Massive twistor particle with spin generated by Souriau–Wess–Zumino term and its quantization

Sergey Fedoruk a,1, Jerzy Lukierski b,∗

a Bogoliubov Laboratory of Theoretical Physics, JINR, Joliot-Curie 6, 141980 Dubna, Moscow region, Russia
b Institute for Theoretical Physics, University of Wrocław, pl. Maxa Borna 9, 50-204 Wrocław, Poland

1. Introduction

In order to introduce in geometric way the spin degrees of freedom one has to enlarge the space–time description of relativistic point particles. A well-known introduction of spin degrees of freedom is provided by superspace extension of space–time, with anticommuting Grassmann algebra attached to each space–time point. Another way of introducing the geometric spin degrees of freedom is to consider twistorial particle models, with primary spinorial coordinates. The single twistor space has the degrees of freedom describing massless particles with arbitrary helicity [1–3]. In order to describe in twistor space the massive particles with arbitrary spin one should consider particle models in two-twistor space [2–5].

The Penrose twistor approach [6,1,2,7] has been shown to be a powerful tool for the analysis of different point-like and extended objects. Recently some renaissance of the twistor method was connected with successful application of twistors in description of amplitudes in (super)Yang–Mills and (super)gravity theories [4,5,15–21]. It should be added that the twistor approach has been considered mainly for massless (super)particles (see e.g. [14] for approximately complete list of more references on this subject), but its application to massive particles, especially with non-zero spin, were investigated in rather limited number of papers [4,5,15–21].

The description of particles with a nonconformal mass parameter and nonvanishing spin requires additional degrees of freedom which has been studied in space–time as well as in the twistorial approach. In space–time formalism one introduces an additional Pauli–Lubanski spin fourvector \( w_\mu \) which satisfies the subsidiary conditions [22,2,5]

\[
\begin{align*}
    w_\mu p^\mu = 0, \\
    w^2 \equiv w_\mu w^\mu = -m^2 j^2, \\
    \epsilon_\mu
\end{align*}
\]

(1.1)

with relativistic spin-shell described by \( j^2 \) and fourmomenta satisfying the mass-shell condition \( p^2 = m^2 \). Alternatively, in twistor approach the two-twistor space is required to describe the phase space of massive particle with arbitrary spin, and one constructs from two twistors the composite spin fourvector \( w_\mu \) satisfying the constraints (1.1).

In our presentation we shall generalize from \( D = 3 \) to \( D = 4 \) the arguments of Mezincescu, Routh and Townsend [21], who demonstrated that for \( D = 3 \) massive particle the nonvanishing spin is generated in phase space \( (x^\mu, p_\nu) \) by adding the term in the action described by the pullback to the world-line of the following symplectic \( D = 3 \) two-form

\[
\Omega^2_{(D=3)} = \frac{s}{2(p^2)^{3/2}} e^{H_{1D}} p_\mu dp_\nu \wedge dp, \\
\]

(1.2)

satisfying \( d\Omega^2_{(D=3)} = 0 \). It appears that such a term describes in \( D = 3 \) action the Lorentz–Wess–Zumino (LWZ) term \( \Omega^4_{(D=3)} \) which
is the solution of the equation \( \Omega_2^{(D=3)} = d\Omega_1^{(D=3)} \) \cite{23,24}. Calculating in two-twistor formulation the LWZ term one can see that it generates the twistor shift which modifies standard Penrose incidence relations as follows

\[
\alpha^j_a = x_a \beta^j + \frac{5}{m} \gamma^j_a. \tag{1.3}
\]

Using modified incidence relations (1.3) one can obtain in the twistorial action of \( D = 3 \) massive particle with spin, the kinetic term for twistors, which implies standard twistor Poisson brackets (PB). Moreover as was shown in \cite{21}, the eigenvalues of the Casimir operators of \( D = 3 \) Poincare algebra correspond to massive states with the \( D = 3 \) counterpart of spin \( s \). Note that twistorial shift in twistorial models is more important for massive particles, because for \( D = 3, 4 \) massless particles it does not produce any change of the particle helicities \cite{25}.

In this paper we provide an analogous scheme by introducing in place of (1.2) for \( D = 4 \) the symplectic two-form introduced by Souriau \cite{26,22,27}

\[
\Omega_2^{(D=4)} = \frac{1}{2m^2} \varepsilon_{\mu
u\rho\sigma} \omega^{\rho\sigma} \times \left( \frac{1}{m^2} dP_\mu \wedge dP_\nu + \frac{1}{w^2} dw_\mu \wedge dw_\nu \right). \tag{1.4}
\]

where \( \omega^{\rho\sigma} \) is the Pauli–Lubanski vector satisfying the relations (1.1). In Section 2 we consider first the \( D = 4 \) spinless massive particle and we recall that such a model can be formulated in three equivalent ways (see e.g. \cite{20})

- by using relativistic phase space description \((x_\mu, p_\mu)\)
- by employing mixed space–time/spinorial description (Shirafuji formulation \cite{28})
- by using two-twistor framework.

We obtain that in \( D = 4 \) two-twistor space our model is described by free action with added six constraints: two related with mass-shell condition, three describing vanishing spin and sixth introducing vanishing \( U(1) \) charge.

In Section 3 we add in \( D = 4 \) space-time formulation the Souriau–Wess–Zumino (SWZ) topological term which depends on the spin four-vector \( w_\mu \) (see (1.1)). After passing to the spinorial description one can calculate the SWZ term by the pullback to the world-line of the Souriau symplectic two-form (1.4). Subsequently, using spin dependent twistor shift we obtain the model depending on two-twistor coordinates and auxiliary spin three-vector which spans the coordinates of two-sphere. We review how to derive from topological action such semi-dynamical spin variables, which satisfy \( SU(2) \) PB bracket relations. In two-twistor description the model is described by free bilinear action with four first class and two second class constraints imposed by Lagrange multiplier method, what leaves eight unconstrained physical degrees of freedom. Further, in Section 4, using the two-twistor formulation of our particle model, we obtain the relativistic wave functions with mass and properly quantized spin values. In final Section 5 we summarize main results and point out some possible generalizations of presented scheme.

