Lorentz violation in weak decays
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Chapter 1

Introduction

1.1 Lorentz and CPT invariance

Since the discovery of its importance to physics in 1905, Lorentz symmetry has become an essential part of all our fundamental theories of nature [1, 2, 3]. It becomes exceedingly important when the regime of validity of these theories extends to high and relativistic energies. Lorentz symmetry is therefore an integral part of the two pillars of contemporary physics: the Standard Model (SM) of particle physics and general relativity.

The recognition of the significance of Lorentz symmetry of course originates from Einsteins brilliant insights that led to a radical change in our views of space and time, summarized in his theory of special and general relativity [1, 4, 5]. Einstein realized that if the widely accepted principle of relativity was to be reconciled with the experimental fact that the speed of light is the same for any observer, then time cannot be absolute. Put differently, if Maxwell’s equations, which predict the speed of light, were to hold in any inertial reference frame, i.e. are subject to the principle of relativity, then Lorentz transformations should be used to move between reference frames, instead of Galilean transformations.

Lorentz transformations are mathematical operations that we use to relate the physical observations made in reference frames that move at different constant velocities and have different orientations, i.e. inertial reference frames. These operations are called boosts and rotations. Rotations obviously account for the angle changes between reference frames while boosts relate frames that move with differing constant velocities. Lorentz symmetry means that the laws of physics should look the same in all reference frames that can be related by a combination of such transformations.

That Lorentz transformations are the correct way of implementing velocity differences became clear only after Einsteins work and the validity of this assertion has been investigated experimentally ever since. In particular in the last two decades or so, when theories of quantum gravity sprouted renewed interest in the possibility that Lorentz symmetry might not be an exact symmetry of nature.

Strictly speaking, the Lorentz transformations we will study mostly – the restricted Lorentz transformations – are only part of the full set of possible Lorentz transformations, i.e. the full (real) Lorentz group. As a definition of a Lorentz transformation we can take that it leaves the spacetime interval $ds^2 = dx_\mu dx^\mu$ invariant (note that we use
the Einstein summation convention throughout this thesis). For a four-vector like \( x^\mu \), a Lorentz transformation is implemented by a matrix \( \Lambda^\mu_\nu \), which transforms \( x^\mu \) to a new reference frame: \( x^\mu \mapsto x'^\mu = \Lambda^\mu_\nu x^\nu \). We see that if \( ds^2 \) is to be invariant, the defining property of a Lorentz transformation becomes

\[
g_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta = g_{\alpha\beta},
\]

where \( g_{\mu\nu} \) is the Minkowski metric, for which we use the ‘mostly minus’ convention. The full Lorentz group can be split up in four disconnected components, where the distinction is based on two properties. A Lorentz transformation can be proper or improper, meaning that \( \Lambda^\mu_\nu \) has a determinant equal to one, or minus one, respectively. In addition, a Lorentz transformation can be orthochronous or non-orthochronous, which tells us that \( \Lambda^0_0 \geq 1 \), meaning orthochronous, or \( \Lambda^0_0 \leq -1 \), in which case the Lorentz transformation is non-orthochronous. The restricted Lorentz group consists of the proper orthochronous Lorentz transformations and is the only one of the four disconnected components that forms a group, since it contains the identity element and the other sets do not.

There are two special elements of the Lorentz group that are physically important and have been given their own names. They are space inversion, or the parity transformation and the time-reversal transformation. These are denoted by \( P \) and \( T \), respectively, and their action on a spacetime four-vector can be written as

\[
P : (x^0, x) \mapsto (x^0, -x) \tag{1.2}
\]

\[
T : (x^0, x) \mapsto (-x^0, x) \tag{1.3}
\]

Incidentally, these transformations can be used to relate different components of the full Lorentz group and are thus not part of the restricted Lorentz group (since they lead out of this subgroup). Together with a third important transformation in physics, \( P \) and \( T \) comprise another symmetry operation. The third transformation is called charge conjugation, denoted by \( C \). Under this operation all particles are transformed into their corresponding antiparticles and vice versa. Nature is believed to be invariant under the combined operation of \( C, P \) and \( T \). Besides Lorentz symmetry, this CPT symmetry is another important fundamental symmetry in particle physics.

That CPT symmetry is an exceedingly fundamental symmetry in particle physics can be understood because of its tight relationship to Lorentz symmetry \([6]\). An important result in this context is the CPT theorem \([7]\). This theorem states that any Lorentz-symmetric field theory will, under some mild assumptions, be CPT invariant. In other words, for very general relativistic field theories, Lorentz symmetry implies CPT symmetry. However, the converse is not true, \( i.e. \) CPT symmetry does not imply Lorentz symmetry. In 2002 O.W. Greenberg proved the anti-CPT theorem \([33]\), which states that for any local interacting field theory CPT violation implies Lorentz violation. This means that we cannot have CPT violation without Lorentz violation, but violation of Lorentz invariance while preserving CPT symmetry is a possibility.

### 1.2 Lorentz violation and observables

Our presently most fundamental understanding of nature consists of two separate parts. On the one hand there is the Standard Model (SM) of particle physics. It describes all the
known elementary particles and three of the four ways they can interact. On the other hand we have general relativity, which describes the fourth force, gravity. Both theories withstood the scrutiny of many experimental tests and some features of in particular the Standard Model have been tested up to astonishing precision. However, there is also convincing evidence that the Standard Model is not complete, e.g. neutrino masses, dark matter and the need for more CP violation. In addition, many theoretical physicists believe that the Standard Model of particle physics and general relativity should be unified in one theory of everything, which describes even the most extreme conditions in our universe, such as black holes and the big bang. There are many attempts to accomplish such a unification, such as string theory, loop quantum gravity, noncommutative geometry and spin-foam models. However, non of these attempts are completely satisfactory as of yet. A major problem in this respect is the energy scale at which such theories become relevant. At energies that are presently attainable in the laboratory, it is perfectly fine to use general relativity and the Standard Model of particle physics as two separate theories, both applying to their own scales. Only at energies comparable to the Planck mass ($M_P = 10^{19}$ GeV) are quantum-gravity effects expected to become apparent. Since these kinds of energies are far out of reach of present-day experiments, it is hard to test or distinguish different theories of quantum gravity.

The breakdown of Lorentz and/or CPT symmetry might be a way to remedy this problem. It is known that many theories of quantum gravity predict the possibility of some kind of Lorentz violation [8, 9, 10, 11], even though none of them predict Lorentz-symmetry breaking conclusively. Over the last few decades it has become clear that it is possible to constrain such a breaking of symmetries with enormous precision. With such a precision, in fact, that some have coined the relevant effects ‘windows on quantum gravity’.

An obvious reason as to why we can constrain the breakdown of Lorentz and CPT symmetry this precise, is the distinct experimental signals it produces. Lorentz and CPT symmetry are build into all our theories at a fundamental level, and the background of the signals their breakdown will produce is minimal in many cases.

Since we do not know the fundamental theory that unifies general relativity and the standard model of particle physics and we also do not know if and how it breaks Lorentz symmetry, we can consider Lorentz violation in any sector of physics and try to constrain it. This produces a broad range of possible signals that are potentially experimentally testable. Examples of which are sidereal variations, vacuum birefringence, kinematically forbidden reactions, anomalous boost dependencies, and many others. To systematically investigate all the possibilities and to catalogue the experimental results, an effective-field-theory approach to Lorentz violation was developed in 1997 and 1998 [12, 13, 14]. This general framework was called the Standard-Model Extension (SME). It contains many coefficients that parametrize the breakdown of Lorentz and CPT symmetry and many experimental constraints have been placed on these coefficients [15].

When we look at all the work that has been done in the context of constraining Lorentz-symmetry breaking, we notice two things. First of all, not a lot of work has been done in the context of the weak interaction, an interaction that is famous for its violation of parity. Secondly, the study of cross sections and decay rates with Lorentz violation has not been as extensive as one might like. We can point to a few reasons for these
gaps in the theoretical and experimental studies of the SME. Most importantly, the QED limit of the SME has received much attention than other sectors, since the conventional Lorentz-symmetric version of QED is understood better and in many cases much higher precision can be achieved in experiments with only electrons, positrons, or photons. This kind of precision can typically not be achieved in measurements of cross sections or decay rates, which is one of the reasons that they have not received that much attention in the literature. Exceptions are decays that are kinematically forbidden in the Lorentz-symmetric case, but become possible with Lorentz violation (since Lorentz violation can change the kinematics of the process). However, in most cases it is then not needed to do a full calculation of the decay rate, since one can rely on a purely kinematical approach. Nevertheless, as will become clear in this thesis, it is worthwhile to look at the weak interaction and cross sections and decay rates. First of all, if we want to fully test Lorentz invariance, we should also look closely at the weak interaction. Secondly, even though one might not be able to achieve very high precision for cross sections and decay rates, there are coefficients in the SME that are still best approached through measurement of such processes.

1.3 Outline of the thesis

In this thesis we study Lorentz violation in the context of the Standard-Model Extension (SME) [12, 13, 14]. We focus especially on the weak interaction and decay rates. We calculate a set of different weak-decay processes, with different Lorentz-violating coefficients, taken from the SME. In particular, we develop a general framework to approach Lorentz violation in processes involving a $W$ boson.

The thesis is organized in the following way. In Chapter 2 we introduce the Standard-Model Extension. We first motivate its use by pointing to an argument presented by Kostelecký. This argument makes clear that the SME is the only available approach to Lorentz violation available, that is coordinate independent, general, and incorporates all the known and established physics. After that we make the distinction between observer Lorentz invariance and particle Lorentz invariance and point out that the SME only violates the latter of the two. We then proceed to introduce the Lagrangian of the minimal SME (minimal means power-counting renormalizable in this context) and briefly discuss some of the properties of the coefficients. In the last two sections of Chapter 2 we develop the basic quantum field theory for a scalar and a Dirac field. In doing this we highlight some stability and causality issues in the theory, which necessitates the concept of a concordant frame.

In Chapter 3 we address the issue of calculating cross sections and decay rates in the SME. We start by introducing the $S$-matrix and the general formula for a cross section in terms of $S$-matrix elements. We point out some differences with the conventional case. We then turn to the LSZ reduction formula, which relates $S$-matrix elements to correlation functions and Feynman diagrams. In this context we discuss a recent paper, which pointed out some unusual features of the LSZ reduction formalism in the context of Lorentz violation [16]. We continue with a section on decay rates, which is a delicate issue even in the Lorentz-symmetric case, since one cannot discuss unstable particles in
terms of asymptotic states. We will discuss these issues using the optical theorem and the largest-time equation [17].

In Chapter 4 we will introduce the $\chi^{\mu\nu}$ framework. This entails adding a general Lorentz-violating tensor to the $W$-boson propagator, in terms of which many observables can be defined. We discuss what kind of coefficients can contribute to $\chi^{\mu\nu}$ and calculate the general $\beta$-decay rate of a nucleus that decays through allowed $\beta$ decay. Using our result for the allowed $\beta$-decay rate, an experiment was designed that studied the decay of $^{20}\text{Na}$ [18]. In this experiment, developed and performed at the Kernfysisch Versneller Instituut (KVI) in Groningen (the Netherlands), the decay rate of $^{20}\text{Na}$, dependent on the absolute polarization direction, is measured. Using data from this experiment bounds on $\chi^{\mu\nu}$ were derived.

Chapter 5 is dedicated to forbidden $\beta$ decay with Lorentz violation in terms of $\chi^{\mu\nu}$. We calculate the general Lorentz-violating decay rate of nuclei. The transitions can have any degree of forbiddenness. We identify some cases where the Lorentz violation can be enhanced. This enhancement hinges on the fact that the forbiddenness of a transition is usually dictated by angular-momentum conservation. Since rotational invariance is broken by most components of $\chi^{\mu\nu}$, angular momentum conservation and the Lorentz-violating contribution to the transition rate might be less suppressed by the conventional mechanism than the Lorentz-symmetric part. By enhancement we mean that if the dimensionless Lorentz-violating coefficients were of order 1, then they would constitute the dominant contribution to the decay rate. Using the general formula for the decay rate we will derive the first strict direct bounds on $\chi^{\mu\nu}$.

In Chapter 6 we will discuss two standard weak-decay processes: pion and muon decay. We will calculate the decay rates for a set of Lorentz-violating coefficients, both CPT odd and CPT even. We give some interesting observables for future experiments and derive bounds on the discussed coefficients from existing experimental data. We also identity a data set pertaining the total muon-decay rate, that is not compatible with Lorentz symmetry. This concerns the total muon-decay rate, measured at different boost factors.

