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Yukawa couplings in string theory: the case for F-theory GUT’s

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Abstract. We study the pattern of Yukawa couplings in local F-theory SU(5) GUT’s. Couplings for the third family of quarks and leptons appear at the perturbative level, but to reproduce the observed couplings for the lighter families requires non-perturbative dynamics. We show that corrections due to instanton effects do lead to a Yukawa matrix with a hierarchical structure. Our results apply to both down-like $10 \times 5 \times 5$ and up-like $10 \times 10 \times 5$ couplings. The models include magnetic fluxes needed for a chiral spectrum and for symmetry breaking down to the Standard Model. We compute the holomorphic couplings via residues and then obtain the physical couplings taking into account the normalization of wavefunction profiles. Combining non-perturbative corrections and magnetic fluxes allows to fit the measured masses and hierarchies of the third and second generations in the Standard Model.

1. Introduction
An outstanding open question in the Standard Model (SM) is how to explain the structure of fermion masses and mixings. If the SM arises as a low-energy limit of a fundamental string theory [1], it should be possible to understand the observed pattern in terms of basic parameters characterizing the string vacuum. In particular the values of Yukawa couplings in the string context are determined by the geometric properties of extra compactified dimensions. In principle an explicit computation of Yukawa coupling constants seems then rather complicated since we would need a complete knowledge of the underlying compact space. However, due to the localization properties of branes, certain quantities such as Yukawa couplings, do not depend on the full geometry of the compactification space but rather on local data around the region in which the SM fields are localized. This key feature allows to follow a bottom-up road to reproduce the SM in the framework of string theory [2].

The bottom-up idea can be implemented in type IIB compactifications with D3 and/or D7-branes. In these local constructions gauge coupling unification emerges naturally because all gauge interactions originate from the same region of the internal space and all SM gauge couplings typically depend on the same closed string modulus. Moreover, the chiral spectrum of the resulting local GUT models, such as SU(5) GUT’s, can be systematically realized with D-branes. However, in type IIB SU(5) GUT’s the up-like coupling $10 \times 10 \times 5$, denoted $Y_U$, is forbidden at perturbative level by charge conservation [1]. This particular Yukawa coupling could be induced by D-brane instantons [3], but the large experimental value for the top Yukawa rather suggests to consider a set-up, such as F-theory [4], in which both up-like $Y_U$, as well as down-like Yukawas $10 \times 5 \times 5$, denoted $Y_D$, can appear on equal footing.
F-theory provides a promising starting point to construct realistic models and a comprehensive formalism to build F-theory GUT’s has been developed in the last years [5–8] (see [9–13] for recent reviews). F-theory and type IIB models share properties such as localization of gauge and matter fields, gauge coupling unification, and moduli stabilization. Moreover, in F-theory, unlike in type IIB, exceptional gauge groups and matter representations such as spinorials of SO(10) are allowed, thereby enlarging the possibilities for spectra and interactions. In local F-theory SU(5) GUT’s the gauge degrees of freedom live in a 4-cycle $S_{\text{GUT}}$ of the internal space. The chiral matter fields, transforming as $10$, $\bar{5}$, or $5$, are supported at certain 2-cycles of $S_{\text{GUT}}$. At these curves the gauge symmetry is enhanced, for instance to SU(6) or SO(10), and at points where the curves intersect there is further enhancing, for instance to SO(12) or E$_6$. The triple intersection of quarks/leptons/Higgs curves, gives rise to Yukawa couplings that are given by the overlap integral of the internal wavefunctions for the chiral matter fields. Since this integral is dominated by the wavefunction profiles around the intersection point $p$, only data in a small region around $p$ is necessary to determine Yukawa couplings in local F-theory models. In fact, calculation of holomorphic couplings can be achieved via computation of residues at $p$ [14].

One general result established earlier is that fermion mass hierarchies appear automatically by restricting the number of intersection points. Indeed, as shown in [14] (see also [15–17]) the matrix of down-like Yukawa couplings $Y_D$ will have rank one if there is a single intersection point $p_{\text{down}}$, with a similar statement for up-like Yukawas. This ensures a flavor structure in which one family is much heavier, while the remaining two acquire smaller masses through non-perturbative effects. Indeed, such masses can be induced by D3-brane instantons or a gaugino condensate supported on a different 4-cycle of the compactification, as proposed in [18].

A systematic analysis of this scenario was initiated in [23] where a toy example with U(1) gauge symmetry was considered. It was continued in [24] where a more realistic local F-theory model with SU(5) symmetry enhanced to SO(12) at the intersection point was explored in depth. In this model D-quark, or lepton, Yukawa couplings $Y_D$ are allowed. It was found that non-perturbative effects distort the wavefunction profiles near the intersection point in such a way that a hierarchy of fermion mass eigenvalues of the form $(1, \epsilon, \epsilon^2)$ is generated. Here $\epsilon$ is a small parameter that measures the size of the non-perturbative effects. The hierarchy is already manifest at the level of holomorphic Yukawa couplings which are independent of the magnetic fluxes in local F-theory models. In contrast, the physical Yukawa couplings do depend on fluxes, and in particular on the hypercharge flux needed to trigger further symmetry breaking down to SU(3) $\times$ SU(2) $\times$ U(1)$_Y$. This hypercharge dependence then allows to explain the difference between D-quark and lepton Yukawas at the unification scale.

U-quark $Y_U$ couplings require an SU(5) model with enhancing to E$_6$ at the intersection point $p_{\text{up}}$. This case is more subtle because a non-trivial local geometry is necessary to reproduce the observed pattern. Indeed, if the fields in the coupling come from three distinct matter curves there can be no diagonal U-quark couplings. This problem could be avoided with a self-intersecting matter curve associated to the $10$, as suggested in [6, 7, 15]. However, as noticed in [26], if the two branches of the $10$ wavefunction are independent the Yukawa matrix would be rank two with no automatic hierarchy. In order to obtain the desired rank one structure one must either introduce 7-brane monodromy [25] or describe the matter curves via non-Abelian 7-brane profiles [26, 27] called T-branes. In [28] the T-brane approach was taken to compute up-like Yukawa couplings in presence of the kind of non-perturbative effects described in [18]. A hierarchical structure $(1, \epsilon, \epsilon^2)$ occurs again at the level of holomorphic Yukawa couplings and translates into the eigenvalues of the physical Yukawa matrix. The local model parameters can be chosen to attain a $O(1)$ Yukawa for the top quark, thus justifying the initial motivation that led to consider F-theory instead of type IIB SU(5) models.

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1 For different mechanisms to generate fermion mass hierarchies in F-theory see e.g. [19–22].
This paper is based on the work in [23,24,28]. We will report the main results, but will skip many details that can be found in the original articles. The organization is as follows. In section 2 we review the construction of local F-theory models. The SU(5) models are presented in section 3, emphasizing the geometrical and field theoretical data behind the Yukawa couplings. In section 4 we first describe the non-perturbative effects that can induce corrections to the Yukawa couplings. The holomorphic couplings are then computed by means of a residue formula. Section 5 is devoted to discussing the physical couplings. Some concluding remarks are collected in section 6.

2. Review of local F-theory models
In this section we briefly outline the standard approach to F-theory model building [5–8] (see [9–13] for recent reviews). We first describe the geometry behind the matter content and their interactions, and then discuss the corresponding effective theory.

In F-theory GUT’s the gauge theory arises from 7-branes wrapping a compact four-dimensional surface $S_{GUT}$ in the six-dimensional base $B$ of an elliptically-fibered eight-dimensional Calabi-Yau. The gauge group $G_{GUT}$ depends on the singularity type of the elliptic fiber. Together with the stack of 7-branes on $S_{GUT}$, a semi-realistic F-theory model contains extra 7-branes wrapping other surfaces $S_i$, which intersect $S_{GUT}$ on certain curves $\Sigma_i$. On these curves the singularity type of the elliptic fiber is enhanced to a higher rank group $G_i \supset G_{GUT}$. As expected, there will be additional degrees of freedom due to open strings stretching between the intersecting 7-branes. More precisely, there will be chiral matter multiplets charged under $G_{GUT}$, localized at the so-called matter curves $\Sigma_i$. The charged representations can be found decomposing the adjoint of the enhanced group $G_i$ under $G_{GUT}$. For example, in the case $G_{GUT} = SU(5)$, one can have curves $\Sigma_{10}$ with enhancement to SO(10) and matter in the $10$ or $\bar{10}$ representations, or curves $\Sigma_5$ with enhancement to SU(6) and matter in the $5$ or $\bar{5}$. Finally, when two or more matter curves meet at a point $p$ there will be further enhancement to a group $G_p$, which signals the appearance of interactions among the matter fields localized on the curves meeting at $p$. The couplings allowed at $p$ depend on the the enhanced group $G_p$ which contains $G_{GUT}$ and each of the $G_i$ involved. For $G_{GUT} = SU(5)$, down-like $10 \times 5 \times 5$ Yukawa couplings arise at points of SO(12) enhancement, whereas for up-like $10 \times 10 \times 5$ Yukawas enhancement to $E_6$ is expected.

