A SHORT REVIEW OF RESULTS FROM CONTINUOUS MOMENT SUM RULES

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In recent years much information on two-body or quasi two-body processes at high energy has been accumulating. This has stimulated a quantity of phenomenological work by theoreticians, mainly in the frame of Regge-pole and diffractive models. In the typical case, however, not one but several and rather different theoretical pictures have been found to "explain" correctly the experimental data. This unsatisfactory situation derives mainly from the scarcity of good polarization data at high energy, which causes difficulties in obtaining a full determination of the scattering amplitude. The experimental reason for this is the smallness of high-energy cross-sections, and the consequent low statistics available.

In contrast to high-energy, low energy presents itself in a much more favourable situation. Cross-sections are much higher and therefore polarizations are easier to measure. Moreover, the scattering amplitude can be described by relatively few parameters in a model-independent way, because the centrifugal barrier factors depress high partial waves. A full determination of the scattering amplitude then becomes possible here.

Because of the analyticity properties of the scattering amplitude one is able in principle to extract useful pieces of high-energy information from this wealth of low-energy information. Continuous Moment Sum Rules (CMSR) are just a convenient way to accomplish this.

After a few words on the derivation, the properties and the analysis of CMSR we present below a short
review of some relevant phenomenological results which have been obtained by them.

CMSR $^{1}$ are a simple generalization of conventional Finite Energy Sum Rules $^{2}$. If $F(v)$ is a scattering amplitude, one considers the product $v^\gamma F(v)$ ($\gamma \geq 0$) which is regular in the upper half-plane of $v$. Therefore by integration over a semicircle, like that in Fig. 1, zero is obtained because of Cauchy theorem. The condition

$$\oint d\nu \, \nu^\gamma F(v) = 0$$  \hspace{1cm} (1)

establishes a relation between the contribution of the semicircle, which can be expressed in terms of the parameters appearing in a Regge expansion for $F(v)$, and the integral along the real axis which can be calculated by some low-energy partial-wave analysis.

Supposing, for simplicity, that $F(v)$ is antisymmetric, its Regge expansion appears:

$$F(v) = \sum_i \beta_i \frac{1 - e^{-i\pi\alpha_i}}{\xi_i \pi\alpha_i} \nu^{\alpha_i}$$  \hspace{1cm} (2)

where the sum is to be intended (see Fig. 2) over a finite number of Regge-pole terms, a possible continuous distribution along the real axis of $\alpha$ (i.e. a Regge cut), and then a background contribution as indicated in Fig. 2 (wavy vertical curve).
Then Eq. 1 becomes

\[ \phi(y) = -\int_{\nu_0}^{\nu} d\nu \nu^\gamma I_{\nu} \left[ e^{-i\frac{\pi}{2}y}, F(\nu) \right] \]

\[ = \sum_{i'} \frac{\beta_{i'}}{\cos \frac{\pi}{2} \alpha_{i'}} \frac{s_i u_1 \left[ \frac{\pi}{2} (\alpha_{i'} + \gamma + 1) \right]}{\alpha_{i'} + \gamma + 1} \nu_0 \alpha_{i'} \quad (3) \]

In order to appreciate the properties of this relation let us compare it with the following expression, in which only information at the energy \( \nu_0 \) is exploited:

\[ I_{\nu} \left[ e^{-i\frac{\pi}{2}y}, F(\nu) \right] = \sum_{i'} \frac{\beta_{i'}}{\cos \frac{\pi}{2} \alpha_{i'}} s_i u_1 \left[ \frac{\pi}{2} (\alpha_{i'} + \gamma + 1) \right] \nu_0 \alpha_{i'} \quad (4) \]

The difference in information content between the two relations is reflected only in the presence in the r.h.s. of Eq. 3 of the denominator \((\alpha_{i'} + \gamma + 1)\). Because of this denominator the high-\( \alpha \) tails of the background integral (see Fig. 2), which are mostly responsible for the low-energy oscillations in \( F(\nu) \), are strongly depressed. The part of the background integral which can still appreciably contribute to the r.h.s. of Eq. 3 is localized near the real axis. It can be parametrized as an effective Regge-pole contribution with \( \alpha \sim -1 \) and included in the analysis.

