)}{q \cosh \Delta_{1a}} \right|.
\]
As we have now solved for \(\Lambda_{1a}\) and \(\Lambda_{2a}\) in terms of \(\eta_a, \eta_b\), and \(\Lambda_{1b}\), we do not bother
to eliminate these two parameters from future expressions.
We next solve for \( \lambda_a \). Starting from eq. (5.34) and using the continuity conditions to simplify further, we arrive at the expression

\[
\lambda_a = \frac{1}{\tanh \Lambda_{1a}} \left( \lambda_1 \tanh \Delta_{1a} - \frac{\lambda_3}{2} \right) + \left[ 1 - \frac{\varepsilon \varepsilon_a g_a \lambda_2}{g_1 \lambda_1} \cosh \Delta_{1a} \cosh \Lambda_{2a} \right] \left[ \frac{q k_a}{e} - \frac{\varepsilon_a \lambda_1 e^{\Lambda_{1a}}}{\cosh \Delta_{1a}} \right].
\] (5.47)

It is important to note that by choosing the integer \( k_a \) appropriately, we can ensure \( \lambda_a > 0 \). Again, as we have solved for \( \lambda_a \) in terms of the three required parameters, we will not need to eliminate it from future equations. Finally, the 3 continuity equations at brane \( a \) directly give

\[
p_a = \lambda_3 \eta_a, \quad (5.48)
\]
\[
q_a = \left( \frac{\varepsilon q \lambda_a}{\lambda_1} \right) \left( \frac{2F}{1 + F^2} \right) \cosh \Delta_{1a}, \quad (5.49)
\]
\[
\sigma_a = \sigma + \lambda_3 \eta_a + \frac{1}{2} \ln \left| \frac{g_a \lambda_2 \cosh \Lambda_{2a}}{g_1 \lambda_a \cosh \Delta_{2a}} \right|. \quad (5.50)
\]

The analysis at brane \( b \) is similar, for which we find

\[
\Lambda_{2b} = \Lambda_{1b} + \ln \left| \frac{q \cosh \Delta_{1b}}{2 \lambda_1 g_b \cosh \Lambda_{1b}} \right|, \quad (5.51)
\]
\[
\lambda_b = \frac{1}{\tanh \Lambda_{1b}} \left( \lambda_1 \tanh \Delta_{1b} - \frac{\lambda_3}{2} \right) + \left[ 1 - \frac{\varepsilon \varepsilon_b g_b \lambda_2}{g_1 \lambda_1} \cosh \Delta_{1b} \cosh \Lambda_{2b} \right] \left[ \frac{q k_b}{e} + \frac{\varepsilon_b \lambda_1 e^{-\Lambda_{1b}}}{\cosh \Delta_{1b}} \right],
\] (5.52)

---

\(^6\) One might worry that this is no longer true if the first term in square brackets is zero, but a little work shows that the condition for this term being nonzero (for arbitrary \( k_a \)) is equivalent to the condition \( U_a \neq 0 \), which we assume.
and

\[ p_b = \lambda_3 \eta_b, \]  
\[ |q_b| = \left( \frac{|q| \lambda_b}{\lambda_1} \right) \cosh \Delta_1 \lambda_b, \]  
\[ \sigma_b = \sigma + \lambda_3 \eta_b + \frac{1}{2} \ln \left| \frac{g_b \lambda_2 \cosh \Lambda_2}{g_1 \lambda_b \cosh \Delta_2} \right|. \]

By using the previous expressions for \( \Lambda_2 b \) and \( \lambda_b \), we see that we have solved for \( |q_b| \) and \( \sigma_b \) in terms of the required 3 parameters. The sign of \( q_b \) can be determined by the gauge field jump condition at brane \( b \).

This exhausts all of the matching conditions which do not involve the 4-brane coupling functions. The value of these functions, \( U \) and \( V \), at each brane is then easily obtained by solving the Israel junction conditions, eqs. (5.31) and (5.32), leaving only the dilaton jump condition to be solved. If \( U \) and \( V \) contain enough parameters to allow them and their derivatives to be varied independently for each brane, then this last condition can be solved without adding further constraints on the parameters of the cap geometry.

### 5.3.2 Bulk Geometries Sourced by Given Branes

In the previous section the bulk geometry is considered to be given, and we ask whether regularizing caps can be constructed. This section adopts a different point of view, wherein the characteristics of the caps — \( i.e. \) all the integration constants that define the cap geometry and the quantities \( U \) and \( V \) — are given, and we seek the properties of the bulk which results. In particular, our interest is to see whether and how the two caps must be related to one another, and to check whether the bulk
configuration is always of the form of a GGP solution, with flat four-dimensional slices.

Our goal in doing so is two-fold. First, in this section, we wish to see whether this reduced problem is over-determined, and if so what is required in detail of the branes in order to ensure a solution. Secondly, in §5.3.3 we set out to understand how the volume of the bulk geometry is related to the brane properties, and, by doing so, to exhibit a stabilization mechanism for the bulk volume. Of particular interest is then to understand what 4-brane/cap properties are required to ensure the volumes of the capped regions are much smaller than that of the intervening bulk (as is required if the 4-branes and caps describe the microscopic structure of more macroscopic 3-branes).

Parameter counting and junction conditions

We now show that counting equations and parameters suggests we are not completely free to specify the 4-brane action for brane \( a \) arbitrarily if we ask that it interpolate between 4\( D \)-flat cap and bulk geometries. This can be done only if the 4-brane action is subject to one constraint equation (as was argued in ref. [71]), but once this is satisfied there is sufficient information to determine the parameters describing both the bulk geometry and the properties of brane \( b \).\(^7\)

\(^7\) To be precise, we find a two parameter family of solutions for the bulk and cap \( b \), corresponding to where we choose to embed the two branes in the bulk. Once this choice is made, then the bulk and cap \( b \) are unique.
To this end, imagine we specify the cap geometry and 4-brane action at a given position $\eta = \eta_a$. Next recall that there are 7 integration constants characterizing the bulk geometry — $\lambda_1$, $\lambda_2$, $\xi_1$, $\xi_2$, $p$, $q$ and $\sigma$. (Notice that, although previously we have removed two of these quantities — $\xi_1$ and $p$ — by suitably adjusting coordinates, this is typically no longer possible without altering the specified parameters for cap $a$.) These 7 parameters are subject to a total of 7 conditions at $\eta_a$, consisting of 3 continuity conditions (metric and dilaton) and 4 jump conditions (Israel, Maxwell and dilaton), suggesting that the bulk parameters are completely specified in terms of those of the cap and 4-brane at $\eta_a$.

As we show in the next section, however, one of these seven equations which is supposed to determine one of the bulk parameters turns into a constraint equation amongst cap and brane parameters. Thus, what we find is that for any given cap and brane which satisfies the constraint, there is a one-parameter family of flat bulks to which we can match. Physically, it is easiest to interpret this one parameter in the coordinate system where $\xi_1 = \xi_1a = 0$. Recall that in this coordinate system the brane location in the bulk and cap is $\eta_a$ and $\hat{\eta}_a$, respectively, where these two numbers are generically not the same. Here, we again imagine fixing the cap and brane properties at $\hat{\eta}_a$, and then solving for six of the seven bulk parameters: $\lambda_1$, $\lambda_2$, $\xi_2$, $\sigma$, $p$, $q$, and $\eta_a$. Thus, this one-parameter family of bulk solutions corresponds to where we choose to place the brane in the bulk coordinate system. If $\eta_a$ is fixed, then we find a unique solution for the bulk.

Continuing to use the coordinate system where $\xi_1 = \xi_1a = \xi_{1b} = 0$, we see that once the bulk geometry is thus inferred, there remain 10 parameters associated with
cap $b$, consisting of 6 integration constants — $\lambda_b$, $q_b$, $\xi_{2b}$, $\sigma_b$, $p_b$, and $\dot{\eta}_b$ — plus the brane position $\eta_b$ in the bulk coordinate system, the two 4-brane parameters, $U_b$ and $V_b$, and one linear combination of their derivatives. These 10 parameters are then subject to 9 conditions, consisting of the 7 continuity and jump conditions at the brane location, the smoothness condition at $\eta \to \infty$ for cap $b$ and the topological constraint on the Maxwell field. Provided there are no obstructions to solving these equations, this shows that once we choose the properties of one brane (subject only to the Hamiltonian constraint), together with the location of the two branes in the bulk coordinate system, $\eta_a$ and $\eta_b$, then properties of the other brane and the intervening 4D-flat bulk are precisely dictated. If the properties of brane $b$ are not adjusted in this way in terms of those of brane $a$ then the intervening bulk solution cannot be 4D flat, and instead must either be 4D maximally symmetric but not flat [1] or time-dependent and not Lorentz invariant [2].

This counting bears out, and make more precise, expectations based on earlier studies of the general properties of bulk solutions to 6D supergravity. In particular, for 4D maximally-symmetric solutions [1] (including those which are not 4D flat) the bulk geometry depends nontrivially only on $\eta$, and so we may imagine integrating the bulk field equations in the $\eta$ direction, starting at brane $a$ and ending at brane $b$. Since the $\eta$-$\eta$ Einstein equation does not involve second derivatives of the metric, it represents a ‘Hamiltonian’ constraint on those ‘initial’ conditions at brane $a$ which can be consistently used for such an integration. In this language, the above-mentioned constraint on the allowed 4-brane parameters corresponds to requirements imposed on the 4-brane by matching to the Hamiltonian constraint in
the bulk, restricted to 4D-flat geometries [71]. Furthermore, since the bulk geometry is completely specified by integrating forward in \( \eta \) using the ‘initial’ conditions at brane \( a \), its asymptotic form at brane \( b \) is seen to be completely determined, in agreement with what we find here for explicit 4-brane/cap regularizations of this asymptotic form.

**Explicit solutions**

To better see if parameter and equation counting provides the whole story, we next solve the matching to see whether obstructions to their solutions can exist.

*The Bulk*

The continuity equations, eqs. (5.13) – (5.15), read in this case:

\[
|q| = |q_a| \left( \frac{\lambda_1 \cosh \Lambda_{1a}}{\lambda_a \cosh \Delta_{1a}} \right),
\]

\[
e^{-2\sigma} = e^{-2(\sigma_a - \lambda_3 \eta_a)} \left( \frac{g_a \lambda_2 \cosh \Lambda_{2a}}{g_1 \lambda_a \cosh \Delta_{2a}} \right),
\]

\[
p = p_a - \lambda_3 \eta_a,
\]

and can be thought as fixing the bulk parameters \( q, \sigma, \) and \( p \) (recall the definitions of the parameters \( \Lambda \) and \( \Delta \) in formulae (5.41) and (5.42)). Note that the sign of \( q \) is not yet fixed. These solutions are given in terms of the four bulk quantities \( \lambda_1, \lambda_2, \Delta_{1a}, \) and \( \Delta_{2a} \), for which we now solve.
Before proceeding it is convenient to first define four combinations of brane and cap parameters:

\[
C_1 = \left( \frac{g_a \cosh \Lambda_{2a}}{2g_1 \lambda_a} \right) \left[ \dot{A} \dot{W}^4 (V_a - 2V_a') - \frac{\dot{W}^4}{2\dot{A}} (U_a + 2U_a')(k_a - e\dot{A}_\psi)^2 \right], \tag{5.59}
\]

\[
C_2 = \left( \frac{g_a \cosh \Lambda_{2a}}{4g_1 \lambda_a} \right) \left[ \dot{A} \dot{W}^4 (5V_a - 2V_a') \right. \\
+ \frac{\dot{W}^4}{2\dot{A}} (3U_a - 2U_a')(k_a - e\dot{A}_\psi)^2 + 4\lambda_a \tanh \Lambda_{2a} \left] \right., \tag{5.60}
\]

\[
C_3 = \left( \frac{\varepsilon_a g_a \cosh \Lambda_{2a}}{g_1 \cosh \Lambda_{1a}} \right) \left[ -\frac{eU_a}{q_a} \left( \frac{\dot{W}^4}{\dot{A}} \right) (k_a - e\dot{A}_\psi) + 1 \right], \tag{5.61}
\]

\[
C_4 = \left( \frac{g_a \cosh \Lambda_{2a}}{4g_1 \lambda_a} \right) \left[ \dot{A} \dot{W}^4 (V_a - 2V_a') \right. \\
+ \frac{\dot{W}^4}{2\dot{A}} (7U_a - 2U_a')(k_a - e\dot{A}_\psi)^2 + 4\lambda_a \tanh \Lambda_{1a} \left] \right., \tag{5.62}
\]

where primes here denote differentiation with respect to \( \phi \). These four parameters will take the place of \( U_a(\eta_a), V_a(\eta_a) \), their derivatives (which appear in only one linear combination), and \( k_a \). The action for brane \( a \) can therefore be equally well characterized by these four quantities, as by our original parameterization in terms of \( U_a(\eta_a), V_a(\eta_a) \), and derivatives. With these definitions in hand, the remaining four matching conditions reduce to the following equations:

\[
C_1 = \cosh \Delta_{2a} \left[ 1 - \left( \frac{\lambda_1}{\lambda_2} \right)^\frac{1}{2} \right], \tag{5.63}
\]

\[
C_2 = \sinh \Delta_{2a} \tag{5.64}
\]

\[
C_3 = \varepsilon \left( \frac{\lambda_1}{\lambda_2} \right) \cosh \Delta_{2a} \cosh \Delta_{1a} \tag{5.65}
\]

\[
C_4 = \left( \frac{\lambda_1}{\lambda_2} \right) \tanh \Delta_{1a} \cosh \Delta_{2a}. \tag{5.66}
\]
Recalling that both $\lambda_1$ and $\lambda_2$ are positive, we see immediately that $\varepsilon \equiv \text{sign } q = \text{sign } C_3$.

We note that this system of equations is over-determined, since there are four equations but only three unknowns: $\Delta_{1a}$, $\Delta_{2a}$, and $\lambda_1/\lambda_2$. In fact, by squaring the above equations it is straightforward to check this constraint is given by

$$C_1^2 - C_2^2 + C_3^2 + C_4^2 = 1.$$  \hspace{1cm} (5.67)

When the above equation is satisfied, then it can be shown that the bulk fields satisfy the Hamiltonian constraint which ensures $4D$ flatness. Henceforth, we assume that the brane properties are chosen such that the Hamiltonian constraint is satisfied. In this case, the solution to eqs. (5.63) - (5.66) is

$$\Delta_{1a} = \text{sign}(C_4) \text{arcosh} \left[\left(1 + \frac{C_4^2}{C_3^2}\right)^{\frac{1}{2}}\right],$$  \hspace{1cm} (5.68)

$$\Delta_{2a} = \text{arsinh}(C_2),$$  \hspace{1cm} (5.69)

$$\frac{\lambda_1}{\lambda_2} = \left(1 - \frac{C_4^2}{C_1^2 + C_3^2 + C_4^2}\right)^{\frac{1}{2}}.$$  \hspace{1cm} (5.70)

where the range of arcosh is taken be $\{x \in \mathbb{R} : x \geq 0\}$. It is easy to see that solutions to these equations exist for any values of the $C_i$, subject only to the constraint that they obey eq. (5.67).

As expected from the arguments in the previous section, we indeed find a one-parameter family of possible bulks. Once this parameter is fixed — corresponding

---

8 This Hamiltonian constraint is given by eq. (34) in reference [1].
to choosing where in the bulk we wish to embed the brane — then the bulk solution becomes unique. Henceforth, we assume that this choice has been made (as can be accomplished by making a specific choice for $p$ in eq. (5.58)).

Cap $b$

Having uniquely determined the bulk solution, it remains to determine the properties of the 4-brane and cap at brane $b$. In order to find a unique solution, we first specify the location where we wish to cap the bulk, $\eta_b$. Since this analysis is identical to that of §5.3.1, we do not repeat it in detail here, however the three continuity conditions, the smoothness condition, and the combination of the jump conditions which is independent of $U_b$ and $V_b$ provide 5 constraints on the 5 cap integration constants $p_b$, $\Lambda_{2b}$, $\lambda_b$, $q_b$ and $\sigma_b$ (see eqs. (5.51) - (5.55)). Then, the two Israel junction conditions fix $U_b$ and $V_b$, and the dilaton jump condition provides the constraint which fixes the one relevant combination of derivatives $U_b'$ and $V_b'$. The only cap parameter which is not fixed by these conditions is $\Lambda_{1b}$, and this can be determined from the topological equation (5.43). As expected [1], both the properties of the bulk and those of the 4-brane and cap at $\eta = \eta_b$ are dictated by those of the brane and cap at $\eta = \eta_a$.

5.3.3 Volume Stabilization and Large Hierarchy

The previous analysis fixing the seven bulk integration constants in terms of given cap parameters fixes in particular the integration constant, $\sigma$, that parameterizes the bulk volume. This provides a natural 6D mechanism for stabilizing this bulk volume. In this section we explore this stabilization in more detail, focussing on the conditions which are required to obtain a large hierarchy between the volumes
of the bulk and the caps. In the next section we identify the low-energy 4D effective potential which is generated in this way for $\sigma$.

**Conditions for a hierarchy**

We now ask for the conditions the brane and cap actions should satisfy to ensure that the cap volumes are much smaller than those of the bulk. Our point of view here is that the bulk geometry is given, and so would like to phrase the conditions for a hierarchy in terms of only those parameters over which we have control: the bulk parameters and the three cap parameters, $\eta_a$, $\eta_b$, and $\Lambda_{1b}$.

In order to have branes whose circumference is small, we seek to ensure $A(\eta_a)$ and $A(\eta_b)$ are much less than one. We see from eq. (5.4) that it is natural to examine for this purpose the limit $\eta_a \to -\infty$ and $\eta_b \to \infty$, although in general this need not be sufficient in itself to have small cap volumes. However, we now argue that sufficient conditions for obtaining small cap volumes are given by

$$\Lambda_{1a} = \lambda_a (\eta_a - \xi_{1a}) \ll -1$$
$$\Lambda_{2a} = \lambda_a (\eta_a - \xi_{2a}) \ll -1$$

with similar conditions for brane $b$. Large, negative $\eta_a$ is not sufficient for small cap volumes because it does not in itself ensure that these conditions are satisfied. Under these conditions we may use the asymptotic form for the hyperbolic functions and so obtain the following expression for volume of cap $a$

$$\Omega_a = 2\pi \int_{-\infty}^{\eta_a} d\eta A^2 \hat{W}^4$$
$$\simeq \frac{\pi}{\lambda_a} e^{2(\sigma - \sigma_a + p_a)} (A^2 \hat{W}^4)\big|_{\eta_a}$$

(5.72)
In arriving at the second line we have used the continuity equations, (5.11), to relate cap functions to bulk functions. The cap volume must be compared with the bulk volume, given by the expression

\[ \Omega_{\text{bulk}} = \frac{2 \pi}{(2g_1)^3 q} \int_{\eta_a}^{\eta_b} d\eta \frac{A^2 W^4}{\cosh^{\frac{3}{2}} \left[ \lambda_2 (\eta - \xi_2) \right] \cosh^{\frac{3}{2}} \left[ \lambda_1 (\eta - \xi_1) \right]} \]

(5.73)

It is simple to check that the integral in the previous expression is always finite. Then, it is enough to choose the parameters in the bulk of order one, to obtain \( \Omega_{\text{bulk}} \sim \mathcal{O}(1) \). In order to obtain a hierarchy between bulk and cap volumes, it is necessary to demand that \( \Omega_a \ll \mathcal{O}(1) \).

We now evaluate the cap volume using the general solutions found earlier. If we also use the hierarchy assumptions, eq. (5.71), and the continuity equation (5.14), we calculate the cap volume to be

\[ \Omega_a \simeq 2\pi \left( \frac{g_1 \lambda_1 \cosh \Delta_{2a}}{|q| \lambda_2 \cosh \Delta_{1a}} \right) (A^2 W^4)_{|\eta_a|} \]

\[ = \pi A^2_{|\eta_a|} \cdot \]

(5.74)

For the generic situation of \( \mathcal{O}(1) \) bulk parameters, we see from eq. (5.4) that \( A^2_{|\eta_a|} \ll 1 \) in the limit of large \( |\eta_a| \). Thus, we obtain the desired result: \( \Omega_a \ll \Omega_{\text{bulk}} \sim \mathcal{O}(1) \).

Alternatively, if we instead wish to have cap volumes which are \( \mathcal{O}(1) \) and bulk volumes which are much larger, we simply need to choose \( \sigma \gg 1 \) while keeping all other bulk parameters fixed.
It remains now to show what conditions must be imposed on the bulk parameters and cap parameters in order to ensure that conditions (5.71) are satisfied. To simplify this discussion, we only consider the case $\lambda_3 = 0$. We accomplish this by adjusting the background gauge coupling, $e$, so that it is approximately equal to its value, $e_0 = qN/(2\lambda_1)$, in the absence of caps. More precisely, if we define

$$\epsilon = \frac{1}{2}e^{-\lambda_1(\eta_a - \xi_1)} \left[ \frac{qN}{\Lambda_1 e} - 2 \right],$$

(5.75)

then we should take

$$\epsilon \ll 1 \quad \text{and} \quad \Lambda_{1b} \gg 1$$

(5.76)

and, for definiteness, take $\eta_a \approx -\eta_b$. In this case, the general cap solutions found earlier satisfy the desired hierarchy conditions (5.71). The analogous hierarchy conditions at brane $b$ are much simpler to satisfy since we get to choose freely $\Lambda_{1b}$. For example, choosing $\eta_b$ large and $\Lambda_{1b} \sim \Delta_{1b} \gg 1$ guarantees that $\Lambda_{2b} \gg 1$ and so the two hierarchy constraints are satisfied.

To summarize, we see here how to obtain regularizing caps which are much smaller than the bulk volume, by appropriately tuning the gauge coupling $e$ and by choosing large coordinate values for the brane positions. We have also shown that requiring such a hierarchy at only a single brane is not difficult to achieve in the sense that it involves no tuning of any bulk parameters.

### 5.3.4 Low-energy 4D Effective Potential

We next dimensionally reduce the capped bulk to 4 dimensions in order to identify more explicitly how the 4-brane action influences the stabilization of the
would-be flat direction parameterized by $\sigma$. In this section we restrict ourselves to evaluating the effective 4D potential for $\sigma$ within the classical approximation.

To this end we identify the effective 4D action $S_{\text{eff}} = \int d^4 x \mathcal{L}_{\text{eff}}$ by computing the 6D action at a one-parameter family of classical solutions labelled by the constant $\sigma$:

$$S_{\text{eff}} = S_a + S_b + S_{\text{cap} a} + S_{\text{bulk}} + S_{\text{cap} b}, \quad (5.77)$$

where $S_a = \int d^5 x \mathcal{L}_a$ and $S_b = \int d^5 x \mathcal{L}_b$ represent the 4-brane action for caps $a$ and $b$, given by eq. (5.21), while $S_M = \int_M d^6 x \mathcal{L} + S_{GH}(\partial M)$ represents the 6D bulk action, including the Gibbons-Hawking boundary contribution, defined by eq. (5.7). The three last terms correspond to dividing the integration over the 2 extra dimensions into the three intervals defining the bulk, cap $a$ or cap $b$.