The above geometric picture has a corresponding formulation in terms of the 8d supersymmetric action on the worldvolume of the 7-branes [5–8]. The observable 4d theory emerges upon compactification of the 8d theory on the surface $S$ wrapped by the 7-branes. The complex coordinates of $S$ are denoted $x$ and $y$. The 8d bosonic fields are the gauge field $A$ and the $(2,0)$ form $\Phi$, namely $\Phi = \Phi_{xy} dx \wedge dy$. $\Phi$ is associated to motion transverse to the 7-branes. Both $A$ and $\Phi$ transform in the adjoint of a non-Abelian gauge group $G$ that contains $G_{GUT}$ and $G_{\Sigma_i}$, which for the purposes of analyzing Yukawa couplings at $p$ can be taken to be $G_p$. The 4d bosonic fields are the zero modes of $A$ and $\Phi$ and the 4d action is obtained by dimensional reduction. In particular, the Yukawa couplings among 4d matter fields can be computed from the superpotential

$$ W = m^4 \int_S \text{tr} \left( F \wedge \Phi \right) $$

where $m_4$ is the F-theory characteristic scale, and $F = dA - iA \wedge A$. The dynamics also depends on the D-term

$$ D = \int_S \omega \wedge F + \frac{1}{2} [\Phi, \bar{\Phi}] $$

where $\omega$ is the fundamental form of $S$. The equations of motion that follow from
superpotential and the D-term turn out to be
\[ \bar{\partial}_{A} \Phi = \bar{\partial} \Phi - i[A_{0,1}, \Phi] = 0 \]
\[ F_{0,2} = 0 \]
(3a)

for the F-term equations and
\[ \omega \wedge F + \frac{1}{2}[\Phi, \bar{\Phi}] = 0 \]
(4)

for the D-term equation. The equations of motion do apply to the background and the fluctuations.

To derive the equations of motion for the bosonic fluctuations we define
\[ \Phi_{xy} = \langle \Phi_{xy} \rangle + \varphi_{xy} \]
\[ A_{\bar{m}} = \langle A_{\bar{m}} \rangle + a_{\bar{m}} \]
(5)

and then expand (3) and (4) to first order in the fluctuations \( (\varphi, a) \). In this way we find
\[ \bar{\partial}_{(A)} \varphi + i[\langle \Phi \rangle, a] = 0 \]
\[ \bar{\partial}_{(A)} a = 0 \]
\[ \omega \wedge \partial_{(A)} a - \frac{1}{2}[\langle \bar{\Phi} \rangle, \varphi] = 0 \]
(6a)
(6b)
(6c)

where \( a = a_{x}d\bar{x} + a_{y}d\bar{y} \), and \( \varphi = \varphi_{xy}dx \wedge dy \). In the local analysis we can take the Kähler form to be
\[ \omega = \frac{i}{2}(dx \wedge d\bar{x} + dy \wedge d\bar{y}) \]
(7)

A crucial result of the theory is that the equations (6) admit zero mode solutions that are localized on the matter curves.

The possible matter curves are in turn determined by the background of \( \Phi \) which in absence of fluxes depends holomorphically on the complex coordinates of \( S \), as implied by (3a). A non-trivial vev \( \langle \Phi \rangle \) such that its rank changes at curves \( \Sigma_{i} \) implies that instead of a single \( S \) there are intersecting surfaces \( S_{GUT} \) and \( S_{i} \). At a generic point on \( S \) \( \langle \Phi \rangle \) breaks \( G_{GUT} \) to the SM group.

In a local model we can also include a background \( \langle A \rangle \), or equivalently a worldvolume flux \( \langle F \rangle \). A flux can produce a chiral spectrum by choosing it so that at a given matter curve only zero modes of a definite 4d chirality are fully localized. Besides, turning extra flux along the hypercharge generator breaks \( G_{GUT} \) to the SM group.

It is instructive to look at a simple example to illustrate the formalism we have been describing. To this end we consider the toy model with \( G_{p} = U(3) \) discussed in detail in [23] in which
\[ \langle \Phi_{xy} \rangle = \frac{1}{3}m^{2}\text{diag}(-2x + y, x + y, x - 2y) \]
(8)

where \( m \) is a mass parameter. We see that this vev breaks \( U(3) \) to \( U(1)^{3} \) at generic points in \( S \). It is also clear that the group is enhanced to \( \text{SU}(2) \times U(1)^{2} \) at the curves \( \Sigma_{a} = \{ x = 0 \} \), \( \Sigma_{b} = \{ y = 0 \} \), and \( \Sigma_{c} = \{ x = y \} \). For instance, the generators \( E_{a^{\pm}} \) given by
\[ E_{a^{+}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ; \quad E_{a^{-}} = E_{a^{+}}^{\dagger} \]
(9)
modes are indeed localized at $x_f$ with $a_\pm$ commuting localized zero mode solutions [23]. For instance, the zero modes localized on $\Sigma_a$ and (4).

The solutions in real gauge, in which $\langle A \rangle$ is real, are easily found by means of a gauge transformation. Notice that the backgrounds (8) and (10) satisfy the equations of motion (3) and (4).

Given the backgrounds (8) and (10) it can be shown that the equations of motion (6) admit localized zero mode solutions [23]. For instance, the zero modes localized on $\Sigma_a = \{x = 0\}$ have $a_\pm = 0$, $a_\pm \propto \varphi_{xy}$, and $\varphi_{xy} = \chi_a \pm E_{a_\pm}$, where in real gauge the scalar wavefunction $\chi_a$ becomes

$$\chi_a = e^{-\sqrt{\frac{M}{4}}|x|^2} e^{\pm \frac{M}{4} |y|^2} f_{a_\pm}(y)$$

with $f_{a_\pm}$ an arbitrary holomorphic function of the intersecting coordinate $y$. Therefore, the zero modes are indeed localized at $x = 0$, whereas convergence in the $y$ direction depends on the flux. To select a definite chirality we can choose $M > 0$ so that only the zero modes in sector $a_\pm$ are fully normalizable.

Summarizing, in the language of the 8d field theory, to specify an F-theory local model we need to choose vevs $\langle \Phi \rangle$ and $\langle A \rangle$ such that there are localized zero modes on the matter curves $\Sigma_i$ determined by $\langle \Phi \rangle$. Concerning Yukawa couplings, the tree-level superpotential (1) includes the trilinear term

$$W_{Yuk} = -im^4 \int_S \text{tr} (A A \Phi)$$

that leads to 4d couplings among the zero modes of $A$ and $\Phi$. Notice that the couplings are given by an overlap integral of the zero mode wavefunctions. Since the zero modes are localized, the integrals can be restricted to a region of $S$ near the point $p$ where the matter curves intersect. In fact, as found in [14] and explained in section 4.2, the couplings can be computed by residues evaluated at $p$. We will verify that the resulting Yukawa matrix has rank one, thus implying that two families of quarks and leptons would be massless. To evade this rank one problem we will introduce non-perturbative corrections to the superpotential that in turn contribute to the couplings.

In order to obtain a hierarchical pattern of fermion masses we follow the proposal of [15] and consider a setup in which all up-like Yukawas are generated from a single Yukawa point $p_{up}$, and all down-like Yukawas from $p_{down}$. Thus, to compute down-like or up-like Yukawa couplings we only need to describe the F-theory GUT model in the vicinity of a single point. It would be interesting to engineer a model in which both up-like and down-like Yukawas are present but we have only studied each case separately. In section 3 we will describe two local F-theory models with $G_{GUT} = SU(5)$ but with different $G_p$. In one model $G_p = SO(12)$ and down-like couplings $10 \times \bar{5} \times \bar{5}$ are allowed [24]. In the other model $G_p = E_6$ and up-like couplings $10 \times 10 \times 5$ are generated [28].

3. Local F-theory SU(5) GUT's

In this section we present the F-theory local models with $G_p = SO(12)$ and $G_p = E_6$ which contain down-like and up-like Yukawa couplings [24, 28]. In each case we first define the
background $\langle \Phi \rangle$ that determines the matter curves and break $G_p$ to SU(5). We then specify the flux $\langle F \rangle$ responsible for 4d chirality and breaking to SU(3) $\times$ SU(2) $\times$ U(1)$_Y$.

Since $\langle \Phi \rangle$ and $\langle A \rangle$ transform in the adjoint of $G_p$ they can be written in terms of generators that can be decomposed as $\{ H_i, E_i \}$, where the $H_i$, $i = 1, \cdots, 6$, belong to the Cartan subalgebra and the $E_i$ are step generators. Recall that

$$[H_i, E_p] = \rho_i E_p$$

where $\rho_i$ is the $i$-th component of the root $\rho$. The 60 non-trivial roots of SO(12) read

$$\pm(1, \pm 1, 0, 0, 0, 0)$$

Here and below underlying means all possible permutations of the vector entries. On the other hand, the 72 non-zero roots of E$_6$ are given by

$$(0, \pm 1, \pm 1, 0, 0, 0)$$

$$\pm(\pm \sqrt{3}, \pm 1, \pm 1, \pm 1, \pm 1)$$

with even number of $+$'s.