2. Massive spinless particle

The three equivalent descriptions of massive spinless particle are known but we present them here in order to prepare the ground for the generalization in Section 3 to the case of the massive particle with spin.

Relativistic phase space formulation of massive spinless particle is defined by well-known action

\[
\tilde{S}_1 = \int d\tau \left[ p_\mu \dot{x}_\mu + e(p_\mu p_\mu - m^2) \right]. \tag{2.1}
\]

Here, \( x_\mu(t), \mu = 0, 1, 2, 3 \) are the coordinates of position, \( \dot{x}_\mu = dx_\mu/d\tau \) and \( p_\mu \) is fourvector of momenta. We use the metric with plus time signature, \( g_{\mu\nu} = \delta_{\mu\nu}(+-+--) \).

In order to pass to mixed space–time/spinorial Shirafuji formulation we should use the Cartan–Penrose formula expressing the relativistic fourmomenta by a pair of Weyl commuting spinors \((k = 1, 2)\)

\[
p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}, \tag{2.2}
\]

where

\[
\lambda_\alpha = (\lambda_1, \lambda_2), \quad \tilde{\lambda}_{\dot{\alpha}} = (\tilde{\lambda}_{\dot{1}}, \tilde{\lambda}_{\dot{2}}). \tag{2.3}
\]

Massive spinless particle dynamics is described by an extension of Shirafuji approach \cite{28}

\[
\tilde{S}_2 = \int d\tau \left[ \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} e_{\alpha\dot{\alpha}} + g(\lambda_\alpha \lambda_{\alpha\dot{\alpha}} - 2M) + \tilde{g}(\tilde{\lambda}_{\dot{\alpha}} \tilde{\lambda}_{\dot{\alpha}} - 2\tilde{M}) \right], \tag{2.4}
\]

where \( e_{\alpha\dot{\alpha}} \) is a complexified mass parameter. In action (2.4) there are incorporated the mass-shell constraints\(^3\)

\[
\lambda_\alpha \lambda_{\alpha\dot{\alpha}} = M^2, \quad \tilde{\lambda}_{\dot{\alpha}} \tilde{\lambda}_{\dot{\alpha}} = \tilde{M}^2 \tag{2.5}
\]

or equivalently

\[
\lambda_\alpha \lambda_{\alpha\dot{\alpha}} = M^2, \quad \tilde{\lambda}_{\dot{\alpha}} \tilde{\lambda}_{\dot{\alpha}} = \tilde{M}^2 \tag{2.6}
\]

Due to the constraints (2.5) we have the following real mass-shell condition \((p_{\alpha\dot{\alpha}} p_{\alpha\dot{\alpha}} = 2|M|^2)\)

\[
p_{\alpha\dot{\alpha}} p_{\alpha\dot{\alpha}} = 2|M|^2 \tag{2.7}
\]

and comparing with (2.1) we get

\[
m = \sqrt{2|M|}. \tag{2.8}
\]

The pair of spinors \( \lambda_\alpha, \tilde{\lambda}_{\dot{\alpha}} \) describe one-half of two-twistor components. Remaining twistorial components are defined by the Penrose incidence relations (see e.g. \cite{6,17})

\[
\mu_{\alpha\dot{\alpha}} = \tilde{\alpha}_{\dot{\alpha}} \tilde{\lambda}_{\dot{\alpha}} \lambda_\alpha, \quad \mu_{\alpha\dot{\alpha}} = \tilde{\lambda}_{\dot{\alpha}} \tilde{\lambda}_{\dot{\alpha}} \lambda_\alpha. \tag{2.9}
\]

The relations (2.2) and (2.9) link the Poisson brackets (PB) of space–time and twistor space approaches. Namely, when the relations (2.9) are satisfied then

\[
p_{\alpha\dot{\alpha}} \tilde{\alpha}_{\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} \tilde{\lambda}_{\dot{\alpha}} \tilde{\lambda}_{\dot{\alpha}} + \tilde{\lambda}_{\dot{\alpha}} \tilde{\lambda}_{\dot{\alpha}} \tilde{\lambda}_{\dot{\alpha}} \tilde{\lambda}_{\dot{\alpha}} + \text{(total derivative)} \tag{2.10}
\]

and we get the kinematic terms which lead to canonical PB in relativistic phase space as well as in two-twistor space.

If space–time coordinates are real twistor incident relations (2.9) lead to the following conditions

\[
\lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} = \tilde{\lambda}_{\dot{\alpha}} \tilde{\lambda}_{\dot{\alpha}} \tilde{\lambda}_{\dot{\alpha}} = 0. \tag{2.11}
\]

