Schwinger effect and negative differential conductivity in holographic models

Shankhadeep Chakrabortty a,b,*, B. Sathiapalan b

a Indian Institute of Science Education and Research, Pune 411008, India
b Institute of Mathematical Sciences, Taramani, Chennai 600113, India

Received 10 September 2014; accepted 10 November 2014
Available online 13 November 2014
Editor: Herman Verlinde

Abstract

The consequences of the Schwinger effect for conductivity are computed for strong coupling systems using holography. The one-loop diagram on the flavor brane introduces an $O(\frac{1}{N_c})$ imaginary part in the effective action for a Maxwell flavor gauge field. This in turn introduces a real conductivity in an otherwise insulating phase of the boundary theory. Moreover, in certain regions of parameter space the differential conductivity is negative. This is computed in the context of the Sakai–Sugimoto model.

© 2014 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/3.0/). Funded by SCOAP3.

1. Introduction

The phenomenon of electron positron pair production in the presence of an electric field was described quantitatively in the classic work of Schwinger [1]. He wrote down an expression for the probability of particle production (per unit space–time volume) by solving the Dirac equation in a uniform and constant background electric field $E$ and obtaining the electron Green function. The expression is

$$P(E, m) = 1 - e^{-\Gamma VT}$$

---

* Corresponding author.
E-mail addresses: shankhadeep.chakrabortty@iiserpune.ac.in, shankha@imsc.res.in (S. Chakrabortty), bala@imsc.res.in (B. Sathiapalan).

http://dx.doi.org/10.1016/j.nuclphysb.2014.11.010
0550-3213/© 2014 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/3.0/). Funded by SCOAP3.
with
\[
\Gamma = \frac{E^2}{4\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-\frac{m^2}{|E|}}.
\]

A noteworthy feature is that there is an exponential damping with the mass of the electron which signifies that it is a tunneling phenomenon. Thus for laboratory electric fields the magnitude is negligible and unobservable.

Schwinger’s derivation treats the electric field as classical and also ignores higher order effects such as the Coulomb force between the electrons. As shown in [2], the presence of the Coulomb potential modifies the solution for large values of the electric field. It is found that for \( E > E_c \) (critical field), there is no potential barrier and hence no exponential suppression. In this approximation the potential has the form \( 2m - Ed - \frac{\alpha}{d} \) [2], where \( \alpha \) is proportional to the electric charge. When

\[
E = E_c = \frac{m^2}{\alpha},
\]

the maximum of the potential is zero and there is no barrier.

It is clear that this is non-perturbative in \( \alpha \). One can then pose the same question in \( \mathcal{N} = 4 \) Super Yang–Mills theory which is also in the Coulomb phase. Here in the planar strong coupling limit one can resort to the AdS/CFT correspondence and get an exact answer. Again one finds a critical electrical field [2].

This is also what one expects from string theory in flat space: An open string in an electric field is a quark–antiquark pair in an electric field. When the force due to the electric field is stronger than the string tension, the effective string tension becomes zero and one can expect unsuppressed production of quark antiquark pairs from the vacuum. The signal of this is that the Dirac–Born–Infeld action becomes zero (and starts to become imaginary) at the critical electric field.

One can ask whether this effect can be observed in condensed matter systems by its consequences on conductivity [3]. The effect of Schwinger pair production and its influences on conductivity was shown using holographic calculations in [4,5]. Probe branes representing flavor quarks were introduced in an AdS–Schwarzschild background and placed in an electric field and the conductivity was calculated [6]. Similar systems were also studied in [7,12,8]. The electric field induced conductivity was found to be a non-linear function of the electric field, and more importantly the critical field turns out to be zero[12]. In the bulk calculation the absence of a critical field can be traced to the warp factors. For an arbitrarily small electric field the warp factors reduce the effective tension of the string and at some value of the radial coordinate \( u = u^* \), the DBI action threatens to turn complex. It can be shown then that this unphysical behavior is resolved if one assumes that a non-zero current is induced. This current is present even at zero temperature and can be attributed to the Schwinger effect without, however, any exponential suppression. In the boundary theory this can be attributed to the strong coupling effects which reduce the critical field to zero.\(^1\)

It has also been shown that this effect can be described in terms of an effective event horizon at \( u = u^* \), not in the bulk metric, but in the open string metric on the brane [9,10]. In fact many

\(^1\) Naively, if one let \( \alpha \to \infty \) in (3), \( E_c \to 0 \).
of the results derived for black hole backgrounds, such as the proof of the fluctuation–dissipation theorem [11] can be shown to be directly applicable in this situation also.

There are situations, such as the one described in the Sakai–Sugimoto model [14], where the branes do not reach as far as $u^*$. In this case this effect is not there and the material is insulating even when there is an electric field [12]. For the boundary theory this implies that the quarks are too heavy and the strong coupling interactions are not sufficient to remove the barrier.

In the case that there is no barrier (due to strong coupling effects), the pair production takes place “classically” i.e. no quantum mechanical tunneling is involved. One doesn’t really need the Schwinger formula to calculate conductivity. This is the situation dealt with in the classical bulk calculations of [4,10,8]. However, one can ask the question whether, in the case where there is a barrier, as in the Sakai–Sugimoto model, where chiral symmetry breaking induces a quark mass and puts it in an insulating phase [8], one can really use the Schwinger formula to calculate the conductivity. Schwinger’s calculation is a one-loop calculation and since the quarks circulating in the loop transform in the fundamental of the color group rather than the adjoint, this is a $1/N$ effect – beyond the planar approximation. If one includes the Schwinger correction (exponentially suppressed) to the Maxwell action (as calculated in [17]) this would induce corrections to conductivity that is proportional to the electric field.

In this paper we address this question. The Maxwell action on the brane (which is the leading term obtained in expanding the DBI action) is modified by the addition of the one-loop effective action, which in the presence of a background electric field has an imaginary part explicitly. If one calculates conductivity using the Kubo formula, from the Green’s function, which can be evaluated by the usual AdS/CFT prescription, one expects a non-zero real part. Indeed, this is what we find. Thus we can conclude that what was thought to be an insulating phase actually has a small conductivity.

Whether this is large enough to be observable experimentally depends crucially on the mass of the fermion. In metals the role of particle and antiparticle is played by fermionic excitations at the Fermi surface which is ungapped and one can expect such effects. In ordinary metals the dispersion relation is non-relativistic and the Schwinger formula would not be directly applicable although. Moreover, external electric fields are screened in metals. It has been suggested that systems such as graphene one can look for this effect [18]. In graphene the electrons and holes are effectively massless at the “Dirac points” and obey a relativistic dispersion relation. Nevertheless the density of states is small enough that screening effect is not strong. Also the rate may be large enough to be measurable [18]. The holographic calculation would then be applicable in a strong coupling version of this.

Another interesting feature that arises in this calculation is that there are regions where the real part of the conductivity is negative. While this is unphysical ordinarily and would be a sign that the approximations are breaking down, this need not be so when there is a background electric field, from which energy can be pumped into the system. Thus in semiconductor physics it is well known that Gunn diodes and tunnel diodes display negative differential resistivity in the presence of large external fields. A holographic calculation showing this phenomena has been done recently [20,21,19]. Our main point of departure is that we are working in a region where the phenomenon is due to a one-loop effect in the flavor brane in the bulk.