### 3.1. The SO(12) model

At the intersection point of matter curves the group is $G_p = SO(12)$ but away from this point the symmetry is broken to SU(5) $\times$ U(1)$_2$ by the vev $\langle \Phi \rangle$. Moreover, this vev must be such that at some curves there is an enhancement to either SO(10) $\times$ U(1) or SU(6) $\times$ U(1). In this way, we can identify $G_S = SU(5)$ as the GUT gauge group and the enhancement curves as matter curves where chiral matter wavefunctions are localized.

To obtain the above symmetry breaking pattern the vev of the transverse position field $\Phi = \Phi_{xy} dx \wedge dy$ is chosen to be

$$\langle \Phi_{xy} \rangle = m^2 (xQ_x + yQ_y)$$

where $m$ is a mass parameter, and $(x, y)$ are the complex coordinates of the 4-cycle $S$. The charge operators $Q_x$ and $Q_y$ are the following combinations of generators of elements of the SO(12) Cartan subalgebra

$$Q_x = -H_1 \quad ; \quad Q_y = \frac{1}{2} (H_1 + H_2 + H_3 + H_4 + H_5 + H_6)$$

The implications of (18) can be understood from the commutator

$$[\langle \Phi_{xy} \rangle, E_p] = m^2 q_\Phi(\rho) E_p$$

with $q_\Phi$ a holomorphic function of $x$ and $y$. The subgroup of SO(12) not broken by the presence of this vev corresponds to those generators that commute with $\langle \Phi \rangle$ at any point in $S$. This set includes the Cartan subalgebra of SO(12) and those step generators $E_p$ such that $q_\Phi(\rho) = 0$ for all $(x, y)$. It is easy to see that such unbroken roots are given by

$$(0, 1, -1, 0, 0, 0)$$

together with the Cartan generators. Therefore, from the symmetry group SO(12) only the subgroup SU(5) $\times$ U(1)$_2$ remains as a gauge symmetry, and we can identify $G_S = SU(5)$ as our GUT gauge group.
The remaining generators of SO(12), that have \( q_\Phi \neq 0 \) for generic \( (x, y) \), precisely describe the pattern of matter curves and the charged matter localized on them. The broken roots and their charges \( q_\Phi \) are displayed in table 1. From this table we see that there are three complex curves within \( S \) where the bulk symmetry \( SU(5) \times U(1)^2 \) is enhanced, in the sense that there \( q_\Phi = 0 \) for an additional set of roots. Concretely, for \( x = 0 \) there are 10 additional roots that together with those in (21) complete the \( SU(6) \) root system. We have labeled such matter curve as \( \Sigma_a \), so that in the language of the previous section we would have that \( G_{\Sigma_a} = SU(6) \times U(1) \). These extra set of roots whose \( q_\Phi \) vanishes at \( \Sigma_a \) can be split into subsets that have different \( q_\Phi \) away from \( \Sigma_a \). It is easy to convince oneself that each of these subsectors must fall into complete weight representations of \( SU(5) \), which in turn correspond to the matter localized at the curve.

In the case of \( \Sigma_a \), there are two sectors \( a^+ \) and \( a^- \) that correspond to the representations \( \bar{5} \) and \( 5 \) of \( SU(5) \), respectively, as shown in table 1. Similarly to \( \Sigma_a \), at the curve \( \Sigma_b = \{ y = 0 \} \) there are 20 extra unbroken roots and \( G_{\Sigma_b} = SO(10) \times U(1) \), giving rise to the representations \( 10 \) and \( \bar{10} \). The third matter curve is given by \( \Sigma_c = \{ x = y \} \), where there is also an enhancement to \( SU(6) \times U(1) \). Finally, notice that the three curves meet at \( p_{down} = \{ x = y = 0 \} \).

### Table 1. SO(12) broken generators

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>root</th>
<th>( q_\Phi )</th>
<th>( SU(5) ) rep.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a^+ )</td>
<td>( (1, -1, 0, 0, 0, 0) )</td>
<td>( -x )</td>
<td>( \bar{5} )</td>
</tr>
<tr>
<td>( a^- )</td>
<td>( (-1, 1, 0, 0, 0, 0) )</td>
<td>( x )</td>
<td>( 5 )</td>
</tr>
<tr>
<td>( b^+ )</td>
<td>( (0, 1, 1, 0, 0, 0) )</td>
<td>( y )</td>
<td>( 10 )</td>
</tr>
<tr>
<td>( b^- )</td>
<td>( (0, -1, -1, 0, 0, 0) )</td>
<td>( -y )</td>
<td>( \bar{10} )</td>
</tr>
<tr>
<td>( c^+ )</td>
<td>( (-1, -1, 0, 0, 0, 0) )</td>
<td>( x - y )</td>
<td>( \bar{5} )</td>
</tr>
<tr>
<td>( c^- )</td>
<td>( (1, 1, 0, 0, 0, 0) )</td>
<td>( -(x - y) )</td>
<td>( 5 )</td>
</tr>
</tbody>
</table>

Let us now discuss the fluxes needed to achieve 4d chirality and trigger symmetry breaking to the Standard Model group. It is convenient to assemble the flux in various steps. We first add a flux \( \langle F_1 \rangle \) to obtain a chiral spectrum on the curves \( \Sigma_a \) and \( \Sigma_b \), selecting the sectors \( a^+ \) and \( b^+ \), as opposed to \( a^- \) and \( b^- \). Then we add a piece \( \langle F_2 \rangle \) such that the matter curve \( \Sigma_c \) also has a chiral spectrum including a Higgs. Neither of these two fluxes breaks \( SU(5) \) so that we must also include a flux \( \langle F_Y \rangle \) along the hypercharge generator. This last flux further enforces doublet-triplet splitting in the Higgs sector.

To proceed we then consider the flux

\[
\langle F_1 \rangle = -iM (dx \wedge d\bar{x} - dy \wedge d\bar{y}) Q_F
\]

where

\[
Q_F = \frac{1}{2}(H_1 - H_2 - H_3 - H_4 - H_5 - H_6)
\]

To simplify we take the parameter \( M \) to be constant. Since the unbroken roots in (21) are neutral under \( \langle F_1 \rangle \) this flux does not break \( SU(5) \). On the contrary, the roots in sectors \( a \) and \( b \) are charged. Thus, if the flux permeates the curves \( \Sigma_a \) and \( \Sigma_b \), namely if the integral of (22) over each of these curves does not vanish, they will each support zero modes of definite chirality.
We will assume that this occurs so that \( \langle F_1 \rangle \) leads to a net chiral spectrum of three \( 5 \)'s in the curve \( \Sigma_a \) and three \( 10 \)'s in the curve \( \Sigma_b \). In fact, in [24] it is shown that only zero modes in the sectors \( a^+ \) and \( b^+ \) are fully localized as long as \( M > 0 \), in agreement with the general arguments in [29].

The spectrum in the \( c \) sector is not affected by \( \langle F_1 \rangle \) because the associated roots are neutral under (23). Now, to generate a down-like Yukawa coupling at the intersection of three matter curves we do need one Higgs \( 5 \) in \( \Sigma_c \), but no \( 5 \) to avoid unwanted \( 5 5 \) mass terms. These conditions can be met by adding the extra flux

\[
\langle F_2 \rangle = i (dx \wedge d\bar{y} + dy \wedge d\bar{x}) (N_a Q_x + N_b Q_y)
\]

where \( Q_x \) and \( Q_y \) are given in (19). The matter on \( \Sigma_c \) is charged under this flux and the required Higgs \( 5 \) appears in the \( c^+ \) sector provided that \( N_a > N_b \), as shown in appendix B of [24]. Although the matter on the curves \( \Sigma_a \) and \( \Sigma_b \) is also charged under (24) it can be proved that the number of (local) families in these curves is independent of the flux \( \langle F_2 \rangle \) [24].

Unlike (22) and (24) the last piece of worldvolume flux to be added does break SU(5) down to the MSSM gauge group. As usual, such flux should be turned along the hypercharge generator \( Q_Y \), and it can generically be taken to be

\[
\langle F_Y \rangle = i \left[ (dx \wedge d\bar{y} + dy \wedge d\bar{x}) N_Y + (dy \wedge d\bar{y} - dx \wedge d\bar{x}) \tilde{N}_Y \right] Q_Y
\]

where

\[
Q_Y = \frac{1}{3} (H_2 + H_3 + H_4) - \frac{1}{2} (H_5 + H_6)
\]

The component proportional to \( \tilde{N}_Y \) could in principle upset the net chirality in sectors \( a^+ \) and \( b^+ \) but this is prevented choosing \(-1 < \tilde{N}_Y / M < 3 \) [24]. A final constraint comes from the condition of doublet-triplet splitting in the \( c^+ \) sector. It can be verified that the triplets are vector-like, thus generically massive, if \( N_Y = 3(N_a - N_b) \) [24].