\(^2\) We shall use \( D = 4 \) two-spinor notation, i.e. \( p_{\alpha\dot{\alpha}} = \frac{1}{2} \varepsilon_{\alpha\dot{\alpha}} p_{\mu\nu} \), \( p_\mu = \frac{1}{2} \varepsilon_{\alpha\dot{\alpha}} p_{\mu\nu} \), where \( (\sigma^\mu)_{\alpha\dot{\alpha}} = (1, \bar{\sigma})_{\alpha\dot{\alpha}}, (\bar{\sigma}^\mu)_{\alpha\dot{\alpha}} = e^{\sigma^\mu} (\sigma^\mu)_{\alpha\dot{\alpha}} \), \( \bar{\sigma}^\mu = \frac{1}{2} i \bar{\sigma} (\sigma_5)\bar{\sigma}^\mu, (\sigma^\mu)_{\alpha\dot{\alpha}} = e^{\sigma^\mu} (\sigma^\mu)_{\alpha\dot{\alpha}} \), \( \bar{\sigma}^\mu = \bar{\sigma}^\mu (\sigma^\mu)_{\alpha\dot{\alpha}} \). So, we have \( p^{\alpha\dot{\alpha}} p_{\gamma\delta} = p^{\alpha\dot{\alpha}} p_{\gamma\delta} \). We use weight coefficient in (anti)symmetrization, i.e.

\[
A_{\alpha\dot{\alpha}} B_{\gamma\delta} = \frac{1}{2} (A_{\alpha\dot{\alpha}} B_{\gamma\delta} + A_{\gamma\delta} B_{\alpha\dot{\alpha}}), A_{\alpha\dot{\alpha}} B_{\gamma\delta} = \frac{1}{2} (A_{\alpha\dot{\alpha}} B_{\gamma\delta} - A_{\gamma\delta} B_{\alpha\dot{\alpha}}).
\]

\(^3\) We go up and down the indices \( \alpha, \beta, \gamma, \ldots \) and \( i, j, k, \ldots \) by antisymmetric tensors \( \epsilon_{\alpha\beta\gamma}, \epsilon_{ij}, \epsilon_{ij}, \epsilon_{ij}, \epsilon^{ij} \), \( A_{\alpha\beta} = \epsilon_{\alpha\beta\gamma} A^\gamma, A^{ij} = \epsilon^{ij} A_{\gamma\delta}, A_{\alpha\dot{\beta}} = \epsilon_{\alpha\dot{\beta}ij}, A_{\alpha\dot{\beta}} = \epsilon_{\alpha\dot{\beta}ij} \). We take \( \epsilon_{ij} = \epsilon^{ij} = 1 \).
Due to Eq.(2.8) and the constraints (2.13) the vectors (2.21) satisfy
\[ \tilde{S}_3 = \int d\tau \left[ \tilde{\alpha}_a \tilde{A}_k^a + \tilde{\delta}_{ak} \tilde{\mu}_k^a + gT + \tilde{g} \tilde{T} + \lambda^a S' + \Lambda S \right] \] (2.12)

incorporating the mass constraints (see also (2.5))
\[ T \equiv \lambda^{ak} \lambda_{ak} - 2M \approx 0, \quad \tilde{T} \equiv \lambda^{ak} \lambda_{ak} - 2\tilde{M} \approx 0 \] (2.13)
and the \( U(2) \) constraints
\[ S' = -i \left( \frac{1}{2} \left( \lambda^{ij}_a \tilde{\mu}^{ij}_k - \tilde{\delta}_{ak} \mu^{ij}_k \right) \right) (\tau')_k^{ij} \approx 0, \quad r = 1, 2, 3 \] (2.14)
\[ S = i \left( \lambda^{ij}_a \tilde{\mu}^{ij}_k - \tilde{\delta}_{ak} \mu^{ij}_k \right) \approx 0, \] (2.15)
which are the traceless and trace parts of the conditions (2.11) (in (2.14) the 2 \times 2 matrices \((\tau')_k^{ij} , i, k = 1, 2, r = 1, 2, 3 \) are the usual Pauli matrices).

The action (2.12) yields canonical twistor Poisson brackets
\[ \{ \lambda^{ij}_a, \lambda^{kl}_b \}_p = \delta^{il}_{a} \delta^{jk}_{b}, \quad \{ \mu^{ij}_a, \tilde{\delta}_{ak} \}_p = \delta^{il}_{a} \delta^{jk}_{b}, \] (2.16)

Then, nonvanishing Poisson brackets of the constraints (2.13), (2.14), (2.15) are
\[ \{ S^p, S' \}_p = 2iMT + 4iM, \quad \{ S, \tilde{T} \}_p = -2i\tilde{T} - 4i\tilde{M}, \] (2.17)
where the constraints \((S^p, S') \) describe \( U(2) \) PB algebra. One can check easily that we can choose four real constraints \( S', (MT + \tilde{M}) \) as first class constraints whereas two real constraints \( S \) and \( i(T - \tilde{T}) \) are second class. We get therefore six unconstrained degrees of freedom which coincides with number of degrees of freedom in standard space–time formulation (2.1) of massive particle.

In twistor formulation the Poincare generators \( p_{\mu} \) and \( m_{\mu \nu} = x_{\mu} p_{\nu} - x_{\nu} p_{\mu} \) are represented by the expressions (2.2) and
\[ m_{\mu \nu} = -\sigma_{\mu \nu} m_{a \beta} + \gamma_{\mu \nu} \bar{m}_{a \beta}, \quad m_{a \beta} = \lambda^{a}_{(\alpha} \mu_{\beta)k}, \]
\[ \bar{m}_{a \beta} = \lambda^{a}_{(\alpha} \bar{\mu}_{\beta)k}. \] (2.19)

Then, Pauli–Lubanski vector \( \mu_{\mu} = \frac{i}{2} \epsilon_{\mu \nu \rho} p^{\nu} m^{\rho} \) has the following twistor representation
\[ w_{\mu \nu} = S' u_{\mu \nu}, \] (2.20)
where \( S' \) are defined by (2.14) and (see e.g. [16])
\[ u_{\mu \nu} = \lambda^{a}_{(\nu} \left( \tau' \right)_{k}^{a} \lambda_{\nu} \] (2.21)
Due to Eq. (2.8) and the constraints (2.13) the vectors (2.21) satisfy
\[ u_{\mu}^{i} u_{\mu}^{j} = -m^2 S'. \] (2.22)
Therefore due to the constraints (2.14) and formulæ (2.20)–(2.22) in consistency with (1.1) we have
\[ p^{a}_{\mu} w_{\mu} = 0, \quad w^{a}_{\mu} w_{\mu} = -m^2 S' S', \] (2.23)
\[ S' S' = j^2. \] (2.24)
In conclusion the spin of the massive particle described by the twistor action (2.12) vanishes, i.e. we should put \( j = 0 \).