Following [51], we see that using the 6D field equations, (2.4), to simplify the 6D bulk action in a region $M$ with boundaries leads to the simple expression (with $\kappa^2 = 1$)

$$S_{\text{cl}} = \frac{1}{2} \int_M d^6 x \sqrt{-g} \Box \phi_{\text{cl}} - \int_{\partial M} d^5 x \sqrt{-\gamma} K_{\text{cl}}, \quad (5.78)$$

which, together with Gauss’ Law, allows the last three terms in eq. (5.77) to be written

$$S_{\text{cap} a} + S_{\text{bulk}} + S_{\text{cap} b} = -\frac{1}{2} \int d^5 x \left( [\sqrt{-g} \partial^\eta \phi + 2 \sqrt{-\gamma} K]_{\eta a} + [\sqrt{-g} \partial^\eta \phi + 2 \sqrt{-\gamma} K]_{\eta b} \right), \quad (5.79)$$

where as before $[f(\eta)]_{\eta a} = f(\eta_a + \epsilon) - f(\eta_a - \epsilon)$ (and similarly for $\eta_b$).

Writing $S_{\text{eff}} = \int d^4 x \mathcal{L}_{\text{eff}}$, and evaluating the right-hand-side of this last expression using the Israel and dilaton jump conditions, (5.31), (5.32) and (5.36) finally
\[ \mathcal{L}_{\text{eff}} = 2\pi \sum_{i=a,b} A W^4 e^{2(\sigma - p)} \left[ \left(-V_i + \frac{5V_i}{4} - \frac{1}{2} \frac{dV_i}{d\phi} \right) \right. \\
\left. + \frac{1}{2\mathcal{A}^2} \left(-U_i + \frac{3U_i}{4} - \frac{1}{2} \frac{dU_i}{d\phi} \right) (k_i - eA_{\psi})^2 \right] \\
= \pi \sum_{i=a,b} A W^4 e^{2(\sigma - p)} \left[ \left(\frac{V_i}{2} - \frac{dV_i}{d\phi} \right) - \frac{1}{2\mathcal{A}^2} \left(\frac{U_i}{2} + \frac{dU_i}{d\phi} \right) (k_i - eA_{\psi})^2 \right]. \] 

Finally, to make the \( \sigma \)-dependence explicit we write \( \mathcal{A} = A_0 e^{\sigma/2}, \phi = \phi_0 - \sigma \), and choose for concreteness \( V(\phi) = v e^{s\phi} \) and \( U(\phi) = u e^{t\phi} \). Identifying \( V_{\text{eff}} = -\mathcal{L}_{\text{eff}} \), we find

\[ V_{\text{eff}}(\sigma) = \sum_{i=a,b} \left[ C_{V_i} e^{(5/2-s_i)\sigma} + C_{U_i} e^{(3/2-t_i)\sigma} \right], \] 

where

\[ C_{V_i} = \pi \left[ \mathcal{A}_0 W^4 \left(\frac{1}{2} - s_i \right) u_i e^{s_i\phi_0 - 2p} \right]_{\eta = \eta_i}, \]
\[ C_{U_i} = -\frac{\pi}{2} \left[ \mathcal{W}^4 \left(\frac{1}{2} + t_i \right) u_i e^{t_i\phi_0 - 2p} (k_i - eA_{\psi})^2 \right]_{\eta = \eta_i}. \]

It is clear that this potential generically only has runaway solutions when both \( C_{U_i} \) and \( C_{V_i} \) and both of the coefficients of \( \sigma \) in the exponents have the same sign, but has nontrivial minima when some of these signs differ. Given the explicit relative sign appearing in eqs. (5.82), and positive \( u_i \) and \( v_i \), we expect that stabilization of \( \sigma \) to be fairly generic.
The scale invariant case

Of particular interest is the case of scale-invariant branes, for which we have \( s_i = 1/2 \) and \( t_i = -1/2 \). In this case, not only do we recover the generic scale-invariant form for the potential

\[
V_{\text{eff}}(\sigma) = C e^{2\sigma} \quad \text{with} \quad C = \sum_{i=a,b} \left( C_{U_i} + C_{V_i} \right),
\]

(5.83)

but we also learn that \( C = C_{U_i} = C_{V_i} = 0 \). This agrees, and makes more precise, the arguments of ref. [51], wherein the same conclusion was drawn when scale-invariant branes were characterized as delta-function sources.

5.4 Discussion

In this chapter we present a regularization procedure for resolving the singularities in the most general axially symmetric, 4\(D\)-flat solutions to 6\(D\) gauged, chiral supergravity. This procedure resolves the singularities of these geometries using an explicit, but broad, class of cylindrical 4-branes that couple with the bulk Maxwell, dilaton and gravitational fields. The space interior to these 4-branes is capped off using the most general smooth, 4\(D\)-flat, and axially symmetric solutions to the same 6\(D\) supergravity equations that were used in the bulk between the two branes. Our analysis provides the necessary tools required to precisely explore the connections between properties of the bulk field configurations and the structure of the branes which source them.

We keep our analysis very general, with the goal of being able to map out these connections with as few restrictions as possible. We show, in particular, that the class of caps and 4-brane actions we consider contain sufficient numbers of parameters
to cap an arbitrary axially-symmetric and 4D-flat bulk geometry. We also show that once the properties of one of the 4-brane caps is specified, there are sufficient parameters in the bulk geometry and in the other cap to complete the geometry. This both identifies the properties of the bulk sourced by a given brane, and precisely identifies how the properties of the brane at the other end of the bulk are dictated by those of the source brane with which one starts.

Knowing the properties of the caps shows that the presence of regularizing branes has important consequences on the properties of the bulk solutions. In particular, we show how the classical degeneracy amongst bulk geometries having different volumes can be lifted by the coupling of the 4-branes with the 6D dilaton. This provides a stabilization mechanism for the bulk, which relates the size of the extra dimensions with brane properties. By performing a dimensional reduction we also identify the effective 4D potential which captures this stabilization mechanism in the low-energy limit. We are able to do because our regulated 6D configurations are smooth everywhere, with the bulk fields not diverging at the brane positions (as they do for the effective codimension-2 3-branes obtained in the thin-brane limit when the circumference of the 4-brane is taken to zero).
CHAPTER 6
Ultraviolet Sensitivity in Higher Dimensions

In this chapter, we calculate the size of the one-loop quantum corrections induced by bulk loops for various supergravities in higher dimensions. As explained in the introductory chapter, the success of the SLED proposal relies on these one-loop effects being of order $1/r_c^4 \sim m_{sb}^4$ in the effective 4D potential, where we recall $r_c$ is the compactification radius and $m_{sb} \sim M_{EW}^2/M_{Pl} \sim 10^{-3}$ eV is the supersymmetry-breaking scale in the bulk. At this point, it is important to emphasize again that within the SLED proposal, the Casimir energy on the brane itself is not required to be small. This is a direct consequence of the fact that such an energy does not give rise to a 6D cosmological constant, but instead a localized energy density $\int d^4x \sqrt{g} \rho$ which is indistinguishable from a brane tension.

Thus, when we integrate out the SM brane fields to obtain the low-energy cosmological constant, this has the effect of simply renormalizing various brane tensions. The marvel of the SLED proposal occurs when we proceed to integrate out the bulk fields in the presence of the above pure tension branes. At the classical level this amounts to substituting their equations of motion into the action, yielding the classical contribution of the bulk fields to the effective 4D cosmological constant. As we saw in chapter 2, for a certain class of theories the bulk fields dynamically adjust such that their contribution exactly cancels the contribution coming from the brane tensions.
The final stage of the calculation involves the quantum contribution of the bulk fields to the effective cosmological constant. Given the cancellations in the first two steps, the quantum effects due to bulk loops will be the sole contribution to the effective 4D cosmological constant. Generically, this contribution is expected to be of order $M_{EW}^2 m_{sb}^2$, and so our task is to determine under what conditions this term vanishes. Assuming this leading term does vanish, the remaining contribution would be the correct order of magnitude to account for the observed dark energy.

As pointed out in [73], this effective potential can have the correct properties to provide a natural quintessence-type description of dark energy. Once quantum effects are taken into account, the resulting 4D effective potential has the general form (cf. eq. (1.26))

$$V(r) = \frac{1}{r^4} \left[ c_2 (M_* r)^2 + c_3 + c_3' \log(M_* r) + \cdots \right]. \quad (6.1)$$

Denoting the minimum of this potential by $r_0$, we see that if $c_2 \neq 0$ then obtaining a minimum at the phenomenologically desirable value $r_0 \sim 10^{-3}$ eV will inevitably require a large fine-tuning among the coefficients $c_i$. For $c_2 = 0$ on the other hand, this potential can have a minimum at $r_0 \sim 10^{-3}$ eV for $c_3 < 0$ and a reasonable hierarchy $c_3/c_3' \sim 35$. Happily, potentials of this form have been studied in other works and were shown to lead to acceptable cosmologies [73]. Of course, this argument relies on the vanishing of the coefficient $c_2$, something that must be checked on a model-by-model basis.
The goal of this chapter is therefore to determine whether this coefficient does indeed vanishes for certain supergravity theories. In this regard, we have only partial success to report. The difficulty lies in the fact that for the Salam-Sezgin and GGP solutions, we have not yet been able to disentangle various mixings between particles of different spins, which is a prerequisite in order to apply the results of this chapter. Research on this problem is still on-going. Nonetheless, the results we obtain can be directly applied to the case where the extra dimensions are Ricci-flat, and show that the dangerous terms in the effective potential vanish when effects for all particles in a massive supermultiplet are taken into account. Thus, we find that these results support the interpretation of supersymmetric large extra dimensions in 6 dimensions as a potential solution to the cosmological constant problem, but do not yet completely clinch the case.

6.1 General Results

Our goal is to compute explicitly (as functions of background fields) the ultraviolet sensitive part of the one-loop vacuum energy for compactifications to $4D$ of various $6D$ field theories. Before embarking on the full calculation it is worth first collecting a few general results concerning the kinds of ultraviolet divergences which can be encountered in calculations of this sort. Because the ultraviolet behavior only depends on the very short-distance limit of the theory these divergences can always be absorbed into renormalizations of local functions of the background fields, with coefficients which can be computed very generally for arbitrary background geometries [96, 97, 98].
6.1.1 The Gilkey-DeWitt Coefficients

This section collects the results for the ultraviolet-divergent parts of the one-loop action obtained by integrating out various kinds of particles in 6 dimensions. To this end, consider a collection of $N$ fields, $\Psi^z$ with $z = 1, \ldots, N$, coupled to a collection of background fields, possibly including a 6-dimensional spacetime metric, $g_{MN}$, scalars, $\varphi^i$, and form fields, $A^a_{M_1\ldots M_p}$. For each $z$, $\Psi^z$ can carry a gauge and/or Lorentz index, although for simplicity of notation the Lorentz index is often suppressed in this section. We suppose that the background-covariant derivative, $D_M$, appropriate to $\Psi^z$ is:

$$D_M \Psi^z = \partial_M \Psi^z + \omega_M \Psi^z - i A^a_M (t_a)^z_y \Psi^y,$$

where $\omega_M$ is the appropriate matrix-valued spin connection, and the gauge group is represented by the hermitian matrices $(t_a)^z_y$. For real fields the $t_a$ are imaginary antisymmetric matrices, and (for canonically-normalized gauge bosons) we take the gauge group generators to include a factor of the corresponding gauge coupling, $g_a$. The commutator of two such derivatives defines a generalized matrix-valued curvature, $(Y_{MN})^z_y \Psi^y = [D_M, D_N] \Psi^z$, which has the following form:

$$(Y_{MN})^z_y = -\frac{i}{2} R_{MN}{}^{AB} J_{AB} \delta^z_y - i F^a_{MN} (t_a)^z_y,$$

where the $J_{AB}$ are the usual (field-appropriate) generators of Lorentz transformations.

Quite generally the result of integrating out the fields $\Psi^z$ at one-loop leads to the following contribution to the effective quantum action

$$i \Sigma = -(-)^F \frac{1}{2} \text{Tr} \log \Delta,$$

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where \((-\)^F = 1\) for bosons and \(-1\) for fermions, and the differential operator $\Delta^z_y$ has the following form

$$\Delta^z_y = -\delta^z_y \Box + X^z_y + (m^2)^z_y.$$  

(6.5)

Here $\delta^z_y$ is the Kronecker delta, $\Box = g^{MN} D_M D_N$ and $X^z_y$ is a local quantity built from the background fields whose form depends on the kind of field under consideration (explicit examples are given below for the usual fields of interest). The mass matrix, $m^2$, can either be regarded as being physical masses which are extracted from within $X$, or as a regulator mass, $(m^2)^z_y = \mu^2 \delta^z_y$, which is to be taken to zero (or to infinity) at the end of the calculation.

Our interest for this section is in two parts of $\Sigma$ which are very closely related to one another. One of these is the ultraviolet divergent part of $\Sigma$, and the other is that part of $\Sigma$ which depends most strongly on the mass of any massive 6D fields which are integrated out. We collect here the very general results which can be obtained for both of these quantities using the Gilkey-DeWitt heat-kernel methods [96, 97, 98]. When identifying the divergent part we work within dimensional regularization and so continue the spacetime dimension to complex values, $n$, which are slightly displaced from the actual integer spacetime dimension, 6, which is of interest: $n = 6 - 2\epsilon$. We then follow the poles in $\Sigma$ as $\epsilon \to 0$, in the usual fashion. These may be related to the logarithmic divergences which would be obtained from an ultraviolet cutoff, $\Lambda$, through the usual relation

$$\frac{1}{\epsilon} \leftrightarrow \ln (\Lambda^2).$$  

(6.6)
For $6D$ spaces without boundaries and singularities the ultraviolet-divergent terms (and heavy-mass-dependent terms) are simply characterized. In $n$ dimensions they may be written as [96, 99]

$$
\Sigma_\infty = \frac{1}{2} \left( -\frac{1}{4\pi} \right)^{n/2} \int d^n x \sqrt{-g} \sum_{k=0}^{[n/2]} \Gamma(k - n/2) \text{Tr} \left[ m^{n-2k} a_k \right], \quad (6.7)
$$

which for $n = 6 - 2\epsilon$ specializes to

$$
\Sigma_\infty = \frac{1}{2(4\pi)^3} \left( -\frac{1}{4\pi} \right)^{3} \sum_{k=0}^{3} \Gamma(k - 3 + \epsilon) \int d^6 x \sqrt{-g} \text{Tr} \left[ m^{6-2k} a_k \right]. \quad (6.8)
$$

Here $\Gamma(z)$ denotes Euler’s gamma function. The divergence as $\epsilon \to 0$ is contained within the gamma function, which has poles at non-positive integers of the form $\Gamma(-r + \epsilon) = (-)^r / (r! \epsilon) + \cdots$, for $\epsilon$ an infinitesimal and $r$ a non-negative integer.

The coefficients, $a_k$, are known $N' \times N'$ matrix-valued local quantities constructed from the background fields, and are given explicitly in appendix C. Here $N' = N d$ with $N$ counting the number of fields and $d$ being the dimension of the appropriate Lorentz representation. The trace is over the $N'$ matrix indices of the $a_k$.

The above expression shows that for massless fields ($m = 0$) in compact spaces without boundaries and singularities in 6 dimensions the divergent contribution is proportional to $\text{tr} \left[ a_3 \right]$ in dimensional regularization, so the problem of identifying these divergences reduces to the construction of this coefficient.\footnote{Notice that the freedom to keep $m^2$ within or separate from $X$ implies that the divergence obtained from computing just $a_3$ using $X_m = X + m^2$ gives the same result as computing $a_0$ through $a_3$ using only $X$.} By contrast, for

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massive fields there are divergences proportional to $\text{tr} [m^6 a_0]$, $\text{tr} [m^4 a_1]$, $\text{tr} [m^2 a_2]$ and $\text{tr} [a_3]$, and these are also the terms in $\Sigma$ which involve the highest powers of $m$. For example, it turns out that $a_0$ is proportional to the unit matrix, $I$, and so the term involving $a_0$ represents a divergent and strongly $m$-dependent contribution to the $6D$ cosmological constant, proportional to $\text{tr} [m^6]$. Similarly, since $a_1$ contains a term proportional to $RI$, where $R$ is the background metric’s Ricci scalar, $\text{tr} [m^4 a_1]$ contains an $m$-dependent renormalization of the Einstein-Hilbert action, and so on.

What is attractive about the above results is that an algorithm for constructing the coefficients $a_k$ is known for general $X$ and $D_M$, and the result for the first few has been computed explicitly [96, 99] and can be given as a closed form in terms of $X$, background curvatures and the generalized curvature $(Y_{MN})^y_z$. The explicit results for the quantities $a_0$ through $a_3$ are given in their general form in appendix C. These allow the calculation of the most ultraviolet-sensitive contributions from quantum loops for arbitrary theories in the presence of very general background field configurations.

The remainder of this section specializes these results to the various fields of interest for $6D$ supergravity theories. We take the bosonic part of the action for these theories to be

$$S = - \int d^6x \sqrt{-g} \left[ \frac{1}{2} g^{MN} G_\ij(\Phi) D_M \Phi^i D_N \Phi^j + V(\Phi) \right. \left. + \frac{1}{2} U(\Phi) R + \sum_p \frac{1}{2p!} W_p(\Phi) F_{aM_1...M_p}^a F^{aM_1...M_p} \right], \quad (6.9)$$

where $\Phi^i$ denote the theory’s scalar fields, and $F_{(p)} = dA_{(p-1)} + \omega_p$ is a $p$-form field strength for a $(p-1)$-form gauge potential, and $\omega_p$ is an appropriate Chern-Simons
form whose details are not important in what follows. The coefficient functions $U$, $V$, $W$ and $G_{ij}$ are known functions of the $\Phi^i$ which differ for different choices for the 6D supergravity of interest.

As usual, we are always free to use the classical equations of motion obtained from this action to simplify the one-loop quantity $\Sigma$, because anything which vanishes with the classical field equations may be removed from $\Sigma$ by performing an appropriate field redefinition \cite{14, 100, 101}.

6.1.2 Dimensional Reduction from 6D to 4D

When using the Gilkey coefficients in 6-dimensional theories compactified to 4 dimensions one might be tempted to ask whether we should take $n = 4$ or $n = 6$ when evaluating formulae like eq. (6.7). In this section we show that it makes no difference, inasmuch as the sum over the result for each $4D$ KK mode reproduces the full $6D$ expression.

**Dimensional reduction on $S^2$**

To establish this point we take the simplest nontrivial example: the reduction of a 6D scalar field theory to 4D on a 2-sphere.\footnote{Compactification on a torus is too trivial for the present purposes, since all of the Gilkey coefficients except $a_0$ tend to vanish for flat manifolds like tori. Refs. \cite{102, 103, 104, 105} include more recent explicit examples.} For these purposes we start with the 6D action

$$S = -\int d^4x d^2y \sqrt{-g} (-\Phi^* \Box_6 \Phi + M^2 \Phi^* \Phi),$$  \hspace{1cm} (6.10)
where $\Phi$ is a minimally-coupled complex scalar field with a $6D$ mass $M$, and $\Box_6 = g^{MN}D_M D_N$ is the $6D$ d’Alembertian.

We further assume that the background metric takes the product form $ds_6^2 = g_{\mu \nu}(x)dx^\mu dx^\nu + g_{mn}(y)dy^m dy^n$, where $g_{mn}$ is the metric on the internal two dimensions and $g_{\mu \nu}$ is the metric of the ‘large’ $4D$ dimensions. In this case the $6D$ d’Alembertian is related to its $4D$ counterpart, $\Box_4 = g^{\mu \nu}D_\mu D_\nu$, and the $2D$ Laplacian, $\Box_2 = g^{mn}D_mD_n$, by $\Box_6 = \Box_4 + \Box_2$. Finally, we specialize to an internal $S^2$ by taking $g_{mn}dy^m dy^n = r_c^2 \gamma_{mn} dy^m dy^n = r_c^2 (d\theta^2 + \sin^2 \theta d\phi^2)$, where $r_c$ denotes the $2$-sphere’s radius.

The dimensional reduction is performed by writing $\Phi$ as a mode sum in terms of the eigenfunctions of the scalar Laplacian on a $2$-sphere. Our ansatz therefore becomes:

$$\Phi(x, y) = \frac{1}{r_c} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \phi^m_l(x) Y^m_l(y),$$  \hspace{1cm} (6.11)

where $Y^m_l(y)$ are the standard spherical harmonics, and $\phi^m_l(x)$ are the corresponding $4D$ fields. Using $\Box_6 = \Box_4 + \Box_2$ we find

$$\Phi^* \Box_6 \Phi = \frac{1}{r_c^2} \sum_{m,l,m',\nu} \left[ \left( \phi^m_{l*}(x) \Box_4 \phi^{m'}_{l*}(x) \right) Y^m_l Y^{m'}_{l*} + \phi^m_{l*}(x) \phi^{m'}_{l*}(x) Y^m_l \Box_2 Y^{m'}_{l*} \right].$$  \hspace{1cm} (6.12)

Finally, using the results $-\Box_2 Y^m_l = [l(l+1)/r_c^2] Y^m_l$, $\sqrt{-g_6} = r_c \sqrt{-g_4} \sqrt{\gamma}$ as well as the orthonormality relations $\int d^2 y \sqrt{\gamma} Y^m_{l*} Y^{m'}_{l*} = \delta_{l \nu} \delta_{m \nu}$, the $6D$ action becomes

$$S = -\sum_{m,l} \int d^4 x \sqrt{-g_4} \left[ -\phi^{m*}_l \Box_l \phi^m_l + \left( M^2 + \frac{l(l+1)}{r_c^2} \right) \phi^{m*}_l \phi^m_l \right],$$  \hspace{1cm} (6.13)

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where all quantities are now functions only of $x$. This is the standard manipulation which expresses the theory of one complex 6D scalar in terms of an infinite tower of 4D Kaluza-Klein (KK) modes, consisting of complex 4D scalars whose 4D masses are $\mu^2 = M^2 + l(l + 1)/r_c^2$. Notice that for scalars on the 2-sphere each KK mass level has degeneracy $d_l = (2l + 1)$.

**UV sensitivity**

We now check that the UV sensitive terms are identical when computed in 6D or as the sum over a series of 4D results for each KK mode. Recall for these purposes that the divergent part of the one-loop quantum action can be written in $n$ dimensions as

$$\Sigma_\infty = \frac{1}{2} (-)^F \left( \frac{1}{4\pi} \right)^{n/2} \int d^n x \sqrt{-g} \sum_{k=0}^{[n/2]} \Gamma(k - n/2) \text{Tr} [M^{n-2k} a_k],$$

(6.14)

where $M$ is the $n$-dimensional mass of the particle which traverses the loop.