In summary, the total worldvolume flux on the local SO(12) model is given by

\[
\langle F \rangle = i(dy \wedge d\bar{y} - dx \wedge d\bar{x})Q_P + i(dx \wedge d\bar{y} + dy \wedge d\bar{x})Q_S
\]

where we have defined

\[
Q_P = MQ_F + \tilde{N}_Y Q_Y
\]

\[
Q_S = N_a Q_x + N_b Q_y + N_Y Q_Y
\]

The vev of the corresponding vector potential \( A \) can be written in the holomorphic gauge, defined in [16], as

\[
\langle A_{1,0} \rangle^{\text{hol}} = i (\bar{x} Q_P - \bar{y} Q_S) dx - i (\bar{y} Q_P + \bar{x} Q_S) dy
\]

This is the quantity that enters into the D-term equation of motion for the zero mode wavefunctions at the curves \( \Sigma_a, \Sigma_b \) and \( \Sigma_c \). Finally, notice that compared to table 1 the matter in the sectors \( a, b \) and \( c \) splits into SU(3) \( \times \) SU(2) \( \times \) U(1) \( \times \) representations once the total flux is taken into account. The resulting content of charged particles is displayed in table 2 of [24].

Notice that the backgrounds for \( \langle \Phi \rangle \) and \( \langle F \rangle \) fulfill the equations of motion. Since both vevs are Abelian the F-terms (3) just require \( \partial \langle \Phi \rangle = 0 \) and indeed (18) depends holomorphically on the coordinates. Concerning the flux, observe that (27) has \( \langle F_{0,2} \rangle = 0 \). Moreover, the total flux is primitive, i.e. \( \omega \wedge \langle F \rangle = 0 \), so that the D-term (4) is satisfied because \( \langle [\langle \Phi \rangle, \langle \Phi \rangle] \rangle = 0 \).

To end this section we wish to stress that provided the backgrounds (18) and (27), and the constraints on the flux parameters discussed above, it can be shown that the equations of motion (6) admit zero mode solutions of definite chirality localized on the matter curves. The fluctuations are of the form \( \varphi = \varphi_\rho E_\rho, a = a_\rho E_\rho \), where \( E_\rho \) is the generator corresponding to a root \( \rho \) in a given sector. We refer the reader to appendix D of [24] for the explicit computation of the \( \langle \varphi_\rho, a_\rho \rangle \) and their normalizations.
3.2. The $E_6$ model

We will follow the approach of [27] and realize the breaking pattern by means of a non-Abelian background for $\langle \Phi \rangle$. In this way the correct hierarchy of up-quark masses can be obtained as argued in [27], and verified in the next sections.

Let us consider the T-brane background

$$\langle \Phi_{xy} \rangle = m(E^+ + m x E^-) + \mu^2(x - y)Q$$

(31)

where $m$ and $\mu$ are mass parameters. The charge $Q$ is given by the combination of Cartan generators

$$Q = \frac{1}{2} \left( \frac{5}{\sqrt{3}} H_1 - H_2 - H_3 - H_4 - H_5 - H_6 \right)$$

(32)

The main new ingredients are the non-commuting generators $E^{\pm}$ whose corresponding roots, also denoted $E^{\pm}$, are defined as

$$E^{\pm} = \pm \frac{1}{2} (\sqrt{3}, 1, 1, 1, 1)$$

(33)

These generators satisfy the relation $[E^{+}, E^{-}] = P$, where

$$P = \frac{1}{2} (\sqrt{3} H_1 + H_2 + H_3 + H_4 + H_5 + H_6)$$

(34)

More precisely, the triplet $\{E^{+}, E^{-}, P\}$ spans the $\text{su}(2)$ factor of a $\text{su}(5) \oplus \text{su}(2) \oplus \text{u}(1)$ maximal Lie subalgebra of $\mathfrak{e}_6$, under which the $E_6$ adjoint decomposes as

$$78 \rightarrow (24, 1)_0 \oplus (1, 3)_0 \oplus (1, 1)_0 \oplus (10, 2)_{-1} \oplus (\overline{10}, 2)_1 \oplus (5, 1)_2 \oplus (\overline{5}, 1)_{-2}$$

(35)

where the subindex is the $Q$ charge. This decomposition shows that there is a pair of $10$'s transforming as a doublet of the $\text{SU}(2)$ generated by $\{E^{+}, E^{-}, P\}$. In particular, if we define

$$E^{10+}_{10+} = (0, 1, 1, 0, 0, 0) \quad E^{10-}_{10-} = \frac{1}{2} (-\sqrt{3}, 1, 1, -1, -1, -1)$$

(36)

we have the relations

$$[E^{\pm}, E^{10\pm}] = E^{10\pm}, \quad [E^{\pm}, E^{10\pm}] = 0, \quad [P, E^{10\pm}] = \pm E^{10\pm}$$

(37)

We now analyze the resulting gauge symmetry group and the structure of matter curves due to (31) by looking at the commutant of $\langle \Phi \rangle$ as a function of the coordinates $(x, y)$. The gauge group is the commutant at generic points while changes in its rank signal the matter curves [27]. In this case one can easily check that the set of roots of the subalgebra $\text{su}(5) \oplus \text{u}(1) \subset \mathfrak{e}_6$ do commute at generic points in $S$ and so we can identify the GUT gauge group with $\text{SU}(5)$.

Regarding the matter curves, we find that at $\Sigma_5 = \{x - y = 0\}$ there are additional roots commuting with $\langle \Phi \rangle$ given by

$$E_5 = \frac{1}{2} (\sqrt{3}, 1, -1, -1, -1, -1) \quad E_6 = \frac{1}{2} (-\sqrt{3}, -1, 1, 1, 1, 1)$$

(38)

which correspond respectively to $(\overline{5}, 1)_2$ and $(\overline{5}, 1)_{-2}$ in the decomposition (35). Hence, at $\Sigma_5$ the symmetry group enhances to $\text{SU}(6) \times \text{U}(1)$. On the other hand, the action of $\langle \Phi \rangle$ on the sector $(10, 2)_{-1}$ is given by

$$[\langle \Phi \rangle, R_+ E^{10+} + R_- E^{10-}] = \left( \begin{array}{cc} -\mu^2(x - y) & m \\ \frac{m}{m^2 x} & -\mu^2(x - y) \end{array} \right) \left( \begin{array}{c} R_+ E^{10+} \\ R_- E^{10-} \end{array} \right)$$

(39)
where $R_\pm$ are functions on $S$. For the conjugate sector $(\overline{10}, 2)_1$ there is an analogous result. At $\Sigma_{10} = \{ \mu^4 (x - y)^2 = m^2 \}$ the matrix (39) has vanishing determinant. Hence, there are additional roots commuting with $(\Phi)$, and therefore a jump in the rank of the symmetry group. Therefore $\Sigma_{10}$ is identified with the $10$ curve of this T-brane background. Observe that the curves $\Sigma_5$ and $\Sigma_{10}$ meet at the Yukawa point $p_{up} = \{ x = y = 0 \}$.

The non-Abelian T-brane background is such that $[\langle \Phi \rangle, \langle \Phi \rangle] \neq 0$. In consequence, a non-primitive flux $\langle F_{np} \rangle$, i.e. $\omega \wedge \langle F_{np} \rangle \neq 0$, is required to fulfill the D-term equation (4). Proceeding as in [27] we deduce that the F-terms (3) are still verified by

$$\langle \Phi_{xy} \rangle = m(e^i E^+ + m x e^{-i E^-}) + \mu^2 (b x - y) Q, \quad \langle A_{0,1} \rangle = -i \partial f P, \quad \langle F_{1,1} \rangle = -i \partial \bar{\partial} f P \ (40)$$

It then transpires that to satisfy the D-term the function $f$ must satisfy a non-linear differential equation of the Painlevé III type. This complicates the task of solving for the zero mode wavefunctions, but as shown in [28] approximate solutions can still be found.

Besides the non-primitive flux in (40), we can still add contributions to the background worldvolume flux $\langle F \rangle$ provided they do not spoil the F-term and D-term conditions. The simplest possibility is to introduce primitive $(1, 1)$ fluxes in the Cartan of $E_6$. Such fluxes are generically present and are important for the phenomenology of the model. Indeed they can be chosen to generate 4d chirality for the SU(5) spectrum, and to further break SU(5) to SU(3) $\times$ SU(2) $\times$ U(1)$_Y$. Let us then consider consider the worldvolume flux

$$\langle F_Q \rangle = i \left[ -M (dy \wedge d\bar{y} - dx \wedge d\bar{x}) + N (dx \wedge d\bar{y} + dy \wedge d\bar{x}) \right] Q \ (41)$$

where $Q$ is given in (32), and $M$ and $N$ are flux densities near the Yukawa point that we approximate by constants. It is easy to check that the equations of motion are still satisfied for any value of $M$, $N$, which can be considered as real parameters of the model. Moreover, $\langle F_Q \rangle$ induces 4d chirality in the matter curves because, unlike the background in (40), it does distinguish the modes of opposite chirality $5, \overline{5}$ and $10, \overline{10}$. Thus, it can select normalizable modes of one chirality or the other according to the sign of $M$ and $N$. A more detailed discussion of the local chirality index can be found in appendix B of [24].

We also switch on a flux along the hypercharge generator. Concretely

$$\langle F_Y \rangle = i \left[ \tilde{N}_Y (dy \wedge d\bar{y} - dx \wedge d\bar{x}) + N_Y (dx \wedge d\bar{y} + dy \wedge d\bar{x}) \right] Q_Y \ (42)$$

where $N_Y, \tilde{N}_Y$ are local flux densities and

$$Q_Y = \frac{1}{3} (H_2 + H_3 + H_4) - \frac{1}{2} (H_5 + H_6) \ (43)$$

is the hypercharge generator. As in the SO(12) model the hypercharge flux splits the particles within the same SU(5) multiplet according to their hypercharge.

