Is string interaction the origin of quantum mechanics?

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A B S T R A C T

String theory was developed by demanding consistency with quantum mechanics. In this paper we wish to reverse the reasoning. We pretend that open string field theory is a fully consistent definition of the theory – it is at least a self-consistent sector. Then we find in its structure that the rules of quantum mechanics emerge from the non-commutative nature of the basic string joining/splitting interactions. Thus, rather than assuming the quantum commutation rules among the usual canonical variables we derive them from the physical process of string interactions. Morally we could apply such an argument to M-theory to cover quantum mechanics for all physics. If string or M-theory really underlies all physics, it seems that the door has been opened to an explanation of the origins of quantum mechanics from the physical processes point of view.

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1. Introduction

Quantum mechanics (QM) works amazingly well in all known parts of microscopic physics. One can deduce classical physics as the limit of QM for large quantum numbers (or equivalently the small ℏ limit). Hence the general belief is that QM is the only rule for all types of mechanics. Despite the tremendous success of QM, the fundamental commutation rules from which all QM is derived, namely \([ x, p ] = iℏ \) for every degree of freedom, need to be put in mysteriously “by hand” without any underlying reasoning. It is well established that, if the quantization rule is accepted, then all of the amazing and correct consequences of quantum mechanics follow. The success of QM is of course a justification to accept the mysterious rule as correct, but it leaves us begging for an underlying explanation.

In this paper we will present arguments that there may be a physical explanation for where the QM rules come from. We will show that there is a clear link between the commutation rules of QM operators and the non-commutative string joining/splitting interactions [1] that were expressed in the language of the Moyal star formulation of strong field theory (MSFT) [2] in a recently improved and more intuitive version [3]. Except for the mathematical similarity, the Moyal \( \ast \) in MSFT has nothing to do with the Moyal product [4] that reproduces\(^1\) QM, because the basic non-commuting quantities in the string \( \ast \) in MSFT are very different than the canonical conjugates indicated by quantum mechanics. Nevertheless, we found how to link the basic QM commutators to the string \( \ast \) and derive the QM rules only from the rules of string joining/splitting. This link suggests that there is a deeper physical phenomenon, namely string interactions, underlying the usual quantum rules of QM, thus providing a possible explanation for where they come from.

The essential arguments for the thesis of this paper can be adequately presented in a simplified model that captures the necessary ingredients of MSFT. The simplified model, which we call mini-MSFT, consists basically of the phase space system of two particles, rather than the full phase space of an infinite number of particles that make up all the points on a string. The two particles may be thought of as the end points of an open string, but it is also possible not to think of the string concept at all to discuss the main ideas. This is because only the properties of phase space, rather than the property of the dynamics of the two particles enter in the main part of the discussion. Hence to keep our discussion as simple as possible, we will define the mini-MSFT system in Section 3 and discuss how to derive the QM properties from “string” interactions. The mini-MSFT may be a useful model in its own right to discuss some physically interesting systems, as in the examples we outline at the end of Section 3.

Even though we will not use the full machinery of MSFT in this paper, we begin our discussion in Section 2 with a brief description of its setup so that the reader, even without knowing much about string theory, can see the connection between the full string field theory and the simplified 2-particle model in Section 3, and be able to deduce easily that the arguments for the thesis

\(^1\)For the explanation of how the well known Moyal product [4] for classical phase space functions reproduces all the details of quantum mechanics, read Section III in [3] which summarizes the essentials of this correspondence.

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of this paper given in the context of the simple model in Section 3 apply equally well to the full string theory in our preferred MSFT language for the full string. The full string theory (and its M-theory extension) is needed to be able to apply the argument to all physics, provided one is willing to make the assumption that string- or M-theory actually underlies all physics.