In supersymmetric BPS configurations of branes, the low energy action is fixed by the symmetries. However, when supersymmetry is spontaneously broken by finite temperature effects one can expect finite loop corrections. Loop corrections can involve open strings with $(A)$ both ends on the flavor brane or $(B)$ with one end on the flavor brane and one end on the colored brane.
In case (B) the loop diagram with an infinite number of massive modes running around the loop dualizes to a tree diagram involving closed string modes connecting the flavor and color branes (see Fig. 1). One can see that they are of $O(\lambda)$. These are the same class of diagrams that generate the AdS background in the first place. So the question arises as to whether we are double counting. Are these diagrams (of type (B)) to be considered or are they already included when the flavor brane is placed in an AdS space? In general it is the UV limit of the open string loop that is reproduced by the graviton tree diagram. The finite part of the open string loop requires all the massive closed string tree diagrams. This continues to be true in the decoupling limit taken for the AdS/CFT correspondence. On the open string side, on the brane, in the $\alpha' \to 0$ limit only the massless open string particles traverse the loop. However, the closed string side still requires the full string theory and the massive modes also contribute and their effect survives in this limit [16]. Thus one is tempted to conclude that one has to explicitly calculate the finite part of a gauge theory loop diagram (with colored particle in the loop) on the brane placed in an AdS background. The finite contribution in the presence of an electric field has an imaginary part corresponding to Schwinger pair production. In situations where the open string metric develops a horizon such effects are seen, in the gravity dual, already at the tree level on the brane. But in general it is hard to imagine how an imaginary part can develop in an effective action except through loop effects. We leave the issue of type (B) diagrams as an open question for the moment.

However, diagrams of type (A) are not included in the dual picture and must be computed – they are of order $\lambda \times \frac{1}{N_c} \times N_f$. As an example if we have two flavor branes separated by a small amount, we have $SU(2)$ broken to $U(1)$. The effective action for the $U(1)$ photon will get corrected by the loop with $W^\pm$ or its fermionic partners running around. This kind of symmetry breaking occurs for instance in the holographic description of strong coupling BCS theory [15].

The actual calculation of these classes of diagrams is not very different – they differ only by an overall numerical factor which is of $O(N_c/N_f)$. Furthermore while the original Schwinger calculation involved only fermions in the loop in the supersymmetric theory there are fermions as well as bosons. The expressions for arbitrary spin are available in the literature [2]. In this paper we are interested in probing the qualitative effects of the Schwinger effect, so we restrict ourselves to the fermionic contribution.

---

2 In some situations the massive mode contributions cancel and the duality holds for the massless modes on both sides [16].
Having discussed an overview and the motivation for our holographic study of Schwinger pair production in strongly coupled system, here we briefly mention about the specific strategies we follow in the subsequent sections. Our aim is to capture the effect of pair production process on the electrical conductivity in the context of Sakai–Sugimoto model. In Section 2, we briefly discuss the holographic aspects of Sakai–Sugimoto model. The bulk gravity theory of Sakai–Sugimoto model is built on the consideration of $N_f$ number of flavor D8/$\overline{D}8$ branes probing the near horizon geometry of type IIA supergravity background corresponding to the $N_c$ number of color D4 branes. Using the D4–D8–$\overline{D}8$ brane picture, we describe how the model holographically allows a deconfined boundary phase with broken chiral symmetry. In Section 3, we study the response of the Sakai–Sugimoto model to a constant electric background by turning it on inside the world volume of the probe D8/$\overline{D}8$ brane. The DBI action of the probe D8/$\overline{D}8$ brane is modified accordingly. Moreover, we systematically consider some time dependent fluctuation over the constant electric background and construct the associated effective classical Lagrangian as a leading Maxwell term in the $\alpha'$ expansion of the modified DBI action. Our aim is to add a one-loop effective action to this classical Maxwell action and study the gauge field dynamics governed by the total action. It is important to mention that the quantum action we are interested in can be obtained by systematically covariantizing the one in four-dimensional flat space–time. Therefore to accommodate the quantum action we need to dimensionally reduce the classical action to a four-dimensional subspace of the D8/$\overline{D}8$ world volume. The systematic recipe for adding the one-loop effective Lagrangian capturing the physics of Schwinger pair production is described in Section 4. The Schwinger pair of our interest is the quark antiquark pair associated with end points of the fundamental string joining either D4–D8/$\overline{D}8$ or D8–$\overline{D}8$ brane stacks (modulo the comments of the previous paragraph). Furthermore, in Section 5 by analyzing the quantum corrected effective Lagrangian, we demonstrate the holographic method to compute the conductivity in the strongly coupled boundary theory. For the conductivity computation we require the information of retarded Green’s function. In this work we calculate the Green’s function using numerical methods. Finally we plot the real part of the conductivity with the frequency of the fluctuating gauge field. We summarize our results and conclude with some remarks on our analysis in Section 6.

2. Sakai–Sugimoto model

Sakai–Sugimoto model provides a very successful holographic method to construct the type IIA supergravity background, dual to the large $N_c$ QCD theory [14]. The IIA background includes a stack of $N_c$ color D4 branes wrapping a $S^4$ circle. In addition to that, within probe approximation, $N_f$ number of flavor D8/$\overline{D}8$ branes at two different points on the circle are also considered, accounting no back reaction on the D4 geometry. The four-dimensional boundary theory of the D4 branes is realized as a QCD-like theory with the $N_c$ colors of gluons and the $N_f$ flavors of massless chiral fermions ($N_f \ll N_c$) with anti-periodic boundary condition. The left (right) handed fermions come from the spectrum of D8 ($\overline{D}8$) flavor branes and they transform in the fundamental representation of both the color gauge group $U(N_c)$ and the flavor gauge group $U(N_f)_L \times U(N_f)_R$. The above realization of the Sakai–Sugimoto model is valid at an energy scale far below the Kaluza–Klein mass scale.

In more detail, we consider the near horizon limit of the D4 branes in the type IIA string theory. If we set the curvature of space–time $R = (\pi g s N_c)^{1/3} l_s = 1$ and impose the limit $N_c \gg 1$, the metric structure takes the following form:
\[ ds^2 = u^{3/2} \left( -dt^2 + dx^2 + dy^2 + dz^2 + f(u)dx_4^2 \right) + u^{-3/2} \left( \frac{du^2}{f(u)} + u^2 d\Omega_4^2 \right), \]
\[ f(u) = 1 - \left( \frac{u_{KK}^3}{u^3} \right), \quad e^\phi = g_s u^{\frac{1}{4}}, \quad F_4 = 3\pi l_s^3 N_c d\Omega_4, \]

where \( u_{KK}^{-1/2} = \frac{3}{2} R_4 \) signifies the supersymmetry breaking scale of the theory, \( x_4 \) is the circular direction with the periodicity, \( x_4 \sim x_4 + 2\pi R_4 \) and \( \Omega_4 \) is the volume element of \( S^4 \). The \((u,x_4)\) subspace turns out to be topologically a cigar. The tip of this \((u,x_4)\) subspace is localized at \( u = u_{KK} \). The AdS/CFT correspondence maps this particular background (4) to a confined phase of the boundary gauge theory at zero temperature with the 't Hooft coupling parameter,

\[ \lambda_{\text{Sakai-Sugimoto}} = 4\pi g_s N_c l_s. \]

Possible embeddings of D8 and \( \overline{D8} \) probe branes consistent with the topology of the background (4) correspond to the various chiral phases in the boundary QCD like theory. For example, one can construct a \( \mathcal{U} \) shaped embedding by smoothly joining the D8 and \( \overline{D8} \) branes at a radial distance \( u_0 \geq u_{KK} \). The \( \mathcal{U} \) shaped embedding holographically conceives a very simple geometrical picture of the spontaneous breaking of the chiral gauge group \( SU(N_f)_R \times SU(N_f)_L \) to the diagonal one \( U(N_V) \) [14]. The form of \( \mathcal{U} \) shaped embedding can be extracted by solving the equation of motion for \( x_4 \) obtained from the DBI action,

\[ S_{D8} = -T_8 \int d^9 X e^{-\phi} \text{Tr} \sqrt{-\det(G + 2\pi l_s^2 F)}, \]

where \( T_8 \) is the tension of D8 brane, \( G \) is the induced world volume of metric and \( F \) is the world volume gauge field. The boundary condition for the solution is fixed by specifying that the asymptotic separation between the stack of D8 and \( \overline{D8} \) branes at boundary is \( l_s \). Using the background metric (4), the DBI action of D8 and \( \overline{D8} \) stacks reduces to

\[ S_{D8} = -\mathcal{W} \int du u^4 \left[ f(u)x_4'^2 + \frac{1}{u^2 f(u)} \right]^{-\frac{1}{2}}. \]

