Causal structure in the scalar–tensor theory with field derivative coupling to the Einstein tensor

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We investigate the causal structure in the scalar–tensor theory with field derivative coupling to the Einstein tensor, which is a class of the Horndeski theory in the four-dimensional spacetime. We show that in general the characteristic hypersurface is non-null, which admits the superluminal propagations. We also derive the conditions that the characteristic hypersurface becomes null and show that a Killing horizon can be the causal edge for all the propagating degrees of freedom, if the additional conditions for the scalar field are satisfied. Finally, we find the position of the characteristic hypersurface in the dynamical spacetime with the maximally symmetric space, and that the fastest propagation can be superluminal, especially if the coupling constant becomes positive. We also argue that the superluminality itself may not lead to the acausality of the theory.

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1. Introduction

After the investigation of many models [1], it has turned out that the successful modification of the Einstein gravity can be rewritten into a class of the Horndeski scalar–tensor theory [2,3], which is known as the most general scalar–tensor theory with the second order equations of motion despite the existence of the various derivative interactions. On the other hand, it is also well known that in the spacetime with more than five dimensions the most general tensor gravitational theory is not the Einstein gravity, but the gravitational theory with the correction of the Lovelock terms [4], for example, in the five-dimensional spacetime the Einstein gravity with the correction of the Gauss–Bonnet term, which does not give rise to the higher derivative terms in the gravitational equations of motion. In superstring/M theories the Lovelock terms appear as a typical form of the quantum corrections [5]. The relation between the Horndeski and Lovelock theories has been argued in the recent works [6] and essentially the Horndeski theory can be derived via the dimensional reduction from the higher-dimensional Lovelock theory. Thus, to find the fundamental aspects of quantum gravity, the investigation of the general properties of the Horndeski theory will be very important.

The causality and well-posedness of the initial value problem are the fundamental issues in any gravitational theory. For example, the well-posedness of the initial value problem in the Einstein gravity has been proven (see e.g. [7]). In the Einstein gravity coupled to the fields with the canonical kinetic terms, it is well known that all the speeds of propagation are less than or equal to the speed of light. On the other hand, if a gravitational theory admits a superluminal degree of freedom, its propagation can become spacelike and hence the discussion on the causality based on the metric does not make sense, because the Cauchy development is fixed by this fastest propagation.

A superluminal propagation is a typical feature in the theory with noncanonical kinetic terms [8–10]. In the case where all the fields have canonical kinetic terms, taking the high frequency mode, the equation of motion of the \( i \)-th canonical field \( \psi_1 \) in the Fourier space reduces to \( g^{\mu\nu}k_\mu k_\nu \psi_1 = 0 \), where \( g^{\mu\nu} \) is the (inverse) gravitational metric, \( \hat{\psi}_1 \) is the Fourier component of \( \psi_1 \) and \( k_\mu \) is the covariant momentum vector, which gives the solution that \( k_\mu \) is a null vector. Thus the fastest propagation speeds are the same and coincide with the speed of light. On the other hand, if the degrees of freedom have noncanonical kinetic terms, the above equation is modified as \( g^{\mu\nu}(I) k_\mu k_\nu \hat{\psi}_1 = 0 \), where \( g^{\mu\nu}(I) \) represents the effective metric for the \( i \)-th field, which in general nonlinearly depends on the fields and differs from \( g^{\mu\nu} \). Thus the fastest propagation may not be along the null hypersurface but along the spacelike one. If \( \Sigma \) is the hypersurface beyond which the evolution is not unique, \( \Sigma \) is called the characteristic hypersurface (see e.g. [11]). The characteristic hypersurface gives the edge

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of the Cauchy development and the fastest propagation must be tangent to it. The superluminality does not necessarily mean acausality, but the fastest propagation could characterize the Cauchy development. The causal structure should be defined as the chronological past set by this fastest propagation. The Cauchy problem in the modified gravity theories has been investigated, so far in the Lovelock theory [8, 9, 12, 13], the scalar–tensor theories [14, 15], the \( f(R) \) gravity [16], the nonlinear massive gravity [10] and the string-inspired gravitational theories [17].