We finish the thesis with an overview and conclusions in Chapter 7. Some technical issues and a few formulas that are convenient for future calculations are collected in the appendices.
Chapter 2

The Standard-Model Extension

In this chapter we introduce the framework that is used in most of the literature to study Lorentz-symmetry breaking in a general way: the Standard-Model Extension (SME). This framework is based on an effective-field-theory approach and incorporates all the known physics of the Standard Model of particle physics. Except for Lorentz and CPT symmetry, it aims to keep all the conventional features of the Standard Model, such as gauge invariance and the Higgs mechanism.

First we motivate why the SME is a good theoretical approach to study Lorentz-symmetry breaking. After that we briefly discuss the important conceptual difference between what are usually called observer Lorentz transformations and particle Lorentz transformations. We then describe (part of) the SME Lagrangian and in particular the properties of the coefficients that parametrize the Lorentz violation. Finally we develop the quantum-field-theory basics that follow from the SME Lagrangian for a scalar field and a Dirac field.

In this thesis we restrict ourselves to the extension of the Standard Model of particle physics, without including gravity. Although an approach similar to the one described in the following sections has been developed for gravity and matter-gravity couplings [19], it lies outside the scope of this thesis.

Large parts of this chapter describe work which was not originally done by the author. They rather represent a selection of the results described in Refs. [12, 13, 14, 20, 21]. We include a summary of these results in this thesis, because of their importance for what follows.

2.1 Motivation

Throughout this thesis we will use the Standard-Model Extension (SME) [12, 13, 14] to study the breaking of Lorentz invariance theoretically. The SME is an effective-field-theory approach to Lorentz-symmetry breaking and it is obtained by adding to the Standard-Model Lagrangian all Lorentz-violating terms one can build (as we describe in Section 2.3) out of the usual Standard-Model fields coupled to coefficients that parametrize the Lorentz violation. These terms violate Lorentz and in some cases CPT symmetry but are required to obey all the other symmetries of the Standard Model.
Its generality is one of the merits of the SME. It gives a systematic parametrization of all the possible Lorentz-violating effects and provides a set of parameters that can be bounded by experiment. It is therefore very well suited for phenomenological analyses and many experiments have been performed to put bounds on the Lorentz-violating parameters in the SME [15].

There are also other approaches to study Lorentz-symmetry breaking. Two of the most important alternative approaches are modified dispersion relations and modified Lorentz transformations. The former is a phenomenological approach that is useful in studying kinematical properties of particles by modifying the usual relation between energy and momentum:

\[
(p^0)^2 - p^2 = m^2 + \delta f(p^0, p),
\]

where \(\delta f(p^0, p)\) controls the Lorentz violation. In most cases it is used in threshold analyses of decay processes or to study particle velocities. Modified dispersion relations also occur in the SME, however, not every ad hoc postulated form of a dispersion relation can originate from the SME Lagrangian [22]. Modified Lorentz transformations, on the other hand, are an entirely different approach to Lorentz violation. Here the Lorentz-symmetry breaking is not added perturbatively to the Lagrangian, but the Lorentz transformations themselves are changed or replaced (see e.g Ref. [23]).

To motivate the use of the SME, instead of some of the other approaches to Lorentz violation, we recapitulate an argument presented by Kostelecký in Ref. [24]. He argues that an approach to investigate the breakdown of Lorentz symmetry should be

- Coordinate independent,
- General,
- Realistic.

Implementing modified dispersion relations or modified Lorentz transformations is troublesome at best, if we want to stick to these conditions. In what follows, we will discuss the conditions one by one.

First of all we have the condition of coordinate independence. The way we describe physics should be coordinate independent. This is actually quite obvious. An experiment has a definite outcome, and two observers that are smart enough and have enough knowledge about the laws of physics, should reach the same conclusions. In other words, with or without Lorentz violation, nature should not care about the observer and the coordinates he/she uses to describe a physical process. In the context of the SME, this will give rise to what is called observer Lorentz invariance, which is explained more extensively in Section 2.2. When building the SME, one demands that all the terms in the Lagrangian are invariant under observer Lorentz transformations, which translates into the fact that all Lorentz indices should be contracted. Some models in the literature are not observer Lorentz invariant. Some are restricted to a particular frame such that the expressions look simpler. For example, sometimes a model with isotropic Lorentz violation (i.e. leaving rotational invariance intact) is used. However, this can be true only in a particular set of frames that are related through rotations. In frames boosted with respect to the frame
with isotropic Lorentz violation, the symmetry breaking will also extend to the breaking of rotational invariance.

Secondly, we have the requirement of generality. To date, no convincing evidence for Lorentz violation has been found. Also no definite model for quantum gravity (that incorporates Lorentz violation) has been satisfyingly confirmed by experimental results. These two facts entail that we do not know if Lorentz violation exists and neither can we point to a preferred sector to look for it. This means that one should perform broad searches, to be sure not to miss anything. Only when Lorentz-symmetry breaking is found, should one start to think about model building to explain the observed effect in detail. This is why we say that the theory should be general. We mean that it should (in principle) include all Lorentz-violating effects that one can think of. Arguably the best way to realize this wide field of view, is an effective-field-theory approach. It allows for a systematic approach to building all the Lorentz-violating terms in the Lagrangian that obey a certain set of symmetries or other restrictions (e.g. gauge symmetry and mass dimension of the Lagrangian density). As far as one believes that (at the relevant energies) nature can be described by a quantum field theory, one is certain to include all the possible Lorentz-violating effects. These will for example include dynamical effects as well as kinematical effects. In this context, we mention that modified dispersion relations are good for doing analyses of kinematical properties of particles, such as velocities or reaction thresholds. However, to study, for example, decay rates or cross sections, one should take into account that also the dynamics of the particles can be influenced by Lorentz violation. In fact, if one desires that the dispersion relation originates from an action (which is certainly not the case for all modified dispersion relations [22]) and one demands Standard-Model-gauge invariance, the same parameters that modify the dispersion relation also modify the dynamics and this fact should be included in a calculation of a dynamical process.

The final argument in favor of using the SME is realism. This basically means that the approach we use should include all the well-established physical laws we have today. We know that the Standard Model of particle physics is correct up to very high precision. Therefore, the SME is build on the Standard-Model Lagrangian and keeps many of its usual features, such as the $SU(3) \times SU(2) \times U(1)$ gauge group, the Higgs mechanism and energy and momentum conservation. In particular theories with modified Lorentz transformations have trouble complying with this condition of realism. It is awkward at best to produce the Standard Model when one uses a different set of Lorentz transformations.

The arguments above should make clear why we consider the SME to be the preferred approach to study the breakdown of Lorentz invariance. That this view is shared by many others is attested by the large body of theoretical and experimental work there has been done in the context of the SME. The results of this work, in the form of limits on SME coefficients, are collected in a set of data tables [15], that is updated on a yearly basis. We will refer to Ref. [15] many times in this thesis, since in contains the most complete and up to date overview of the status of tests of Lorentz symmetry.
2.2 Observer and particle Lorentz invariance

In the previous section we mentioned briefly that the way we describe physics should be independent of coordinate changes. One of the consequences of this is that the SME Lagrangian, and therefore all physical observables we can derive from it, is observer Lorentz invariant. This is what we mean by observer Lorentz transformations: they are just coordinate changes. Observer Lorentz invariance thus follows from the requirement of coordinate independence, which is more general than Lorentz symmetry.

Observer Lorentz invariance should be contrasted with particle Lorentz invariance. When we say that we study the breakdown of Lorentz invariance in this thesis, we actually mean that we study the breakdown of particle Lorentz invariance. In the SME Lagrangian the breakdown of Lorentz invariance is introduced through tensor coefficients coupled to operators build out of Standard-Model fields. An example of this is

\[ \mathcal{L} \supset i c_{\mu\nu} \bar{\psi} \gamma^\mu \partial^\nu \psi, \tag{2.2} \]

where \( c_{\mu\nu} \) is the Lorentz-violating coefficient. All Lorentz-violating coefficients in the SME can be viewed as background fields with constant values. When applying an observer Lorentz transformation, one transforms all quantities, including the background fields. In the case of the example in Eq. (2.2) we get

\[ i c_{\mu\nu} \bar{\psi} \gamma^\mu \partial^\nu \psi \rightarrow \Lambda^\rho_\mu \Lambda^\sigma_\nu L^\rho_\lambda \Lambda^\nu_\kappa i c_{\rho\sigma} \bar{\psi} \gamma^\lambda \partial^\kappa \psi = i c_{\mu\nu} \bar{\psi} \gamma^\mu \partial^\nu \psi, \tag{2.3} \]

where \( \Lambda^\rho_\mu \) is a Lorentz transformation, for which Eq. (1.1) holds by definition. Observer Lorentz invariance thus implies that all Lorentz indices are contracted. Doing a particle Lorentz transformation, however, transforms all quantities, except the background fields. A particle Lorentz transformation thus gives us

\[ i c_{\mu\nu} \bar{\psi} \gamma^\mu \partial^\nu \psi \rightarrow \Lambda^\rho_\mu \Lambda^\nu_\nu i c_{\rho\nu} \bar{\psi} \gamma^\nu \partial^\nu \psi. \tag{2.4} \]

This corresponds to physically boosting or rotating the experiment with respect to the background. This happens, for example, when the Earth moves through space and the experiment moves with it. As seen from the laboratory frame, the values of \( c_{\mu\nu} \) will change as the Earth changes its orientation. These values will thus oscillate with the rotation frequency of the Earth (or twice that frequency).

Because a priori there is no preferred frame (except perhaps that defined by the cosmic microwave background) and values of coefficients differ from frame to frame, we have to choose a standard reference frame. In this frame we then compare bounds on Lorentz-violating parameters from different experiments. The customary choice is the Sun-centered inertial reference frame [15, 25] (see Appendix A). An observable that is measured in the laboratory frame will depend on Lorentz-violating coefficients defined in the lab frame. We thus have to express these Lorentz-violating coefficients, which are measured in the laboratory frame (denoted here by \( c_{\mu\nu} \)), in terms of Lorentz-violating coefficients in the Sun-centered inertial reference frame (denoted here by \( C_{\mu\nu} \)). This is accomplished by

\[ c_{\mu\nu} = R^\mu_\rho R^\nu_\sigma C^{\rho\sigma}, \tag{2.5} \]
with $R$ given by

$$R(\zeta, t) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \zeta \cos \Omega t & \cos \zeta \sin \Omega t & -\sin \zeta \\
0 & -\sin \Omega t & \cos \Omega t & 0 \\
0 & \sin \zeta \cos \Omega t & \sin \zeta \sin \Omega t & \cos \zeta
\end{pmatrix}, \tag{2.6}$$

where $\zeta$ is the colatitude of the experiment and $\Omega \simeq 2\pi/(23\text{h} 56\text{m})$ is the Earth’s sidereal rotation frequency. This form of $R$ is correct if the laboratory coordinate axes ($\hat{x}, \hat{y}, \hat{z}$) are defined such that $\hat{z}$ is perpendicular to the Earth’s surface, $\hat{x}$ points in the north-south direction, and the laboratory $\hat{y}$-direction completes the right-handed coordinate system by pointing from west to east. Further details on the Sun-centered inertial reference frame are given in Appendix A.

It is easy to see that observables that depend on the vector-like quantities such as $c_{0j}$ and $c_{j0}$, with $j$ a space-like index, will oscillate with a frequency equal to the rotation frequency of the Earth, while quantities depending on $c_{jk}$ in addition have parts that oscillate with twice this frequency. These observables will in general look like $O_1 = c_{0j} \hat{p}^j$ or $O_2 = c_{jk} \hat{p}^j \hat{p}^k$, where $\hat{p}$ is some vector associated with the experiment, e.g. the momentum or spin of a particle. The way the observables oscillate depends on the direction of this vector with respect to the Earth and its axis. In Fig. 2.1 the behavior of $O_1$ and $O_2$ is depicted for three examples of orientations of $\hat{p}$. In the examples we used $C^{0X} = C^{XX} = C^{YY} = C^{ZZ} = 0.1$, $C^{0Y} = C^{XY} = C^{YX} = C^{YZ} = C^{ZY} = 0.2$, and $C^{0Z} = C^{XZ} = C^{ZX} = 0.3$, while we took $\zeta = 41^\circ$ for the colatitude. For both $O_1$ and $O_2$ we see that an orientation of $\hat{p}$, parallel to the rotation axis of Earth results in no oscillation. When $\hat{p}$ is perpendicular to the rotation axis, e.g. points east, $O_1$ will oscillate around zero, while $O_2$ will still have a nonvanishing offset. An observable with a vanishing offset can be advantageous in view of systematic effects. One is then sure that such an offset does not originate from Lorentz violation.

