Solutions to the Equations of Motion of Classical Relativistic Strings

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In this report some solutions to the equations of motion of classical relativistic strings are presented. Rotating solutions of open and closed strings are found, including intermediate forms between open strings that are folded $k$ and $3k$ times.

1 Introduction

In classical relativistic string theory the Howe and Tucker form of the action can be reduced to

$$ S = -\frac{T}{2} \int d^2 \sigma \eta^{\alpha \beta} \partial_{\alpha} x^\mu \partial_{\beta} x_\mu, $$

where $T$ is the string tension, in combination with constraints

$$ g_{\alpha \beta} - \frac{1}{2} (h \cdot g) h_{\alpha \beta} = 0. $$

This action leads to separate solutions for open and closed strings.

1.1 Open strings

For open strings we have the Neumann boundary conditions:

$$ \partial_{\sigma} x^\mu (\tau, 0) = \partial_{\sigma} x^\mu (\tau, \pi) = 0. $$

The equations of motion for open strings is given by

$$ x^\mu (\tau, \sigma) = q^\mu + \frac{1}{\pi T} P^\mu \tau + i \ell \sum_{n \neq 0} \frac{1}{n} \alpha_n e^{-in\tau} \cos n\sigma \tag{1} $$
such that \( \alpha_n^\mu = (\alpha_{-n}^\mu)^* \), where * denotes the complex conjugate.

The constraints can be written as
\[
\frac{1}{2} \dot{x}^2 + \frac{1}{2} (x')^2 = 0, \quad \dot{x} \cdot x' = 0, \tag{2}
\]
or equivalently, as shown in [1],
\[
L_n \equiv \frac{1}{2} \sum_m \alpha_m^\mu \alpha_{n-m,\mu} = 0, \text{ for all } n, \tag{3}
\]
where
\[
\alpha_0^\mu = \frac{1}{\pi T \ell} P^\mu .
\]

### 1.2 Closed Strings

For closed strings we have the boundary conditions
\[
x^\mu(\tau, \sigma) = x^\mu(\tau, \sigma + \pi).
\]
The equations of motion are
\[
x^\mu(\tau, \sigma) = q^\mu + \frac{1}{\pi T} P^\mu \tau + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} e^{-2 i n \sigma} (\alpha_n^\mu e^{2 i n \sigma} + \bar{\alpha}_n^\mu e^{-2 i n \sigma}). \tag{4}
\]

The constraints are equivalent to
\[
L_n \equiv \frac{1}{2} \sum_m \alpha_m^\mu \alpha_{n-m,\mu} = 0, \quad \alpha_0^\mu = \frac{1}{2 \pi \ell T} P^\mu, \tag{5}
\]
\[
\bar{L}_n \equiv \frac{1}{2} \sum_m \bar{\alpha}_m^\mu \bar{\alpha}_{n-m,\mu} = 0, \quad \bar{\alpha}_0^\mu = \alpha_0^\mu.
\]

### 2 The constraints

A simple solution of the equations of motion is given by
\[
\left. \begin{array}{l}
x^0 = \ell \tau \\
x^1 = \ell \cos \sigma \cos \tau \\
x^2 = \ell \cos \sigma \sin \tau
\end{array} \right\} . \tag{6}
\]
This solution corresponds to a rotating open string of which the endpoints move with the speed of light. To verify that this solution indeed obeys the constraints we can use the equations (2) or we can calculate the coefficients \( \alpha_m^\mu \). The only nonzero coefficients \( \alpha_m^\mu \) for this case are:

| \( \alpha_0^0 = 1 \) | \( \alpha_1^1 = -\frac{i}{2} \) | \( \alpha_1^2 = \frac{1}{2} \) |
| \( \alpha_{-1}^1 = \frac{i}{2} \) | \( \alpha_{-1}^2 = \frac{1}{2} \) |
2.1 General properties of the constraints

Before calculating the \( L_n \) it is convenient to make some general statements to see which of them actually have to be calculated. The first one indicates that it is only necessary to consider the \( L_n \) with positive \( n \).

