Deformations of Conformal Field Theories to Models with Noncommutative World Sheets

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Vienna, Preprint ESI 931 (2000)  
September 1, 2000
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\textsuperscript{*}Part of project P11783-PHY of the Fonds zur Förderung der wissenschaftlichen Forschung in Österreich
Abstract

We discuss deformations of conformal field theories to bosonic models where both - the target space and the world sheet - become noncommutative spaces (in the sense of noncommutative geometry). We give an action for such models which in the commutative limit is on the classical level equivalent to the usual string action. When a special choice of noncommutative target is made and for the world sheet a kind of point particle limit is taken, one gets the bosonic part of the action of the BFSS matrix model of $M$-theory. Besides this, we briefly discuss the question of the underlying symmetries (comparable to the quantum group symmetries of usual two dimensional conformal field theories) and argue that trialgebraic deformations of Hopf algebras should appear, here. A brief look at dualities concludes the discussion.
1 Introduction

In this paper, we present the idea of models where both - the target space and the world sheet - are noncommutative spaces. There are several results showing that string theory is intimately linked to a noncommutative deformation of the target space structure (see e.g. [CF], [FG], [Kon 1997], [Pol]). The question therefore arises if the world sheet could not be deformed to a noncommutative space, as well. We will show that this is possible by explicitly giving an action for such noncommutative world sheet models which in the commutative limit is on the classical level equivalent to the usual string action. Indeed, one can give arguments that such noncommutative deformations of the world sheet should be relevant for string theory (so, our approach, here, might hopefully be more than just an abstract exercise) which we will present in the final section.

The content of the paper is as follows: In section 2, we construct the action. Besides the limit where target space and world sheet become commutative, we also consider the noncommutative case where for the world sheet a kind of point particle limit is taken. We will see that in this case one can get the bosonic terms of the action of the BFSS matrix model of $M$-theory (see [BFSS]) from a noncommutative string model. In section 3, we consider the question of what replaces the quantum group symmetries of the conformal field theory string models in the noncommutative world sheet setting. We argue that trialgebras as defined in [CrFr] - which can be understood as an aneued quantum deformation of quantum groups, see [GS] - should take this rôle. Section 4 contains a brief look at dualities. We completely restrict ourselves to the bosonic case in this paper. The final section contains some concluding remarks.

Throughout this paper, we assume that the reader is acquainted with some of the basics of noncommutative geometry (along the lines of e.g. [Con] or [Lan]).

We should mention that after completing the main part of this paper, we learned about the work of [CDP2000] where noncommutative world sheet models are discussed from a different perspective. There is also an approach to braided quantum field theory using Feynman graph methods (see [Oec]).

2 The model

We start from the case of a two dimensional world sheet $M$ (which we will assume to be given by a cylinder in the sequel) and a target space $T$ (which
is usually taken to be a ten dimensional space-time manifold). We will in this paper consider the case where both spaces carry Euclidean signature. Recall that the bosonic string action is given by

\[ S(X) \sim \int \sqrt{g} g_{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} d\sigma \]  

(1)

where the integral is taken over the world sheet, \( \eta_{\mu\nu} \) is the target space metric, \( g_{ab} \) the world sheet metric (and \( g \) the determinant of \( g_{ab} \)), and

\[ X : M \to T \]

a mapping of the world sheet into \( T \) with components \( X^\mu \). The basic idea of noncommutative geometry is to give an equivalent description of a geometric space in terms of a suitable algebra of complex valued functions on it (where addition and multiplication are taken pointwise, i.e. the algebra is automatically commutative). E.g. the Gelfand-Naimark theorem tells us that we can describe any compact Hausdorff topological space equivalently by the commutative \( C^* \)-algebra of continuous functions on it. For the case of a compact \( C^\infty \)-manifold, one can give a more refined description (including even the differentiable structure beyond the topological one) by using the algebra of \( C^\infty \)-functions (which is dense in the continuous ones and therefore trivially determines the latter, i.e. the topological structure). For special cases - where the manifold is algebraically defined - we can get rid of the compactness requirement by using the still more restrictive algebra of polynomial functions (see e.g. [KS] and the literature cited therein). We will restrict to such a setting, here, by assuming that \( M \) is given by a two dimensional cylinder and \( T \) is flat ten dimensional Euclidean space but the basic idea of our approach can be adopted to the more general setting.