**The 6D Calculation:** For the 6D calculation we use the general result specialized to a minimally-coupled scalar field in $n = 6$ dimensions. For simplicity we also assume the 6D complex scalar to be massless — so $M = 0$ — and take $Y_{MN} = X = 0$. Because the scalar has been taken to be massless in 6D, the only relevant Gilkey coefficient is $a_3$, which we must evaluate. For this evaluation we specialize the general result to the product geometry, for which $R_6 = R_4 + R_2$, $R_{MN} R^{MN} = R_{\mu\nu} R^{\mu\nu} + R_{mn} R^{mn}$ etc., where the 2-sphere curvatures satisfy $R_{mnpq} = (1/r_c^2)(g_{mq} g_{np} - g_{mp} g_{nq})$, $R_{mn} = -(1/r_c^2) g_{mn}$ and $R_2 = -2/r_c^2$, so $R_{mnpq} R^{mnpq} = 2 R_{mn} R^{mn} = R_2^2 = 4/r_c^4$. 

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Remembering the overall factor of 2 because the scalar is complex, and expanding \( a_3 \) in powers of the 4D curvature tensor we find

\[
\text{Tr} [a_3] = \frac{8}{315 r^6} - \frac{1}{45 r^4} R_4 + \ldots ,
\]

(6.15)

and so integrating over the 2-sphere, and using \( \Gamma(0) \sim 1/\epsilon \), we find

\[
\Sigma_\infty = \frac{1}{2} \left( \frac{1}{4\pi} \right)^2 \frac{1}{\epsilon} \int d^4 x \sqrt{-g} \left( \frac{8}{315 r^4} - \frac{1}{45 r^2} R_4 + \ldots \right).
\]

(6.16)

**The 4D Calculation:** In the 4D theory we may similarly take \( X = 0 \) provided we separate explicitly the KK mass terms from \( X \), and up to linear order in \( R_4 \) we need keep only the contributions to \( a_1 \) and \( a_0 \). Using \( \Gamma(-k + \epsilon) \sim (-)^k/(k! \epsilon) \) we find for each KK mode (remembering again the factor of 2 for complex scalars),

\[
\Sigma_{lm}^\infty = \frac{1}{2} \left( \frac{1}{4\pi} \right)^2 \int d^4 x \sqrt{-g} \left( \text{Tr} [\mu^4 a_0] \frac{1}{2\epsilon} - \text{Tr} [\mu^2 a_1] \frac{1}{\epsilon} + \ldots \right)
\]

\[
= \frac{1}{2} \left( \frac{1}{4\pi} \right)^2 \int d^4 x \sqrt{-g} \left( \frac{\mu^4}{\epsilon} + \frac{\mu^2}{3\epsilon} R_4 + \ldots \right).
\]

(6.17)

We now sum over the KK modes, and interpret the resulting divergent sums using \( \zeta \)-function regularization [106, 107, 108]. Recalling that each mass eigenvalue \( \mu_l^2 = l(l + 1)/r_c^2 \) has degeneracy \( (2l + 1) \) we have

\[
\sum_{lm} \mu_l^4 = \sum_{l=0}^{\infty} \left( \frac{l(l + 1)}{r_c^2} \right)^2 (2l + 1)
\]

\[
= \frac{1}{r_c^4} \sum_{l=1}^{\infty} (2l^5 + 5l^4 + 4l^3 + l^2)
\]

\[
= \frac{1}{r_c^4} \left[ 2\zeta(-5) + 5\zeta(-4) + 4\zeta(-3) + \zeta(-2) \right],
\]

(6.18)
where $\zeta(s) = \sum_{n=1}^{\infty} (1/n^s)$ is the Riemann zeta-function. Using the results [109]
\[
\zeta(-5) = -\frac{1}{252}, \quad \zeta(-3) = \frac{1}{120}, \quad \zeta(-4) = \zeta(-2) = 0, \quad \zeta(-1) = -\frac{1}{12},
\]
(6.19)
we find that
\[
\sum_{lm} \mu_l^4 = \frac{8}{315r_c^4}. \quad (6.21)
\]
Similarly
\[
\sum_{lm} \mu_l^2 = \sum_{l=0}^{\infty} \frac{l(l+1)}{r_c^2}(2l + 1)
= \frac{1}{r_c^2} \sum_{l=1}^{\infty} (2l^3 + 3l^2 + l)
= \frac{1}{r_c^2} \left[ 2\zeta(-3) + 3\zeta(-2) + \zeta(-1) \right],
\]
(6.22)
Finally, combining these results we obtain the following expression for the divergent piece, as computed in 4 dimensions:
\[
\Sigma_{\infty} = \sum_{lm} \Sigma_{\infty}^{lm}
= \frac{1}{2} \left( \frac{1}{4\pi} \right)^2 \frac{1}{\epsilon} \int d^4 x \sqrt{-g} \left( \frac{8}{315r_c^4} - \frac{1}{45} R_4 + \ldots \right). \quad (6.23)
\]
As expected, we obtain the same result for $\Sigma_{\infty}$ regardless of whether we do the calculation in the 6D or the 4D theory, provided we sum over all of the KK modes in the lower-dimensional case. It is therefore a matter only of convenience whether or not to use the higher- or lower-dimensional formulation.
Dimensional reduction in supersymmetric models

The previous calculations are useful when computing the UV sensitivity of 6D supersymmetric theories, particularly when 6D supersymmetry breaks due to the compactification down to 4 dimensions. Seen from the 4D point of view it might appear that supersymmetry is badly broken, making the cancellations due to 6D supersymmetry hard to follow. However, the freedom to perform computations in the higher dimensions makes it easier to see the cancellations which follow from higher-dimensional supersymmetry. Physically these cancellations still hold because it is the UV sensitive part of the one-loop result which we compute, and this is only sensitive to the very short wavelengths for which the higher-dimensional symmetries apply.

We now turn to a discussion those UV-sensitive effects which are localized near the position of any codimension-2 branes.

6.1.3 Brane-Localized Terms

For supersymmetric large extra dimensions we require the Casimir energy in the presence of brane sources, which typically introduces either boundaries or singularities into the bulk geometry, depending on the dimension of the brane involved. Since the presence of boundaries and singularities permit the appearance of more complicated divergences in the Casimir energy, additional local counterterms are required in order to renormalize them. Since all of these are localized at the brane positions, they can be regarded as renormalizations of the effective brane actions.

Unfortunately, results with the generality described above are not yet available in the presence of codimension-2 brane sources. Some things are known, however,
and we summarize those which are most relevant to the SLED proposal here. The main calculations which have been done assume the geometry near the branes to be described by a conical singularity, for which the $2D$ bulk metric can be written in the form $dr^2 + c_b^2 r^2 d\theta^2$ near the singularity ($r = 0$), where $\theta$ is a periodic coordinate with period $2\pi$ and $c_b$ is a constant. This geometry has a defect angle at the brane position, whose size is given by $\delta = 2\pi(1 - c_b)$. As discussed in chapter 1, this introduces a delta-function-type divergence into the curvature at the brane position which is proportional to $\delta$. This kind of singularity is often (but not always [53, 110]) what is produced by 3-branes which are aligned within the 6 dimensions to be parallel with the large 4 dimensions.

Some explicit results are known for the types of ultraviolet divergences which arise in this case. This includes explicit results for the heat kernel coefficients for specific types of particles in the presence of these singularities [99, 111] as well as more general expressions which apply to the limit of small defect angles, which are obtained by interpreting the cone to be the limit of a sequence of ‘blunted’ cones for each of which the tip is smoothed off [112, 113, 114]. According to this line of argument, the leading contributions (for small defect angles) to the brane counterterms may be found by taking the limit of the bulk terms obtained for each of the blunted cones.
Applied to quadratic order in the background curvatures, this leads to the relations:

\[
\begin{align*}
\int_{\tilde{B}} d^6 x \sqrt{-g} \tilde{R} & \approx \int_{B'} d^6 x \sqrt{-g} R - \sum_b 4\pi (1 - c_b) \int_b d^4 x \sqrt{-h} \\
\int_{\tilde{B}} d^6 x \sqrt{-g} \tilde{R}^2 & \approx \int_{B'} d^6 x \sqrt{-g} R^2 - \sum_b 8\pi (1 - c_b) \int_b d^4 x \sqrt{-h} R \\
\int_{\tilde{B}} d^6 x \sqrt{-g} \tilde{R}_{MN} \tilde{R}^{MN} & \approx \int_{B'} d^6 x \sqrt{-g} R_{MN} R^{MN} - \sum_b 4\pi (1 - c_b) \int_b d^4 x \sqrt{-h} R_{aa} \quad (6.24) \\
\int_{\tilde{B}} d^6 x \sqrt{-g} \tilde{R}_{MNPQ} \tilde{R}^{MNPQ} & \approx \int_{B'} d^6 x \sqrt{-g} R_{MNPQ} R^{MNPQ} - \sum_b 8\pi (1 - c_b) \int_b d^4 x \sqrt{-h} R_{abab},
\end{align*}
\]

where the approximate equality indicates that terms of order \((1 - c_b)^2\) are neglected on the right-hand side. In these expressions \(\tilde{B}\) denotes the limit of the sequence of blunted cones (having curvature \(\tilde{R}_{MNPQ}\)) which approach the bulk, \(B\), including the conical singularity, and \(b\) denotes the 4D world-surface of the brane, defined by the position of the conical singularity. \(B'\) denotes the bulk with the positions of the conical singularities removed: \(B' = B - \sum_b b\), and \(R_{MNPQ}\) is the curvature of this bulk in the limit of no singularity. \(h_{\mu\nu}\) denotes the induced metric on \(b\), which we also suppose to have no extrinsic curvature, and

\[
R_{aa} = \sum_{a=1}^{2} R_{MN} n_a^M n_a^N, \quad \text{and} \quad R_{abab} = \sum_{a,b=1}^{2} R_{MNPQ} n_a^M n_b^N n_a^P n_b^Q \quad (6.25)
\]

where \(n_i^M\) denote two mutually-orthogonal unit normals to the appropriate brane world surface.
Because these expressions for the brane-localized contributions to the heat kernel are obtained as limits of a sequence of bulk contributions, they permit an easy generalization of the expressions given in appendix C to include brane-localized terms in the limit of small defect angles. We now summarize the results which are obtained in this way, which give the following brane-localized counterterms:

\[
\begin{align*}
\int_B d^6 x \sqrt{-g} \text{Tr} [m^4 a_1] & \approx \int_{B'} d^6 x \sqrt{-g} \text{Tr} [m^4 a_1] + \sum_b \frac{2\pi}{3} (1 - c_b) \int_B d^4 x \sqrt{-h} \text{Tr} [m^4] \\
\int_B d^6 x \sqrt{-g} \text{Tr} [m^2 a_2] & \approx \int_{B'} d^6 x \sqrt{-g} \text{Tr} [m^2 a_2] - \frac{2\pi}{3} (1 - c_b) \int_B d^4 x \sqrt{-h} \left\{ \text{Tr} [m^2 \hat{X}] + \frac{1}{30} (2R_{abab} - R_{aa} + 5R) \text{Tr} [m^2] \right\}.
\end{align*}
\]

Here \(\text{Tr} [m^2 \hat{X}]\) is defined as follows: if \(\text{Tr} [m^2 X] = aR + b\) then \(\text{Tr} [m^2 \hat{X}] = 2aR + b\).

These expressions assume that the defect angles are small and that the only singular bulk fields near the brane positions are the curvatures. This latter assumption is a natural consequence of our assumption of the vanishing of background quantities like \(X\) and \(F_{a MN} F^{a MN}\), since this guarantees that these quantities remain smooth there. They predict (for small defect angles) the new brane-localized ultraviolet divergences which arise once branes are inserted into the bulk space. Where the explicit results can be compared with this small-defect limit they agree.\(^3\)

\(^3\) It is claimed in ref. [114] that the small-defect result does not agree with explicit calculations for the divergences produced by integrating out spin-3/2 and spin-2 particles. However we regard these conclusions to be suspect inasmuch as the obstruction they find explicitly involves the contributions of pure-gauge modes — \(i.e.\)
Finally, since our present interest is in whether the Gilkey coefficients vanish when summed over the elements of a 6D supermultiplet, it is useful to notice here that — to linear order in \(1 - c_b\) — the vanishing of the UV sensitive terms on the boundary is an automatic consequence of the vanishing of all of the corresponding bulk terms from which they arise. It would clearly be very useful to have more general calculations of these quantities.

### 6.2 Heat Kernel Coefficients for Various Spin Fields

We now turn to the calculation of the first few heat kernel coefficients, \(\text{tr} (a_k)\) \((k = 0, 1, 2)\), in \(n\) spacetime dimensions for particles having spin zero, one-half, one, three-halves and two, as well as for the rank-two antisymmetric gauge potential which appears in supergravity models. Although our real interest is to applications with massive fields, we provide the results for massless fields which are required as intermediate steps in the calculation.

#### 6.2.1 Spin 0

The Lagrangian for a set of \(N_0\) real scalar fields, denoted collectively by \(\phi\), is given by

\[
e^{-1} \mathcal{L}_0 = -\frac{1}{2} \phi (\Box + m^2 + \xi R) \phi
\]

(6.27)

where in general both \(m^2\) and \(\xi\) are arbitrary constant \(N_0 \times N_0\) matrices, and as usual \(e = \sqrt{-g}\). We here assume for simplicity that \(m^2\) and \(\xi\) commute with one another, so a basis of fields exists for which both are diagonal. A case of particular interest

---

conformal Killing vectors and spinors — and this reference does not treat properly the contributions of the ghosts which would be expected to cancel such modes.
is the massless, minimally-coupled case, $\xi = m^2 = 0$, such as would be enforced by a Goldstone-boson symmetry $\phi \rightarrow \phi + \text{constant}$. Alternatively, the case $m^2 = 0$ and

$$\xi = -\frac{(n-2)}{4(n-1)} I$$

(6.28)

describes a conformally-invariant coupling for all $N_0$ scalars.

For scalars we have $Y_{MN} = -iF^a_{MN}t_a$, where $t_a$ is the gauge-group generator acting on the scalars of any background gauge group, under which the scalars are assumed to transform in a representation $\mathcal{R}_0$. If this representation contains $N_0$ real scalars, then we have $\text{tr} (I) = N_0$. For $X = \xi R$ we find

$$\text{tr}_0(a_0) = N_0$$

$$\text{tr}_0(a_1) = -\left( \text{tr} \left( \xi + \frac{N_0}{6} \right) R \right)$$

$$\text{tr}_0(a_2) = \frac{N_0}{180} \left[ R_{ABMN}R^{ABMN} - R_{MN}R^{MN} \right] + \frac{1}{2} \text{tr} \left[ \left( \xi + \frac{1}{6} \right)^2 \right] R^2$$

$$-\frac{1}{6} \text{tr} \left( \xi + \frac{1}{5} \right) \Box R - \frac{g_a^2}{12} C(\mathcal{R}_0) F^a_{MN} F_a^{MN}. \quad (6.29)$$

Here $\text{tr} \xi^k = N_0 \xi^k$ if all scalars share the same coupling to $R$ (i.e. if $\xi = \xi_0 I$), and $\text{tr} [t_a t_b] = g_a^2 C(\mathcal{R}_0) \delta_{ab}$, where $C(\mathcal{R}_0)$ is the Dynkin index for the scalar representation $\mathcal{R}_0$. (Our normalization is such that $C(F) = k/2$ or $C(A) = Nk$, respectively, for $k$ fields in the fundamental or adjoint representations of $SU(N)$.)

6.2.2 Spin 1/2

We take the Lagrangian for $N_{1/2}$ spin-half particles to be

$$e^{-1} \mathcal{L}_{1/2} = -\frac{1}{2} \bar{\psi}(i\not{\partial} + m)\psi.$$  (6.30)
where $D = \Gamma M D M$ with $\Gamma M$ denoting the $d' \times d'$ Dirac matrices in $n$ dimensions. In $n$ dimensions $d' = 2^{[n/2]}$ where $[n/2]$ is the largest integer which is less than or equal to $n/2$. Since different kinds of spinors are possible in different spacetime dimensions, it proves useful to define a new quantity, $d_s = 2d'/\zeta$, where the pre-factor of 2 comes because we count real fields, and $\zeta = 1, 2, \text{or } 4$ depending on whether the spinors in question are Dirac, Majorana or Weyl, or Majorana-Weyl.\footnote{For a discussion on the allowed spinors in spacetimes of arbitrary dimension and signature, see for example [115].}

In order to put the operator $\Delta$ into a form for which eq. (C.1) applies, we use the fact that\footnote{We assume there are no gauge or Lorentz anomalies.} 

$$\log \det (D + m) = \frac{1}{2} \log \det (m^2 - D^2),$$

which implies

$$i \Sigma_{1/2} = \frac{1}{4} \text{Tr} \log \left( m^2 - D^2 \right) = \frac{1}{4} \text{Tr} \log \left( -\Box + m^2 - \frac{1}{4} R + \frac{i}{2} \Gamma^{AB} F_{a}^{a} t_{a} \right), \quad (6.31)$$

where we use the spin-half result $J_{AB} = -\frac{i}{2} \Gamma_{AB}$, with $\Gamma_{AB} = \frac{1}{2} [\Gamma_{A}, \Gamma_{B}]$. Thus, we see that eq. (C.1) may be applied if we use $X = -\frac{1}{4} R I + \frac{i}{2} \Gamma^{AB} F_{a}^{a} t_{a}$, and divide the overall result by 2 (because of the extra factor of $1/2$ in eq. (6.31) relative to eq. (6.4)). Here $I$ denotes the $N_{1/2} \times N_{1/2}$ unit matrix, with $N_{1/2} = N_{1/2} d_s$.

Using eq. (6.3), we find in this way

$$\text{tr} \left( Y_{MN} Y^{MN} \right) = -d_s g_{a}^{2} C(\mathcal{R}_{1/2}) F_{a}^{a} F^{MN} - \frac{1}{8} N_{1/2} R_{ABMN} R^{ABMN}. \quad (6.32)$$

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This leads to the following values for $a_k$:

\[
\begin{align*}
\text{tr} \frac{1}{2}(a_0) &= \frac{\mathcal{N}_{1/2}}{2} \\
\text{tr} \frac{1}{2}(a_1) &= \frac{\mathcal{N}_{1/2}}{24} R \\
\text{tr} \frac{1}{2}(a_2) &= \frac{\mathcal{N}_{1/2}}{360} \left[ -\frac{7}{8} R_{ABMN} R^{ABMN} - R_{MN} R^{MN} + \frac{5}{8} R^2 + \frac{3}{2} \Box R \right] \\
&\quad + \frac{d s g_a^2}{12} C(R_{1/2}) F_{aMN} F_a^{MN}.
\end{align*}
\] (6.33)

### 6.2.3 Spin 1

For spins higher than 1/2 the massless and massive cases must be handled separately, due to the different number of spin states which are involved in these two cases. This is also related to the need for gauge symmetries for these higher spins [116, 117], and the possibility of mixing between higher-spin and lower-spin fields (i.e. the Anderson-Higgs-Kibble mechanism). In order to be explicit we first present the massless case.

#### Massless Spin 1

We start by dividing the total gauge field into a background component, $A^a_M$, and a fluctuation, $\mathcal{A}^a_M$, according to $a^a_M = A^a_M + \mathcal{A}^a_M$. In terms of these fields the gauge field strength for the full field, $a^a_M$, becomes

\[
f_{MN}^a = F_{MN}^a + D_M A_N^a - D_N A_M^a + \epsilon^{a}_{\;bc} A^b_M A^c_N,
\] (6.34)

where $D_M$ is the background covariant derivative built from the background gauge connection, $A_M^a$, and Christoffel symbol, and as before $F_{MN}^a$ is the background field-strength tensor. As usual, the fluctuation $\mathcal{A}^a_M$ is chosen to transform in the adjoint
representation under background gauge transformations — and so \((t_a)_{bc} = -ic_{abc}\) — as well as transforming as a vector under background coordinate transformations.

It is convenient to fix the spin-1 gauge invariance using a background-covariant gauge-averaging term,

\[
e^{-1} \mathcal{L}_V^{gf} = -\frac{1}{2\xi_1} (D^a A^a_m)^2,
\]

where \(D_m\) denotes the background-covariant derivative built from the background gauge field and Christoffel symbols. Then expanding the gauge-field Lagrangian,

\[
e^{-1} (\mathcal{L}_V + \mathcal{L}_V^{gf}) = -\left[ \frac{1}{4} f^a_{MN} f^M_{ab} + \frac{1}{2\xi_1} (D^a A^a_m)^2 \right],
\]

to second order in \(A^a_m\) and choosing the background-covariant Feynman gauge \((\xi_1 = 1)\), the part of the Lagrangian which is quadratic in the fluctuations, \(\mathcal{L}_A\), becomes

\[
e^{-1} \mathcal{L}_A = -\frac{1}{2} A^M_a \left[ -\Box g^a_{MN} \delta^{ab} - Y^a_{MN} + c^a_c F^c_{MN} \right] A^N_b,
\]

where as before \([D_M, D_N] A^a_N = Y^a_{MN} A^N_b\).

For a vector field the Lorentz generators are \((J^{AB})_{CD} = -i(\delta^{AC} \delta^{BD} - \delta^{AD} \delta^{BC})\), and so we see that \([D_M, D_N] A^a_b = R_{MN} A^a_N = iF^a_{MN} (t_a)^b_c A^N_c\). The one-loop contribution due to vector loops is then given by:

\[
i \Sigma_V = -\frac{1}{2} \log \det \left[ \Delta^M_a M^a_N \right] = -\frac{1}{2} \log \det \left[ -\Box \delta^{a}_{b} + R^a_{N} \delta^a_{b} + 2i F^a_{MN} (t_c)^a_{b} \right].
\]

We can now see that \(X^M_{N} a^a_{b} = -\eta R^a_{N} \delta^a_{b} + 2i (t_c)^a_{b} F^a_{MN}\), where \(\eta = \pm 1\) is a useful constant to include for later purposes. For the case considered here we see that
η = 1, whereas when we consider the ghosts associated with spin-2 particles we will find that η = −1.

For $N_1$ vector fields, we therefore find that $\text{tr}_V(X) = -\eta N_1 R$ and

$$\text{tr}_V(X^2) = N_1 R_{MN} R^{MN} + 4 g_a^2 C(A) F^a_{MN} F^{MN}_a, \quad (6.39)$$

where $C(A)$ is the Dynkin index for $N_1$ fields in the adjoint representation. Similarly,

$$\text{tr}_V(Y_{MN} Y^{MN}) = -N_1 R_{ABMN} R^{ABMN} - n g_a^2 C(A) F^a_{MN} F^{MN}_a \quad (6.40)$$

and $\text{tr}_V(I) = n N_1$. These imply the following results for vector fields in $n$ spacetime dimensions:

$$\text{tr}_V(a_0) = n N_1$$

$$\text{tr}_V(a_1) = \left( \eta - \frac{n}{6} \right) N_1 R$$

$$\text{tr}_V(a_2) = \frac{N_1}{360} \left[ (2n - 30) R_{ABMN} R^{ABMN} + (180 - 2n) R_{MN} R^{MN} + (5n - 60\eta) R^2 
+ (60\eta - 12n) \Box R \right] + \frac{g_a^2}{12} (24 - n) C(A) F^a_{MN} F^{MN}_a. \quad (6.41)$$

Since we work in a covariant gauge, to this result must be added the contributions of the ghosts. For the gauge chosen, the gauge fixing condition $f^a = D^a A^a_M$ varies under gauge transformations according to $\delta f^a = \Box \epsilon^a$. Consequently, the Lagrangian for the gauge ghosts is

$$e^{-1} L_{Vgh} = -\omega^*_a (\Box) \omega^a, \quad (6.42)$$

where the $\omega^a$ are complex fields obeying Fermi statistics. Since this has the same form as the spin zero Lagrangian discussed above (specialized to $\xi = 0$), for the ghosts we
may simply adopt the spin-0 results for the $a_k$, with $N_0 \to N_1$ and multiplied by an overall factor of $-2$.