### 3. Massive particle with spin and twistor shift

We define \( D = 4 \) massive spin particle in space–time formulation with help of the action
\[ S_1 = \tilde{S}_1 + \int \Omega_1^{(D=4)} + \int d\tau \left[ (p^{a}_{\mu} w_{\mu} + I_2 (w^{a}_{\mu} w_{\mu} + m^2 j^2) \right], \] (3.1)
where first term \( \tilde{S}_1 \) is given by (2.1), one-form \( \Omega_1^{(D=4)} \) is defined by Souriau symplectic two-form (1.4) as follows
\[ \Omega_1^{(D=4)} = d\Omega_1^{(D=4)}, \] (3.2)
and the constraints on Pauli–Lubanski four-vector \( w^{a}_{\mu} \) are imposed by Lagrange multipliers.

Using the expressions (2.2), (2.20) and the property that \( M, \tilde{M} \) are constants we obtain the following twistorial expression for Souriau two-form
\[ \Omega_1^{(D=4)} = -\frac{i}{2MM} \tilde{S}' (\tau')_k^{ij} (\tilde{M} d\lambda^{ai} \wedge d\lambda_{ak} + M d\bar{\lambda}^{ai} \wedge d\bar{\lambda}_{ak}), \] (3.3)
where the three-vector \( S' \) satisfies the constraint (2.24).

We recall here that in the theory of massive relativistic free fields with spin the Pauli–Lubanski four-vector satisfies the relations (2.23) with \( S' \) described by the nondynamical matrix realization of \( SU(2) \) algebra (see e.g. [19,20]). Further, because \( \Omega_1^{(D=4)} \) in relation (3.3) due to (3.2) satisfies the condition \( d\Omega_1^{(D=4)} = 0 \), we can postulate that
\[ \dot{S}' = 0 \quad \rightarrow \quad S' = \tilde{S}', \quad s' \tilde{s}' = j^2 \] (3.4)
with the variables \( s', \tilde{s}' \in S^2 \) as classical counterparts of quantum spin components endowed with \( SU(2) \) PB relation
\[ \{ s'^p, \tilde{s}'_p \} = \epsilon^{pq} s'^q. \] (3.5)
Using (3.4) one sees easily that Liouville one-form \( \Omega_1^{(D=4)} \) satisfying (3.2) takes the form
\[ \Omega_1^{(D=4)} = -\frac{i}{2MM} s' (\tau')_k^{ij} (\tilde{M} d\lambda^{ai} \wedge d\lambda_{ak} + M d\bar{\lambda}^{ai} \wedge d\bar{\lambda}_{ak}), \] (3.6)
and the action (3.1) becomes the following Shirafuji-like action
\[ S_2 = \int d\tau \left[ \lambda^{a}_{(\nu} \lambda_{\nu)k} \tilde{s}' \right] \]
\[ + g(\lambda^{a}_{(\nu} \lambda_{\nu)k} - 2\tilde{M}) + \bar{g}(\bar{\lambda}^{a}_{(\nu} \bar{\lambda}_{\nu)k} - 2\tilde{M}) \]
\[ - \frac{i}{2MM} \int d\tau s' (\tau')_k^{ij} (M d\lambda^{ai} \wedge d\lambda_{ak} + M d\bar{\lambda}^{ai} \wedge d\bar{\lambda}_{ak}). \] (3.7)

It appears that due to relation (2.20) the constraint \( p^{a}_{\mu} w_{\mu} = 0 \) is valid as identity, thus the action (3.7) becomes the sum of the action (3.4) and the twistorial Souriau–Wess–Zumino topological term, represented by second integral in (3.7).

It should be stressed that the postulated PB relations (3.5) can be derived from the dynamical formulation if we supplement the action (3.7) with the following topological (Chern–Simons) coupling term (see e.g. [29–31])
\[ \Delta S_2 = \int d\tau [A'(s) \tilde{s}' + l(s' \tilde{s}' - j^2)], \] (3.8)
where three-potential \( A'(S) \) is such that
\[ F^{tr} = \tilde{A}', A'^{q} - \tilde{\alpha} A_{q} = -j e^{tr}[\tilde{s}' / |s'|^3. \] (3.9)
In order to derive the conditions $\tilde{s} = 0$ one should then pass to twistor formulation and fix the local $SU(2)$ gauge which are generated by first class constraints defined below (see (3.15)).

Let us eliminate the space–time variables $x^{\mu}$ and pass to pure twistorial formulation in two-twistor space. This requires to define second twistorial spinors. As first attempt one can use the relations (2.9) as defining the second pair of Weyl twistors $\mu^{\hat{\alpha}k}$, but if we use the spinor variables $\lambda^k_\beta$, $\mu^{\hat{\alpha}k}$ the terms with derivatives in the action (3.7)

$$\hat{\lambda}^k_\beta \hat{\chi}^{\hat{\beta}k} \hat{\lambda}^{\hat{\beta}k} - \frac{i}{2MM} s^r (\tau^r)^k \left( M \lambda^{\alpha i} \lambda_{\alpha k} + M \lambda^{\alpha i} \lambda_{\alpha k} \right)$$  \hspace{1cm} (3.10)

take the form

$$\hat{\lambda}^k_\beta \hat{\chi}^{\hat{\beta}k} \hat{\lambda}^{\hat{\beta}k} - \frac{i}{2MM} s^r (\tau^r)^k \left( M \lambda^{\alpha i} \lambda_{\alpha k} + M \lambda^{\alpha i} \lambda_{\alpha k} \right)$$ \hspace{1cm} (3.11)

The kinetic terms given by (3.11) show that the variables $\lambda^k_\beta$, $\mu^{\hat{\alpha}k}$ and their complex conjugates do not satisfy the canonical twistorial Poisson brackets.

