Bound States in the Continuum in Nuclear and Hadron Physics

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Abstract
The population of bound states in the continuum and their spectral properties are studied on the nuclear and hadronic scale. The theoretical approach is presented and realizations in nuclear and charmonium spectroscopy are discussed. The universality of the underlying dynamical principles is pointed out. Applications to nuclear systems at the neutron dripline and for charmonium spectroscopy by $e^-e^+ \rightarrow D\bar{D}$ production are discussed.

1. Introduction
A general feature observed in quantum systems at all scales from atomic to hadron physics is the occurrence of long-lived states embedded into the spectral region of continuum states. The internal structure and dynamical properties of a discrete state coupled to a continuum are encoded in its line shape. Only in simple potential problems the line shapes of spectral distribution come close to the widely used Lorentz- or Breit-Wigner shapes. Under more realistic physical conditions the line shape of resonance is modified by the interaction between the discrete and continuum components of the spectra. This kind of interference among states of different configuration type is ubiquitous in quantum physics and leads to a plethora of interesting phenomena in nuclear, atomic, condensed matter and quantum optical physics. The foundations of quantum mechanical continuum spectroscopy was laid as early as 1961 by the pioneering work of Fano [1]. The relevance of that approach for nuclear physics was soon recognized and the concept of bound states in the continuum was formulated in a systematic manner by Mahaux and Weidenmüller in their book [2] published in the late 1960ties. An exciting recent progress on continuum interactions in atomic physics was the control of line shapes by the direct manipulation of spectral distributions with intense laser light with different frequencies [3]. In that paper a representative list of other work in atomic, molecular, and laser physics is found. Obviously, this kind of immediate external influence is out of reach for nuclear and hadron physics. An early application of the Fano-approach to nuclear physics is found in [4] and in Ref. [5] the formalism was used to investigate pionic resonances. The basically different situation in nuclear and hadronic systems is their much shorter lifetime because of the strong interaction with neighbouring states. Hence, in sub-atomic systems the line shapes are often found to be convoluted by overlapping contributions and, additionally, are influenced by coupled-channels effects [6, 7, 8]. In section 2, we discuss continuum spectroscopy at the neutron dripline by the representative example of $^{10}\mathrm{Li}$. Configuration interactions in the $D\bar{D}$ continuum and their influence on charmonium line shapes are discussed in section 3. A summary and an outlook is given in section 4.

2. Nuclear Bound States in the Continuum
Away from shell closures pairing and core polarization effects from particle-particle and particle-hole interactions, respectively, are playing an increasingly significant role. A full picture of the spectral properties in weakly bound nuclei should account for bound and continuum interactions on the same
Fig. 1: Spectral distributions seen in elastic partial wave cross sections for the scattering of a neutron on $^9$Li by the HFB mean-field and Gorkov-coupling. Full HFB Gorkov-pairing and bare mean-field results are compared for partial waves up to d-waves. The attractive pairing self-energy produces resonances in the p-wave channels.

footing. In our previous work \cite{6,7} we have described interplay of closed and open channels in nuclear continuum spectra. In \cite{7} it was pointed out that pairing interactions lead in weakly bound nuclei already on the mean-field level to a mixing of particle- and hole-type states. We denote their respective wave functions by $u_\alpha q$ and $v_\alpha q$ carrying quantum numbers $\alpha = (n \ell jm)$ and where $q = p, n$ denotes protons and neutrons. In addition to the overall single particle mean-field potential $U_q$, close to the particle threshold the pairing fields $\Delta q$ are of special importance because they lead to coupled channels effects already on the mean-field level. The problem is properly described by the Gorkov equations \cite{7}:

$$
\begin{pmatrix}
T_q + U_q - 2\lambda_q + e_\alpha & \Delta_q(r) \\
-\Delta_q^*(r) & -(T_q + U_q - e_\alpha)
\end{pmatrix}
\begin{pmatrix}
u_{\alpha q}(r) \\
u_{\alpha q}(r)
\end{pmatrix} = 0
$$

