Cosmological perturbations of non-minimally coupled quintessence in the metric and Palatini formalisms

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\section*{ABSTRACT}

Cosmological perturbations of the non-minimally coupled scalar field dark energy in both the metric and Palatini formalisms are studied in this paper. We find that on the large scales with the energy density of dark energy becoming more and more important in the low redshift region, the gravitational potential becomes smaller and smaller, and the effect of non-minimal coupling becomes more and more apparent. In the metric formalism the value of the gravitational potential in the non-minimally coupled case with a positive coupling constant is less than that in the minimally coupled case, while it is larger if the coupling constant is negative. This is different from that in the Palatini formalism where the value of gravitational potential is always smaller. Based upon the quasi-static approximation on the sub-horizon scales, the linear growth of matter is also analyzed. We obtain that the effective Newton’s constants in the metric and Palatini formalisms have different forms. A negative coupling constant enhances the gravitational interaction, while a positive one weakens it. Although the metric and Palatini formalisms give different linear growth rates, the difference is very small and the current observation cannot distinguish them effectively.

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1. Introduction

Our Universe is undergoing an accelerated expansion, and this has been confirmed by various observations including the type Ia supernovae [1,2], the baryonic acoustic oscillation [3] and the cosmic microwave background radiation anisotropy [4], and so on. This observed phenomenon can be accounted for by the existence of an exotic energy with negative pressure, called dark energy (see [5] for recent reviews), or a modification of the theory of general relativity on the cosmic scale (see [6,7] for recent reviews). The simplest and most popular candidate of dark energy is the cosmological constant [8–10]. On the one hand, it fits almost all observational data very well; on the other hand, however, it suffers from two seriously theoretical problems: \textit{coincidence problem} and \textit{fine tuning problem}. Thus, some minimally coupled scalar fields, such as quintessence, phantom and quintom, are proposed for the explanation of the accelerating cosmic expansion. Quintessence [11] is a normal scalar field with the equation of state being larger than $-1$, while phantom [12] has a negative kinetic term, which leads to a less than $-1$ value for its equation of state. Quintom [13], a combination of quintessence and phantom, can realize a crossing of the phantom divide line ($-1$ line) for the equation of state.

Besides the quintom scalar field dark energy, a non-minimal coupling between quintessence and gravity can also realize the crossing of the $-1$ line for the effective equation of state [14–17]. The advantage of the latter is that the phantom matter is not needed. Furthermore, it has been found that non-minimal couplings are generated naturally when quantum corrections are considered and they are essential for the renormalizability of the scalar field theory in curved space. As a result, a scalar field with a non-minimal coupling to gravity has been suggested to be responsible for both the early cosmic inflation [22] and the present accelerated expansion [14–21]. Quintessence with a non-minimal coupling between the scalar field and the curvature scalar is the special case of the scalar–tensor theories [23–37], and it is called extended quintessence.

It is well known that the curvature scalar $R$, on the one hand, can be defined in the metric formalism in which only the metric...
is the independent variable, and on the other hand, it can also be obtained from the Palatini formalism where the metric and connection are two independent variables. In the general relativity limit, these two different formalisms give the same field equation [38]. This may however change if the gravity is modified. For example, in the $f(R)$ theory of gravity, where $f$ is an arbitrary function of $R$, the metric formalism gives a fourth order differential equation, while the Palatini formalism leads to a second order one [6,7]. Thus, the latter can automatically pass the solar system tests.

Since the $f(R)$ theory of gravity gives rise to different field equations in the metric and Palatini formalisms and the field equations in the latter are simpler, it is natural and interesting to consider the extended quintessence with the Ricci scalar defined in the Palatini formalism. In [39], Wang, Wu and Yu have studied this new extended quintessence, in which the Ricci scalar is defined in the Palatini formalism rather than in the metric one. Through the dynamical analysis, they found that the effective equation of state of the new extended quintessence can cross the phantom divide line, and moreover, it can realize an oscillation around the $-1$ line. This oscillatory behavior is a new feature different from that of the previous extended quintessence with the Ricci scalar defined in the metric formalism [14–17].

In this paper, we plan to analyze the cosmological perturbations and the linear growth of matter in the extended quintessence with a coupling between the scalar field and the Ricci scalar defined both in the metric and Palatini formalisms, and try to study the possibility of distinguishing these two different formalisms in the non-minimal case. The paper is organized as follows. In Section 2, we provide a short review of the background equations of motion in both metric and Palatini formalisms. The linear perturbations are studied in Section 3 and the corresponding numerical results are given in Section 4. We give a conclusion in Section 5. Throughout this paper, unless specified, we adopt the metric signature $(-, +, +, +)$. Latin indices run from 0 to 3 and the Einstein convention is assumed for repeated index.