In order to obtain the canonical twistorial PB we should redefine the half of the spinorial variables by the following modified incidence relations

$$\omega^{\hat{\alpha}k} = \mu^{\hat{\alpha}k} + \frac{i}{2M} s^r (\tau^r)^k \lambda^{\hat{\alpha}k} \lambda_k^r + \frac{i}{2M} s^r (\tau^r)^k \lambda^{\hat{\alpha}k} \lambda^{\hat{\beta}k}$$ \hspace{1cm} (3.12)

The formulae (3.12) describe the spin-dependent twistor shift from Weyl spinors $\lambda^k_\beta$, $\mu^{\hat{\alpha}k}$ to $\lambda^r_k$, $\omega^{\hat{\alpha}k}$. It appears that subsequently the kinetic terms (3.10) take (even without (3.4)) the standard form

$$\lambda^k_\beta \hat{\lambda}^{\hat{\beta}k} + \lambda_k^r \omega^{\hat{\alpha}k} + \text{(total derivative)} \hspace{1cm} (3.13)$$

We see that the variables $\lambda^k_\beta$, $\omega^{\hat{\alpha}k}$ and $\lambda_k^r$ are the canonical twistorial variables for particle with spin and they are obtained by the twistor shift applied to standard Penrose incidence relations for spinless particle (compare with (2.9)).

If the space–time coordinates are real, the twistor incidence relations (3.12) lead to the following conditions

$$\lambda^i_{\tau k} \hat{\lambda}^{\hat{k}i} - \lambda_{\hat{k}i} \omega^{\hat{i}k} = -i s^r (\tau^r)^k,$$ \hspace{1cm} (3.14)

which generalize the constraints (2.14) in the presence of nonvanishing spin variables $s^r$. Thus, in two-twistor formulation we have the constraints (2.13) and the modified constraints (2.14)–(2.15)$^4$

$$\mathcal{V} = V' + s^f$$

$$\mathcal{V} = \frac{i}{2} (\lambda^i_{\tau k} \hat{\lambda}^{\hat{k}i} - \lambda_{\hat{k}i} \omega^{\hat{i}k}) (\tau^r)^k + s^r \approx 0, \quad r = 1, 2, 3,$$ \hspace{1cm} (3.15)

$$\mathcal{V} = i (\lambda^i_{\tau k} \hat{\lambda}^{\hat{k}i} - \lambda_{\hat{k}i} \omega^{\hat{i}k}) \approx 0,$$ \hspace{1cm} (3.16)

which traceless and trace parts of the conditions (3.14) supplemented by the condition (2.24). Thus, pure twistorial formulation with semi-dynamical spinning variables is described by the action

$$S_3 = \int d\tau \left[ \lambda^i_{\tau k} \hat{\lambda}^{\hat{k}i} + \lambda_{\hat{k}i} \omega^{\hat{i}k} + gT + \hat{g}T + \mathcal{A} (V' + s^f) + \Delta V \right].$$ \hspace{1cm} (3.17)

Semi-dynamical variables $s^r$ which satisfy due to (3.14) describe the conformal-invariant scalar products. We recall that PB (3.5) for $s^r$ can be described if we add to (3.17) the nonwistorial action (3.8); all the constraints in the model are introduced by using Lagrange multipliers.

In the formulation (3.17) of our model Poincare generators are given by the expressions (2.2) and Lorentz generators are

$$M_{\alpha\beta} = \left( \epsilon_{\alpha}(\tilde{a}) \omega^{\beta k} \right), \quad \tilde{M}_{\alpha\beta} = \left( \lambda_{(\hat{\alpha}k)} \omega^{\alpha k} \right).$$ \hspace{1cm} (3.18)

The Pauli–Lubanski vector $W_{\alpha\beta} = i\omega^{\beta k} \tilde{M}_{\alpha k} - \omega^{\alpha k} M_{\beta k}$ has the following twistor representation

$$W_{\alpha\beta} = V' u^r_{\alpha\beta},$$ \hspace{1cm} (3.19)

where $V'$ are defined in (3.15) and $u^r_{\alpha\beta}$ by (2.21). Further due to the constraints (3.15) and relation (2.24) we get

$$W_{i\beta} W_{\alpha k} = -m^2 \left( V' V' \right) = -m^2 (s^f s^f) = -m^2 s^2.$$ \hspace{1cm} (3.20)

The action (3.17) yields the canonical twistorial Poisson brackets

$$\{ \hat{\omega}^{\alpha k}, \hat{\lambda}_k^r \} = \hat{\delta}_k^r \hat{\lambda}_k^r; \quad \{ \hat{\omega}^{\hat{\alpha}k}, \hat{\lambda}^i_{\tau k} \} = \hat{\delta}_k^r \hat{\lambda}^i_{\tau k}.$$ \hspace{1cm} (3.21)

The twistorial PB of the quantities $V'$ are the same as these for $s^f$ in (3.5)

$$\{ V^p, V' q \} = e^{pq} V^q,$$ \hspace{1cm} (3.22)

what will provide the relations (3.15) as first class constraints. Because twistor coordinates and variables $s^r$ are kinematically independent, nonvanishing Poisson brackets of all constraints (2.13), (3.15), (3.16) are the following

$$\{ V^p, V' q \} = e^{pq} V^q,$$ \hspace{1cm} (3.23)

$$\{ V, T \} = 2iT + 4iM; \quad \{ V, \tilde{T} \} = -2i\tilde{T} - 4i\tilde{M}.$$ \hspace{1cm} (3.24)

We see that in present model four constraints are first class: three constraints $V'$ and the constraint $(MT + \tilde{M})$. Other two constraints $V$ and $i(TM - \tilde{MT})$ are second class. In comparison with spinless case, we have additional two degrees of freedom in $s^f$, describing spin degrees of freedom and the number of unconstrained degrees is $18 - 10 = 8$.