Let \( \mathcal{F}(M) \) and \( \mathcal{F}(T) \) denote the algebra of polynomial functions on \( M \) and \( T \), respectively. As is well known, a map \( X \) as above determines an algebra morphism

\[ u : \mathcal{F}(T) \to \mathcal{F}(M) \]

by

\[ u(\varphi) = \varphi \circ X \]

for \( \varphi \in \mathcal{F}(T) \) and every algebra morphism form \( \mathcal{F}(T) \) to \( \mathcal{F}(M) \) uniquely determines a map \( X \) relating to it in this way. So, the information on the fields of the conformal field theory model defined by (1) can completely be stored in such algebra morphisms. It is therefore natural to search for an action which is equivalent to (1) but uses only the data of the function algebras instead of

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the geometric spaces $M$ and $T$ themselves. One can then hope that such an
action would be amenable to a generalization to the case where $\mathcal{F}(M)$ and
$\mathcal{F}(T)$ are deformed into noncommutative algebras. This is the approach we
are going to follow, here.
Let us work in a gauge fixed setting where $g_{ab}$ and $\eta_{\mu\nu}$ are just the usual flat
space Euclidean metrics, i.e. we have
\[ S(X) \sim \int \left[ (\partial_1 X^\mu)^2 + (\partial_2 X^\mu)^2 \right] d\sigma \]  
(2)
Variation of $S$ with respect to $X$ gives
\[ \partial_1^2 X^\mu + \partial_2^2 X^\mu = 0 \]  
(3)
for $\mu = 0, \ldots, 9$ as the equation of motion.
From now on let $\varphi$ denote a generator of the algebra $\mathcal{F}(T)$, i.e. $\varphi$ is one of
the coordinate functions of $T$. Since we can regard $\partial_1, \partial_2$ as derivations on the
algebra $\mathcal{F}(M)$ it makes sense to consider expressions like $\partial_1(u(\varphi))$ or $\partial_2(u(\varphi))$
for an algebra morphism $u$ as above. One calculates that
\[ \partial_i(u(\varphi)) = \nabla_\varphi \partial_i X \]  
(4)
for $i = 1, 2$ and (denoting the second derivative of $\varphi$ by $D^2 \varphi$)
\[ \partial_i^2(u(\varphi)) = \partial_i X D^2 \varphi \partial_i X + \nabla_\varphi \partial_i^2 X \]  
(5)
In equations (4) and (5), $\partial_i X$ denotes the vector with components $\partial_i X^\mu$. Since
for coordinate functions $\varphi$ of $T$
\[ D^2 \varphi = 0 \]
we get
\[ \partial_i^2(u(\varphi)) = \nabla_\varphi \partial_i^2 X \]  
(6)
Define for $i = 1, 2$ the linear operator $\hat{\partial}_i$ on the space of algebra morphisms
form $\mathcal{F}(T)$ to $\mathcal{F}(M)$ by
\[ (\hat{\partial}_i u)(\varphi) = \partial_i(u(\varphi)) \]  
(7)
for generators $\varphi$ of $\mathcal{F}(T)$. We have the following result, then:
Lemma 1 If
\[ u : \mathcal{F}(T) \to \mathcal{F}(M) \]
is an algebra morphism, linked to $X$ as above, $u$ satisfies the equation
\[ \hat{\partial}_1^2 u + \hat{\partial}_2^2 u = 0 \]
if and only if $X$ satisfies equation (3).
Proof: Use equation (6).

It is now straightforward to get the corresponding action.

Lemma 2 For the action

\[ \hat{S}(u) = \sum_\varphi \int \left[ (\partial_1(u(\varphi)))^2 + (\partial_2(u(\varphi)))^2 \right] \, d\sigma \]

where the sum is taken over the (finite) set of generators of \( \mathcal{F}(T) \) variation of \( \hat{S} \) with respect to \( u \) leads to

\[ \hat{\partial}_1^2 u + \hat{\partial}_2^2 u = 0 \]

as the equation of motion.

Though the definition of the action \( \hat{S} \) uses integration over the classical world sheet, it has a direct generalization to the noncommutative setting by remembering that we understand \( \hat{S} \) to be defined as an action which by variation with respect to algebra morphisms \( u \) leads to the equation of motion as given in Lemma 1 (which is really only dependent on data on the level of the function algebras). Let us explain this at an example.