Adding the results for vector fields ($\eta = +1$) and ghosts gives the contribution of physical spin-1 states. Thus, we obtain for massless spin-1 particles:

\[
\begin{align*}
\text{tr}_1(a_0) &= N_1(n - 2) \\
\text{tr}_1(a_1) &= \frac{N_1}{6} (8 - n) R \\
\text{tr}_1(a_2) &= \frac{N_1}{180} [(n - 17) R_{ABMN} R^{ABMN} + (92 - n) R_{MN} R^{MN}] + \frac{N_1}{72} (n - 14) R^2 \\
&\quad+ \frac{N_1}{30} (7 - n) \Box R + \frac{g_a^2}{12} (26 - n) C(A) F_{MN}^a F^{aMN}.
\end{align*}
\]

(6.43)

**Massive Spin 1**

If the gauge symmetry is spontaneously broken by the expectation of a scalar field, $\langle \phi^i \rangle = v^i$, then the previous discussion is complicated because the part of the Lagrangian quadratic in fluctuations acquires cross terms between the vector and scalar fields of the form $A^a_M t_a \partial^M \phi$. These terms reflect the physical process whereby the spin-1 particles acquire masses by absorbing the scalar fields through the Anderson-Higgs-Kibble mechanism.

In this case the same analysis as above can be performed provided we average over a more general gauge condition: $f^a = D^M A^a_M + c v \cdot t^a \phi$, with the constant $c$ chosen to remove the cross terms between $A^a_M$ and $\partial_M \phi$. This simply results in the addition of the same mass matrix $\mu^2$ to the differential operator $\Delta = -\Box + X$ for the vector fields and the ghost fields. This process also results in the would-be Goldstone
bosons \((i.e.\) the scalar fields which mixed with the gauge fields) acquiring the same mass matrix, \(\mu^2\) as also appears in the vector-field and ghost actions \([118, 119]\).

The upshot for massive spin-1 particles is therefore to add the result for \(N_1\) massless spin-1 particles to that of \(N_1\) massless scalar fields, with \(\xi = 0\). This leads to the following contributions if the mass \(\mu^2\), is not included in \(X\):

\[
\begin{align*}
\text{tr}_1 m(a_0) &= N_1(n - 1) \\
\text{tr}_1 m(a_1) &= \frac{N_1}{6}(7 - n)R \\
\text{tr}_1 m(a_2) &= \frac{N_1}{180} \left[(n - 16)R_{ABMN}R^{ABMN} + (91 - n)R_{MN}R^{MN}\right] + \frac{N_1}{12}(n - 13)R^2 \\
&+ \frac{N_1}{30}(6 - n)\Box R + \frac{g^2}{12}(25 - n)C(A)F^a_{MN}F^{MN}_a. \\
\end{align*}
\]

\eqref{6.44}

### 6.2.4 Antisymmetric Tensors

We next consider in detail the antisymmetric rank-2 gauge potential, \(B_{MN}\), which appears in supergravity models. As before we first treat the massless case, and then move on to massive particles. We also quote the results for massless antisymmetric tensors of arbitrary rank, as taken from ref. \([120, 121]\). Since the calculations in the subsequent sections parallel those for the spin-1 field, we omit the detailed calculations from the main text and instead include these in appendix C.

#### Massless Antisymmetric Tensors

The appropriate Lagrangian for this field is

\[
e^{-1} \mathcal{L}_B = -\frac{1}{12} H_{MNP}H^{MNP},
\]

\eqref{6.45}
where $H_{MNP} = D_{[M}B_{NP]} = 2(D_MB_{NP} + D_NB_{PM} + D_PB_{MN})$, and to this we add the gauge-fixing term $e^{-1} \mathcal{L}_B^{gf} = -\frac{1}{2\xi_B}(D_MB^{MN})^2$. Choosing the gauge parameter to be $\xi_B = 1/4$, we obtain the Lagrangian

$$e^{-1}(\mathcal{L}_B + \mathcal{L}_B^{gf}) = -B_{MN}\left(-\Box\delta^{MN}_{PQ} + 2R^M_{\,P}N - 2R^M_{\,P}\delta^N_{Q}\right)B^{PQ}. \quad (6.46)$$

Here, $\delta^{MN}_{PQ} = \frac{1}{2}(\delta^M_P\delta^N_Q - \delta^M_Q\delta^N_P)$ is the appropriate identity matrix for a rank-2 anti-symmetric tensor.

As in the spin-1 case, we must also consider the ghost Lagrangian which results from our chosen gauge-fixing term. The antisymmetric tensor gauge transformations are $\delta B_{MN} = D_M\Lambda_N - D_N\Lambda_M$, where $\Lambda_M$ is itself only defined up to a gauge transformation: $\Lambda_M \rightarrow \Lambda_M + D_M\Phi$. We therefore average over the secondary gauge-fixing condition $f = D_M\Phi^M$, where $D_M$ is the appropriate background-covariant derivative. Introducing ghosts and ghost-for-ghosts for these symmetries, we acquire the ghost counting of ref. [122], which states that each initial tensor gauge potential gives rise to a complex, fermionic vector ghost, $\omega^M$, and three real, scalar, bosonic ghosts-for-ghosts, $\phi^i$. Their Lagrangians are given by

$$e^{-1}\mathcal{L}_{BVgh} = -\omega^*_M(-\Box\delta^M_N - R^M_{\,N})\omega^N,$$

$$e^{-1}\mathcal{L}_{BSgh} = -\frac{1}{2}\phi^i(-\Box)\phi^i. \quad (6.47)$$

\footnote{The reason we do not obtain four scalar ghosts, as a naive ghost counting would imply, has to do with the fact the gauge-fixing function $G_N = D_MB_{MN}$ satisfies the constraint $D^NG_N = 0$. A more detailed discussion of this point can be found in [122].}
The contributions of the vector ghosts to $a_k$ is therefore obtained by replacing $N_1 \rightarrow -2N_a$ in the result given above for vector fields (with $\eta = +1$). Similarly, the scalar ghosts are obtained from the spin-0 result quoted above, with the replacements $N_0 \rightarrow 3N_a$ and $\xi \rightarrow 0$.

Summing the contribution of the rank-2 tensor and its ghosts leads to the following expression for the physical massless particles associated with these antisymmetric tensor fields:

$$
\text{tr}_a(a_0) = \frac{N_a}{2}(n - 2)(n - 3)
$$
$$
\text{tr}_a(a_1) = -\frac{N_a}{12}(n^2 - 17n + 54)R
$$
$$
\text{tr}_a(a_2) = \frac{N_a}{360} \left[ (n^2 - 35n + 306)R_{ABMN}R^{ABMN} - (n^2 - 185n + 1446)R_{MN}R^{MN} \right]
+ \frac{N_a}{144}(n^2 - 29n + 174)R^2 - \frac{N_a}{60}(n^2 - 15n + 46)\Box R.
$$

(6.48)

**Massive Particles**

The particles associated with antisymmetric tensor fields can also acquire mass through an Anderson-Higgs-Kibble mechanism, in which the antisymmetric tensor particle ‘eats’ an ordinary gauge field, $V_M$ [123]. As before, a modification of the gauge choice is required in this case in order not to have mixing terms of the form $B^{MN}\partial_M V_N$. As we now show, the contribution of each massive tensor particle is given by adding the above result for a massless particle to the result for an $\eta = +1$ massless abelian — so with $C(A) = 0$ — gauge field (including its ghosts).

To demonstrate this explicitly, we start with the Lagrangian

$$
e^{-1} L_{mB} = -\frac{1}{12} H_{MNP}H^{MNP} - \frac{1}{4}(V_{MN} - 2m B_{MN})^2,
$$

(6.49)
where \( V_{MN} = D_M V_N - D_N V_M \) is the field strength of the abelian gauge field \( V_M \), and \( m \) is a constant with dimensions of mass. This Lagrangian is invariant under

\[
\delta B_{MN} = D_M \Lambda_N - D_N \Lambda_M \\
\delta V_M = 2m \Lambda_M + 2 \partial_M \omega,
\]

where \( \Lambda_M \) and \( \omega \) are arbitrary gauge parameters. As in the massless case, this set of gauge transformations is itself invariant under a gauge transformation,

\[
\delta \Lambda_M = \partial_M \epsilon \\
\delta \omega = -m \epsilon,
\]

for an arbitrary function \( \epsilon \). As before, therefore, we find that the ghosts themselves have ghosts. Note that in the limit \( m \to 0 \) the Lagrangian decouples into the Lagrangian for a massless antisymmetric tensor and a massless vector.

To fix the two gauge freedoms in eq. (6.50), and to remove unwanted mixing terms, we add to the Lagrangian the gauge-fixing term

\[
e^{-1} L_{\text{gf}}^{mB} = -2 \left( D^M B_{MN} - \frac{m}{2} V_N \right)^2 - \frac{1}{2} (D^M V_M)^2.
\]

After adding this term to eq. (6.49) we find

\[
e^{-1} (L + L_{\text{gf}}^{mB}) = -B_{MN} (\Delta^{MN}_{\text{PQ}} + m^2 \delta^{MN}_{\text{PQ}}) B^{PQ} - \frac{1}{2} V_m (\Delta^M_N + m^2 \delta^M_N) V^N,
\]

where \( \Delta \) is the differential operator appropriate for the field it operates on; specifically, \( \Delta^M_N = -\Box \delta^M_N - R^M_N \), and \( \Delta^{MN}_{\text{PQ}} \) is given by eq. (C.3).
The Lagrangian for the ghosts is obtained by varying the gauge-fixing conditions appearing in eq. (6.52), and we thus find

\[ \mathcal{L}_{mBgh} = -\xi^* N (-\Box_N^M + D_M D^N + m^2 \delta^N_M) \xi^M - \omega^* (-\Box) \omega \\
- m \xi^* M D^M \omega + m \omega^* D_M \xi^M. \quad (6.54) \]

Here, \( \xi_M \) and \( \omega \) are the ghost fields associated with \( \Lambda_M \) and \( \omega \), respectively. To fix the gauge freedom implied by eq. (6.51), we add to the ghost Lagrangian the term

\[ \mathcal{L}_{mBgh}^{gf} = -(D_M \xi^M + m \omega)^*(D_N \xi^N + m \omega) \]

and so we find

\[ \mathcal{L}_{mBgh} + \mathcal{L}_{mBgh}^{gf} = -\xi^* N \left[ (-\Box + m^2) \delta^N_M + R^N_M \right] \xi^M - \omega^* (-\Box + m^2) \omega. \quad (6.56) \]

Notice that the complex scalar ghost, \( \omega \), combines with the vector, \( V_N \), to form the field content of a physical massless spin-1 particle.

The ghosts-for-ghosts Lagrangian is similarly obtained, and as in the massless case we find three bosonic scalar ghosts-for-ghosts, with Lagrangian

\[ \mathcal{L}_{mBSgh} = -\frac{1}{2} \phi_i (-\Box + m^2) \phi^i. \quad (6.57) \]

Except for the presence of mass terms, the Lagrangian for a massive antisymmetric tensor is therefore the sum of a massless spin-1 Lagrangian and a massless antisymmetric tensor Lagrangian (including their ghosts). Thus, in calculating the \( a_k \) for a massive antisymmetric tensor, we simply need to add to the massless result given in
the previous section the result for a massless spin-1 field. It is important to empha-
size that such a sum — where we factor all mass terms out of $X$, as described in the
§6.1 — makes sense only because in the gauge we have chosen all particles share the
same mass.

The result of this sum, for massive rank-2 tensor fields in $n$ spacetime dimensions,
is

\[
\begin{align*}
\text{tr}_{am}(a_0) &= \frac{N_a}{2} (n - 2)(n - 1) \\
\text{tr}_{am}(a_1) &= -\frac{N_a}{12} (n^2 - 15n + 38) R \\
\text{tr}_{am}(a_2) &= \frac{N_a}{360} \left[ (n^2 - 33n + 272) R_{ABMN} R^{ABMN} - (n^2 - 183n + 1262) R_{MN} R^{MN} \right] \\
&\quad + \frac{N_a}{144} (n^2 - 27n + 146) R^2 - \frac{N_a}{60} (n^2 - 13n + 32) \Box R .
\end{align*}
\] (6.58)

**Higher-Rank Antisymmetric Tensors**

The result for a higher-rank massless skew-tensor gauge potential in $n$ dimensions
has been worked out in a similar fashion to the above [120, 121]. This leads to the
following results for the first few Gilkey coefficients for a massless 3-form gauge field
(for $n > 4$ dimensions), specialized to Ricci-flat background geometries ($R_{MN} = 0$):

\[
\begin{align*}
\text{tr}_{3a}(a_0) &= \frac{N_{3a}}{3!} (n - 2)(n - 3)(n - 4) \\
\text{tr}_{3a}(a_1) &= 0 \\
\text{tr}_{3a}(a_2) &= \frac{N_{3a}}{1080} (n^3 - 54n^2 + 971n - 4164) R_{ABMN} R^{ABMN} .
\end{align*}
\] (6.59)
The analogous results for a massless 4-form gauge field (in $n > 5$ Ricci-flat dimensions) are given by:

$$
\text{tr}_{4a}(a_0) = \frac{N_{4a}}{4!} (n - 2)(n - 3)(n - 4)(n - 5)
$$

$$
\text{tr}_{4a}(a_1) = 0
$$

$$
\text{tr}_{4a}(a_2) = \frac{N_{4a}}{4320} (n^4 - 74n^3 + 2051n^2 - 18634n + 52680) R_{ABMN} R^{ABMN}.
$$

(6.60)

These results for massive 1- and 2-forms suggest a short-cut for extending our results to the case of a massive $p$-form for arbitrary $p$, since they show that the Gilkey coefficients for a massive $p$-form are obtained by summing the contributions of a massless $(p - 1)$-form to that of a massless $p$-form. It can also be readily seen that the Gilkey coefficients for a massive spin-1 field are obtained by the replacement $n \to (n + 1)$ in the massless formulae, and similarly for the antisymmetric 2-form. One way to see why this should give the correct result is to reason as follows. It is clear that (for a Minkowski-space background) a massless $p$-form in $(n+1)$ dimensions and a massive $p$-form in $n$ dimensions share the same little group, $SO(n - 1)$, and transform in the same representation of this group. This connection can also be made more explicit by dimensionally reducing an $(n+1)$-dimensional massless $p$-form on $S^1$ to obtain a Kaluza-Klein tower of massive $p$-forms in the lower-dimensional theory. Each massive field is thereby seen to contain the spin content of an $n$-dimensional massless $p$- and $(p - 1)$-form. A final check on this reasoning can be had using the results of ref. [120, 121], which show that the first few Gilkey coefficients for a massless $(n + 1)$-dimensional $p$-form — and hence a massive $n$-dimensional $p$-form
— are the same as the sum of the coefficients for a massless \( p \)- and \((p - 1)\)-form in \( n \) dimensions.

### 6.2.5 Spin 3/2

Before proceeding with spin-3/2 and spin-2 particles, we first pause to establish a few of our supergravity conventions. Our starting point is the coupled Einstein/Rarita-Schwinger system. We take the spin-2 field to be described by the standard Einstein-Hilbert action, which in our conventions is

\[
e^{-1} \mathcal{L}_{EH} = -\frac{1}{2\kappa^2} R,
\]

with \( \kappa^2 = 8\pi G_N \). For the moment, we do not include a cosmological term; the generalization of the massless and massive spin-3/2 particle to the case of a nonzero cosmological constant is given in appendix D.

The spin-3/2 particle is described by a vector-spinor field, \( \psi_M \), with a kinetic term given by the Lagrangian

\[
e^{-1} \mathcal{L}_{VS} = -\frac{1}{2} \overline{\psi}_M \Gamma^{MNP} D_N \psi_P.
\]

As before, we use indices \( A, B, \ldots \) for the tangent frame, \( M, N, \ldots \) for world indices and lower-case indices to label gauge-group generators. Conversion between tangent and world indices is accomplished using the vielbein, \( e^A_M \). Here, \( \Gamma^{ABC} = \frac{1}{6} [\Gamma^A, \Gamma^B, \Gamma^C] \) and \( \Gamma^{AB} = \frac{1}{2} [\Gamma^A, \Gamma^B] \) are normalized completely antisymmetric combinations of gamma matrices.

The covariant derivative appearing in eq. (6.62) can involve background gauge fields in addition to the Christoffel connection, but only if the corresponding gauge
symmetry does not commute with supersymmetry. Such transformations are parti-
cularly rich when there is more than one supersymmetry in the problem. Gravitini
cannot carry charges for internal symmetries which commute with supersymmetry,
because for these the gravitino must share the charge of the graviton, which is neutral
under all gauge transformations.

When there are no gauge fields in \( D_M \psi_N \), it is straightforward to verify that the
combination \( \mathcal{L}_{VS} + \mathcal{L}_{EH} \) is invariant under the linearized supersymmetry transfor-
mations
\[
\delta e^A_M = -\frac{\kappa}{4} \psi_M \Gamma^A \epsilon + \text{c.c.}, \quad \delta \psi_M = \frac{1}{\kappa} D_M \epsilon .
\]
(6.63)
When background gauge fields \emph{are} present in \( D_M \psi_N \), the combination \( \mathcal{L}_{VS} + \mathcal{L}_{EH} \)
varies into terms involving these gauge fields. These terms then cancel against vari-
ations of the gauge-field kinetic terms and with gauge-field-dependent terms in the
gravitino transformation law. This shows that gauge fields for symmetries which do
not commute with supersymmetry are special in that they are intimately related to
the gravitini by supersymmetry.

**Massless Spin 3/2**

In order to put the spin-3/2 Lagrangian into a form for which the general expressions
for the Gilkey coefficients apply, it is convenient to use the following gauge-averaging
term,
\[
e^{-1} L_{VS}^{gf} = -\frac{1}{2 \xi_{3/2}} (\Gamma \cdot \psi) \bar{\mathcal{D}}(\Gamma \cdot \psi) .
\]
(6.64)
With this term, and after making the field redefinition \( \psi_M \to \psi_M + A \Gamma_M \Gamma \cdot \psi \), we
find that the Lagrangian simplifies in the desired way when we make the following
choices for $A$ and $\xi_{3/2}$:

$$A = \frac{1}{2 - n} \quad \text{and} \quad \frac{1}{\xi_{3/2}} = \frac{2 - n}{4}.$$  \hspace{1cm} (6.65)

These choices allow the vector-spinor Lagrangian to be written as

$$e^{-1} (L_{VS} + L^{gf}_{VS}) = -\frac{1}{2} \bar{\psi} \gamma_{\mu} \gamma_{\nu} \psi,$$  \hspace{1cm} (6.66)

and so give the one-loop contribution

$$i \Sigma = \frac{1}{2} \log \text{det} \left[ (\nabla \right) A_{\mu} \left. \right] = \frac{1}{4} \log \text{det} \left[ (-\nabla^2)^A \right].$$  \hspace{1cm} (6.67)

We next consider the contribution from the ghost fields. From the supersymmetry transformation rules, we see that $\delta (\Gamma \cdot \psi) = \frac{1}{\kappa} \nabla \epsilon$, and so there are two bosonic, Faddeev-Popov spinor ghosts with the Lagrangian

$$e^{-1} L_{LVFPgh} = -\bar{\omega} \gamma_{\mu} \omega,$$  \hspace{1cm} (6.68)

where $i = 1, 2$ labels the two ghosts. Since this has the same form as the spin-1/2 Lagrangian used earlier, eq. (6.30), the Faddeev-Popov ghost result for $\text{tr} [a_k]$ is obtained by multiplying the massless spin-1/2 result by $-2$.

In addition to the Faddeev-Popov ghosts, there is also a bosonic, Nielsen-Kallosh ghost [124, 125] coming from the use of the operator $\nabla$ in the gauge-fixing Lagrangian, eq. (6.64). The Nielsen-Kallosh ghost Lagrangian is given by

$$e^{-1} L_{LVNKgh} = -\bar{\eta} \gamma_{\mu} \eta.$$  \hspace{1cm} (6.69)
This ghost therefore has a contribution to $\text{tr} [a_k]$ given by $-1$ times the massless spin-1/2 result.

