Extended Standard Model in multi-spinor field formalism: Visible and dark sectors

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1. Introduction

Quarks and leptons exist in threefold family modes with color and electroweak symmetries. Recent observations by WMAP [1] and Planck [2] have established that our universe consists of more dark matter of an unknown nature than visible matter composed of ordinary quarks and leptons. To investigate the origin of such a rich spectrum of visible fermions and the real identity of dark matter, we generalize the Dirac theory of spinor fields and develop a theory of multi-spinor fields on the basis of an algebra that consists of triple-tensor products of elements of the Dirac algebra. Chiral combinations of multi-spinor fields form reducible representations of the Lorentz group possessing component fields with spin 1/2, which we interpret as expressing three ordinary families and an additional fourth family of quarks and leptons. Apart from the gauge and Higgs fields of the Standard Model interacting with the fermions of the three ordinary families, we assume the existence of additional gauge and Higgs fields interacting exclusively with the fermions of the fourth family. While the fields of the Standard Model organize the “visible sector” of our universe, the fields related with the fourth family are presumed to generate a “dark sector” that can contain dark matter. The two sectors possess a channel of communication through the bi-quadratic interaction between visible and dark Higgs fields. After experiencing a common inflationary phase, the two sectors follow a reheating period and weak-coupling paths of thermal histories. We propose scenarios for dark matter that have a tendency to take relatively broad interstellar distributions and examine methods for the detection of the main candidate particles of dark matter. The exchange of superposed fields of the visible and dark Higgs bosons induces weak reaction processes between the fields of the visible and dark sectors, which enables us to have a glimpse of the dark sector.

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su(4), which have very fine substructures with “su(3) plus su(1)” conformations that are no longer reducible under the group $S_3$.

Reflecting the structure of the triplet algebra $A_T$, the triplet field makes up a reducible representation of the Lorentz group, including sixteen component fields with spin $\frac{1}{2}$, which have degrees of freedom of four families and four colors. The family mode and the color symmetry of the triplet field have substructures with “three plus one” formations. Namely, the triplet field possesses the modes of three families and an additional fourth family of tricolor and colorless fermions. Hereafter, we call the three-family mode triple mode and the fourth family mode single mode. The existence of the single mode is a unique characteristic of the current theory of multi-spinor fields.

The electroweak symmetry of the Standard Model (SM) is incorporated by introducing two types of compound fields, called the L-field and R-field, which consist of left-handed triplet fields and right-handed triplet fields, respectively. We demand that the triple mode of the L-field (R-field) is composed of left-handed doublets (right-handed singlets) of the electroweak symmetry $SU_L(2)$ and that the electroweak hypercharges $Y$ of the gauge group $U_Y(1)$ are assigned so as to cancel chiral anomalies in each family. It is necessary, however, to go beyond the SM in order to determine the physical interpretation of the single mode of the L- and R-fields.

There is no experimental evidence for the existence of fermions other than three families of ordinary quarks and leptons. This means that, if the additional fermions belonging to the single mode exist in the range of energy that is presently attainable by experiment, they are sterile with respect to the interactions mediated by the gauge and Higgs fields related to the SM symmetry $G = SU_c(3) \times SU_L(2) \times U_Y(1)$. Accordingly, we hypothesize that the single mode of the R-field (L-field) contains right-handed doublets (left-handed singlets) of an L–R twisted symmetry $SU_R(2)$ and that hypercharges $Y_\star^1$ of a new gauge group $U_{Y_\star}(1)$ are assigned so that chiral anomalies are canceled in the family.

To qualify the interactions of quarks in the triple and single modes, we have to take the observed characteristics of hadron spectra into account. If the ordinary mechanism of confinement based on the color $SU_c(3)$ symmetry were applied to both family modes, there might emerge exotic hadrons bearing hybrid quantum numbers of gauge symmetries $G_{EW} = SU_L(2) \times U_Y(1)$ and $G_{EW_\star} = SU_R(2) \times U_Y(1)$. So far no such hadrons have been found. Therefore, the quarks in the single mode are required to interact exclusively with confining gauge fields of another color symmetry, expressed hereafter as $SU_\star(3)$.

These suppositions lead us to the viewpoint that, while the fermion fields of the triple mode and the gauge and Higgs fields of the SM symmetry $G$ give birth to our visible sector including baryonic matter, the fermion fields of the single mode and the gauge and Higgs fields of the symmetry $G_\star = SU_c(3) \times SU_R(2) \times U_Y(1)$ work to create a dark sector that can comprise dark matter. To develop the renormalizable gauge field theory describing the structure of the two sectors and their mutual relations, we postulate that no basic field can share both attributes characterizing each sector. For example, the field with “charges” of both gauge symmetries $G$ and $G_\star$ is predicted not to exist, since observation of the effects of such a field entails a denial of the darkness of the dark sector.

For the present formalism to give a realistic theory for a unified description of the universe, it should have effective ways and means, apart from gravity, to observe dark phenomena from the visible sector. The present theory possesses a natural channel for weak communication between the

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1 We express physical quantities and entities in the dark sector by suffixing the $\star$ symbol.
two sectors, which is opened by means of the bi-quadratic interaction of the visible and dark Higgs fields related, respectively, with the $G_{EW}$ and $G_{EW*}$ gauge symmetries.

The two sectors are presumed to experience a common inflationary phase in the primordial universe and then follow weak-coupling paths of thermal histories after a reheating period. The similarity of the gauge groups $G$ and $G_*$ enables us to assume that the symmetry $G_*$ is also broken after the Weinberg–Salam (WS) mechanism. Breakdowns of the symmetries $G_{EW}$ and $G_{EW*}$ are specified, respectively, by the scales $\Lambda$ and $\Lambda_*$ ($\Lambda < \Lambda_*$). The symmetry $G_*$ is broken down at $\Lambda_*$ to the low energy symmetry $SUc(3) \times UQ(1)$, leaving the same number of bosonic fields as in the SM. The $UQ(1)$ gauge field induces phenomena similar to electromagnetism. This suggests that the dark sector consists of dark radiations and dark materials analogous to ordinary atoms and molecules in the visible sector.

We examine two scenarios for the emergence of dark matter, which tends to have relatively extensive interstellar distributions. The channel for communication between the two sectors is opened through exchanges of superpositions of the fields of visible and Higgs bosons. We inquire into possible ways to observe effects that can prove the existence of the dark sector.

In Sects. 2 and 3, the triplet algebra and its subalgebras are described in detail. We introduce the triplet field in Sect. 4. The $G_{EW}$ and $G_{EW*}$ gauge symmetries are incorporated in terms of the chiral sets of the triplet fields and breakdowns of these symmetries are examined in Sect. 5. We investigate the emergence of the dark matter and its detection in Sect. 6 and discuss future problems in Sect. 7.

2. Triplet algebra and external subalgebra

Let us call the triple-tensor products of the bases\(^2\) $1, \gamma_\mu, \sigma_{\mu\nu} = \frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu), \gamma_5\gamma_\mu$, and $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \gamma^5$ of the Dirac algebra $A_\gamma = \langle \gamma_\mu \rangle$ primitive triplets [3], and define the triplet algebra $A_T$ by all of the linear combinations of primitive triplets. In other words, the triplet algebra is generated in terms of the 12 primitive triplets $\gamma_\mu \otimes 1 \otimes 1, 1 \otimes \gamma_\mu \otimes 1, 1 \otimes 1 \otimes \gamma_\mu$ as follows:

\[
A_T = \{a \otimes b \otimes c : a, b, c \in A_\gamma \}
= \langle \gamma_\mu \otimes 1 \otimes 1, 1 \otimes \gamma_\mu \otimes 1, 1 \otimes 1 \otimes \gamma_\mu \rangle. \tag{1}
\]

The transpose (Hermite conjugate) of the primitive triplet is defined by the triple-tensor product of its transposed (Hermite conjugate) elements of $A_\gamma$, and the trace of the primitive triplet is given by the product of the traces of its $A_\gamma$ elements as

\[
\text{Tr}(a \otimes b \otimes c) = \frac{1}{16}((\text{tr } a)(\text{tr } b)(\text{tr } c)). \tag{2}
\]

2.1. External algebra

We introduce a set of four primitive triplets defined by [4,5]

\[
\Gamma_\mu = \gamma_\mu \otimes \gamma_\mu \otimes \gamma_\mu \quad (\mu = 0, 1, 2, 3), \tag{3}
\]

which satisfy the anti-commutation relations

\[
\Gamma_\mu \Gamma_\nu + \Gamma_\nu \Gamma_\mu = 2\eta_{\mu\nu}I, \quad I = 1 \otimes 1 \otimes 1. \tag{4}
\]

\(^2\)The elements $\gamma_\mu$ ($\mu = 0, 1, 2, 3$) satisfy the anti-commutation relations $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu}I$ with $(\eta_{\mu\nu}) =$ diagonal$(1, -1, -1, -1)$. The Hermite conjugate of $\gamma_\mu$ is defined by $\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0$. 