Example: Replace \( \mathcal{F}(M) \) by the algebra \( \mathcal{A} \) of a noncommutative plane as introduced in [CDP]. (For simplicity, we use the plane instead of the cylinder, here. But there is a straightforward generalization to the - slightly more complicated - case of the noncommutative cylinder - as also introduced in [CDP],) and use instead of \( \mathcal{F}(T) \) a noncommutative deformation \( \mathcal{B} \) of the ten dimensional plane (see e.g. [KS]). One can show that there are elements \( D_1, D_2 \) of \( \mathcal{A} \) such that the derivations \( \partial_1, \partial_2 \) on \( \mathcal{A} \) are given by the commutator, i.e.

\[ \partial_i A = [D_i, A] \]

for \( A \in \mathcal{A} \). It follows that the equation of motion as given in Lemma 1 is equivalent to

\[ \forall \varphi \left[ D_1, [D_1, u(\varphi)] \right] + [D_2, [D_2, u(\varphi)]] = 0 \]  \hspace{1cm} (8)

where \( \varphi \) runs, again, over the generators of \( \mathcal{B} \). One verifies that the action

\[ \hat{S}(u) = \sum_\varphi \text{tr} \left( (u(\varphi))^{\dagger} \left( [D_1, [D_1, u(\varphi)]] + [D_2, [D_2, u(\varphi)]] \right) \right) \]  \hspace{1cm} (9)
leads to (8) as the equation of motion.

What we can learn from this example for the general case is that it is the explicit knowledge of the derivations of the noncommutative world sheet algebra which we need in order to be able to write down the action $\hat{S}$ corresponding to the equation of motion of Lemma 1. As a consequence of Lemma 1, the actions $\hat{S}$ and $S$ are classically equivalent (i.e. they lead to equivalent equations of motion).

Next, consider the case where $\mathcal{F}(T)$ is replaced by a noncommutative deformation $\mathcal{B}$ but the world sheet algebra remains commutative in a certain sense. We take a special limit of this situation where $M$ becomes one dimensional, i.e. we replace $M$ by the real line $\mathbb{R}$ (physically speaking, we consider the limit where the string becomes a point particle). Since there is only one derivation on the algebra of functions on the line (which we write, alternatively, by placing a dot over the symbol), we can rewrite the equation of motion as

$$\forall \varphi \hat{u}(\varphi) = 0$$  \hspace{1cm} (10)

Assume the ten coordinate functions of $\mathcal{F}(T)$ have in $\mathcal{B}$ been replaced by ten coordinate functions with values in $N \times N$ matrices as in the BFSS matrix model of $M$-theory (see [BFSS]).

We should explain at this point in more detail what we mean by the expression that the world sheet algebra remains commutative “in a certain sense”. Actually, we make it noncommutative by replacing the algebra on the line by the algebra of $N \times N$ matrices depending on a real parameter (which we can imagine as time). So, we replace the algebra $\mathbb{C}$ where the functions take their values in by the algebra $M_N(\mathbb{C})$ of $N \times N$ matrices. But $M_N(\mathbb{C})$ is Morita equivalent to $\mathbb{C}$. So, this is only a very mild noncommutative deformation. We still keep the dot for denoting the only derivation we consider on this algebra (which is due to the “time” parameter).

Then $u$ can be rewritten as a function $Y$ on $\mathbb{R}$ with ten matrix valued components $Y^\mu(\mu = 0, \ldots, 9)$. It follows that the action

$$\hat{S}(Y) = \sum_\mu \int tr(\dot{Y}^\mu)^2 dt$$  \hspace{1cm} (11)

leads by variation to (10) as the equation of motion. But this is precisely one of the terms in the bosonic part of the BFSS action. Remembering that - up to a constant of normalisation - the bosonic part of the BFSS action is

$$S_{BFSS}(Y) = \sum_\mu \int tr(\dot{Y}^\mu)^2 dt - \sum_{\mu,\nu} \int \frac{1}{2} tr[Y^\mu, Y^\nu]^2 dt$$  \hspace{1cm} (12)
we have the following result:

**Lemma 3** For \( \mathcal{B} \) the noncommutative algebra given by the matrix valued coordinate functions of the BFSS matrix model and the world sheet algebra taken to the point particle limit as described above, the action

\[
\tilde{S}(u) = \dot{S}(u) - \frac{1}{2} \sum_{\varphi_1, \varphi_2} \int tr [u(\varphi_1), u(\varphi_2)]^2 \, dt
\]