The Lorentz transformation $R$ in Eq. (2.6) contains no boosts. In principle the Earth’s velocity when going around the Sun, or the velocity of the laboratory due to the rotation of the Earth are nonzero. However, these effects are negligible in most cases (see Appendix A). This approximate boost-independence of $R$ means that isotropic coefficients like $c^{00}$ are not influenced by $R$. These parameters are thus only accessible if one measures observables involving velocities high enough for the relativistic gamma factor to become significant ($R$ then no longer applies) or if one compares measurements at different energies of the particles involved.

Finally, we observe that since the Lorentz-violating coefficients transform under the Lorentz group, their size is in principle unbounded. A parameter with two indices, for example, will roughly grow with the relativistic gamma factor squared. This means that the coefficients cannot be treated perturbatively in every observer frame. It is an experimental fact, however, that if Lorentz-violation exists, the associated parameters must be tiny in frames where experiments have been performed, since no Lorentz-violation has been detected. The set of frames where the size of the coefficients is indeed small, such that they can be treated as a perturbation, are called concordant frames. Since no Earth-bound experiments have detected Lorentz violation, the Earth must be in one of these concordant frames.
2.3 The SME Lagrangian

The idea of the SME is that it is a low-energy effective description of (particle) physics that incorporates Lorentz violation. The physics at high energies, on the other hand, is presumed to be described by a theory that unifies the Standard Model of particle physics and General Relativity. In many candidate theories for such a high-energy description of nature, mechanisms have been found that cause Lorentz symmetry to be spontaneously broken at some (high) energy scale. In principle the SME allows for the possibility of explicit breaking of Lorentz symmetry. However this has been shown to be incompatible with Riemannian geometry [26]. When the breaking occurs spontaneously, tensor fields get vacuum expectation values, which causes Lorentz symmetry to be broken. Since the high-energy theory is assumed to be dynamically Lorentz invariant, these tensor fields are coupled to Standard-Model fields\(^1\) in a way that leaves no uncontracted spacetime indices.

The Lagrangian of the Standard-Model Extension is thus built out of the conventional Standard-Model fields, together with coefficients that control the Lorentz violation, which we can view as originating from vacuum expectation values of Lorentz tensors in an underlying fundamental theory. We saw an example of this in Eq. (2.2). Being an effective field theory in four spacetime dimensions, all the terms in the corresponding Lagrangian are restricted to have mass dimension four, making the action dimensionless. Any term with a combination of fields and derivatives with a mass dimension greater than four, must therefore be suppressed by negative powers of some mass scale \(M\) that is large compared to the scale \(m\) of the effective field theory. In some scenarios, such as Lorentz-symmetry breaking with its origin in string theory, this large mass scale \(M\) can be expected to be the Planck mass.

Considering these points, a typical Lorentz-violating term looks like

\[
\mathcal{L} \supset \frac{\lambda}{M^k} \langle T \rangle \cdot \bar{\psi} \Gamma(k) \psi + \text{H.c.} \quad \text{,} \tag{2.7}
\]

\(^1\)And possibly fields that describe other new physics, but we do not consider that here.
where $\lambda$ is a dimensionless coupling constant, $\Gamma$ is some $\gamma$-matrix structure and we suppressed the Lorentz indices for simplicity. It should be kept in mind, however, that all Lorentz indices are fully contracted to guarantee observer Lorentz invariance (even if the Lorentz-symmetry breaking is not assumed to be spontaneous). Furthermore, $\langle T \rangle$ is the vacuum expectation value of a Lorentz tensor, whose expected dependence on different mass scales can be determined from $k$ and the condition that its value should be small to comply with experimental constraints. For $k = 0$, for example, we expect $\langle T \rangle \propto m^2/M$.

When building the Lagrangian of the Standard-Model Extension, the terms that we can add to the Standard-Model Lagrangian are required to be singlets under the $SU(3) \times SU(2) \times U(1)$ gauge group of the Standard Model. When we limit ourselves to power-counting renormalizable terms, i.e. operators that have mass dimension four or less, the resulting model is called the minimal Standard-Model Extension (mSME). It is the mSME Lagrangian that was formulated in the original papers on the SME [12, 13]. There is also an extensive body of work on nonminimal terms in the SME [27, 28, 29, 30], but in the next part we only describe the mSME for later reference.

We use the same conventions and notations as in Ref. [13], where the full mSME Lagrangian was first given. The Lagrangian is build out of the conventional Standard-Model fields, which we denote by

\begin{align}
L_A &= \left( \begin{array}{c}
\nu_A \\
l_A
\end{array} \right)_L, \quad R_A = (l_A)_R, \\
Q_A &= \left( \begin{array}{c}
u_A \\
d_A
\end{array} \right)_L, \quad U_A = (u_A)_R, \quad D_A = (d_A)_R,
\end{align}

where $L_A$ is the conventional left-handed lepton doublet with $A$ running over particle generation, $R_A$ is the right-handed lepton singlet. Similarly, $Q_A$ is the left-handed quark doublet and $U_A$ and $D_A$ are the right-handed singlets. We define left-handed and right-handed fields as usual by

\begin{align}
\psi_L &= \frac{1}{2} \left(1 - \gamma^5\right) \psi, \quad \psi_R = \frac{1}{2} \left(1 + \gamma^5\right) \psi.
\end{align}

We first give the Lorentz-invariant Standard-Model Lagrangian for reference. The lepton and quark parts are given by

\begin{align}
L_{\text{lepton}} &= i \tilde{L}_A \gamma^\mu D_\mu L_A + i \tilde{R}_A \gamma^\mu D_\mu R_A, \\
L_{\text{quark}} &= i \tilde{Q}_A \gamma^\mu D_\mu Q_A + i \tilde{U}_A \gamma^\mu D_\mu U_A + i \tilde{D}_A \gamma^\mu D_\mu D_A,
\end{align}

where $D_\mu$ denotes the conventional covariant derivative. The Yukawa terms that generate the fermion masses are given by

\begin{equation}
L_{\text{Yukawa}} = - \left[ (G_L)_{AB} \tilde{L}_A \phi R_B + (G_U)_{AB} \tilde{Q}_A \phi U_B + (G_D)_{AB} \tilde{Q}_A \phi D_B + \text{H.c.} \right],
\end{equation}

where $(G_L)_{AB}$, $(G_U)_{AB}$, and $(G_D)_{AB}$ are general complex-valued matrices, that are not necessarily symmetric or Hermitian; $\phi$ is the Higgs doublet with its conjugate $\phi^\dagger$. The Higgs Lagrangian, including the usual Higgs potential, is

\begin{equation}
L_{\text{Higgs}} = (D_\mu \phi)^\dagger D^\mu \phi + \mu^2 \phi^\dagger \phi - \frac{\lambda}{3!} (\phi^\dagger \phi)^2.
\end{equation}
Finally, we give the gauge parts of the SM Lagrangian

\[ \mathcal{L}_{\text{gauge}} = -\frac{1}{2} \text{Tr}(G_{\mu\nu} G^{\mu\nu}) - \frac{1}{2} \text{Tr}(W_{\mu\nu} W^{\mu\nu}) - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}, \]  

(2.10e)

where \( G_{\mu\nu}, W_{\mu\nu} \) and \( B_{\mu\nu} \) are the \( SU(3), SU(2) \) and \( U(1) \) gauge-field strengths, respectively. We omit any \( \theta \) terms, since we will not consider them in this thesis.

We now turn to the Lorentz-violating part of the mSME Lagrangian. The lepton and quark parts are given by

\[ \mathcal{L}_{\text{LV lepton}} = \bar{L}_A \left[ i(c_L)_{\mu\nu} A^\mu D^\nu - (a_L)_{\mu\nu} A^\mu \right] L_B + \bar{R}_A \left[ i(c_R)_{\mu\nu} A^\mu D^\nu - (a_R)_{\mu\nu} A^\mu \right] R_B , \]  

(2.11a)

\[ \mathcal{L}_{\text{LV quark}} = \bar{Q}_A \left[ i(c_Q)_{\mu\nu} A^\mu D^\nu - (a_Q)_{\mu\nu} A^\mu \right] Q_B + \bar{U}_A \left[ i(c_U)_{\mu\nu} A^\mu D^\nu - (a_U)_{\mu\nu} A^\mu \right] U_B + \bar{D}_A \left[ i(c_D)_{\mu\nu} D^\mu A^\nu - (a_D)_{\mu\nu} A^\mu \right] D_B , \]  

(2.11b)

where all the Lorentz-violating parameters are Hermitian matrices in generation space. The \( a \) parameters are CPT odd and have mass dimension one, while the \( c \) parameters are CPT even and dimensionless. The \( c \) parameters can be assumed to be traceless over their spacetime indices (i.e. \( g_{\mu\nu} c^{\mu\nu} = 0 \)). Any trace part is not Lorentz violating and can be absorbed by a field normalization.

The Yukawa couplings with Lorentz violation are

\[ \mathcal{L}_{\text{Yukawa}}^{\text{LV}} = -\frac{1}{2} (H_L)_{\mu\nu} A_{\mu\nu} \phi^\mu D^\nu R_B + \frac{1}{2} (H_U)_{\mu\nu} A_{\mu\nu} Q^\mu D^\nu U_B + \frac{1}{2} (H_D)_{\mu\nu} A_{\mu\nu} D^\mu Q^\nu U_B + \frac{1}{2} \bar{Q} A_{\mu\nu} Q^\mu D^\nu + \text{H.c.} , \]  

(2.11c)

with \( \sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu] \). This is the only \( \gamma \)-matrix structure that gives Lorentz violation for such Yukawa terms of mass dimension smaller than, or equal to, four. As the matrices in Eq. (2.10c), also the matrices here do not have to be symmetric or Hermitian in generation space. All the terms are CPT even and the coefficients have mass dimension one.

For the Higgs part of the Lagrangian the following terms can be written:

\[ \mathcal{L}_{\text{Higgs}}^{\text{LV}} = \left[ \frac{1}{2} (k_{\phi})^{\mu\nu} (D_{\mu} \phi)^\dagger D_{\nu} \phi + \text{H.c.} \right] - \frac{1}{2} (k_{\phi B})^{\mu\nu} \phi^\dagger \phi B_{\mu\nu} - \frac{1}{2} (k_{\phi W})^{\mu\nu} \phi^\dagger W_{\mu\nu} \phi + [i(k_{\phi})^{\mu\nu} \phi^\dagger D_{\mu} \phi + \text{H.c.}] . \]  

(2.11d)

Of the parameters in this part of the Lagrangian, only \( k_{\phi} \) is CPT odd, the others are CPT even. This \( k_{\phi} \) parameter is a set of arbitrary complex numbers and it is also the only parameter in Eq. (2.11d) of mass dimension one, the others are dimensionless. Furthermore, from the requirement of Hermiticity, it follows that \( k_{\phi}^{\mu\nu} \) can have a symmetric and real part and a antisymmetric and imaginary part. The other CPT even parameters must be real and antisymmetric.
The final part of the mSME Lagrangian is formed by the Lorentz-violating gauge terms. They are given by

\[ L_{\text{LV gauge}} = -\frac{1}{2}(k_G)_{\kappa\lambda\mu\nu} \text{Tr}(G^{\kappa\lambda} G^{\mu\nu}) - \frac{1}{2}(k_W)_{\kappa\lambda\mu\nu} \text{Tr}(W^{\kappa\lambda} W^{\mu\nu}) - \frac{1}{4}(k_B)_{\kappa\lambda\mu\nu} B^{\kappa\lambda} B^{\mu\nu}. \]  

All the parameters here are dimensionless and real. They also have a vanishing double trace (since it does not violate Lorentz symmetry) and must obey the same symmetries as the Riemann tensor, \textit{i.e.}

\[ (k_W)_{\kappa\lambda\mu\nu} = -(k_W)_{\lambda\kappa\mu\nu} = -(k_W)_{\kappa\lambda\nu\mu} = (k_W)^{\mu\nu\kappa\lambda}. \]  

There are also CPT odd Lorentz-violating gauge terms [13], but we omit them here, since they generate instabilities in the theory (also loop corrections do not seem to reintroduce these coefficients, at least not to one loop order).