In this letter, we investigate the causal properties in a class of the scalar–tensor theory with the field derivative coupling to the Einstein tensor:

\[
S = \frac{1}{2} \int d^Dx \sqrt{-g} \left[ M_D^{D-2} R - \left( \frac{\mu \nu}{M_D^2} G_{\mu \nu} \right) \partial_\mu \phi \partial_\nu \phi - 2V(\phi) \right].
\]

where the indices \( \mu, \nu = 0, 1, 2, \ldots, D-1 \) run the \( D \)-dimensional spacetime, \( g_{\mu \nu} \) is the metric tensor, \( g := \text{det}(g_{\mu \nu}) \) is its determinant, and \( R \) and \( G_{\mu \nu} \) are the Ricci scalar and the Einstein tensor computed from the metric \( g_{\mu \nu} \), respectively. \( M_D \) and \( z \) represent the \( D \)-dimensional Planck mass and the dimensionless parameter characterizing the field derivative coupling to the Einstein tensor, respectively. The scalar field \( \phi \) has the mass dimension \( \frac{D-2}{2} \) and \( V(\phi) \) is its potential. Despite the existence of the derivative coupling, the highest derivative terms in the equations of motion are of the second order because of the contracted Bianchi identities \( \nabla_\mu \Gamma_{\mu \nu \rho} = 0 \), where \( \nabla_\mu \) is the covariant derivative with respect to the metric \( g_{\mu \nu} \). In this letter, we do not consider the matter sector, as we focus on the causal properties of the pure gravity sector which is composed of the metric and the scalar field. Also, we will set \( M_D = 1 \), unless it should be given explicitly.