(where \( \dot{S} \) as defined by (11)) gives the action \( S_{BFSS} \) (here, \( \varphi_1, \varphi_2 \) both run over the generators of \( \mathcal{B} \)).

\[\square\]

It is clear that the action \( \tilde{S} \) as defined above can be introduced for any noncommutative world sheet model action \( \dot{S} \) (the trace and the integral over \( dt \) can be seen as giving together a "trace" over the whole algebra). Since

\[
\tilde{S}(u) - \dot{S}(u) \to 0
\]

in the commutative limit, \( \tilde{S} \) is in the commutative limit also classically equivalent to the usual string action \( \dot{S} \).

We should at this point shortly comment on the special world sheet algebra one has to use to derive the action \( S_{BFSS} \) from \( \dot{S} \). Blowing up the commutative algebra on the line by using Morita equivalence can be understood as blowing up the world sheet algebra enough in order to allow for morphisms \( u \) from the noncommutative algebra \( \mathcal{B} \). But this is certainly a point which should deserve additional investigation in the future.

In conclusion, we can get the bosonic part of the BFSS matrix model of \( M \)-theory as a special case of a noncommutative model where the world sheet algebra stays still largely commutative and we consider a point particle limit.

**Remark:**

The well known fact that the BFSS model introduces basically a noncommutative deformation of the target space is in accordance with the results mentioned at the beginning which show that string theory is very deeply linked to noncommutative deformations of the target space. Since these results are tied to considerations of the structure of the moduli space of string theory (see [Kon 1999]), rederiving the bosonic part of the BFSS action from a more general approach to noncommutative world sheet models suggests that these models should be linked to an extension of moduli space. This is a point we will come back to in the final section.
Though $\hat{S}$ and $\tilde{S}$ have the same commutative limit, the fact that $\hat{S}$ reproduces the BFSS action in a special case (which is known to be a candidate for a nonperturbative definition of string theory) suggests that on the quantum level $\tilde{S}$ might be the more suitable noncommutative generalization of the string action.

3 The quantum symmetries

In the case of two dimensional conformal field theory models, we know that there are quantum symmetries (i.e. symmetries not given by a group but, in this case, by a Hopf algebra) behind these. What happens if we include noncommutative deformations of the world sheet? We give an argument showing that trialgebras can appear in this case. Trialgebras were first suggested in [CrFr] and are algebraic structures with a coassociative coproduct and two associative products where all three operations are pairwise compatible (and one of the products is allowed to be partially defined, only). More precisely:

Definition 1 A trialgebra $(A, *, \Delta, \cdot)$ with * and \cdot associative products on $A$ (where * may be partially defined, only, and for reasons to be seen below, we allow for the case that \cdot is quasi-associative, only, i.e. the rebracketing of three factors involves an associator) and $\Delta$ a coassociative coproduct on $A$ is given if both $(A, *, \Delta)$ and $(A, \cdot, \Delta)$ are bialgebras and the following compatibility condition between the products is satisfied for arbitrary elements $a, b, c, d \in A$:

$$(a * b) \cdot (c * d) = (a \cdot c) * (b \cdot d)$$

whenever both sides are defined.

Trialgebras can be seen as an aneued quantum deformation of quantum groups. For the details, we refer the reader to [GS] where also simple examples of trialgebras are explicitly constructed. We do not claim at all to have results showing that trialgebras always appear in the fusion structure of noncommutative world sheet models. Such results (as given by the Doplicher-Roberts theorem for the higher dimensional or by [FK] for the conformal case) have to await future investigation. Our arguments only lead to the conclusion that trialgebras can appear in such models.

As we have seen in the previous section, the knowledge of the derivations on the world sheet algebra is decisive. The physically relevant information of a noncommutative space is not contained in the algebra alone but one has to know also the differential calculus on it. So, as a very simple toy model one could
study the case where we have still the usual commutative function algebras \( \mathcal{F}(M) \) and \( \mathcal{F}(T) \) but on \( \mathcal{F}(M) \) we use \( q \)-derivatives (see [KS]) instead of the classical ones. Since in the completely classical case the equation of motion induced from the string action \( S \) is the wave equation (or its Euclidean version), in the \( q \)-calculus setting we will get \( q \)-deformed vibrational modes, closely related to the \( q \)-oscillator (see [KS], [Wes] and the literature cited therein). Now, to arrive at the quantum group symmetries of conformal field theories, the Virasoro algebra is an important intermediate step. What happens to the Virasoro algebra in this \( q \)-calculus setting?