2. The background field equations

We consider a non-minimal coupling between the scalar field $\varphi$ and gravity with the action taking the form

$$ S = \int d^4x \sqrt{-g} \left[ \frac{1}{2k^2} f(\varphi, \tilde{R}) + L_\varphi + L_m \right], $$

(1)

where $k^2 = 8\pi G$, with $G$ being the Newton constant, $g$ is the determinant of metric $g_{\mu\nu}$, and $L_\varphi$ and $L_m$ represent the Lagrangians of the scalar field and matter, respectively. The Ricci scalar is obtained from $R = g^{\mu\nu} \tilde{R}_{\mu\nu}$, where $\tilde{R}_{\mu\nu}$ is the Ricci tensor and it is defined in terms of the connection: $\tilde{R}_{\mu\nu} = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu,\alpha} \Gamma^\nu_{\alpha\nu} + \Gamma^\alpha_{\alpha\nu} \Gamma^\mu_{\alpha\nu} - \Gamma^\alpha_{\mu \alpha} \Gamma^\nu_{\alpha\nu}$. In the metric formalism, the metric tensor $g_{\mu\nu}$ is the only variable and the connection is the Levi-Civita connection which is determined by $g_{\mu\nu}$. Thus, $R = R(g_{\mu\nu})$.

While, in the Palatini formalism, since the metric and the connection are two independent variables, $R = R(g_{\mu\nu}, \Gamma^\mu_{\alpha\nu})$. The basic equations of the Palatini formalism can be found in Appendix A.

In the above action, $f$ is an arbitrary function of the Ricci scalar $\tilde{R}$ and the scalar field $\varphi$. Here, we consider a special coupling between scalar field and gravity

$$ f = F(\varphi) \tilde{R}, \quad F(\varphi) = 1 + \omega k^2 \varphi^2 $$

(2)

with $\omega$ being the coupling constant. If $\tilde{R}$ is defined in the metric formalism, then the nonminimal coupling corresponds to the usual extended quintessence [16]. If $\tilde{R}$ is defined in the Palatini formalism however, the so-called new extended quintessence is obtained [39]. When $\omega = 0$, the action reduces to that in general relativity.

Varying the action (1) with respect to the metric tensor $g_{\mu\nu}$, we get the field equations

$$ F \tilde{R}_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} - (1 - \xi) (\nabla_\mu \nabla_\nu F - g_{\mu\nu} \nabla_\sigma \nabla^\sigma F) $$

$$ = k^2 \left[ T^{(\psi)}_{\mu\nu} + T^{(m)}_{\mu\nu} \right], $$

(3)

where $\xi = 0$ or 1 corresponds to the metric formalism or the Palatini one, respectively, $\nabla_\mu$ is the usual covariant derivative related to the Levi-Civita connection, $T^{(m)}_{\mu\nu}$ is the energy–momentum tensor of matter

$$ T^{(m)}_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} L_m)}{\delta g^{\mu\nu}}, $$

(4)

and $T^{(\psi)}_{\mu\nu}$ is the energy–momentum tensor of the scalar field, which has the form

$$ T^{(\psi)}_{\mu\nu} = \nabla_\mu \varphi \nabla_\nu \varphi - g_{\mu\nu} \left[ \frac{1}{2} (\partial \varphi)^2 + V \right], $$

(5)

according to $L_\varphi = -\frac{1}{2} (\partial \varphi)^2 - V(\varphi)$, where $V(\varphi)$ is the potential of the scalar field. Using the metric Einstein tensor $G^{\mu\nu}_{\mu} = R^{\mu\nu}_{\mu} - \frac{1}{2} R g^{\mu\nu}$, where $R^{\mu\nu}_{\mu}$ and $R$ are defined in the metric formalism, Eq. (3) can be rewritten in the standard form

$$ G^{\mu\nu}_{\mu} = k^2 \left[ T^{(\psi)}_{\mu\nu} + T^{(m)}_{\mu\nu} + T^{(eff)}_{\mu\nu} \right], $$

(6)

where $T^{(eff)}_{\mu\nu}$ is an effective energy–momentum tensor defined as

$$ T^{(eff)}_{\mu\nu} = \frac{1}{k^2} \left[ (1 - F) R^{\mu\nu}_{\mu} + \nabla^\mu \nabla_\nu F + \frac{1}{2} \delta^{ij}_{\mu} (FR - R - 2 \nabla_\sigma \nabla^\sigma F) \right] $$

$$ + \xi \left( - \frac{3}{2} \nabla^\mu F \nabla_\nu F + \frac{3}{4} \delta^{ij}_{\mu} \nabla_\sigma \nabla^\sigma F \right), $$

(7)

It is easy to see that in the limit of general relativity $(\omega \rightarrow 0)$ and $F \rightarrow 1$, $T^{(eff)}_{\mu\nu} = 0$ and two different formalisms give the same field equations.