2. Degrees of freedom in MSFT

The open string position degrees of freedom $X^M(\sigma)$, at a fixed value of the worldsheet parameter $\tau$, are parametrized by the worldsheet parameter $\sigma$, with $0 \leq \sigma \leq \pi$. Witten [1] suggested to imagine the string field $\psi(X(\sigma))$ as an infinite dimensional matrix $\psi_1(X)$

$$\psi(X(\sigma)) = \psi_{X_0(\sigma)}, X_0(\tilde{x}), \tag{2.1}$$

whose left/right halves $i \sim X^M_0(\sigma)$, and $j \sim X^M_0(\sigma)$ are the left/right halves of the string relative to the midpoint at $\sigma = \pi/2$, namely $X^M_0(\sigma) = X^M(\sigma)$ for $0 \leq \sigma < \pi/2$ and $X^M_0(\sigma) = X^M(\sigma)$ for $\pi/2 < \sigma \leq \pi$, while $X^M = X^M(\pi/2)$ is the location of the midpoint. It was then suggested in [1] that the products of fields $\psi_1(X) \star \psi_2(X)$ in open string field theory are the non-commutative matrix product of matrices of the form (2.1), and that the action is similar to the Chern–Simons theory

$$S = \int \left( \frac{1}{2} \psi_1 \star (\hat{Q} \psi) + \frac{g}{3} \psi_1 \star \psi_1 \star \psi \right), \tag{2.2}$$

where $\hat{Q}$ is the BRST operator of a conformal field theory on the worldsheet (CFT). This proposal worked and produced correctly the Veneziano model type perturbative string scattering amplitudes [5].

The matrix product in $\psi \star \psi$ was implemented by going back to the worldsheet conformal field theory to perform computations, which proved to be prohibitively complicated and took away much of the simplicity and elegance of the matrix-like setup of the product and the action. Seeking a way of avoiding the complicated CFT maps, while keeping the elegant algebraic structure, the Moyal star product formulation of string field theory (MSFT) was suggested in [2], and computations were performed in [6–8], showing that this was a more efficient approach to compute and correctly recover the perturbative Veneziano amplitudes, including a higher degree of accuracy for the off-shell versions of the amplitudes [8]. The MSFT formalism has been reformulated recently in [3] in a new basis of degrees of freedom in which all expressions, especially the product and computations greatly simplify. It is the new form of the star product displayed below that suggests the connection between string joining and quantum mechanics.

In the new version of MSFT the string field is taken to be a functional $A(x_+, p_-)$ of half of the phase space of the string, where $X^M(\sigma)$ is the symmetric part of $X^M_0(\sigma)$ under reflections relative to the midpoint, $X^M_0(\sigma) = \frac{1}{2}(X^M(\sigma) + X^M(\pi - \sigma))$, while $p_{-M}(\sigma) = \frac{1}{2}(P_{M}(\sigma) + P_{M}(\pi - \sigma))$ is the antisymmetric part of the momentum density. Note that $p_{-M}(\sigma)$ is the canonical conjugate to $X^M_0(\sigma)$ and commutes with $X^M_0(\sigma)$ in the first quantization of the string. The symmetric/antisymmetric $X^M_0(\sigma)$ are related to $(x_\perp, \tilde{x}_\perp)$ in the Witten version by $x_\perp = \frac{1}{2}(x_\perp \pm \tilde{x}_\perp)$, and including the midpoint $\tilde{x}$ as part of $x_\perp$. Thus the MSFT field $A(x_+, p_-)$ is related to the field $\psi(X)$ as $\psi(x_\perp, \tilde{x}_\perp, \tilde{x}) = \psi(x_\perp, \tilde{x})$ by a Fourier transform from $x_\perp$ and $\tilde{x}_\perp$ to $x_\perp$. With this choice of half-phase-space degrees freedom to label the string field $A(x_+, p_-)$, the matrix-like product for string joining in position space $\psi_{12}(X) = \psi_1(X) \star \psi_2(X)$ is mapped to the Moyal product in the half-phase-space, $A_{12}(x_+, p_-) = A_1(x_+, p_-) \star A_2(x_+, p_-)$ with

$$* = \exp \left[ \frac{g}{2} \int \sigma \partial^{-\sigma} \left( \frac{\partial_{p_{-M}}}{\partial_{X^M}} \right) \right]. \tag{2.3}$$

A very important property of the new star product is that it is background independent because phase space does not care which conformal field theory on the worldsheet underlies the string action $S_{\text{string}}$ or which background fields it contains. The sum over the $M$ indices in (2.3) does not involve a metric because $X^M$ is defined with an upper index and then $P_M$, which is derived from the action according to the canonical procedure, $P_M = \partial S_{\text{string}}(\partial X^M)$, automatically has a lower index.