From these triplets, the external algebra $A_\Gamma$ is constructed as follows:

$$A_\Gamma = \{ I, \Gamma_\mu, \Sigma_{\mu\nu}, \Gamma_5 \}$$  \hspace{1cm} (5)

where

$$\Sigma_{\mu\nu} = -\frac{i}{2}(\Gamma_\mu \Gamma_\nu - \Gamma_\nu \Gamma_\mu) = \sigma_{\mu\nu} \otimes \sigma_{\mu\nu} \otimes \sigma_{\mu\nu}$$  \hspace{1cm} (6)

and

$$\Gamma_5 = -i\Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3 = \gamma_5 \otimes \gamma_5 \otimes \gamma_5.$$  \hspace{1cm} (7)

The algebra $A_\Gamma$ is, evidently, isomorphic to the original Dirac algebra $A_\gamma$ and all its elements are severally invariant under the action of the group $S_3$. The Hermite conjugate of $\Gamma_\mu$ is defined by $\Gamma_\mu^\dagger = \eta_{\mu\nu} \Gamma_\nu$.

Accordingly, we can postulate that the operators $M_{\mu\nu} = \frac{1}{2} \Sigma_{\mu\nu}$ generate the Lorentz transformations for the triplet field in the 4D Minkowski space-time where we exist as observers. The subscripts of operators $\Gamma_\mu$ are related and contracted with the superscripts of the space-time coordinates $x^\mu$.

### 2.2. Centralizer of the external algebra

To explore the structure of the triplet algebra, it is relevant to introduce the centralizer of the external algebra $A_\Gamma$ as follows:

$$C_\Gamma = \{ X \in A_T : [X, \Gamma_\mu] = 0 \}.$$  \hspace{1cm} (8)

The primitive triplets of the centralizer are the triple-tensor products of even numbers of the elements $\gamma_\mu$ for arbitrary $\mu$ [3]. Namely, the centralizer is given by

$$C_\Gamma = \langle 1 \otimes \gamma_\mu \otimes \gamma_\mu, \gamma_\mu \otimes 1 \otimes \gamma_\mu \rangle.$$  \hspace{1cm} (9)

With the mutually commutative subalgebras $A_\Gamma$ and $C_\Gamma$, we have the following decomposition of the triplet algebra as

$$A_T = A_\Gamma C_\Gamma, \quad A_\Gamma \cap C_\Gamma = \emptyset.$$  \hspace{1cm} (10)

Apparently, the centralizer $C_\Gamma$ is commutative with arbitrary generators $M_{\mu\nu}$ of the Lorentz transformation. Note that the internal properties of fundamental fermions must be fixed independently of the inertial frame of reference in which observations are made. This means that the elements of the centralizer $C_\Gamma$ satisfy a necessary condition for the generators specifying internal attributes of fundamental fermions.

### 3. Internal algebras for family and color degrees of freedom

Let us examine substructures of the centralizer $C_\Gamma$ and inquire what sorts of internal attributes of the fundamental fermions can be inscribed on it. For its purpose, we notice that the Dirac algebra $A_\gamma$.
possesses two sets of mutually commutative $su(2)$ subalgebras as follows:

$$A_\sigma = \{ \sigma_\alpha : \sigma_1 = \gamma_0, \sigma_2 = i \gamma_0 \gamma_5, \sigma_3 = \gamma_5 \}$$  \hspace{1cm} (11)

and

$$A_\rho = \{ \rho_\alpha : \rho_1 = i \gamma_2 \gamma_3, \rho_2 = i \gamma_3 \gamma_1, \rho_3 = i \gamma_1 \gamma_2 \}.$$  \hspace{1cm} (12)

By taking the triple-tensor products of elements of the respective subalgebras $A_\sigma$ and $A_\rho$ so as to be included in the centralizer $C_\Gamma$, we are able to construct two types of $su(4)$ algebras that are commutative and isomorphic with each other. From those algebras, we select the appropriate subalgebras to describe the family and color degrees of freedom of fundamental fermions.

3.1. Subalgebra for family degrees of freedom

Taking the sums and differences of primitive triplets made out of the subalgebra $A_\sigma$, we can construct 15 elements belonging to the centralizer $C_\Gamma$ as follows [3,4]:

$$\begin{align*}
\pi_1 &= \frac{1}{2}(\sigma_1 \otimes \sigma_1 \otimes 1 + \sigma_2 \otimes \sigma_2 \otimes 1), \\
\pi_2 &= \frac{1}{2}(\sigma_1 \otimes \sigma_2 \otimes \sigma_3 - \sigma_2 \otimes \sigma_1 \otimes \sigma_3), \\
\pi_3 &= \frac{1}{2}(1 \otimes \sigma_2 \otimes \sigma_3 - \sigma_3 \otimes 1 \otimes \sigma_3), \\
\pi_4 &= \frac{1}{2}(\sigma_1 \otimes 1 \otimes \sigma_1 + \sigma_2 \otimes 1 \otimes \sigma_2), \\
\pi_5 &= \frac{1}{2}(\sigma_1 \otimes \sigma_3 \otimes \sigma_2, -\sigma_1 \otimes \sigma_3 \otimes \sigma_2), \\
\pi_6 &= \frac{1}{2}(\sigma_1 \otimes 1 \otimes \sigma_1 + \sigma_3 \otimes \sigma_2 \otimes \sigma_2), \\
\pi_7 &= \frac{1}{2}(\sigma_1 \otimes \sigma_1 \otimes \sigma_2 - \sigma_3 \otimes \sigma_2 \otimes \sigma_1), \\
\pi_8 &= \frac{1}{2\sqrt{3}}(1 \otimes \sigma_3 \otimes \sigma_3 + \sigma_3 \otimes 1 \otimes \sigma_3 - 2\sigma_3 \otimes \sigma_3 \otimes 1)
\end{align*}$$  \hspace{1cm} (13)

and

$$\begin{align*}
\pi_9 &= \frac{1}{2}(1 \otimes \sigma_1 \otimes \sigma_1 - 1 \otimes \sigma_2 \otimes \sigma_2), \\
\pi_{10} &= -\frac{1}{2}(\sigma_3 \otimes \sigma_1 \otimes \sigma_2 + \sigma_3 \otimes \sigma_2 \otimes \sigma_1), \\
\pi_{11} &= \frac{1}{2}(\sigma_1 \otimes 1 \otimes \sigma_2 + \sigma_2 \otimes 1 \otimes \sigma_2), \\
\pi_{12} &= -\frac{1}{2}(\sigma_3 \otimes \sigma_3 \otimes \sigma_2 + \sigma_2 \otimes \sigma_3 \otimes \sigma_1), \\
\pi_{13} &= \frac{1}{2}(\sigma_1 \otimes \sigma_1 \otimes 1 - \sigma_2 \otimes \sigma_2 \otimes 1), \\
\pi_{14} &= -\frac{1}{2}(\sigma_1 \otimes \sigma_2 \otimes \sigma_3 + \sigma_2 \otimes \sigma_1 \otimes \sigma_3), \\
\pi_{15} &= -\frac{1}{2\sqrt{6}}(1 \otimes \sigma_3 \otimes \sigma_3 + \sigma_3 \otimes 1 \otimes \sigma_3 + \sigma_3 \otimes \sigma_3 \otimes 1).
\end{align*}$$  \hspace{1cm} (14)

These elements are proved to satisfy the commutation relations

$$[\pi_j, \pi_k] = 2f_{jkl}^{(4)}\pi_l$$  \hspace{1cm} (15)

and the anti-commutation relations

$$\{\pi_j, \pi_k\} = \delta_{jk}I + 2\delta_{jkl}^{(4)}\pi_l$$  \hspace{1cm} (16)

of the Lie algebra $su(4)$, where $f_{jkl}^{(4)}$ and $\delta_{jkl}^{(4)}$ are the antisymmetric and symmetric structure constants characterizing the algebra. The elements $\pi_j$ are self-adjoint and have the traces

$$\text{Tr} \pi_j = 0, \quad \text{Tr} \pi_j \pi_k = 2\delta_{jk}.$$  \hspace{1cm} (17)

By inspecting the explicit forms of the elements in Eqs. (13) and (14), we can confirm that the algebra

$$A_{\pi}^{(4)} = \{ I, \pi_1, \pi_2, \ldots, \pi_{15} \} \subset C_\Gamma$$  \hspace{1cm} (18)

is closed under the action of the $S_3$ group.
To examine the substructure of $A^{(4)}_t$ in detail, it is relevant to introduce the projection operators

$$\Pi_{(t)} = \frac{1}{4} (3I - 1 \otimes \sigma_3 \otimes \sigma_3 - \sigma_3 \otimes 1 \otimes \sigma_3 - \sigma_3 \otimes \sigma_3 \otimes 1)$$

and

$$\Pi_{(s)} = \frac{1}{4} (I + 1 \otimes \sigma_3 \otimes \sigma_3 + \sigma_3 \otimes 1 \otimes \sigma_3 + \sigma_3 \otimes \sigma_3 \otimes 1),$$

which satisfy the relations $\Pi_{(t)} + \Pi_{(s)} = I$ and

$$\Pi_{(a)} \Pi_{(b)} = \delta_{ab} \Pi_{(a)}$$

for $a, b = t, s$. Although it is laborious, we can prove by direct calculations the following equations

$$\Pi_{(a)} \pi_j = \delta_{aj} \pi_j$$

for the operators $\Pi_{(a)}$ and $\pi_j$ ($j = 1, 2, \ldots, 8$). These identities imply that $\pi_j$ are simultaneous eigen-operators of $\Pi_{(t)}$ and $\Pi_{(s)}$ with the respective eigenvalues 1 and 0.