Remember that the Virasoro algebra is a central extension of the Witt algebra and that for the Witt algebra the operators \( L_n \) are defined as

\[
L_n = - z^{n+1} \partial_z
\]

Replacing \( \partial_z \) by the \( q \)-derivative \( \partial_q z \), defined as

\[
(\partial_q z f)(z) = \frac{f(qz) - f(z)}{qz - z}
\]

one calculates the commutators of

\[
L_{q,n} = - z^{n+1} \partial_q z
\]

to be

\[
([L_{q,n}, L_{q,m}] \varphi)(z) = q^{n+1} [[m - n]] z^{n+m+1} (\partial_{q,z} \varphi)(qz) \tag{13}
\]

for \( \varphi \in \mathcal{F}(M) \). Here, \( [[a]] = \frac{q^a - 1}{q - 1} \) is the \( q \)-number corresponding to \( a \). So, at first sight the deformed algebra does not seem to close but this problem can be solved by introducing an operator \( Q \) with

\[
(Q \varphi)(z) = \varphi(qz)
\]

since we have

\[
[L_{q,n}, L_{q,m}] = - q^n [[m - n]] L_{q,n+m} Q \tag{14}
\]

then and

\[
[L_{q,n}, Q] = (1 - q^n) L_{q,n} Q \tag{15}
\]

which can be rewritten as

\[
Q L_{q,n} = q^n L_{q,n} Q \tag{16}
\]

Obviously, \( Q \to 1 \) for \( q \to 1 \). A natural guess is that a central extension of this \( q \)-deformed Witt algebra should be given by replacing integers by \( q \)-integers in the central term of the classical Virasoro commutator, i.e. (14) should go to

\[
[L_{q,n}, L_{q,m}] = - q^n [[m - n]] L_{q,n+m} Q + \frac{c}{12} [[n - 1]] [[n]] [[n + 1]] \delta_{n,-m} \tag{17}
\]

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which is confirmed by the proofs in [CILPP] and [AS]. Equation (15) remains unchanged which is best be seen by observing that instead of (14) and (15) we could also use (14) and a version of (14) with $L_{q,n+m}$ and $Q$ in reversed order on the right hand side. Since (15) is then gained by basically taking a difference between these two equations and the central term in (17) is not dependent on the order in which $L_{q,n+m}$ and $Q$ appear, the conclusion is that there is no central term for equation (15). So, (15) and (17) together constitute a $q$-deformation of the Virasoro algebra.

This was a heuristic argument for the form a $q$-deformation of the Virasoro algebra should take. For a detailed study (including the relation to the $q$-oscillator) the reader should consult [CK] and the literature cited therein.

In the case of the classical Virasoro algebra, it is the central term which is linked to the fact that quantum groups instead of classical group symmetries arise in two dimensional conformal field theories (one can show that classical group symmetries are not compatible with a nonvanishing central charge). E.g. in the WZW model one can explicitly see the link between the quantum group deformation parameter and the central charge. But in the case of the $q$-deformed Virasoro algebra given by (15) and (17), we have already a deformation parameter $q$ present before performing the central extension, so, together with the central charge two deformation parameters should be present in a theory which incorporates this $q$-deformed Virasoro algebra. Since in the limit $q \to 1$ we get a quantum group, the symmetries of a noncommutative world sheet toy model, using $q$-calculus as above, should be given by a quantum deformation of a quantum group.