Adding the results for the Faddeev-Popov and Nielsen-Kallosh ghosts to that of the vector-spinor, we obtain the following results for the contribution to $\text{tr} [a_k]$ by physical massless spin-3/2 states:

$$
\begin{align*}
\text{tr}_{3/2}(a_0) &= \frac{N_{3/2}}{2} (n - 3) \\
\text{tr}_{3/2}(a_1) &= \frac{N_{3/2}}{24} (n - 3) R \\
\text{tr}_{3/2}(a_2) &= \frac{N_{3/2}}{360} \left[ \left( 30 - \frac{7}{8}(n - 3) \right) R_{ABMN} R^{ABMN} - (n - 3) R_{MN} R^{MN} \\
&\quad\quad + \frac{5}{8} (n - 3) R^2 + \frac{3}{2} (n - 3) \Box R \right] + \frac{dg_a^2}{12} (n - 3) C(R_{3/2}) F_{MN}^a F^{MN}_a.
\end{align*}
$$

(6.70)

**Massive Spin 3/2**

A spin-3/2 state acquires a mass through the existence of an off-diagonal coupling of the form $\bar{\chi} \Gamma \cdot \psi$ with a spin-1/2 Goldstone fermion state, $\chi$. Choosing a gauge for which this term vanishes causes the super-Higgs mechanism to occur, through which the spin-3/2 particle ‘eats’ the fermion $\chi$. Although $\chi$ vanishes in a unitary gauge, it remains in the theory in a covariant gauge much as does the would-be Goldstone boson for the massive spin-1 case.
To show explicitly how this process occurs, we assume that the part of the fermionic Lagrangian which is quadratic in the fluctuations has the general form\(^7\)

\[
e^{-1} \mathcal{L}_{\text{mVS}} = -\overline{\psi}_M \Gamma^{MN}D_N \psi_P - \overline{\chi} \not{D} \chi - \left( \overline{\psi} \cdot \Gamma (a \not{D} + b) \chi \right) + \text{c.c.} \\
- \left( c \overline{\psi}_M D^M \chi + \text{c.c.} \right) - m_{1/2} \overline{\chi} \chi - \mu_{3/2} \overline{\psi}_M \psi^M \\
+ m_{3/2} \overline{\psi}_M \Gamma^{MN} \psi_N ,
\]

(6.71)

where the parameters \(a, b, c, m_{1/2}, m_{3/2}, \) and \(\mu_{3/2}\) are constrained by demanding that the action be invariant under linearized supersymmetry transformations. For simplicity we assume these parameters to be real, although in general some or all of these parameters may be complex, depending on whether the fermions are Majorana or Weyl in the supergravity of interest. Requiring invariance under the supersymmetry transformations

\[
\delta \psi_M = \frac{1}{\kappa} D_M \epsilon + \mu \Gamma_M \epsilon \quad \text{and} \quad \delta \chi = f \epsilon ,
\]

(6.72)

then imposes the following constraints on the various parameters:

\[
a = c = \mu_{3/2} = 0 \quad b = \kappa f \quad f^2 = (n - 1)(n - 2)\mu^2 \\
m_{1/2} = n\kappa \mu \quad m_{3/2} = (n - 2)\kappa \mu .
\]

(6.73)

This leaves one free parameter which we can take to be \(\mu, f, \) or \(b.\)

\(^7\) We follow here the approach of refs. [126, 127, 128] to identify the form of these couplings to quadratic order in a model-independent way.
With these choices, the variation of the gravitino/goldstino Lagrangian is

\[ e^{-1} \delta \mathcal{L}_{mVS} = \frac{1}{2\kappa} G^{MN} \bar{\psi}_M \Gamma_N \epsilon + \text{c.c.}, \]  

(6.74)

where \( G^{MN} = R^{MN} - \frac{1}{2} R g^{MN} \) is the Einstein tensor. This term is cancelled in the usual way by the variation of the Einstein-Hilbert action under the graviton transformation

\[ \delta e^A_M = -\frac{\kappa}{4} \bar{\psi}_M \Gamma^A \epsilon + \text{c.c.} \]  

(6.75)

To this Lagrangian we add the gauge-fixing term

\[ \frac{1}{e} \mathcal{L}_{mVS}^{gf} = -\bar{F}(\bar{D} + \gamma)F, \]  

(6.76)

where

\[ F = \alpha \Gamma \cdot \psi + \beta \chi. \]  

(6.77)

The constants \( \alpha, \beta, \) and \( \gamma \) are chosen to ensure that the gauge-fixed Lagrangian has the form

\[ e^{-1} (\mathcal{L}_{mVS} + \mathcal{L}_{mVS}^{gf}) = -\bar{\psi}_M' (\bar{D} + m'_{3/2}) \psi'_M - \bar{\chi}' (\bar{D} + m'_{1/2}) \chi' \]  

(6.78)

where \( \psi'_M \) and \( \chi' \) are given by

\[ \chi' = A \chi + B \Gamma \cdot \psi \quad \text{and} \quad \psi'_M = \psi_M + C \Gamma_M \Gamma \cdot \psi + D \Gamma_M \chi, \]  

(6.79)

where we again take the parameters \( A, B, C, \) and \( D \) to be real for simplicity. Note that the transformation of \( \psi_M \) is non-singular provided \( C \neq -1/n \). Using eq. (6.79) to evaluate the right-hand side of eq. (6.78) while using eqs. (6.71), (6.73), and (6.76)
to evaluate the left-hand side, leads to the conditions

\[ A = \left(\frac{n-1}{n-2}\right)^{1/2}, \quad B = C = -\frac{1}{2}, \quad D = 0, \quad m'_{3/2} = m'_{1/2} = (n-2)\kappa\mu, \]

\[ \alpha = -\frac{1}{2}\sqrt{n-1}, \quad \beta = \frac{1}{\sqrt{n-2}}, \quad \gamma = -(n-2)\kappa\mu. \]  

(6.80)

The ghost action consists of a Nielsen-Kallosh ghost, with Lagrangian

\[ e^{-1} \mathcal{L}_{mVSNK} = -\overline{\psi}(\overline{D} + \gamma)\psi, \quad (6.81) \]

as well as two Faddeev-Popov ghosts, with Lagrangian

\[ e^{-1} \mathcal{L}_{mVSFP} = -\overline{\xi}_i \left[ \overline{D} + (n-2)\kappa\mu \right] \xi^i, \quad (6.82) \]

where \( i = 1, 2 \) labels the two ghosts. Dropping the primes, and defining \( m = (n-2)\kappa\mu \), the complete Lagrangian, eqs. (6.78), (6.81) and (6.82), becomes

\[ e^{-1} \mathcal{L}_{m3/2} = -\overline{\psi}_m(\overline{D} + m)\psi^m - \overline{\chi}(\overline{D} + m)\chi - \overline{\omega}(\overline{D} - m)\omega - \overline{\xi}_i(\overline{D} + m)\xi^i. \]  

(6.83)

Since the heat-kernel coefficients are even under \( m \to -m \), we see from this that \( a_k \) for a massive gravitino are given by the sum of the corresponding coefficients for a massless gravitino (including ghosts) plus those of a massless fermion. Summing the massive spin-1/2 result, eq. (6.33), with the spin-3/2 result, eq. (6.70), we obtain
the following Gilkey coefficients for a massive spin-3/2 particle

\[
\begin{align*}
\text{tr}_{m3/2}(a_0) &= \frac{N_{3/2}}{2}(n - 2) \\
\text{tr}_{m3/2}(a_1) &= \frac{N_{3/2}}{24}(n - 2)R \\
\text{tr}_{m3/2}(a_2) &= \frac{N_{3/2}}{360} \left[ \left( 30 - \frac{7}{8}(n - 2) \right) R_{ABMN} R^{ABMN} - (n - 2)R_{MN} R^{MN} \\
&\quad + \frac{5}{8}(n - 2)R^2 + \frac{3}{2}(n - 2)\Box R \right] + \frac{g_a^2}{12}(n - 2)d_s C(R_{3/2}) F^a_{MN} F^{MN}.
\end{align*}
\]

(6.84)

### 6.2.6 Spin 2

Finally, we turn to spin-2 particles. In order to maximize the utility of this section, we do so for the case where the Lagrangian includes a cosmological constant, as is typically true for non-supersymmetric theories (and for supersymmetric theories in four dimensions), and so start with the following action

\[
\begin{align*}
e^{-1} \mathcal{L}_{EH} &= -\frac{1}{2\kappa^2}(R + 2\Lambda).
\end{align*}
\]

(6.85)

For situations where \( \Lambda \) represents the value of a scalar potential, \( V \), evaluated at the classical background, we see from the above that \( \Lambda = +\kappa^2 V \).

Although it is usually true that only a single spin-2 particle is massless in any given model, we include a parameter \( N_2 \) which counts the massive spin-2 states. We do so because there is typically more than one massive spin-2 state in the models of interest, typically arising as part of a Kaluza-Klein tower or as excited string modes.
Massless Spin 2

The Lagrangian for a massless rank-two symmetric field is the Einstein-Hilbert action, eq. (6.85). As usual we write the metric as $g_{MN} + 2\kappa h_{MN}$, where $g_{MN}$ is the background metric and $h_{MN}$ are the fluctuations. Expanding to quadratic order in these fluctuations, and adding the gauge-fixing term

$$\frac{1}{\epsilon} L_{EH}^{gf} = - \left( D^M h_{MN} - \frac{1}{2} D_N h^M_M \right)^2,$$

we obtain the standard result [129]

$$\frac{1}{\epsilon} (L_{EH} + L_{EH}^{gf}) = \frac{1}{2} h^{MN} \left[ \Box h_{MN} + (R + 2\Lambda)h_{MN} - \left( h_{MA} R^A_N + h_{NA} R^A_M \right) 
- 2R_{MANB} h^{AB} \right] + h^{MN} R_{MN} h - \frac{1}{4} h \left[ \Box h + (R + 2\Lambda)h \right],$$

(6.87)

where $h = g^{MN} h_{MN}$.

It is useful to try to decouple the scalar, $h$, from the traceless symmetric tensor $\phi_{MN} = h_{MN} - \frac{1}{n} h g_{MN}$, in this expression. In terms of these variables the Lagrangian is

$$\frac{1}{\epsilon} (L_{EH} + L_{EH}^{gf}) = \frac{1}{2} \phi^{MN} \left[ \Box \phi_{MN} + (R + 2\Lambda)\phi_{MN} - \left( \phi_{MA} R^A_N + \phi_{NA} R^A_M \right) 
- 2R_{MANB} \phi^{AB} \right] + \left( \frac{n-4}{n} \right) \phi^{MN} R_{MN} h

- \left( \frac{n-2}{4n} \right) \left[ h \Box h + \left( \frac{n-4}{n} \right) R h^2 + 2\Lambda h^2 \right],$$

(6.88)

which shows that these fields decouple in four dimensions, and in arbitrary dimensions if we make the assumption that the background metric is an Einstein space.
\[ R_{MN} = \frac{1}{n} R g_{MN} \]. It is straightforward to consider the case when this mixing term does not vanish, and we do so at the end of the calculation. In the interim, however, we ignore this term.

Canonically normalizing the scalar mode by taking \( \phi = [(n - 2)/(2n)]^{1/2} h \), we arrive at the desired expression:

\[
\frac{1}{e}(\mathcal{L}_{EH} + \mathcal{L}_{EH}^f) = -\frac{1}{2} \phi^{MN} \left[ -\Box \delta^{AB}_{MN} + 2 R^A_{MN} B + (R^A_{MN} \delta^B_N + R^A_{MN} \delta^B_M) \\ -(R + 2 \Lambda) \delta^{AB}_{MN} \right] \phi_{AB} - \frac{1}{2} \phi \left[ \Box + \left( \frac{n - 4}{n} \right) R + 2 \Lambda \right] \phi + \cdots ,
\]

(6.89)

where \( \delta^{MN}_{AB} = \frac{1}{2} (\delta^M_A \delta^N_B + \delta^M_B \delta^N_A) - \frac{1}{n} g^{MN} g_{AB} \) is the unit matrix appropriate for a traceless symmetric tensor, and the ellipsis denotes the mixing term. Notice the presence of the well-known ‘wrong’ sign for the kinetic term of the scalar mode \( \phi \). We may now separately compute the contributions of \( \phi \) and \( \phi_{MN} \) to the heat-kernel coefficients, \( a_k \) (see appendix C for details of the \( \phi_{MN} \) calculation).

The scalar part of the spin-2 Lagrangian is given by

\[
\frac{1}{e} \mathcal{L}_{EHs} = \frac{1}{2} \phi \left[ -\Box - \left( \frac{n - 4}{n} \right) R - 2 \Lambda \right] \phi ,
\]

(6.90)

which, apart from an overall sign, has the same form as eq. (6.27) if we make the substitution \( \xi R \rightarrow -\left( \frac{n - 4}{n} \right) R - 2 \Lambda \). Since the overall sign of \( \Delta \) contributes a background-field-independent phase to the action which is cancelled by a similar contribution from the ghost action (see below), we may ignore it for the present purposes. With these comments in mind, we may then use the previous results for spin-0 fields to compute the contribution of \( \phi \) to the Gilkey coefficients, \( a_k \).
We now consider the ghosts for the graviton field. Since the gauge-fixing term is \( f_N = D^M h_{MN} - \frac{1}{2} D_N h \), and the gauge transformations are \( \delta h_{MN} = D_M \xi_N + D_N \xi_M \), we find the transformation property

\[
\delta f_N = \Box \xi_N - R^M_N \xi_M ,
\]

leading to a complex, fermionic, vector ghost \( \omega_M \) with Lagrangian

\[
e^{-1} L = -\omega^*_M ( - \Box^M \delta^M_N + R^M_N ) \omega^N .
\] (6.92)

The contribution of the vector ghost to the Gilkey coefficients is therefore obtained by multiplying the results found earlier for the real spin-1 field by an overall factor of \(-2\) (and using the choice \( \eta = -1 \) in eq. (6.41)).

Finally, we return to the issue of the mixing-term between the symmetric-traceless and the trace components of the metric. Using the symmetries of \( \phi_{MN} \), this mixing term can be written as \( 2 Z_{MN} \phi^{MN} \phi \), where

\[
Z_{MN} = \frac{n - 4}{2n} \sqrt{ \frac{2n}{n - 2} } \left( R_{MN} - \frac{1}{n} g_{MN} R \right) .
\] (6.93)

The effect of this term can be included by adding an off-diagonal term to the matrix \( X \) of the field \( \Phi = (\phi_{MN}, \phi)^T \). Omitting indices, we have

\[
\begin{pmatrix}
X_{\text{symtr}} & Z \\
Z & X_{\text{tr}}
\end{pmatrix}
\]

where \( X_{\text{symtr}} \) corresponds to the mass matrix for \( \phi_{MN} \), and \( X_{\text{tr}} \) the mass matrix for \( \phi \). Note that since the Gilkey coefficients only contain traces of polynomials of \( X \), the effect of this mixing first occurs in the term \( \text{Tr} (X^2) \). It is easy to calculate now
that
\[ \text{Tr} (X^2) = \text{Tr} (X^2)|_{z=0} + \frac{n-4}{n^2(n-2)} \left( nR_{MN}R^{MN} - R^2 \right). \] (6.94)

Collecting all these results, we thus obtain the first few Gilkey coefficients for the massless graviton in \( n \) dimensions:

\[
\begin{align*}
\text{tr}_2(a_0) &= \frac{N_2}{2} n(n-3) \\
\text{tr}_2(a_1) &= N_2 \left[ \frac{1}{12} (5n^2 - 3n + 24)R + n(n+1)\Lambda \right] \\
\text{tr}_2(a_2) &= N_2 \left[ \frac{1}{360} (n^2 - 33n + 540)R_{ABMN}R^{ABMN} \\
&\quad - \frac{1}{360n(n-2)} (n^4 - 185n^3 + 1086n^2 + 4140n - 10800)R_{MN}R^{MN} \\
&\quad + \frac{1}{144n^2(n-2)} (25n^5 - 149n^4 - 48n^2 + 2232n - 4320)R^2 \\
&\quad + \frac{1}{30} (2n^2 - n + 10)\Box R + \frac{n}{6} (5n - 7)\Lambda R + n(n+1)\Lambda^2 \right].
\end{align*}
\] (6.95)

**Massive Spin 2**

We next derive the Lagrangian for the massive graviton. In order to do so we require an expression for the quadratic part of the massive spin-2 Lagrangian, such as might be obtained from a Kaluza-Klein reduction or as a massive string mode. To keep the analysis as background-independent as possible, we work with the most general such action for which the spin-2 state acquires its mass by mixing with the appropriate Goldstone field, as in the Anderson-Higgs-Kibble mechanism. We believe that by making this requirement we capture quite generally the contributions of the massive spin-2 states which arise in dimensional reduction and as heavy string modes [130, 131].
We start, therefore, with the Lagrangian

\[ e^{-1} \mathcal{L}_{mEH} = e^{-1} \mathcal{L}_{EH} - \frac{1}{4} F_{MN} F^{MN} - a h^{MN} D_M V_N - b V^M D_M h \]
\[ -c R^{MN} V_M V_N - \frac{1}{2} m_1^2 V_M V^M - \frac{1}{2} m_2^2 h_{MN} h^{MN} - \frac{1}{2} \mu_2^2 h^2, \tag{6.96} \]

where the coefficients \( a, b, c, m_1, m_2 \) and \( \mu_2 \) are to be determined by demanding the presence of a non-linearly realized gauge symmetry (which would correspond to the diffeomorphisms which do not preserve the background geometry within the Kaluza-Klein context). \( F_{MN} \) is the field strength \( D_M V_N - D_N V_M \), where we take \( V_M \) to have the spin content of a massive spin-1 particle. From the previous sections we see that this should consist of a specific combination of a massless vector field, \( A_M \), and a would-be Goldstone scalar, \( \sigma \). Accordingly, we make the definition

\[ V_M = A_M + p D_M \sigma, \tag{6.97} \]

where the coefficient \( p \) is also to be determined in what follows. Notice that, as defined, any Lagrangian built from the vector field \( V_M \) automatically has the gauge invariance

\[ \delta A_M = D_M \epsilon \quad \text{and} \quad \delta \sigma = -\frac{1}{p} \epsilon. \tag{6.98} \]

If we desire we may use unitary gauge for this symmetry to remove \( \sigma \) completely from the theory, however this is not a convenient gauge for our purposes and so in what follows we instead gauge-fix using a more convenient covariant gauge.
In order to implement the underlying gauge invariance which any such a spin-2 field must manifest we ask the above Lagrangian to be invariant under the usual spin-2 gauge transformation \( \delta h_{MN} = D_M \xi_N + D_N \xi_M \), supplemented by the Goldstone-type transformation \( \delta V_M = f \xi_M \). This leads to the following Lagrangian\(^8\)

\[
e^{-1} \mathcal{L}_{mEH} = e^{-1} \mathcal{L}_{EH} - \frac{1}{4} F_{MN} F^{MN} + f h^{MN} D_M V_N + f V^M D_M h - R^{MN} V_M V_N - \frac{1}{4} f^2 h_{MN} h^{MN} + \frac{1}{4} f^2 h^2,
\]

(6.99)
corresponding to the choices

\[
a = b = -f, \quad c = 1,
\]

\[
m_1 = 0,
\]

and \( m_2^2 = -\mu_2^2 = \frac{f^2}{2} \).

(6.100)

We now fix the two gauge freedoms of this action in such a way as to remove the mixings between the various fields having differing spins. To do so we take for the spin-2 gauge-fixing Lagrangian

\[
e^{-1} \mathcal{L}_{mEH2}^{gf} = - \left( f_N - \frac{1}{2} f V_N \right)^2,
\]

(6.101)

---

\(^8\) As a check on this result, we note that by choosing the gauge where \( V_M = 0 \), we recover the Pauli-Fierz Lagrangian of massive gravity [132]. Also, in flat space, this result agrees (after a suitable field redefinition) with the one given in [133].
where $f$ is the parameter appearing in the Lagrangian (6.99), and as before $f_N$ is defined as $f_N = D^M h_{MN} - \frac{1}{2} D_N h$. This gauge choice removes the $h^{MN} D_M V_N$ term from the action and introduces a mass term, $m$, for the vector field, $V_M$, with $m^2 = \frac{1}{2} f^2$.

To fix the other gauge freedom, eq. (6.98), we add the following gauge-fixing term

$$e^{-1} \mathcal{L}_{mEH1}^{gf} = -\frac{1}{2} (D_M A^M + \lambda h + r \sigma)^2,$$

(6.102)

with $\lambda$ and $r$ being parameters which are chosen to remove the remaining vector-gravity mixing terms in the quadratic action. In order to simplify the analysis, we specialize to the case where the background spacetime is an Einstein space, which we also take to be a solution to the Einstein equations of the form $G_{MN} - \Lambda g_{MN} = 0$, or $R_{MN} = -[2\Lambda/(n-2)] g_{MN}$. Using this we see that the removal of cross terms between $A_M$, $h_{MN}$, and $\sigma$ requires the choices

$$\lambda = -\frac{f}{2}, \quad \text{and} \quad r = pq^2,$$

(6.103)

where $q^2$ is defined as

$$q^2 = m^2 - \frac{4\Lambda}{n-2},$$

(6.104)

since in this case the gauge-fixed Lagrangian can be written as

$$\mathcal{L}_{mEH} + \mathcal{L}_{mEH1}^{gf} + \mathcal{L}_{mEH2}^{gf} = \mathcal{L}_{mEH0} + \mathcal{L}_{mEH1} + \mathcal{L}_{mEH2},$$

(6.105)

with the decoupled Lagrangians, $\mathcal{L}_{mEH0}$, $\mathcal{L}_{mEH1}$, and $\mathcal{L}_{mEH2}$, defined as follows.
$\mathcal{L}_{mEH2}$ denotes the $\phi_{MN}$ Lagrangian, which takes the form

$$e^{-1} \mathcal{L}_{mEH2} = \frac{1}{e} \mathcal{L}_{EH} - f_{N} f^{N} - \frac{1}{4} f^{2} h_{MN} h^{MN} + \frac{1}{8} f^{2} h^{2}$$

$$= -\frac{1}{2} \phi_{MN} (\Delta^{MN}_{PQ} + m^{2} \delta^{MN}_{PQ}) \phi^{PQ}$$

$$+ \left( \frac{n-2}{4n} \right) h \left( -\Box + m^{2} + \frac{1}{n} (4-n) R - 2\Lambda \right) h$$

$$+ \left( \frac{n-4}{n} \right) R_{MN} \phi^{MN} h$$

$$= -\frac{1}{2} \phi_{MN} (\Delta^{MN}_{PQ} + m^{2} \delta^{MN}_{PQ}) \phi^{PQ}$$

$$+ \frac{1}{2} \phi \left( -\Box + m^{2} - \frac{4\Lambda}{n-2} \right) \phi,$$  \hspace{1cm} (6.106)

where $\phi$, $\phi_{MN}$, $\delta^{MN}_{AB}$ and $\Delta^{MN}_{PQ}$ are as defined above for the massless spin-2 case. The mass $m$ is related to the symmetry-breaking parameter $f$ by $m^{2} = \frac{1}{2} f^{2}$.

We similarly find the following vector Lagrangian, $\mathcal{L}_{mEH1}$:

$$e^{-1} \mathcal{L}_{mEH1} = -\frac{1}{4} F_{MN} F^{MN} - \frac{1}{2} (D^{M} A_{M})^{2} - R^{MN} A_{M} A_{N} - \frac{1}{2} m^{2} A_{M} A^{M}$$

$$= -\frac{1}{2} A_{M} \left[ (-\Box + m^{2}) \delta^{M}_{N} + R^{M}_{N} \right] A^{N},$$  \hspace{1cm} (6.107)

where $m$ is the same as for $\phi_{MN}$.