It should be kept in mind that not all parameters in the mSME are observable in all physical processes. This can be seen at the level of the Lagrangian by the fact that some parameters can be removed from the Lagrangian by field redefinitions (mostly when restricting to some part of the Lagrangian that is relevant for the physical problem at hand). Since the physics should be invariant under such field redefinitions, the removable parameters cannot appear in physical observables. A extensive treatment of this issue can be found in Ref. [31]. We will also use of field and coordinate redefinitions when we treat quark parameters in pion decay in Section 6.4. In that case we do not remove the parameters but use the redefinitions to move them from one sector to another in order to facilitate a tractable calculation.

Finally, we mention that there exists a very useful QED limit of the SME. It is described in Ref. [13] and it only contains electrons, positrons and photons and their (Lorentz-violating) interactions. This model has been used very successfully to obtain very tight constraints on Lorentz-symmetry breaking. We do not discuss it here, since, in this thesis, we focus on the weak interaction for which much less stringent bounds exist.

### 2.4 QFT basics of a scalar field with Lorentz violation

To develop the basics of quantum field theory in the context of Lorentz violation, we start by treating the case of a scalar field. We do this to highlight the most important differences with the Lorentz-symmetric treatment for this simple case. In the next section we will deal with the more complicated Dirac field.

The most general Lorentz-violating quadratic Lagrangian for a complex scalar field with operators of mass dimension four or less, is

\[ \mathcal{L} = (\partial_{\mu} \phi)(\partial^{\mu} \phi^\dagger) + 2K_{\mu\nu}(\partial^{\mu} \phi)(\partial^{\nu} \phi^\dagger) - m^2 \phi \phi^\dagger, \]  

where \( K^{\mu\nu} \) is a Lorentz-violating CPT-even coefficient that is symmetric and real. We included a factor two in front of the \( K^{\mu\nu} \) term for convenience and for easy comparison
with the case of a $\epsilon^{\mu\nu}$ fermion coefficient, which we will encounter in the next section. The $K^{\mu\nu}$ coefficient can be compared to the $k^{\phi\phi}$ Higgs parameter in the SME (see Eq. (2.11d)).

It is known that a parameter such as $K^{\mu\nu}$ is physically unobservable in a system that only contains the scalar field. This can be seen, because it can be removed by a coordinate redefinition [31]. However, as soon as we include interactions with other fields the coefficient is observable, because the coordinate redefinitions will also affect the Lagrangian of the fields with which the scalar field interacts. In effect, a coordinate transformation then moves $K^{\mu\nu}$ from one sector of the Lagrangian to another. Which sector of the Lagrangian is conventional and contains no Lorentz violation is therefore merely a choice of coordinates. We will thus treat the scalar field here with Lorentz violation, keeping in mind that the $K^{\mu\nu}$ has no physically observable consequences without considering other fields.

The Lorentz-violating equation of motion that follows from the Lagrangian in Eq. (2.13) is the Lorentz-violating Klein-Gordon equation and is given by

$$\left(\partial^2 + 2K_{\mu\nu}\partial^\mu\partial^\nu + m^2\right)\phi = 0.$$  

Making a plane-wave ansatz $\phi \propto e^{-i\lambda \cdot x}$ gives a dispersion relation

$$\tilde{\lambda}^2 - m^2 = 0,$$

with $\tilde{\lambda}^2 = \lambda^2 + 2K_{\mu\nu}\lambda^\mu\lambda^\nu$. To first order in Lorentz violation we can define $\lambda^\mu = \lambda^\mu + K^{\mu\nu}\lambda^\nu$. The dispersion relation has two roots, which are given by

$$\lambda_\pm^0 = -\frac{(K_{0i} + K_{i0})\lambda^i \pm \sqrt{(K_{0i} + K_{i0})^2 + (E^2 - 2K_{ij}\lambda^i\lambda^j)(1 + 2K_{00})}}{1 + 2K_{00}} \approx \pm \bar{E} \left(1 - K_{00}^{00} \pm \frac{(K_{0i} + K_{i0})\lambda_i}{E} - \frac{K_{ij}\lambda^i\lambda^j}{E^2}\right),$$

where the approximation sign signals that the result is to first order in Lorentz violation with $\bar{E} = \sqrt{\lambda^2 + m^2}$ being the conventional relativistic solution for the energy. For large values of the Lorentz-violating coefficients the energy can become imaginary. We will assume that this does not happen in the frames relevant for our analyses, i.e. that all these frames are concordant frames, as defined in the last paragraph of Section 2.2. We redefine the negative-energy solution $\lambda_0^-$ as usual, by defining the energy of the antiparticle as $E_v = -\lambda_0^-(\lambda)$, while the energy of the particle is $E_u = \lambda_0^+(\lambda)$. We find that $E_u = E_v = E$ with

$$E \approx \bar{E} - \frac{K_{\mu\nu}p^\mu p^\nu}{E}.$$  

Here, we defined $p^\mu$ to be the four-momentum of the scalar particle.

An important distinction between the Lorentz-symmetric and the Lorentz-violating dispersion relation is that the latter allows for spacelike momenta [20]. To see this, consider the case where $K^{00}$ is the only nonvanishing component of $K^{\mu\nu}$ and that $K^{00} > 0$. The energy is then given by $\sqrt{(p^2 + m^2)/(1 + 2K^{00})}$ and the four momentum becomes spacelike when $|p| \gtrsim m/\sqrt{2K^{00}} \gtrsim \sqrt{mM_P}$. The final inequality follows from the assumption that $K^{00} \sim m/M_P$, which originates from arguments presented below Eq. (2.7).
When momenta become spacelike, energy positivity is no longer guaranteed in all observer frames. This causes instabilities in the theory [20]. However this only happens for large momenta and high energies or for frames that are highly boosted with respect to concordant frames.

From the fact that the dispersion relation for a Lorentz-violating field is unconventional, it follows that also the notion of velocity changes. The appropriate measure of velocity is taken to be the group velocity \( v_g = \frac{\partial E}{\partial p} \) [32]. For the scalar field this becomes

\[
\mathbf{v}_g = \frac{K^{0i} + K^{i0}}{1 + 2K^{00}} + \frac{(p^i + 2K^{ij}p_j)(1 + 2K^{00}) - ((K^{0i} + K^{i0})p_i)(K^{00} + K^{i0})}{(1 + 2K^{00})\sqrt{((K^{0i} + K^{i0})p_i)^2 + (E^2 - 2K^{lm}p_lp_m)(1 + 2K^{00})}}. \tag{2.18}
\]

This group velocity is in general not parallel to the momentum and neither is it zero when \( p = 0 \). The latter means that the term restframe and zero-momentum frame become two different things. We will reserve the term restframe for the frame where \( v_g = 0 \). Notice also that the magnitude of the velocity can exceed the causality bound of \( |v_g| = 1 \). For example, if \( K^{00} \) is the only component of \( K^{\mu\nu} \) that is nonzero, the magnitude of the velocity is given by

\[
|v_g| = \frac{|p|}{\sqrt{(1 + 2K^{00})(p^2 + m^2)}}. \tag{2.19}
\]

Going back to our plane-wave ansatz and renaming \( \lambda \) to \( p \), we get a general solution for the field in terms of creation and annihilation operators, given by

\[
\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{N_p} \left( a_p e^{-ipx} + b_p^\dagger e^{ipx} \right), \tag{2.19}
\]

where \( N_p \) is a normalization factor and \( p^0 = E \). From the Lagrangian in Eq. (2.13) we can determine the momentum density conjugate to \( \phi \). It is given by

\[
\pi(x) = \frac{\partial L}{\partial (\partial_0 \phi)} = \partial_0 \phi^\dagger + 2K_{0\mu} \partial^\mu \phi^\dagger = \int \frac{d^3p}{(2\pi)^3} \frac{i}{N_p} (E + 2K_{0\mu}p^\mu) \left( a_p^\dagger e^{ipx} - b_p e^{-ipx} \right). \tag{2.20}
\]

We assumed here that the normalization factor \( N_p \) is real, which can always be accomplished by a phase redefinition of the field. We take the value of the normalization factor to be

\[
N_p = \sqrt{2(E + 2K_{0\mu}p^\mu)} = \sqrt{(1 + 2K^{00})(\lambda_0^0 - \lambda_0^-)} \tag{2.21}
\]

and impose the usual commutation relations

\[
[a_p, a_k^\dagger] = [b_p, b_k^\dagger] = (2\pi)^3 \delta^{(3)}(p - k), \tag{2.22}
\]

with the other commutators equal to zero. We then get the desired equal-time commutation relation between the conjugate variables \( \phi \) and \( \pi \), given by

\[
[\phi(x), \pi(y)]_{x^0 = y^0} = i\delta^{(3)}(x - y). \tag{2.23}
\]
The normal-ordered Hamiltonian that follows from the Lagrangian in Eq. (2.13) is given by
\[
H = \int d^3 x : \left[ (\partial_0 \phi)(\partial_0 \phi^\dagger) (1 + K_{00}) - (\partial_i \phi)(\partial^i \phi) - K_{ij}(\partial_i \phi)(\partial^j \phi^\dagger) + m^2 \phi \phi^\dagger \right]:
\]
\[
= \int \frac{d^3 p}{(2\pi)^3} E \left[ a_p^\dagger a_p + b_p^\dagger b_p \right],
\]
while the normal-ordered momentum operator comes out correctly as
\[
P = \int \frac{d^3 p}{(2\pi)^3} p \left[ a_p^\dagger a_p + b_p^\dagger b_p \right].
\]
These operators are the conserved charges that are associated with translations in time and space, respectively. Energy and momentum are thus conserved, as anticipated. The conserved charge that is associated with the transformation \( \phi \to e^{i\alpha} \phi \) is given by
\[
Q = i \int d^3 x : \left[ \left( (\partial_0 + 2K^{0\nu} \partial_\nu) \phi \right) \phi - \left( (\partial_0 + 2K^{0\nu} \partial_\nu) \phi \right) \phi^\dagger \right]:
\]
\[
= - \int \frac{d^3 p}{(2\pi)^3} \left[ a_p^\dagger a_p - b_p^\dagger b_p \right].
\]
We can now set up a Fock space with particles with energy \( E \) and momentum \( p \). For example, we define a one-particle state to be
\[
|p\rangle = N_p a_p^\dagger |0\rangle.
\]
This normalization of the one-particle state gives us a one-particle completeness relation given by
\[
(1)_{1\text{-particle}} = \int \frac{d^3 p}{(2\pi)^3} |p\rangle \frac{1}{N_p} \langle p|.
\]
An important observation is that the integral
\[
\int \frac{d^3 p}{(2\pi)^3} \frac{1}{N_p^2} = \int \frac{d^4 p}{(2\pi)^4} \delta(p^2 - m^2) \Theta(p^0),
\]
is an observer Lorentz invariant integral. At least in frames where the Lorentz-violating coeffic
Using the observer Lorentz invariance of the integral in Eq. (2.29), we can establish microcausality in the Lorentz-violating scalar theory. Consider the commutator of \( \phi(x) \) and \( \phi^\dagger(y) \). Since it is a c-number we can sandwich it between vacuum states without changing the result. This can be calculated to be
\[
\langle 0 | [\phi(x), \phi^\dagger(y)] | 0 \rangle = \langle 0 | \phi(x) \phi^\dagger(y) | 0 \rangle - \langle 0 | \phi^\dagger(y) \phi(x) | 0 \rangle
\]
\[
= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{N_p^2} (e^{-ip(x-y)} - e^{ip(x-y)}).
\]
The latter expression is observer Lorentz invariant for transformations between concordant frames. This is apparent from the fact that the integral in Eq. (2.29) is observer Lorentz invariant for such transformations. If \( x - y \) is spacelike we can always transform it to \( y - x \) by a continuous Lorentz transformation and therefore the integral vanishes. For transformations beyond concordant frames, causality might break down [20]. The integral in Eq. (2.29) might no longer by observer Lorentz invariant because one or two of the roots of the dispersion relation have changed sign or have obtained an imaginary part.