4. Quantization and field twistor transform

We obtained the system, which is described in phase space by the variables $\lambda^i_{\tau k}$, $\lambda_k^r$, $\omega^{\hat{i}k}$, $\omega^{\hat{i}k}$, $s^r$, with canonical brackets (3.21), (3.5) and the constraints $T$, $\tilde{T}$ (see (2.13)), $V'$ (see (3.15)) and $V$ (see (3.16)). The constraints $V$ and $i(TM - \tilde{MT})$ are second class. We shall introduce the gauge fixing condition

$$G = \lambda^i_{\tau k} \hat{\lambda}^{\hat{k}i} + \lambda_{\hat{k}i} \omega^{\hat{i}k} \approx 0$$ \hspace{1cm} (4.1)

for the local gauge transformations generated by the constraint $MT + \tilde{M}$, i.e. we get second pair of second class constraints. After introducing Dirac bracket for the second class constraints $(V', i(TM - \tilde{MT}))$, $(MT + \tilde{M}, G)$ will should only impose three first class constraints $\mathcal{V}'$.

Nonvanishing PBs of the constraint (4.1) are

$$\{ G, T \} = 2T + 4M; \quad \{ G, \tilde{T} \} = 2\tilde{T} + 4\tilde{M}.$$ \hspace{1cm} (4.2)

Then, the Dirac brackets (DB) for second class constraints $V$, $G$ and

$$F_1 = \tilde{M} + MT, \quad F_2 = i\tilde{MT} - MT \hspace{1cm} (4.3)$$

are given by formula

4 We denote by $V'$, $V$ the expressions (2.14)–(2.15) for $s^r$, $s$ with the replacement of $\mu^{\alpha k}$ by $\omega^{\hat{\alpha}k}$ (see (3.12)). The constraints (2.14) are additionally modified by inhomogeneous terms proportional to $s^r$.}
\(\{A, B\}_D = \{A, B\}_p + \frac{1}{8MM} [\{A, G\}_p F_1, B\}_p - \{A, F_1\}_p [G, B\}_p - \{A, V\}_p [F_2, B\}_p + \{A, F_2\}_p [V, B\}_p \). \hspace{1cm} (4.4)

The DBs for twisted spinor components take the form
\(\{\lambda^\alpha, \lambda^\beta\}_D = \{\tilde{\lambda}^\alpha, \tilde{\lambda}^\beta\}_D = 0\), \hspace{1cm} (4.5)
\(\alpha^\alpha, \lambda^\beta\}_D = \delta^\alpha_\beta \delta^\alpha_\beta + \frac{1}{2M} \lambda^\alpha \lambda^\beta\), \hspace{1cm} (4.6)
\(\omega^{\alpha k}, \tilde{\alpha}^\beta\}_D = \omega^{\alpha k} \tilde{\alpha}^\beta - \frac{1}{2M} \tilde{\alpha}^{\alpha k} \tilde{\alpha}^\beta\), \hspace{1cm} (4.7)
\(\omega^{\alpha k}, \tilde{\beta}^\alpha\}_D = 0\), \hspace{1cm} (4.8)
\(\alpha^\alpha, \alpha^\beta\}_D = \frac{1}{2M} (\tilde{\alpha}^{\alpha k} \alpha^\beta - \tilde{\alpha}^{\beta k} \alpha^\alpha\), \hspace{1cm} (4.9)

Further we consider \((\lambda, \tilde{\lambda})\)-coordinate representation. In such spinorial Schrödinger representation for the commutator algebra obtained by quantization of DB (4.5)-(4.9) the spinorial momentum operators under suitable ordering \((\lambda, \tilde{\lambda})\)'s on the left, \(\omega, \tilde{\omega}\)'s on the right) are realized in the following way
\(\hat{\omega}^\alpha_k = i \frac{\partial}{\partial \lambda^\alpha_k} + \frac{i}{2M} k^\alpha \lambda^\beta \frac{\partial}{\partial \lambda^\beta}\), \hspace{1cm} (4.10)
\(\hat{\omega}^\alpha_k = i \frac{\partial}{\partial \lambda^\alpha_k} - \frac{i}{2M} \tilde{\alpha}^{\alpha k} \tilde{\alpha}^\beta \frac{\partial}{\partial \lambda^\beta}\). \hspace{1cm} (4.10)

It is important that second terms in the operators (4.10) do not contribute to the realization of quantum counterpart \(\hat{V}'\) of the quantities \(V'\) (see (3.15)):
\(D' = \hat{V}' = \frac{1}{2} (\gamma^i \gamma^j \frac{\partial}{\partial \lambda^i} - \tilde{\lambda}^\alpha_k \frac{\partial}{\partial \tilde{\lambda}^\alpha_k})(\gamma^j)^i\). \hspace{1cm} (4.11)

After quantization \(s' \rightarrow \tilde{s}'\) of the classical PB algebra (3.5) we get the SU(2) algebra
\(\{\tilde{s}'^\alpha, \tilde{s}'^\beta\} = i e^{pq} \tilde{s}'^p\), \hspace{1cm} (4.12)
with classical constraint (2.24) becoming an operator identity
\(\tilde{s}'^2 = 1\). \hspace{1cm} (4.13)

Because the quantum constraint (4.13) describe the eigenvalue condition of SU(2) Casimir operator, for the unitary finite-dimensinal representations of spin algebra (4.12) the value of \(j^2\) are quantized in known way
\(j^2 = J(J + 1)\). \hspace{1cm} (4.14)

where \(J\) is a non-negative half-integer number, i.e. \(2J \in \mathbb{N}\).