Summarizing, the total flux of this model is given by

$$\langle F \rangle = \langle F_p \rangle + \langle F_{np} \rangle \ (44)$$

Here $\langle F_{np} \rangle$ is the non-primitive flux provided in (40). The primitive piece $\langle F_p \rangle$ is the sum of (41) and (42) that can be conveniently cast as

$$\langle F_p \rangle = i Q_R (dy \wedge d\bar{y} - dx \wedge d\bar{x}) + i Q_S (dx \wedge d\bar{y} + dy \wedge d\bar{x}) \ (45)$$

$^2$ Note that for this local model $\langle \Phi_{xy} \rangle \neq 0$ at $p_{up}$, and so the symmetry group is no longer $E_6$ at the Yukawa point. As discussed in [27] this is a general feature of T-brane configurations. By abuse of terminology, we will still refer to this point as the $E_6$ point of the local model.
where we have defined

\[ Q_R = -MQ + \tilde{N}Y, \quad Q_S = NQ + N_Y Q_Y \]

(46)

Given the corresponding roots it is straightforward to compute the charges of the fields localized
on the matter curves (c.f. table 1 in [28]). The flux parameters are constrained by the conditions
of chirality and doublet-triplet splitting. In [28] it was found that chirality in the 10 sector
requires \(-3/2 < \tilde{N}_Y/M < 6\). In the 5 sector doublet-triplet splitting imposes \(N_Y = -6N\),
whereas chirality demands \(N > 0\).

The equations of motion with given backgrounds (40) and (45) have been studied in [28] and
shown to have localized solutions on the matter curves \(\Sigma_5\) and \(\Sigma_{10}\). In the 5 sector, which is
neutral under SU(2), \(\langle \Phi \rangle\) is effectively Abelian and the zero modes that are found are analogous
to those in the \(c\) sector in the SO(12) model. The 10 sector is more complicated because the
zero modes belong in a SU(2) doublet. Hence, these modes feel the non-Abelian piece of \(\langle \Phi \rangle\)
and are charged under the non-primitive flux that depends on the Painlevé transcendent \(f\).
Nevertheless, in [28] we were able to find approximate solutions in the limit \(m \gg \mu\). With
the zero modes at hand one can then compute the normalizations that appear in the physical
Yukawa couplings.

4. Non-perturbative effects and Yukawa couplings

In this section we will first review how the non-perturbative effects proposed in [18] can be
incorporated in local F-theory models. We will then study corrections to the Yukawa couplings
due to the non-perturbative effects. Following [14,27] we will obtain the holomorphic couplings
by means of a residue formula.

4.1. Non-perturbative effects

Besides the surfaces \(S_{GUT}\) and \(S_i\) that enter in the local GUT model, a global F-theory
compactification will contain other divisors of the six-dimensional base \(B\) that are also wrapped
by branes. Typical examples are hidden sector 7-branes that develop a gaugino condensate, or
Euclidean 3-branes. In both cases the additional branes generically wrap a surface, denoted
\(S_{np}\), defined by a function \(h\), namely

\[ S_{np} = \{h = 0\} \]

(47)

where \(h, A\) are proportional to the \(n^{th}\) derivative of \(h\) normal to \(S = \{z = 0\}\) and \(\epsilon\) is a small
parameter that measures the strength of the non-perturbative effect. More precisely,

\[ \epsilon = A e^{-T_{np} h_0 N_{D3}} \]

(48)

where \(T_{np}\) is the complexified Kähler modulus of \(S_{np}\), \(h_0 = \int_S h\), \(A\) depends on the bulk moduli
of the base \(B\), and \(N_{D3} = (8\pi^2)^{-1} \int_S \text{tr}(F \wedge F) \in \mathbb{N}\). We refer to Appendix C of [24] for more
details about these quantities, the derivation of (47), and the proof that these non-perturbative
effects do not modify the 7-brane D-term.

Observe that (47) is written in terms of the GUT 7-brane fields \(\Phi\) and \(A\), just like the
tree-level superpotential (1). Hence, we can add up both expressions and apply the ultra-local
approach to compute 7-brane zero mode wavefunctions near a Yukawa point. This program was
first carried out in a toy model with \(G_p = U(3)\) but considering only \(\theta_1 \neq 0\). The reason for such
simplification was the assumption made in [18] that the two 4-cycles $S$ and $S_{np}$ do not intersect, so that $h|s \equiv h_0$ and $\theta_0 = 0$, while other $\theta_i$ are more suppressed. Now, for $G_p = \text{SO}(12)$ or $G_p = \text{E}_6$ these assumptions have to be revisited because the corrections to Yukawa couplings due to the term proportional to $\theta$ in $W_{np}$ are proportional to the symmetric trace of three group generators which vanishes identically for SO(12) and E$_6$.

In [24] it was argued that $\theta_0 = (4\pi^2m^*_s)^{-1}\log h/h_0|_{x=0}$ can be non-trivial and it was included together with $\theta_2$ in the computation of down-type Yukawa point blown. In fact, it was found that just $\theta_0$ was enough to generate a hierarchical rank 3 matrix of Yukawas. In [28] it was also shown that the least suppressed parameter $\theta_0$ solves the rank 1 problem for up-type Yukawas. Hence, in order to simplify the discussion we only consider the first term of $W_{np}$ and work with the total superpotential

$$W_{\text{total}} = m^4_s \int_S \text{tr} (F \wedge \Phi) + \epsilon \frac{\theta_0}{2} \text{Tr} (F \wedge F)$$

In the following we will base our local analysis on the superpotential (49) and the D-term (2) which does not receive non-perturbative corrections.

The specific form of $\theta_0$ depends on $S_{np}$ and thus on the global completion of the local model but we can still assume it to be a general holomorphic function of $(x, y)$. Near the intersection point, that can be chosen to be $x = y = 0$, we can make the linear approximation

$$\theta_0 = i(\theta_{00} + \theta_x x + \theta_y y)$$

where $\theta_{00}$, $\theta_x$ and $\theta_y$ are constant parameters. Notice that for $\theta_0$ non-constant the second term in (49) is not a total derivative, and we can expect a non-trivial effect on the 7-brane fields. In fact, the non-perturbative correction changes the F-term equations (3) to

$$\bar{\partial}_A \Phi + \epsilon \partial \theta_0 \wedge F = 0$$
$$F_{0,2} = 0$$

(51a)
(51b)

The $\theta_0$ dependent contribution already indicates that the backgrounds $\langle \Phi \rangle$ and $\langle F \rangle$ must be shifted from their original values in order to satisfy the new F-terms. Clearly, the equations of motion for the fluctuations will also be modified as discussed below.

To solve the equations we perform a perturbative expansion in the small parameter $\epsilon$ of the form

$$\Phi = \Phi^{(0)} + \epsilon \Phi^{(1)} + \cdots$$

(52)

and similarly for the gauge field. To begin let us solve the F-terms at the level of the background. To zeroth order in $\epsilon$ we use a holomorphic gauge $\langle A_{0,1} \rangle^{(0)} = 0$. The F-terms are then satisfied as long as $\bar{\partial} \langle \Phi \rangle^{(0)} = 0$, so that $\langle \Phi \rangle^{(0)} = 0$ is a holomorphic function of the coordinates. To first order in $\epsilon$ the F-terms still allow to maintain a holomorphic gauge, namely $\langle A_{0,1} \rangle^{(1)} = 0$. However, $\langle \Phi \rangle$ must be shifted according to

$$\langle \Phi \rangle = \langle \Phi \rangle^{(0)} + \epsilon \partial \theta_0 \wedge \langle A_{1,0} \rangle^{(0)} + \mathcal{O}(\epsilon^2)$$

(53)

Recall that in general $\langle A_{1,0} \rangle^{(0)} \neq 0$, for instance in the SO(12) model it is given in (30).

Let us now derive the equations of motion for the bosonic fluctuations $(\varphi, a)$ defined by (5). In turn $(\varphi, a)$ have a perturbative expansion in $\epsilon$ of the form (52). From the F-terms, and taking into account the shift in the background (53), we find the zero mode equations

$$\bar{\partial} a = 0 + \mathcal{O}(\epsilon^2)$$
$$\bar{\partial} \varphi - i[a, \langle \Phi \rangle^{(0)}] = -\epsilon \partial \theta_0 \wedge \partial a + \mathcal{O}(\epsilon^2)$$

(54a)
(54b)
Here we have disregarded the fluctuations $a^\dagger$ since they do not contribute to the Yukawa couplings as shown in appendix D of [24]. The D-term does not receive non-perturbative corrections, meaning that the fluctuations still satisfy the equation (6c). However, this D-term equation will acquire a $\theta_0$ dependence once the shift in the background (53) is taken into account.