Before starting with the explicit analyses, we should add more explanations about why we focus on the field derivative coupling \( G^{\mu \nu} \partial_\mu \partial_\nu \phi \). Among the field derivative couplings to the curvature which are of the quadratic order with respect to \( \phi \), argued in the earlier works [18], \( G^{\mu \nu} \partial_\mu \partial_\nu \phi \) is unique coupling that gives the second order equations of motion. In the four-dimensional spacetime, in addition to the fact that the theory (1) corresponds to a class of the Horndeski theory, the coupling \( G^{\mu \nu} \partial_\mu \partial_\nu \phi \) is of the quadratic order with respect to \( \phi \), provides the simplest class of the derivative couplings to the curvature, because the other derivative couplings in the Horndeski theory are typically of higher order with respect to \( \phi \). Moreover, from the cosmological point of view, Ref. [19] argued that among the various couplings in the Horndeski theory, the coupling \( F_1(\phi) G^{\mu \nu} \partial_\mu \partial_\nu \phi \) is one of the special ones which could exhibit the self-tuning mechanism of the cosmological constant. The other couplings obtained in [19] are the nonminimal coupling to the Ricci scalar \( F_2(\phi) R \), that to the Gauss–Bonnet term \( F_3(\phi) (R^2 - 4 R_{\mu \nu} R^{\mu \nu} + R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma}) \) and also the field derivative coupling to the double-dual of the Riemann tensor \( F_4(\phi) \bar{R}^{\alpha \beta \mu \nu} (\partial_\mu \phi \partial_\nu \phi) \bar{V}_\alpha \bar{V}_\beta \), where \( \bar{R}^{\alpha \beta \mu \nu} \) is the Riemann tensor [20]. Then, Ref. [19] also argued that among them the field derivative couplings to the spacetime curvature, \( F_1(\phi) G^{\mu \nu} \partial_\mu \partial_\nu \phi \) and/or \( F_2(\phi) G^{\mu \nu} \partial_\mu \partial_\nu \phi \bar{V}_\alpha \bar{V}_\beta \), should always be included into the theory for obtaining the phenomenologically viable self-tuning mechanism. As the operator \( \partial_\mu \partial_\nu \phi \partial_\alpha \partial_\beta \phi \) is typically higher-dimensional than \( G^{\mu \nu} \partial_\mu \partial_\nu \phi \), the coupling \( G^{\mu \nu} \partial_\mu \partial_\nu \phi \) would be less important at the low energy scales. Therefore, among the couplings argued in [19], as the first step it is reasonable to focus on \( G^{\mu \nu} \partial_\mu \partial_\nu \phi \) in (1). On the other hand, also in the proxy theory of the nonlinear massive gravity [21] both the couplings \( G^{\mu \nu} \partial_\mu \partial_\nu \phi \partial_\alpha \partial_\beta \phi \) and \( \partial_\mu \partial_\nu \phi \partial_\alpha \partial_\beta \phi \) should be included into the theory for obtaining the phenomenologically viable self-tuning mechanism. As the operator \( \partial_\mu \partial_\nu \phi \partial_\alpha \partial_\beta \phi \) is typically higher-dimensional than \( G^{\mu \nu} \partial_\mu \partial_\nu \phi \), the coupling \( G^{\mu \nu} \partial_\mu \partial_\nu \phi \partial_\alpha \partial_\beta \phi \) appears in the low energy effective action of string theory [22] and can be embedded into supergravity [23]. Finally, from the phenomenological points of view, the theory (1) has been extensively applied to cosmology [18, 24, 25] and black hole physics [26–28]. In the context of the inflationary cosmology, the coupling \( G^{\mu \nu} \partial_\mu \partial_\nu \phi \partial_\alpha \partial_\beta \phi \) could realize the inflationary expansion and a graceful exit from inflation in the early universe without introducing a potential (for \( z < 0 \)) [18]. The shift symmetry and the modified scalar field dynamics with the enhanced friction term due to this kinetic coupling could also provide a UV protected framework for slow-roll inflation (for \( z > 0 \)) [25], which could give the predictions consistent with the observational data more easily. In the context of the black hole physics, the exact solution found in the theory (1) represents the stealth accretion of the scalar field onto a Schwarzschild black hole [27], which could circumvent the no-hair arguments and may provide an interesting playground to test the Horndeski theory in the astrophysical environment (see also [26–28], for the other black hole solutions). In summary, \( G^{\mu \nu} \partial_\mu \partial_\nu \phi \partial_\alpha \partial_\beta \phi \) is one of the most important derivative couplings in the Horndeski theory, in the sense that it could be dominant at the low energy scales and is motivated very well by the various aspects of more fundamental physics, and has very interesting applications to the problems in cosmology and black hole physics. It has also been reported that the perturbation could exhibit the superluminal propagation in the inflationary and black hole backgrounds [25, 29]. Therefore, as the next step, it will be very important to clarify more general properties of the theory (1) beyond the particular background solutions, and in this letter we will focus on the causal properties in the theory (1).

Our purpose is to clarify the general conditions that the fastest propagation speed can be superluminal and also all the propagation speeds coincide with the speed of light. We believe that our results can reveal some of the essential causal properties in the Horndeski theory, and also the similarity/difference between the Horndeski and the Lovelock theories studied in [8, 9, 13]. While the Lovelock terms can be nontrivial in the spacetime up to five dimensions, the theory (1) becomes nontrivial even in the four-dimensional spacetime and hence the properties pointed out here may also be important in the problems in astrophysics and cosmology.

2. The dynamical equations and characteristics

Varying the action (1) with respect to the metric \( g_{\mu \nu} \) gives the gravitational equation of motion

\[
G_{\mu \nu} = T_{\mu \nu} + z E_{\mu \nu},
\]

where we defined

\[
T_{\mu \nu} := \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu \nu} \nabla^2 \phi \nabla^2 \phi - V(\phi) g_{\mu \nu},
\]
where

\[ E_{\mu
u} := \nabla_\mu \nabla_\nu \phi \Box \phi - \nabla_\mu \nabla_\nu \phi \nabla^\lambda \phi - \frac{1}{2} g_{\mu\nu} \left( (\Box \phi)^2 - \nabla_\alpha \nabla_\beta \phi \nabla^\alpha \nabla^\beta \phi \right) - R_{\mu
u} \nabla^\alpha \phi \nabla^\beta \phi + \frac{1}{2} R \nabla_\mu \phi \nabla_\nu \phi \]

\[ - 2 \nabla^\gamma \phi R_{\gamma(\mu \nu)} \phi + g_{\mu\nu} R^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi + \frac{1}{2} \nabla^\alpha \phi \nabla_\alpha \phi \nabla_\mu \phi \nabla_\nu \phi. \]

Similarly, varying the action (1) with respect to the scalar field \( \phi \) gives the scalar field equation of motion

\[ (g^{\mu\nu} - z G^{\mu\nu}) \nabla_\mu \phi - V'(\phi) = 0. \]

As mentioned previously, both the gravitational and scalar field equations of motion, (2) (with (3)) and (5), contain at most the second order derivative terms.