To make this more precise, consider a theory where \( K^{00} \) is the only nonzero component of \( K^{\mu\nu} \) in a concordant frame \( S \). In a frame \( S' \), boosted with respect to \( S \), we have that \( K^{00} = \gamma^2 K^{00} \), with \( \gamma \) the relativistic \( \gamma \) factor. Consider a particle that is at rest in \( S' \). The roots of the dispersion relation are then given by \( \lambda^\pm_0 = \pm m/\sqrt{(1 + 2\gamma^2 K^{00})} \). These roots become imaginary\(^\text{2}\) for negative \( K^{00} \) and \( \gamma^2 \gtrsim 1/2|K^{00}| \). At this point we can no longer consider both terms in Eq. (2.30) to be observer Lorentz invariant and we cannot transform \( x - y \) to \( -(x - y) \). To see what this means, define \( z = x - y \) and consider the case where \( z = (z_0, 0, 0, z_0) \). A boost in the \( z_z \)-direction transforms \( z_0 \) to \( -z_0 \) if \( z_0/z_0 = (1 + \gamma)/\sqrt{\gamma^2 - 1} \). A rotation can then transform \( z_z \) to \( -z_z \). When \( \gamma^2 \sim -1/2K^{00} \), i.e. at the point where the roots of the dispersion relation become imaginary, \( z_0/z_0 \sim 1 - 2K^{00} \). We see that for a spacetime interval with \( (x - y)^2 \lesssim 4(x^0 - y^0)^2K^{00} \) the Lorentz transformation needed to get from \( x - y \) to \( -(x - y) \) is so large that it makes the energy imaginary and the expression in Eq. (2.30) observer Lorentz noninvariant. The latter is therefore possibly nonvanishing. For negative \( K^{00} \) there is therefore a region outside the usual lightcone where signals can propagate, violating causality (see Fig 2.2).

We see that this happens when \( \gamma^2 \sim -1/2K^{00} \), i.e. at energies of the order of \( \sqrt{m\tilde{M}_P} \), if \( K^{00} \sim m/M_P \). However, in concordant frames and correspondingly small energies and boosts, the complex-scalar theory with Lorentz violation is microcausal. Notice,\(^\text{2}\) We do not discuss what this means exactly. We can create a similar situation where one root of the dispersion relation changes sign, instead of becoming imaginary.

\[\text{Figure 2.2: The dark-gray area represents the usual lightcone. For the light-gray area, the commutator in Eq. (2.30) is possibly nonvanishing, which violates causality.}\]
THE STANDARD-MODEL EXTENSION

by looking at Eq. (2.30), that this happens as usual by cancellation between a particle propagating from \( x \) to \( y \) and an antiparticle propagating from \( y \) to \( x \). These particles must have the same mass. It is known that introducing a different mass for a particle and an antiparticle will result in a nonlocal theory, also for higher-spin fields [33]. This is therefore not a way CPT violation is introduced in the SME.

In the above we noticed that causality and stability in the Lorentz-violating scalar theory can break down at an energy scale of \( \mathcal{O}(\sqrt{mM_P}) \). This energy scale is specific to \( K_{\mu\nu} \) (and \( c_{\mu\nu} \) for fermions). Other Lorentz-violating coefficients in the fermion sector will cause these issues only at an energy scale of \( \mathcal{O}(M_P) \). Both of these scales are expected to be well out of reach of any experiments. In other words, it is expected not to be a problem in concordant frames. However, it also signals that the theory must be supplemented by higher-dimensional terms in the Lagrangian that regulate its high-energy behavior. For a more extensive discussion of this issue see Ref. [20].

In the same way as in Lorentz-symmetric theories, we can define the retarded and advanced Green’s function of the Lorentz-violating Klein-Gordon operator. However, the quantity one needs in practical calculation is the Feynman propagator. The Feynman propagator of the complex scalar field with Lorentz violation is given by

\[
D_F(x - y) = \langle 0 | T \phi(x) \phi^\dagger(y) | 0 \rangle = \int \frac{d^4 \lambda}{(2\pi)^4} \frac{i}{\lambda^2 - m^2 + i\epsilon} e^{-i\lambda(x-y)}, \tag{2.31}
\]

where \( T \) is the usual time-ordering operator.

The treatment of interactions proceeds just as in the usual Lorentz-symmetric case. One can define fields in the interaction picture and do the perturbative expansion of correlation functions in terms of these fields. Also Wick’s theorem holds as usual and we can calculate correlation functions in terms of Feynman diagrams. The Feynman rules themselves will be altered by the Lorentz-violating coefficients, as is exemplified by Eq. (2.31). A point that is worth mentioning here, is that one should treat a \( K_{\mu\nu} \) term, or any Lorentz-violating term quadratic in the fields, as part of the free theory. Treating these terms as interaction terms, to be able to handle them perturbatively, is not the correct approach. Perhaps the easiest way to see this, is by looking at external legs. The expression for an external leg with one insertion of \( K_{\mu\nu} \) will have the form

\[
\frac{1}{p^2 - m^2} \cdot 2K_{\mu\nu} p^\mu p^\nu \cdot \frac{1}{p^2 - m^2}.
\]

After amputating the diagram, one factor of \( 1/(p^2 - m^2) \) remains. When putting the corresponding particle on shell (since it is an external leg), the expression will blow up, making the diagram divergent. Therefore we will treat all Lorentz-violating terms that are quadratic in the fields as part of the free theory.

2.5 QFT basics of a Dirac field with Lorentz violation

We now treat a Dirac field with Lorentz violation. To do this we limit ourselves to the quadratic Lagrangian of a single spin-\( \frac{1}{2} \) fermion. We will also constrain the discussion
to the mSME. For a discussion of higher-dimensional terms, see Ref. [30]. The relevant Lagrangian is then given by

$$\mathcal{L} = \bar{\psi} (i\Gamma^\mu \partial_\mu - M) \psi \, ,$$

(2.33)

where

$$\Gamma^\mu = \gamma^\mu + c^{\mu\nu} \gamma_\nu + d^{\mu\nu} \gamma^5 \gamma_\nu + e^\mu + f^\mu \gamma^5 + \frac{1}{2} g^{\rho\mu\sigma} \sigma_{\rho\sigma} \, ,$$

(2.34)

$$M = m + a_\mu \gamma^\mu + b_\mu \gamma^5 \gamma^\mu + \frac{1}{2} H^{\mu\nu} \sigma_{\mu\nu} \, .$$

(2.35)

All the Lorentz-violating coefficients here are real, since we demand the Lagrangian to be Hermitian. The coefficients in $\Gamma^\mu$ are dimensionless, while the parameters in $M$ have the dimension of mass. The Lorentz-violating parameters $c^{\mu\nu}$ and $d^{\mu\nu}$ are CPT even and traceless. They arise from the $c_R$ and $c_L$ terms in Eq. (2.11a). The coefficients $e^\mu$, $f^\mu$ and $g^{\rho\mu\nu}$ are CPT odd and they do not follow from any terms in the mSME Lagrangian, since they are not compatible with the SME gauge structure. They can however be induced by radiative corrections and we keep them here for completeness. The $a^\mu$ and $b^\mu$ parameters are CPT odd and they follow from the $a_L$ and $a_R$ terms in Eq. (2.11a). The $H^{\mu\nu}$ parameter is CPT even and follows from the Yukawa terms in Eq. (2.11c). Just as the $K^{\mu\nu}$ term of the scalar field in the previous section, the symmetric part of $c^{\mu\nu}$ can be removed by a coordinate redefinition and is unobservable in a system without other fields. Also some (parts of) the other coefficients are unobservable, as is shown in Ref. [31] by using field redefinitions. For generality, we will treat the full set coefficients.

The Lagrangian in Eq. (2.33) has extra, unconventional, time derivatives. One of the consequences of this is that the apparent Hamiltonian $\tilde{H}$, in the Dirac equation

$$(i\partial^0 - \tilde{H})\psi = 0 \, ,$$

(2.36)

is not Hermitian, since it is given by

$$\tilde{H} = (\Gamma^0)^{-1} (i\Gamma \cdot \partial + M) \, .$$

(2.37)

This causes problems in the theory, notably nonunitary time evolution. A well known solution [34] to remove the extra time derivatives is to do a field redefinition

$$\psi = A \chi \, ,$$

(2.38)

where $A$ is a matrix chosen such that

$$A^\dagger \gamma^0 \Gamma^0 A = 1 \, .$$

(2.39)

In terms of $\chi$ the Hamiltonian will turn out to be Hermitian and we will treat $\chi$ as the physical field. A drawback of this way of dealing with the nonhermitian Hamiltonian is that we lose manifest observer Lorentz covariance. Although $A$ does not change the physics, leaving observer Lorentz covariance of the observables intact, the redefined spinors will no longer transform Lorentz covariantly. Also the Feynman rules for propagators and vertices and phase space factors will no longer be Lorentz covariant. In most cases we will therefore have to calculate everything in one observer frame and will not be able to easily
switch frames for different parts of the calculation, which are usually separately Lorentz invariant.

The existence of $A$ is discussed in Ref. [20], where it was concluded that $A$ always exists in a concordant frame. The form of $A$ can be determined to any order in the Lorentz violation by

$$A = I + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} (I - \gamma^0 \Gamma^0)^n,$$

(2.40)

where $I$ is the $4 \times 4$ unit matrix. The matrix $A$ is Hermitian due to the Hermiticity of $\gamma^0 \Gamma^0$. In most cases we will only need the first order $n = 1$ term. To that order $A$ is given by

$$A = I + \frac{1}{2} \left( c_{\mu0} \gamma^0 \gamma^\mu + d_{\mu0} \gamma^0 \gamma^5 \gamma^\mu + e_0 \gamma^0 + if_0 \gamma^0 \gamma^5 + \frac{1}{2} g_{\lambda\nu0} \sigma^{\lambda\mu} \right).$$

(2.41)

An alternative method to deal with the unconventional time derivative and the non-hermitian Hamiltonian is to use an unconventional inner product in spinor space [19, 35]

$$\langle u | v \rangle \equiv \bar{u} \Gamma^0 v.$$

(2.42)

The Hamiltonian is then Hermitian with respect to this inner product and will have real eigenvalues. Quantization in terms of this inner product is worked out in Ref. [16]. However, we will use the spinor redefinition in Eq. (2.39) in this thesis.

In terms of the redefined field $\chi$ we can write the Lagrangian as

$$L = \bar{\chi} \tilde{A} (i \Gamma_\mu \partial^\mu - M) A \chi = \bar{\chi} \left( i \tilde{\Gamma}_\mu \partial^\mu - \tilde{M} \right) \chi,$$

(2.43)

with $\tilde{A} = \gamma^0 A \gamma^0$ and

$$\tilde{\Gamma}^\mu = \tilde{A} \Gamma^\mu A \approx \gamma^\mu + C^{\mu\nu} \gamma^\nu + D^{\mu\nu} \gamma^5 \gamma^\nu + E^{\mu} + i F^{\mu} \gamma^5 + \frac{1}{2} G^{\lambda\mu} \sigma_{\lambda\nu}$$

(2.44)

$$\tilde{M} = \tilde{A} MA \approx \tilde{m} + A_\mu \gamma^\mu + B_\mu \gamma^5 \gamma^\mu + \frac{1}{2} H^{\mu\nu} \sigma_{\mu\nu}.$$ (2.45)

Here the approximation signs signal a solution to first order in Lorentz-violating coefficients and

$$\tilde{m} = m(1 - e^{00}),$$

(2.46a)

$$A^\mu = a^\mu - me^0 g^{\mu0},$$

(2.46b)

$$B^\mu = b^\mu - \frac{1}{2} me^0 g_{\lambda\nu0} g^{\mu\lambda\nu},$$

(2.46c)

$$C^{\mu\nu} = c^{\mu\nu} - c^{\mu0} g^{\nu0} + e^{\lambda0} e^{00} - e^{00} g^{\mu\nu},$$

(2.46d)

$$D^{\mu\nu} = d^{\mu\nu} - d^{\mu0} g^{\nu0} + g^{\mu0} d^{\nu0} - d^{00} g^{\mu\nu},$$

(2.46e)

$$E^{\mu} = e^\mu - g^{\mu0} e^0,$$

(2.46f)

$$F^{\mu} = f^\mu - g^{\mu0} f^0,$$

(2.46g)

$$G^{\lambda\mu0} = g^{\lambda\mu0} - g^{\mu0} g^{\lambda0} + 2 g^{\mu0} g^{\lambda0} - 2 g^{\lambda\mu} g^{000},$$

(2.46h)

$$H^{\mu\nu} = H^{\mu\nu} - me^{0\rho\mu} d_{\rho0}. $$

(2.46i)

The fact that $C^{\mu0} = D^{\mu0} = C^0 = F^0 = G^{\lambda00} = 0$ shows that the time-derivative term is conventional in the new spinor basis. The $\chi$ fields now have unitary time evolution and
we will take them to be the physical fields to calculate observables like cross sections and decay rates. They are not Lorentz-covariant objects however, due to the observer-frame dependence of the matrix $A$. The physical observables are still observer Lorentz invariant however, as they should be, because the physics is independent of a spinor redefinition.