For fixed \(J\) the operators \(\tilde{s}'\) are realized as \((2J + 1) \times (2J + 1)\) matrices. \hspace{1cm} (5)

Therefore, twistor wave function of massive particle of spin \(J\) has \((2J + 1)\) components which are functions of \(\lambda^\alpha_k, \tilde{\lambda}^\alpha_k\), constrained by strong conditions (2.5). Because \(\tilde{s}'\tilde{s}'\) commutes with \(\tilde{s}'\), the wave function for fixed spin \(J\) still depends on eigenvalues \(J = -J, \ldots, J - 1, J\) of the spin projection \(s'\). The wave function \((\lambda, \tilde{\lambda})\) of \(J = 1\) is scalar field. Integral transform (4.20) gives us the scalar space–time field
\(\Phi^{(0)}(x) = \int d\mu_6(\lambda, \tilde{\lambda}) e^{ikx^\alpha \lambda^\beta \tilde{\lambda}^\beta} \psi(0)(\lambda, \tilde{\lambda})\), \hspace{1cm} (4.21)
which due to (2.5)–(2.8) satisfies the Klein–Gordon equation
\[ (\hat{\alpha}^\mu \partial_\mu + m^2) \Phi^{(0,0)}(x) = 0, \] (4.22)
i.e. describes in space–time the relativistic particle with mass m and spin zero.

**Spin 1/2:** In this case due to integral transformations (4.20) we obtain two Weyl spinor fields
\[ \Phi^{(1,0)}_a(x) = \int d\mu \hat{\lambda}_\mu \hat{\lambda}_\beta \Phi^{(0,1)}_a(\lambda, \hat{\lambda}), \]
\[ \Phi^{(0,1)}_a(x) = \int d\mu \hat{\lambda}_\mu \hat{\lambda}_\beta \Phi^{(0,1)}_a(\lambda, \hat{\lambda}). \] (4.23)
These space–time fields due to algebraic properties of Weyl spinors satisfy the following generalized Dirac equations with complex mass M
\[ i\hat{\gamma}^\mu \Phi^{(1,0)}_a + M \Phi^{(0,1)}_a = 0, \quad i\hat{\gamma}^\mu \Phi^{(0,1)}_a + M \Phi^{(1,0)}_a = 0. \] (4.24)

We note however that phase \( e\imath \theta \) of \( M = |M|e\imath \theta \) can be absorbed into space–time spinor fields by the redefinition \( \Phi^{(0,0)}_a \rightarrow (e\imath \theta/2) \Phi^{(0,0)}_a, \) \( e\imath \theta/2 \Phi^{(0,1)}_a \rightarrow (\Phi^{(0,1)}_a) \). Thus, the fields (4.23) provide four-component Dirac spinor \( \Phi^{(1,0)}_a, \Phi^{(0,1)}_a \) providing standard Dirac equation with real mass m and describe spin 1/2 massive particle. Finally it can be shown that even for complex mass M Eqs. (4.24) imply Klein–Gordon equations
\[ (i\hat{\gamma}^\mu \partial_\mu + M) \Phi^{(1,0)}_a = 0, \quad (i\hat{\gamma}^\mu \partial_\mu + M) \Phi^{(0,1)}_a = 0. \] (4.25)