**Theorem 2.1** \( L_n = 0 \) if and only if \( L_{-n} = 0 \).

**Proof** We use the fact that in the equations of motion (1) \( \alpha^\mu_n = (\alpha^{\mu}_{-n})^* \) where the * denotes the complex conjugate.

\[
L_n = \sum_m \alpha^\mu_m \alpha_{n-m,\mu} \\
= \sum_{m>0} \alpha^\mu_m \alpha_{n-m,\mu} + \sum_{m<0} \alpha^\mu_m \alpha_{n-m,\mu} + \alpha^\mu_0 \alpha_{n,\mu} \\
= \sum_{m>0} (\alpha^\mu_m)^* (\alpha^{\mu}_{-n+m,\mu})^* + \sum_{m<0} (\alpha^\mu_m)^* (\alpha^{\mu}_{-n+m,\mu})^* + (\alpha^\mu_0)^* (\alpha_{-n,\mu})^* \\
= \sum_{m<0} (\alpha^\mu_m)^* (\alpha^{\mu}_{n-m,\mu})^* + \sum_{m>0} (\alpha^\mu_m)^* (\alpha^{\mu}_{n-m,\mu})^* + (\alpha^\mu_0)^* (\alpha_{n,\mu})^* \\
= \sum_m (\alpha^\mu_m)^* (\alpha^{\mu}_{n-m,\mu})^* = \left( \sum_m \alpha^\mu_m \alpha_{n-m,\mu} \right)^* = (L_n)^*
\]

If the complex conjugate of \( L_{-n} \) equals zero, then \( L_{-n} \) is zero as well, and also the converse is true.

Another useful theorem is the next one. It states that we do not have to evaluate the conditions \( L_n \) for any \( n > 2n_{\text{max}} \), where \( n_{\text{max}} \) is the highest \( n \) of all non-zero \( \alpha^\mu_n \), since they all vanish.

**Theorem 2.2** For any set of non-zero coefficients \( \alpha^\mu_n \) let \( \alpha^\mu_{n_{\text{max}}} \) be the coefficient with highest \( n \). Then \( L_k = 0 \) if \( k > 2n_{\text{max}} \).

**Proof** It is convenient to split the sum in a part with \( m < 0 \) and a part with \( m \geq 0 \), so

\[
L_k = \sum_m \alpha^\mu_m \alpha_{k-m,\mu} = \sum_{m \geq 0} \alpha^\mu_m \alpha_{k-m,\mu} + \sum_{m < 0} \alpha^\mu_m \alpha_{k-m,\mu}.
\]

First consider the part where \( m \) runs negative:

\[
\sum_{m < 0} \alpha^\mu_m \alpha_{k-m,\mu} = \alpha^\mu_{-1} \alpha_{k+1,\mu} + \alpha^\mu_{-2} \alpha_{k+2,\mu} + \cdots
\]

Since \( k > n_{\text{max}} \) all factors are zero and so this part does not contribute to the sum.
Next $m \geq 0$.

$$
\sum_{m \geq 0} \alpha^\mu_m \alpha_{k-m,\mu} = \alpha^\mu_0 \alpha_{k,\mu} + \alpha^\mu_1 \alpha_{k-1,\mu} + \cdots + \alpha^\mu_{n_{\text{max}}} \alpha_{k-n_{\text{max}},\mu} + \cdots
$$

The contravariant components, starting from $m = 0$, can be nonzero up to $m = n_{\text{max}}$. In that region the index of the covariant components runs from $k$ down to $k-n_{\text{max}}$. If $k > 2n_{\text{max}}$ it follows that $k-n_{\text{max}} > n_{\text{max}}$, so all covariant components that are multiplied by possible nonzero contravariant components indeed vanish.

Furthermore there is a condition for the coefficients $\alpha^\mu_n$ with highest $n$.