Here we have introduced the normalization constant \( \mathcal{W} \) as follows:

\[ \mathcal{W} = 2N_f V_{\mathcal{M}_4} \Omega_4 T_8, \]

where \( V_{\mathcal{M}_4} \) is the volume of 4-dimensional space–time \((t,x,y,z)\). Finally the \( \mathcal{U} \) shaped profile is obtained by solving the equation of motion for \( x_4(u) \) coordinate.

\[ x_4(u) = \frac{1}{u^{3/2} f(u)} \left[ u_0^{9/2} f(u) - 1 \right]^{-1/2}. \]

A finite temperature extension of the confining phase can be holographically realized by considering the Euclidean continuation of (4). In the Euclidean signature, the time coordinate, \( t^E \) develops a periodicity \( t^E \sim t^E + 1/T \), where \( T \) is identified with the temperature of the gauge theory. Moreover, the gravity background dual to the deconfined phase at finite temperature is achieved with interchanging the role of \( t \) and \( x_4 \) [13]. In Minkowski signature, the corresponding black hole background reads as

\[ ds^2 = u^{3/2} \left( -f(u)dt^2 + dx^2 + dy^2 + dz^2 + dx_4^2 \right) + u^{-3/2} \left( \frac{du^2}{f(u)} + u^2 d\Omega_4^2 \right). \]
Fig. 2. Here we show the embeddings of the D8 brane corresponding to appropriate boundary phases. (a) Corresponds to a deconfined boundary phase with broken chiral symmetry. (b) Depicts a deconfined boundary phase with chiral symmetry restored.

The background develops a horizon at \( u = u_T = (\frac{4\pi T}{3})^2 \) and the blackening function is defined as

\[
f(u) = 1 - \frac{u^3}{u^3_T}. \tag{11}\]

The induced metric of D8 and \( \overline{\text{D}8} \) world volume becomes

\[
ds_{\text{D8}}^2 = u^{3/2}(-f(u)dt^2 + dx^2 + dy^2 + dz^2) + u^{-3/2}\left(\frac{1 + u^3 f(u)x_4'^2}{f(u)}du^2 + u^2 d\Omega_4^2\right). \tag{12}\]

In this black hole background, the \((u, x_4)\) subspace is topologically a cylindrical and the possible embeddings of D8/\( \overline{\text{D}8} \) brane consistent with this topology turn out to the \( \mathcal{U} \) shaped (broken chiral symmetry, \( T < \frac{0.154}{L} \)) and the parallel embeddings (restoration of the chiral symmetry, \( T > \frac{0.154}{L} \)) \[14\]. Again, both of them can be obtained from the DBI action with the boundary condition previously mentioned.

\[
S_{\text{D8}} = -\mathcal{W} \int du u^4\left[f(u)x_4'^2 + \frac{1}{u^3}\right]^\frac{1}{2}. \tag{13}\]

The form of \( \mathcal{U} \) shaped embedding is fixed by the following equation of motion of the \( x_4 \) coordinate (Fig. 2),

\[
x_4'(u) = \frac{1}{u^{3/2}\sqrt{f(u)}}\left[\frac{u^8 f(u)}{u_0^8 f(u_0)} - 1\right]^{-1/2}. \tag{14}\]

The allowed parallel D8/\( \overline{\text{D}8} \) embedding satisfies,

\[
x_4'(u) = 0. \tag{15}\]

The transition between the chiral symmetry broken phase to the chiral symmetry restored phase is first order. In the dual gravity theory, the curvature of D8/\( \overline{\text{D}8} \) brane plays the role of order
parameter. Following these holographic aspects of the Sakai–Sugimoto model, in the next section we elaborate upon the response of the deconfined phase with broken chiral symmetry to an external electric field.

3. External electric field and the construction of effective Lagrangian

Recently in [12], authors have thoroughly studied the response Sakai–Sugimoto model to the external electric as well as magnetic field. Without the loss of generality we consider the electric field is directed along $z$ axis. The response to the external field is studied by turning on an Abelian component of the unbroken $U(N_V)$ gauge field in the world volume of probe D8/ $\overline{D8}$ brane.

$$\mathcal{A} = \frac{1}{N_f} \text{tr} \mathcal{A}_{N_V}. \tag{16}$$

In this analysis, assuming a current flowing along the direction of the electric field one can set a suitable choice of ansatz as

$$\mathcal{A}_z = E t + \mathcal{E}(u), \tag{17}$$

where $E$ is the constant electric background and $\mathcal{E}(u)$ is the time-independent fluctuation encoding the holographic notion of boundary current [4]. With the above choice of gauge field, the modified DBI action for the D8/$\overline{D8}$ brane embedded in the gravity background dual to the deconfined phase of finite temperature gauge theory, is given as

$$S_{D8} = -\mathcal{W} \int du u^4 \sqrt{f(u)x_4^2 + \frac{1}{u^3}} \left( 1 - \frac{e^2}{f(u)u^3} \right) + \frac{f(u)\xi'^2}{u^3}, \tag{18}$$

where we have introduced the dimensionless quantities like $e = 2\pi l_s^2 E$ and $\xi = 2\pi l_s^2 \mathcal{E}$.

It is important to note that the DBI action does not explicitly depend on the gauge field fluctuations. Therefore the equation of motion for $\xi$ evokes a constant of motion $\mathcal{J}$.

$$\frac{u f(u)\xi'}{\sqrt{f(u)x_4^2 + \frac{1}{u^3}}(1 - \frac{e^2}{f(u)u^3})} = \mathcal{J}. \tag{19}$$

Using (19) and substituting $\xi'$ in terms of the constant of motion $\mathcal{J}$ in (18) we get

$$S_{D8} = -\mathcal{W} \int du u^4 \sqrt{f(u)x_4^2 + \frac{1}{u^3}} \left( f(u) - \frac{e^2}{u^3} \right) \left( f(u) - \frac{\mathcal{J}^2}{u^3} \right)^{-1}. \tag{20}$$

Moreover, by demanding the reality of the DBI action (20) the response coefficient like conductivity is holographically computed in the boundary theory. By integrating (19) with respect to $u$ and then taking the boundary limit, the result gives the leading asymptotic behavior of the gauge field,

$$2\pi l_s^2 \mathcal{A}_z = e t - \frac{2}{3} \frac{\mathcal{J}}{u^3}, \tag{21}$$

where $\mathcal{J}$ can be physically interpreted as the conserved current associated with $\mathcal{A}_z$. To obtain the $\mathcal{U}$ shaped embedding of D8/$\overline{D8}$ brane we solve the equation motion for $x_4'$ derived from the
DBI action (18).

\[
\chi_4'(u) = \frac{1}{u^{3/2} \sqrt{f(u)}} \left[ \frac{u^8 f(u) \left( f(u) - \frac{\pi^2}{u} \right) \left( f(u) - \frac{\pi^2}{u} \right)^{-1}}{u_0^8 f(u_0) \left( f(u_0) - \frac{\pi^2}{u_0} \right) \left( f(u_0) - \frac{\pi^2}{u_0} \right)^{-1}} - 1 \right]^{-1/2}.
\]  

(22)

It is straightforward to get an on-shell DBI action \( S_{\mathcal{J} \neq 0}^{U} \) by substituting the above solution into (18). However, in [12], it has been shown that the physically favored configuration allows no current as the on-shell DBI action always satisfies \( S_{\mathcal{J} \neq 0}^{U} > S_{\mathcal{J} = 0}^{U} \), where \( S_{\mathcal{J} = 0}^{U} \) can be constructed by setting \( \mathcal{J} = 0 \) in (22) and then substituting it back to (18). Therefore, in the boundary theory the deconfined phase with broken chiral symmetry turns out to be insulating.