The Dirac equation that follows from the Lagrangian in Eq. (2.43) is given by

$$\left( i\bar{\Gamma}_\mu \partial^\mu - \tilde{M} \right) \chi = \gamma^0 (i\partial^0 - H) \chi = 0 ,$$

(2.47)

where $H$ is a Hermitian Hamiltonian given by

$$H = A_\alpha \gamma^0 \left( i \Gamma \cdot \partial + M \right) A = \gamma^0 \left( i \bar{\Gamma} \cdot \partial + \tilde{M} \right) ,$$

(2.48)

To solve the Dirac equation, we use the usual plane-wave ansatz $\chi(x) = w(\lambda) e^{i\lambda \cdot x}$. The dispersion relation that follows then is

$$\det \left( \bar{\Gamma}^\mu \lambda_\nu - \tilde{M} \right) = \det (A) \det (\Gamma^\mu \lambda_\nu - M) \det (A) = 0 ,$$

(2.49)

which has in general four nondegenerate roots. It does not matter in what spinor basis we solve the dispersion relation, since the nonsingular matrix $A$ will only give overall factors that do not change the solutions. As is shown in Ref. [20], in concordant frames, there are two positive and two negative roots. In general the two positive (or negative) roots are nondegenerate and we will denote them by $(\lambda^0)^\pm_\alpha$, where the $\pm$ sign labels a positive or a negative root and $\alpha = 1, 2$ labels what are usually the spin or helicity states. In general, however, we will not have an operator such as a spin component or helicity operator that commutes with the Hamiltonian. The labeling of the different $\alpha$ states then essentially becomes arbitrary. One can, however, define a labeling scheme based on the desired behavior of $\alpha$ under charge conjugation [21].

We redefine the negative energy solutions in the usual way, i.e. $E^\alpha_u = (\lambda^0)^+_{\alpha}(\lambda)$ and $E^\alpha_v = -(\lambda^0)_{\alpha}(-\lambda)$. Since these solutions are different from the conventional case, also the group velocity $v_\mathbf{g} = \partial E/\partial \mathbf{p}$ will differ from the conventional case [32]. As for the scalar-field case, given in Eq. (2.18), the velocity will in general not be parallel to the momentum and the $\mathbf{p} = 0$ frame will not always be equal to the $v_\mathbf{g} = 0$ frame.

For each of the solutions of the dispersion relation, there is a corresponding spinor solution $w^\alpha_{\mathbf{p}}(\lambda)$ of the Dirac equation. In most cases the complete spinor is fixed by the Dirac equation and a normalization condition. The degree of freedom that is conventionally the direction of the spin is fixed by the Lorentz violation. After redefinition of the negative energy solutions we have the spinors $u^\alpha_u = w^\alpha_{\mathbf{p}}(\lambda)$ and $v^\alpha_v = w^\alpha_{\mathbf{p}}(-\lambda)$. We rename $\lambda$ to $p$ and define the quantities $p^\alpha_u = (E^\alpha_u, \mathbf{p})$ and $p^\alpha_v = (E^\alpha_v, \mathbf{p})$. The complete solution for the Dirac field can then be written as

$$\chi(x) = \int \frac{d^3p}{(2\pi)^3} \sum_{\alpha=1}^{2} \left( \frac{1}{N^\alpha_u(\mathbf{p})} b^\alpha_u(p) e^{-ip^\alpha_u \cdot x} + \frac{1}{N^\alpha_v(\mathbf{p})} d^\alpha_v(p) e^{ip^\alpha_v \cdot x} \right) ,$$

(2.50)

where $b^\alpha_u$ and $d^\alpha_v$ are complex weights that will become the annihilation operators for particles and antiparticles respectively, when we quantize the theory. The normalization
factor $N_a(p)$ corresponds to the normalization of the spinors, \(i.e.
\)

$$
u^{\alpha\dagger}(p)\nu^{\alpha}(p) = \delta^{\alpha\alpha'}(N_{a}^a(p))^2, \quad (2.51a)$$

$$
u^{\alpha\dagger}(p)\nu^{\alpha}(p) = \delta^{\alpha\alpha'}(N_v^a(p))^2, \quad (2.51b)$$

$$
u^{\alpha\dagger}(p)\nu^{\alpha}(-p) = 0, \quad (2.51c)$$

$$
u^{\alpha}(p)\nu^{\alpha}(-p) = 0. \quad (2.51d)$$

The spinors then satisfy the completeness relation

$$
\sum_\alpha \left( \frac{1}{(N_{a}^a(p))^2} u^{\alpha}(p)u^{\alpha\dagger}(p) + \frac{1}{(N_v^a(p))^2} v^{\alpha}(p)v^{\alpha\dagger}(p) \right) = I, \quad (2.52)
$$

with \(I\) the \(4 \times 4\) identity matrix. Except for the restriction that \((\chi')^c = \chi\), where \(c\) means charge conjugation, the normalization factor $N^a_{(u,v)}(p)$ is arbitrary. In some cases it might be convenient to choose the normalization such that the phase space factor in $\int d^3p/N^2$ is observer Lorentz invariant in concordant frames, as in Eq. (2.29), but we leave it arbitrary for now.

An expression that is very useful in calculating cross sections and decay rates, is the expression for $u^{\alpha}(p)\bar{u}^{\alpha}(p)$ and $v^{\alpha}(p)\bar{v}^{\alpha}(p)$. Usually these are called projection operators, but since the usual orthogonality relations of $\bar{u}^{\alpha}(p)$ and $u^{\alpha}(p)$ no longer hold (and similarly for $\bar{v}^{\alpha}(p)$), this is a misnomer in the Lorentz-violating case. However, we can still find the appropriate expressions [21]. If all the roots of the dispersion relation are nondegenerate, they are given by

$$
u^{\alpha}(p)\bar{u}^{\alpha}(p) = \frac{(N_{u}^\alpha(p))^2\text{Adj}\left(\bar{\Gamma} \cdot p_u - \bar{M}\right)}{(E_{u}^{\alpha}(p) - E_{u}^{\alpha}(p))(E_{v}^{\alpha}(p) + E_{v}^{\dagger}(- p))(E_{v}^{\alpha}(p) + E_{v}^{\dagger}(- p))}, \quad (2.53a)$$

$$
u^{\alpha}(p)\bar{v}^{\alpha}(p) = \frac{(N_{v}^\alpha(p))^2\text{Adj}\left(\bar{\Gamma} \cdot p_v + \bar{M}\right)}{(E_{v}^{\alpha}(p) - E_{u}^{\alpha}(p))(E_{v}^{\alpha}(p) + E_{v}^{\dagger}(- p))(E_{v}^{\alpha}(p) + E_{v}^{\dagger}(- p))}, \quad (2.53b)$$

where $\alpha' \neq \alpha$ and $\text{Adj}(X) = \text{det}(X)X^{-1}$ is the adjugate matrix of $X$. If the dispersion relation has degenerate roots, these expressions seem to diverge. However the factors that give zero in the denominator will also be present in the adjugate matrix. So we have to find a way to factor them out of the adjugate matrix and cancel them with one of the factors in the denominator (see Ref. [21] for more details). It follows from the derivation in Ref. [21] that after doing this, we obtain the sum of $u^{\alpha}(p)\bar{u}^{\alpha}(p)$ or $v^{\alpha}(p)\bar{v}^{\alpha}(p)$ over $\alpha$ and not half of this. So, if, for example, the two positive roots and the two negative roots are degenerate we obtain

$$
\sum_{\alpha=1}^2 u^{\alpha}(p)\bar{u}^{\alpha}(p) = \frac{(N_{u}(p))^2\text{Adj}\left(\bar{\Gamma} \cdot p_u - \bar{M}\right)}{(E_{u}(p) + E_{u}(- p))^2}, \quad (2.54a)
$$

$$
\sum_{\alpha=1}^2 v^{\alpha}(p)\bar{v}^{\alpha}(p) = \frac{(N_{v}(p))^2\text{Adj}\left(\bar{\Gamma} \cdot p_v + \bar{M}\right)}{(E_{v}(p) + E_{u}(- p))^2}, \quad (2.54b)
$$
where \( \text{Adj} \left( \tilde{\Gamma} \cdot p_u \mp \tilde{M} \right) \) denotes the adjugate matrix with the appropriate factor, containing the degenerate roots, divided out. This expression is only valid if the normalization of the spinors is chosen such that \( N^1_{(u,v)} = N^2_{(u,v)} \) if \( E^1_{(u,v)} = E^2_{(u,v)} \). The results in Eq. (2.54) are relevant if \( b^\mu = d^{\mu\nu} = q^{\mu\nu\rho} = H^{\mu\nu} = 0 \), since under this condition the same sign energies are equal, as is shown in Ref. [21].

We could also sum the expressions in Eq. (2.53) over \( \alpha \), even though the roots with different \( \alpha \) are nondegenerate, but in general this is not useful for the calculation of cross sections or decay rates, since other quantities (e.g. the delta function and therefore the integration limits) will also depend on \( \alpha \) and should be included in the sum. We will encounter this in Chapter 3.

The expressions in Eqs. (2.53) and (2.54) are quite formal and are only useful in calculations of observables if we can actually determine the explicit form of the adjugate matrix and the roots of the dispersion relation. The latter have been found to first order in Lorentz violation by block diagonalizing the Hamiltonian through a Foldy-Wouthuysen transformation in Ref. [36] and we can in principle take them from there. The adjugate matrix can be found by using that the adjugate matrix of a general 4 \( \times \) 4 matrix can be written as [21]

\[
\text{Adj} \left( S + i\gamma^5 P + V_\mu \gamma^\mu + A_\mu \gamma^5 \gamma^\mu + T_{\mu\nu} \sigma^{\mu\nu} \right) = \left( \hat{S} - \hat{P} \gamma^5 - \hat{V}_\mu \gamma^\mu \right) \left( S + i\gamma^5 P + V_\mu \gamma^\mu - A_\mu \gamma^5 \gamma^\mu - T_{\mu\nu} \sigma^{\mu\nu} \right),
\]

with

\[
\hat{S} = S^2 - P^2 + V^2 + A^2 - 2T^2,
\]

\[
\hat{P} = 2 \left( iPS - V \cdot A - iT \cdot \tilde{T} \right),
\]

\[
\hat{V}_\mu = 2 \left( SV^\mu + iPA^\mu - 2iT^{\mu\nu} V_\nu + 2\tilde{T}^{\mu\nu} A_\nu \right),
\]

and \( \tilde{T}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} T_{\rho\sigma} \). The explicit expression for the adjugate matrix of \( \tilde{\Gamma} \cdot p - \tilde{M} \) is complex in general and can be found in Appendix B. In Appendix B we also give a useful expression for the denominator of Eq. (2.53) without having to plug in the separate results for the roots of the dispersion relation. Furthermore, we work out some explicit examples of the matrices \( u^\alpha(p) \bar{u}^{\alpha'}(p) \) and \( v^\alpha(p) \bar{v}^{\alpha'}(p) \) in Appendix B.