**Theorem 2.3** Let $n_{\text{max}}$ be the index of the non-zero coefficient $\alpha^\mu_n$ with the highest value of $n$. Then

$$
\alpha^\mu_{n_{\text{max}}} \alpha_{n_{\text{max}},\mu} = 0.
$$

**Proof** Consider $L_{2n_{\text{max}}}$ and again split it in two parts.

$$
L_{2n_{\text{max}}} = \sum_m \alpha^\mu_m \alpha_{2n_{\text{max}}-m,\mu} = \sum_{m \geq 0} \alpha^\mu_m \alpha_{2n_{\text{max}}-m,\mu} + \sum_{m < 0} \alpha^\mu_m \alpha_{2n_{\text{max}}-m,\mu}.
$$

In the sum where $m \geq 0$ we have

$$
\sum_{m \geq 0} \alpha^\mu_m \alpha_{2n_{\text{max}}-m,\mu} = \alpha^\mu_0 \alpha_{2n_{\text{max}},\mu} + \alpha^\mu_1 \alpha_{2n_{\text{max}}-1,\mu} + \cdots + \alpha^\mu_{n_{\text{max}}} \alpha_{n_{\text{max}}+1,\mu} + \alpha^\mu_{n_{\text{max}}} \alpha_{n_{\text{max}},\mu} + \cdots
$$

where the only nonzero term is given by $\alpha^\mu_{n_{\text{max}}} \alpha_{n_{\text{max}},\mu}$. In the part where $m$ runs negative we have

$$
\sum_{m < 0} \alpha^\mu_m \alpha_{2n_{\text{max}}-m,\mu} = \alpha^\mu_{-1} \alpha_{2n_{\text{max}}+1,\mu} + \alpha^\mu_{-2} \alpha_{2n_{\text{max}}+2,\mu} + \cdots = 0
$$

So the condition $L_{2n_{\text{max}}} = 0$ is equivalent to

$$
\alpha^\mu_{n_{\text{max}}} \alpha_{n_{\text{max}},\mu} = 0.
$$

Finally the number of $L_n$ that have to be calculated can be decreased by the following theorem:

**Theorem 2.4** In the constraints $L_n$ the term $\alpha^\mu_k \alpha_{l,\mu}$ is contained only in $L_{l+k}$.

**Proof** The sum can be written as

$$
L_n = \alpha^\mu_{-m} \alpha_{n+m,\mu} + \cdots + \alpha^\mu_{-k} \alpha_{k,\mu} + \cdots + \alpha^\mu_{n-k} \alpha_{n-k,\mu} + \cdots + \alpha^\mu_{n-m} \alpha_{n-m,\mu}
$$

so in the whole sum $\alpha^\mu_k$ appears twice (If $n > 2k$ the two terms in the middle should be interchanged). In both cases it is multiplied by $\alpha^\mu_{n-k}$. We have $\alpha^\mu_l = \alpha^\mu_{n-k}$ only when $n = l + k$. 


From this theorem it is possible to consider only the terms that can contribute to the $L_n$. For example, if the coefficients $\alpha_n^\mu$ exist for only the spatial coordinates of $n = \pm 1$, the only way they can contribute to the $L_n$ is when $\alpha_1^\mu \alpha_{1,\mu}$ or $\alpha_{-1}^\mu \alpha_{1,\mu}$ appear in the sum. This happens only in $L_0$ and $L_2$.

The condition $L_0$ can be seen as a normalization of the components. If $\alpha_0^0 = 1$ it can be shown that

$$\sum_{n \neq 0} |\alpha_n^\mu|^2 = \frac{1}{2}$$

(8)

2.2 Calculating the constraints

For calculating the constraints on the coefficients of the equations (6) we only need to calculate $L_0$ and $L_2$ as can be seen from the above theorems.

$$L_0 = \sum_m \alpha_m^\mu \alpha_{-m,\mu} = \alpha_0^\mu \alpha_{0,\mu} + \alpha_1^\mu \alpha_{-1,\mu} + \alpha_{-1}^\mu \alpha_{1,\mu}$$

$$= -1 - \frac{i}{2} - \frac{i}{2} + \frac{1}{2} = 0$$

$$L_2 = \alpha_1^\mu \alpha_{1,\mu} = (-\frac{i}{2})^2 + (\frac{1}{2})^2 = 0$$

So by the constraints it can be seen that solution (6) is indeed valid. The same result is obtained when using the constraints given by (2).