It is now possible to constitute two subalgebras as

$$A_{(t)} = \{ \Pi_{(t)}, \pi_1, \pi_2, \ldots, \pi_8 \}$$

and

$$A_{(s)} = \{ \Pi_{(s)} \}.$$  

The elements of $A_{(t)}$ are confirmed to satisfy the commutation relations

$$[\pi_j, \pi_k] = 2 f^{(3)}_{jkl} \pi_l$$

and the anti-commutation relations

$$\{ \pi_j, \pi_k \} = \frac{4}{3} \delta_{jk} \Pi_{(t)} + 2 d^{(3)}_{jkl} \pi_l$$

where $f^{(3)}_{jkl}$ and $d^{(3)}_{jkl}$ are the structure constants of the Lie algebra $su(3)$. Accordingly, the sets $A_{(t)}$ and $A_{(s)}$ form, respectively, the Lie algebras $su(3)$ and $su(1)$.

By using the equations for $\Pi_{(a)}$ and $A_{(a)}$, we can prove the relations

$$\Pi_{(a)} A_{(b)} = \delta_{ab} A_{(b)}$$

and

$$A_{(a)} A_{(b)} = \delta_{ab} A_{(b)}$$

for $a, b = t, s$. Direct inspection of all elements of Eq. (13) ascertains that the algebra $A_{(t)}$ is irreducible under the action of the permutation group $S_3$. We interpret that the algebras $A_{(t)}$ and $A_{(s)}$ have functions to classify, respectively, the triple and single modes of family degrees of freedom.

Note that the projection operator $\Pi_{(t)}$ on to the triple mode can be subdivided as follows:

$$\Pi_{(t)} = \sum_{i=1}^{3} \Pi_i$$

where

$$\begin{cases} 
\Pi_1 = \frac{1}{4} (I + 1 \otimes \sigma_3 \otimes \sigma_3 - \sigma_3 \otimes 1 \otimes \sigma_3 - \sigma_3 \otimes \sigma_3 \otimes 1), \\
\Pi_2 = \frac{1}{4} (I - 1 \otimes \sigma_3 \otimes \sigma_3 + \sigma_3 \otimes 1 \otimes \sigma_3 - \sigma_3 \otimes \sigma_3 \otimes 1), \\
\Pi_3 = \frac{1}{4} (I - 1 \otimes \sigma_3 \otimes \sigma_3 - \sigma_3 \otimes 1 \otimes \sigma_3 + \sigma_3 \otimes \sigma_3 \otimes 1). 
\end{cases}$$

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These operators $\Pi_i$ ($i = 1, 2, 3$) work to sort each family out of the triple mode in the so-called interaction states, to which the triple modes of mass eigenstates are related by the action of the algebra $A_t(\ell)$. By renaming the operator $\Pi(\omega)$ as

$$\Pi_4 = \Pi(\omega) = \frac{1}{4}(I + 1 \otimes \sigma_3 \otimes \sigma_3 + \sigma_3 \otimes 1 \otimes \sigma_3 + \sigma_3 \otimes \sigma_3 \otimes 1) \quad (31)$$

we obtain the four set of operators $\Pi_i$ ($i = 1, \ldots, 4$) satisfying the relations

$$\Pi_i \Pi_j = \delta_{ij} \Pi_i, \quad \sum_{i=1}^{4} \Pi_i = I. \quad (32)$$

### 3.2. Subalgebra for color degrees of freedom

The isomorphism between the commutative subalgebras $A_\pi$ and $A_\rho$ results in another $su(4)$ algebra consisting of the triple-tensor products of the elements of $A_\rho$. First, replacing $\sigma_a$ and $\tau_j$ with $\rho_a$ and $\lambda_j$ in Eqs. (13)–(18), we find a new set of generators forming the Lie algebra $su(4)$ in the centralizer $C_\Gamma$ as

$$A_\lambda^{(4)} = \{I, \lambda_1, \lambda_2, \ldots, \lambda_{15}\} \subset C_\Gamma. \quad (33)$$

Then, similar procedures of replacement and relabeling can be applied to all equations in the previous subsection. From Eqs. (30) and (31), there follow the four elements

$$\begin{align*}
\Lambda_r &= \frac{1}{4}(I + 1 \otimes \rho_3 \otimes \rho_3 - \rho_3 \otimes 1 \otimes \rho_3 - \rho_3 \otimes \rho_3 \otimes 1), \\
\Lambda_g &= \frac{1}{4}(I - 1 \otimes \rho_3 \otimes \rho_3 + \rho_3 \otimes 1 \otimes \rho_3 - \rho_3 \otimes \rho_3 \otimes 1), \\
\Lambda_b &= \frac{1}{4}(I - 1 \otimes \rho_3 \otimes \rho_3 - \rho_3 \otimes 1 \otimes \rho_3 + \rho_3 \otimes \rho_3 \otimes 1), \\
\Lambda_\ell &= \frac{1}{4}(I + 1 \otimes \rho_3 \otimes \rho_3 + \rho_3 \otimes 1 \otimes \rho_3 + \rho_3 \otimes \rho_3 \otimes 1),
\end{align*} \quad (34)$$

which we interpret as projection operators for the extended color degrees of freedom including tricolor and colorless fermion states. Similarly, from Eqs. (19) and (20), we obtain the projection operators for the tricolor quark state and the colorless lepton state$^3$ as follows:

$$A^{(q)} = \sum_{c=r,b}^{\lambda} A_c = \frac{1}{4}(3I - 1 \otimes \rho_3 \otimes \rho_3 - \rho_3 \otimes 1 \otimes \rho_3 - \rho_3 \otimes \rho_3 \otimes 1) \quad (35)$$

and

$$A^{(\ell)} = \Lambda_\ell = \frac{1}{4}(I + 1 \otimes \rho_3 \otimes \rho_3 + \rho_3 \otimes 1 \otimes \rho_3 + \rho_3 \otimes \rho_3 \otimes 1), \quad (36)$$

which are invariant under the action of the group $S_3$. The operator for baryon number minus lepton number is determined by

$$Q_{B-L} = \frac{1}{3} A^{(q)} - A^{(\ell)} = -\frac{1}{3}(1 \otimes \rho_3 \otimes \rho_3 + \rho_3 \otimes 1 \otimes \rho_3 + \rho_3 \otimes \rho_3 \otimes 1), \quad (37)$$

which has eigenvalues 1/3 and $-1$.

For the sets of elements $\pi_j$, $\Pi_i$, and $\Pi(\omega)$ describing the family degrees of freedom, there exist the sets of elements $\lambda_j$, $\Lambda_i$, and $A^{(a)}$ for the color degrees of freedom, in a one-to-one correspondence, in the centralizer $C_\Gamma$. Both of them have the isomorphic structure of the Lie algebra $su(4)$ with

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$^3$ For the sake of simplicity, the concepts of quark and lepton are used here prior to specifying their attributes concerning the electroweak symmetry.
the “su(3) plus su(1)” subalgebras whose elements show the same behavior under the action of the group $S_3$.

Correspondingly to Eqs. (23) and (24), we obtain the subalgebras $su(3)$ and $su(1)$ out of $A^{(4)}_\lambda$ for the quark and lepton degrees of freedom as follows:

$$A^{(q)} = \{\lambda^{(q)}, \lambda_1, \lambda_2, \ldots, \lambda_8\}$$ (38)

and

$$A^{(\ell)} = \{\Lambda^{(\ell)}\}.$$ (39)

The elements of $A^{(q)}$ satisfy the commutation relations

$$[\lambda_j, \lambda_k] = 2f^{(3)}_{jkl}\lambda_l$$ (40)

and the anti-commutation relations

$$\{\lambda_j, \lambda_k\} = \frac{4}{3}\delta_{jk}\Lambda^{(q)} + 2d_{jkl}\lambda_l.$$ (41)

Analogously to Eqs. (27) and (28), there hold the relations

$$\Lambda^{(a)} A^{(b)} = \delta^{ab} A^{(b)}$$ (42)

and

$$A^{(a)} A^{(b)} = \delta^{ab} A^{(b)}$$ (43)

for $a, b = q, \ell$.