Now, we consider a spatially flat, homogeneous and isotropic universe described by the Friedmann–Lemaître–Robertson–Walker (FLRW) metric

$$ ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j, $$

(8)

where $t$ is the cosmic time and $a(t)$ is the cosmic scale factor. Thus, the background equations can be found from (0, 0) and (i, j) components of Eq. (6) [19,20,28,39]:

$$ H^2 = \frac{k^2}{2 \rho_m + \frac{\omega k^2}{3} \left( \frac{1}{2} \varphi^2 + V \right)} - H \dot{F} - \omega k^2 H \dot{\varphi}^2 - \frac{\dot{\varphi}^2}{4 F}, $$

(9)

$$ -3 H^2 - 2 \dot{H} = k^2 \left( \frac{1}{2} \dot{\varphi}^2 + V \right) + 2 H \ddot{F} + \ddot{F} $$

$$ + \omega k^2 \dot{\varphi} (3 H^2 + 2 \dot{H}) - \frac{\dot{\varphi}^2}{4 F}. $$

(10)

Here, $H = \frac{\dot{a}}{a}$ is the Hubble parameter, and a dot denotes the derivative with respect to the cosmic time. In the above equations, we neglect the tiny radiation in the present universe and only consider the pressureless matter including baryonic matter and dark matter, and use $\rho_m$ to describe the corresponding energy density, which satisfies the canonical continuity equation.
\[ \dot{\rho}_m + 3H\rho_m = 0 . \]  \hfill (11)

The dynamical equation of the scalar field has the form [30,31]
\[ \ddot{\phi} + 3H\dot{\phi} + V,\phi = - \frac{F,\phi}{2} \dot{R} = 0 , \]  \hfill (12)

where \( \dot{R} = 12H^2 + 6H + \xi( - \frac{3\dot{F}^2}{2F^2} + \frac{3\ddot{\phi}}{2\dot{\phi}} + 3H\ddot{\phi} ) \), and the subscript \( \phi \) represents the derivative with respect to the scalar field, i.e., \( V,\phi = \frac{dV}{d\phi} \). Using Eqs. (9) and (10), we obtain a useful relation
\[ \frac{\dot{H}}{H^2} = - \frac{\kappa^2}{2FH^2}( \rho_m + \dot{\phi}^2 ) + \frac{1}{2} \frac{\dot{F}}{F} - \frac{1}{2} \frac{\ddot{F}}{F^2} + \frac{\xi}{4} \frac{\dot{\phi}^2}{F^2} \frac{\dot{H}}{H^2} . \]  \hfill (13)

To analyze the cosmic background evolution, we introduce four dimensionless variables
\[ x_1 = \frac{\kappa\dot{\phi}}{\sqrt{6H}}, \quad x_2 = \frac{\kappa\sqrt{\ddot{V}}}{\sqrt{3H}}, \quad x_3 = \frac{\kappa\sqrt{\rho_m}}{\sqrt{3H}}, \quad x_4 = \kappa\phi . \]  \hfill (14)

Here, for simplicity, we consider an exponential potential and assume \( \frac{dH}{dt} = \gamma \) with \( \gamma \) being a constant. Therefore the Friedmann equation (Eq. (9)) can be re-expressed as
\[ 1 = x_1^2 + x_2^2 + x_3^2 - 2\sqrt{6}\omega x_4 - \omega x_2^2 - 6\xi\omega - \frac{2x_1^2}{1 + \omega x_4^2} . \]  \hfill (15)

and this indicates that only three of four new dimensionless variables are independent. Differentiating these new dimensionless variables with respect to the number of e-foldings \( N = \ln a \) gives:
\[ \frac{dx_1}{d\ln a} = \frac{c}{\sqrt{6}} - bx_1 , \]  \hfill (16)
\[ \frac{dx_2}{d\ln a} = \frac{\sqrt{6}}{2} \gamma x_1 x_2 - bx_2 , \]  \hfill (17)
\[ \frac{dx_3}{d\ln a} = - \frac{3}{2} x_3 - bx_3 , \]  \hfill (18)
\[ \frac{dx_4}{d\ln a} = \sqrt{6} x_1 . \]  \hfill (19)

Here \( c \) and \( b \) are defined as
\[ c = -3\sqrt{6}x_1 - 3\gamma x_2^2 + 3\omega x_4 (x_2^2 - 2x_1^2 + 4x_2^2) \]
\[ + \frac{3\omega(1 - \xi)x_4}{1 + \omega x_4^2} (8\sqrt{6}\omega x_1 x_4 + 6\gamma\omega x_2^2 x_4 - 3x_2^2) \]
\[ - 6x_1^2 - 12\omega x_2^2 + 44 + 46\omega x_4^2) \]  \hfill (20)
and
\[ b = \frac{1}{2(1 + \omega x_4^2)} (3x_1^2 + 6x_2^2 + 12\omega x_1^2 + 2\sqrt{6}\omega x_1 x_4 + 2\omega x_4) \]
\[ + 18\xi\omega^2 \frac{(x_1 x_4)}{1 + \omega x_4^2} . \]  \hfill (21)

Solving numerically any three of Eqs. (16), (17), (18), (19), we can obtain the background evolution of our Universe with a non-minimally coupled scalar field dark energy.