An elegant aspect of MSFT that will be centrally relevant for our discussion in this paper is that the quantum canonical operators for any point on the string $X(\sigma)$, $P(\sigma)$ are represented on the string field $A(x_+, p_-)$ only by string joining/splitting operations, namely by either left or right star-multiplication, depending on whether the point is to the left or to the right of the midpoint at $\sigma = \pi/2$

$$\hat{X}^M(\sigma, e) A(x_+, p_-) = \begin{cases} X^M(\sigma, e) A(x_+, p_-), & \text{if } 0 \leq \sigma \leq \pi/2, \\ A(x, p) \star x_+ (\sigma, e) (-1)^{MA}, & \text{if } \pi/2 \leq \sigma \leq \pi \end{cases}, \tag{2.4}$$

$$\hat{P}_M(\sigma) A(x_+, p_-) = \begin{cases} (e^{-\epsilon \sigma} P_{-M}(\sigma)) A(x_+, p_-), & \text{if } 0 \leq \sigma \leq \pi/2, \\ A(x_+, p_-) \star (e^{-\epsilon \sigma} P_{-M}(\sigma)) (-1)^{MA}, & \text{if } \pi/2 \leq \sigma \leq \pi \end{cases}. \tag{2.5}$$

Note that on the right hand side the string fields that are being joined are $A(x_+, p_-)$ and $x_+$, or $A$ and $p_-$, where $x_+$, $p_-$ are special cases of a more general string field $A(x_+, p_-)$.

This product includes a small parameter $\epsilon$ which is a regulator to avoid notorious midpoint anomalies, and the label $M = (\mu, b, c)$ includes spacetime ($\mu$) and ghost ($b, c$) degrees of freedom, all of which are necessary and insure a well defined theory. For the reader interested in the details we suggest [3]. None of these complications will be needed to discuss the main points of this paper. We will switch to the mini-MSFT that imitates in a simplified way only the star product for string splitting/joining using only two particles. The remainder of this paper should be understandable to the reader without having to know anything about strings or string field theory.

3. Toy model with two particles (mini-MSFT)

We begin with the phase space of two particles named $l$ (left) and $r$ (right). It may be helpful to imagine that these correspond to the two end points of a string, however this picture is not necessary and the setup below may apply to more general physical circumstances. The particles are located at arbitrary positions $(x_l, \tilde{x}_l)$, and have canonical conjugate momenta $(\tilde{p}_l, \tilde{p}_r)$. Their center of mass and relative coordinates are, $r_l = \frac{1}{2} (x_l + \tilde{x}_l)$, and $r_r = \frac{1}{2} (x_r - \tilde{x}_r)$, while the momenta canonically conjugate to $(r_l, r_r)$ are the total momentum $P_l = (p_{1L} + p_{1R})$, and the relative momentum $P_l = \frac{1}{2} (p_{1L} - p_{2R})$. The dynamics is controlled by some Hamiltonian $H(P, p, R, r)$ whose details are unimportant for now.2

2 The Hamiltonian $H$ in the toy model is the analog of the Virasoro operator $L_0$ for a string in any background, that plays a role of the kinetic energy operator in the quadratic term in string field theory in the Siegel gauge. More generally, the kinetic operator in string field theory is the BRST operator as in (2.2).
Independent of the Hamiltonian, the phase space \((P, p, R, r)\) has the standard canonical properties, namely we may define classical Poisson brackets or quantum commutators based on the canonical pairs \((R^i, P_i)\) and \((r^i, p_j)\). In particular, the classical Poisson bracket between any two phase space functions, \(U(P, p, R, r)\), \(V(P, p, R, r)\), is

\[
\{U, V\} = \frac{\partial U}{\partial P_i} \frac{\partial V}{\partial p_i} - \frac{\partial U}{\partial p_i} \frac{\partial V}{\partial P_i} + \frac{\partial U}{\partial R^i} \frac{\partial V}{\partial r^i} - \frac{\partial U}{\partial r^i} \frac{\partial V}{\partial R^i}.
\]