Here we postulate that the algebras $A^{(q)}$ and $A^{(\ell)}$ carry functions to classify the quark and lepton states, in parallel with the algebras $A^{(t)}$ and $A^{(s)}$ sorting the triple and single family modes. Note, however, that this parallelism, which holds at the algebraic level, cannot be retained at the level of the Lie group, as discussed in the next section.

4. Triplets fields

After formation of the external and internal algebras in the preceding sections, it is now possible to introduce the triplet field $\Psi(x)$ on the space-time point $(x^\mu)$, which behaves like the triple-tensor product of the Dirac spinor field. Under the proper Lorentz transformation $x'^\mu = \Omega^\mu_v x^v$, where $\Omega_{\mu\nu} \Omega^\lambda_v = \eta_{\mu\nu}$ and $\det\Omega = 1$, the triplet field $\Psi(x)$ and its adjoint field $\overline{\Psi}(x) = \Psi^\dagger(x)\Gamma_0$ are transformed as follows:

$$\Psi'(x') = S(\Omega)\Psi(x), \quad \overline{\Psi}'(x') = \overline{\Psi}(x) S^{-1}(\Omega)$$ (44)

where the transformation matrix is given by

$$S(\Omega) = \exp\left(-\frac{i}{4}\Sigma_{\mu\nu} \omega^{\mu\nu}\right)$$ (45)

with the generators $\Sigma_{\mu\nu}$ in Eq. (6) and the angles $\omega^{\mu\nu}$ in the $\mu$–$\nu$ planes. The Lorentz invariant scalar product is formed as

$$\overline{\Psi}(x)\Psi(x) = \sum_{abc} \overline{\Psi}_{abc}(x)\Psi_{abc}(x).$$ (46)

For discrete space-time transformations such as space inversion, time reversal, and charge conjugation, the present scheme retains exactly the same structure as the ordinary Dirac theory.
The “three plus one” constructions of the internal algebras permit the triplet field $\Psi(x)$ to possess the orthogonal decompositions as follows:

$$\Psi(x) = \Psi_{(t)}(x) + \Psi_{(s)}(x) = \Psi_{(q)}^{(q)}(x) + \Psi_{(q)}^{(q)}(x)$$

$$= \Psi_{(t)}^{(q)}(x) + \Psi_{(q)}^{(q)}(x) + \Psi_{(s)}^{(q)}(x) + \Psi_{(s)}^{(q)}(x)$$

(47)

with $\Psi_{(a)}^{(b)} = \Lambda_{(b)}^{(b)} \Psi_{(a)} = \Lambda_{(b)}^{(b)} \Pi_{(a)} \Psi$ $(a = t, s; b = q, \ell)$, where $\Psi_{(t)}$ and $\Psi_{(s)}$ represent the triple and single modes, and $\Psi_{(q)}^{(q)}$ and $\Psi_{(q)}^{(q)}$ express the quark and lepton states.

To embody how the algebras $A_{(q)}^{(q)}$ and $A_{(q)}^{(q)}$ for quark and lepton states act on the triple and single modes, it is necessary to make up the algebras for each state by

$$\Pi_{(a)} A_{(q)} = \{ \Pi_{(a)} \Lambda_{(q)}^{(q)}, \Pi_{(a)} \lambda_{j} : j = 1, \ldots, 8 \}$$

(48)

and

$$\Pi_{(a)} A_{(q)}^{(q)} = \{ \Pi_{(a)} \Lambda_{(q)}^{(q)} \}$$

(49)

where $a = t, s$. The elements of the algebras $\Pi_{(a)} A_{(q)}^{(q)}$ satisfy the commutation relations

$$[\Pi_{(a)} \lambda_{j}, \Pi_{(a)} \lambda_{k}] = 2 f_{jk}^{(3)} \Pi_{(a)} \lambda_{l}$$

(50)

and the anti-commutation relations

$$\{ \Pi_{(a)} \lambda_{j}, \Pi_{(a)} \lambda_{k} \} = \frac{4}{3} \delta_{jk} \Pi_{(a)} \Lambda_{(q)}^{(q)} + 2 d_{jk}^{(3)} \Pi_{(a)} \lambda_{l}$$

(51)

for $a = t, s$ and $j, k, l = 1, 2, \ldots, 8$.

In the present formalism, the quarks in the triple and single modes are presumed to be confined separately by different interactions associated with the color gauge groups $SU_{c}(3)$ and $SU_{c*}(3)$. Those symmetry groups are defined by the exponential mappings of the algebras $\Pi_{(t)} A_{(q)}^{(q)}$ and $\Pi_{(s)} A_{(q)}^{(q)}$ as follows:

$$SU_{c}(3) \times SU_{c*}(3) = \left\{ \exp \left( -\frac{i}{2} \sum_{j} \lambda_{j} \Pi_{(t)} \theta^{j}(x) - \frac{i}{2} \sum_{j} \lambda_{j} \Pi_{(s)} \theta^{j}(x) \right) \Lambda_{(q)}^{(q)} \right\}$$

(52)

where $\theta^{j}(x)$ and $\theta^{j}(x)$ are arbitrary real functions of space-time. The scalar product for the triplet field in Eq. (46) is invariant under the action of these groups on the quark states in the triple and single modes, i.e., $\Psi_{(t)}^{(q)} = \Pi_{(t)} \Psi_{(q)}^{(q)}$ and $\Psi_{(s)}^{(q)} = \Pi_{(s)} \Psi_{(q)}^{(q)}$. In addition to the ordinary gauge fields $A_{(q)}^{(3)(3)}(x)$ with coupling constant $g_{(3)}^{(3)}$ of the $SU_{c}(3)$ symmetry, our theory necessitates the new gauge fields $A_{(q)}^{(3)(3)}(x)$ with coupling constant $g_{(3)}^{(3)}$ of the $SU_{c*}(3)$ symmetry.

As stated in the previous section, the present theory shows parallelism between the color and family degrees of freedom at the algebraic level. If this parallelism persists up to the group level, there arises the theoretical possibility that claims to gauge also the family degrees of freedom. Namely, it is necessary to gauge the $SU_{c}(3)$ symmetries induced through the exponential mappings of the algebras $\Lambda_{(q)}^{(q)} A_{(t)}$ and $\Lambda_{(q)}^{(q)} A_{(t)}$ defined by

$$\Lambda_{(b)}^{(b)} A_{(t)} = \{ \Lambda_{(b)}^{(b)} \Pi_{(t)}^{(t)}, \Lambda_{(b)}^{(b)} \pi_{j} : j = 1, \ldots, 8 \}$$

(53)

which are isomorphic to Eq. (48). These family groups act on the quark and lepton states in the triple mode, i.e., $\Psi_{(t)}^{(q)} = \Lambda^{(q)} \Psi_{(t)}^{(q)}$ and $\Psi_{(t)}^{(q)} = \Lambda^{(q)} \Psi_{(t)}^{(q)}$.

Evidently such a parallelism is not realized in nature. While the color symmetry $SU_{c}(3)$ holds exactly over whole energy scales, the family symmetry is totally broken in the low energy region.
It is appealing and theoretically possible for us to postulate that both the color and family gauge symmetries hold in a sufficiently high energy regime. At the present stage of the theory, however, it is markedly difficult to formulate whole processes of breakdown of the family symmetry so as to describe the various phenomena of flavor physics in low energy scales. Therefore, we follow here the ordinary scheme of the SM and refrain from gauging the family symmetry.

The left-handed and right-handed fields of the triplet field are given as follows:

$$\Psi_L(x) = L\Psi(x), \quad \Psi_R(x) = R\Psi(x)$$  \hspace{1cm} (54)

with the chirality operators $h = L, R$ defined by

$$L = \frac{1}{2}(I - \Gamma_5), \quad R = \frac{1}{2}(I + \Gamma_5).$$  \hspace{1cm} (55)

Combining them with the projection operators of family modes, we can also introduce the operators $\Pi_{ih}(h = L, R)$ where $\Pi_{iL} = \Pi_i L$ and $\Pi_{iR} = \Pi_i R$.