Finally, the part of the quadratic action depending on $\sigma$ is

$$e^{-1} \mathcal{L}_{mEH0} = \frac{1}{2} \left[ (p r) \sigma \Box \sigma - (p f) \sigma \Box h - (r^{2}) \sigma^{2} + (f r) h \sigma \right],$$  \hspace{1cm} (6.108)

which contains terms which mix $\sigma$ and $h$. However, since $p$ is as yet unspecified we may choose its value to remove these cross terms. This may be done by choosing
\[ p = -f/(4q^2) \] and making the field redefinition \( \tilde{\sigma} = \frac{\sqrt{2m}}{4q} (\sigma + 2h) \), after which we find

\[
e^{-1} \mathcal{L}_{mEH0} = -\frac{1}{2} \tilde{\sigma} \left( -\Box + m^2 - \frac{4\Lambda}{n-2} \right) \tilde{\sigma} + \left( \frac{m^2}{4q^2} \right) h \left( -\Box + m^2 - \frac{4\Lambda}{n-2} \right) h. \quad (6.109)
\]

Notice that in this form the last term in \( \mathcal{L}_{mEH0} \) (involving \( h \)) has the same form as the last term in \( \mathcal{L}_{mEH2} \), and so these can both be combined into \( \mathcal{L}_{mEH2} \) by appropriately rescaling the scalar \( \phi \). Once this is done, and dropping the tilde on \( \sigma \), the remaining term becomes

\[
e^{-1} \mathcal{L}_{mEH0} = -\frac{1}{2} \sigma \left[ -\Box + m^2 - \frac{4\Lambda}{n-2} \right] \sigma. \quad (6.110)
\]

Finally, the action for the ghosts can be easily calculated from the gauge-fixing conditions. The spin-2 gauge-fixing term introduces a complex, fermionic, vector ghost with Lagrangian

\[
e^{-1} \mathcal{L}_{mEHVgh} = -\omega^* \left[ (-\Box + m^2) \delta_M^N + R_N^M \right] \omega^N. \quad (6.111)
\]

Similarly, the spin-1 gauge-fixing term introduces a complex scalar ghost with Lagrangian

\[
e^{-1} \mathcal{L}_{mEHSgh} = -\omega^* \left( -\Box + m^2 - \frac{4\Lambda}{n-2} \right) \omega. \quad (6.112)
\]

The complete Lagrangian, including all ghosts, for the massive graviton is thus the sum

\[
\mathcal{L}_{m2} = \mathcal{L}_{mEH0} + \mathcal{L}_{mEH1} + \mathcal{L}_{mEH2} + \mathcal{L}_{mEHSgh} + \mathcal{L}_{mEHVgh}. \quad (6.113)
\]

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We are now in a position to assemble the results for $a_k$. To this end, notice that all fields have been decoupled in the kinetic terms and all now have the same mass, $m^2 = \frac{1}{2} f^2$. This allows us to sum the separate contributions to $a_k$ from each of these fields. It is also interesting to note that the scalar fields $h$, $\sigma$ and the complex scalar ghost all have precisely the same Lagrangian, and so their net effect is to completely cancel one another in the one-loop action. Similarly, the vector boson $A_M$ and the complex vector ghost also share the same Lagrangian, and so for our purposes these two together contribute the equivalent of one real vector ghost.

In summary, the one-loop divergences for the massive graviton are given by the sum of the divergences of a symmetric traceless field and one real vector ghost (for which $\eta = -1$). Thus, we find

\[
\begin{align*}
\text{tr}_{2m}(a_0) &= \frac{N_2}{2} (n + 1)(n - 2) \\
\text{tr}_{2m}(a_1) &= -N_2 \left[ \frac{(6 - n)(n + 4)(n + 1)\Lambda}{6(n - 2)} \right] \\
\text{tr}_{2m}(a_2) &= N_2 \left[ \frac{1}{360} (n^2 - 31n + 508) R_{ABMN} R^{ABMN} \right. \\
&\quad + \left. \frac{5n^4 - 7n^3 - 248n^2 - 596n - 1440}{180(n - 2)^2} \Lambda^2 \right]
\end{align*}
\]

(6.114)

for $n$-dimensional massive gravitons on background metrics satisfying $G_{MN} - \Lambda g_{MN} = 0$.

### 6.3 Supergravity Models

In supergravity theories the ultraviolet sensitivity of the low-energy theory is often weaker than in non-supersymmetric models. This weaker sensitivity arises due
to cancellations between the effects of bosons and fermions in loops. The purpose of this section is to illustrate the utility of the previous section’s results by using them to exhibit this cancellation explicitly for supergravities in various dimensions. Some of the results we obtain — particularly those for massless particles in higher-dimensional supergravities — are computed elsewhere, and we use the agreement between these earlier calculations and our results as a check on the validity of our computations.

We proceed by summing the above expressions over the particles appearing in the appropriate supermultiplets. The result for the ultraviolet-sensitive part of the one-loop action obtained by integrating out a supermultiplet is given by

$$\Sigma_{UV} = \frac{1}{2} \left( \frac{1}{4\pi} \right)^{n/2} \int d^n x \sqrt{-g} \sum_{k=0}^{[n/2]} \sum_p (-)^{F(p)} m_p^{n-2k} \Gamma(k-n/2) \text{tr}_p [a_k], \quad (6.115)$$

where the sum on $p$ runs over the elements of a supermultiplet. As is clear from this expression, it is the weighted sum $\sum_p (-)^{F(p)} m_p^{n-2k} \text{tr}_p [a_k]$ which is of interest in supersymmetric theories.

In Minkowski space the strongest suppression of UV sensitivity arises when supersymmetry is unbroken, in which case all members of a supermultiplet share the same mass (so that $m_p = m$ for all $p$). In this case, eq. (6.115) can be written as

$$\Sigma_{UV} = \frac{1}{2} \left( \frac{1}{4\pi} \right)^{n/2} \int d^n x \sqrt{-g} \sum_{k=0}^{[n/2]} m_k^{n-2k} \Gamma(k-n/2) \text{Tr} [a_k], \quad (6.116)$$

where

$$\text{Tr} [a_k] \equiv \sum_p (-)^{F(p)} \text{tr}_p [a_k] \quad (6.117)$$

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is the relevant combination of heat-kernel coefficients for a supermultiplet. Since \( \text{tr} \left[ a_0 \right] \) simply counts the spin states of the corresponding particle type, the cancellation of the leading UV sensitivity occurs for a mass-degenerate supermultiplet simply because each supermultiplet contains equal numbers of bosons and fermions:

\[
\text{Tr} \left[ a_0 \right] = \sum_p (-)^{F(p)} \text{tr}_p[a_0] = N_B - N_F = 0.
\] (6.118)

This ensures the absence of a dependence of the form \( m^n \) in \( \Sigma_{UV} \).

The story is more complicated when there is a nonzero cosmological constant, and this is because mass itself is more delicate to define in de Sitter or anti-de Sitter spacetimes. For Minkowski space mass can be defined for particle states as a Casimir invariant of the Poincaré group, but this definition is no longer appropriate when \( \Lambda \) is nonzero because Poincaré transformations are then not the relevant spacetime isometries. Rather, for de Sitter space the relevant isometry group in four dimensions is \( \text{SO}(4,1) \), while the isometries of anti-de Sitter space fill out the group \( \text{SO}(3,2) \). For these geometries it only makes sense to inquire about the implications of unbroken supersymmetry for the anti-de Sitter case. This is because supersymmetry is always broken in de Sitter spacetime, whereas there is a supersymmetric generalization of \( \text{SO}(3,2) \) for which one can find particle supermultiplets which represent the unbroken supersymmetry.

In our previous calculations of the Gilkey coefficients we have defined \( m^2 \) to be that piece in the operator \( -\Box + X \) which is a constant for arbitrary background
fields.\footnote{This statement requires appropriate modification in the case of spin 2, where we include a cosmological constant term in the Lagrangian.} We nevertheless must still grapple with the above ambiguities as to the meaning of mass in de Sitter and anti-de Sitter spacetimes, due to the freedom of absorbing into $m^2$ contributions coming from the background curvature for constant-curvature spacetimes. One can try to restrict this freedom by demanding masslessness to correspond to conformal invariance or (for higher-spin fields) to unbroken gauge invariance, bearing in mind that these choices need not imply propagation along the light cone [134].

The upshot of this discussion is that it need not be true that all of the particles within a supermultiplet share the same mass even when working about a supersymmetric AdS background. In such cases one cannot pull a common mass out of the sum over particles within a supermultiplet, as was done in going from eq. (6.115) to eq. (6.116).

To see this concretely, consider the specific example of a Wess-Zumino multiplet in $n = 4$ spacetime dimensions expanded about a supersymmetric AdS background. Such a multiplet consists of a scalar, pseudoscalar, and spinor field: $(S, P, \chi)$, and taking the scalar and pseudoscalar to have a conformal coupling parameter, $\xi = -1/6$, their mass terms can be written as $m^2_S = m^2 - \delta m^2$, $m^2_P = m^2 + \delta m^2$ and $m^2_\chi = m^2$.

Unbroken supersymmetry implies that these mass terms are related to one another by $m^2 = -\mu^2 \Lambda / 12$ and $\delta m^2 = -\mu \Lambda / 6$, where $\Lambda$ is the AdS cosmological constant and $\mu$ is a dimensionless parameter which classifies the massive supersymmetric particle.
representations. In this case, we find

\[ \sum_{p} (-)^{F(p)} m_p^4 \text{tr}\, p[a_0] = m_S^4 \text{tr}\, S[a_0] + m_p^4 \text{tr}\, p[a_0] - m_\chi^4 \text{tr}\, \chi[a_0] = 2 \delta m^4 = \frac{\mu^2 \Lambda^2}{18} \]

\[ \sum_{p} (-)^{F(p)} m_p^2 \text{tr}\, p[a_1] = m_S^2 \text{tr}\, S[a_1] + m_p^2 \text{tr}\, p[a_1] - m_\chi^2 \text{tr}\, \chi[a_1] = \frac{2 m^2 \Lambda}{3} = -\frac{\mu^2 \Lambda^2}{18} \]

\[ \sum_{p} (-)^{F(p)} m_p^0 \text{tr}\, p[a_2] = \text{tr}\, S[a_2] + \text{tr}\, p[a_2] - \text{tr}\, \chi[a_2] = \frac{R_{MNPQ}^2}{48} - \frac{\Lambda^2}{9}. \quad (6.119) \]

The above complication keeps us from quoting general expressions for the sum of the Gilkey coefficients over arbitrary supermultiplets in general dimensions, since for AdS backgrounds these must be computed with the specific dependence of the relevant masses on \( \Lambda \). Notice however that the last expression in eq. (6.119) contains no dependence on the individual particle masses (since \( \text{tr}\, p[a_2] \) is multiplied by \( m_p^0 = 1 \)). Terms which are only present in the mass invariant piece of \( \Sigma_{UV} \), such as \( R_{MNPQ}^2 \) and \( F_{MN}^2 \), can be calculated once and for all in a model-independent way because their coefficients do not depend on the details of the particle masses involved.

As can be seen from the above example, however, calculating the complete answer for \( \Sigma_{UV} \) is not difficult once the individual particle masses are known. Similar considerations hold for dimensions other than four, with some terms in \( \Sigma_{UV} \) being mass independent and others requiring more detailed knowledge of the particle spectrum about a given background.

### 6.4 Application to Ricci Flat Backgrounds in 6D

We now apply the previous results to the case where the background spacetime is Ricci flat, satisfying \( R_{MN} = 0 \). Since we are particularly interested in the UV sensitivity of supersymmetric theories in six dimensions, we will need to calculate
Table 6–1: Results for massless particles in six dimensions, including ghost contributions and specialized to Ricci-flat background metrics. Note that the boson fields in this table are real and the fermion fields are $6D$ symplectic-Weyl spinors.

<table>
<thead>
<tr>
<th>Field</th>
<th>$(-)^F \text{Tr } a_0$</th>
<th>$(-)^F \text{Tr } a_1$</th>
<th>$(-)^F \text{Tr } 720 \text{Tr } a_2$</th>
<th>$(-)^F \text{Tr } 45360 \text{Tr } a_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi$</td>
<td>1</td>
<td>0</td>
<td>$4 R^2_{ABMN}$</td>
<td>$-17 I_1 + 28 I_2$</td>
</tr>
<tr>
<td>$\psi$</td>
<td>-2</td>
<td>0</td>
<td>$7 R^2_{ABMN}$</td>
<td>$-29 I_1 + 70 I_2$</td>
</tr>
<tr>
<td>$A_M$</td>
<td>4</td>
<td>0</td>
<td>$-44 R^2_{ABMN}$</td>
<td>$184 I_1 - 392 I_2$</td>
</tr>
<tr>
<td>$A_{MN}$</td>
<td>6</td>
<td>0</td>
<td>$264 R^2_{ABMN}$</td>
<td>$-7158 I_1 + 14280 I_2$</td>
</tr>
<tr>
<td>$\psi_M$</td>
<td>-6</td>
<td>0</td>
<td>$-219 R^2_{ABMN}$</td>
<td>$-591 I_1 + 1218 I_2$</td>
</tr>
<tr>
<td>$g_{MN}$</td>
<td>9</td>
<td>0</td>
<td>$756 R^2_{ABMN}$</td>
<td>$8919 I_1 - 17892 I_2$</td>
</tr>
</tbody>
</table>

the Gilkey coefficient $a_3$ for the various spin fields. Fortunately, this is a matter of simply plugging in our previous results into the equation for $a_3$, eq. (C.2). To simplify the equations to come, we make use of the following result for a Ricci flat space (we neglect total derivatives)

$$D_E R_{ABCD} D^E R^{ABCD} = -I_1 - 4I_2,$$  \hspace{1cm} (6.120)

where $I_1$ and $I_2$ are defined as

$$I_1 = R_{ABCD} R_{EF}^{AB} R_{CDEF}^{CDE}$$
$$I_2 = R_{ABCD} R_{E}^{AC} R_{F}^{BEDF}.$$  \hspace{1cm} (6.121)

It should be noted that the integrand of the Euler number, $\chi$, in 6 dimensions is proportional to the combination $-I_1 + 2I_2$. We now summarize the results of this calculation for spin 0 through to spin 2. In the following formulae, $N$ counts the number of fields of a given spin. Tables (6–1) and (6–2) summarize the results which these calculations imply respectively for massless and massive particles in 6 dimensions.
Table 6–2: Results for massive particles in six dimensions, specialized to Ricci-flat backgrounds. Note that in this table the bosonic fields are real and the spinors are symplectic, but not Weyl.

<table>
<thead>
<tr>
<th>Multiplet</th>
<th>$(-)^F \text{Tr}[a_0]$</th>
<th>$(-)^F \text{Tr}[a_1]$</th>
<th>$(-)^F 360 \text{Tr}[a_2]$</th>
<th>$(-)^F 45360 \text{Tr}[a_3]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi$</td>
<td>1</td>
<td>0</td>
<td>$2 R^2_{ABMN}$</td>
<td>$-17 I_1 + 28 I_2$</td>
</tr>
<tr>
<td>$\psi$</td>
<td>$-4$</td>
<td>0</td>
<td>$7 R^2_{ABMN}$</td>
<td>$-58 I_1 + 140 I_2$</td>
</tr>
<tr>
<td>$A_M$</td>
<td>5</td>
<td>0</td>
<td>$-20 R^2_{ABMN}$</td>
<td>$167 I_1 - 364 I_2$</td>
</tr>
<tr>
<td>$A_{MN}$</td>
<td>10</td>
<td>0</td>
<td>$110 R^2_{ABMN}$</td>
<td>$-6974 I_1 + 13888 I_2$</td>
</tr>
<tr>
<td>$\psi_M$</td>
<td>$-16$</td>
<td>0</td>
<td>$-212 R^2_{ABMN}$</td>
<td>$-1240 I_1 + 2576 I_2$</td>
</tr>
<tr>
<td>$g_{MN}$</td>
<td>14</td>
<td>0</td>
<td>$358 R^2_{ABMN}$</td>
<td>$9086 I_1 - 18256 I_2$</td>
</tr>
</tbody>
</table>

6.4.1 Supersymmetric Multiplets

The next few sections record the analogous result for the coefficients $a_0$ through $a_3$ once summed over the particles within various 6D supermultiplets, under the assumption that all elements of the supermultiplet have the same 6D mass. Notice that our discussion of the equivalence between performing the sum over the UV sensitive part of the 4D contribution of each KK mode and the UV sensitivity as computed using the full 6D fields shows that the assumption of equal masses within a 6D supermultiplet relies only on there being unbroken (2,0) supersymmetry at the high-energy scale, $M$, appropriate to the compactification from higher dimensions down to 6D. Provided that $M \gg m_{KK} \sim 1/r$, this assumption does not require that some supersymmetry remains unbroken below the scale of the KK masses encountered in the compactification from 6D down to 4D.

Massless 6D Supermultiplets

We first summarize the particle content of the simplest 6D supermultiplets, starting first with massless multiplets and then moving on to massive multiplets. Our discussion follows that of ref. [135].
Massless multiplets are partially characterized by their representation properties for the ‘little group’ which preserves the form of a standard light-like energy-momentum vector. In 5+1 dimensions, the light-like little group contains the rotations, \(SO(4)\), of the 4 spatial dimensions transverse to the direction of motion of the standard light-like momentum. The representations of \(SO(4) \sim SU(2) \times SU(2)\) corresponding to the simplest particle types are

\[
\begin{align*}
(1, 1) : & \quad \text{scalar (}\phi\text{)} \\
(2, 1), (1, 2) : & \quad \text{Weyl spinors (}\psi_\pm\text{)} \\
(2, 2) : & \quad \text{gauge potential (} A_M \text{)} \\
(3, 1), (1, 3) : & \quad \text{(anti) self-dual 2-form potentials (} A^\pm_{MN}\text{)} \\
(3, 2), (2, 3) : & \quad \text{Weyl gravitino (}\psi^\pm_M\text{)} \\
(3, 3) : & \quad \text{graviton (} g_{MN}\text{)}
\end{align*}
\]

where we denote the particle type by the field with which it is usually represented. The two representations listed for the Weyl fermions correspond to the two types of chiralities these fermions can have. Similarly, the two representations listed for the 2-form potential correspond to the self- and anti-self-dual pieces, which are defined to satisfy \(G_{MNP} = \pm \epsilon_{MNPQRS} G^{QRS}\).

Massless supermultiplets are also characterized by the action of the graded automorphism symmetry which is generated by the supercharges, which transform as spinors of \(SO(4)\). Since the fundamental spinor representations — \(2_+ = (2, 1)\) and \(2_- = (1, 2)\) — of \(SO(4) \sim SU(2) \times SU(2)\) are pseudo-real, the graded automorphism group in 6 dimensions is \(USp(N_+) \times USp(N_-)\), where \(N_+\) and \(N_-\) (which must both

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Table 6–3: Massless representations of (2, 0) supersymmetry in 6 dimensions, labelled by the dimension of the representation of the corresponding little-group algebra, and by the corresponding 6D field content. The fermions are taken to be symplectic-Weyl, and the 2-form potentials are similarly self (or anti-self) dual.

<table>
<thead>
<tr>
<th>Multiplet</th>
<th>Representation</th>
<th>Field Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$2^2 = (2,1;1) + (1,1;2)$</td>
<td>$(\psi,2\phi)$</td>
</tr>
<tr>
<td>8</td>
<td>$(1,2;1) \times 2^2 = (2,2;1) + (1,2;2)$</td>
<td>$(A,2\psi)$</td>
</tr>
<tr>
<td>8’</td>
<td>$(2,1;1) \times 2^2 = (3,1;1) + (1,1;1) + (2,1;2)$</td>
<td>$(A_{MN},2\psi,\phi)$</td>
</tr>
<tr>
<td>12</td>
<td>$(1,3;1) \times 2^2 = (2,3;1) + (1,3;2)$</td>
<td>$(\psi,2A_{MN})$</td>
</tr>
<tr>
<td>16</td>
<td>$(2,2;1) \times 2^2 = (3,2;1) + (1,2;1) + (2,2;2)$</td>
<td>$(\psi_{M},2A_{MN},\psi)$</td>
</tr>
<tr>
<td>24</td>
<td>$(2,3;1) \times 2^2 = (3,3;1) + (1,3;1) + (2,3;2)$</td>
<td>$(g_{MN},2\psi_{M},A_{MN})$</td>
</tr>
</tbody>
</table>

Table 6–4: Results for the statistics-weighted sum over Gilkey coefficients for massless 6D supermultiplets, specialized to vanishing gauge fluxes and Ricci-flat backgrounds.

<table>
<thead>
<tr>
<th>Multiplet</th>
<th>$(-)^F , Tr , [a_0]$</th>
<th>$(-)^F , Tr , [a_1]$</th>
<th>$(-)^F , 48 , Tr , [a_2]$</th>
<th>$(-)^F , 720 , Tr , [a_3]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>$R^2_{ABMN}$</td>
<td>$(-I_1 + 2I_2)$</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>0</td>
<td>$-2 R^2_{ABMN}$</td>
<td>$-2 (-I_1 + 2I_2)$</td>
</tr>
<tr>
<td>8’</td>
<td>0</td>
<td>0</td>
<td>$10 R^2_{ABMN}$</td>
<td>$58 (-I_1 + 2I_2)$</td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td>0</td>
<td>$3 R^2_{ABMN}$</td>
<td>$123 (-I_1 + 2I_2)$</td>
</tr>
<tr>
<td>16</td>
<td>0</td>
<td>0</td>
<td>$-20 R^2_{ABMN}$</td>
<td>$4 (-I_1 + 2I_2)$</td>
</tr>
<tr>
<td>24</td>
<td>0</td>
<td>0</td>
<td>$30 R^2_{ABMN}$</td>
<td>$-66 (-I_1 + 2I_2)$</td>
</tr>
</tbody>
</table>

be even) characterize the number of independent, pseudo-real chiral supersymmetries. The minimal supersymmetry algebra therefore corresponds to $(N_+, N_-) = (2, 0)$, and so $N = N_+ + N_- = 2$ [135]. The relevant little group characterizing the massless supermultiplets is then $G = SO(4) \times USp(2) \sim SU(2) \times SU(2) \times USp(2)$, under which the active supercharges transform in the representation $(2,1;2)$.

The minimal representation of this little algebra has dimension $2^2 = 4$, and transforms under $G$ like [135]

$$4 = (2,1;1) + (1,1;2).$$

(6.122)
This consists of 2 real (1 complex) scalars and a single symplectic-Weyl fermion, and so consists of 2 bosonic and 2 fermionic states.

Higher-dimensional representations may be obtained from this minimal one by taking direct products of it with an irreducible representation of the bosonic part of the little group. Table (6–3) lists some possible representations which are obtained in this way, including the hyper-multiplet (4), gauge multiplet (8), tensor multiplet (8'), two types of gravitino multiplet (12, 16) and the graviton multiplet (24). To derive the results in this table, we use the standard results for $SU(2)$: $2 \times 2 = 1 + 3$, $2 \times 3 = 2 + 4$ and $2 \times 4 = 3 + 5$.

We may now sum the previous expressions for the Gilkey coefficients over the particle content of these massless 6D supermultiplets. The results obtained for $\text{Tr}_{\text{sm}}[a_k] = \sum_{p \in \text{sm}} (-)^{F(p)} \text{Tr}_p[a_k]$ if the particles all share the same mass are summarized in table (6–4). Notice that the resulting expressions for $\text{Tr}_{\text{sm}}[a_3]$ are proportional to the combination $-I_1 + 2I_2$ which gives the Euler number density for compact 6D manifolds.

**Massive 6D supermultiplets**

The massive representations of (2, 0) 6D supersymmetry are found in a similar manner, except in this case the little group for the time-like energy-momentum vector appropriate to massive fields is $SO(5)$. The particle types and fields corresponding
Table 6–5: Massive representations of \((2,0)\) supersymmetry in six dimensions, labelled by their dimension. Note that the fermions are not chiral and the 2-form potentials are not self-dual or antiself-dual.