3 More general rotating solutions

The solutions to the equations of motions (6) can be made more general. If we consider a system with 2 spatial dimensions and 1 time dimension such that there are only five non-zero complex coefficients the system is almost the same as the above example. The time is given by $x^0 = \ell \tau$ so $\alpha_0^0 = 1$. It is convenient to use the polar form for the complex coefficients. The non-zero coefficients are given below.

<table>
<thead>
<tr>
<th>$\alpha_0^0 = 1$</th>
<th>$\alpha_1^1 = ae^{i\phi}$</th>
<th>$\alpha_2^2 = a_2e^{i\phi_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\alpha_1^1)^* = ae^{-i\phi}$</td>
<td>$(\alpha_2^2)^* = a_2e^{-i\phi_2}$</td>
<td></td>
</tr>
</tbody>
</table>

Since the only non-zero values of $\alpha_n$ are for $n = \pm k$, by theorem (2.4) we can see that $\alpha_{\pm k}^\mu \alpha_{\pm k,\mu}$ only appears in $L_{\pm 2k}$ and $\alpha_{k}^\mu \alpha_{-k,\mu}$ only in $L_0$. By theorem (2.1) we only have to make sure that $L_{2k}$ and $L_0$ vanish.

From theorem (2.3) it follows that $\alpha_{k}^\mu \alpha_{k,\mu} = 0$, so $\alpha_k^2 = \pm i \alpha_k^1$, and this satisfies the constraint $L_{2k}$. If we take $\alpha_k^1 = ae^{i\phi}$ we have, using that $i = e^{i\pi/2}$, the following set of $\alpha_k^\mu$. 

5
\[ \begin{array}{c|c|c}
\alpha_0^0 = 1 & \alpha_k^1 = ae^{i\phi} & \alpha_k^2 = ae^{i(\phi + \frac{\pi}{2})} \\
\alpha_{-k}^1 = ae^{-i\phi} & \alpha_{-k}^2 = ae^{-i(\phi + \frac{\pi}{2})} & 
\end{array} \]

where \( a \) is a positive real number.

For \( L_0 \) to vanish we need

\[
L_0 = \sum_m \alpha_m^\mu \alpha_{-m,\mu} = \alpha_0^\mu \alpha_0 + \alpha_k^\mu \alpha_{-k,\mu} + \alpha_{-k}^\mu \alpha_k = -1 + 4a^2 = 0
\]

With \( a = \frac{1}{2} \) this leads to the following solution:

\[
\begin{align*}
x^0 &= \ell \tau \\
x^1 &= \ell \left( \sin k\tau \cos \phi - \cos k\tau \sin \phi \right) \cos k\sigma \\
x^2 &= \pm \ell \left( -\sin k\tau \sin \phi - \cos k\tau \cos \phi \right) \cos k\sigma
\end{align*}
\]

(9)

This solution corresponds to a rotating string. It is an open string for \( k = 1, 3, 5 \ldots \) For \( k > 1 \) the string is folded \( k \) times on itself. At \( \tau = 0 \), the angle the string makes with the \( x^2 \)-axis is given by \( \phi \). If the sign of the \( x^2 \)-part of solution (9) is positive the string rotates counter-clockwise.

When \( \phi = 0 \) the equations of motion are given by

\[
\begin{align*}
x^0 &= \ell \tau \\
x^1 &= \ell \sin k\tau \cos k\sigma \\
x^2 &= \pm \ell \cos k\tau \cos k\sigma
\end{align*}
\]

(10)

4 More complicated solutions of open strings

A more complicated solution of an open string can be obtained when we have a solution like (9) with an additional function of order \( l > k \). First we consider the case in which \( l \) and \( k \) are chosen arbitrarily.