To order $\epsilon$ the zero mode equations (54) have the general solution

$$a = \bar{\partial}\xi$$

$$\varphi = h - i[\langle \Phi \rangle^{(0)},\xi] - \epsilon \bar{\partial}\theta_0 \wedge \partial\xi$$

where $\xi$ is a 0-form and $h$ is a $(2,0)$-form such that $\bar{\partial}h = 0$. Both $\xi$ and $h$ transform in the adjoint representation of $G_p$. As in [14] $h$ can be fixed up to normalization by exploiting the invariance of the tree-level superpotential (1) under the gauge transformations

$$a \rightarrow a + \bar{\partial}_A \chi$$

$$\varphi \rightarrow \varphi - i[\langle \Phi \rangle,\chi]$$

Concretely, to fix $h$ we can perform transformations with $\chi$ holomorphic.

The function $\xi$ will enter prominently in the computation of the Yukawa couplings. It can be found in terms of $\varphi$ as follows. First we introduce the holomorphic matrix $\Psi$ defined by

$$[\langle \Phi \rangle^{(0)},\xi] = \Psi\xi \, dx \wedge dy$$

Then we rewrite (55b) as

$$\Psi\xi \, dx \wedge dy = i(\varphi - h + \epsilon \bar{\partial}\theta_0 \wedge \partial\xi)$$

This last equation can be solved perturbatively to obtain

$$\xi = \xi^{(0)} + i\epsilon \Psi^{-1} \left( \partial_x\theta_0\partial_y\xi^{(0)} - \partial_y\theta_0\partial_x\xi^{(0)} \right) + O(\epsilon^2)$$

$$\xi^{(0)} = i\Psi^{-1} \left( \varphi^{(0)}_{xy} - h_{xy} \right)$$

The function $\xi$ is required to be regular at the matter curves.

Substituting $\xi$ in (59) in $a = \bar{\partial}\xi$ we can find $a$ in terms of $\varphi$ and then plug in the D-term equation (6c) to obtain the solution for $\varphi$. In this way the non-perturbative zero modes are derived. We could then insert these zero modes in the superpotential to deduce the Yukawa couplings. However, we will see that the couplings can be more efficiently determined by a residue formula which is independent of the exact form of the corrected zero modes. We only need to know that the corrected zero modes are localized and this is indeed the case as shown in [24] for the SO(12) model and in [28] for the $E_6$ model. Although the zero modes receive corrections at order $\epsilon$ it can be proved that the contributions to the norms appear only at $O(\epsilon^2)$ and would only affect the physical Yukawa couplings at this order. This means that to compute the couplings at $O(\epsilon)$ it suffices to use the normalization of the tree-level zero modes.

### 4.2. Holomorphic Yukawa couplings

The Yukawa couplings are found by expanding the superpotential to cubic order in fluctuations, substituting the zero mode solutions, and finally performing the integration. In principle there are contributions due to the tree-level superpotential plus the non-perturbative term. In the case of the total superpotential (49), with a non-perturbative correction depending only on $\theta_0$, it can be shown that expanding to cubic order gives

$$Y = -im^4 \int_S \text{Tr}(\varphi \wedge a \wedge a)$$

(60)
Although there is no additional piece proportional to $\epsilon$, the couplings will depend on $\theta_0$ through the corrected zero modes.

To perform the integration we first use the solution to the F-terms (55) to arrive at

$$ Y = -\frac{im^4}{3} \int S \{ h \wedge a \wedge a - \partial(\varphi \wedge [a, \xi]) - \epsilon \partial(\theta_0 \partial(a \wedge a \xi)) \} $$

(61)

The last two terms are total derivatives that vanish upon integration because they involve localized fields $(\varphi, a)$. Using (54a) once more then yields

$$ Y = -\frac{im^4}{3} \int S \{ h \wedge \bar{\partial} \xi \wedge \bar{\partial} \xi \} $$

(62)

Since $h$ is holomorphic the integrand is still a total derivative. Thus, upon integration, $Y$ can be converted into a surface integral around the intersection point. As in [14, 27] the localized modes $\varphi_{xy}$ that appear in (59) do not contribute, and we deduce that

$$ Y = m^4 f_{abc} \int_{\mathcal{R}} (\eta^a \eta^b h_{xy}^c) \, dx \wedge dy $$

(63)

where $f_{abc}$ are structure constants of the symmetry group $G_p$ at the Yukawa point $p$, $\mathcal{R}$ is diffeomorphic to the product of two circles surrounding $p$, and $\eta$ are the auxiliary holomorphic functions

$$ \eta = -i \Psi^{-1} h_{xy} + \epsilon \Psi^{-1} (\partial_x \theta_0 \partial_y (\Psi^{-1} h_{xy}) - \partial_y \theta_0 \partial_x (\Psi^{-1} h_{xy})) + \mathcal{O}(\epsilon^2) $$

(64)

related to $\xi$ by removing the dependence on $\varphi_{xy}$. Finally, we can express (63) as a residue formula evaluated at the Yukawa point $p$

$$ Y = m^4 \pi^2 f_{abc} \text{Res}_p (\eta^a \eta^b h^c) $$

(65)

where for simplicity we have dropped the subindices to $h_{xy}$. In the following we will apply this residue formula to the SO(12) and $E_6$ local models constructed in section 3.

It is important to realize that the residue formula was obtained inserting only the solution of the F-terms in the superpotential as advocated in [14]. In this way only the holomorphic piece of the Yukawa couplings is computed. To evaluate the physical Yukawa couplings we also need to solve the D-term equations to find the zero mode wavefunctions and impose that they give canonically normalized kinetic terms. This last condition will fix the arbitrary constants that appear in the holomorphic couplings.

4.2.1. Holomorphic couplings in the SO(12) model

To determine the couplings we first have to find the auxiliary functions $\eta$ defined in (64). In the SO(12) model the matrix $\Psi$ is proportional to the identity. To see this we write $\xi = \xi_\rho E_\rho$ where $\rho$ is a root in one of the $a^+, b^+$, or $c^+$ sector. From (20) and (57) it follows that we can replace $\Psi$ by $m^2 q_\Phi(\rho)$, where the $q_\Phi(\rho)$ are given in table 1. Hence, in the SO(12) case (64) reduces to

$$ \eta_\rho = -\frac{i}{m^2 q_\Phi(\rho)} h_\rho + \frac{i \epsilon}{m^2 q_\Phi(\rho)} \left[ \theta_x \partial_y \left( \frac{h_\rho}{q_\Phi(\rho)} \right) - \theta_y \partial_x \left( \frac{h_\rho}{q_\Phi(\rho)} \right) \right] + \mathcal{O}(\epsilon^2) $$

(66)

where we have already used (50) for $\theta_0$.

We also need to know the so called holomorphic representatives $h_\rho$. Consider for example the $a^+$ sector in which $q_\Phi(\rho) = -x$. In principle $h_{a^+}$ is an arbitrary function of $(x, y)$ but
by selecting \( \chi \) appropriately in (56) we can gauge away all \( x \) dependence. Hence, \( h_{a+} \) is a holomorphic function of \( y \) and following standard practice [15] we can choose a basis \( \{1, y, y^2\} \) to represent 3 families. The \( b^+ \) and \( c^+ \) sectors are analogous. Concretely, \( h_{b+} \) is a holomorphic function of \( x \) and we pick a basis \( \{1, x, x^2\} \), whereas \( h_{c+} \) depends on \( (x + y) \) and we take it to be a constant since in this sector there is only one zero mode corresponding to the Higgs. Altogether we have

\[
\begin{align*}
    h_{a+} &= \gamma_a y^{3-i} \quad ; \quad h_{b+} = \gamma_b x^{3-j} \quad ; \quad h_{c+} = \gamma_c
\end{align*}
\]

(67)

where \( i, j = 1, 2, 3 \).

At this stage the normalization constants in (67) are arbitrary, but they are actually related to the normalization of the zero mode wavefunctions through (59b). For example,

\[
\xi^{(0)}_a = -\frac{i}{m^2 x} \left( \varphi^{(0)}_a - \gamma_a y^{3-i} \right)
\]

(68)

Regularity at the curve \( \Sigma_a = \{x = 0\} \) where the modes are localized requires that

\[
\varphi^{(0)}_a \rightarrow \gamma_a y^{3-i}
\]

(69)

as \( x \rightarrow 0 \).

Inserting (67) in (66) yields

\[
\begin{align}
    \eta^i_{a+} &= \frac{i\gamma^i}{m^2 x} \left( y^{3-i} + \frac{\epsilon}{m^2 x} \left[ (3 - i) \theta_x y^{2-i} + \frac{\theta_y}{x} y^{3-1} \right] \right) + O(\epsilon^2) \quad (70a) \\
    \eta^j_{b+} &= -\frac{i\gamma^j}{m^2 y} \left( x^{3-j} + \frac{\epsilon}{m^2 y} \left[ (3 - j) \theta_y x^{2-j} + \frac{\theta_x}{y} x^{3-j} \right] \right) + O(\epsilon^2) \quad (70b)
\end{align}
\]

We can now plug these functions, together with \( h_{c+} = \gamma_c \), in the formula (65) and then extract the residue at \( (x, y) = (0, 0) \) to evaluate the couplings to \( O(\epsilon^2) \). The resulting couplings are collected in the Yukawa matrix \( Y_{a^+b^+c^+} \)

\[
Y_{a^+b^+c^+} = \pi^2 \frac{\gamma_c}{\rho_m^2} f_{a^+b^+c^+} \begin{pmatrix}
0 & \epsilon(\theta_x + \theta_y) m^2 \gamma_2 \gamma_0 \gamma_1 \\
0 & \epsilon(\theta_x + \theta_y) m^2 \gamma_1 \gamma_0 \gamma_1 \\
0 & \epsilon(\theta_x + \theta_y) m^2 \gamma_2 \gamma_1 \gamma_0 \\
\end{pmatrix} + O(\epsilon^2)
\]

(71)

where \( \rho_m = m^2 / m_z^2 \).