Eq. 3 is the fundamental equation which allows us to extract high-energy information from low-energy partial-wave analysis. After evaluating \( \phi(y) \) through the second member of the equation, one must analyse it in terms of the contributions of the \( \alpha \)-plane singularities as they appear in the third member of the equation. A pure one-Regge-pole contribution has a
shape \( s i\omega [\frac{1}{2}(\alpha + \gamma + 1)]/(\alpha + \gamma + 1) \). If such a contribution dominates \( \Phi(\gamma) \), then from the position of the curve along the \( \gamma \)-axis and the height of its maxima it is possible to determine the \( \alpha \) and the weight, respectively, of the corresponding Regge pole. If \( \Phi(\gamma) \) has not the one-pole shape, this means that the structure of the \( \alpha \)-plane singularities is more complicated. One must then proceed through least-square fits, as for experimental data.

I shall consider now the main results which have emerged so far from the CMSR analyses of near-forward \( \pi N \) scattering \(^3\),\(^4\) and pion-photoproduction \(^5\),\(^6\),\(^7\). A short list of them could be as follows.

1) Evidence for the existence of Regge-pole conspiracies (in particular \( \pi-\pi_c \) and \( \rho-\rho \) conspiracies).

2) Systematics on the existence of Regge-pole residue zeros near \( \hat{t} = 0 \) in forward contribution amplitudes, and their absence in the other amplitudes.

3) Evidence for the presence of Regge-cut contributions, and indications on the shape of their discontinuities.

4) A rather large amount of quantitative data on Regge singularities (trajectories, residues, etc.).

Let us now consider in more detail the single points.

1) The existence of conspiracies in pion-photoproduction can be realized, for instance, looking at the relations which link invariant amplitudes to the regu-
larized t-channel helicity amplitudes. In particular for one of the invariant amplitudes one has

\[ A_2 = \frac{F_1}{t-4\mu^2} + \frac{1}{t} \left[ \frac{F_2}{t-\mu^2} + \frac{2\mu^2 F_3}{t-4\mu^2} \right] \]  

(5)

In order to avoid a kinematical singularity of \( A_2 \) at \( t=0 \), one must then have

\[ F_2 (s, t=0) = -\frac{\mu^2}{2\mu} F_3 (s, t=0) \]  

(6)

which is a conspiracy relation.

Let us consider in particular the amplitudes \( F_2^{(-)} \) and \( F_3^{(-)} \) (the superscript denotes exchange of \( I^G = 1^- \) quantum numbers in t-channel). The first receives contribution from \( \pi^- \)-exchange, while the second has a parity opposite to that of the \( \pi^- \).

Because of the large forward peak in \( \pi^\pm \)-photoproduction (see Fig. 3) they surely do not satisfy the constraint of Eq. 6 by evasion. A possibility is that \( F_2^{(-)} \) and \( F_3^{(-)} \) are dominated near \( t=0 \) by Regge-poles (8),5),6). This would imply a conspiracy between Regge poles and would support the relevance of \( O(4) \) symmetry in the treatment of the near-forward region.

Another possibility which would not have such relevant theoretical consequences is a Regge-cut dominance in the two amplitudes 10).