### 3. Linear perturbations

In this section, we study the linear perturbations in both the metric and Palatini formalisms by assuming the perturbed FRW metric to be
\[ ds^2 = -(1 + 2\Phi)dt^2 + a(t)^2(1 + 2\Phi)\delta_{ij}dx^idx^j , \]  \hfill (22)

where the Newton gauge is taken, \( \Psi \) is the so-called Bardeen potential, and \( \Phi \) represents the perturbation to the spatial curvature. In addition, we use \( \delta \) to represent the perturbation of the scalar field, which satisfies
\[ \delta\ddot{\phi} + 3H\delta\dot{\phi} + \left[ \frac{k^2}{a^2} + \frac{1}{2} \left( 2V - \frac{f}{\dot{\phi}^2} \right) \right] \delta\phi \]
\[ = \ddot{\phi}(\Psi - 3\dot{\Phi}) - 3(2V - \frac{f}{\dot{\phi}^2}) \dot{\phi} \dot{\Phi} + \frac{f}{2\dot{\phi}^2} \delta\dot{R} , \]  \hfill (23)

in the Fourier space, where \( k \) represents the wavenumber and
\[ \delta\ddot{R} = -2\dddot{A} - 8HA + \left[ \frac{k^2}{a^2} - 6H \right] \dot{\Phi} + \frac{k^2}{a^2} \dot{\Phi} \]
\[ + \frac{\xi}{2} \left[ -6\frac{\ddot{F}}{F} - 18H\frac{\dot{F}}{F} + 3\frac{\dot{F}^2}{F^2} \right] \dot{\Phi} + 9\frac{\ddot{F}}{F} \dot{\Phi} \]
\[ - 3\frac{\ddot{F}}{F} \dot{\Phi} + 3 \left( \frac{\ddot{F}}{F^2} - 3H\frac{\dot{F}}{F} - \frac{\dot{F}}{F} + \frac{k^2}{a^2} \right) \dot{F} \frac{\delta F}{F} \]
\[ + 3 \left( 3H - \frac{\dot{F}}{F} \right) \delta\frac{\dot{F}}{F} + \frac{3}{2}\frac{\delta F}{F} \]  \hfill (24)

Here for convenience we define
\[ A = 3(\dot{H} - \dot{\Phi}) . \]  \hfill (25)

The perturbed energy–momentum tensor of pressureless matter can be expressed as
\[ T^{0(m)}_0 = - (\rho_m + \delta\rho_m), \quad T^{1(m)}_0 = - \tau^{(mm)}_0 = \rho_m v^i . \]  \hfill (26)

with \( v^i = a\frac{dx^i}{dt} \) being the peculiar velocity of matter. The density contrast \( \delta_m \equiv \frac{\delta\rho_m}{\rho_m} \) satisfies the same equation as that in general relativity
\[ \ddot{\delta}_m + 2H\dot{\delta}_m = - \frac{k^2}{a^2} \Psi - 3\dot{\Phi} - 6H\dot{\Phi} . \]  \hfill (27)

Perturbation equations can be achieved by perturbing Eq. (6) in a similar way as that in the metric formalism [16,19,28,30], and its (0, 0) component gives
\[ \left( 2H + \frac{\dot{F}}{F} \right) \dot{A} = - \frac{k^2}{a^2} \Phi + \left( 3H\frac{\dot{F}}{F} - \frac{k^2}{a^2} \frac{\dot{\phi}^2}{\dot{\phi}^2} \right) \Psi + \frac{\dot{F}^2}{2F} \Psi \]
\[ = \frac{1}{F} \left[ -k^2 \delta\rho_m - k^2 \dot{\phi}\delta\phi - \frac{1}{2}(2V - f,\phi,\phi,\phi) \right] \dot{\phi} \delta\phi \]
\[ + \left( \frac{k^2}{a^2} + 3H^2 - \frac{1}{2} \frac{\dot{F}^2}{F} \right) \delta\dot{F} + 3H\delta\dot{F} + \frac{\dot{F}}{2F} \frac{\delta F}{F} \]
\[ - \frac{1}{2} \left( \frac{3\dot{F}}{F} + 9H\frac{\dot{F}}{F} \right) \delta\ddot{F} . \]  \hfill (28)

The (0, 1) component represents the momentum density
\[ A + \frac{3\dot{F}}{2F} \Psi = \frac{3}{2F} \left[ k^2 \dot{\phi}\rho_m + k^2 \dot{\phi}\delta\phi - H\delta F + \delta\dot{F} - \frac{3\dot{F}}{2F} \delta F \right] \]  \hfill (29)

The \( (i, j) \) components can be divided into two parts. One comes from \( i \neq j \) and has the form
\[ -\Psi - \Phi = \frac{1}{F} \delta F . \]  \hfill (30)

The other, corresponding to \( i = j \), represents the momentum flux.