To proceed with usual quantization in quantum mechanics (QM) we may define the eigenspace basis for a complete set of commuting operators, such as position space, \(\{\tilde{x}_i, \tilde{x}_k\}\) or \(\{\tilde{R}, \tilde{r}\}\), and express the probability amplitude for an arbitrary quantum state \(\langle \psi \rangle\) in any such basis as the dot product in the Hilbert space, e.g. \(\psi(x, \bar{x}_R) = \langle x, \bar{x}_R | \psi \rangle = \langle x, \bar{r} | \psi \rangle\). We will be interested in the Fourier transform of the latter

\[
A(R, p) = \int \frac{d^n \hat{p}}{(2\pi \hbar)^{n/2}} \psi(R, \hat{p}),
\]

where \(\langle R, p \rangle\) is the complete eigenbasis for the space of the commuting operators \(\{\hat{P}_i, \hat{p}_i, \hat{R}^i, \hat{r}^i\}\). We will think of the probability amplitude \(A(R, p) = \langle R, p | \psi \rangle\), as a field in a field theory as a function of the classical half-phase-space \((\hat{R}, \hat{p})\). This setup is motivated by MSFT that was briefly outlined in Section 2. We will call the toy model in this section "mini-MSFT". The parallels between the full MSFT and mini-MSFT are

\[
R^i \sim x^M, \quad i^j \sim x^M,
\]

\[
P_i \sim p + M, \quad p_i \sim p - M
\]

and we did not care to make parallels between \(i\) and \(M\), which permit many possibilities including bosons and fermions (see [3]), but to keep the discussion simple it is sufficient to consider bosonic Euclidean space for \(i\).

To quantize this 2-particle system in a new way we will take the approach inspired by MSFT. We will not a priori assume the quantum commutation rules of the operators \(\{\hat{P}_i, \hat{p}_i, \hat{R}^i, \hat{r}^i\}\) that describe nature so well, but whose fundamental origin remains mysterious. Rather, as the primary physical origin of QM we will begin from a non-commutative product that has physical significance as interactions of strings by joining/splitting. Only from the algebra of string joining/splitting will we derive the quantum algebra of the operators \(\{\hat{P}_i, \hat{p}_i, \hat{R}^i, \hat{r}^i\}\) without assuming it. String joining/splitting was formulated for open strings in [1] as a matrix-like product for the field as in (2.1). For the present toy model with only two particles we define a similar matrix-like product of fields in position space in the form

\[
\psi_{12}(x_L, x_R) = \int \frac{d^n z}{(2\pi \hbar)^{n/2}} \psi_1(x_L, z) \psi_2(z, x_R),
\]

where each field \(\psi(x_L, x_R)\) is regarded as an infinite dimensional matrix whose rows and columns are labeled by the continuous indices \(x_L, x_R\) that correspond to the locations of the two particles. The matrix-like rule (3.3) is interpreted as a prescription for computing the probability amplitude \(\psi_{12}(x_L, x_R)\) when two 2-particle clouds, described by \(\psi_1(x_L, x_R)\) and \(\psi_2(x_L, x_R)\), join together into a single cloud \(\psi_{12}(x_L, x_R)\) by annihilating a pair of particles, one from each cloud, when they meet locally at all possible points \(z\) in the full volume. This is similar to the picture for joining/splitting worldsheets, but in the present case there are dynamical degrees of freedom only at the ends of the string. It was shown in [2,3] that this string-like joining/splitting can equivalently be formulated as a Moyal-type product, \(A_{12} = A_1 \star A_2\), in the half-phase-space \((R^i, p_i)\) related to position space \((x_L, x_R)\) through the Fourier transform indicated in (3.2).