In order to extract the internal component fields out of the triplet field, we conveniently import Dirac's bra-ket symbols for the projection operators $\Lambda_a$ and $\Pi_{ih}$ as follows:

$$\Lambda_a = \langle a \mid, \Pi_{ih} = \mid i h \rangle \langle ih \mid.$$  \hspace{1cm} (56)

For the triplet field and its conjugate, the projection operators act as

$$\Lambda_a \Pi_{ih} \Psi(x) = \Lambda_a \langle ih \rangle \langle ih \mid \Psi(x) \rangle = |aih\rangle \langle aih\mid \Psi(x) \rangle = |aih\rangle \Psi_{aih}(x)$$  \hspace{1cm} (57)

and

$$\bar{\Psi}(x) \Lambda_a \Pi_{ih} = \langle \bar{\Psi}(x) \rangle \Lambda_a \langle ih\rangle \langle ih\mid = \langle \bar{\Psi}(x) \rangle |aih\rangle \langle aih\mid = \bar{\Psi}_{aih}(x) \langle aih\mid$$  \hspace{1cm} (58)

where $\Psi_{aih}(x) = \langle aih\mid \Psi(x) \rangle$ and $\bar{\Psi}_{aih}(x) = \langle \bar{\Psi}(x) \rangle |aih\rangle$ are chiral component fields, and $\bar{h}$ implies that $\bar{L} = R$ and $\bar{R} = L$. Then, the decomposition of the bilinear scalar and vector forms of the triplet fields can be achieved as follows:

$$\bar{\Psi}(x) \Psi(x) = \sum_a \sum_{ih} \bar{\Psi}(x) \Lambda_a \Pi_{ih} \Psi(x) = \sum_a \sum_{ih} \bar{\Psi}_{aih}(x) \Psi_{aih}(x)$$  \hspace{1cm} (59)

and

$$\bar{\Psi}(x) \Gamma_\mu \Psi(x) = \sum_a \sum_{ih} \bar{\Psi}(x) \Gamma_\mu \Lambda_a \Pi_{ih} \Psi(x) = \sum_a \sum_{ih} \bar{\Psi}_{aih}(x) \Gamma_\mu \Psi_{aih}(x).$$  \hspace{1cm} (60)

5. **Gauge symmetries $G$ and $G_\ast$ in multi-spinor field formalism**

In the WS theory of electroweak interaction, the left-handed chiral components of electron and neutrino fields constitute the doublet representation of the electroweak $SU_L(2)$ symmetry and the right-handed component of the electron field forms its singlet. Since the discovery of neutrino oscillation, the right-handed component of the neutrino field has also been added as another singlet. To integrate the WS scheme for the $G_{EW}$ and $G_{EW_\ast}$ symmetries into our theory, we have to introduce the new Pauli $su(2)$ algebras as follows:

$$A_L = \{\tau_{Lj}\}, \quad A_R = \{\tau_{Rj}\},$$  \hspace{1cm} (61)

which act, respectively, on the triple and single modes of the triplet fields and generate the $SU_L(2)$ and $SU_R(2)$ groups.
5.1. Multi-spinor field theory for \( G \) and \( G_\ast \) symmetries

Fundamental representations of the \( SU_L(2) \) and \( SU_R(2) \) groups are given by two types of compound fields, L-field \( \Psi_L \) and R-field \( \Psi_R \), which are respectively composed of left-handed and right-handed triplet fields. The L-field contains the triple mode consisting of left-handed doublets of \( SU_L(2) \) and the single mode consisting of left-handed singlets of \( SU_R(2) \). In contrast, the R-field possesses the triple mode composed of the right-handed doublets of \( SU_L(2) \) and the single mode composed of the right-handed doublet of \( SU_R(2) \).

The L-field \( \Psi_L \) and R-field \( \Psi_R \) constitute fundamental representations of the \( G \) and \( G_\ast \) symmetries in the forms

\[
\Psi_L = \begin{pmatrix} u & c & t \\ d & s & b \end{pmatrix}_L, \quad \Psi_R = \begin{pmatrix} U(t) \\ D(t) \end{pmatrix}_R
\]

in which \( \Psi \) and \( U \) (\( D \)) are used, respectively, to express the doublet and the up (down) singlet. The operation of transpose \( t \) is applied to line up the family modes in the horizontal direction.

In order to name the fermions in the single mode, we assign new symbols \( u_\ast \) and \( d_\ast \) for up and down quark states, and \( v_\ast \) and \( e_\ast \) for up and down lepton states. Then, by refraining to specify the color degrees of freedom, we can display the components of the quark and lepton parts of the L-field, \( \Lambda^{(q)} \Psi_L \) and \( \Lambda^{(\ell)} \Psi_L \), as follows:

\[
\Psi^{(q)}(t) = t \left( \begin{array}{ccc} u & c & t \\ d & s & b \end{array} \right)_L, \quad U^{(q)}(t) = (u_\ast)_L, \quad D^{(q)}(t) = (d_\ast)_L
\]

and

\[
\Psi^{(\ell)}(t) = t \left( \begin{array}{ccc} v_\ell & v_\mu & v_\tau \\ e & \mu & \tau \end{array} \right)_L, \quad U^{(\ell)}(t) = (v_\ast)_L, \quad D^{(\ell)}(t) = (e_\ast)_L.
\]

Likewise, the quark and lepton parts of the R-field, \( \Lambda^{(q)} \Psi_R \) and \( \Lambda^{(\ell)} \Psi_R \), are expressed by the components as

\[
U^{(q)}(t) = t \left( \begin{array}{c} u \\ c \\ t \end{array} \right)_R, \quad D^{(q)}(t) = t \left( \begin{array}{c} d \\ s \\ b \end{array} \right)_R, \quad \Psi^{(q)}(t) = \left( \begin{array}{c} u_\ast \\ d_\ast \end{array} \right)_R
\]

and

\[
U^{(\ell)}(t) = t \left( \begin{array}{cccc} v_\ell & v_\mu & v_\tau \end{array} \right)_R, \quad D^{(\ell)}(t) = t \left( \begin{array}{ccc} e & \mu & \tau \end{array} \right)_R, \quad \Psi^{(\ell)}(t) = \left( \begin{array}{c} v_\ast \\ e_\ast \end{array} \right)_R.
\]

We are now able to write down the kinetic and gauge part of the Lagrangian density of all fermions in terms of the L- and R-fields as follows:

\[
L_{kg} = \bar{\Psi}_L i \gamma^\mu D_\mu \Psi_L + \bar{\Psi}_R i \gamma^\mu D_\mu \Psi_R
\]

where the covariant derivatives take the forms

\[
i D_\mu \Psi_L = \left\{ i \partial_\mu - \left[ g^{(3)} A^{(3)}_\mu \frac{1}{2} \lambda_j + g^{(2)} A^{(2)}_\mu \frac{1}{2} \tau_{LJ} + g^{(1)} A^{(1)}_\mu \frac{1}{2} Y \right] \Pi(t) \right\} \Psi_L
\]

and

\[
i D_\mu \Psi_R = \left\{ i \partial_\mu - \left[ g^{(3)} A^{(3)}_\mu \frac{1}{2} \lambda_j + g^{(2)} A^{(2)}_\mu \frac{1}{2} \tau_{RJ} + g^{(1)} A^{(1)}_\mu \frac{1}{2} Y \right] \Pi(t) \right\} \Psi_R
\]
in which \( A^{(2)j}_\mu (x) \) and \( A^{(1)j}_\mu (x) \) \((A^{(2)j}_\mu (x) \) and \( A^{(1)j}_\mu (x) \)\) are gauge fields of the \( G_{EW} \) \((G_{EW*})\) symmetry with coupling constants \( g^{(2)} \) and \( g^{(1)} \). The hypercharges for the triple and single modes, \( Y \) and \( Y_* \), can be expressed by

\[
Y = Q_{B-L} + y, \quad Y_* = Q_{B-L} + y_.*
\]  

(70)

in which \( y \) and \( y_* \) take 0, 1, and \(-1\), respectively, for the doublet \( \Psi \), the up singlet \( U \), and the down singlet \( D \).

The Lagrangian density for the gauge fields is, as usual, constructed by summing up all the Lorentz invariant bilinear forms of the separate field strengths of \( G \) and \( G_* \) symmetries. For the sake of brevity, we leave out its analysis here and will give brief notes on the results of the WS mechanism at the end of this section.

To break down the gauge symmetries \( G_{EW} \) and \( G_{EW*} \), we require two types of Higgs doublets as

\[
\phi = \left( \begin{array}{c} \phi^+ \\ \phi^0 \end{array} \right), \quad \phi_* = \left( \begin{array}{c} \phi_+^* \\ \phi_0^* \end{array} \right)
\]  

(71)

where \( \phi(x) \) and \( \phi_*(x) \) have the hypercharges \((Y = 1, Y_* = 0)\) and \((Y = 0, Y_* = 1)\), respectively. The Lagrangian density of the Yukawa interaction is given as follows:

\[
L_Y = \bar{\Psi}_L [\text{Higgs}] \Psi_R + \text{h.c.}
\]

\[
= \bar{\Psi}_{(i)} \tilde{D} Y U_{(i)} + \bar{\Psi}_{(i)} \phi Y_D D_{(i)} + y_{u*} \bar{U}_<(i) \phi^+_< \phi^0_+ + y_{d*} \bar{D}_<(i) \phi^+_< \phi^0_+ + \text{h.c.}
\]

(72)

where \( \tilde{D} = i \tau L_2 \phi^* \) and \( \phi^* = i \tau R_2 \phi^* \). The operators \( Y_U \) and \( Y_D \), consisting of elements of the algebra \( A_{(i)} \) in Eq. (23), determine the patterns of Yukawa interactions among the fermions in the triple mode [4], and \( y_{u*} \) and \( y_{d*} \) are the Yukawa coupling constants of the fermions in the single mode.