Though we used a simple $q$-calculus toy model, it is known that results gained in $q$-calculus are often directly related to ones on truely noncommutative spaces (see [Koo]). In general, one finds that a larger family of deformed Virasoro algebras occurs (see [Oda]). Detailed studies of models realizing such deformed Virasoro algebras have shown that so called elliptic quantum groups describe the fusion structure in these cases (see [Fel], [Oda]). Elliptic quantum groups are quasi Hopf algebras which are gained from a usual Hopf algebra by the operation of twisting. If we have a quasi Hopf algebra with algebra $A$, coproduct $\Delta$ and coassociator $\phi$ ($\phi = 1$ for a Hopf algebra), a twist is given by an invertible element

$$F \in A \otimes A$$

and the twisted quasi Hopf algebra has the underlying algebra $A$ and coproduct and coassociator given by

$$\bar{\Delta} = F \Delta F^{-1}$$

$$\bar{\phi} = (F^{(23)}(id \otimes \Delta)F)\phi(F^{(12)}(\Delta \otimes id)F)^{-1}$$
Lemma 4 Every twisted Hopf algebra defines a trialgebra. So, as a special case, elliptic quantum groups define trialgebras.

Proof: 
The twist can be understood as a twist in the tensor product. One can define a trialgebra from a Hopf algebra by passing to the symmetric tensor algebra over the underlying vector space of the Hopf algebra (so, the tensor product becomes the second product of the trialgebra) and extending the product and coproduct of the Hopf algebra in the usual way. The twist then leads to a deformed trialgebra which is, in general, noncocommutative and noncommutative in both products. The compatibility conditions of a trialgebra hold by the definition of a twist. 

The proof of the above lemma (using the tensor algebra construction of a trialgebra, as discussed in detail in [GS]) shows the relationship between the elliptic quantum group and the trialgebraic descriptions: They correspond to each other as does a first quantized description to one using second quantization, instead.

We should remark at this point that a general trialgebra can not be rewritten as a twisted Hopf algebra.

Why should one be interested in the trialgebraic description if in the known examples one can equivalently use the elliptic quantum group one? Conceptually, the trialgebraic description stresses the fact that we are dealing with an aneved quantum deformation. Besides this, trialgebras were suggested in [CrFr] to be linked to four dimensional topological field theories since their representation categories are so called bialgebra categories. It was proved in [CKS] that one can indeed construct four dimensional topological field theories from certain bialgebra categories. So, the trialgebraic view on the symmetries of noncommutative world sheet models points towards a link between these and four dimensional topological field theories (generalizing the known one between the conformal case and three dimensional topological field theories, see [FFFS]).

4 Dualities

Dualities play a prominent rôle in the modern treatment of conformal field theory. What happens to these on passing to a deformation to a noncommutative world sheet model? We discuss two cases in turn, mirror symmetry and T-duality.
Let us start with mirror symmetry. Since the deformation of the classical conformal field theory model is described in terms of both, a deformation of target space and a deformation of the world sheet, it follows that infinitesimally such deformations are described by cohomological data of the conformal field theory model. But mirror symmetry - in its full formulation - is a complete equivalence between two conformal field theories. But this induces an isomorphism on cohomologies. Since the conditions on infinitesimal deformations which assure that they can be integrated to full deformations are also cohomological, it follows that there is an equivalence between deformations of two conformal field theories linked by mirror symmetry. So, by the fact that it is a full equivalence of conformal field theories mirror symmetry even extends to noncommutative world sheet deformations.

Heuristically, one expects that in the same way $T$-duality extends to the setting of noncommutative world sheets. If two theories are linked by $T$-duality, there should be also a dual pairing between the deformations of these theories. Taking the simple setting of $q$-calculus, once again (and a closed string with a cylinder as target space-time as the classical model), one can see at a quite simple observation how $T$-duality will extend to this case. Denote the radius of the cylinder by $r$. Classically, $T$-duality can be seen as an exchange of vibrational and winding modes of the string under an exchange $r \leftrightarrow r^{-1}$, i.e. the energy levels of the two types of modes exchange when replacing the radius by its inverse. Now, the energy levels of the vibrational modes are a consequence of the dispersion relation of the wave equation together with the periodic boundary conditions. In the setting of $q$-calculus, the wave equation has to be replaced by

$$\partial^2_{q,x} f = \partial^2_{q,t} f$$

(18)

where

$$\partial_{q,x} f(x,t) = \frac{f(qx,t) - f(x,t)}{qx - x}$$

(19)

and, similarly,

$$\partial_{q,t} f(x,t) = \frac{f(x,qt) - f(x,t)}{qt - t}$$

(20)

Using the fact that for

$$\sin_q(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(q;q)_{2n+1}}$$

(21)

and

$$\cos_q(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(q;q)_{2n}}$$

(22)
where

\[ (a; q)_0 = 1 \]
\[ (a; q)_n = (1 - a)(1 - aq)\ldots(1 - aq^{n-1}), \quad n \geq 1 \]  