We now give the details of the \(\star\) product in the half-phase space for this simplified mini-MSFT. It is physically different but mathematically analogous to the usual Moyal product:

\[
A_{12}(R, p) = (A_1 \star A_2)(R, p) = A_1(R, p) \exp\left(\frac{i\theta}{2} (\bar{p}_R \bar{p}_1 - \bar{p}_1 \bar{p}_R)\right) A_2(R, p).
\]

It is the parallel of the string star product in (3.3). The parameter \(\theta\) must have the dimensions of the Planck constant \(\hbar\), so it must be a multiple of \(\hbar\) up to a dimensionless constant. In fact, we will show that it is identically the Planck constant. The arrows in (3.4) instruct the reader to apply the derivatives on the functions to the left (\(A_1\)) or right (\(A_2\)). For example, expanding in powers of \(\theta\) this \(\star\) product gives

\[
A_{12} = A_1 A_2 - \frac{i\theta}{2} \left( \partial_{R^1} A_1 A_2 \partial_p p_1 - \partial_{R^1} A_1 A_2 \partial_p p_1 \right) + \cdots
\]

The first order term in \(\theta\) looks like a Poisson bracket, but this is clearly different than the canonical Poisson bracket of classical mechanics in Eq. (3.1) since it does not involve the traditional canonical conjugates exhibited in (3.1). Instead, the center of mass position \(\bar{R}^i\) and the relative momentum \(p_i\), which belong to different conventional canonical pairs, are set to play a new role analogous to canonical conjugates in the half-phase-space \((R^i, p_i)\).

Using (3.4) we compute \(A_1 \star A_2\) for the special cases when \(A_1\) or \(A_2\) is just \(R^i\) or \(p_i\), thus obtaining the left or right multiplication of the general \(A\) by the elementary degrees of freedom in the half-phase-space

\[
R^i \star A = \left( R^i + \frac{i\theta}{2} \frac{\bar{p}_1}{\bar{p}_R} \right) A(R, p),
\]

\[
A \star R^i = A(R, p) \left( R^i - \frac{i\theta}{2} \frac{\bar{p}_1}{\bar{p}_R} \right),
\]

\[
p_i \star A = \left( p_i - \frac{\bar{p}_1}{\bar{p}_R} \right) A(R, p),
\]

\[
A \star p_i = A(R, p) \left( R^i + \frac{i\theta}{2} \frac{\bar{p}_1}{\bar{p}_R} \right)
\]

There are no higher powers of \(\theta\) because the higher derivatives in the expansion of the exponential in (3.4) vanish for this computation. Other useful equivalent ways of writing the general \(\star\) product are

\[
A_1 \star A_2 = A_1((R^i + \frac{i\theta}{2} \bar{p}_R), (p^i - \frac{i\theta}{2} \bar{p}_R)) A_2(R, p)|_{R^i=R, p^i=p}
\]

\[
A_1(R^i, p^i) A_2((R - \frac{i\theta}{2} \bar{p}_R), (p + \frac{i\theta}{2} \bar{p}_R))|_{R^i=R, p^i=p}
\]

Just like the well known Moyal star product [4], which is related to the Poisson bracket (3.1) in the full phase space \((P, p, R, r)\), reproduces all aspects of ordinary quantum mechanics (see footnote 1), the string-joining Moyal star product in (3.4) will evidently produce a quantum-like system in the half-phase space \((R, p)\), which we call induced quantum mechanics (iQM). This induced iQM has the following properties:

- The product is associative \(A_1 \star (A_2 \star A_3) = (A_1 \star A_2) \star A_3 = A_1 \star A_2 \star A_3\), just as should be expected for the associative product of operators in the induced iQM, where any product
\( \hat{A}_1 \hat{A}_2 \hat{A}_3 \cdots \) does not require parentheses to be computed unambiguously.