We list and discuss here the main evidences from CMSR which support Regge-pole conspiracy against Regge-cut dominance.
i) The $\phi(t)$'s calculated from the Walker fit \(^{11}\) for $F_2^{(-)}$ and $F_J^{(-)}$ satisfy near $t = 0$ all the quantitative relations required for one-pole curves (see Fig. 4). Taking errors into account, this means that they represent Regge singularities much localized in the $\alpha$-plane. Recently a paper by Jackson and Quigg \(^{12}\) appeared in which it is concluded that these curves are compatible with a Regge-cut interpretation. This is, of course, true allowing for cut discontinuities much peaked near the value of $\alpha$ indicated by CMSR. It is clear that it will never be possible to conclude from the shape of $\phi(t)$ at some particular value of $t$ that the singularity distribution in the $\alpha$-plane at this value of $t$ is just a $\delta$-function.

ii) Let us consider the Regge contribution which dominates $F_3^{(-)}$ and in particular its, let us say, effective Regge trajectory $\alpha_{\text{eff}}(t)$. The errors involved in the determination of $\alpha_{\text{eff}}(t)$ are rather small because $F_3^{(-)}$ practically dominates the whole forward peak. Now the slope of $\alpha_{\text{eff}}(t)$ is about $\sim 1.5 \text{ GeV}^{-2}$ \(^{5}\) which is rather a high value for an absorptive Regge cut, for which we should expect something of the order of $1/2 - 1/3 \text{ GeV}^{-2}$.

iii) Evidence, which would become crucial if confirmed by other multipole analyses, is offered by the appearance of the pion-conspirator $\pi_\pi$ in the $F_4^{(-)}$ amplitude \(^{6}\). In the $\phi(t)$'s calculated from the Walker fit the $\pi_\pi$ emerges because the $A_2$ contribution vanishes at $t \approx -5$ (see Fig. 6). It can be identified by its trajectory which is just the same
as that which has been determined from $F_3^{(-)}$ (see Fig. 5).

Besides the pion-conspiracy, evidence has also been found for the existence of $j(-\beta)$ conspiracy\(^7\).

This conspiracy, however, does not produce a pronounced forward peak in $\pi^0$-photoproduction because of the presence of two dynamical zeros near $t=0$ in the $j'$ and $\mathcal{B}$ residues (see point 2) which strongly depress the contribution near $t=0$ of these Regge poles (see Fig. 7).

2) A very interesting systematics has emerged from both $\pi N$ scattering and pion-photoproduction analyses about the presence and absence of Regge-pole residues zeros near $t=0$. While these zeros are present in forward-contributing amplitudes ($F_1$ and $F_3$ in pion-photoproduction, and $A'$ in $\pi N$ scattering) they are absent in the other amplitudes. There are only two exceptions: $A^{(+)}$ in $\pi N$ and $F_3^{(-)}$ in pion-photoproduction. The former is not a real exception because it is still not clear which kind of Regge singularity corresponds to vacuum exchange. As to the latter, we observe that in $F_3^{(-)}$ also occurs the notable fact that no trace of pole is present at $t\simeq 0.01$ when $\alpha_{\pi c} \simeq 0$.

We should like to recall a simple argument by Toller\(^{13}\), based on $O(4)$ symmetry, which fits rather well all this systematics and also the pion-conspirator exception. A bias of this argument is that it also requires the possibility of an analytic continuation of the scattering amplitude in the particle masses. This might find some resistance from those who believe
in the bootstrap philosophy. The fundamental fact is that $O(4)$ symmetry seems to prevent the existence of poles at $t=0$ in the scattering amplitude of equal-equal processes \(^{13}\). Consider now the $t$-exchange contribution of a particle $B$ to a forward-contributing amplitude of some particular process (Fig. 8), and suppose thus this contribution continues analytically to the configuration $\omega_{LB} = 0$. The pole must then disappear. Because of analyticity there must be a dynamical zero of the $B$ contribution near $t = 0$ which moves to exactly $t = 0$ when $\omega_{LB} = 0$. The absence in $F_3(t)$ of both the zero and the $\omega_{LB} = 0$ pole is simply explained in this argument by their coincidence.