**Spin 1:** As the result of twistor transform (4.20) we obtain the following three-space–time fields
\[ \Phi^{(2,0)}_{a_1 a_2}(x) = \int d\mu \hat{\lambda}_\mu \hat{\lambda}_\beta \Phi^{(2,0)}(\lambda_1, \lambda_2), \]
\[ \Phi^{(1,1)}_{a_3}(x) = \int d\mu \hat{\lambda}_\mu \hat{\lambda}_\beta \Phi^{(1,1)}(\lambda_1, \lambda_2), \]
\[ \Phi^{(0,2)}_{a_4}(x) = \int d\mu \hat{\lambda}_\mu \hat{\lambda}_\beta \Phi^{(0,2)}(\lambda_1, \lambda_2). \] (4.26)
From these definition it follows that these fields satisfy Dirac-like equations
\[ i\hat{\gamma}^\mu \Phi^{(2,0)}_{a_1 a_2} + M \Phi^{(1,1)}_a = 0, \]
\[ i\hat{\gamma}^\mu \Phi^{(1,1)}_a + M \Phi^{(0,2)}_a = 0. \] (4.27)
\[ i\hat{\gamma}^\mu \Phi^{(0,2)}_a + M \Phi^{(0,1)}_a = 0. \] (4.28)
Further, the formulae (4.27), (4.28) even for complex M lead to the Klein–Gordon equations for all fields (4.26)
\[ (i\hat{\gamma}^\mu \partial_\mu + m^2) \Phi^{(2,0)}_{a_1 a_2} = 0, \quad (i\hat{\gamma}^\mu \partial_\mu + m^2) \Phi^{(1,1)}_a = 0, \]
\[ (i\hat{\gamma}^\mu \partial_\mu + m^2) \Phi^{(0,2)}_a = 0. \] (4.29)
Eqs. (4.28) imply transversality of four-vector field \( \Phi^{(1,1)}_{a_1} = \frac{1}{2} \gamma^\mu \partial_\mu A_\mu \)
\[ \hat{\alpha}^\mu \Phi^{(1,1)}_a = 0 \leftrightarrow \hat{\alpha}^\mu A_\mu = 0. \] (4.30)
We can consider vector field \( \Phi^{(1,1)}_{a_1} \) as primary field with spin 1 and remaining two fields \( \Phi^{(2,0)}_{a_1 a_2}, \Phi^{(0,2)}_{a_1} \) as derivable from \( \Phi^{(1,1)}_{a_1} \) by the formulae (4.28) defining selfdual and anti-selfdual \( J = 1 \) field strengths. The masses in Eqs. (4.27), (4.28) can be made real after the redefinition \( \Phi^{(0,0)}_{a_1 a_2} \rightarrow (e\imath \phi/2) \Phi^{(0,0)}_{a_1 a_2}, \)
\[ e\imath \phi/2 \Phi^{(0,1)}_{a_1} \rightarrow (\Phi^{(0,1)}_{a_1}) \), where \( e\imath \phi \) is the phase of complex mass M. If we define the \( J = 1 \) field strength (see also [20])
\[ F_{\mu \nu} = \frac{\imath}{\sqrt{2}} (\sigma_{\mu \nu} \Phi^{(2,0)}_{a_1 a_2} + \sigma_{\mu \nu} \Phi^{(2,0)}_{a_1 a_2}). \] (4.31)
due to Eqs. (4.27), (4.28), (4.30) the fields (4.31) satisfy the Proca equations
\[ \hat{\alpha}^\mu A_\mu = 0, \quad \hat{\alpha}^\mu A_\mu - \hat{\alpha}^\mu A_\mu = F_{\mu \nu}, \quad \hat{\alpha}^\mu F_{\mu \nu} + m^2 A_\nu = 0 \] (4.32)
and the Bianchi identity \( \hat{\alpha}^\mu F_{\mu \nu} = 0 \).

For arbitrary \( J \) one can derive in analogous way the general form of the Bargmann–Wigner equations for massive fields with arbitrary spin \( J \).

**5. Outlook**

Twistor theory aims at providing a new geometric framework for the description of classical and quantum dynamical models, and one of its basic aims is to formulate the twistor theory of free and interacting particles. The theory in single \( D = 4 \) twistor space describes conformal space–time geometry and provides six-dimensional phase space of massless particles with remaining two degrees of freedom describing \( U(1) \) gauge and discrete set of helicities. After quantization the twistor theory via so-called twistor transform provides new method for solving the field equations for massless fields with arbitrary helicity (see e.g. [33]). These techniques were further extended to curved twistor theory and provided new way of solving Einstein and Yang–Mills equations for selfdual and anti-selfdual cases (see e.g. [34,35]).

The subject studied in this paper is the twistor description of free massive particles with arbitrary spin. In order to introduce in twistor theory time-like fourmomentum vector it is necessary to consider the two-twistor geometry, with sixteen real degrees of freedom. Relativistic spin is described by the Pauli–Lubanski fourvector which carries for definite mass and spin two new degrees of freedom. These new degrees we describe as parametrizing two-dimensional fuzzy sphere \( S^2 \) with nonAbelian \( SU(2) \) Poisson brackets. In this paper we did show that

- in space–time framework the particle dynamics with nonvanishing spin is obtained adding Souriau–Wess–Zumino term;
- in order to get pure twistorial formulation of massive particles with spin we should modify the standard Penrose incidence relations, which can be obtained by suitable shift of the twistor components;
- in two-twistor space the model is described by free two-twistor Lagrangian with suitable chosen six constraints bilinear in twistor variables;
- the degrees of freedom described by the three-vector \( s^\mu \) due to the constraints (3.15) can be treated as specifying the choice of conformal-invariant scalar products of the pair of twistors, i.e. in such a way in physical phase space the variables \( s^\mu \) are determined as well by the twistor components;
- in order to obtain Pauli–Lubanski spin fourvector one should multiply (see (2.20) and (3.4)) the three-vector \( s^\mu \) with internal three-vector indices by three fourvectors \( u^\mu_{sr} \) describing the soldering between internal and space–time descriptions of spin degrees of freedom.
The methods presented in this paper can be extended in several ways, in particular to particle models which generalize the presented here $D = 4$ case. In particular:

- One can consider the theory of supersymmetric particles and study the supertwistor description [36] of free massive superparticles with nonvanishing superspin. The superspin should be described by supersymmetric extension of Pauli–Lubanski fourvector [37]. The formalism after using the first quantization rules will provide various known $D = 4$ free massive superfields.

- It should be recalled that infinite higher spin multiplets have been obtained by spinorial and twistorial formulations of the free particle models in extended space–time with tensororial coordinates generated by tensorial central charges (for $D = 4$ the extended tensorial space–time is ten-dimensional [38–40]). These models used only the set of single twistorial variables and were describing massless higher spin fields. It is interesting to consider the massive two-twistor models linked with tensorially extended space–time which can be obtained by dimensional reduction of higher-dimensional massless spinorial theory in extended tensorial space–time. This idea has been already outlined in our previous paper [41] (see also [15]), with the description of two-twistor $D = 3$ massive spinorial model as obtained by the dimensional reduction from $D = 4$ massless spinorial model.

Acknowledgements

We acknowledge a support from the grant of the Bogoliubov–Infeld Programme and RFBR grants 12–02–00517, 13–02–91330 (S.F.), as well as Polish National Center of Science (NCN) research projects No. 2011/01/ST2/03354 and No. 2013/09/B/ST2/02205 (J.L.). S.F. thanks the members of the Institute of Theoretical Physics at Wrocław University for their warm hospitality.

References