- By using (3.6) we compute the products of the half-phase-space elementary degrees of freedom \((R, p)\)

\[
R^i \ast R^j = R^i R^j, \quad p_j \ast p_j = p_j p_j,
\]

\[
R^i \ast p_j = R^i (p_j + i0 \delta^i_j), \quad p_j \ast R^i = p_j R^i - i0 \delta^i_j.
\] (3.8)

This leads to the star commutator

\[
[R^i, p_j] = R^i \ast p_j - p_j \ast R^i = i0 \delta^i_j.
\] (3.9)

Hence \((R^i, p_j)\) behave just like quantum mechanical degrees of freedom. But this is not quantum mechanics since in ordinary QM the corresponding operators compute \([\hat{R}^i, \hat{p}_j] = 0\). Instead, this is the basic commutation property in the induced iQM that comes from the non-commutative interactions in string theory.

We now show that this induced iQM is a seed for constructing the usual QM in the full operator space \((\hat{x}^i, \hat{p}_L, \hat{x}^j, \hat{p}_R),\) A map between operators in QM and their representative in iQM is an elegant and intuitive property of MSFT as given in Eqs. (2.4), (2.5).

Translated to mini-MSFT, his map is given only in terms of the star between two fields in the half-phase-space, as follows

\[
\hat{x}_L^i A = R^i \ast A \quad \hat{p}_L = p_i \ast \hat{a} \quad \hat{x}_R^j A = A \ast R^j \quad \hat{p}_R = A \ast (\hat{p}_j)
\]

(3.10)

The reason for the \((-\) sign in the last line is naturally explained in the stringy version of the star in the full MSFT: it is because for strings \(\chi^i(\sigma)\) is symmetric with respect to reflections at the midpoint, while \(p\cdot(\sigma)\) is antisymmetric, leading to \(p \cdot \sigma \leftrightarrow \sigma_{z \rightarrow -z}\). Using this map, let us now check the consistency between the commutation rules for various QM vs.

versus the iQM representations above. We compute the commutators by using only the star rules in (3.10), associativity of the \(\ast\), and the result for the star commutator in (3.9).

\[
[x_L^i, p_L] = R^i, \quad [x_L^i, p_R] = A \ast p_j + R^i \ast A \ast p_j = 0,
\]

\[
[x_R^j, p_L] = A \ast (p_j + R^i), \quad [x_R^j, p_R] = A \ast p_j - p_j A \ast R^i = 0.
\] (3.14)

For this iQM result to match the QM commutators of operators, \([{\hat{x}}_L^i, \hat{p}_L] = i\hbar \delta^i_j\), we must identify the parameter \(\theta\) with the Planck constant

\[
\theta = \hbar.
\] (3.15)

Continuing with mini-MSFT, next we investigate some operators constructed from the basic ones. From the basic properties in (3.10) we may extract the representation of each operator \((\hat{p}_L, \hat{p}_R, \hat{R})\), in terms of only the \(*\) product of fields, and then evaluate the star products in each line below by using (3.6), after inserting \(\theta = \hbar\), as follows

\[
\hat{R}^i = \frac{1}{2} (\hat{x}_L^i + \hat{x}_R^i) A = \frac{1}{2} (R^i \ast A + A \ast R^i) = R^i A,
\]

\[
\hat{p}_L = (\hat{p}_L + \hat{p}_R) A = (p_L \ast A - A \ast p_L) = -i\hbar \partial_{\hat{p}_R} A,
\]

\[
\hat{p}_R = \frac{1}{2} (\hat{p}_R - \hat{p}_L) A = \frac{1}{2} (p_R \ast A + A \ast p_R) = i\hbar \partial_{\hat{p}_L} A.
\] (3.16)

The end result in terms of differential operator representation is fully consistent with the corresponding well known differential operator representation of operators in QM. But the point here is that this result follows from only the string joining/splitting interactions via the \(*\) product of fields given in (3.4) and (3.10).