In this letter, we consider the problem of the time evolution in the theory (1). For this purpose, we introduce the coordinate system \((t, x^i) \ (i = 1, \ldots, D - 1)\), so that the hypersurface \( \Sigma \) is located at a constant \( t \) slice. The coordinates \( t \) and \( x^i \) represent the directions off and tangential to \( \Sigma \), respectively. For a given set of the data on \( \Sigma \), which are composed of the metric, the scalar field, and their first order derivatives with respect to \( t \), and satisfy the constraint relations derived from the non-dynamical components of (2), the dynamical equations obtained from the field equations (2) and (5) can fix their second order derivatives with respect to \( t \) on \( \Sigma \), which then determine the data on the hypersurface infinitesimally neighboring to \( \Sigma \). On the other hand, if the second order derivatives with respect to \( t \) cannot be uniquely determined via the dynamical equations on a particular hypersurface \( \Sigma \), the evolution beyond \( \Sigma \) cannot be fixed. Then \( \Sigma \) is called the characteristic hypersurface which gives the edge of the Cauchy development (see e.g. [11] and also [8–10]).

In this coordinate system, the \((t, t)\) and \((t, i)\) components of (2) do not contain the second order time derivative terms with respect to \( t \) and give the constraint relations for the data on \( \Sigma \). On the other hand, the \((i, j)\) components of (2) reduce to the dynamical equations

\[ \left( \alpha A^{i,k fine} + z B^{i,k fine} \right) g_{i,k fine} + 2 z (A^{i,mn} \nabla_m \nabla_n \phi) \phi, i = \nabla^j, \]

where \( \nabla^j \) represents the terms of at most the first order derivative with respect to \( t \), and \( A^{i,k fine} \) and \( B^{i,k fine} \) are defined by

\[ A^{i,k fine} := \alpha g^{i,k fine} + g^{i,l} g^{j,k fine} + g^{i,l} g^{j,k fine} \]

\[ B^{i,k fine} := g^{i,m} g^{j,n fine} g^{j,k fine} - g^{i,m} g^{j,n fine} g^{j,k fine} + A^{j,m} g^{i,n fine} g^{j,k fine} - A^{j,m} g^{i,n fine} g^{j,k fine} \]

\[ - g^{i,m} g^{j,n fine} g^{j,k fine} + g^{i,m} g^{j,n fine} g^{j,k fine} + g^{i,m} g^{j,n fine} g^{j,k fine} - g^{i,m} g^{j,n fine} g^{j,k fine} \]

\[ + g^{i,m} g^{j,n fine} g^{j,k fine} - g^{i,m} g^{j,n fine} g^{j,k fine} \]

\[ \alpha := 1 + \frac{z}{2} \left( (g^{i,j} \nabla_i \phi)^2 + 2 g^{i,j} \nabla_i \phi \nabla_j \phi - g^{i,j} \nabla_i \phi \nabla_j \phi \right). \]

By construction, \( A^{i,k fine} \) and \( B^{i,k fine} \) are symmetric with respect to the exchange of the indices \( i \) and \( j \), and also that of \( k \) and \( \ell \). They are also symmetric under the exchange of the pairs \((i, k)\) and \((k, \ell)\), namely \( A^{k,i fine} = A^{i,k fine} \) and \( B^{k,i fine} = B^{i,k fine} \).