(1)

derived from an underlying energy density functional \cite{7}. $T_q$ is the kinetic energy operator and the chemical potentials $\lambda_q$ account for particle number conservation. The quasiparticle energy is expressed in terms of the quasi-hole energy $e_\alpha < \lambda_q$. For weakly bound exotic nuclei with separation energies well below 1 MeV the continuum gap is easily overcome by residual interactions. Thus, the dominance of the static nuclear mean-field is broken and is competing with induced interactions in the particle-particle, hole-hole and the particle-hole channels. As discussed in \cite{6,7} the increased polarizability of dripline nuclei leads to the dissolution of shell structures in the bound state region and initiates a rich spectrum of sharp particle resonances above the particle threshold. They are produced by complex many-body states containing interfering contributions from closed channels due to multi-particle-hole core excitations and open channels given by single particle scattering states. The closed channel states cannot decay by themselves but only through the coupling to the single particle continua. That mechanism is an universal feature of all open quantum systems. In Fig. 4 the dynamical principles are depicted schematically. Since close to the dripline, the usual BCS approach is no longer meaningful, we solve Eq. 1 as a coupled...
channels problem thus accounting properly for the asymptotics of bound and unbound states \[.\] Core polarization is described by an extended system of coupled equation. Integrating out the core degrees of freedom we end up at a set of equations describing single particle motion under the influence of residual particle-hole interactions \[.\]:

\[
(H_{MF}^{(\alpha)} - \varepsilon_\alpha) \phi_{\alpha k} + \sum_{\beta} F_{\alpha\beta} \phi_{\beta n} = 0 \tag{2}
\]

The mean-field (HFB) Hamiltonian for the core state \(\alpha\) is denoted by \(H_{MF}^{(\alpha)}\) and the effective single particle energy \(\varepsilon_\alpha \equiv E - E_\alpha\) is given by subtracting the energy \(E_\alpha\) of the core state \(|\alpha\rangle\) from the total energy \(E\). The transition form factors are given by matrix elements of the residual interaction, \(F_{\alpha\beta} = \langle \alpha | V_{res} | \beta \rangle\). For further details we refer to ref. \[.\]. Both Eq.(1) and Eq.(2) can be recast formally into an effective single particle equation

\[
(H_{MF}^{(\alpha)} + \Sigma^{\alpha}(\omega) - \omega) \phi_{\alpha k} = 0 \tag{3}
\]

with an energy dependent, non-local complex self-energy \(\Sigma^{\alpha}(\omega)\) \(U_r G(\omega) U_r\) and the transition potentials \(U_r\) are given either by the pairing field or the core polarization form factors, respectively. The real and imaginary part of the self-energy modify the spectral distributions substantially by energy shifts and decay and damping widths, thus expressing the finite life time of single particle states in an interacting quantum many-body system.

The spectral distributions obtained by Eq. (1) for the \(^{10}Li = ^9Li + n\) system are displayed in Fig. 1. Two distinct regions can be identified: 1) the region of discrete bound states for \(2\lambda_q < \varepsilon_\alpha < \lambda_q\) with \(2\lambda_q - \varepsilon_\alpha < 0\) (negative particle energies); 2) the continuum region \(\varepsilon_\alpha < 2\lambda_q\) with \(2\lambda_q - \varepsilon_\alpha > 0\) (positive particle energies) \[.\]. Although the hole wave functions are still bound states they obtain continuous spectral distributions with peak structures slightly shifted from the bare mean-field positions. These effects are much more enhanced when core polarization is taken into account. In Fig. 2 the p-wave single particle continuum spectral distributions for \(^{10}Li = ^9Li + n\) are shown. In both the \(\frac{3}{2}^-\) and the \(\frac{1}{2}^-\) partial wave sharp resonances occur, mainly caused by the coupling to the \(^9Li(2^+, 2.691)\) and a few other core states. The single particle and the core spectrum are obtained fully microscopically by HFB and QRPA calculations, respectively. The effect is especially pronounced in the \(\frac{3}{2}^-\) channel where the mean-field resonance couples strongly to the core excitations and is fragmented considerably. The width and energy shifts are determined by the channel coupling, reflecting the real and imaginary parts.
Fig. 3: Angle-integrated cross section for the \( d(^9\text{Li},^{10}\text{Li})p \) reaction at 2.36 AMeV \([7]\), including the experimental energy resolution and compared with data from Refs. \([10]\). Two resonances are seen in the \( 1/2^- \) and \( 3/2^- \) partial waves.

of the dynamical self-energies discussed above. These interaction effects lead to a suppression of the spectral strengths, given by the residues of the spectral distributions at the pole positions, to values well below unity. In \([6]\) corresponding results for \(^{15}\text{C}\) have been used to describe sharp resonances, experimentally observed well above the neutron threshold. Pairing spectral distributions corresponding to Fig. 1 were used in \([7, 9]\) to analyse \(^{10}\text{Li}\) data, observed the first time in a \( d(^9\text{Li},^{10}\text{Li})p \) REX-ISOLDE experiment \([10]\). As seen in Fig. 3 the data are well described by the Gorkov spectral functions, Fig. 1 and DWBA transfer calculations, treating the transfer kernel by the Vincent-Fortune method. Recently, the same reaction has been remeasured at TRIUMF with a much better energy-resolution. The new data are presently being analysed and theoretical calculations including core polarization are in preparation.