2\ddot{A} + \left(6\dot{H} + 2\ddot{\dot{F}}/F\right)A - 2\frac{k^2}{a^2}\Phi
+ \left(6\dot{H} + 6\dot{F}/F + 6\frac{\dot{\dot{F}}}{F} + 3\kappa^2 \dot{\psi}^2 - 2\frac{k^2}{a^2}\right)\psi
+ 3\frac{\dot{F}}{F} + \xi\frac{\dot{F}^2}{2F^2}\psi
= \frac{1}{F}\left[3\kappa^2 \dot{\psi}\dot{\psi} - \frac{3}{2}(2\kappa^2 V - f)\dot{\psi}\dot{\psi}
+ \left(2\frac{k^2}{a^2} - 9H^2 - 3\dot{H}\right)\delta F + 6H\delta\dot{F} + 3\dot{\delta} F
- \xi\left(\frac{9\dot{F}}{2F}\delta F + \frac{27H\dot{F}}{2F}\delta F + \frac{9\dot{F}}{2F}\dot{\delta} F\right)\right].
\quad (31)

4. Numerical results

From Eq. (30), one can obtain
\Phi = -\Psi - \frac{F}{F}\dot{\psi}\dot{\psi}.
\quad (32)

Thus, in the perturbation equations, there are only three independent variables ($\Phi$, $\dot{\psi}$, and $\delta_m$). Substituting Eq. (32) into Eqs. (23), (27), (31) gives a set of equations to be solved together. Since these equations are dependent on the background evolution, we first solve numerically Eqs. (16), (17), (19). In our calculation, we set $\gamma = -1$, and choose suitable initial conditions to make sure that the current matter density is $\Omega_m = 0.30$. Figs. 1, 2 show the results. In Fig. 1, we plot the evolutionary curves of the gravitational potential $\Psi/\Psi_i$, where $\Psi_i$ is the initial value of $\Psi$, with different coupling constants and different wavenumbers. The dashed and solid lines show the results of the metric and Palatini formalisms, respectively, with the black dotted line representing the minimally coupled case. The red, green and purple lines correspond to $\omega = -0.5$, -0.2 and 0.2, respectively, and the left and right panels correspond to $k = 0.01h$ Mpc$^{-1}$ and $0.001h$ Mpc$^{-1}$, respectively. As expected, in the matter dominated era ($z > 1$) the gravitational potential is a constant. With the energy density of dark energy becoming more and more important at the low redshift, the gravitational potential becomes smaller and smaller and the effect of non-minimal coupling becomes more and more apparent. In the metric formalism, from the dashed line of Fig. 1, we find that for a positive coupling constant the gravitational potential is smaller than that of the minimal coupled case, and for a negative one, it is larger. While in the Palatini formalism, this property disappears, for example, in the case of $k = 0.01h$ Mpc$^{-1}$ both a positive and a negative coupling constant lead to a smaller value of gravitational potential than that of minimally coupled case. At the scale ($k = 0.01h$ Mpc$^{-1}$), the deviation from the minimal coupling in the Palatini formalism is larger than that in the metric formalism, and it is just the opposite at the larger scale ($k = 0.001h$ Mpc$^{-1}$). Fig. 2 shows the evolutions of matter density perturbation with respect to the redshift at different scales, in which red solid and black dashed lines represent the results of the metric and Palatini non-minimal coupling with $\omega = -0.2$. This figure indicates that the metric and Palatini formalisms almost give the same behavior.

Now, we study the matter growth on small scales limit $k \gg aH$. Using the quasi-static approximation [27]
\[ |\dot{X}| \ll |HX|, \quad \text{for } X = \Phi, \Psi, \dot{\psi}, F, \dot{F}, \]
Eq. (27) reads
\[ \ddot{\delta}_m + 2H\dot{\delta}_m + \frac{k^2}{a^2}\Psi \simeq 0. \]
And, from Eqs. (23), (24) and (28) we obtain the following relations
\[ \frac{k^2}{a^2}\dot{\psi}\dot{\psi} \simeq \frac{1}{2}F, \dot{\psi}\delta R, \]
\[ \delta R \simeq \frac{k^2}{a^2}\left(\Psi + 2\Phi + \xi\frac{3F}{2F}\dot{\psi}\dot{\psi}\right). \]
\[-\frac{2k^2}{a^2} F \Phi \simeq \frac{k^2}{a^2} F_\varphi \delta \varphi - k^2 \rho_m \delta m. \tag{37} \]