3) In correspondence with the near-forward zeros discussed in Section 2, the vanishing of the dominant Regge-pole contribution allows secondary Regge contributions to emerge. A relevant fact about these contributions is that the corresponding $\varphi(\bar{s})$'s have the same characteristic shape (see Fig. 9) in $\pi \Lambda$ and pion-photoproduction. Thus the possibility that this shape is caused by some alterations due to errors is unlikely. A two Regge-pole fit to the $\varphi(\bar{s})$ plotted in Fig. 9 gives unacceptable results; on the other hand, a three-Regge-pole interpretation is not very exciting from a phenomenological point of view. It is more natural to attribute this kind of curve to a continuous discontinuity in the $\alpha'$-plane, i.e. to a Regge cut. We have tried several shapes for the cut-discontinuity, and we have found that good fits are provided by discontinuities of exponential shape,
whose contribution to the scattering amplitude (supposed antisymmetric for simplicity) is of the form

$$ F(u) \sim i e^{-i \pi \beta \epsilon} \frac{\int_{-\infty}^{\infty} C(e^{-i \pi \alpha \epsilon} \epsilon^{-i \pi (x-\epsilon)} \nu e^{i \beta \epsilon}}{L_x (v/v_L) - i \pi \frac{1}{2}} $$

where

$$ v_1 = e^{\frac{1}{2}} $$

A Regge-cut like this is what emerges in correspondence with the vanishing of the $\rho$ contribution at $\ell \approx -15$ GeV$^2$ in the $A^{(\prime)}$ amplitude of $\pi N$ scattering. Its interference with the $\rho$ contribution to $B^{(\prime)}$ gives rise to a finite polarization in $\pi \gamma \rightarrow \pi^0 \gamma$. The prediction from CMSR for this polarization is compared with the available experimental data in Fig. 10. We must point out that Barger and Phillips have also made a similar analysis for $\pi N$ scattering and have concluded that the secondary contribution which emerges at $\ell \approx -15$ GeV$^2$ in $A^{(\prime)}$ is to be identified with a conspiring $\rho^\prime$. We list here the evidence from CMSR which support the Regge-cut against the $\rho^\prime$ interpretation.

i) At $\ell \approx -15$ GeV$^2$, i.e. where the $\rho$ contribution vanishes in $A^{(\prime)}$, the secondary contribution must clearly show up in $\phi(\ell)$, but the curve one finds there cannot be absolutely interpreted as due to one Regge pole. At $\ell \lesssim -15$ GeV$^2$ $\phi(\ell)$ should admit, in the $\rho^\prime$ interpretation, a two Regge-pole fit, one of the two poles being the $\rho$. However if one tries to force such a fit a very large value for $\alpha_\rho$ results, of the
order of .7 - 1. , which is unacceptable in this region of \( t \). As we said above, it would be unjustified to call errors on the \( \phi (f) \) into the game, because exactly the same kind of curves have been found in the analysis of pion-photoproduction.

ii) A conspiring \( \rho^\prime \) cannot give any contribution to \( A^{(1)}(f) \) at \( t=0 \). Using CMSR applied to the much more precise total cross-section data, it was possible to establish that besides the \( \rho \) another appreciable contribution is present in the Regge spectrum of \( A^{(1)}(f) \) at \( t=0 \) (see Della Selva et al., ref. 1)\(^{14} \). Indeed, making a two Regge-pole least-square fit to the \( \phi (f) \) in this case one finds besides the \( \rho \) -pole another pole with \( \alpha \approx -1 \)\(^{15} \). Its weight is \( \sim 5\% \) at 5 GeV and (consistently with \( \alpha \approx -1 \) \( \sim 10\% \) at 2 GeV. It has been possible to put it in evidence because in some intervals of \( \gamma \) its contribution to \( \phi (f) \) is enhanced by a factor \( 5 \) with respect to that of the \( \rho \), thus causing clear deviations from the pure one-Regge-pole shape. While the presence of this secondary contribution at \( t=0 \) causes great difficulties for the conspiring- \( \rho^\prime \) interpretation\(^{16} \), the result of the two-pole best-fit can easily be explained in the Regge-cut interpretation. Indeed, in the \( \alpha \) -plane the \( \rho \) -pole and the Regge-cut singularities overlap (assuming that Regge-cut results from an absorptive correction to the \( \rho \) -pole)\(^{17} \). Therefore in a two-pole fit to \( \phi (f) \) one of the poles will describe the contribution of the singularities near \( \alpha \rho \), and the other the contribution of the tail of the Regge-cut. It this tail damps exponentially for decreasing \( \alpha \)'s, the \( \alpha \)
of this second pole will result not too far from $\alpha'$. On the other hand the role of this second pole in the fit is just to fill the deviations of the $\varphi(t)$ from the first pole contribution. Therefore it must essentially describe those $\alpha$'s of the tail which mainly contribute to these deviations, that is, the ones whose maxima in their $\hat{s}_t \pi^2 (\alpha' + \delta \pi')/(\alpha + \delta \pi')$ contributions are roughly near the zeros of the first pole contribution. This makes us understand why in the best-fit the second pole chooses an $\alpha$ spaced by $\sim 1.5$ from $\alpha'$.