<table>
<thead>
<tr>
<th>Mult.</th>
<th>Representation</th>
<th>Field Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>(16_m)</td>
<td>(2^4 = (5,1) + (1,3) + (4,2))</td>
<td>((A^m_M,2\psi^m,3\phi^m))</td>
</tr>
<tr>
<td>(64_m)</td>
<td>((4,1) \times 2^4 = (16,1) + (4,1) + (4,3)) + ((10,2) + (5,2) + (1,2))</td>
<td>((\psi^m_M,2A^m_{MN},2A^m_M,4\psi^m,2\phi^m))</td>
</tr>
<tr>
<td>(80_m)</td>
<td>((5,1) \times 2^4 = (14,1) + (10,1) + (1,1)) + ((5,3) + (16,2) + (4,2))</td>
<td>((g^m_{MN},2\psi^m_M,4A^m_{MN},3A^m_M,2\psi^m_M,2\phi^m))</td>
</tr>
</tbody>
</table>

To the representations of \(SO(5)\) are as follows:

1: massive scalar \(\phi^m\)

4: massive spinor \(\psi^m\)

5: massive gauge potential \(A^m_M\)

10: massive 2-form \(A^m_{MN}\)

16: massive gravitino \(\psi^m_M\)

14: massive graviton \(g^m_{MN}\).

The little algebra for the supersymmetry representations is therefore \(SO(5) \times USp(2)\). The irreducible spinor representations of \(SO(5)\) are not chiral, and are 4-dimensional, and so the number of supercharges doubles in going from light-like to time-like representations.\(^{10}\) It follows that the dimensionality of the minimal representation is the square of what it was in the light-like situation: \(2^4 = 16\), with the

---

\(^{10}\) We assume here vanishing central charges — and so no ‘short’ multiplets.
Table 6–6: Results for massive 6D multiplets (Ricci flat).

<table>
<thead>
<tr>
<th>Multiplet</th>
<th>$(-)^F \text{Tr}[a_0]$</th>
<th>$(-)^F \text{Tr}[a_1]$</th>
<th>$(-)^F \text{Tr}[a_2]$</th>
<th>$(-)^F \text{Tr}[a_3]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$16_m$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$64_m$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{3}(-I_1 + 2I_2)$</td>
</tr>
<tr>
<td>$80_m$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

following decomposition under $SO(5) \times USp(2)$ [135]:

$$16_m = (5, 1) + (1, 3) + (4, 2).$$ (6.123)

Again, we find all other representations by taking appropriate direct products. To do so, we use the following standard results for $SO(5)$: $4 \times 4 = 10 + 5 + 1$ and $4 \times 5 = 16 + 4$, leading to the massive supermultiplets given in table (6–5). It is now straightforward to compute the results for the statistics-weighted sum of the Gilkey coefficients over the massive supermultiplet particle content. The results obtained in this way are summarized in table (6–6). What is striking about this table is the vanishing of the contributions to $a_0$ through $a_2$, which ensures the absence within the effective action of positive powers of the common 6D mass $M$ of a supermultiplet of degenerate massive particles.

6.4.2 Higher-Dimensional Field Content

Of particular interest are those massive 6D supermultiplets which are obtained by dimensionally reducing the various 10D supergravities to six dimensions. The simplest such compactifications are obtained by reducing the higher-dimensional theories on a 4-torus. Since the IIA theory can be obtained from 11-dimensional supergravity via dimensional reduction on $S^1$, the results we obtain for the Type IIA
theory dimensionally reduced on a 4-torus are equivalent to what is obtained from 11-dimensional supergravity dimensionally reduced on a 5-torus.

For the purposes of the present argument all that matters is the total number of each type of massive $6D$ field which is produced by such a reduction. Our purpose in this section is to show that the massive field content which is obtained by such a dimensional reduction is the same as would be obtained by combining a small number of the massive $16_m$, $64_m$ and $80_m$ representations of $(2,0)$ $6D$ supersymmetry.\textsuperscript{11} Since we know that each of these multiplets gives a vanishing contributions to the first 3 heat-kernel coefficients, the same must also be true of the contributions of the massive $6D$ states which are obtained by dimensional reduction.

\textbf{Type IIA and type IIB supergravities}

The field content of Type IIA supergravity in ten dimensions consists of: a graviton; a 3-form, 2-form, 1-form, and 0-form; two majorana-Weyl gravitini having opposite $10D$ chiralities; and two majorana-Weyl dilatini having opposite $10D$ chiralities. The IIB theory is obtainable from the IIA theory by giving all fermions the same chirality, and by trading the 3-form and the 1-form for a self-dual 4-form, a 2-form, and a 0-form.

Since each majorana-Weyl spinor in $10D$ has 16 real components, reduction on a torus (in the absence of any Scherk-Schwartz supersymmetry-breaking twists [136])

\footnote{This need not imply that these massive states actually transform in these representations under $(2,0)$ supersymmetry, for example due to possible presence of central charges and short multiplets.}
gives $(8,8)\ 6D$ supersymmetry. The massless sector of the resulting $6D$ theory can therefore be described in terms of $(2,0)$ supersymmetry multiplets, and arranges itself into the following collection of massless $(2,0)$ supermultiplets,

$$\text{Type IIA/B : } 24 + 4(16) + 2(12) + 5(8') + 8(8) + 10(4), \quad (6.124)$$

with both Type IIA and IIB supergravities giving the same multiplet content when dimensionally reduced on a 4-torus. In deriving these results, we use the equivalence (in 6 dimensions) of a 3-form and a 1-form gauge potential.

Using the results from table (6–4), we find the following statistics-weighted sum for the Gilkey coefficients produced by the massless sector of the $6D$ theory:

$$\begin{align*}
\text{Tr} [a_0] &= \text{Tr} [a_1] = \text{Tr} [a_2] = 0, \\
\text{Tr} [a_3] &= \frac{2}{3} (-I_1 + 2I_2).
\end{align*} \quad (6.125)$$

The only UV-sensitive quantity is a topological term proportional to the Euler-number density. If we had not used the duality relationship to exchange the 3-form potential for a 1-form potential, even the coefficient of the Euler-number term would have vanished, because in 6 dimensions [120],

$$\text{Tr}_{3f}[a_3] - \text{Tr}_{1f}[a_3] = \frac{2}{3} (-I_1 + 2I_2). \quad (6.126)$$

\footnote{This is an example of the breakdown of naive equivalences between different field representation descriptions of the same particle [137, 138].}
A similar statement can be made for the massive KK modes produced by any such \((8,8)\)-supersymmetric dimensional reduction on a 4-torus. In this case the massive states have the same field content as do the following 6\(D\) massive \((2,0)\) supersymmetry representations:

\[
\text{Type IIA/B : } \quad 80_m + 2(64_m) + 3(16_m) .
\] (6.127)

The vanishing of \(\text{Tr}[a_0]\) through \(\text{Tr}[a_2]\) for the massive multiplets ensures the vanishing of any UV sensitive contributions from the KK modes obtained when reducing from 10 to 6 dimensions (provided these are not in short multiplets).

### 6.5 Discussion

Our final results are as follows. We find that the Casimir energy due to any one 6\(D\) field typically does involve terms proportional to \(M^6\), \(M^4\) or \(M^2\) — cf. eq. (1.26). However our main result is to find that all three of these kinds of terms cancel once they are summed over a massive supermultiplet of \((2,0)\) supersymmetry in 6 dimensions. Furthermore, this cancellation is independent of the details of the compactification from 6\(D\) to 4\(D\) (provided it is Ricci flat), and in particular does not require that the compactification be supersymmetric. This result applies in particular to the contributions of the massive Kaluza-Klein (KK) modes which would be obtained if the 6\(D\) supergravity of interest were itself obtained from a 10- (or 11-) dimensional theory by dimensional reduction on 4 (or 5) small dimensions, since these have the same particle content as do the massive \((2,0)\) 6\(D\) supermultiplets we examine.
As such, we believe these results to be encouraging for the success of the SLED program, inasmuch as they show how bulk supersymmetry can stop very heavy KK particles from contributing too large one-loop amounts to the low-energy effective cosmological constant, both through bulk and brane-localized effective interactions. These results leave one type of one-loop contribution which could still be dangerously large from the SLED point of view: dimension-two interactions localized on the brane (such as a brane-localized Einstein-Hilbert action) which are generated by loops of brane-bound particles (for which supersymmetry is badly broken). Indeed, such effective interactions are likely to be generated by brane loops, and it is not yet clear whether the bulk response need cancel their $4D$ effects in the same way as it would do for an effective brane tension.
CHAPTER 7
Conclusions

In this thesis we examined critical aspects of Supersymmetric Large Extra Dimensions, a proposal whose goal is to address both the cosmological constant and the hierarchy problems. Since the SLED proposal relies critically on $6D$ supergravity, part of our investigation focused on finding new solutions to the relevant equations of motion. When work began on this thesis, very little was known about the solutions to the supergravity of interest to us, Nishino-Sezgin supergravity. Progress in this direction was considerably improved thanks to Gibbons, Guéven and Pope (GGP), who found the most general solution to the equations of motion, subject to the requirements of having $4D$ Minkowski space and axial symmetry in the extra dimensions. As discussed in the introductory chapter, they also showed that the unique singularity-free solution consistent with maximal symmetry of the four large dimensions is the one found by Salam and Sezgin, corresponding to $4D$ Minkowski space and spherical extra dimensions.

One of the questions we answer in chapter 3 is what happens when we relax the requirement that solutions be $4D$-flat. In that case, we show that Minkowski space is no longer the only maximally-symmetric solution allowed; in particular, we explicitly find solutions having both de Sitter and anti-de Sitter symmetry which necessarily involve non-conical singularities. This result is important since it shows that in the
presence of branes — which generically induce singularities at their location — 4D-flat solutions are not guaranteed in the same way that they are in the absence of branes. In this chapter we also find time-dependent solutions to the 6D equations of motion. These are ‘scaling’ solutions since the time dependence factors out of all equations of motion, which corresponds to a considerable simplification. Ultimately, it must be determined whether the SLED proposal yields acceptable cosmologies, and so finding time-dependent solutions is a necessary first step in this direction.

It is also important to determine whether the various solutions of the supergravity equations of motion are stable against small perturbations. Obviously, a solution who timescale for decay is much less than the age of the universe does not yield a good cosmological model. Thus, in chapter 4 we examine the perturbative stability of a large class of solutions to Nishino-Sezgin supergravity. We do so by linearizing the relevant equations of motion about the general time-independent solutions found by GGP. We solve the resulting linearized equations of motion exactly in the case where the background GGP solution is chosen to be conical, and then show that for appropriate boundary conditions all modes have masses greater than or equal to zero. For the more general case where the background is nonconical, we can no longer write closed-form solutions; nonetheless, we show that the solutions are marginally stable, subject again to appropriate boundary conditions. Here, the marginal stability is expected since scale invariance ensures that there is a massless mode. Finally, we conclude by closing a possible loophole in the above analysis by showing that there are no ghosts present.
In chapter 5 we turn to the issue of how to regularize singularities in the case where the metric is not conical in the vicinity of brane sources since, as it turns out, the majority of singularities encountered are not conical in nature. For example, in the case of the GGP solutions we see that the conical solutions satisfy \( \lambda_1 = \lambda_2 \), and so make up only a tiny fraction of the parameter space of solutions. Thus, it is important to understand what objects source these more exotic singularities. To smooth out these singularities, we excise small regions of space in the neighbourhood of the singular points, and on the boundaries of these excised regions we place codimension-1 branes. The interior regions are then filled with non-singular solutions to the equations of motion. Our analysis shows in detail how these branes dictate various bulk properties; for example, we show that there is a ‘Hamiltonian constraint’ which the branes must satisfy if there are to source a flat bulk solution. We further identify the brane properties required to ensure unbroken scale invariance. Finally, we calculate the low-energy effective potential for the mode corresponding to this scaling symmetry, and as expected we find that it is zero in the case where the scale invariance is unbroken.

The purpose of chapter 6 is to address the all-important question of the size of the Casimir energy induced by bulk loops. In order that these results be as general as possible, we do not restrict ourselves to working only in six dimensions and so keep the number of dimensions arbitrary. We calculate the first few heat kernel coefficients for the fields up to spin 2, including the contributions of the ghosts. While similar calculations have been performed in the past for massless particles [120], our main extension is to provide \( n \)-dimensional results for massive fields rather than massless
fields, as well as to extend these results to spacetimes having more general background geometries. This includes calculating the contributions of the various ghosts and would-be Goldstone particles which participate in the generalized higher-spin mass-acquisition (Higgs) mechanism. We then assemble these results into supermultiplets in order to obtain the heat kernel coefficients for general supergravity theories. It is shown how once assembled into massive supermultiplets, these coefficients vanish in a number of cases.

**Future directions**

While much progress has been made in determining the viability of the SLED proposal, there is work that still needs to be done. In chapter 3 we present a number of new exact solutions to the $6D$ supergravity equations of motion, although it is obvious that these solutions describe only a tiny subset of possible behaviours of this system. For example, all known solutions currently assume that the extra dimensions are axially symmetric, and so one possible avenue of research is to search for solutions where this assumption is relaxed. As suggested in [139], such solutions may represent the endpoint of a decay from one of the unstable solutions of NS supergravity.

Another area for future research is in improving our understanding of brane-induced singularities. While chapter 5 focuses on regularizing these singularities by modeling the brane as having a finite thickness, such a procedure requires solving complicated algebraic equations for many unknowns. It would be therefore be beneficial to have a more direct handle on how the renormalization of these singularities should proceed. For example, in the case of a fixed background geometry which
is conical, references [88] and [89, 90] show in detail how the various brane couplings renormalize. A generalization of these results to more general backgrounds is currently underway.

Finally, given the critical importance to the SLED proposal of determining the size of the one-loop quantum corrections, it certainly behooves us to determine the answer to this question in the case of the general GGP solutions. As mentioned in chapter 6, this involves disentangling the mixings between fields of different spin, induced by the nontrivial background geometry. Once this is accomplished, the general results of chapter 6 can be used to determine these one-loop corrections. Work on this is currently in progress.
APPENDIX A
Newton’s Law in Higher Dimensions

In this appendix, we show how Newton’s law is derived from the Einstein-Hilbert action. This justifies the claim that the usual inverse square law (ISL) of gravity is modified to a “$1/r^{2+d}$ law” when there are $d$ extra dimensions. We also show how the ISL is recovered at low energies, for the special case where the extra dimensions are compact. A more detailed treatment of this transition can be found in [140].

To see how Newton’s law is derived in higher-dimensional gravity, we recall that in the Newtonian limit the only non-trivial metric component is $g_{00} = -(1 + 2\phi)$, and so the only nonzero component of the Einstein tensor is $G_{00} \approx \nabla^2 g_{00} = -2\nabla^2 \phi$. Thus, using also that $T_{00} = \rho$ is the energy density of nonrelativistic matter, we obtain

$$\nabla^2 \phi = \frac{\rho}{2M_z^{2+d}}$$  \hspace{1cm} (A.1)

and so recognize $\phi$ as the usual gravitational potential. Applying Gauss’ law we see that for $d$ extra dimensions the gravitational force is given by

$$F = \frac{G_* m_1 m_2}{r^{2+d}},$$  \hspace{1cm} (A.2)

which defines the higher-dimensional Newton constant by $G_*^{-1} = 2 \Omega_{2+d} M_z^{2+d}$, where $\Omega_n$ is the surface area of the unit $n$-sphere. Using this last result together with eq. (1.17), we obtain the expression for the 4D Newton constant in terms of the
higher-dimensional one,
\[ G_N = \frac{\Omega_{2+d} G_s}{4\pi V_d}. \] (A.3)

Since we are interested in the case where the \( d \) dimensions are compact with
typical size \( R \), the above equation applies only when \( r \ll R \). In the opposite limit,
the net force can be obtained by the method of images and one recovers \( 4D \) Newton
gravity in the limit \( r \gg R \). For example, assuming one extra dimension with periodic
boundary conditions, and if we take \( y \) to be the location in the extra dimensions and
\( r \) the perpendicular distance to the source in the large dimensions, then the method
of images gives
\[ F_\perp = \frac{G_s m_1 m_2}{r^3} \sum_{n=-\infty}^{\infty} \left[ 1 + \left( \frac{y + nR}{r} \right)^2 \right]^{-2} \] (A.4)
where \( F_\perp \) is the force perpendicular to the extra dimensions. In the limit \( y, r \ll R \)
we recover eq. (A.2) specialized to \( d = 1 \). On the other hand, in the limit \( r \gg R \) we
obtain
\[ F_\perp \approx \left( \frac{\pi G_s}{2R} \right) \frac{m_1 m_2}{r^2}. \] (A.5)
Notice that we recover the ISL, and furthermore this result implies the same rela-
tionship between \( G_N \) and \( G_s \) as given by eq. (A.3) with \( d = 1 \).

Thus, we see that for distances \( r \ll R \) gravity appears higher dimensional,
whereas for \( r \gg R \) the gravitational flux begins to ‘saturate’ the extra dimensions
at which point the force versus distance relation begins to transition from \( r^{-2-d} \) to
$r^{-2}$. In the SLED proposal, this dilution of gravitational flux into the extra dimensions is the ultimate reason why gravity is perceived to be so much weaker than the electroweak force even though we assume $M_s \sim M_{EW}$.
APPENDIX B
Some Special Functions

In this appendix we record some useful properties of the special functions which are used in the main text. In our conventions the Legendre functions are related to Hypergeometric functions by

\[ P_\nu(z) = F\left[-\nu, 1 + \nu; 1; \frac{1}{2}(1 - z)\right] \]

\[ Q_\nu(z) = \frac{\sqrt{\pi} \Gamma(1 + \nu)}{(2z)^{1+\nu} \Gamma\left(\frac{3}{2} + \nu\right)} F\left[1 + \nu, \frac{1}{2}, \frac{1}{2} + \nu; 3 + \nu, z^{-2}\right], \quad \nu \neq -\frac{3}{2}, -\frac{5}{2}, \ldots. \]

The asymptotic form of the Legendre functions as \( z \to \pm 1 \) can be found from well-established properties of the hypergeometric functions, most notably its series definition

\[ F[a, b; c; z] = 1 + \frac{ab}{c} z + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{z^2}{2!} + \cdots, \quad \text{(B.1)} \]

provided \( c \neq 0, -1, -2, \ldots \). For \( |z| < 1 \) the hypergeometric function has the property

\[ F[a, b; c; z] = (1 - z)^{c-a-b} F[c - a, c - b; c; z]. \quad \text{(B.2)} \]

The potential singularities of \( F[a, b; c; z] \) lie at \( z = \pm 1 \) and \( z = \infty \), and our interest is in particular its behaviour at \( z = \pm 1 \). A useful identity, valid when \( |\arg(1 - z)| < \pi \),
for identifying these behaviours is

\[
F[a, b; c; u] = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - b)\Gamma(c - a)} \cdot \frac{1}{1 - u} \\
+F[a, b; 1 + a + b - c; 1 - u] + (1 - u)^{c-a-b} \left[ \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} \right] \cdot \frac{F[c - a, c - b; c - a - b + 1; 1 - u]}{1 - u}.
\]

This expression shows that \( F[a, b; c; z] \) is finite as \( z \to 1 \) provided \( c > a + b \), it diverges there logarithmically if \( c = a + b \), and it can be more singular if \( c < a + b \):

\[
F[a, b; c; 1] = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}, \quad \text{if} \quad c > a + b, \quad a, b \notin \mathbb{Z}_-
\]

\[
F[a, b; a + b; z] \xrightarrow{z \to 1} - \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} \log(1 - z), \quad \text{if} \quad a, b \notin \mathbb{Z}_-
\]

\[
F[-n, b; c; 1] = \frac{(c - b)_n}{(c)_n}, \quad n \in \mathbb{N}.
\]

Here \( \Gamma \) is Euler’s Gamma function \( \Gamma(n) = (n - 1)! \), defined for complex arguments by \( \Gamma(x) = \int_0^\infty du \ u^{x-1} e^{-u} \). \((\cdots)_n\) is the Pochhammer symbol, defined by \( (x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} \).

Using eq. (B.3) with \( a = -\nu, b = 1 + \nu, c = 1 \) and \( u = \frac{1}{2}(1 - z) \) leads (provided \( \nu \neq 0, 1, 2, \ldots \)) to the following asymptotic expression

\[
P_{\nu}(z) = - \ln \left[ \frac{\frac{1}{2}(1 + z)}{\Gamma(-\nu)\Gamma(1 + \nu)} \right] F\left[ -\nu, 1 + \nu; 1; \frac{1}{2}(1 + z) \right] + O(1)
\]

\[
= - \ln \left[ \frac{\frac{1}{2}(1 + z)}{\Gamma(-\nu)\Gamma(1 + \nu)} \right] + O(1) \quad \text{as} \quad z \to -1.
\]

If, on the other hand, \( \nu = \ell = 0, 1, \ldots, \) then the hypergeometric series terminates and \( P_{\ell}(z) \) is bounded at both \( z = \pm 1 \). This leads in the usual way to the Legendre polynomials, for which with \( P_0(z) = 1 \) and:

\[
P_{\ell}(z) = \frac{1}{2\ell!} \left( \frac{d}{dz} \right)^\ell (z^2 - 1)^\ell, \quad \ell = 1, 2, \ldots,
\]
so that $P_1(z) = z$, $P_2(z) = \frac{1}{2}(3z^2 - 1)$, $P_3(z) = \frac{1}{2}z(5z^2 - 3)$, and so on.