For the general scalar-tensor dark energy the contributions of \( V_\varphi \) and \( V_{\varphi \varphi} \) can be neglected [27,40,41]. Substituting Eq. (36) into Eq. (35) gives

\[ \delta \varphi = \frac{2 F F_\varphi}{2 F - 3 \xi F_\varphi^2} (\Psi + 2 \Phi). \tag{38} \]

Using Eq. (30), we obtain

\[ \delta \varphi = \frac{2 F F_\varphi}{2 F - 3 \xi F_\varphi^2 + 2 F_\varphi^2} \Phi = -\frac{2 F F_\varphi}{2 F - 3 \xi F_\varphi^2 + 4 F_\varphi^2} \Psi. \tag{39} \]

Substituting the above expression into Eq. (37), one can get

\[ \frac{k^2}{a^2} \left( 1 + \frac{F_\varphi^2}{2 F - 3 \xi F_\varphi^2 + 2 F_\varphi^2} \right) \Phi \]
\[ = \frac{k^2}{2 F} \rho_m \delta m = \frac{k^2}{a^2} \left( 1 - \frac{F_\varphi^2}{2 F - 3 \xi F_\varphi^2 + 4 F_\varphi^2} \right) \Psi. \tag{40} \]

Rewriting the modified density perturbation equation as

\[ \delta m + 2H \delta m - 4\pi G_{\text{eff}} \rho_m \delta m = 0, \tag{41} \]

where \( G_{\text{eff}} \) is the effective Newton’s constant, and using Eq. (40), we obtain

\[ Q \equiv \frac{G_{\text{eff}}}{G} = \frac{1}{F} \frac{2 F - 3 \xi F_\varphi^2 + 4 F_\varphi^2}{2 F - 3 \xi F_\varphi^2 + 3 F_\varphi^2}. \tag{42} \]

Thus, in the metric formalism, the effective Newton constant has the form

\[ G_{\text{eff}} = \frac{1}{F} \frac{2 F + 4 F_\varphi^2}{2 F + 3 F_\varphi^2} G. \tag{43} \]

which is the same as that obtained in [27,28,36]. While, in the Palatini formalism,

\[ G_{\text{eff}} = \frac{1}{F} \frac{2 F + F_\varphi^2}{2 F} G. \tag{44} \]

It is easy to see that the metric and Palatini formalisms give different effective Newton gravitational constants. In the minimal coupling case \( \omega = 0 \), the effective gravity constant \( G_{\text{eff}} \) reduces to the Newton’s constant \( G \) as expected.

Fig. 3 shows the evolutions of \( G_{\text{eff}}/G \) with respect to \( z \). Dashed and solid lines represent the results of the metric and Palatini formalisms, respectively. Red and purple lines correspond to \( \omega = -0.5 \) and 0.2, respectively. It is easy to see that \( G_{\text{eff}} \) are almost the same in both the metric and Palatini formalisms although they have different expressions (see Eqs. (43) and (44)). The properties of \( G_{\text{eff}} \) are independent of the formalisms. A negative coupling will enhance the gravitational interaction, while a positive one will weaken it. At the redshift \( z > 2 \), \( G_{\text{eff}}/G \) reaches 1, which means that deep in the matter dominated era the effective Newton’s constant equals to \( G \). Thus, deep in the matter dominated era, the solution of Eq. (41) indicates that the matter density perturbation goes like \( \delta_m \propto a \). Thus, we have \( \delta \rho_m = \rho_m \delta m \propto a^{-2} \) and \( \Phi \propto \Psi \propto \text{constant} \). The latter can also be found in Fig. 1.

To study the linear growth of matter perturbations, one can define the growth as \( \delta = \delta_m(a)/\delta_m(a_0) \), which is the ratio of the perturbation amplitude at some scale factor relative to the initial one. Since \( \delta(a) \sim a \) during the matter dominated era, we introduce an \( a \)-independent variable \( g(a) = \frac{\delta}{a} \) in the matter era. Then, Eq. (41) can be rewritten as

\[ \frac{d^2 g}{d \ln a^2} + \left( \frac{\dot{H}}{H^2} + 4 \right) \frac{dg}{d \ln a} + \left( 3 + \frac{\dot{H}}{H^2} - 2 \frac{3}{\Omega_m} \right) g = 0. \tag{45} \]

with the initial conditions being \( g(a_0) = 1 \) and \( dg/d \ln a|_{a=a_0} = 0 \).

Fig. 4 shows the evolutionary curves of the growth rate \( \delta \). The dashed and solid lines correspond to the results from the metric and Palatini formalisms, respectively. The red, blue and purple lines correspond to \( \omega = -0.5, -0.2, \) and 0.2, respectively. The dotted line shows the result of the minimal coupling (\( \omega = 0 \)). One can see that a negative coupling constant leads to a larger growth than a positive one. This property becomes more apparent in the Palatini formalism. The inserted figure in the right panel shows that the metric and Palatini formalisms give different linear growth rates, but the difference is very small and the current observation cannot distinguish them.