4) The CMSR analyses of $\pi N$ scattering and pion-photoproduction have provided much quantitative information on Regge parameters. In Table I are listed some Regge trajectories, determined from the pion-photoproduction CMSR analysis, and on which very little information previously existed. To have an idea of the quantitative reliability of the method one can look at Fig.s 3, 10, 11, where some high-energy predictions are compared with the existing experimental data.

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References


2) For references see, for instance, Chan Hong-Mo, Proceedings of the XIV Int. Conf. on Elementary Particles, Vienna (1968).


14) The result obtained by Olsson (ref. 1) in a similar analysis is not correct. For a discussion on this point see A. Della Selva, L. Masperi and R. Odorico: Nuovo Cimento 55A, 602 (1968).

15) CMSR applied to the less precise phase-shifts leads also to the same result.

16) And also to a non-conspiring- $\rho'$ interpretation, because the too low value of $\chi (\sim -1)$ would make it difficult to describe the magnitude and the energy dependence of the $\pi N$ CEX polarization. Moreover such interpretation would still not match with the $\phi(t)$ shape at $t \geq -15$.

17) This is the reason why it is difficult to make a $\rho +$ Regge-cut fit to the $\phi(t)$'s.
### Table I

<table>
<thead>
<tr>
<th>TC</th>
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<td>(+1)</td>
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Figure Captions

1) See Eq. (1) and the text.

2) See Eq. (2) and the text.

3) Experimental data on $\pi^+\pi^-$ photoproduction (Ref. 9). Solid lines are the no-free parameter predictions from CMSR applied to the Walker 11) fit (Ref. 5).

4) Plots of the $\Phi(z)'s$ corresponding to $F_2^{(-)}$ and $F_3^{(-)}$ at $\xi = 0 \ (*)$ and their one Regge-pole best-fit ($R$, at alternate values of $\gamma$) (Ref. 5).

5) Trajectories and residues of the pion and the con-spirator in $\pi^+\pi^-$ photoproduction as result from CMSR (Ref. 5).

6) Residues of $\vec{A}_2$ and $\pi_c$ in $F_1^{(-)}$ as result from CMSR (Ref. 6).

7) Residues (a) and trajectories (b) of $J$ and $J'$ as result from CMSR.

8) See the text.

9) Type of curves which appear in correspondence of the vanishing of the residues of the $\vec{p}$ in $A_{\pi NN}$, $\pi$ in $F_2^{(-)}$, $p'$ in $F_3^{(0)}$, $g$ in $F_2^{(0)}$ (Ref. 3, 7).

10) Comparison of CMSR predictions with differential cross-section and polarization data of $\pi^- p \rightarrow \pi^0 n$ (Ref. 3).

11) Comparison of CMSR predictions with $\pi^- p$ and $\pi^+ p$ elastic scattering data (Ref. 3).
Fig. 1
Fig. 3
Fig. 4
Fig. 6
Fig. 7
Fig. 11