The Lagrangian density of the visible and dark Higgs fields takes the form

\[
L_H = (D^\mu \phi)^+ (D^\mu \phi) + (D^\mu \phi_*)^+ (D^\mu \phi_*) - V_H
\]  

(73)

in which the covariant derivatives act as follows:

\[
i D^\mu \phi = \left( i \partial^\mu - g^{(2)} A^{(2)a}_\mu \frac{1}{2} \tau^a \gamma^\mu - g^{(1)} A^{(1)}_\mu \frac{1}{2} \right) \phi
\]  

(74)

and

\[
i D^\mu \phi_* = \left( i \partial^\mu - g^{(2)}_* A^{(2)a}_\mu \frac{1}{2} \tau^a \gamma^\mu - g^{(1)}_* A^{(1)}_\mu \frac{1}{2} \right) \phi_*
\]  

(75)

The Higgs potential is generally given by

\[
V_H = V_0 - \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2 - \mu^2 \phi_*^\dagger \phi_* + \lambda^* (\phi_*^\dagger \phi_*)^2 + 2\lambda_l (\phi^\dagger \phi_*) (\phi^* \phi)
\]  

(76)

where \( \lambda, \lambda_*, \) and \( \lambda_l \) are the constants of self-coupling and mutual interaction.

The bi-quadratic interaction term, \( 2\lambda_l (\phi^\dagger \phi_*) (\phi^* \phi) \), plays the key role of relating the otherwise independent visible and dark sectors. It is crucial to recognize that there is no reason to exclude this cross-interaction term from \( V_H \). The invariance under the \( G_{EW} \) and \( G_{EW*} \) symmetries and the condition of renormalizability allow this term to exist. We have to grasp the \( \textit{raison d’être} \) of this interaction term in both the lower and higher energy regions than the scale \( \Lambda_* \). In the next subsection, we will see its role in breaking both symmetries \( G_{EW} \) and \( G_{EW*} \), leaving effective interactions between the resulting visible and dark Higgs bosons in the lower energy region. In Sect. 7, we will discuss the virtual quantum effects arising from the bi-quadratic term that cause interactions between the fermions in triple and single modes and also between the visible and dark gauge fields. In an early reheating period, those interactions lead all quanta of the fields of the universe to a state of thermal equilibrium.
5.2. Breakdowns of $G_{EW}$ and $G_{EW*}$ symmetries

Reflecting the similarity of the gauge symmetries $G$ and $G_*$, the field contents of the bosonic parts possess one-to-one correspondence in the visible and dark sectors. In the lower energy regime, where the symmetries $G_{EW}$ and $G_{EW*}$ are broken, we can assume that massive vector fields $W_{*\mu}^{\pm}(x)$ and $Z_{*\mu}(x)$ emerge in the dark sector, similarly to the well-established SM vector fields $W_{\mu}^{\pm}(x)$ and $Z_{\mu}(x)$. Likewise, the massive scalar field $h_*(x)$ comes forth in the dark sector, corresponding to the massive SM scalar field $h(x)$ in the visible sector.

In order to apply the WS mechanism, we have to first find a vacuum state

$$
\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad \phi_* = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_* \end{pmatrix}
$$

at which the Higgs potential takes the minimum value. The vacuum expectation values (VEVs), $v$ and $v_*$, are determined by solving the stationary conditions:

$$
\begin{cases}
\lambda v^2 + \lambda_I v_*^2 = \mu^2, \\
\lambda_I v^2 + \lambda_* v_*^2 = \mu_*^2,
\end{cases}
$$

which lead to the non-vanishing values

$$
v = \sqrt{\frac{\lambda_* \mu^2 - \lambda_I \mu_*^2}{\lambda \lambda_* - \lambda_I^2}}, \quad v_* = \sqrt{\frac{\lambda \mu_*^2 - \lambda_I \mu^2}{\lambda \lambda_* - \lambda_I^2}},
$$

provided that the quantities in both square roots are positive definite. By using the stationary conditions to eliminate $\mu^2$ and $\mu_*^2$, we can calculate the Hessian to be $4(\lambda \lambda_* - \lambda_I^2)v^2v_*^2$ at the stationary state. Consequently, under the condition that the coefficient satisfies the restrictions $\lambda \lambda_* - \lambda_I^2 > 0$, the Higgs potential possesses the stable vacuum state with the VEVs in Eq. (79).

Around this vacuum state, we can decompose the Higgs fields as follows:

$$
\phi(x) = \frac{1}{\sqrt{2}} U(\vartheta(x)) \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}, \quad \phi_*(x) = \frac{1}{\sqrt{2}} U_*(\vartheta_*(x)) \begin{pmatrix} 0 \\ v_* + h_*(x) \end{pmatrix}
$$

where $U(\vartheta(x))$ ($U_*(\vartheta_*(x))$) is the unitary group element of the $G_{EW}$ ($G_{EW*}$) symmetry including the local phase functions $\vartheta(x)$ ($\vartheta_*(x)$), which are destined to be gauged away so as to transform the gauge fields $A_{\mu}^{(2)}(x)$ and $A_{\mu}^{(1)}(x)$ ($A_{*\mu}^{(2)}(x)$ and $A_{*\mu}^{(1)}(x)$) to the massive vector fields $W_{\mu}^{\pm}(x)$ and $Z_{\mu}(x)$ ($W_{*\mu}^{\pm}(x)$ and $Z_{*\mu}(x)$). By substituting this decompositions into the Higgs potential, we obtain

$$
V_H(h, h_*) = V_H(0, 0) + \lambda h^2 + \lambda_* h_*^2 + 2\lambda_I v v_* h h_* + \lambda v h^2 + \lambda_* v_* h_*^2 + \lambda_I v v_* h_*^2 + \frac{1}{4} \lambda h^4 + \frac{1}{4} \lambda_* h_*^4 + \frac{1}{2} \lambda_I h^2 h_*^2
$$

with

$$
V_H(0, 0) = V_0 - \frac{1}{4} \lambda v^4 - \frac{1}{4} \lambda_* v_*^4 - \frac{1}{2} \lambda_I v^2 v_*^2.
$$

Note that the bosonic fields $h(x)$ and $h_*(x)$ are in the interaction modes. The former describes the SM Higgs boson interacting exclusively with the fermions in the triple mode and the gauge fields of the $G_{EW}$ symmetry. Owing to the cross-interaction term $2\lambda_I v v_* h h_*$ in Eq. (81), it mixes with the
bosonic field $h_*(x)$ of the dark sector. To obtain the mass eigenmodes, we introduce their superposed fields as follows:

$$\begin{align*}
H &= \cos \theta h - \sin \theta h_* \\
H' &= \sin \theta h + \cos \theta h_*.
\end{align*}$$

(83)

By fixing the mixing angle $\theta$ so as to satisfy

$$\tan 2\theta = \frac{2 \lambda_I v v_*}{\lambda_* v_*^2 - \lambda v^2},$$

(84)

the Higgs potential is diagonalized as

$$V_H(H, H') = V_H(0, 0) + \frac{1}{2} m_H^2 H^2 + \frac{1}{2} m_{H'}^2 H'^2 + \{\text{cubic and bi-quadratic coupling terms of } H \text{ and } H'\}$$

(85)

where the squared masses of the superposed fields $H$ and $H'$ are given by

$$\begin{pmatrix} m_H^2 \\ m_{H'}^2 \end{pmatrix} = (\lambda_* v_*^2 + \lambda v^2) \mp (\lambda_* v_*^2 - \lambda v^2) \sec 2\theta$$

$$= (\lambda v^2 + \lambda_* v_*^2) \mp \sqrt{(\lambda_* v_*^2 - \lambda v^2)^2 + (2 \lambda_I v v_*)^2}. \quad (86)$$

It is the superposed field $H(x)$ with the lower mass $m_H$ that we should interpret as expressing the Higgs-like particle whose existence has been confirmed by the recent LHC experiments [6–8].

Here we can determine the scales for the symmetry breakings by $\Lambda \approx m_H$ and $\Lambda_* \approx m_{H'}$. Breakdowns of the two gauge symmetries to the low energy symmetries $SU_\epsilon(3) \times U_Q(1)$ and $SU_\epsilon(3) \times U_Q(1)$ and mass acquisitions of various fields take place through multi-stage processes of phase transitions around the scale zone ($\Lambda \sim \Lambda_*$). From Eqs. (84) and (86), we see that the energy scales depend critically on the value of the coupling constant $\lambda_I$.

To see the effects of the WS mechanism on gauge fields, we have to introduce the visible and dark Weinberg angles by $\tan \theta_W = g^{(1)} / g^{(2)}$ and $\tan \theta_{W*} = g_*^{(1)} / g_*^{(2)}$. The visible and dark massive vector fields satisfy the same mass relations. The massless fields of visible and dark electromagnetism, $A_\mu(x)$ and $A_{\mu*}(x)$, have the coupling constants specified, respectively, by $e = g^{(2)} \sin \theta_W$ and $e_* = g_*^{(2)} \sin \theta_{W*}$.