3. Charmonium Line Shapes

Next we turn our attention to configuration mixing effects of the same kind as discussed in the previous section but with a quite different realization in charmonium spectroscopy. Following closely the Fano-formulation, we assume a pre-diagonalization of the confined \( c\bar{c} \) states and of the continua given by the \( D\bar{D} \) meson channels. As before, the two types of configurations are coupled by residual interactions, e.g. giving rise to a finite life time to the confined \( c\bar{c} \) states. As in \([8]\) we investigate specifically the \( \psi(3770) \) state, considered as a bare \( 1^3D_1 \) \( c\bar{c} \) charmonium state interacting with the \( D\bar{D} \) continuum, as schematically depicted in Fig. 4. Of course, this ansatz is easily extended to higher lying states and other open charm channels, e.g. the \( \psi(4040) \) state and \( D^*\bar{D} + \text{c.c.} \) channel. The confined \( c\bar{c} \) states define the set of closed channels with respect to \( c\bar{c} \) motion. Because of confinement, the QCD-type \( c\bar{c} \) channels remain closed channels at all energies while sub-threshold hadronic \( D\bar{D} \) channels eventually change to open channels.

We assume that the bare \( c\bar{c} \) states, their wave functions \( \phi_c \) and mass \( m_c \) are known, as well as the hadronic \( DD \) scattering states and their relative motion wave functions \( \phi_d \). At total center-of-mass
energy $\omega = \sqrt{s}$ we expanded the wave function into the basis of these states,

$$\Psi_\omega = z_c(\omega) \phi_c(\omega) + \int d\omega' \phi_d(\omega') \phi_d(\omega')$$  \hspace{1cm} (4)

For simplicity, we consider the simplest case given by a single closed channels $c$ and a single open channel $d$. We separate intermediate propagators into pole and principle values parts and consider that the coupling to closed channels lead to dispersive, but not absorptive self-energies, as shown in the original Fano paper [1]. Interactions modify the closed channel wave functions by dressing by a cloud of virtual $D\bar{D}$ states

$$\chi_c(\omega) = \phi_c(\omega) + P \int d\omega' \frac{V_{cd}(\omega')}{\omega - \omega'} \phi_d(\omega')$$  \hspace{1cm} (5)

and the correlated state vector is obtained as

$$\Psi_\omega = x_c(\omega) \chi_c(\omega) + x_d(\omega) \phi_d(\omega)$$  \hspace{1cm} (6)

The channel interaction may lower one or a few eigenstates below the particle emission threshold, and one may speculate whether $\psi(3686)$ is of such a nature, see e.g. Ref. [11]. The amplitudes $x_{c,d}$ are obtained for the solution of a set of coupled equations and by the proper normalization of the state vector [1, 6, 8]. Their detailed forms are of no special interest here. A more important message of Eq. (4) is that the observed charmonium states like $\psi(3770)$ have to be considered as varying mixtures of $c\bar{c}$ and $DD$ configurations. While the bare $c\bar{c}$ closed channel by itself lives indefinitely long, the configuration interactions $V_{cd}$ induce a spectral distribution of a width $\Gamma_c(\omega) = 2\pi|V_{cd}(\omega)|^2$ and a related energy dependent mass shift $\Delta m_c(\omega)$. Here, $V_{cd}(\omega)$ denotes the matrix element of the configuration mixing interaction. The coupling to closed channels induces in the open channels an additional configuration mixing phase shift, derived in the present context as

$$\tan \delta_{cd}(\omega) = \frac{x_c(\omega) x_d(\omega)}{x_d(\omega)} = \frac{m_c \Gamma_c(\omega)}{m^2 - \omega^2}$$  \hspace{1cm} (7)

where $m_c = m^0_c + \Delta m_c$ includes the mass shift. However, because of the extremely small width, $\Gamma_c \ll m_c$, both $m_c$ and $\Gamma_c$ can be taken as constant in the resonance region. The phase shift $\delta_{cd}$ varies rapidly with energy on a scale set by $V_{cd} \sim \sqrt{\Gamma_c}$. $\delta_{cd}$ has to be added to bare $DD$ channel phase shift varying on
a much larger energy scale, given by $t$- and $u$-channel interactions from the exchange of light mesons. Hence, in (hypothetical) $D\bar{D}$ scattering one would observe a sharp resonance around $\omega \sim m_c$, superimposed on and interfering with a slowly varying background.