4.1 Arbitrary higher order terms

For arbitrary \( l \) the nonzero coefficients \( \alpha_n^\mu \) are as follows:
<table>
<thead>
<tr>
<th>$\alpha_0^0 = 1$</th>
<th>$\alpha_1^1 = ae^{i\phi}$</th>
<th>$\alpha_2^2 = a_2e^{i\phi_2}$</th>
<th>$\alpha_3^3 = b_3e^{i\theta_3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1^1 = be^{i\theta}$</td>
<td>$\alpha_2^2 = b_2e^{i\theta_2}$</td>
<td>$\alpha_3^3 = b_3e^{i\theta_3}$</td>
<td></td>
</tr>
<tr>
<td>$\alpha_{-1}^1 = be^{-i\theta}$</td>
<td>$\alpha_{-2}^2 = b_2e^{-i\theta_2}$</td>
<td>$\alpha_{-3}^3 = b_3e^{-i\theta_3}$</td>
<td></td>
</tr>
</tbody>
</table>

The solution is completely general if the $\alpha_{\pm l}$ is chosen to be zero. Such a term inevitably leads to a rotating solution, since it is build up of sine and cosine functions and we can always choose a coordinate system such that this motion is restricted to the $x_1, x_2$-plane. The coefficients of highest order are easiest fixed by theorem (2.3) if they are restricted to two spatial dimensions. We use theorem (2.4) to see which of the constraints $L_n$ have to be calculated. The only way that non-zero multiplications of the coefficients $\alpha_n^\mu$ occur is in $\alpha_1^1\alpha_{l,\mu}$, $\alpha_1^1\alpha_{-k,\mu}$, $\alpha_1^1\alpha_{-L,\mu}$, $\alpha_k^k\alpha_{k,\mu}$, $\alpha_k^k\alpha_{-k,\mu}$, and they emerge in the constraints $L_{2l}, L_{l+k}, L_{l-k}, L_0, L_{2k}, L_0$, respectively.

First we calculate $L_{2l}$. It follows from theorem (2.3) that

$$a_2e^{i\phi_2} = \pm i\alpha_1^1 = ae^{i(\phi \pm \pi/2)}$$

so $a_2 = a$ and $\phi_2 = \phi \pm \pi/2$.

Next we consider $L_{l+k}$:

$$L_{l+k} = \alpha_l^\mu\alpha_{k,\mu} = abe^{i\phi + \theta} + ab_2e^{i(\phi + \theta_2 \pm \pi/2)} = 0$$

It follows that $b_2 = b$ and $\theta_2 = \theta \pm \frac{\pi}{2}$, where the sign is the same as in equations (11). So we have

$$\alpha_1^1 = be^{i\theta} \text{ and } \alpha_2^2 = be^{i(\theta \pm \pi/2)}, \quad (12)$$

This means that if condition $L_{2k}$ is to be satisfied, $\alpha_3^3$ must vanish. Furthermore, if we consider condition $L_{k-l}$ we see that

$$L_{l-k} = \alpha_l^\mu\alpha_{-k,\mu} + \alpha_{-k}^\mu\alpha_{l,\mu} = 2(\alpha_1^1\alpha_{-k,1} + \alpha_2^2\alpha_{-k,2}) = 2(ab\hat{e}^{\phi}e^{i\theta} + ab\hat{e}^{i(\phi \pm \pi/2)}e^{-i(\theta \pm \pi/2)}) = 4ab\hat{e}^{i(\phi - \theta)}$$
wich can be zero only if

\[ a = 0 \text{ or } b = 0. \]

This implies that either the coefficients \( \alpha^\mu_k \) vanish, or the coefficients \( \alpha^\mu_l \). This shows that an arbitrary combination of \( l \) and \( k \) is not possible.

### 4.2 Reducing the number of constraints

Above we saw that the constraints for a system with coefficients of order \( l \) and \( k \) are

\[
L_{2l}, L_{2k}, L_{l+k}, L_{l-k} \text{ and } L_0. \]

If we choose \( l \) such that two of the constraints become the same, we can effectively remove one of them. By choosing \( l = 3k \), the constraint \( L_{l-k} \) becomes identical to \( L_{2k} \). There is no other way in which two of the constraints can be combined. The constraint \( L_{6k} \) still leads to equation (11), and \( L_{4k} \) leads to (12), where \( 3k \) must be substituted for \( l \).