6. Dark matter

In the visible sector, interstellar gases, which consist of a variety of atoms, molecules, and ions, can lose kinetic energy by electromagnetic radiations with various wavelengths and contract by gravitational interaction into relatively dense distributions so that rich and active processes, such as the formation of galaxies and stars, take place. In contrast, observations of gravitational lensing infer that the dark matter has a tendency to spread out rather monotonically over broad spatial regions. This feature must be a necessary reflection of the characteristics of particle physics with broken $G_\epsilon$ symmetry in the low energy regime.

6.1. Scenarios for dark matter

If the quarks $u_*$ and $d_*$ acquire close masses like the $u$ and $d$ quarks of the first family after the spontaneous breakdown of the $G_{\epsilon W*}$ symmetry à la the WS mechanism, there must exist many kinds of dark hadrons and a variety of dark elements, which are, thereby, destined to follow a rich
and intricate path of evolution just as our visible sector. Therefore, we presume that the masses of quarks in the single mode, \( m_{u^*} \) and \( m_{d^*} \), are largely different. Here it is supposed also, for the sake of simplicity, that the masses of the dark electron and neutrino, \( m_{e^*} \) and \( m_{\nu^*} \), are much smaller than those of dark quarks.

First, let us examine the scenario that, just like the case of the \( t \) and \( b \) quarks of the third family, the \( u^* \) quark is much heavier than the \( d^* \) quark as

\[
\mu^* \gg m_{d^*} + m_{e^*} + m_{\nu^*}.
\]

(87)

In such a case, the \( u^* \) quark disappears quickly through the process

\[
u^* \to d^* + \bar{e}^* + \nu^*.
\]

(88)

leaving the \( d^* \) quark as the main massive component of the dark sector.

Consequently, the gauge fields of \( SU_c(3) \) symmetry act to confine the \( d^* \) quark into the dark hadron

\[
\Delta^- = [d^*_n d^*_m d^*_o] = \frac{1}{\sqrt{6}} \epsilon_{ijk} d^*_n d^*_m d^*_o,
\]

(89)

which has \( Q^* = -1 \) and the spin angular momentum \( \frac{3}{2} \) due to the Fermi statistics. The confined systems of the quark and anti-quark \( (\bar{d}_n d^*_n) \) come out as dark neutral mesons in the pseudo-scalar and vector states.

In this scenario, the dark sector, where no dark baryon other than \( \Delta^- \) is stable and no nuclear reaction takes place, follows a rather meager history of thermal evolution. Nevertheless, however, there must exist a subtle and important stage of coupling and decoupling of the two sectors. Deferring this issue for future investigation, we point out that the stable dark atom allowed to exist in the matter-dominant stage of the dark sector is limited to being

\[
\hat{H} = (\Delta^- + \hat{e}^*).
\]

(90)

This hydrogen-like atom has a spectrum of excited states that can absorb and emit the dark radiations \( \gamma^* \) through quantum transitions. The dark molecule

\[
(\hat{H}^*)_2 =!\hat{H}_n \hat{H}_n
\]

(91)

can be a stable entity prevailing over a broad region of the late universe. The constituents of the late dark sector are these atoms and molecules, their ions, the leptons \( e^* \) and \( \nu^* \), and the radiations \( \gamma^* \).

There exists another almost equivalent scenario for dark matter in which the \( d^* \) quark has a larger mass than the \( u^* \) quark as

\[
m_{d^*} \gg m_{u^*} + m_{e^*} + m_{\nu^*}.
\]

(92)

In this case, the \( d^* \) quarks disappear rapidly through the process \( d^* \to u^* + e^* + \bar{\nu}^* \) and the remaining \( u^* \) quarks are confined into the spin \( \frac{3}{2} \) hadron

\[
\Delta^{++} = [u^*_n u^*_m u^*_o] = \frac{1}{\sqrt{6}} \epsilon_{ijk} u^*_n u^*_m u^*_o.
\]

(93)

with \( Q^* = 2 \). The quark \( u^* \) and anti-quark \( \bar{u}^* \) are confined to form the dark neutral mesons of the pseudo-scalar and vector states of \( (\bar{u}^*_n u^*_n) \).
Fig. 1. Scattering between the dark $d_\star$ quark in the $\Delta^-_\star$ hadron and the nucleon $N$ by exchanges of the superposed fields $H$ and $H'$ of the visible and dark Higgs bosons $h$ and $h_\star$.

With the dark hadron $\Delta^{++}_\star$, the helium-like atom is formed as

$$\text{He}_\star = (\Delta^{++}_\star + e_\star + e_\star),$$

(94)

which produces the molecule

$$(\text{He}_\star)_2 = \text{He}_\star \text{He}_\star.$$

(95)

In the matter-dominant phase of the universe, the dark materials consisting of dark atoms, molecules, and their ions coexist with the baryonic materials and reinforce formations of astronomical compact objects as well as cosmological large-scale structure through gravitational interactions. It is crucial to recognize that the dark materials are stable as a whole and no dark nuclear reaction takes place in all those astronomical and cosmological processes in our scenario of the dark matter.

6.2. Detection of the effects of the dark hadron

The visible and dark sectors possess a channel for mutual communication through exchanges of the superposed fields $H$ and $H'$. We have to inquire how to open the channel and contrive methods for observations of the effects of the dark sector. In the scenario specified by the condition in Eq. (87), we consider a method to detect the effect of the $\Delta^-_\star$ hadron constituting the main component of dark matter. There are direct and indirect ways to detect the effects induced by the $d_\star$ quark in the $\Delta^-_\star$ hadrons.

Let us examine first the interaction phenomena between the dark and visible matter. Figure 1 shows a scattering process between the dark quark $d_\star$ and the nucleon $N$ induced by exchanges of the superposed fields $H$ and $H'$ in Eq. (83), which consist of the visible and dark Higgs bosons $h$ and $h_\star$. When the dark hadron $\Delta^-_\star$ encounters a heavy element, such as xenon or germanium, situated in a low noise environment on the ground and penetrates deeply into its nucleus, the dark quark $d_\star$ and the nucleons $N$ interact through the exchange of $H$ and $H'$. The amplitude of this process is roughly estimated to be proportional to

$$\frac{1}{2} y_{d_\star} y_{\text{eff}} \sin \theta \cos \theta \left[ \frac{1}{m_H^2} - \frac{1}{m_{H_\star}^2} \right] = \frac{1}{4} \frac{\lambda_I - \lambda_2}{\lambda_1} y_{d_\star} y_{\text{eff}} v_{h_\star} v_h,$$

(96)

where $y_{d_\star}$ is the Yukawa coupling constant for the dark quark and $y_{\text{eff}}$ is an effective coupling constant of the Higgs boson with the nucleon [9,10], which has the dominant contribution of the top quark loop-correction produced by gluons inside the nucleon.

In the low energy regime, where the energy of the dark quark $d_\star$ is less than the masses of the fields $H$ and $H'$, the elastic scattering process of Fig. 1 can be described approximately by an effective four-Fermi interaction with the coupling constant given by Eq. (96), and the cross section is proportional
Fig. 2. Decay of the light and heavy superposed fields $H$ and $H'$ radiating from the dark quark $d_*$ of the accelerated dark hadron $\Delta^-_*$. 

to the square of its energy. Rare phenomena induced by this sort of scattering process of the dark matter with the nucleon can be observed by the ground experiments designed for the direct detection of weakly interacting particles [11–14].

The cross section of the elastic scattering process between the dark hadrons $\Delta^-_*$ in the dark matter wind and the nucleon $N$ inside the target element with mass number $A$ is enhanced approximately by the factor $3^2 \times A^2$, provided that the effective coupling constants $\gamma_{\text{eff}}$ to the proton and neutron are approximately equal. Direct detection enables us to estimate the values of the coupling constants, $\lambda_I/(\lambda_\lambda - \lambda^2)$, and the product of the VEVs $v$ and $v_*$. For theoretical analysis of the experimental data, it is necessary to take into account the bound state effects of the nucleon wave functions in the target element.

Corresponding to the scattering process induced by virtual exchanges of the fields $H$ and $H'$ in Fig. 1, it is theoretically possible to picture the decay processes of the $H$ and $H'$ fields produced through Brehmsstrahlung from the accelerated dark hadron $\Delta^-_*$, as shown in Fig. 2. This kind of production process can take place only when the dark hadron $\Delta^-_*$ has sufficiently high energy, as realized in the LHC experiment for the Higgs search. To utilize such processes for experiments of indirect detection of dark matter, it is also necessary to acquire reliable techniques from the LHC experiments to precisely identify the process from its decay products.