In our work, we accomplish a holographic reanalysis of the aforementioned boundary phase but with an explicit time dependence in the gauge field fluctuation inside the D8/\( \overline{D8} \) world volume. For further computation, we consider an approximation that the generalization of the time independent gauge field fluctuation into the time-dependent one will not modify the \( \mathcal{U} \) shaped classical embedding for \( \mathcal{J} = 0 \). The field ansatz in the Minkowski signature we are considering reads as

\[
\mathcal{A}_z = E t + \tilde{\mathcal{E}}(t, u).
\]  

(23)

Using this time-dependent gauge field ansatz (23), it is straightforward to write down the DBI action for the D8/\( \overline{D8} \) brane. We will purposefully factorize the action for our future convenience. In particular, we shall keep track of the electric field by redefining the term under first square root as the modified world volume metric and then derive the fluctuation Lagrangian.

\[
S_{\mathcal{D}8} = -\frac{2N_f T_{D8}}{g_s} \int d^9 x u^{-3/4} \sqrt{-\text{Det}(\mathcal{G} + (\mathcal{E} + \tilde{\mathcal{X}}(t, u)))}
= -\frac{2N_f T_{D8}}{g_s} \int d^9 x u^{-3/4} \sqrt{-\text{Det}(\mathcal{G} + \mathcal{E})} \sqrt{\text{Det}(1 + \tilde{\mathcal{X}}(t, u)(\mathcal{G} + \mathcal{E})^{-1})}.
\]  

(24)

In our notation, \( \mathcal{G} \) is again the world volume metric tensor for the D8/\( \overline{D8} \) brane. \( \mathcal{E} \) and \( \mathcal{X}(t, u) \) are the world volume field strength tensors for the constant background and the time-dependent fluctuation respectively. The non-zero components of \( \mathcal{E} \) are

\[
\mathcal{E}_{t z} = -\mathcal{E}_{xz} = e,
\]  

(25)

whereas, the same for \( \mathcal{X}(t, u) \) are

\[
\mathcal{X}_{t z} = -\mathcal{X}_{xz} = \tilde{\xi}(t, u), \quad \mathcal{X}_{t u} = -\mathcal{X}_{zu} = \tilde{\xi}'(t, u).
\]  

(26)

Similar to the time independent case, here we also introduce the dimensionless quantities, \( e = 2\pi l_s^2 E \) and \( \tilde{\xi} = 2\pi l_s^2 \tilde{\xi} \). For small values of gauge field fluctuation, we expand the DBI action using the following matrix identity,

\[
\sqrt{\text{Det}(1 + M)} = 1 + \frac{1}{2} \text{Tr} M + \frac{1}{8} (\text{Tr} M^2) - \frac{1}{4} \text{Tr} (M)^2,
\]  

(27)

where \( M \) stands for \( \tilde{\mathcal{X}}(t, u)(\mathcal{G} + \mathcal{E})^{-1} \). In this expansion, it is sufficient for our purpose to keep the terms up to the quadratic fluctuations.

Neglecting those terms which are either constant or linear in fluctuation, thus not contributing to the equation of motion and putting everything else together up to quadratic order we get the
classical action for fluctuation,

\[ S_{cl} = -\frac{2N_f T_{D8}}{g_s} \int d^9x \, u^{-3/4} \]

\[ \times \sqrt{f(u)u^{3/2}\left(1 - \frac{e^2}{u^3 f(u)}\right)} \left(\frac{u^{3/2}}{f(u)}\right)^3 \left(\frac{1}{u^3} + f\dot{x}_4^2\right) g_{\Omega_4\Omega_4} \]

\[ \times \left(-\frac{1}{2f(u)u^3 (1 - \frac{e^2}{f(u)u^3})^2} + \frac{f(u)\dot{\xi}^2}{2u^3 (1 - \frac{e^2}{f(u)u^3})(\frac{1}{u^3} + f(u)\dot{x}_4^2)}\right). \]  

(28)

Our aim is to write \( S_{cl} \) obtained by expanding DBI action up to quadratic order of fluctuation in a canonical four-dimensional Maxwell form. The motivation for imposing the restriction on space–time dimension comes from the fact that at the end we aim to study the effect of Schwinger pair production on electrical conductivity in a strong coupling phase by analyzing the dynamics of gauge field fluctuation governed by a quantum corrected action and the quantum contribution for the action we are interested in is originally constructed in [17] for four-dimensional flat space–time. Therefore to successfully accommodate the quantum correction we need the above mentioned dimensional reduction. To do so we proceed in the following way. First, by observing the factors under the square root in Eq. (28) we make an ansatz for a modified induced metric \( \tilde{G} \). This modified metric absorbs the extra factor coming due to the presence of constant electric background \( E \) and reduces to the original one, \( G \) in the \( E \to 0 \) limit. Then we execute a dimensional reduction on the DBI action realized over nine-dimensional world volume coordinates \((t, x, y, z, u, \theta_1, \theta_2, \theta_3, \theta_4)\) and write it as an effective action in four-dimensional space–time \((t, y, z, u)\), where \( \theta_i \)'s are the angular variables in \( \Omega_4 \). Finally we redefine the coupling parameter in the four-dimensional effective action in such a way the final expression takes the desired canonical form. The ansatz for the modified induced metric \( \tilde{G} \) of our interest is set as

\[ \tilde{G}_{tt} = \left(1 - \frac{e^2}{f(u)u^3}\right) G_{tt}, \]

\[ \tilde{G}_{ij} = G_{ij} \quad \forall i, j. \]  

(29)

With this modified metric (29), we rewrite the \( S_{cl} \) in (28) in the following way,

\[ S_{DBI} = -\frac{2N_f T_{D8}}{g_s} \int d^9x \, u^{-3/4} \sqrt{-\text{Det} \tilde{G}} \left(\frac{1}{1 - \frac{e^2}{f(u)u^3}}\right) \]

\[ \times \frac{1}{2} \{ \tilde{G}^{tt} \tilde{G}^{zz} \chi_{t}^2 + \tilde{G}^{uu} \tilde{G}^{zz} \chi_{u}^2 \}. \]  

(30)

Moreover, we perform the dimensional reduction from nine to four dimensions. It is important to note that in doing so we make another assumption that one of the spatial boundary coordinates is wrapped in a circle and introduces another energy scale \(< u_{KK} \). For all subsequent part of this paper we restrict our analysis far below this newly introduced energy scale. Redefining the coupling constant in a suitable way we finally land up with a four-dimensional effective action in a canonical Maxwell form.

\[ S_{\text{Classical}} = -\int d^4x \, \frac{1}{8\pi g_s^{\text{eff}}} \sqrt{-\text{Det} \tilde{G}_4} \frac{1}{4} F_4^2, \]  

(31)
where, \( g_s^{\text{eff}} = \frac{g_s(1 - \frac{e^2}{f_{\text{u}u}})}{2N_f T_D^8 V_4 \sqrt{4\pi} \ell_P^4 u} = \tilde{g}_s^{\text{eff}} \frac{1 - \frac{e^2}{a}}{u} \). \( \tilde{G}_4 \) is the four-dimensional effective metric having non-zero diagonal element,

\[
(G_4)_{tt} = \tilde{G}_{tt}, \quad (G_4)_{uu} = \tilde{G}_{uu}, \quad (G_4)_{yy} = \tilde{G}_{yy}, \quad (G_4)_{zz} = \tilde{G}_{zz}.
\]

(32)

Since \( u \) has been chosen to have zero length dimensions, the running Yang–Mills coupling constant consistently turns out to be dimensionless. The non-zero components of the four-dimensional field strength tensor \( F_4 \) are the following:

\[
F_{4t\tau} = \tilde{\Sigma}(t, u), \quad F_{4u\tau} = \tilde{\Sigma}'(t, u).
\]

(33)

Up to this stage, we construct an effective Maxwell action for gauge field fluctuation around a constant electric background in four-dimensional subspace of D8/D8 brane world volume characterized by the modified metric \( \tilde{G}_4 \). In the next section, we discuss how to add a four-dimensional effective action arising from the one-loop correction.