Now we have the following set of coefficients:

<table>
<thead>
<tr>
<th>( \alpha_0^0 = 1 )</th>
<th>( \alpha_{3k}^1 = ae^{i\phi} )</th>
<th>( \alpha_{3k}^2 = \pm iae^{i\phi} )</th>
<th>( \alpha_{3k}^3 = b_3 e^{i\theta_3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_k^1 = be^{i\theta} )</td>
<td>( \alpha_k^2 = \pm ibe^{i\theta} )</td>
<td>( \alpha_k^3 = b_3 e^{i\theta_3} )</td>
<td>( \alpha_{-k}^3 = b_3 e^{-i\theta} )</td>
</tr>
<tr>
<td>( \alpha_{-k}^1 = be^{-i\theta} )</td>
<td>( \alpha_{-k}^2 = \mp ibe^{-i\theta} )</td>
<td>( \alpha_{-3k}^1 = ae^{-i\phi} )</td>
<td>( \alpha_{-3k}^2 = \mp iae^{-i\phi} )</td>
</tr>
<tr>
<td>( \alpha_{-3k}^3 = b_3 e^{i\theta_3} )</td>
<td>( \alpha_{-3k}^3 = b_3 e^{-i\theta} )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The value of \( \alpha_{3k} \) can be found using \( L_{2k} \).

\[
\begin{align*}
L_{2k} &= \sum_{m} \alpha^\mu_m \alpha_{2k-m,\mu} \\
&= \alpha^\mu_k \alpha_{k,\mu} + \alpha^\mu_{3k} \alpha_{-k,\mu} + \alpha^\mu_{-k} \alpha_{3k,\mu} \\
&= \left( \alpha_{2k}^3 \right)^2 + 2(abe^{-i\theta} e^{i\phi} + abe^{i(\phi \pm \frac{\pi}{2})} e^{-i(\theta \pm \frac{\pi}{2})}) \\
&= \left( \alpha_{3k}^3 \right)^2 + 4abe^{i(\phi-\theta)}
\end{align*}
\]

It follows that

\[
\alpha_{3k}^3 = \pm 2i \sqrt{abe^{i(\phi-\theta)}}/2 = 2\sqrt{abe^{i(\phi-\theta \pm \pi)}/2}.
\]

Finally, using \( L_0 \), we can calculate that

\[
L_0 = -1 + 2(2b^2 + 4ab + 2a^2) = -1 + 4(a + b)^2,
\]

so \((b + a)^2 = \frac{1}{4}\), where both \( a \) and \( b \) are positive real numbers. We have \( b = \frac{1}{2} - a \)

where \( 0 \leq a, b \leq \frac{1}{2} \).
4.3 The equations of motion

Now we can write down the equations of motion. Putting together all values of the $\alpha_n^\mu$, remembering that the sign in front of $\alpha_{3k}^2$ should be the same as the one in front of $\alpha_k^2$, we get the following system of coefficients.

$$
\begin{array}{|c|c|c|}
\hline
\alpha_0^0 & \alpha_{3k}^1 = ae^{i\phi} & \alpha_{3k}^2 = \pm i ae^{i\phi} \\
\alpha_k^1 & \alpha_k^2 = \pm i be^{i\theta} & \alpha_{-k}^3 = \mp 2i \sqrt{ab} e^{-i(\phi - \theta)/2} \\
\alpha_{-k}^1 = be^{-i\theta} & \alpha_{-k}^2 = \mp i e^{-i\phi} & \\
\alpha_{-3k}^1 = ae^{-i\phi} & & \\
\hline
\end{array}
$$

and putting the coefficients in equation (1) we obtain:

$$
\begin{align*}
\ell \tau & \\
\frac{\ell}{k} & \left( 2b(\cos \theta \sin k \tau - \sin \theta \cos k \tau) \cos k \sigma + \frac{2}{3} a (\cos \phi \sin 3k \tau - \sin \phi \cos 3k \tau) \cos 3k \sigma \right) \\
\pm \frac{\ell}{k} & \left( 2b(\cos \theta \cos k \tau + \sin \theta \sin k \tau) \cos k \sigma + \frac{2}{3} a (\cos \phi \cos 3k \tau - \sin \phi \sin 3k \tau) \cos 3k \sigma \right) \\
\pm \frac{\ell}{k} & 4 \sqrt{ab} (\cos \frac{\phi - \theta}{2} \cos k \tau + \sin \frac{\phi - \theta}{2} \sin k \tau) \cos k \sigma
\end{align*}
$$