In order to work out the role played by $c\bar{c}$ states in the resonances observed in $e^+ e^-$-annihilation reactions we need to combine properly the reaction model on the one side and the configuration model on the other side. Starting from an initial reaction channel $|\tau\rangle$ at total energy $\omega$, let be $M_\tau$ the transition operator for the production of the state $\Psi_\omega$. In the following formulae we omit partial wave indices because we are studying the production and decay of $1^{--}$ charmonium vector states which couple to $D\bar{D}$ $P$-waves. Obviously, the formalism is easily extended to any other partial waves. The charmonium production amplitude out of the incident channel $|\tau\rangle$ is described by the matrix element of the corresponding operator $M_\tau$

$$\langle \Psi_\omega | M_\tau | \tau \rangle = x_c(\omega) \langle \chi_c | M_\tau | \tau \rangle + x_\bar{c}(\omega) \langle \phi_\bar{c} | M_\tau | \tau \rangle .$$

The reaction amplitude is given by a production form factor which we express as

$$|F_\tau|^2 = |\langle \phi_\bar{c} | M_\tau | \tau \rangle|^2 \frac{|q_{c\bar{c}} - \epsilon|^2}{1 + \epsilon^2} ,$$

where $\epsilon = \cot \delta_{c\bar{c}} = (-\omega^2 + m_c^2)/m_c \Gamma_c$ is due to configuration mixing. The parameter

$$q_{c\bar{c}}(\omega) = \frac{\langle \chi_c | M_\tau | \tau \rangle}{\langle \phi_\bar{c} | M_\tau | \tau \rangle}$$

plays a central role in our approach. Obviously, $|q_{c\bar{c}}|^2$ is the a measure of the population probability of the (dressed) closed $c\bar{c}$ QCD-channel relative to the population probability of the purely hadronic $D\bar{D}$ channel. To a very good approximation we are allowed to use $q = q_{c\bar{c}}(m_c) = \text{const.}$. Eq. (9) shows that $q$ is controlling the line shape of the spectral distribution: a dip, eventually down to zero, will appear at an energy $\omega_0$ where $q = \epsilon(\omega_0)$. For $q = 0$ an inverted resonance line shape with a minimum at $\omega = m_c$ will occur. The widely used Breit-Wigner profile is recovered if $q \gg \epsilon$ over the whole resonance region:

$$F_c(s) = \frac{A_c}{s - m_c^2 + i m_c \Gamma_c} .$$

The latter two cases correspond to the limiting scenarios of the reaction, namely for $q = 0$ exclusive annihilation into the hadronic $D\bar{D}$ channel and, as the other extreme, exclusive annihilation into the $c\bar{c}$ channel for $q \to \infty$. The latter case describes, by the way, sub-threshold charmonium production for which the form factor, Eq. (9) reduces to the $c\bar{c}$ amplitude. Thus, besides fixing the line shape, $q$ provides information on the reaction mechanism. As such, it depends naturally on the type of reaction and we have to expect different line shapes when populating the same final state in different reactions. Thus, spectral distribution of different shapes have to be expected in charmonium production in leptonic $\ell \bar{\ell}$ and hadronic $hh$ annihilation reactions. If many open channels are contributing, the interference minimum will be superimposed on a finite background. The structural properties of charmonium are imprinted in $\epsilon$, given by the configuration mixing phase shift $\delta_{c\bar{c}}$. Hence, $\epsilon$ contains the full spectral information on the mass and the width of the resonance. In the specific case of $c = \psi'(3770)$ the width is given by the $P$-wave relation

$$\Gamma_{\psi}(s) = \frac{8G_{DD}^2}{6\pi s} \left( p_0(s) + p_\pm(s) \right)$$

(12)
where the c.m. momenta are \( p_i(s) = \sqrt{s/4 - m_{D_i}^2} \) for neutral \((i = 0)\) and charged \((i = \pm 1)\) \(D\bar{D}\) channels.

Finally, the yet missing population probability of the hadronic \(D\bar{D}\) component in Eq. (9) is defined by the ansatz

\[
|\langle \phi_d | M_\tau | \tau \rangle|^2 = |A_{\psi'} F_d|^2
\]  

(13)

The energy dependence is described by the form factor

\[
F_d(s) = \frac{1}{s - m_d^2 + i m_d \Delta_d},
\]  

(14)

mimicking the weak energy dependence induced by the non-resonant \(D\bar{D}\) interactions. The quantities \(m_d\) and \(\Delta_d\), which are purely numerical parameters, are determined by the data, as listed in Tab.1.