In the \((t, x')\) coordinate system, the scalar field equation of motion (5) gives

\[ 4 g^{i,j} \phi, i = z (A^{i,mn} \nabla_m \nabla_n \phi) g_{i,k fine} \phi, i = \nabla^j, \]

where \( \nabla^j \) represents the terms of at most the first order derivative with respect to \( t \), and \( A^{i,k fine} \) is defined in (7). We stress that the dynamical equations (6) and (9) are quasilinear in the coordinate \( t \), namely they depend linearly on \( g_{i,k fine} \) and \( \phi, i \). This property is the same as that in the Lovelock theory [8,9,13]. Thus, starting from the \( \frac{D(D + 1)}{2} + 1 \) variables, namely \( \frac{D(D + 1)}{2} \) components of the metric \( g_{i,k fine} \) and the scalar field \( \phi \), the second order derivatives of \( g_{i,k fine} \) and \( \phi, i \) with respect to \( t \) do not appear in (6) and (9), and hence they can be eliminated by the appropriate gauge conditions. Other \( D \) variables among the remaining are related to the rest via the constraint relations, leaving the \( \frac{D(D - 1)}{2} + 1 \) propagating degrees of freedom in the \( D \)-dimensional scalar–tensor theory.

Eqs. (6) and (9) can be shortly expressed as

\[ G \cdot g, i = \nabla^j, \]

where the vector \( g \) collectively represents all the variables appearing in the dynamical equations,

\[ g := \left( g_{i,k fine} \phi \right), \]

\[ G \]

do the coefficient matrix of the second order derivatives with respect to \( t \)

\[ G := \left( \alpha A^{i,k fine} + z B^{i,k fine} \right) \left( 2 z A^{i,mn} \nabla_m \nabla_n \phi \right) \]

\[ \left( 2 z A^{i,mn} \nabla_m \nabla_n \phi \right) \]

\[ \phi \]

and \( G \) represents the other terms in the dynamical equations. By writing in this way, the conditions for the characteristic hypersurface surface can be clarified. Assuming that \( g \) and \( g_{i,k fine} \) are known on \( \Sigma \) and \( g_{i,k fine} \) is also known by taking the derivative with respect to \( x' \), the value of \( g, i \) on \( \Sigma \) is uniquely determined by the dynamical equations (10), if the matrix \( G \) is invertible. In other words, if

\[ \det(G) = 0. \]

\( \Sigma \) becomes the characteristic hypersurface.
3. The conditions for the null characteristic hypersurface

Now, we will see that in the theory (1) the characteristic hypersurfaces are non-null in general, but under the certain conditions they still become null in as the case of the Einstein gravity with the canonical scalar field. Here, we clarify under which conditions a null hypersurface becomes characteristic. When \( \Sigma \) approaches a null hypersurface, assuming that among the coordinates \( x^i \) tangential to \( \Sigma \), \( x^i \) represents the null direction and \( x^a (a = 2, 3, \ldots, D - 1) \) does the others,

\[
g^{rr} \to 0, \quad g^{ta} \to 0, \quad g_{11} \to 0, \quad g_{1a} \to 0, \quad (14)
\]

and the remaining components remain finite, where \( g^{rr} \) and \( g^{11} \) approach zero with the same convergence. In this limit, \( A^{ij;kl} \) defined in (7) approaches

\[
A^{11;11} = A^{11;1b} = A^{1c;11} = A^{1c;ab} = A^{cd;1a} = A^{cd;ab} \to 0,
\]

\[
A^{ab;11} = A^{11;ab} = -2A^{1a;1b} \to (g^{11})^2 g^{ab}. \tag{15}
\]

Similarly, \( B^{ij;kl} \) defined in (7) approaches

\[
B^{11;11} = B^{11;1b} = B^{11;1a} \to 0, \quad B^{11;ab} = -2B^{1a;1b} \to (g^{11})^2 g^{ab}. \tag{16}
\]

Firstly, we confirm that in the Einstein gravity with the canonical scalar field, \( z = 0 \), the characteristic hypersurface becomes null for all the propagating degrees of freedom. If \( \Sigma \) is the null hypersurface where (14) is satisfied, for \( g \) defined in (11) rewritten in the \((t, x^1, x^3)\) coordinate system as

\[
g = \begin{pmatrix}
g_{11} \\ g_{1b} \\ g_{ab} \\ \phi
\end{pmatrix},
\tag{17}
\]

the matrix (12) reduces to

\[
G = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -g^{cb}(g^{11})^2 & 0 & 0 \\
g^{cd}(g^{11})^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}. \tag{18}
\]