We now continue to quantize the theory that follows from the Lagrangian given in Eq. (2.33). The momentum density conjugate to \( \chi \) is

\[
\pi(x) = \frac{\partial \mathcal{L}}{\partial (\partial_0 \chi)} = i\tilde{\chi}(x)\tilde{\Gamma}^0 = i\chi^\dagger(x).
\]

We promote the complex weights \( b^\alpha_p \) and \( d^\alpha_p \) in Eq. (2.50) to operators on a Hilbert space and impose the usual anticommutation relations

\[
\{ b^\alpha_p, b^{\alpha'}_{p'} \} = (2\pi)^3 \delta^{\alpha\alpha'} \delta^{(3)}(p - p'),
\]

\[
\{ d^\alpha_p, d^{\alpha'}_{p'} \} = (2\pi)^3 \delta^{\alpha\alpha'} \delta^{(3)}(p - p'),
\]
with other anticommutators equal to zero. Using the completeness relation in Eq. (2.52), we then get the desired equal-time anticommutator relation between the conjugate variables $\chi(x)$ and $\pi(x)$, given by
\[
\{\chi_a(t, x), \pi_b(t, x')\} = i\delta^{(3)}(x - x')\delta_{ab},
\] (2.59)
where $a$ and $b$ are spinor indices. Other anticommutators are zero:
\[
\{\chi_a(t, x), \pi^\dagger_b(t, x')\} = \{\chi^\dagger_a(t, x), \pi_b(t, x')\} = 0.
\] (2.60)

The vacuum $|0\rangle$ and the one-particle state $|p, \alpha\rangle$ are defined by
\[
\begin{align*}
b^\alpha_p |0\rangle &= 0, & N^\alpha_u(p)b^\alpha_p |0\rangle &= |p_u, \alpha\rangle, \\
d^\alpha_p |0\rangle &= 0, & N^\alpha_v(p)d^\alpha_p |0\rangle &= |p_v, \alpha\rangle.
\end{align*}
\] (2.61)

Since the theory is invariant under spacetime translations, energy and momentum are conserved. Using Noether’s theorem we can thus construct the canonical energy-momentum tensor $T^{\mu\nu}$, which, in terms of the physical fields, is given by
\[
T^{\mu\nu} = i\bar{\chi}\tilde{\Gamma}^{\mu}\partial^{\nu}\chi.
\] (2.62)

The conserved four-momentum operator is then given by
\[
P^\mu = \int d^3x : T^0^\mu : = \int \frac{d^3p}{(2\pi)^3} \frac{2}{i} \sum_{\alpha=1}^2 \left[ (p^\alpha_u)^\mu b^\alpha_p + (p^\alpha_v)^\mu d^\alpha_p \right],
\] (2.63)
with the Hamiltonian $H$ being the zeroth component of $P^\mu$. The Noether current that corresponds to the transformation $\chi(x) \rightarrow e^{i\alpha}\chi(x)$, i.e. corresponding to $U(1)$ symmetry of the Lagrangian, is given by
\[
\begin{align*}
\mathcal{J}^\mu(x) &= \bar{\chi}(x)\tilde{\Gamma}^\mu\chi(x),
\end{align*}
\] (2.64)
which is easily verified to have vanishing divergence, when $\chi$ satisfies the Lorentz-violating Dirac equation in Eq. (2.47). The associated conserved charge is
\[
Q = \int d^3x : J^0(x) : = \int \frac{d^3p}{(2\pi)^3} \frac{2}{i} \sum_{\alpha=1}^2 \left[ b^\alpha_p + d^\alpha_p \right].
\] (2.65)

Acting with $P^\mu$ or $Q$ on a one-particle state defined in Eq. (2.61), we see that these indeed represent one-particle states with the correct energy, momentum, $\alpha$-state and charge.

The issue of microcausality and stability of a quantum field theory with Lorentz violation can also be discussed in the context of Dirac fields. It is investigated in detail in Ref. [20]. The results are analogous to the results for the scalar field, described in the previous section, and we will not redo the full analysis here. Instead, we will recapitulate some of the important points.

In the context of microcausality, we point out that the anticommutator of the fields can be written as [20]
\[
\{\psi(x), \bar{\psi}(y)\} = \int \frac{d^4\lambda}{(2\pi)^4} i\text{Adj}(\Gamma \cdot \lambda - M) e^{-i\lambda(x-y)},
\] (2.66)
where the integral over $\lambda^0$ is along a contour that encircles all poles in a clockwise direction. Microcausality is best investigated in terms of the fields $\psi(x)$, instead of the redefined fields $\chi$, since the latter are not observer Lorentz covariant. It is shown in Ref. [20] that, as long all poles in Eq. (2.66) are on the real axis, they are encircled by the contour and their contributions cancel. However, for large Lorentz-violating coefficients some of the poles might obtain an imaginary part and may no longer be encircled by the contour, giving a nonzero result. The theory then fails to be microcausal. Conventionally the theory is microcausal because the contributions of particles and antiparticles cancel. In this case the cancellation occurs not only between particles and antiparticles, but also between different $\alpha$-states.

Concerning stability, the dispersion relation allows for spacelike momenta, as it did for the scalar field. When this happens, the theory can become unstable.

As pointed out in the previous section, breakdown of microcausality and stability can occur at an energy scale of $O(\sqrt{mM_P})$ for $c_{\mu\nu}$ and at a scale of $O(M_P)$ for the other coefficients. This signals that the theory is a low-energy approximation and should include higher-dimensional operators that become important at high energies or large boosts. For a detailed analysis of this issue, we refer the interested reader to Ref. [20]. Here, we just conclude that, as for the scalar case, the theory is microcausal and stable in concordant frames and for experimentally attainable energies.

We now proceed to derive the Feynman propagator for a fermion field. It is defined by

$$S_F(x-y) \equiv \langle 0|T\chi(x)\bar{\chi}(y)|0\rangle = \theta(x^0-y^0) \langle 0|\chi(x)\bar{\chi}(y)|0\rangle - \theta(y^0-x^0) \langle 0|\bar{\chi}(y)\chi(x)|0\rangle$$

(2.67)

where $T$ is the usual time-ordering operator. For four nondegenerate roots we use Eq. (2.53) to calculate that

$$\langle 0|\chi(x)\bar{\chi}(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \sum_\alpha \frac{\text{Adj}\left(\bar{\Gamma} \cdot p^\alpha - \tilde{M}\right)e^{-ip_0(x-y)}}{(E_\alpha^0(p) - E_{\alpha'}^0(p))(E_{\alpha'}^0(p) + E_\alpha^1(-p))(E_\alpha^0(p) + E_\alpha^2(-p))},$$

(2.68a)

$$\langle 0|\bar{\chi}(y)\chi(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \sum_\alpha \frac{\text{Adj}\left(\bar{\Gamma} \cdot p^\alpha + \tilde{M}\right)e^{-ip_0(y-x)}}{(E_\alpha^0(p) - E_{\alpha'}^0(p))(E_{\alpha'}^0(p) + E_\alpha^1(-p))(E_\alpha^0(p) + E_\alpha^2(-p))}.$$  

(2.68b)

This means that we can write the Feynman propagator as

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i \text{Adj}(\bar{\Gamma} \cdot p - \tilde{M})}{\det(\bar{\Gamma} \cdot p - M)} e^{-i(p-x)\cdot y),}$$

(2.69)

where the integral over $p^0$ is along the contour that goes above the two positive poles and below the two negative poles. This contour follows from the Fourier transform of the Heaviside stepfunction given by

$$\theta(z) = \frac{1}{2\pi i} \int d\tau \frac{e^{i\tau z}}{\tau - i\epsilon}$$

(2.70)
For degenerate eigenvalues we have to use, instead of Eq. (2.53), the expressions where the appropriate factors have been canceled between numerator and denominator, such as in Eq. (2.54). The result will be identical, with the exception that the $p^0$ integration contour now encircles less poles. It still goes above positive poles and below negative poles.

The perturbative treatment of the correlation functions is generalized without problems from the usual case to the Lorentz-violating case. This includes Wick’s theorem by which we can transform the time-order products of the fields into normal-ordered products and thus obtain the usual expressions for the $n$-point correlation functions in terms of Feynman diagrams. The Feynman rules are exactly analogous to the conventional Lorentz-symmetric case. The spinors and propagators are replaced by their Lorentz-violating equivalents. The same holds for the vertices, which can be easily read off from the relevant parts of the Lagrangian. We will therefore not give the full set of Feynman rules here, but we will introduce them when needed.
Chapter 3

Cross sections and decay rates

Two of the most relevant observables in quantum field theory are cross sections and decay rates. These observables are related to the $S$-matrix, which can be calculated in terms of Feynman diagrams. The connection between these different quantities is a quite delicate matter even in the Lorentz-symmetric case. It lies outside the scope of this thesis to rigorously derive the different relations, which are not even fully understood in the literature in some of the Lorentz-violating cases. We will, however, sketch the connections, point out some difficulties, and refer to the relevant literature.

3.1 Cross sections and the $S$-matrix

A cross section is a quantity related to a scattering experiment in particle physics. In a typical scattering experiment there will be bunches of particles coming in. They will interact, or scatter, in some interaction region and a number of different particles will be coming out of this collision. These outgoing particles fly through a detector, where some of them will be registered. From the measurement of the outgoing particles one can infer properties of the interactions that played a role in the collision. In particular, by comparing the measurements to a calculation of the cross section in field theory.

A cross section in this context is an effective area for a particular set of colliding particles. We will restrict ourselves to the case where there are two types of incoming particles. The number of scattering events that then occur rises with the number of particles coming in and with the cross-sectional area ($A$) common to the two bunches. From this we conclude that the number of scattering events is

$$N \propto \rho_A \ell_A \rho_B \ell_B A,$$

where $\rho_{AB}$ and $\ell_{AB}$ are the number densities and the lengths of the bunches of the incoming particles of type $A, B$, respectively. We will assume that the particle densities of the initial particles are constant over the interaction region. See Ref. [37] for a more detailed treatment of this latter issue. The proportionality constant related to Eq. (3.1) is what is defined as the cross section:

$$\sigma \equiv \frac{N}{\rho_A \ell_A \rho_B \ell_B A}.$$

(3.2)
We now assume there is one target particle (A) and many incoming particles (B). We want to calculate the number of scattering events with a specific out state (containing \(n\) particles). This is given by

\[
N = \sum_i \mathcal{P}_i ,
\]  

(3.3)

where \(i\) runs over all incoming particles and \(\mathcal{P}\) is the probability of the initial state becoming the specified final state. According to Ref. [37], it is given by

\[
\mathcal{P}(A B \rightarrow 1 2 \cdots n) = \left( \prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{(N_f(p_f))^2} \right) |_{\text{out}} \langle \{ p_1 \} \cdots \{ p_n \} | \phi_A \phi_B |_{\text{in}} \rangle \left|_{\text{out}} \langle \{ p_1 \} \cdots \{ p_n \} | \phi_A \phi_B |_{\text{in}} \rangle \right|^2 ,
\]  

(3.4)

where the product symbol applies to everything in parentheses. The normalization factor \(N(p)\) is the same as the normalization factor of the one-particle states in the free theory in Eq. (2.27) or (2.61). Since Eq. (3.4) is the probability that the final-state particles have a momentum, lying in a region \(d^3 p\), it will give us the differential cross section in the end.

The initial state in Eq. (3.4) is defined in the Heisenberg picture and is represented by wavepackets, such that

\[
|\phi\rangle = \int d^3 k \frac{1}{(2\pi)^3 N(k)} \phi(k) |k\rangle ,
\]  

(3.5)

where \(|k\rangle\) is a one-particle state in the interacting theory and \(\phi(k)\) is the Fourier transform of the spatial wavefunction. The normalization factor \(N(k)\) makes sure that we get a total probability of 1 if the wavepackets are such that \(\int d^3 k \frac{|\phi(k)|^2}{(2\pi)^3} = 1\). Although there are some interesting points to be made about wavepackets in the context of Lorentz violation, pertaining e.g. wavepacket spreading, velocity and wavepacket bifurcation [32], we will ignore them here, since they are not necessary for what follows.

In terms of wavepackets the initial state is written as

\[
|\phi_A \phi_B|_{\text{in}} = \int \frac{d^3 k_A}{(2\pi)^3} \int \frac{d^3 k_B}{(2\pi)^3} \frac{\phi_A(k_A) \phi_B(k_B) e^{-i b \cdot k_B}}{N_A(k_A) N_B(k_B)} |k_A k_B\rangle .
\]  

(3.6)

The factor of \(e^{-i b \cdot k_B}\) is inserted to account for the fact that the momentum of the incoming particles is not necessarily aligned with that of the target particles. So \(b\) is the transverse spatial displacement of the incoming particles \(B\) with respect to the target particles \(A\).

We assume the momenta to be collinear, however. We (as well as the authors of Ref. [37]) use out states of definite momentum, without wavepackets to represent the spreading. This is justified as long as the detectors mainly measure momentum and not position.