Going further, from (3.10) we derive the following additional nice results which were significant in the formulation of MSFT [3]: if we have any quantum operator \(\hat{O}_L(x_L, \hat{p}_L)\) (similarly \(\hat{O}_R(x_R, \hat{p}_R)\)) in usual QM, constructed from only the degrees of freedom of particle \(L, R\) (similarly \(L_R\)), then its representation in the iQM version is given by the same function in which we replace \((\hat{x}_L^i \rightarrow R^i)\) and \((\hat{p}_L \rightarrow p_L)\) and similarly \((\hat{x}_R^j \rightarrow \ast R^j)\) and \((\hat{p}_R \rightarrow \ast (\hat{p}_j))\), where the \(*\) to the right (left) of \(R\) or \(p\).

Namely

\[
\hat{O}_L(x_L, \hat{p}_L) A = O_L(R, p) \ast A \quad \text{from left},
\]

\[
\hat{O}_R(x_R, \hat{p}_R) A = A \ast O_R(R, p) \quad \text{from right},
\] (3.17)

where \(O_L\) means that all \(R, p\) factors within it are star multiplied with each other in the same order that operators appear in the QM version, while in the case of \(O_R\) the same \((\ast)\) factors within it are multiplied in the opposite order of the corresponding operators in \(\hat{O}_R(x_R, \hat{p}_R)\). The expressions for \(O_L\) or \(O_R\) can be reduced to a classical function of \((R, p)\) after using repeatedly the elementary products given in (3.8) to rewrite \(O_L\) or \(O_R\) as classical expressions \(O_L(R, p)\).

In the full MSFT only purely \(L\) or purely \(R\) quantum operators occur, as above, because of the locality in the \(\sigma\) parameter (see footnote 1). More generally, in mini-MSFT one may be interested in writing the QM operator for any Hamiltonian \(\hat{H}_L(x_L, \hat{p}_L, \hat{x}_R, \hat{p}_R)\) in the language of star products in iQM. This is given by representing every elementary \(L/R\) operator as left/right star products according to (3.10). Hence we get the QM representation of any QM Hamiltonian as follows

\[
\hat{H}(x_L, \hat{p}_L, \hat{x}_R, \hat{p}_R) A = H((R, \ast), (p, \ast), (\ast, \ast), (\ast, \ast)) A(R, p),
\]

where the orders of the factors relative to the \(\ast\) must be preserved.

We give two examples. In the first example we have two particles \((L, R)\) interacting with a harmonic oscillator type central force. We can convert the operator \(\hat{H}_L\) for this problem to its iQM version by using the map (3.18) that involves only products of string fields

\[
\hat{H}_L A = \left[\frac{1}{2} (\hat{p}_L^2 + \hat{x}_L^2) + \frac{\omega^2}{2} (\hat{x}_L - x_R)^2\right] A(R, p)
\]

\[
= \frac{1}{2} (\hat{p}_L^2 + \omega^2 \hat{R}_L^2) \ast A = \frac{1}{2} (\hat{p}_L^2 + \omega^2 \hat{R}_L^2) - \omega^2 \hat{R}_L \ast A \ast \hat{R}_L
\]

\[
= \left[\frac{1}{2} \hat{R}_L^2 \hat{p}_L^2 - \frac{\omega^2}{2} \hat{R}_L^2 \hat{p}_L^2\right] A(R, p)
\]

\[
\hat{H}_R A = \left[\frac{1}{2} (\hat{p}_R^2 + \hat{x}_R^2) + \frac{\omega^2}{2} (\hat{x}_R - x_L)^2\right] A(R, p)
\]

\[
= \frac{1}{2} (\hat{p}_R^2 + \omega^2 \hat{R}_R^2) \ast A = \frac{1}{2} (\hat{p}_R^2 + \omega^2 \hat{R}_R^2) - \omega^2 \hat{R}_R \ast A \ast \hat{R}_R
\]

\[
= \left[\frac{1}{2} \hat{R}_R^2 \hat{p}_R^2 - \frac{\omega^2}{2} \hat{R}_R^2 \hat{p}_R^2\right] A(R, p)
\]
In the second line only string field products using the string-joining ∗ appear. The last line follows by evaluating the star products by using (3.7), (3.6). The result in the last line clearly matches the familiar differential operator representation of the Hamiltonian as it would be expressed from QM in the (R, p) basis.