5. Conclusion

In this paper, we have studied in detail the cosmological perturbations of the non-minimally coupled quintessence dark energy in both the metric and Palatini formalisms, which give the same equations in the minimally coupled case. We find that at the high redshift the gravitational potential is a constant. With the energy density of dark energy becoming more and more important in the low redshift region, the gravitational potential becomes smaller and smaller and the effect of the non-minimal coupling becomes more and more apparent. In the metric formalism the value of the gravitational potential in the non-minimally coupled case with a positive coupling constant is less than that in the minimally coupled case, while it is larger if the coupling constant is negative. This is different from that in the Palatini formalism where the value of the gravitational potential is always less than that for the minimally coupled quintessence. At a scale \( (k = 0.01 \text{ Mpc}^{-1}) \) the non-minimal coupling in the Palatini formalism leads to a larger deviation of the gravitational potential from the minimal coupling than in the metric formalism, while at a larger scale \( (k = 0.001 \text{ Mpc}^{-1}) \) it is just opposite.

Using the quasi-static approximation on sub-horizon scales, the matter linear growth is also analyzed. The effective Newton’s constants in the metric and Palatini formalisms are obtained, which have different expressions, as shown in Eqs. (43), (44). We obtain that irrespective of the formalisms a negative coupling enhances the gravitational interaction, while a positive one weakens it. Fig. 4 shows that a negative coupling leads to a larger growth than a positive one. This property becomes more apparent in the Palatini formalism. Although the metric and Palatini formalisms give different linear growth rates, the difference is very small and the current
observation is very hard to distinguish them. Therefore, to distinguish these two different nonminimally coupled quintessences, we need to analyze other effects, such as the integrated Sachs–Wolfe effect, spherical collapse, and so on, which are left for future investigation.

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Appendix A. The basic equations in the Palatini formalism

The action in the Palatini formalism can be found in Eq. (1). Since the metric and the connection are two independent variables, varying the action with respect to the metric $g_{\mu\nu}$ and the connection $\Gamma^\nu_{\mu\nu}$ gives two equations:

$$F \hat{R}_{\mu\nu} = \frac{1}{2} f g_{\mu\nu} = \kappa^2 \left[ T^{(\varphi)}_{\mu\nu} + T^{(m)}_{\mu\nu} \right],$$

(Eq. A.1)

$$\hat{\nabla}_\lambda (\sqrt{-g} g^{\mu\nu}) = 0.$$  

(Eq. A.2)

Eq. (A.2) indicates that one can define a new metric $h_{\mu\nu}$ conformally connected to $g_{\mu\nu}$ by $h_{\mu\nu} = F g_{\mu\nu}$. Then Eq. (A.2) becomes

$$\hat{\nabla}_\lambda (\sqrt{-h} h^{\mu\nu}) = 0,$$

(Eq. A.3)

which means that $\Gamma^\lambda_{\mu\nu}$ can be expressed as the Levi-Civita connection with respect to the new metric $h_{\mu\nu}$

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} h^{\lambda\rho} \left( h_{\mu\rho,\nu} + h_{\nu\rho,\mu} - h_{\mu\nu,\rho} \right).$$

(Eq. A.4)

Thus, from the definitions of the Ricci scalar and Ricci tensor, one can obtain

$$\hat{R}_{\mu\nu}(\Gamma) = R_{\mu\nu} + \frac{3 \nabla_\mu F \nabla_\nu F - \nabla_\mu \nabla_\nu F}{F^2} - g_{\mu\nu} \frac{\nabla_\sigma \nabla^\sigma F}{2F},$$

(Eq. A.5)

$$\hat{R}(\Gamma) = R + \frac{3 \nabla_\sigma \nabla^\sigma F}{F^2} - \frac{3 \nabla_\sigma \nabla^\sigma F}{2F},$$

(Eq. A.6)

$\nabla_\mu$ is the usual covariant derivative related to the Levi-Civita connection, $R_{\mu\nu}$ and $R$ are the Ricci tensor and Ricci scalar in the metric formalism.