### 4. Adding one-loop quantum correction

In this section we discuss how to add a consistent one-loop effective action to the tree level canonical Maxwell action, (31) in four-dimensional space–time defined by \( \tilde{G}_4 \). Following the methodology of QED, we point out that the structure of quantum action, \( S^Q \) solely depends on the loop contribution arising from the vacuum polarization tensor \( M_{\mu\nu} \), generically prescribed as a two-point correlator of the propagating gauge field. In this paper we consider the modification of gauge field propagation in vacuum up to one-loop order. The modification is due to relevant polarization tensor \( M_{\mu\nu} \) encoding the process of virtual quark–antiquark pair creation. In the presence of an external homogeneous electric field \( E \), the polarization tensor develops an explicit dependence on the respective electric field and further restricts the vacuum dynamics. Motivated by Schwinger’s seminal work [1], there has been already an extensive perturbative as well as non-perturbative study to compute the vacuum polarization tensor for the pair creation process in terms of double parameter integral, including the presence of a background electromagnetic field in four-dimensional Minkowski space [17, 22–27].

In this kind of computations the externally set vector parameters are the field vectors and the transferred four momentum vector \( k_\mu \). For the simplification of the computation it is always convenient to decompose the background space–time along the parallel and the perpendicular directions to the external field. Without the loss of generality we consider the electric field is directed along \( z \) axis. The four-dimensional Minkowski space–time coordinates \( (t, y, u, z) \), can be grouped as \( (t, z) \Rightarrow \parallel \), and \( (y, u) \Rightarrow \perp \). Subsequently, the metric can be decomposed in the following way,

\[
\eta^{\mu\nu} = \eta^{\mu\nu}_{\parallel} + \eta^{\mu\nu}_{\perp},
\]

(34)

where \( \parallel \perp \) stands for \( (t, z)/(y, u) \) respectively. Furthermore, the transferred four momentum can be decomposed as

\[
k^\mu = k^\mu_\parallel + k^\mu_\perp.
\]

(35)

With above consideration, following [17], the momentum space representation of the polarization tensor in the presence of a homogeneous external field can be expressed up to one-loop level as
where $C$ is an overall normalization constant and the other parameters $\phi_0$, $N_0$ and $N_1$ are defined as
\begin{align}
N_0 &= \cosh w' v - v \sinh w' v \coth w', \\
N_1 &= 2 \frac{\cosh w' - \cosh w' v}{\sinh^2 w'} - N_0, \\
s\phi_0 &= s m^2 + \frac{k_r^2}{2} \frac{\cosh w' - \cosh w' v}{z' \sinh w'} + \frac{1}{4} s (1 - v^2) k_t^2, \\
w' &= q s E,
\end{align}
where $m$ and $q$ are the mass and charge of the flavor quark. The contact term in (36) can be fixed by demanding the following two conditions,
\begin{equation}
k^2 = 0, \quad \lim_{k^2 \to 0} M_{\mu\nu} = 0.
\end{equation}
The form of contact term turns out to be
\begin{equation}
\text{contact term} = -(1 - v^2) (\eta^{\mu\nu} k^2 - k^\mu k^\nu).
\end{equation}
The constraints $k^2 = 0$ on the regime of transferred four momentum signifies that the derivation of the $M_{\mu\nu}$ is fitted to on-the-light-cone dynamics. There are other possible forms of contact term which can be derived on-the-light-cone. However, we have observed that the symmetry of frequency dependent conductivity due to the pair creation is heavily dependent on the form of contact term as it contributes to the quantum action. Moreover, the holographic realization of the linear response theory suggests that the real part of the frequency dependent conductivity should be symmetric with respect to the associated frequency. By accounting these two facts we fix the contact term contributing to the polarization tensor to the above form and the one-loop action in momentum representation becomes
\begin{equation}
S^Q = -\int \frac{d^4k}{(2\pi)^4} \sqrt{-\text{Det} \eta A^{\mu}(k) M_{\mu\nu}(k) A^{\nu}(-k)},
\end{equation}
where $M_{\mu\nu}$ is specified by Eqs. (36), (37), (38), (39), (40) and (42). Translating from the momentum representation to the space–time representation the quantum action simplifies to
\begin{equation}
S^Q = -\frac{C}{4} \int d^4x \sqrt{-\text{Det} \eta} \left[ \text{Tr} \left( (F_4)_\parallel, \eta_{\parallel}(F_4)_\parallel, \eta_{\parallel} \right) (I_1) + 2 \text{Tr} \left( (F_4)_{\text{mixed}}, \eta_{\parallel}, (F_4)_{\text{mixed}}, \eta_{\perp} \right) (I_2) \right],
\end{equation}
where $F_4$ is the same second rank field strength tensor of propagating gauge field defined in (33). More specifically, $(F_4)_\parallel/(F_4)_\perp$ allows only those space–time indices which signify the parallel/perpendicular direction to the external homogeneous electric field. On the other hand, $(F_4)_{\text{mixed}}$ signifies that one of its space–time indices belongs to $(t, z) \Rightarrow \parallel$ and the other one takes value in $(u, y) \Rightarrow \perp$. Furthermore, $I_1$ and $I_1$ carry the information of the double integrals
in the action.

\[
I_1 = \int \frac{ds}{s} \int \frac{d\nu}{2} e^{-is\phi_0} \frac{w'}{\sinh w'} (N_0 + N_1 - (1 - \nu^2)),
\]

\[
I_2 = \int \frac{ds}{s} \int \frac{d\nu}{2} e^{-is\phi_0} \frac{w'}{\sinh w'} (N_0 - (1 - \nu^2)).
\] (45)

However, we need to construct a one-loop quantum corrected action in the four-dimensional curved space–time denoted by \( \tilde{G}_4 \). Covariantization is to be done by using Riemann normal coordinates (RNC) [28,29]. If we denote by \( y^\mu \) the RNC, then one interprets \( k_\mu \) as \(-i \frac{\partial}{\partial y^\mu} \). Various powers of derivatives can be re-expressed in terms of covariant derivatives and the curvature tensors. If the radius of curvature of the space is large enough, then to a first approximation we can neglect the curvature terms. This is what will be done in this paper. In the large \( N_c \) limit, we can consider only the leading term for simplification of our analysis.

\[
S^Q = -\frac{C}{4} \int d^4x \sqrt{-\text{Det} \tilde{G}_4} [\text{Tr} (F_4)_\| \cdot (\tilde{G}_4)_\| \cdot (\tilde{G}_4)_\|] (I_1) + 2 \text{Tr} [(F_4)_{\text{mixed}} \cdot (\tilde{G}_4)_\| \cdot (\tilde{G}_4)_{\text{mixed}} (I_2)],
\] (46)

where we consider the contravariant components of the metric tensor \( \tilde{G}_4 \) and the covariant component of the field tensor \( F_4 \). It is extremely difficult to evaluate the integrals \( I_1 \) and \( I_2 \) with full generality. One of the ways out is to constrain the full parameter space by imposing weak field approximation,

\[
\frac{qE}{m^2} \ll 1.
\] (47)

Within this weak field approximation if we expand the r.h.s. of (39) in the powers of \( E \), the leading order contribution becomes: (We are interested in the frequency dependence of the conductivity. Currents are assumed to be time varying but uniform in space. So we set \( \tilde{k} = 0 \) and \( k_0 = \omega \neq 0 \).)

\[
s\phi_0 = sm^2 + \frac{1}{48} q^2 \omega^2 |\tilde{G}_{00}| s^3 (1 - \nu^2)^2 E^2.
\] (48)

We can define a relevant dimensionless parameter \( \lambda \) such that it takes both small and large numerical values depending on the magnitude of frequency and the intensity of the field.

\[
\lambda = \frac{3 qE w}{2 m^2 m^2}.
\] (49)

Similarly, under same approximation

\[
(N_0 + N_1 - (1 - \nu^2)) \frac{w'}{\sinh w'} = -\frac{1}{12} E^2 q^2 s^2 (5 - 6 \nu^2 + \nu^4),
\]

\[
(N_0 - (1 - \nu^2)) \frac{w'}{\sinh w'} = -\frac{1}{6} E^2 q^2 s^2 (1 - \nu^2)^2.
\] (50)