7. Discussion

By generalizing the Dirac concept of spinor fields, we have developed a unified theory of multi-spinor fields that can describe the whole spectrum of fields in the ordinary visible sector and the additional sets of fields constructing the dark sector. As shown in Sects. 3 and 4, the triplet algebra with the restriction of the $S_3$ irreducibility has the unique feature of the “three plus one” structure for both the color and family degrees of freedom. The triplet field possesses the component fields of the triple and single modes with the tricolor quarks and colorless leptons. With the chiral representations in Eq. (62), we have formulated a successful unified theory that can describe the flavor physics of the visible sector and astrophysical phenomena related to both the visible and dark sectors.

Here it is relevant to explain why we did not choose a simpler extension of the SM in which the additional fermion multiplet is identified with the sequential 4th family. This sequential model is realized by the following chiral representations of the triplet fields as

$$ \Psi_L = {}^t(\Psi_{(t)} \Psi_{(s)})_L, \quad U_R = {}^t(U_{(t)} \ U_{(s)})_R, \quad D_R = {}^t(D_{(t)} \ D_{(s)})_R, $$

where $\Psi_{(a)}$, $U_{(a)}$, and $D_{(a)}$ ($a = t, s$) are the doublet, up singlet, and down singlet of the Weinberg–Salam symmetry $G_{\text{EW}}$, respectively. In this model, the family structure is described by the Lie algebra $su(4)$ in Eq. (18).
There are two reasons not to adopt the sequential model. First, the model with sequential four families has now been excluded by experiments. The CKM matrix elements of the three-family model are fitted with experimental data [15] to high precision, and the recent LHC experiment for the Higgs search has excluded contributions of the fourth quark up to 790 GeV [16]. Second, there is no room to accommodate candidate fields for dark matter in the sequential four-family model.

Note that, so far as the triple mode is concerned, no essential difference exists between our L–R twisted scheme and the sequential four-family model. With the bra-ket treatment in Sect. 4, we can reproduce the well known results of the SM in a convenient way. The Lagrangian density $L_Y$ in Eq. (72) expressed in terms of the component fields possesses the Yukawa coupling constants in the forms of matrix elements $\langle i | YU | j \rangle$ and $\langle i | YD | j \rangle$ of the generators $Y_U$ and $Y_D$. This method gives us a new means to interpret the Yukawa coupling constants from a unified angle of vision. As an example, we made numerical analyses of the quark mass spectra and CKM matrix elements, and obtained proper results to explain the observed data provided that the generators $Y_D$ and $Y_A$ satisfy a broken cyclic symmetry [4]. To apply this method to the lepton part, we must be careful how to describe the neutrino species, which still possess many unknowns. The visible neutrinos $U^{(\ell)} (\ell)$ and the dark neutrino $\bar{U}_{\ell} (s)$ have no quantum number with respect to both symmetries $G_{EW}$ and $G_{EW\ast}$. They can show up with mixed characters of Dirac and Majorana fermions. The real identities of neutrinos are expected to be clarified by persistent efforts of experiments.

In Sect. 4.2, we have applied the WS mechanism to describe simultaneously the breakdowns of the two gauge symmetries $G_{EW\ast}$ and $G_{EW}$. Multi-stage processes of phase transitions take place around the region specified by the two scales $\Lambda \approx m_H$ and $\Lambda_{\ast} \approx m_{H'}$. The masses of the superposed fields $H$ and $H'$ in Eq. (86) show that crucial features of these scales are determined by the value of the mixing angle $\theta$ defined in Eq. (84).

If the mixing angle $\theta$ takes a sizable value, the breakdowns of the two gauge symmetries occur in inseparable ways and rich processes of phase transitions proceed in multiple steps in the scale zone $(\Lambda \sim \Lambda_{\ast})$. The superposed fields $H$ and $H'$ show unique behaviors in various Higgs-related processes. For example, it may be possible to observe the dark hadron effects in the scattering process in Fig. 1, since the probability being proportional to $(\sin \theta \cos \theta)^2$ becomes non-negligible.

In this connection, we must note that the effects of the dark sector can be found by carefully examining the data for the Higgs search that have been accumulated by the LHC experiments [6–8] or will be acquired by their next-stage experiments. In our scheme, the LHC signal of the Higgs-like scalar should be described by the superposed field $H$ with lower mass. From its configuration in Eq. (83), the branching ratio of the LHC scalar boson into the dark and visible components is estimated to be $(\tan \theta)^2$. All events of the production of dark components occur simply as the disappearance of energy and momentum flow. This is the so-called Higgs portal phenomena predicted many years ago [17–19,22] (for early work, see, e.g., Refs. [20,21] and, for a review, see Ref. [23]).

It may be difficult to discriminate such phenomena of the Higgs portal from the events of neutrino production in the accumulated data of the LHC experiments, which are not designed for such purposes. Probably we must wait for a clean environment of the ILC to perform a precise measurement of the branching ratio of the dark versus visible decays of the scalar field $H$. The ratio can be determined by recoil analyses of the production process $e^+ e^- \rightarrow HZ$ in the ILC experiment [17,24].

If the value of the mixing angle $\theta$ is small, the fields $H$ and $H'$ tend to lose their effects of superposition in Eq. (83) and the breakdowns of the gauge symmetries $G_{EW}$ and $G_{EW\ast}$ take place almost
independently at the separate scales $\Lambda_* \approx \sqrt{2\lambda_* v^2}$ and $\Lambda \approx \sqrt{2\lambda v^2}$. In this case, we find it difficult to discover phenomena showing limits of the SM in the low energy region. The probability of detecting the effect of the $\Delta^-_1$ hadron in the dark matter wind becomes small in the process in Fig. 1. To detect clear signals beyond the SM in this situation, we must observe phenomena of the heavier superposed field $H'$ produced by an accelerated $\Delta^-_1$ dark hadron.

The visible and dark sectors exist in the common universe whose space-time structure evolves with their field contents. The Friedmann equation has the contributions of energy density arising from all species of fields of both sectors. To make a brief consideration on early stages of the universe in the reheating period with energy higher than the scale $\Lambda_*$, we must have some knowledge about interactions between the two sectors.

In the current theory, the two sectors can be related solely through quantum effects. The cross coupling $\lambda_1 (\phi^*_+ \phi^*_-) (\phi^+_+ \phi^+_*)$ in the Higgs potential $V_H$ enables the fields of both sectors to interact with each other. Combining this cross coupling with the Yukawa interactions in Eq. (72), we have two triangular quantum corrections, which give rise to interactions between the fermions in the single and triple modes. Likewise, from the cross coupling and the gauge interactions of the Higgs fields in Eq. (73), we also obtain two-loop quantum corrections, which bring forth interactions between the dark and visible gauge fields.

With those Higgs-induced effects, all fields of both sectors can closely interact among themselves. Hence it is possible to assume that all quanta of the fields of the two sectors exist in a thermal equilibrium state with a common temperature $T (> \Lambda_*)$. All field components make the same relativistic contribution to the energy density. Accordingly, we can calculate the effective number of relativistic degrees of freedom at this stage as follows:

$$g_* = \left(28 + \frac{7}{8} \times 90\right) + \left(28 + \frac{7}{8} \times 30\right) = 106.75 + 54.25 = 161 \quad (98)$$

where the values 106.75 and 54.25 are the contributions from the visible and dark sectors, respectively.

Around the scale $\Lambda_*$, the breakdown of the dark electroweak symmetry takes place through intricate steps of phase transitions. Except for $\gamma_*$ and $\nu_*$, quanta of dark fields become massive step by step, and the Higgs-induced interactions begin inevitably to disappear, leading the dark sector to decouple out of the thermal equilibrium.

The effective number of relativistic degrees of freedom decreases along the processes of mass acquisition of the various fields. During this period, the two sectors start to follow different courses of thermal histories in the common space-time framework of the universe. In the low energy region less than the scale $\Lambda$, the visible and dark fields interact very weakly only through the mediation of the superposed fields $H$ and $H'$, as shown in Fig. 1.

Astrophysics and cosmology have the well established Standard Theory, which can describe all the results of recent observations [1,2]. Therefore, it is necessary for us to examine how the dark sector evolves without causing harmful influences on the thermal history of the visible sector. We leave this crucially important issue for a future investigation.

By adopting the chiral representations in Eq. (62), we have laid the foundation for a unified theory of the visible and dark sectors of our world. It is crucial to recognize that the “three plus one” conformations of family and color degrees of freedom can be realized on the basis of the triplet algebra with restriction of the $S_3$ irreducibility. These algebraic structures enable us to formulate the representation theory for the fundamental fermions, which consist of “visible and dark” sectors.
with “quark and lepton” states. With this successful phenomenology, we should proceed with further considerations to inquire into the possible geometric meanings of the triplet algebra and triplet fields. In this connection, it is suggestive to consider that our world with visible and dark sectors possesses space-time with a hidden multi-sheets structure, similar to the world with double sheets that was proposed by Connes [25] in his unified theory of gauge and Higgs fields.

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