As the third row is degenerate for the \((d-1)(d-2)/2\) components of \((c,d)\) and hence we cannot fix the evolution of the totally \((d-1)(d-2)/2 - 1 = d(d-3)/2\) components of \(g_{ab} \), which agrees with the number of the tensor-type polarizations in the \(D\)-dimensional space-time, and \( \Sigma \) becomes characteristic for them. Similarly, we also cannot fix the evolution of the scalar field and hence \( \Sigma \) also becomes characteristic for it.

Secondly, we turn to the case of \( z \neq 0 \). With (14), (15) and (16), (12) reduces to

\[
G = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \tag{19}
\]

which is in general invertible and hence ensures the continuous evolution beyond \( \Sigma \), unless the particular conditions are imposed. Next, we find these conditions.

On the null \( \Sigma \) satisfying (14), if we further impose

\[
\partial_t \phi = 0, \tag{20}
\]

in order for (20) to be preserved on \( \Sigma \), we also have to impose

\[
\partial_t^2 \phi = \partial_a \partial_t \phi = 0, \tag{21}
\]

as both \( x^1 \) and \( x^2 \) represent the directions tangential to \( \Sigma \). With (14) and \( \partial_t^2 \phi = 0 \), as \( \Gamma^1_{11} \) and \( \Gamma^2_{11} \) vanish on \( \Sigma \), we find \( \nabla_1^2 \phi = 0 \). On the other hand, with (14) and \( \partial_t \partial_a \phi = 0 \), as \( \Gamma^1_{1a} \) vanishes on \( \Sigma \),

\[
\nabla_1 \nabla_a \phi = -\Gamma^1_{1a} \nabla_b \phi = -\frac{1}{2} g^{bc} (\partial_1 g_{ac}) (\partial_b \phi) \neq 0. \tag{22}
\]

Similarly, \( \nabla_a \nabla_b \phi \neq 0 \) on \( \Sigma \). From (16), \( B^{cd;ab} = B^{cd;1b} = B^{cd;1a} = 0 \). Thus (19) reduces to
\[
G = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\alpha A^{11;cd} + zB^{11;cd} & 0 & 2z(A^{11;ab}v_a\phi) \\
2z(A^{11;ab}v_a\phi) & 0 & 0 \\
\end{pmatrix}.
\]

(23)

As the third row is degenerate for the \(\frac{(D-2)(D-3)}{2}\) components of the dynamical equations, we cannot fix the evolution of the independent variables, which is the number of the tensor-type polarizations in the \(D\)-dimensional scalar–tensor theory. If we further impose

\[
g^{bc}(\partial_t g_{\alpha\beta})(\partial_t \phi) = 0,
\]

(24)

for which (22) yields \(\nabla_1 v \phi = 0\), then (23) reduces to

\[
G = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\alpha A^{11;cd} + zB^{11;cd} & 0 & 2z(A^{11;ab}v_a\phi) \\
2z(A^{11;ab}v_a\phi) & 0 & 0 \\
\end{pmatrix}.
\]

(25)

As the last two rows (25) are degenerate for the \(\frac{(D-1)(D-2)}{2}\) components of the dynamical equations, we cannot fix the evolution of the \(\frac{(D-1)(D-2)}{2}\) independent variables, which agrees with the number of the propagating degrees of freedom in the \(D\)-dimensional scalar–tensor theory. Thus (20), (21) and (24) provide the conditions that all the degrees of freedom propagate with the speed of light.