It is important to note that, weak field approximation does not make contradiction with large and small values of \( \lambda \). In both cases the leading contribution for the integral mentioned in (36) comes from \( s \ll 1 \) [22].
Plugging the results of (48) and (50) in (45) we get

\[ I_1 = -\frac{1}{12} E^2 q^2 \int dv \left( 5 - 6v^2 + v^4 \right) \int sds e^{-is\phi_0(\omega)}, \]

\[ I_2 = -\frac{1}{6} E^2 q^2 \int dv \frac{1}{2} (1 - v^2)^2 \int sds e^{-is\phi_0(\omega)}. \]  

(51)

Now we evaluate the integrals \( I_1 \) and \( I_1 \) within the weak field approximation. To do so, first we work out the \( s \) integrals by applying the standard integral representation of the known special function. We introduce the variable \( r \) and the parameter \( \rho \) for simplification of the computation,

\[ r = \frac{1 - v^2}{4} \frac{w}{m q E} \sqrt{|\tilde{g}_{00}|} s, \]

\[ \rho = \frac{4}{\lambda} \frac{1}{1 - v^2} \frac{1}{\sqrt{|\tilde{g}_{00}|}}. \]  

(52)

With the above redefinition, the relation between the \( s \) and \( r \) can be written in a compact form,

\[ s\phi_0 = \frac{3}{2} \rho \left( r + \frac{r^3}{3} \right), \quad s = \left( \frac{3\rho}{2m^2} \right) r. \]  

(53)

Correspondingly the real and the imaginary part of the \( s \) integral are successfully translated into respective components of \( r \) integral.

\[ \int_0^\infty sds \cos (s\phi_0) = \frac{1}{m^4} \left( \frac{3\rho}{2} \right)^2 \int_0^\infty rdr \cos \left( \frac{3\rho}{2} \left( r + \frac{r^3}{3} \right) \right), \]  

(54)

\[ \int_0^\infty sds \sin (s\phi_0) = \frac{1}{m^4} \left( \frac{3\rho}{2} \right)^2 \int_0^\infty rdr \sin \left( \frac{3\rho}{2} \left( r + \frac{r^3}{3} \right) \right). \]  

(55)

Again a suitable change of variable \( r = \left( \frac{3\rho}{2} \right)^{-1/3} p \), recasts the r.h.s. of the equation (54) in the desired form.

\[ \int_0^\infty rdr \cos \left( \frac{3\rho}{2} \left( r + \frac{r^3}{3} \right) \right) = \left( \frac{3\rho}{2} \right)^{-2/3} \int_0^\infty pdp \cos \left( \left( \frac{3\rho}{2} \right)^{2/3} p + \frac{p^3}{3} \right), \]

\[ = \pi \left( \frac{3\rho}{2} \right)^{-2/3} \text{Gi} \left[ \left( \frac{3\rho}{2} \right)^{2/3} \right]. \]  

(56)

where \( \text{Gi}[x] \) is the inhomogeneous Airy function/Scorer function satisfying the following differential equation and the boundary conditions [30].

\[ \text{Gi}''[x] - x \text{Gi}[x] = -\frac{1}{\pi}, \]

\[ \text{Gi}[0] = \frac{1}{3} \text{Bi}[0] = \frac{1}{\sqrt{3}} \text{Ai}[0], \]

\[ \text{Gi}'[0] = \frac{1}{3} \text{Bi}'[0] = -\frac{1}{\sqrt{3}} \text{Ai}'[0]. \]  

(57)
where $A_i(x)$ and $B_i(x)$ are the homogeneous Airy function of the first kind and the second kind respectively. Integral representation of $G_i(x)$ is given as

$$G_i(x) = \frac{1}{\pi} \int_0^\infty dz \sin \left[ xz + \frac{z^3}{3} \right].$$  \hspace{1cm} (58)

Moreover, using integral representation of the modified Bessel function, the r.h.s. of Eq. (55) can be expressed as

$$\int_0^\infty r dr \sin \left[ \frac{3\rho}{2} \left( r + \frac{r^3}{3} \right) \right] = \frac{K_{2/3}[\rho]}{\sqrt{3}},$$ \hspace{1cm} (59)

where $K_{2/3}[\rho]$ is the generalized Bessel function with $2/3$ weight \[31\]. Now combining all the results we can finally write down the result of s integral calculation,

$$\int_0^\infty s ds e^{-is\phi_0} = \frac{\pi}{m^4} \left( \frac{3\rho}{2} \right)^{4/3} G_i \left[ \left( \frac{3\rho}{2} \right)^{2/3} \right] - i \left( \frac{3\rho}{2m^2} \right)^2 \frac{K_{2/3}[\rho]}{\sqrt{3}}. \hspace{1cm} (60)$$

For our present analysis we are interested in the asymptotic region of the associated parameter $\rho$. If we consider $\rho$ is very large ($\rho \gg 1$), the $s$ integral simplifies to

$$\int_0^\infty s ds e^{-is\phi_0} = -\frac{1}{m^4} - i \left( \frac{3\rho}{2m^2} \right)^2 \sqrt{\frac{\pi}{6\rho}} e^{-\rho}. \hspace{1cm} (61)$$

Here we have used the following two asymptotic expansion \[31\],

$$G_i'[x] \approx -\frac{1}{\pi x^2},$$

$$K_{2/3}[x] \approx \sqrt{\frac{\pi}{2x}} e^{-x}. \hspace{1cm} (62)$$

On the other hand, when $\rho$ takes small values ($\rho \ll 1$) the $s$ integral modifies as

$$\int_0^\infty s ds e^{-is\phi_0} = \rho^{4/3} \frac{c_1}{m^4} \left( c_1 = \Gamma \left[ \frac{2}{3} \right] \left( \frac{3}{2^{7/3}} - i \frac{3^{3/2}}{2^{7/3}} \right) \right). \hspace{1cm} (63)$$

In deriving equation (63), we have used the following approximations \[31\],

$$G_i'[x] \approx \frac{3^{-1/3}}{2\pi} \Gamma \left[ \frac{2}{3} \right],$$

$$K_{2/3}[x] \approx 2^{-1/3} \Gamma \left[ \frac{2}{3} \right] x^{-2/3}. \hspace{1cm} (64)$$

Having the $s$ integral done, we aim to work out the $v$ integral within the restricted range of parameters. For small values of $\lambda$ we combine the results mentioned in (61) and (51),
\[ I_1 = \frac{4q^2E^2}{15m^4} + \frac{3i q^2 E^2}{2\sqrt{6m^4}|\tilde{G}_4|_{00}} \sqrt{\frac{\pi \sqrt{|(\tilde{G}_4)_{00}|}}{\lambda^3}} \times \int_0^1 dv \sqrt{1 - v^2} \left( \frac{5 - 6v^2 + v^4}{(1 - v^2)^2} \right) e^{\delta' \sqrt{|(\tilde{G}_4)_{00}|(1 - v^2)}}. \]  

(65)

Implementing the change of variable \[ x = \frac{1}{1 - v^2} \] and applying the useful identity,

\[ \int_{\frac{1}{x}}^{\infty} \frac{dx}{x^2} e^{-\delta x} \approx \sqrt{\frac{\pi}{\delta}} e^{-\frac{\delta}{e \delta}} \left[ 1 + \mathcal{O}\left( \frac{1}{\delta} \right) \right], \]  

(66)

we get the final expression of \( I_1 \) for small \( \lambda \),

\[ I_1 = \left( \frac{4 \omega_0^2}{15m^2} \right) + i \left( \frac{5 \pi}{2 \sqrt{6 \sqrt{\tilde{G}_{00}}}} \left( \frac{\omega}{\omega_0} \right) \right) e^{\frac{-4}{\lambda \sqrt{\tilde{G}_{00}}}} \left( \omega_0 = \frac{qE}{m} \right). \]  

(67)

Similarly the \( I_2 \) integral turns out to be

\[ I_2 = \left( \frac{4 \omega_0^2}{45m^2} \right) + i \left( \frac{\pi}{\sqrt{6 \sqrt{\tilde{G}_{00}}}} \left( \frac{\omega}{\omega_0} \right) \right) e^{\frac{-4}{\lambda \sqrt{\tilde{G}_{00}}}}. \]  

(68)