The magnitude of the \(e^{+}e^{-} \rightarrow D\bar{D}\) production amplitude is fixed by the residue \(A_{\psi'} = m_{\psi'}^2 g_{\psi'\bar{D}D}/g_{\psi'\gamma}\), determined essentially by the \(g_{\psi'\bar{D}D}\), dimensionless coupling constant of \(\psi'(3770)\) to \(D\bar{D}\).

For the present investigations, we determine the photo-vector coupling constant \(g_{\psi'\gamma}\) phenomenologically. The electronic width of vector charmonium states is given by \(\Gamma_{\psi' e^{+}e^{-}} = 4\pi \alpha^2 m_{\psi'}/3 g_{\psi'\gamma}^2\), see e.g. [17] where \(\alpha \approx 1/137\) denotes the electromagnetic fine structure constant. With \(\Gamma_{\psi' e^{+}e^{-}} = 0.265\) MeV from the recent compilation of the Particle Data Group [12] the photo-vector coupling constant \(g_{\psi'\gamma}\) at \(m_{\psi'} = 3770\) MeV could be determined to be 56.35.

We define the hadronic \(e^{+}e^{-} \rightarrow D\bar{D}\) cross section, including the appropriate two-body phase space factor,

\[
\sigma_{D\bar{D}}(s) = \frac{8\pi \alpha^2 p_f^3}{3s^{5/2}} |\langle \phi_d | M_\tau | \tau \rangle|^2
\]  

and obtain the full charmonium production cross section as

\[
\sigma(s) = \frac{8\pi \alpha^2 p_f^3}{3s^{5/2}} |F_\tau|^2 = \sigma_{D\bar{D}}(s) \frac{|q - e|^2}{1 + e^2}\]  

(16)
Hence, the charmonium production cross section is separated into the annihilation cross section populating the hadronic $D\bar{D}$ component and a form factor containing the population and spectroscopy of the confined $c\bar{c}$ component of the full charmonium state vector.

Applying the approach to the BESIII data \[13\], the spectral distributions in the $D_0\bar{D}_0$ and the $D^+D^-$ production cross sections are well described, as seen in Fig. 5. In order to illustrate possible applications to data analyses, the mass, width and line shape parameters, respectively, have been varied freely in a $\chi^2$ minimization process. The resulting parameter sets are shown in Tab. 1. The bare mass and width of $\psi(3770)$ in the neutral and charged channels are consistent with each other. The different behaviour of the two production cross sections results almost totally from the difference in phase space factors of the $D_0\bar{D}_0$ and $D^+D^-$ channels, namely the mass gap of neutral and charged $D$-meson. It should be mentioned that the dip at around 3.82 GeV could be reproduced with above prescription if the Belle data at high energies are included in the fit, as we have shown previously \[8\]. Finally, attempting a fit with a simple Breit-Wigner line shape (i.e. assuming $q \to \infty$) the description deteriorates as reflected by the increased values $\chi^2 = 2.72$ for the $D_0\bar{D}_0$ channel and $\chi^2 = 3.27$ for the fit to the $D^+D^-$ data, respectively.

The results for $q$ obtained from the $D_0\bar{D}_0$ and the $D^+D^-$ data are agreeing within the error bars. The slight differences may be taken as an indication on the reaction-dependence of spectral line shapes observed in production reactions. Despite of the remaining uncertainties due to the relatively large experimental errors the two $q$-values are indicating differences in the reaction mechanism, probably mainly caused by differences in the final states interactions. More precise data from future experiments, either at $e^-e^+$-facilities or from $p\bar{p}$ annihilation as planned at PANDA@FAIR are important for a more detailed analysis.

4. Summary

Universal aspects in nuclear and hadronic spectroscopy have been investigated by general methods, equally well applicable to various kinds of open quantum systems at any scale. The relation to atomic and molecular physics was pointed out. The scale-independent feature is configuration mixing of asymptotically open and closed channels. A theoretical scheme was developed which provides a microscopic approach to spectral distributions in nuclear and hadron spectroscopy. Frequently observed sharp resonances with non-standard, asymmetric line shapes and interference pattern have been explained dynamically in terms of configuration mixing effects. The theory was applied to continuum spectroscopy at the neutron dripline and charmonium production in $e^-e^+$ annihilation reactions.

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<th>$m_d$ (MeV)</th>
<th>$\Delta_d$ (MeV)</th>
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Table 1: Charmonium production in the $e^-e^+ \to D_0\bar{D}_0$ and $e^-e^+ \to D^+D^-$ annihilation reactions: Fano-parameters according to Eq. (9) from the fit to the data shown in Fig. 5.
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References