The case that \(\partial_t g_{ab} = 0\) on \(\Sigma\) (and hence \(\partial_t^2 g_{ab} = \partial_t \partial_t g_{ab} = 0\), so that this is preserved on \(\Sigma\)) is the special case of (24). Thus all the conditions

\[
\begin{align*}
\partial_t \phi &= \partial_t^2 \phi = \partial_t \partial_t \phi = 0, \\
\partial_t g_{ab} &= \partial_t^2 g_{ab} = \partial_t \partial_t g_{ab} = 0,
\end{align*}
\]

(26)

give the null characteristic hypersurface for all the degrees of freedom in the \(D\)-dimensional spacetime. The conditions in the second line of (26) are the same as those obtained in [8], and on a Killing horizon they are satisfied (see also [9]). Thus, in the case that a Killing horizon exists, this can be the causal edge for all the propagating degrees of freedom, if the conditions in the first line of (26) are also satisfied. Thus, in contrast to the case of the Lovelock theory, a Killing horizon may not be always the causal edge.

4. In the dynamical spacetime with the maximally symmetric \((D - 2)\)-dimensional space

In general, the above conditions are not satisfied and then it is important to investigate whether the characteristic hypersurface becomes spacelike. For simplicity, as in [8], we consider the dynamical spacetime with the maximally symmetric \((D - 2)\)-dimensional space, given by

\[
ds^2 = -2f(u, v)\delta t du dv + R(u, v)^2 \gamma_{ab} dx^a dx^b, \quad \phi = \phi(u, v),
\]

(27)

where \(\gamma_{ab} (a, b = 2, 3, \ldots, D - 1)\) represents the metric of the \((D - 2)\)-dimensional space with the constant curvature of +1, 0 and −1. We also assume that \(f > 0\).

We then introduce \((t, x^1)\) coordinate system instead of \((u, v)\) in (27), so that the \((D - 1)\)-dimensional hypersurface \(\Sigma\) is located on a constant \(t\) slice. Now, \(\Sigma\) and hence \(x^1\) tangential to \(\Sigma\) may not be null. The coordinate transformation \(dv = dt - \epsilon dx^1\) and \(du = dx^1\), where \(\epsilon\) is assumed to be a function of \((t, x^1)\), rewrites (27) as

\[
ds^2 = -2f(t, x^1)dt dx^1 + 2\epsilon f(t, x^1)(dx^1)^2 + R(t, x^1)^2 \gamma_{ab} dx^a dx^b, \quad \phi = \phi(t, x^1).
\]

(28)

As the normal vector to \(\Sigma\) is proportional to \(\partial_x t = \delta^t_{x^1}\), its norm becomes \(g^{tt}(\partial_t t)(\partial_t t) = g^{tt} = -\frac{\epsilon}{f}\). Thus \(\epsilon > 0, \epsilon = 0\) or \(\epsilon < 0\) corresponds to the case that \(\Sigma\) is spacelike, null or timelike, respectively.

Now, we investigate the position of the characteristic hypersurfaces in (28). We will explicitly show the dependence on the \(D\)-dimensional Planck mass. The condition (13) gives

\[
\begin{align*}
C &= \frac{D(D-1)}{2}B^2C^2 + (D-2)\alpha BQz^2 + M_D^D - \alpha^2 \epsilon^2 f, \\
\end{align*}
\]

(29)

where \(\alpha\) defined in (8) reduces to

\[
\alpha := 1 - \frac{z}{M_D^D} \left(\epsilon (\partial_\phi \phi)^2 + \partial_1 \phi \partial_\phi \phi\right).
\]

(30)

and we have defined

\[
\begin{align*}
\mathcal{B} &= -\frac{1}{fR} \left(2\epsilon R_\phi + R_{\phi\phi}\right) \partial_\phi \phi - \frac{R_\phi}{fR} \partial_\phi \phi, \\
\mathcal{C} &= -2\epsilon f \alpha + \frac{z}{M_D^D} (\partial_1 \phi)^2, \\
\mathcal{Q} &= \nabla_1^2 \phi + 2(D-3)\epsilon f B,
\end{align*}
\]

(31)
with $\nabla_a \nabla_b \phi = B R^2 \gamma_{ab}$, which gives two possibilities

$$C = 0, \quad \frac{(D-2)(D-3)}{2} B^2 \zeta^2 + (D-2) \alpha B \zeta^2 + M_D^{-2} \alpha^2 \epsilon f = 0.$$  \hspace{1cm} (32)