We clearly observe that at tree-level, \( \epsilon = 0 \), the Yukawa matrix has rank one. Including non-perturbative effects so that \( \epsilon \neq 0 \) increases the rank to three. In fact, the eigenvalues have a hierarchical structure \( (O(1), O(\epsilon), O(\epsilon^2)) \).

4.2.2. Holomorphic couplings in the \( E_6 \) model \quad To find the auxiliary functions it is convenient to treat the \( 5 \) and \( 10 \) sectors separately because the matrix \( \Psi \) has a different form on each. In the \( 5 \) sector, neutral under SU(2), the action of \( (\Phi_{xy})^{(0)} \) in (31) is such that effectively \( \Psi = 2\mu^2 (x - y) \). Hence

\[
i \eta_{5/\gamma_5} = \frac{1}{2 \mu^2 (x - y)} - \epsilon \frac{\theta_x + \theta_y}{4 \mu^4 (x - y)^3} + O(\epsilon^2)
\]

(72)

Here we have used that \( h_5 = \gamma_5 \) which can be attained by a gauge transformation of type (56) that allows to gauge away any dependence on \( (x - y) \). Thus, \( h_5 \) can be taken to be an arbitrary holomorphic function of the orthogonal coordinate \( (x + y) \) and since there is only one Higgs zero mode we choose it to be a constant, according to the usual practice [15].
In the $10$ sector the curve is given by $\Sigma_{10} = \{\mu^4(x-y)^2 = m^3x\}$ and the localized modes live in the root subspace spanned by (36). The fluxes can be chosen so that the chiral zero modes belong in the subspace spanned by $E_{10^+}$ and $E_{10^-}$ [28]. We will assume that there are exactly three chiral families associated with three matter $10$-plets in our SU(5) GUT model. Now $\Psi$ can be represented by a $2 \times 2$ matrix. From (31) we read

$$\Psi \left( \begin{array}{c} E_{10^+} \\ E_{10^-} \end{array} \right) = \left( \begin{array}{cc} -\mu^2(x-y) & m \\ m^2x & -\mu^2(x-y) \end{array} \right) \left( \begin{array}{c} E_{10^+} \\ E_{10^-} \end{array} \right)$$

(73)

since $Q = -1$ acting on the doublet.

We also need to specify $h_{10}$ which is now an SU(2) doublet of arbitrary holomorphic functions. By performing a gauge transformation (56) it can be brought in the form

$$h_{10}^i = \left( \begin{array}{cc} 0 \\ m_{3-i}^\ast \gamma_{10}^i(x-y)^{3-i} \end{array} \right)$$

(74)

with $i = 1, 2, 3$. The reason is that the gauge parameter $\chi$, now a doublet, can be chosen so that the upper component of $h_{10}$ is gauged away while the lower component is made to depend only on $(x-y)$. As usual we take a base of monomials to realize 3 families. Substituting (74) in (64) then yields

$$i\eta_{10}^i/\gamma_{10}^i = -\left[ m_{3-i}^\ast(x-y)^{3-i} \right] \left( \begin{array}{c} m \\ \mu^2(x-y) \end{array} \right) + \mathcal{O}(\epsilon^2)$$

(75)

$$+ \epsilon \left[ \frac{2\mu^4(\theta x + \theta_y)(x-y) + m^3\theta y}{(\mu^4(x-y)^2 - m^3x)^3} m_{3-i}^\ast(x-y)^{3-i} \right] \left( \begin{array}{c} 2m\mu^2(x-y) \\ (m^2x + \mu^4(x-y)^2) \end{array} \right)$$

$$+ \epsilon \left[ \frac{(\theta x + \theta_y)}{(\mu^4(x-y)^2 - m^3x)^2} m_{3-i}^\ast(x-y)^{2-i} \right] \left( \begin{array}{c} 2m\mu^2(x-y)(6-i) \\ m^3x(3-i) + (4-i)\mu^4(x-y)^2 \end{array} \right)$$

The $\mathcal{O}(\epsilon)$ correction to $\eta_{10}^i$ is seemingly complicated but there will be simplifications when inserting in the residue formula.

We can now apply the explicit expressions for $(h_5, \eta_5)$ and $(h_{10}, \eta_{10})$ to the residue formula (65) for the Yukawa couplings. The calculation is made easier from the fact that the structure constants of $E_6$ satisfy

$$\text{Tr}(E_{10^+} E_{10}^M E_{10}^N) = \epsilon_{ijklm} e^{MN}$$

(76)

where $i, j, k, l, m$ are SU(5) indices and $M, N = \pm$ are SU(2) indices. Then, the non-trivial contributions to the $10 \times 10 \times 5$ Yukawa will be of the form

$$Y = m_4^\ast \pi^2 \text{Res}_{(0,0)}(\epsilon_{MN} \eta_5^M h_{10}^N) = m_4^\ast \pi^2 \text{Res}_{(0,0)}(\eta_5^M h_{10}^N)$$

(77)

where the contractions of the SU(5) indices have been left implicit. In the first equality we have used that any other contribution will contain a term of the form $\epsilon_{MN} \eta_{10}^M h_{10}^N$ and thus it will vanish identically, and in the second equality we have used that in our solution (74) $h_{10}^i = 0$. Hence, even if (76) has a complicated expression only the terms proportional to $E_{10^+}$ will be relevant when computing up-like Yukawa couplings.

Let us proceed by computing (77) explicitly. At zeroth order in $\epsilon$ we have a contribution

$$Y_{\text{tree}}^{ij} = m_4^\ast \pi^2 \gamma_{10}^i \gamma_{10}^j \text{Res}_{(0,0)} \left[ \frac{m(m_4^\ast(x-y))^{6-i-j}}{2\mu^2(x-y)(\mu^4(x-y)^2 - m^3x)} \right]$$

(78)

$$= -\frac{m_4^\ast \pi^2}{2m^2\mu^2} \gamma_{10}^i \gamma_{10}^j \delta_{i3} \delta_{j3}$$
Hence at this level only $Y^{33}$ is non-zero. At order $O(\epsilon)$ we find terms of the form

$$Y_{np}^{ij} = \epsilon \frac{m_6^2 \pi^2}{4m_2^2 \mu^4} (\theta_{y} + \theta_x) \gamma_5 \gamma^i_{\bar{10}} \gamma^j_\pi \delta_{(i+j)4}$$

from the $O(\epsilon)$ correction to $\eta_5$. In fact, one can check that the $O(\epsilon)$ correction to $\eta_{10}$ does not contribute to (77). In the end we are left with the following $10 \times 10 \times 5$ Yukawa couplings

$$Y^{ij} = \frac{\pi^2 \gamma_5}{4\rho_\mu \rho_m} \begin{pmatrix}
0 & 0 & 0 & \epsilon(\theta_x + \theta_y) \rho_\mu^{-1} \gamma^1_{10} \gamma^3_{10} & \epsilon(\theta_x + \theta_y) \rho_\mu^{-1} \gamma^2_{10} \gamma^1_{10} \\
0 & 0 & 0 & -2\gamma^3_{10} \gamma^1_{10} & 0
\end{pmatrix} + O(\epsilon^2)$$

(80)

where $\rho_\mu = \mu^2/m_s^2$, and $\rho_m = m^2/m_s^2$.

As anticipated, the Yukawa matrix has rank one in the absence of non-perturbative effects, but when taking them into account increases its rank to three. Moreover, the eigenvalues of this matrix display a hierarchical pattern ($O(1), O(\epsilon), O(\epsilon^2)$).

5. Physical couplings and mass hierarchies

The physical Yukawa couplings are computed via the overlap integral of zero mode wavefunctions which are normalized such that the kinetic terms have canonical form. After considerable work it can be shown that this calculation gives couplings that precisely match the holomorphic couplings up to normalization [24, 28]. We have also explained that the arbitrary $\gamma$ constants introduced in the computation of the holomorphic couplings, e.g. $\gamma^i_a$ in (67), are in turn related to the normalization of the zero modes, e.g. according to (69). We then conclude that the matrices (71) and (80) give the physical couplings provided the $\gamma$ constants are fixed by requiring that the zero modes have canonical norms. Roughly speaking we demand

$$|\gamma^i_a|^{-2} \sim \int_S |\phi^i_{a+}|^2$$

(81)

Moreover, it can also be argued that to obtain the physical Yukawa couplings at order $\epsilon$ order it is enough to use the norms of the tree-level zero modes [24, 28].