Similarly, using $a = 1 + \frac{\nu}{2}$, $b = \frac{1}{2} + \frac{\nu}{2}$, $c = \frac{3}{2} + \nu$ and $u = z^{-2}$ in the identity (B.3), leads to

$$Q_\nu(z) = -\frac{\sqrt{\pi} \Gamma(1 + \nu) \ln(1 - z^{-2})}{(2z)^{1+\nu} \Gamma(1 + \nu) \Gamma(\frac{1}{2} + \frac{\nu}{2})} F\left[1 + \frac{\nu}{2}, \frac{1}{2} + \frac{\nu}{2}; 1; 1 - z^{-2}\right] + O(1)$$

$$= \pm \frac{1}{2(\pm1)^\nu} \left[-\ln(1 - z^{-2}) + O(1)\right] \text{ as } z \to \pm1, \quad \text{(B.6)}$$

where we have used properties of the Gamma function to provide further simplifications. This function clearly has logarithmic singularities at both $z = 1$ and $z = -1$. Notice that in the region of interest, $-1 < z < 1$, we have $1 - z^{-2} < 0$, leading to a function which is complex. Since $\chi$ is given by the real part of $Q_\nu$ we have the following expressions for the near-brane singularities of $\chi$:

$$\chi(z) = -\frac{C_2}{2} \ln(1 - z) + O(1) \quad \text{as } z \to 1 \quad \text{(B.7)}$$

$$= -\left\{\frac{C_1}{\Gamma(-\nu)\Gamma(1 + \nu)} - \frac{C_2}{2(-1)^\nu}\right\} \ln(1 + z) + O(1) \quad \text{as } z \to -1.$$
APPENDIX C
Heat Kernel Calculations

In this appendix, we present the general results for the first few heat kernel coefficients. We also collect some of the intermediate steps involved in the calculation of the heat kernel coefficients for higher-spin fields. Using the heat kernel expansion, it is possible to show that the first few $a_k$ are given by:

$$
\begin{align*}
    a_0 &= I \\
    a_1 &= -\frac{1}{6}(RI + 6X) \\
    a_2 &= \frac{1}{360} \left( 2R_{ABMN}R^{ABMN} - 2R_{MN}R^{MN} + 5R^2 - 12\Box R \right) I \\
        &\quad + \frac{1}{6}RX + \frac{1}{2}X^2 - \frac{1}{6}\Box X + \frac{1}{12}Y_{MN}Y^{MN}
\end{align*}
$$

(C.1)
and

\[
a_3 = \frac{1}{7!} \left( -18 \Box^2 R + 17 D_M R D^M R - 2 D_L R_{MN} D^L R^{MN} - 4 D_L R_{MN} D^N R^{ML} \\
+ 9 D_K R_{MNL} D^K R^{MNL} + 28 R \Box R - 8 R_{MN} \Box R^{MN} + 24 R^M_N D^L D^N R_{ML} \\
+ 12 R_{MNL} \Box R^{MNL} - \frac{35}{9} R^3 + \frac{14}{3} R R_{MN} R^{MN} - \frac{14}{3} R R_{ABMN} R^{ABMN} \\
+ \frac{208}{9} R^M_N R_{ML} R^{NL} - \frac{64}{3} R^{MN} R^{KL} R_{MKNL} + \frac{16}{3} R^M_N R_{MKLP} R^{NKL} \\
- \frac{44}{9} R^{AB}_{MN} R_{ABKL} R^{MNKL} - \frac{80}{9} R^A_M R_{AKMP} R^{BKP} \right) I \\
+ \frac{1}{360} \left( 8 D_M Y_{NK} D^M Y^{NK} + 2 D^M Y_{NM} D_K Y^{NK} + 12 Y^{MN} \Box Y_{MN} \\
- 12 Y^N_M Y_K^M Y^K_M - 6 R^{MNKL} Y_{MN} Y_{KL} + 4 R^M_N Y_{MK} Y^{NK} \\
- 5 R Y^{MN} Y_{MN} - 6 \Box^2 X + 60 X \Box X + 30 D_M X D^M X - 60 X^3 \\
- 30 X Y^{MN} Y_{MN} + 10 R \Box X + 4 R^{MN} D_M D_N X + 12 D^M R D_M X - 30 X^2 R \\
+ 12 X \Box R - 5 X R^2 + 2 X R_{MN} R^{MN} - 2 X R_{ABMN} R^{ABMN} \right),
\]

(C.2)

where \( I \) is the \( N \times N \) identity matrix for the space of fields of interest, and \( Y_{MN} \) is the matrix-valued quantity defined by the expression \( Y_{MN}^i \phi^j = [D_M, D_N] \phi^i \).

C.1 Antisymmetric Tensors

The differential operator which possesses the correct symmetries for the massless antisymmetric tensor field can be read off from eq. (6.46), giving

\[
\Delta_{MN}^{PQ} = -\Box \delta^{MN}_{PQ} + (R^M_P N_Q - R^N_P M_Q) - \frac{1}{2}(R^M_P \delta^N_Q - R^N_P \delta^M_Q + R^N_Q \delta^M_P - R^M_Q \delta^N_P),
\]

(C.3)

and so

\[
X^{MN}_{PQ} = (R^M_P N_Q - R^N_P M_Q) - \frac{1}{2}(R^M_P \delta^N_Q - R^N_P \delta^M_Q + R^N_Q \delta^M_P - R^M_Q \delta^N_P).
\]

(C.4)
Similarly $Y_{MN}$ is given by

$$
(Y_{MN})^{AB}_{CD} = \frac{1}{2}(R^A_{CMN}\delta^B_D \mp R^A_{DMN}\delta^B_C + R^B_{DMN}\delta^A_C \mp R^B_{CMN}\delta^A_D), \tag{C.5}
$$

where for later convenience we also give here the result (bottom sign) for the rank-2 symmetric tensor field. Using these expressions for $X$ and $Y_{MN}$, and taking there to be $N_a$ such antisymmetric gauge potentials, we obtain

$$
\begin{align*}
\text{tr}_B(X) &= N_a(2 - n)R \\
\text{tr}_B(X^2) &= N_a\left[R^{}_{ABMN}R^{ABMN} + (n - 6)R_{MN}R^{MN} + R^2\right] \\
\text{tr}_B(Y_{MN}Y^{MN}) &= N_a(2 - n)R^{}_{ABMN}R^{ABMN}, \tag{C.6}
\end{align*}
$$

and so are led to the following results for $\text{tr}(a_k)$:

$$
\begin{align*}
\text{tr}_B(a_0) &= \frac{N_a}{2}n(n - 1) \\
\text{tr}_B(a_1) &= -\frac{N_a}{12}(n^2 - 13n + 24)R \\
\text{tr}_B(a_2) &= N_a\left[\frac{1}{360}(16 - n)(15 - n)R^{}_{ABMN}R^{ABMN} \\
&\quad - \frac{1}{360}(n^2 - 181n + 1080)R_{MN}R^{MN} + \frac{1}{144}(n^2 - 25n + 120)R^2 \\
&\quad - \frac{1}{60}(n^2 - 11n + 20)\Box R\right]. \tag{C.7}
\end{align*}
$$

C.2 Spin 3/2

For a vector-spinor the Lorentz generators are

$$
(J_{AB})^C_D = -\frac{i}{2}\Gamma_{AB}^C\delta^D_D - iI(\delta^C_A\eta_{BD} - \delta^C_B\eta_{AD}), \tag{C.8}
$$
where $I$ is the $N_{3/2} \times N_{3/2}$ identity matrix, corresponding to the $N_{3/2} = N_{3/2} d_s$ (unwritten) non-vector components of $\psi_M$. (Recall $d_s = 2^{[n/2]+1}/\zeta$, where $\zeta = 1, 2$ and 4 for Dirac, Weyl (or Majorana) and Majorana-Weyl fermions.) Using the identity $\mathcal{D}^2 = \Box + \frac{1}{4} [\Gamma^M, \Gamma^N][D_M, D_N]$, we find

$$[\mathcal{D}^2]_{AB} = \left( \Box + \frac{1}{4} R - \frac{i}{2} F^a_{CD} \Gamma^{CD} t_a \right) \delta^A_B - \frac{1}{2} R^A_{BCD} \Gamma^{CD}. \quad \text{(C.9)}$$

For simplicity of notation, we have suppressed writing the various identity matrices that appear in the above expression. From this we may read off the expression for $X$, given by

$$X^A_B = \left( -\frac{1}{4} R + \frac{i}{2} F^a_{CD} \Gamma^{CD} t_a \right) \delta^A_B + \frac{1}{2} R^A_{BMN} \Gamma^{MN}. \quad \text{(C.10)}$$

Taking appropriate traces, we obtain the results

$$\text{tr}_{VS} (X) = -\frac{n}{4} N_{3/2} R$$

$$\text{tr}_{VS} (X^2) = N_{3/2} \left[ \frac{n}{16} R^2 + \frac{1}{2} R_{ABMN} R^{ABMN} \right] + \frac{n d_s g_a^2}{2} C(\mathcal{R}_{3/2}) F^a_{MN} F^a_{MN}$$

$$\text{tr}_{VS} (Y_{MN} Y^{MN}) = -N_{3/2} \left( 1 + \frac{n}{8} \right) R_{ABMN} R^{ABMN} - n d_s g_a^2 C(\mathcal{R}_{3/2}) F^a_{MN} F^a_{MN}. \quad \text{(C.11)}$$

$\mathcal{R}_{3/2}$ denotes, as usual, the Dynkin index for the representation of the gauge group carried by the spin-3/2 fields.

Combining these results, and remembering to multiply (as for the spin-1/2 case)
eq. (C.1) by an overall factor of $1/2$, we find

\[
\begin{align*}
\text{tr}_{VS}(a_0) &= \frac{n}{2} N_{3/2} \\
\text{tr}_{VS}(a_1) &= \frac{n}{24} N_{3/2} R \\
\text{tr}_{VS}(a_2) &= \frac{N_{3/2}}{360} \left[ \left( 30 - \frac{7n}{8} \right) R_{ABMN} R^{ABMN} - n R_{MN} R^{MN} + \frac{5n}{8} R^2 + \frac{3n}{2} \Box R \right] \\
&\quad + \frac{nd_s g_a^2}{12} C(R_{3/2}) F^a_{MN} F^a_{MN}.
\end{align*}
\]

(C.12)

**C.3 Spin 2**

From eq. (6.89), the symmetric traceless differential operator appropriate for $\phi_{MN}$ is seen to be

\[
\Delta^{MN}_{PQ} = -\left[ \Box + (R + 2\Lambda) \right] \delta^{MN}_{PQ} + (R^M_{\ P \ Q} + R^N_{\ P \ Q}) - \frac{4}{n} (g_{PQ} R^{MN} + g^{MN} R_{PQ}) \\
+ \frac{1}{2} (R^M_{\ P} \delta^N_{Q} + R^N_{\ P} \delta^M_{Q} + R^N_{\ Q} \delta^M_{P} + R^M_{\ Q} \delta^N_{P}) + \frac{4}{n^2} g^{MN} g_{PQ} R,
\]

(C.13)

from which the expression for $X$ can be read off directly. Taking traces of the relevant quantities, we find

\[
\begin{align*}
\text{tr}_{\text{symtr}}(X) &= N_2 \left[ -\frac{1}{2n} (n+2)(n^2-3n+4)R - (n+2)(n-1)\Lambda \right] \\
\text{tr}_{\text{symtr}}(X^2) &= N_2 \left[ 3R_{ABMN} R^{ABMN} + \frac{1}{n} (n^2-2n-32) R_{MN} R^{MN} \\
&\quad + \frac{1}{2n^2} (n^4-3n^3+16n+32) R^2 \\
&\quad + \frac{2}{n} (n+2)(n^2-3n+4) \Lambda R + 2(n+2)(n-1)\Lambda^2 \right] \\
\text{tr}_{\text{symtr}}(Y_{MN} Y^{MN}) &= -N_2 (n+2) R_{ABMN} R^{ABMN}.
\end{align*}
\]

(C.14)
Applying eq. (C.1), we arrive at the following expressions for $\text{tr} [a_k]$:

$$
\begin{align*}
\text{tr}_{\text{symtr}}(a_0) &= \frac{N_2}{2} (n + 2)(n - 1) \\
\text{tr}_{\text{symtr}}(a_1) &= N_2 \left[ \frac{1}{12n} (n + 2)(5n^2 - 17n + 24)R + (n + 2)(n - 1)\Lambda \right] \\
\text{tr}_{\text{symtr}}(a_2) &= N_2 \left[ \frac{1}{360} (n^2 - 29n + 478)R_{ABMN}R^{ABMN} - \frac{1}{360n} (n^3 - 179n^2 + 358n + 5760)R_{MN}R^{MN} \\
&\quad + \frac{1}{144n^2} (25n^4 - 95n^3 + 22n^2 + 480n + 1152)R^2 \\
&\quad + \frac{1}{30n} (n + 2)(2n^2 - 7n + 10)\Box R \\
&\quad + \frac{1}{6n} (n + 2)(5n^2 - 17n + 24)\Lambda R + (n^2 + n - 2)\Lambda^2 \right].
\end{align*}
$$
In this appendix we slightly generalize the treatment of massless and massive spin-3/2 particles given in the main text to include the possibility that the Lagrangian density includes a nonzero cosmological constant (or a nontrivial scalar potential once the background scalar field equations are satisfied). As discussed in §6.3, the nonzero cosmological constant implies particles in a supermultiplet need no longer be degenerate in mass, and so we calculate here how this effect plays out for the gravitino. For instance, this case arises in four dimensions, where an anti-de Sitter (AdS) cosmological constant term in the action is not precluded by supersymmetry itself. Even though the application of most interest is to four dimensions, we carry the spacetime dimension $n$ as a variable in this appendix in case more general applications of the expressions derived here should become of interest.

**Massless Gravitino**

In this case we take the spin-2 field to be described by the Einstein-Hilbert action supplemented by the cosmological term, which in our conventions is

$$ e^{-1} \mathcal{L}_{EH} = -\frac{1}{2\kappa^2}(R + 2\Lambda). \quad (D.1) $$
Supersymmetry then requires the Lagrangian density for the spin-3/2 particle to be described by

\[ e^{-1} \mathcal{L}_S = -\frac{1}{2} \left( \bar{\psi}_M \Gamma^{MNP} D_N \psi_P - m_{3/2} \bar{\psi}_M \Gamma^{MN} \psi_N \right), \]  

(D.2)

where we shall see how the parameter \( m_{3/2} \) is related by supersymmetry to the cosmological constant. The presence of this ‘mass’ term does not mean that supersymmetry is broken; rather it is required in order to ensure that the gravitino/graviton action remains gauge invariant.

The combined gravitino-graviton Lagrangian is invariant under the linearized supersymmetry transformations

\[
\delta e_A = -\frac{\kappa}{4} \bar{\psi}_M \Gamma^A \epsilon + \text{c.c.},
\]

\[
\delta \psi_M = \frac{1}{\kappa} \left( D_M + \frac{1}{(n-2)} m_{3/2} \Gamma_M \right) \epsilon,
\]

(D.3)

provided \( m_{3/2} \) is related to \( \Lambda \) by

\[
\Lambda = -\frac{2(n-1)}{(n-2)} m_{3/2}^2.
\]

(D.4)

Notice that for any \( n > 2 \) this requires \( \Lambda < 0 \), which corresponds to having anti-de Sitter space as the maximally-symmetric background solution. In 4D this reduces to the standard result \( \Lambda_4 = -3 m_{3/2}^2 \) [141, 142].

To put the spin-3/2 Lagrangian into a form for which the general expressions for the Gilkey coefficients apply, we now use the gauge-averaging term

\[ e^{-1} \mathcal{L}^{gf}_{VS} = -\frac{1}{2} \xi_{3/2} \left( \bar{\psi} \right)(\gamma)(\gamma) \left( \Gamma \cdot \psi \right). \]

(D.5)
After making the field redefinition \( \psi_M \rightarrow \psi_M + A \Gamma_M \Gamma \cdot \psi \), we find that the following choices for \( A, \xi, \) and \( \gamma \)

\[
A = \frac{1}{2-n}, \quad \xi^{-1} = \frac{2-n}{4}, \quad \gamma = \left( \frac{n}{2-n} \right) m_{3/2},
\]

lead to the an expression for the vector-spinor Lagrangian given by

\[
e^{-1} (L_{VS} + L_{gF}^{if}) = -\frac{1}{2} \psi_M (\partial + \frac{m_{3/2}}{2}) \psi^M.
\]

Following the analogous procedure in the main text, we obtain the result for the vector-spinor field in the presence of a cosmological constant:

\[
\text{tr}_{VS}(a_0) = \frac{n}{2} N_{3/2}
\]

\[
\text{tr}_{VS}(a_1) = n N_{3/2} \left( \frac{1}{24} R - \frac{1}{2} m_{3/2}^2 \right)
\]

\[
\text{tr}_{VS}(a_2) = \frac{N_{3/2}}{360} \left[ \left( 30 - \frac{7n}{8} \right) R_{ABMN} R^{ABMN} - n R_{MN} R^{MN} + \frac{5n}{8} R^2 + \frac{3n}{2} \Box R \right]
\]

\[
+ \frac{nd_s g_a^2}{12} C(R_{3/2}) F_{MN}^a F^{aMN} + \frac{n}{24} N_{3/2} \left( -m_{3/2}^2 R + 6 m_{3/2}^4 \right)
\]

with \( m_{3/2}^2 \) defined by eq. (D.4).

The ghost action may be read from the supersymmetry transformation rules, from which we see that \( \delta (\Gamma \cdot \psi) = \kappa^{-1} [\partial + \frac{n}{n-2} m_{3/2}] \epsilon \), and so we find two bosonic, Faddeev-Popov spinor ghosts with the Lagrangian

\[
e^{-1} L_{LVFPgh} = -\bar{\omega}^i \left( \partial + \frac{nm_{3/2}}{n-2} \right) \omega_i.
\]

This has the same form as the spin-1/2 Lagrangian, eq. (6.30), although with a \( \Lambda \)-dependent mass. In order to use this we require the following spin-1/2 results for
the Gilkey-DeWitt coefficients quoted in the main text, generalized to include the fermion mass, $m^2$, inside $X$:

\[
\begin{align*}
\text{tr}_{1/2}(a_0) &= \frac{\mathcal{N}_{1/2}}{2} \\
\text{tr}_{1/2}(a_1) &= \frac{\mathcal{N}_{1/2}}{24} (R - 12m^2) \\
\text{tr}_{1/2}(a_2) &= \frac{\mathcal{N}_{1/2}}{360} \left[ -\frac{7}{8} R_{ABMN} R^{ABMN} - R_{MN} R^{MN} + \frac{5}{8} (R - 12m^2)^2 + \frac{3}{2} \Box R \right] \\
&\quad + \frac{d_s g_a^2}{12} C(R_{1/2}) F^a_{MN} F^a_{MN}.
\end{align*}
\] (D.10)

The Faddeev-Popov ghost result for $\text{tr}[a_k]$ is then obtained by multiplying these expressions by $-2$, and specializing to the ‘mass’ $m = n m_{3/2} / (n - 2)$.

The use of the operator $(\not D + \gamma)$ in the gauge-fixing Lagrangian, eq. (D.5), leads to a bosonic, Nielsen-Kallosh ghost. Rewriting $\gamma$ in terms of $m_{3/2}$, we see that the Nielsen-Kallosh ghost has the Lagrangian

\[
\mathcal{L}_{LVNKgh} = -\omega \left( \not D - \frac{n m_{3/2}}{n - 2} \right) \omega.
\] (D.11)

This ghost therefore contributes $-1$ times the spin-$1/2$ result to $\text{tr}[a_k]$, with $m = -n m_{3/2} / (n - 2)$.

Adding the vector-spinor result together with its associated ghosts, we obtain the following contribution to $\text{tr}[a_k]$ by physical spin-$3/2$ states in the presence of a
Massive Gravitino

This section follows closely the procedure outlined in the massive spin-3/2 section of the main text. Starting from eq. (6.71), which was our ansatz for a massive spin-3/2 Lagrangian, we again find that this Lagrangian can be made invariant under the supersymmetry transformations

\[ \delta \psi_M = \frac{1}{\kappa} D_M \epsilon + \mu \Gamma_M \epsilon \quad \text{and} \quad \delta \chi = f \epsilon. \]  

(D.13)

In this case, however, \( f \) is given by

\[ f^2 = (n - 1)(n - 2) \mu^2 + \frac{\Lambda}{2\kappa^2}, \]  

(D.14)

while all other equations in eq. (6.73) remain unchanged. The dependence of \( f \) on \( \Lambda \) is required in order to cancel the variation of the \( \Lambda \) term in the Einstein-Hilbert action.
Again, following the procedure of the main text, we add a gauge-fixing term, eq. (6.76), and perform a field redefinition, eq. (6.79), in order to put the Lagrangian into the form
\[
e^{-1} (\mathcal{L}_{mV_S} + \mathcal{L}^{\rho}_m) = -\overline{\psi} (\partial + m'_{3/2}) \psi^{\mu} - \overline{\chi} (\partial + m'_{1/2}) \chi'. \tag{D.15}
\]
The parameters in the gauge-fixing Lagrangian and in the field redefinitions can be written in terms of \(M\) and \(\hat{M}\), defined as
\[
M = (n - 2)\kappa \mu \quad \text{and} \quad \hat{M} = \sqrt{M^2 - \frac{2\Lambda}{n - 2}}. \tag{D.16}
\]
With these definitions, we find
\[
A = \sqrt{1 + \beta^2}, \quad B = -\frac{\beta}{2} \sqrt{n - 2}, \quad C = -\frac{1}{2}, \quad D = 0,
\]
\[
\alpha = -\frac{1}{2} \sqrt{(n - 2)(1 + \beta^2)}, \quad \beta = \left[\frac{1}{2} \left(\frac{n}{n - 2}\right) \frac{M}{\hat{M}} - \frac{1}{2}\right]^{1/2},
\]
\[
m'_{1/2} = -\gamma = \hat{M}, \quad m'_{3/2} = M. \tag{D.17}
\]
For the case \(\Lambda = 0\), these expressions reduce to those given in eq. (6.80). There is a possible subtlety in the above solution, which comes about because of our simplifying assumption to take all free parameters to be real. We see that for certain choices of \(M\) and \(\Lambda\), it’s possible that some of the parameters will be imaginary. However, in the situations for which our results apply we expect that \(M \gg |\Lambda|\), and so in these cases this problem will not arise.

From the gauge-fixing condition, we see that there are two Faddeev-Popov ghosts, each with mass \(\hat{M}\), and one Nielsen-Kallosh ghost, with mass \(-\hat{M}\). The
one loop effective action for the ghosts is thus given by

\[ i \Sigma_{1/2} = \frac{1}{4} \text{Tr} \log \left( \hat{M}^2 - \hat{\rho}^2 \right) \]

\[ = \frac{1}{4} \text{Tr} \log \left( M^2 - \frac{2\Lambda}{n-2} - \rho^2 \right). \]  

(D.18)

As usual, we factor the \( M^2 \) dependence out of our definition of \( X \), and so obtain

\[ X = -\frac{1}{4} R + \frac{i}{2} \Gamma^{AB} F_{a}^{AB} t_{a} - \frac{2\Lambda}{n-2}. \]  

(D.19)

The contribution to the Gilkey coefficients coming from the three ghosts is thus obtained by multiplying eq. (D.10) by \(-3\), with \( m^2 = -2\Lambda/(n-2) \). Similarly, the Goldstone fermion contribution is also given by eq. (D.10), again with \( m^2 = -2\Lambda/(n-2) \). The contribution from the vector spinor is unchanged from the massless case considered in the main text, and so its Gilkey coefficients are given by eq. (C.12).

Summing these results, we arrive at the expression for a massive gravitino in a background spacetime having nonzero cosmological constant:

\[ \text{tr} m_{3/2}(a_0) = \frac{N_{3/2}}{2} (n-2) \]

\[ \text{tr} m_{3/2}(a_1) = \frac{N_{3/2}}{24} \left( (n-2)R - \frac{48\Lambda}{n-2} \right) \]

\[ \text{tr} m_{3/2}(a_2) = \frac{N_{3/2}}{360} \left[ \left( 30 - \frac{7}{8}(n-2) \right) R_{ABMN} R^{ABMN} - (n-2) R_{MN} R^{MN} 

+ \frac{5}{8}(n-2)R^2 + \frac{3}{2}(n-2)\Box R - \frac{60\Lambda R}{(n-2)} - \frac{720\Lambda^2}{(n-2)^2} \right] 

+ \frac{g_2^2}{12} (n-2) d_s C(R_{3/2}) F_{a}^{a} F_{a}^{MN}. \]  

(D.20)
REFERENCES