(23)

for \( a \in \mathbb{C} \), \( \sin_q(\omega x) \) and \( \cos_q(\omega x) \) are linearly independent solutions of

\[ \partial^2_{t,x} \varphi + \omega^2 (1 - q)^{-2} \varphi = 0 \]  

(24)

(see [KS]), it follows that superpositions of \( \sin_q(kx + \omega t) \) and \( \cos_q(kx + \omega t) \) are solutions of (18) and the dispersion relation is the classical one

\[ \omega^2 = k^2 \]  

(25)

Since \( \sin_q \) and \( \cos_q \) are not really periodic, we can not satisfy strict periodic boundary conditions. But for \( q \) close to 1, one still has an approximate periodicity (up to a \( q \)-controled uncertainty). We therefore can use the classical boundary conditions, too. So, as a consequence of the classical form of (25), the vibrational modes remain unchanged in the \( q \)-calculus setting.

The winding modes can be imagined as being defined by stretching the string against a constant force. So, the basic mathematical operation is an integration over a constant force term. But for a constant function \( f \), the \( q \)-integral

\[ \int_0^c f(x) d_q x = c(1 - q) \sum_{j=0}^{\infty} q^j f(q^j c) \]  

(26)

(see [KS]) does not differ from the classical integral. So, the energy levels of the winding modes remain unchanged, too, and the exchange of vibrational and winding modes under the exchange \( r \leftrightarrow r^{-1} \) persists in the \( q \)-calculus setting. So, in the special case of \( q \)-duality, the reason for \( T \)-duality to extend to this case is simply that the energy levels do not differ from the classical ones.

5 Conclusion

We have seen in this paper that one can give an action for string models where the world sheet becomes noncommutative, too, which reduces in the commutative limit to an action which is classically equivalent to the usual string action. Besides this, we have seen that the bosonic part of BFSS matrix model action of \( M \)-theory can be reproduced from a special case of our approach and have seen that trialgebras occur as the quantum symmetries of noncommutative
world sheet models, generalizing the Hopf algebra symmetries of two dimensional conformal field theories.

Hopefully such noncommutative world sheet models are not just an abstract exercise but they might be related to a still larger moduli space than the extended moduli space of string theory (as introduced in [Wit]). Even extended moduli space includes only deformations generated by part of the operator algebra of the theory. But string field theory (see [RSZ], [Zwi 1993], [Zwi 1996]) can be seen as a quantum field theory on a linearized version of a moduli space which includes all deformations by local observables. Since geometrically the conformal field theory is given in terms of maps

\[ X : M \rightarrow T \]

and the extended moduli space already seems to exhaust the possible deformations of the target space T (see [Kon 1994]), there seems to be only the possibility to look at deformations of M in order to give a geometrical interpretation of the additional deformations suggested by string field theory.

Finally, Green showed in [Gre] that string theory, as given by two dimensional quantum field theories on the world sheet, can itself be seen as a limit of a description where the world sheet itself becomes the target space of another string model. But we know from [Kon 1997] and [CF] that string theory is intimately linked to deformation quantization of the target space. So, the work of [Gre] is a strong hint at noncommutative world sheets. The approach taken here suggests especially that the BFSS matrix model may arise as a kind of point particle limit of the “world sheets for world sheets” scenario suggested by Green.

We did not deal with supersymmetry in this paper but the correspondence to the BFSS action in a limit is suggestive of how to introduce fermionic terms into the action \( \hat{S} \). Also, we restricted completely to considering noncommutative world sheet models on a purely classical level and we restricted to considering the analog of conformal field theories (i.e., we did not deal with the question of summation over world sheets). A more detailed study of the properties of noncommutative world sheet models and, especially, the question of quantization is planned future work.

Acknowledgements:
We would like to thank M. Kreuzer and S. Theisen as the organizers of the ESI workshop on “Dualities in string theory” for creating a stimulating atmosphere and some of the participants of the workshop, especially T. Karki and S.-J. Rey, for discussions on the topics involved. One of the authors (KGS) would like to thank the Erwin Schrödinger Institute for Mathematical Physics (ESI), Vienna
for hospitality and financial support. Both of us thank Mrs. I. Ackermann for her always efficient typing.

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