In the second example we illustrate the Hamiltonian \( \hat{H}_2 \) derived from string theory in 2-dimensions with quarks (0-branes) attached at the ends [10], where the positions of the quarks \( x_{R}^{i} \) (in the lightcone basis) are actually the end points of the string,

\[
\hat{H}_2 A = \left[ \frac{m_{R}^2}{2p_{\perp}} + \frac{m_{R}^2}{2p_{\parallel}} + \gamma [x_{R}^{i} - \hat{x}_{R}^{i}] \right] A(R, p) \\
= \frac{m_{R}^2}{2p} \star A + A \star \frac{m_{R}^2}{2p} + \gamma (R) - (\star R)(R, p) \\
= \frac{m_{R}^2}{2p} \star A(R, p) + A(R, p) \star \frac{m_{R}^2}{2p} + \frac{\gamma h}{\pi k^2} A(R, p + k)
\]

(3.19)

In the second line the map (3.18) is used to connect the ∗ version to the QM operator version. In the third line the prime on \( f \) means the principal value integral which arises from computing the star products in the second line. The last line reproduces exactly the spectrum of large-N QCD in two dimensions (‘t Hooft’s integral equation for a meson [9]), as expected from [10], but we will skip the details here.3

For any choice of Hamiltonian we can define the quadratic term of the field theory for the mini-MSFT, and furthermore we can include “string”-“string” interactions by imitating MSFT as follows

\[
S = \int d^{4}R d^{4}p \left[ \frac{1}{2} \hat{A} \hat{H} A + \frac{g}{3} A \star A \star A + \cdots \right].
\]

(3.20)

Here the dots + ... imply that many mini-MSFT models may be constructed that include higher powers of field interactions beyond the cubic term. The Feynman-like diagrams for this field theory reproduce the joining/splitting of worldsheets as in the old string-like “duality diagrams”. We think that with only the cubic interaction in (3.20), and the 2D string Hamiltonian of Eq. (3.19), it seems that the mini-MSFT approach would parallel the 2D string Feynman diagram computations in [10] that gave correctly the meson interactions by using only strings and branes (quarks at the end) with amplitudes in agreement with planar graphs in 2D large N QCD. Perhaps this successful and exact string-QCD correspondence could now be generalized to four dimensions through mini-MSFT in (3.20) by including the transverse components of \( R^{\mu} \), \( p_{\mu} \) beyond the lightcone components.

This completes the construction of the mini-MSFT field theory model. Time will show if this is a useful approach to discuss some physical systems, such as QCD strings. It is possible to generalize the system further by allowing \( A \) to carry labels that correspond to spin and other quantum numbers and correspondingly choose an appropriate \( \hat{H} \). In this paper the mini-MSFT concept was used mainly as a simplification of the full MSFT to discuss the link between the string-joining star product and the quantization rules of QM. As shown in [3] all facets of our discussions here are also true in the full MSFT as well as subsectors, derivable from it.

4. Outlook

We have shown that in the half-phase-space of iQM we can reproduce all aspects of ordinary QM by relying only on the rules of

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3 In [9,10] the wavefunction is in momentum space \((p_{\perp}, p_{\parallel})\), whereas in (3.19) it is in the mixed phase space \( A(R, p) \). After a Fourier transform \( (R \rightarrow \tilde{R}) \) and appropriate change of variables from \((P, p)\) to \((p_{\perp}, p_{\parallel})\) we find the same integral equation for mesons.
space representation where momentum is represented by derivative. This reasoning is easily extended to other degrees of freedom, including spin, by including fermions in the string field formalism.

Independent of the central thesis in this paper, at a more modest level, we have introduced a new representation space for the quantum mechanical operators through the map in Eq. (3.10) which may find many applications. The mini-MSFT model may be useful in its own right to discuss some perturbative and non-perturbative physics in certain circumstances.

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References

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