The position of the characteristic hypersurface does not depend on the curvature of the $(D - 2)$-dimensional space. As expected, in the case of $z = 0$, $C = -2f\epsilon$ and $\frac{(D-2)(D-3)}{2} B^2 \zeta^2 + (D-2) \alpha B \zeta^2 + M_D^{-2} \alpha^2 \epsilon f = M_D^{-2} \alpha^2 \epsilon f$, and hence the conditions in (32) reduce to $\epsilon f = 0$, which only admits the null characteristic hypersurface $\epsilon = 0$ for $f > 0$. We introduce the dimensionless quantities, $x := \frac{\hat{\phi} M_D}{\sqrt{\epsilon}}, \quad y := \frac{\hat{\phi}}{M_D \sqrt{\epsilon}}$.

In particular, we focus on the limit of $|z| \ll 1$, where the contribution of the field derivative coupling in (1) is subdominant. In this limit the first condition in (32) gives the approximated solution

$$\epsilon = \frac{y^2}{2} z + O(z^2),$$  \hspace{1cm} (33)

approaching null for $z = 0$. For $z > 0$, this characteristic hypersurface can be spacelike and the propagation speed can be superluminal. The second condition in (32) gives the approximated solution for $\epsilon$

$$\epsilon = (D - 2)(y^2 - z z f)(y x_R + x y_R)z^2 + O(z^2),$$  \hspace{1cm} (34)

also approaching null for $z = 0$. The superluminal propagation could appear, if $(y^2 - z z f)(y x_R + x y_R) > 0$.

As (33) and (34) scale as $z$ and $z^2$, respectively, the absolute value of (33) is always greater than that of (34) for $|z| \ll 1$. Thus for $z > 0$, (33) is always positive and the fastest propagation is fixed by (33). On the other hand, if $z < 0$, the fastest propagation is fixed by (34), which can be superluminal if we impose $(y^2 - z z f)(y x_R + x y_R) > 0$.

5. Conclusions

We have investigated the causal structure in the scalar–tensor theory with the field derivative coupling to the Einstein tensor (1), which in the four-dimensional spacetime constitutes a class of the Horndeski theory.

Firstly, we confirmed that in general the characteristic hypersurfaces are non-null, and hence the theory (1) admits the superluminal propagation. We then gave the conditions that the characteristic hypersurface becomes null. In contrast to the case of the Lovelock theory [8,9] where a Killing horizon can always be the causal edge of all the propagating degrees of freedom, in the theory (1) this is not always the case, as the conditions (20) and (21) also have to be satisfied there.

Secondly, we investigated the position of the characteristic hypersurfaces in the dynamical spacetime with the maximally symmetric $(D - 2)$-dimensional space. There are two solutions for characteristic hypersurfaces which are smoothly connected to the null characteristic hypersurface in the case of the Einstein gravity with the canonical scalar field. We showed that the fastest propagation can be spacelike, especially if $z > 0$.

Having the possibility of the superluminal propagations, one may worry about the acausality in the theory (1), for instance, the appearance of the closed timelike curves. Here, we stress that the superluminality itself may not lead to the acausality. As argued in [30], a degree of freedom which exhibits the superluminal propagation in a nontrivial background has its effective metric $\mathcal{G}^{\mu\nu}$, which differs from the original spacetime metric $g^{\mu\nu}$ (i.e., the metric for photons), and hence its own causal structure. Even in the case that the causal cones for $\mathcal{G}^{\mu\nu}$ are wider than those for $g^{\mu\nu}$, so long as there are the spacelike hypersurfaces which can be regarded as the Cauchy surfaces for both the metrics, the causal structure of the spacetime is preserved. See also [31] for the further arguments about the superluminality. Moreover, Ref. [32] argued the possibility that in the models with the noncanonical kinetic terms of the scalar field the closed timelike curves could never arise, because at the onset of the formation of a closed timelike curve the scalar field could be strongly coupled, where the effective field theory is no longer valid. They suggest that the superluminality itself may not mean the acausality, and the further careful studies are needed.

Although we focused on the particular field derivative coupling, the investigation of the causal structure in the scalar–tensor theory with more general field derivative couplings will be a very interesting and important subject. We leave this subject for the future work.

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References