For large \( \lambda \), same type of computation yields,

\[ I_1 = -\left[ \left( \frac{2^\frac{5}{6} \sqrt{\pi} \Gamma[\frac{5}{3}] \Gamma[\frac{1}{3}]}{3^\frac{2}{3} \Gamma[\frac{13}{6}]} \right) \left( \frac{m\omega_0}{\tilde{G}_{00}\omega^2} \right)^{\frac{5}{3}} - i \left( \frac{2^\frac{5}{6} \sqrt{\pi} \Gamma[\frac{5}{3}] \Gamma[\frac{1}{3}]}{3^\frac{2}{3} \Gamma[\frac{13}{6}]} \right) \left( \frac{m\omega_0}{\tilde{G}_{00}\omega^2} \right)^{\frac{5}{3}} \right], \]  

\[ I_2 = -\left[ \left( \frac{2^\frac{5}{6} \sqrt{\pi} \Gamma[\frac{5}{3}]^2}{7 \times 3^\frac{2}{3} \Gamma[\frac{11}{6}]} \right) \left( \frac{m\omega_0}{\tilde{G}_{00}\omega^2} \right)^{\frac{5}{3}} - i \left( \frac{2^\frac{5}{6} \sqrt{\pi} \Gamma[\frac{5}{3}]^2}{7 \times 3^\frac{2}{3} \Gamma[\frac{11}{6}]} \right) \left( \frac{m\omega_0}{\tilde{G}_{00}\omega^2} \right)^{\frac{5}{3}} \right]. \]  

(69)

Combining (31) and (46) we construct the covariant form of Maxwell action with the one-loop quantum correction in a four-dimensional space–time characterized by \( \tilde{G}_4 \). In the next section we study the gauge field dynamics governed by the covariant quantum corrected action. In particular, with the boundary condition set at large \( u \), we solve the equation of motion for gauge field. Furthermore, by utilizing this solution, we follow the Kubo’s prescription to compute a frequency dependent conductivity.

5. Computation of frequency dependent conductivity

We consider the covariant form total effective action constructed in four-dimensional curved space–time,

\[ S_{\text{total}} = \frac{1}{2} \int d^4x \frac{1}{g_{\text{eff}}^2} \left[ \tilde{\mathcal{E}}^2 \varphi_1 - \tilde{\mathcal{E}}^4 \varphi_2 \right]. \]  

(70)

where \( \varphi_1 \) and \( \varphi_2 \) are the following:
\[
\Psi_1(u) = \sqrt{\det \tilde{G}_4} \frac{1}{u^3 f(1 - \frac{e_{10}^2}{f u^2})} \left\{ 1 - g_{s}^{\text{eff}} C(I_1) \right\},
\]
\[
\Psi_2(u) = \sqrt{\det \tilde{G}_4} \frac{f}{u^3 \left( \frac{1}{u^2} + f X_4'^2 \right)} \left\{ 1 - g_{s}^{\text{eff}} C(I_2) \right\}.
\]

(71)

To write the total action consistently we have set $2\pi l_s^2 = 1$. The relative sign between the classical and quantum part is still to be determined. By comparing the standard form $e^{iS_{\text{real}}} e^{-i\Gamma} = e^{i(S_{\text{real}} + i\Gamma)}$ we conclude that the sign of the normalization constant is negative. For simplicity we set $C = -1$.

Assuming the space–time structure of the gauge field $\tilde{\Sigma}(u,t) = \tilde{\Sigma}(u)e^{-i\omega t}$ we get the equation of motion,

\[
\tilde{\Sigma}''(u)\Psi_2(u) + \tilde{\Sigma}'(u)\Psi_2'(u) + \omega^2 \Psi_1(u)\tilde{\Sigma}(u) = 0.
\]

(72)

The solution of the equation of motion with appropriate boundary conditions contains both normalizable and non-normalizable modes. The associated Green’s function $G_R$ is defined as the ratio of normalizable to non-normalizable modes. Using Kubo formula in linear response theory, the working formula for the conductivity can be constructed from the structure of $G_R$.

\[
\sigma = \frac{G_R}{i\omega}.
\]

(73)

Before obtaining a solution of (72) we make a suitable coordinate transformation.

\[
\mathcal{Y}^2 = 1 - \frac{u}{u_0}.
\]

(74)

In this coordinate system, the equation of motion takes the following form,

\[
\left[ \frac{d^2 \tilde{\Sigma}}{d\mathcal{Y}^2} \left( 1 - \mathcal{Y}^2 \right)^4 - \frac{d\mathcal{Y}}{d\mathcal{Y}^2} \left( 3\mathcal{Y}^2 + 1 \right)(1 - \mathcal{Y}^2)^3 \right] \Psi_2 + \left[ \frac{d\tilde{\Sigma}}{d\mathcal{Y}} \frac{d\Psi_2}{d\mathcal{Y}^2} \left( 1 - \mathcal{Y}^2 \right)^4 \right] + \omega^2 \tilde{\Sigma} \Psi_1(u) = 0.
\]

(75)

The range of $\mathcal{Y}$ is fixed as $[0, -1]$. In the nine-dimensional world volume theory of the D8/D8, the above choice of $\mathcal{Y}$ coordinate has a natural interpretation. The location where the D8 and D8 meet is denoted by $\mathcal{Y} = 0$, whereas the intersection points between the D8 and D8 with D4 are specified as $\mathcal{Y} = \pm 1$. For convenient nomenclature we refer $\mathcal{Y} = 1$ as the “boundary” and $\mathcal{Y} = -1$ as the “horizon”. For all subsequent analysis we set $u_T = 1$. The equation of motion for $\tilde{\Sigma}(u)$ near the “horizon” becomes

\[
(1 + \mathcal{Y})\tilde{\Sigma}''(\mathcal{Y}) - \frac{1}{2} \tilde{\Sigma}'(\mathcal{Y}) + \frac{2\omega^2}{u_0} \tilde{\Sigma}(\mathcal{Y}) = 0.
\]

(76)

While deriving Eq. (76) we have used the asymptotic form of $\Psi_1$ and $\Psi_2$.

\[
\Psi_1 \approx \frac{\sqrt{1 - \mathcal{Y}^2}}{\sqrt{u_0}} \frac{1}{g_{s}^{\text{eff}}},
\]
\[
\Psi_2 \approx \frac{u_0^{5/2}}{(1 - \mathcal{Y}^2)^{5/2}} \frac{1}{g_{s}^{\text{eff}}}.
\]