The values of $\phi$ and $\theta$ determine the phase of the different parts of the motion. The motion itself corresponds to a rotating string with a twist, which is rotating as well. The motion when $k = 1$ shown in figure (1) below.
Figure 1: The string as given in equations (13) is drawn here for different values of $a$ and $b$, but all with the same value of $\tau$. In (A) the motion is that of a rotating open string folded three times on itself. In this case $a = \frac{1}{2}$, and $b = 0$. In (B) we have $a = 0.499$ and $b = 0.001$. In the following pictures $a$ is made smaller and smaller, and the string is stretching until it becomes a rotating open string in (H).
5 Closed Strings

For closed strings there are two sets of coefficients, $a_n^\mu$ and $\tilde{a}_n^\mu$ (see equation [4]). They are independent, but $\alpha_0^0 = \tilde{\alpha}_0^0$. Since $\alpha_0^\mu = \frac{1}{2\pi T} P^\mu$, we have $x^0 = \ell \tau$ if $\alpha_0^0 = \tilde{\alpha}_0^0 = \frac{1}{2}$. First the case with the following coefficients is considered:

<table>
<thead>
<tr>
<th>$\alpha_0^0 = \frac{1}{2}$</th>
<th>$\alpha_k^1 = \frac{1}{4} e^{i\theta}$</th>
<th>$\alpha_k^2 = \pm i \frac{1}{4} e^{i\theta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\alpha_{-k}^1 = \frac{1}{4} e^{-i\theta}$</td>
<td>$\alpha_{-k}^2 = \mp i \frac{1}{4} e^{-i\theta}$</td>
</tr>
</tbody>
</table>

and for $\tilde{\alpha}_n^\mu$:

<table>
<thead>
<tr>
<th>$\alpha_0^0 = \frac{1}{2}$</th>
<th>$\alpha_m^1 = \frac{1}{4} e^{i\tilde{\theta}}$</th>
<th>$\alpha_m^2 = \pm i \frac{1}{4} e^{i\tilde{\theta}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\alpha_{-m}^1 = \frac{1}{4} e^{-i\tilde{\theta}}$</td>
<td>$\alpha_{-m}^2 = \mp i \frac{1}{4} e^{-i\tilde{\theta}}$</td>
</tr>
</tbody>
</table>

The factor $\frac{1}{4}$ in front of the coefficients comes from condition $L_0$ with $\alpha_0^0 = \frac{1}{2}$. This results in the following equations of motion:

$$
\begin{align*}
 x^0 &= \ell \tau \\
 x^1 &= \frac{\ell}{4} \left( \frac{1}{k} \sin (2k \tau - \theta - 2k \sigma) + \frac{1}{m} \sin (2m \tau - \tilde{\theta} + 2m \sigma) \right) \\
 x^2 &= \mp \frac{\ell}{4} \left( \frac{1}{k} \cos (2k \tau - \theta - 2k \sigma) \pm \frac{1}{m} \cos (2m \tau - \tilde{\theta} + 2m \sigma) \right)
\end{align*}
$$

When $k = m$ and when we have a (+) in the middle of $x^2$ the motion corresponds to a closed string collapsed to a line that is rotating. If the middle sign is a (−) the motion corresponds to a circle of which the radius is periodically changing between 0 and $\frac{\ell}{2}$. The sign in front of $x^2$ determines the direction of the rotation. The angles $\theta$ and $\tilde{\theta}$ determine the phase of the motion. A few examples of the motion when $k \neq m$ are shown in figure (2).
Figure 2: Some examples for different values of $m$, $k$ and the $\pm$-sign between the two terms of $x^2$. In A and B we have $m = 1$, $k = 2$ with a plus and a minus sign respectively. The same applies for C and D where $m = 10$, $k = 3$. The strings are all rotating with time.