A very important fact is that the zero mode norms depend on the fluxes and the associated charges. For example, in the SO(12) model [24]

$$|\gamma^i_a|^2 \sim \left( \frac{M + q_\nu \bar{N}_\nu}{m^2_s} \right)^{4-i}$$

(82)

where $q_\nu$ is the hypercharge. Hence, as already remarked in [23], the physical couplings depend on the fluxes, and the corresponding charges, through the norms. In particular the couplings $10 \times 5 \times 5$ depend on hypercharge thereby explaining the difference between D-quark and lepton couplings. Explicit formulas for the normalization constants are given in [24] for the SO(12) and in [28] for the E_6 model. The full expressions are necessary to evaluate for instance the third generation couplings. However, mass ratios such as $m_s/m_t$ only involve quotients such as $\gamma^2_a/\gamma^3_a$ which are rather simple.

The Yukawa couplings also depend on the mass parameters $m$ and $\mu$ that enter in the vev $\langle \Phi \rangle$, and on the F-theory scale $m_s$ related to the string scale $m_{st} = 2\pi \alpha'$ according to $m_{st}^4 = g_s(2\pi)^3 m_s^4$, where $g_s$ is the string coupling [24]. To analyze the physical couplings it is convenient to write the quantities $\rho_m$ and $\rho_\mu$ as

$$\rho_m = \left( \frac{m}{m_s} \right)^2 = (2\pi)^{3/2} g_s^{1/2} \sigma_m \quad ; \quad \rho_\mu = \left( \frac{\mu}{m_s} \right)^2 = (2\pi)^{3/2} g_s^{1/2} \sigma_\mu$$

(83)
Here $\sigma_m = (m/m_{st})^2$ and $\sigma_\mu = (\mu/m_{st})^2$ are the 7-brane intersection slopes which are assumed to be small. The value of $g_\sigma$ is constrained by gauge coupling unification $\alpha_G = 2\pi^2 g_s/(m_{st}^4 V_S)$, where $V_S$ is the volume of the 7-brane surface $S$ (see e.g. [1]). Besides, flux quantization requires $M/\sqrt{V_S} \sim 2\pi$. Setting $\alpha_G \simeq 1/24$ leads to the estimate for the fluxes

$$\frac{M}{m_{st}^2} = \left(\frac{2\alpha_G}{g_s}\right)^{1/2} \simeq \frac{0.29}{g_s^{1/2}} \tag{84}$$

Having diluted fluxes imposes $M < m_{st}^2$. We then conclude that $\sqrt{g_s}$ cannot be arbitrarily small. The conditions of small intersection slope and fluxes are needed to justify the effective description of the 7-brane theory.

It is interesting to contrast the theoretical results against experimental values. Our expressions for Yukawa couplings apply at the unification-string scale, presumably of order $10^{16}$ GeV, so that in order to compare with experimental fermion masses it is necessary to run the data up to the unification scale. Our reference will be the updated two-loop analysis for this running within the MSSM performed in [30]. Below we will briefly discuss how the unification scale data can be reproduced in our models. More details can be found in [24, 28].

5.1. Physical couplings in the SO(12) model

The third generation Yukawa can be read from the 33 element in (71). Using (83) and the exact form of the normalization factors yields that $Y_{b/\tau}$ is proportional to $\sigma_m \sqrt{g_s}$ with a small coefficient $O(1)$. Taking a typical value $\sigma_m \sqrt{g_s} \sim 0.1$ indicates that the third generation Yukawa couplings are large, of the same order of the Yukawa coupling of the top quark. This points to a large value of $\tan \beta \simeq 20 - 50$ in the MSSM scheme, according to the data in [30].

We can also consider the ratio $Y_\tau/Y_b$ to see how these couplings differ at the unification scale. Taking into account the dependence on hypercharge of the normalization constants, it can be shown that there is a range of parameters for which $Y_\tau/Y_b \sim 1.37$, in good agreement with the data.

Identifying the first and second eigenvalues with the third and second generations gives to leading order in $\epsilon$ at the unification scale

$$\frac{m_s}{m_b} = \frac{M}{m^2} \left(1 - \frac{\tilde{N}_Y}{3M}\right)^{1/2} \left(1 + \frac{\tilde{N}_Y}{6M}\right)^{1/2} \epsilon (\theta_x + \theta_y) \tag{85a}$$

$$\frac{m_\mu}{m_\tau} = \frac{M}{m^2} \left(1 + \frac{\tilde{N}_Y}{2M}\right)^{1/2} \left(1 + \frac{\tilde{N}_Y}{M}\right)^{1/2} \epsilon (\theta_x + \theta_y) \tag{85b}$$

The ratio of the above quantities is independent of the non-perturbative effects and depends on fluxes only through $\tilde{N}_Y/M$. Experimentally $m_{\mu}/m_{\tau} = 3.3 \pm 1$ [30], which can be fitted taking e.g. $\tilde{N}_Y/M \sim 1.8$. Substituting this value in (85b), and using $m_{\mu}/m_\tau = 5.4 \pm 0.6 \times 10^{-2}$, yields the estimate

$$\frac{M}{m^2} \epsilon (\theta_x + \theta_y) \simeq (2.3 \pm 0.2) \times 10^{-2} \tag{86}$$

Note that this value is quite small, consistent with a non-perturbative origin.

5.2. Physical couplings in the $E_6$ model

The top Yukawa is given in (78). The flux and mass parameters can be chosen so that $Y_t \sim 0.5$ to match the experimental data [30].
Identifying the first and second eigenvalues with the third and second generations of U-quarks gives

\[
\frac{m_c}{m_t} = \frac{M}{2\mu^2} \left( 1 + \frac{2N_Y}{3M} \right)^{1/2} \left( 1 - \frac{N_Y}{6M} \right)^{1/2} \epsilon (\theta_x + \theta_y)
\]

(87)

It is possible to adjust the fluxes to reproduce the hierarchy between the charm and the top quark with a small non-perturbative parameter \(\epsilon\). For example, we may take \(\tilde{N}_Y/M \sim 1.8\) as in the SO(12) model for the down-like Yukawa point \(p_{\text{down}}\), since, if the two Yukawa points \(p_{\text{up}}\) and \(p_{\text{down}}\) are not far away the flux densities should be alike. Substituting this value in (87) and requiring a realistic mass ratio, say \(m_c/m_t \sim 2.5 \times 10^{-3}\), then gives

\[
\frac{M}{\mu^2} \epsilon (\theta_x + \theta_y) \approx 4 \times 10^{-3}
\]

(88)

which is similar to the estimate (86) in the SO(12) model.

It is encouraging that the data can be fitted with similar parameters as in the SO(12) model. However, a more systematic analysis would entail embedding both Yukawa points \(p_{\text{up}}\) and \(p_{\text{down}}\) in the same local model, possibly in a region of \(E_7\) or \(E_8\) enhancement.

6. Final remarks

In this paper we have studied the structure of Yukawa couplings in F-theory SU(5) models, taking into account the contribution of non-perturbative effects. We have considered explicit local models in order to compute down-like and up-like couplings. In one model there is an Abelian background that gives rise to three intersecting curves supporting localized matter transforming as \(10, 5, \) and \(\bar{5}\). At the point where the curves intersect there is enhancing to SO(12) and down-like couplings \(10 \times 5 \times \bar{5}\) emerge. In the second model there is a non-Abelian background such that up-like \(10 \times 10 \times 5\) couplings are generated by the intersection of \(10\) and \(5\) matter curves at a point where SU(5) is enhanced to \(E_6\). The models include worldvolume fluxes needed to guarantee the existence of localized chiral modes, as well as doublet-triplet splitting in the Higgs sector.

From the general results of [14, 27] one expects to obtain a rank one matrix of Yukawa couplings when the couplings arise in a configuration with matter curves intersecting at one point. We have verified that this is indeed the case in our models. To overcome this situation we have then added non-perturbative effects sourced by gaugino condensation or D3-brane instantons. As argued in [18] these effects can be written in terms of the fields in the effective 8d theory and can thus be incorporated in the formalism that allows to compute Yukawa couplings. We have found that in both models the non-perturbative effects do lead to Yukawa matrices of rank three. Therefore, the non-perturbative effects induce non-trivial Yukawas for the two lighter families of quarks and leptons. Moreover, the corrected couplings naturally reproduce a hierarchical mass structure among the three families.

At our level of approximation we are only able to predict the ratios of second to third generation couplings. Adjusting the free parameters in the models allows to match these ratios with the experimental values. We also find that the third generation Yukawa couplings are of the right order of magnitude.

Our results can be extended in various directions. For instance, we could examine the contribution of additional terms in the expansion (47) underlying the non-perturbative effects. From the results of [24] terms proportional to \(\theta_n\) with \(n > 0\) should not alter the hierarchical structure that we have obtained with only \(\theta_0\), but they could be relevant in the computation of the Yukawa couplings of the lightest generation. To this aim, we would have to take our approximations to order \(\epsilon^2\) in the non-perturbative parameter \(\epsilon\). Besides, improving the level
of approximation would be required to determine the CKM matrix for these models. Actually, because the CKM matrix involves considering both Yukawa points $p_{up}$ and $p_{down}$ simultaneously, we could consider implementing our approach in local models of $E_7$ or $E_8$ enhancement [27, 31].

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