(77)
We aim to solve to the above equation of motion near “horizon” in leading order of $\mathcal{Y}$. Considering the newly defined argument $\mathcal{Z}$, satisfying
\[ \Psi_2 \frac{d}{d\mathcal{Y}} = \frac{d}{d\mathcal{Z}}, \] (78)
we rewrite the equation (76) as follows
\[ \frac{d^2 \tilde{\mathcal{S}}}{d^2 \mathcal{Z}} + \omega^2 \Psi_1 \Psi_2 \tilde{\mathcal{S}} = 0. \] (79)
Choosing a suitable ansatz leads to the following set of equations,
\[ \tilde{\mathcal{S}} = e^{-S(\mathcal{Z})}, \]
\[ \tilde{\mathcal{S}}'' = -S''(\mathcal{Z}) e^{-S(\mathcal{Z})} + S'(\mathcal{Z})^2 e^{-S(\mathcal{Z})}. \] (80)
Ignoring $S''(\mathcal{Z})$ for the leading order solution we get
\[ S = \int \mathcal{Z} \pm i \omega \sqrt{\Psi_1 \Psi_2} d\mathcal{Z}'. \] (81)
The form of $\Psi_1$ and $\Psi_2$ at $\mathcal{Y} = -1$ is easy to derive from (77). Therefore, by solving Eq. (76), we get the leading term in $\tilde{\mathcal{S}}$,
\[ \tilde{\mathcal{S}} = e^{ \pm \frac{1}{\sqrt{\mu_0}} \sqrt{1 + \mathcal{Y}} }. \] (82)
For computation of the conductivity usually one has to employ the ingoing boundary condition at some location in the space–time where the flavor brane touches the real horizon of black hole background. In our analysis, neither the flavor branes touch the black hole horizon nor any induced horizon gets developed in the world volume. Similar kind of situation has been discussed in [15], where the background holographically corresponds to a confining phase of a strongly coupled system at finite temperature. Since they have studied a system with real Lagrangian, they make a choice of a complex boundary condition at $\mathcal{Y} = -1$ to get a finite real part of the conductivity. However, in our case the Lagrangian is complex. So the most general choice is the standing wave ansatz, i.e., the real boundary condition,
\[ \tilde{\mathcal{S}} = c_1 e^{ \frac{2\sqrt{2}\omega}{\sqrt{\mu_0}} \sqrt{1 + \mathcal{Y}} } + c_1^* e^{- \frac{2\sqrt{2}\omega}{\sqrt{\mu_0}} \sqrt{1 + \mathcal{Y}} }, \]
\[ = \cos \left[ \frac{2\sqrt{2}\omega}{\sqrt{\mu_0}} \sqrt{1 + \mathcal{Y}} + \phi \right], \] (83)
where $c_1$ is an arbitrary constant and $\phi$ is the phase. We make a choice at $\mathcal{Y} = -1 + \epsilon$, $\phi = - \frac{2\sqrt{2}\omega}{\sqrt{\mu_0}} \sqrt{\epsilon}$ As a result of this choice, we can define the boundary conditions,
\[ \tilde{\mathcal{S}} (\mathcal{Y} = -1) \sim 1, \quad \tilde{\mathcal{S}}' (\mathcal{Y} = -1) \sim 0. \] (84)
At the “boundary”, $\mathcal{Y} = +1$ the equation of motion simplifies as
\[ (1 - \mathcal{Y}) \tilde{\mathcal{S}}'' + \frac{1}{2} \tilde{\mathcal{S}}' + \frac{2\omega^2}{\mu_0} \tilde{\mathcal{S}} = 0. \] (85)
The series solution for $\tilde{\mathcal{S}} (\mathcal{Y})$ near $\mathcal{Y} = +1$ reads as
Fig. 3. (a) and (b) depicts the real part of the conductivity for the set of parametric values \((E = \frac{1}{10^2}, m = 5)\) and \((E = \frac{1}{10^2}, m = 10)\). Here we have chosen \(u_0 = 10\).

Fig. 4. (a) and (b) shows the real part of the conductivity with the choices of the parameters are \((E = \frac{1}{50}, m = 5)\) and \((E = \frac{1}{50}, m = 10)\) respectively. Here we have chosen \(u_0 = 10\).

\[
\tilde{\Xi}(y) = M \left[ 1 + M_1 (1 - y) + M_2 (1 - y)^2 + \cdots \right], \\
+ N (1 - y)^{3/2} \left[ 1 + N_1 (1 - y) + N_2 (1 - y)^2 + \cdots \right],
\]

where \(M\) and \(N\) are the parameters to be determined by boundary conditions and \(M_i\) and \(N_i\) depend on the parameters of the theory. The Green’s function of our interest is given by

\[
G_R \sim \frac{N}{M} = \frac{4}{3} \frac{\sqrt{1 - \mathcal{V} \gamma}}{\mathcal{E}(\gamma) \gamma^{3/2}} d^2 \mathcal{E}.
\]

Finally the conductivity is calculated by inserting the form of \(G_R\) in (73). Here we actually solve Eq. (75) subject to the boundary conditions (84) using numerical methods. Then we apply the full solution together with (73) and (87) to compute the conductivity at the “boundary” \(\mathcal{V} = 1\). Finally, we plot the real part of the conductivity with respect to the frequency for both small and large \(\lambda\). For example, Figs. 3 and 4 show the variation of the real part of \(\sigma\) with respect to frequency for \(\lambda < 1\). We also show the same for \(\lambda > 1\) in Fig. 5. From both cases it is evident that if we increase the mass, the conductivity is less. On the other hand conductivity enhances with the external electric field. For large and small \(\lambda\) real part of the conductivity seems to diverge periodically with respect to \(\omega\). The periodicity of this divergence almost remains constant as we vary the parameters \(m\) and \(E\). For \(\lambda < 1\) we observe that the conductivity takes negative value.
Fig. 5. In this figure we have plotted the real part of the $\sigma$ with respect to $\omega$ for the set of parametric values $(E = \frac{1}{30^3}, m = \frac{1}{100})$ and $(E = \frac{1}{2\times30^3}, m = \frac{1}{100})$. Here we have chosen $u_0 = 1000$.

Fig. 6. (a), (b) and (c) depict the variation of the separation between two consecutive peaks in Re $\sigma$ vs. $\omega$ plot for three different values of $u_0 = 10, 100, 500$. The choice of others parameters are fixed as $(E = \frac{1}{2\times30^3}, m = \frac{1}{100})$.

with respect to the higher values of frequency. For $\lambda > 1$, the region of negative conductivity lies within the lower range of $\omega$. For higher range of $\omega$ it is positive and always periodically divergent. In Fig. 6 we have plotted the conductivity for three different values of $u_0 = 10, 100, 500$ keeping the electric field $E = \frac{1}{2\times30^3}$ and $m = \frac{1}{100}$ fixed. We observe in this plot that for lower values of $u_0 = 10$ the real part of the $\sigma$ takes negative values. As we increase the $u_0 = 100, 500$ the conductivity is always positive. Most importantly, the frequency of the peaks in conductivity
plots scales as $\sqrt{u_0}$. These peaks may arise from the poles of the Green’s function and the corresponding frequency modes are well known as normal/quasinormal modes. In the confined phase of Sakai–Sugimoto model, the normal modes determines the mass of the meson spectrum and the mass parameter turns out to be proportional to the Kaluza–Klein mass ($m_{KK}$) of the theory. Since $m_{KK}$ scales as the square root of $u_{KK}$, the period of normal modes also scales as $\sqrt{u_{KK}}$. In our case, we have set the $u_{KK} = 1$ and in the deconfined phase the role of $u_{KK}$ is played by the radius of horizon $u_T$ (the $u - x_4$ subspace is a topologically cylinder). Therefore the only relevant parameter to set the mass scale is $u_0$. As we already observe that the periodicity of the frequencies where the conductivity (evaluated numerically) seems to diverge scales as $\sqrt{u_0}$ So it is tempting to conclude that these apparent numerical divergences are in fact poles appearing in the Green’s function and actually characterize the quasinormal modes.

6. Conclusion

In this paper we have discussed the effect of pair production in modifying the conductivity in strong coupling situations. The strong coupling theory is described holographically by a system of flavor and color branes. The flavor branes provide flavor quantum numbers to particles. The one-loop Schwinger effect modifies the effective action for a photon on the flavor brane. Since the coupling constant $g_{YM}^2 = g_s$ is small, this one-loop calculation is reliable. The photon is dual to a conserved current in the boundary theory. The current–current correlator in the boundary is modified. This modifies the conductivity of the boundary theory. The interesting point is that because the Schwinger effect introduces an imaginary part to the effective action, the modified conductivity in the boundary acquires a real part. Thus an insulator starts to have a small real conductivity in the presence of an electric field. What is doubly interesting is that the differential conductivity for some range of parameters can be negative. This effect is known in condensed matter contexts in semiconductor diodes such as tunnel diodes and Gunn diodes although the mechanism of negative differential resistance is specific to the device. Here we have a strong coupling version of this. This is similar to the effect discussed in [19–21]. The main difference is that in these papers the effect was already there at the classical level on the flavor brane because of the presence of an open string metric horizon. In our case there is no such horizon and shows up only at one loop.

It would be interesting to identify experiments analogous to those suggested in [18] where the Schwinger effect would be observable.

Acknowledgements

The authors would like to acknowledge Ganapathy Baskaran, S. Kalyana Rama, Nemani V. Suryanarayana, Sudipta Mukherji, Amitav Virmani and Swarnendu Sarkar for various fruitful discussions. S.C. would like to thank Sudipto Paul Chowdhury and Sunanda Patra for useful discussion.

References