More general are the equations of motion following from the following system of coefficients, obtained in the same way as in the open string case:

<table>
<thead>
<tr>
<th>$\alpha_0$</th>
<th>$\alpha_k$</th>
<th>$\alpha_{-k}$</th>
<th>$\alpha_{3k}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_0 = \frac{1}{2}$</td>
<td>$\alpha^1_{3k} = be^{i\beta}$</td>
<td>$\alpha^2_{3k} = \pm ibe^{i\beta}$</td>
<td>$\alpha^3_{3k} = \pm 2i\sqrt{ab}e^{i(\beta - \alpha)/2}$</td>
</tr>
<tr>
<td>$\alpha_k = ae^{i\alpha}$</td>
<td>$\alpha^1_{k} = ae^{i\alpha}$</td>
<td>$\alpha^2_{-k} = \mp ibe^{i\alpha}$</td>
<td>$\alpha^3_{-k} = \mp 2i\sqrt{ab}e^{-i(\beta - \alpha)/2}$</td>
</tr>
<tr>
<td>$\alpha_{-k} = ae^{-i\alpha}$</td>
<td>$\alpha^1_{-3k} = be^{-i\beta}$</td>
<td>$\alpha^2_{-3k} = \mp be^{-i\beta}$</td>
<td></td>
</tr>
</tbody>
</table>

The same applies to the coefficients $\tilde{\alpha}^\kappa_n$:  

12
\[
\begin{array}{c|c|c|c}
\tilde{\alpha}_0 &= \frac{1}{2} & \tilde{\alpha}_{3k} &= be^{i\tilde{\beta}} \\
\tilde{\alpha}_k &= \tilde{a}e^{i\tilde{\alpha}} & \tilde{\alpha}_{3k} &= \pm i\tilde{b}e^{i\tilde{\beta}} \\
\tilde{\alpha}_{-k} &= \tilde{a}e^{-i\tilde{\alpha}} & \tilde{\alpha}_{-3k} &= \mp i\tilde{b}e^{-i\tilde{\beta}} \\
\end{array}
\]

By condition \( L_0 \) we must have \( a + b = \tilde{a} + \tilde{b} = \frac{1}{4} \). This leads to the following solutions for the equations of motion:

\[
\begin{align*}
x^0 &= \ell \tau \\
x^1 &= \ell \left( \frac{1}{8} \left[ \frac{b}{3} \sin (6k\tau - \beta - 6k\sigma) + a \sin (2k\tau - \alpha - 2k\sigma) \right] \\
&\quad + \frac{1}{m} \left[ \frac{b}{3} \sin (6m\tau - \tilde{\beta} + 6m\sigma) + \tilde{a} \sin (2m\tau - \tilde{\alpha} + 2m\sigma) \right] \right) \\
x^2 &= \mp \ell \left( \frac{1}{8} \left[ \frac{b}{3} \cos (6k\tau - \beta - 6k\sigma) + a \cos (2k\tau - \alpha - 2k\sigma) \right] \\
&\quad + \frac{1}{m} \left[ \frac{b}{3} \cos (6m\tau - \tilde{\beta} + 6m\sigma) + \tilde{a} \cos (2m\tau - \tilde{\alpha} + 2m\sigma) \right] \right) \\
x^3 &= \mp \ell \left( \frac{2\sqrt{ab}}{k} \cos (2k\tau - \frac{\beta - \alpha}{2} - 2k\sigma) \\
&\quad \pm \frac{2\sqrt{ab}}{m} \cos (2m\tau - \frac{\tilde{\beta} - \tilde{\alpha}}{2} + 2m\sigma) \right)
\end{align*}
\]

An example of these solutions is given below.

### 6 Summary

Solutions to the equations of classical relativistic strings can be expressed in term of the coefficients \( \alpha_{n\mu} \), and they are subject to the constraints \( L_n \). It is shown that some of the constraints can be combined, and more terms can be added to the equations of motion.
Figure 3: The movement of the string according to equations (15) with \( k = m = 1 \), \( \alpha = \tilde{\alpha} = \beta = \tilde{\beta} = 0 \) and \( a = b = \tilde{a} = \tilde{b} = \frac{1}{8} \). The sign in the middle of \( x^2 \) is \((-)\), all other \((+)\). The pictures have different values of \( \tau \) such that 1 period of its motion is covered.
Figure 4: The same as in figure (3), but this time the box in which the string is drawn is viewed from above.
7 Acknowledgment

I would like to express my gratitude to Dr. M. de Roo for his support and advice.

References