Inserting Eq. (A.5) into Eq. (A.1), we get

$$G^{\mu}_{\nu} = \kappa^2 \left[ T^{\mu}_{\nu}(\varphi) + T^{(m)}_{\nu} + T^{(eff)}_{\nu} \right],$$

(Eq. A.7)

with

$$T^{(eff)}_{\nu} = \frac{1}{k^2} \left[ (1 - F) F_{\nu} - \frac{3}{2} \nabla^\mu F \nabla_\mu F + \nabla^\nu F \nabla_\mu F \right],$$

(Eq. A.8)

The $(0,0)$ and $(i, j)$ components of Eq. (A.7) are:

$$H^2 = \frac{\kappa^2}{3} \rho + \frac{\kappa^2}{3} \left( \frac{1}{2} \dot{\varphi}^2 + \nu \right) - \tilde{H} - \omega \kappa^2 H^2 \varphi^2 - \frac{f^2}{4F},$$

(Eq. A.9)

$$-3H^2 - 2\dot{H} = \kappa^2 \left( \frac{1}{2} \dot{\varphi}^2 - \nu \right) + 2H \dot{F} + \ddot{F}$$

$$+ \omega \kappa^2 \varphi^2 (3H^2 + 2\dot{H}) = \frac{3f^2}{4F}.$$  

(Eq. A.10)

Perturbing Eq. (A.7), we obtain four perturbation equations. The $(0, 0)$ component is

$$\left( 2H + \frac{\ddot{F}}{F} \right) A + \frac{2k^2}{\alpha^2} \Phi + \left( 3H \frac{\ddot{F}}{F} - \kappa^2 \dot{\varphi}^2 + \frac{3f^2}{2F} \right) \Psi$$

$$= \frac{1}{F} \left[ -k^2 \delta \rho_m - \kappa^2 \delta \varphi \dot{\varphi} - \frac{1}{2} \left( 2k^2 V - f \right) \delta \varphi \right] \delta \varphi$$

$$+ \left( \frac{k^2}{\alpha^2} + 3H^2 - \frac{3\dot{F}^2}{4F^2} - \frac{1}{2} \dot{H} \right) \delta \Phi + \left( 3H + \frac{3\ddot{F}}{2F} \right) \delta \dot{F}. $$

(Eq. A.11)

The $(0, i)$ component has the form

$$A + \frac{3\ddot{F}}{2F} \Psi = \frac{3}{2F} \left[ \kappa^2 \rho_m \nu + \kappa^2 \dot{\varphi} \ddot{\varphi} - \left( H + \frac{3\ddot{F}}{2F} \right) \delta F + \delta \dot{F} \right].$$

(Eq. A.12)

The $(i, j)$ components have two equations. If $i \neq j$,

$$-\Psi - \Phi = \frac{1}{F} \delta F.$$  

(Eq. A.13)

When $i = j$, the result is

$$2\lambda + \left( 6H + \frac{\ddot{F}}{F} \right) A - \frac{2k^2}{\alpha^2} \Phi$$

$$+ \left( 6\dot{H} + 6\frac{\dot{F}}{F} + 6\frac{\ddot{F}}{F} + 3\kappa^2 \dot{\varphi}^2 - \frac{2k^2}{\alpha^2} - \frac{9f^2}{2F^2} \right) \Psi + \frac{3\ddot{F}}{F} \Psi.$$

Fig. 4. Evolutions of the matter growth rate $\delta$ in the metric (left panel) and Palatini (right panel) formalisms with respect to the redshift $z$. The red, blue, and purple lines correspond to $\omega = -0.5, -0.2$ and $0.2$, respectively. The dotted line shows the result of minimally coupled quintessence. In the inserted figure of the right panel, a comparison between the metric and Palatini formalisms is given. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)
\[
\begin{align*}
\frac{1}{\mathcal{F}} & \left\{ 3k^2\dot{\phi}^2 - \frac{3}{2}(2k^2V - f)\phi\dot{\phi} \\
+ & \left( \frac{k^2}{a^2} + 9H^2 + 6\dot{H} - \frac{9\dot{F}}{4F} - \frac{3}{2}\frac{\ddot{R}}{R} \right)\delta F \\
+ & \left( 6H - \frac{9\dot{F}}{2F} \right)\dot{\phi} + 3\dot{\phi} \right\}.
\end{align*}
\]

(A.14)

Here \( \ddot{R} = 12H^2 + 6\dot{H} - \frac{3k^2}{a^2} + 3\dot{\phi} + 9H\dot{\phi}. \) Perturbing \( \dot{R} = g^{\mu\nu}\dot{\Omega}_{\mu\nu} \) gives
\[
\dot{\delta R} = -2\dot{A} - 8HA + \left( \frac{k^2}{a^2} - 6H - \frac{6\dot{F}}{F} - 18H\dot{\phi} + \frac{3\dot{F}^2}{F^2} \right)\Phi

+ 4\frac{k^2}{a^2}\Phi + \frac{9\dot{F}}{F}\dot{\phi} - \frac{3\dot{F}}{F}\Psi

+ \frac{3}{F}\left( \frac{\dot{F}^2}{F^2} - 3H\dot{\phi}^2 - \frac{\dot{F}}{F} + \frac{k^2}{a^2} \right)\delta F

+ \left( 3H - \frac{\dot{F}}{F} \right)\dot{\phi} + \dot{\phi} \left( \frac{1}{\mathcal{F}} \right) \right).
\]

(A.15)

References

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