To obtain the cross section we now have to integrate the probability in Eq. (3.4) over the parameter \(b\), which amounts to integrating over all the incoming particles. We then obtain

\[
d\sigma = \left( \prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{(N_f(p_f))^2} \right) \int d^2 b \left( \prod_{i=A,B} \int \frac{d^3 k_i}{(2\pi)^3} \frac{\phi_i(k_i)}{N(k_i)} \int \frac{d^3 \bar{k}_i}{(2\pi)^3} \frac{\phi_i^*(\bar{k}_i)}{N(\bar{k}_i)} \right) \times e^{i b \cdot (k_{AB} - k_{BB})} \left( \text{out} \langle \{ p_f \} | \{ k_i \} \rangle_{\text{in}} \left( \text{out} \langle \{ p_f \} | \{ k_i \} \rangle_{\text{in}} \right)^* \right) ,
\]  

(3.7)
where the product symbol again applies to everything inside parentheses, while now the integral signs also apply to everything outside the parentheses. The overlap of the in and out states of definite momentum can be written in terms of the $S$-matrix as usual:

$$\langle \{ p_f \} | \{ k_i \} \rangle_{\text{in}} = \lim_{T \to \infty} \langle \{ p_f \} | e^{-iH(2T)} | \{ k_i \} \rangle \equiv \langle \{ p_f \} | S | \{ k_i \} \rangle ,$$ \hspace{1cm} (3.8)

where the states after the first and second equality are now defined at a common reference time. The $S$-matrix is split as usual in an interacting and a noninteracting part:

$$S = 1 + iT .$$ \hspace{1cm} (3.9)

In terms of the interacting part $T$ we define the matrix element $M$:

$$\langle \{ p_f \} | iT | \{ k_i \} \rangle = (2\pi)^4 \delta^{(4)} \left( \sum p_f - \sum k_i \right) iM (\{ k_i \} \to \{ p_f \}) ,$$ \hspace{1cm} (3.10)

where we pulled out a delta function that represents conservation of total momentum. The idea of the wavepacket approach is that the wavepackets are concentrated around a definite momentum in the far past (and future). In this way they represent good in (and out) states. We use six of the ten delta functions in Eq. (3.7) to perform the integrals over $\vec{k}_A$ and $\vec{k}_B$ (the integral over $d^2b$ gives a delta function $(2\pi)^2 \delta^{(2)} (k_B^\perp - \vec{k}_B^\perp)$). By using that the wavepackets are localized in momentum space we can then write

$$d\sigma = \left( \prod_f \frac{d^3p_f}{(2\pi)^3 (N_f(p_f))^2} \right) \left| \frac{\partial E_A}{\partial \vec{p}_A} - \frac{\partial E_B}{\partial \vec{p}_B} \right|^{-1} \frac{|M(p_A,p_B \to \{ p_f \})|^2}{(N_A(p_A)N_B(p_B))^2} \times (2\pi)^4 \delta^{(4)} (p_A + p_B - \sum p_f ) .$$ \hspace{1cm} (3.11)

There are a few differences between this expression for the Lorentz-violating case and for the Lorentz-symmetric case, which are not obvious from Eq. (3.11). The most important one comes from the fact that for many of the Lorentz-violating parameters, the degeneracy of the energy eigenvalues is lifted. This means that we have to take care when we sum the expression in Eq. (3.11) over the different spin states. Almost all factors in Eq. (3.11) will depend on these spin states and every relevant integral will have to be performed, before one can perform the polarization sum.

Another difference compared to the conventional case involves the factor $|\partial E_A / \partial \vec{p}_A - \partial E_B / \partial \vec{p}_B|$, which comes from the integral of the energy delta function over momentum. As we saw in the previous chapter we still identify $\partial E / \partial \vec{p}$ with the group velocity of a particle, so this factor still involves the relative velocity of the incoming particles. However, since the dispersion relation is modified, the velocity-momentum relation is not as simple as in the conventional case and therefore we cannot write this prefactor in a manifestly Lorentz-covariant form anymore. The factor

$$F = \left| \frac{\partial E_A}{\partial \vec{p}_A} - \frac{\partial E_B}{\partial \vec{p}_B} \right| (N_A(p_A)N_B(p_B))^2 ,$$ \hspace{1cm} (3.12)

by which we divide in Eq. (3.11), can, however, still be interpreted as representing the properties of the initial beams of particles, i.e. the flux of incoming particles [38].
As we saw in the previous chapter, in general Lorentz violation determines the spin state of the particle. The spinor is completely determined by the Lorentz-violating Dirac equation. Some parameters, like $c_{\mu\nu}$ and $a_\mu$ are an exception, since they leave the energy of the spin states degenerate. Other parameters will lift this degeneracy. As mentioned, this prevents us from summing the expression for a cross section over spin states before doing any relevant integrals. It also has consequences for experiments involving polarized initial or final particles. We can ask what the in and out states are that represent these polarized particles. Since the states following from the Dirac equation in general have different energies, we cannot just use some superpositions of these states. Furthermore, we can ask what the correct direction of the spin is. This will be determined by the Lorentz-breaking coefficients, together with some polarization mechanism. Let us say that the experiment somehow uses a magnetic field to polarize the initial particles. The best way to determine the initial state seems to be to solve the Dirac equation, including the magnetic field. This is, however, far from trivial in most cases. Another possibility is that the polarization of the initial particle is determined by a preceding decay reaction. An example of this is muons created by pion decay. In the Lorentz-symmetric case these are almost fully longitudinally polarized, due to conservation of angular momentum. When we include Lorentz violation, this can be changed and we should in principle calculate the initial state of the muons from Lorentz-violating pion decay. We will deal with these matters for a few Lorentz-violating parameters in Chapter 6.

### 3.2 Unitarity and the largest-time equation

Unitarity is an essential feature of a quantum field theory, since it is necessary for the conservation of probability and therefore the ability of the theory to describe physical reality. We will briefly discuss it here, using the largest-time equation [17], although not in any rigorous way. The main aim is to establish the “cut equation” and the Källén-Lehmann representation of the interacting two-point function, which we will use later on in the discussions of the LSZ reduction formula and decay rates.

Unitarity of the S-matrix can be written as $S^\dag S = 1$, which means for the $T$ matrix, as defined in Eq. (3.9), that

$$-i(T - T^\dag) = T^\dag T . \quad (3.13)$$

Considering the matrix element of this equation between two states $|a\rangle$ and $|b\rangle$ and inserting a complete set of states on the right-hand side we get the schematic equation

$$-i \left( \langle b| T |a\rangle - \langle b| T^\dag |a\rangle \right) = \sum_c \langle b| T |c\rangle \langle c| T^\dag |a\rangle . \quad (3.14)$$

In the sum over all intermediate states $c$ there is an implied integration over phase space, together with the appropriate normalization factors (e.g. see Eq. (2.28)). Taking states $a$ and $b$ to be identical two-particle states, we get a relation that is called the optical theorem, given by

$$2 \text{Im} \mathcal{M} (k_1, k_2 \to k_1, k_2) \propto \sigma_{\text{tot}}(k_1, k_2 \to \text{anything}) . \quad (3.15)$$
3.2 Unitarity and the Largest-Time Equation

Figure 3.1: A random diagram in coordinate space. We assume that $x_2$ has the largest time.

This theorem expresses the relation between the imaginary part of the forward scattering amplitude and the total cross section. The proportionality factor can be determined from Eq. (3.11) and is equal to the flux factor given in Eq. (3.12), which reduces to $4E_{\text{cm}}p_{\text{cm}}$ in the Lorentz-symmetric case.

To proceed, we need a version of the optical theorem, formulated in terms of Feynman diagrams. For this we discuss the largest-time equation \[17\] and the cut equation that follows from it. For simplicity we start with a Lorentz-violating scalar field with mass $m$, as described in Section 2.4, that interacts with itself through vertices that we do not specify, except for the fact that they are proportional to the imaginary unit $i$. Feynman diagrams are built out of these vertices, connected by Feynman propagators, given by

$$D_F(z) = \Theta(z^0)D_+(z) + \Theta(-z^0)D_-(z),$$

where

$$D_{\pm}(z) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot z} (2\pi)\delta(p^2 - m^2)\Theta(\pm p^0),$$

and $p^2 = p^\mu p_\mu$. The expression for the Feynman propagator corresponds to the one given in Eq. (2.31), as checked for example by performing the contour integral over $\lambda^0$ in Eq. (2.31). The Feynman propagator is equal to $D_+(z)$ if $z^0 > 0$ and equal to $D_-(z)$ if $z^0 < 0$, while $D_+^*(z) = D_-(z)$. Now consider a Feynman diagram in coordinate space, for example as in Fig. 3.1. Assume that one of the vertices, say $x_2$, has the largest time, i.e. $(x_2)^0 > x_m$ for any $m$ with $m \neq 2$. Then any propagator $D_F$ with an argument $x_2 - x_m$ may be replaced by $D_+$ with the same argument. Similarly $D_F(x_m - x_2)$ may be replaced by $D_-(x_m - x_2)$. To take this further, we want to put this in terms of Feynman diagrams. To this end, one defines a set of additional Feynman rules introducing certain circled vertices. These rules are given by

- A circled vertex introduces a minus sign (anticipating complex conjugation).

---

\[1\] This is equivalent to the earlier treatment by Cutkosky \[39\], however, we consider the treatment in Ref. \[17\] to be more accessible and intuitive.
Figure 3.2: The sum of a diagram with all its possible circlings vanishes. The diagrams cancel pairwise.

- A line connecting two uncircled (circled) vertices corresponds to a propagator \( D_F \) (\( D_F^* \)).
- A line connecting a uncircled to a circled vertex corresponds to \( D_+ \).
- A line connecting a circled to an uncircled vertex corresponds to \( D_- \).

With this set of rules reconsider the diagram in Fig. 3.1 and again assume that \((x_2)_0^\circ\) is the largest time. If we then circle the vertex corresponding to \( x_2 \) the resulting diagram is equal to minus the original diagram (due to the fact that a circled vertex gets a minus sign). This holds in general when we circle the point with the largest time, regardless of which other vertices are circled. This relation is called the largest-time equation. It states that in any diagram, with any number of vertices circled, the diagram with the largest-time vertex circled equals minus the diagram with that largest-time vertex uncircled.

But what happens when we do not know which vertex corresponds to the largest time? The treatment of that case will give rise to the cutting equation. Consider an arbitrary diagram and circle its vertices in all possible ways and then sum all the resulting diagrams (see Fig. 3.2). We can write this a bit more formally by representing a diagram with \( n \) vertices by a function \( F(x_1, \cdots, x_n) \) and letting underlined coordinates correspond to circled vertices. Then the sum of all possible underlinings (all possible ways of circling the vertices) vanishes. The diagrams in the sum will cancel pairwise, depending on which time is the largest time. This is written as

\[
\sum_{\text{all underlinings}} F(x_1, \cdots, x_n) = 0 , \tag{3.18}
\]

or as

\[
F(x_1, \cdots, x_n) + F(\bar{x}_1, \cdots, \bar{x}_n) = - \sum_{\text{underlinings} \setminus \{0, \text{all}\}} F(x_1, \cdots, x_n) , \tag{3.19}
\]

where the first \( F \) represents the diagram with no underlinings and the second \( F \) corresponds to the one with all coordinates underlined (and thus all vertices circled). These are left out of the sum on the right-hand side. The diagram with all vertices circled is actually the complex conjugate of the diagram with no vertices circled. In Eq. (3.19) we therefore have an expression for the real part of the diagram, which corresponds to the imaginary part of the \( T \) matrix, due to the extra \( i \) in \( S = 1 + iT \).

To consider the \( S \) matrix, we add external legs to some vertices and integrate all coordinates associated to the vertices over all of spacetime. The relation in Eq. (3.19) remains valid, also for the integrated case. Due to the integrations we now have momentum conservation at each vertex. Furthermore, the consequence of the \( \Theta \) functions in the expressions
for $D_\pm$, given in Eq. (3.16), is that they limit the flow of energy. For propagators attached to one circled and one uncircled vertex, positive energy has to flow from the uncircled to the circled vertex. Otherwise the diagram vanishes. For propagators $D_F$ and $D^*_F$, which are connected to two uncircled or circled vertices, there are no restrictions. This means that certain diagrams on the right-hand side of Eq. (3.19) will vanish. If there is an isolated circled or uncircled vertex, for example, it has to act as a sink or a source for energy, which conflicts with energy conservation. This can be extended to regions, i.e. connected sets of circled or uncircled vertices. Also these regions have to be connected to external lines to make energy conservation possible. Diagrams with isolated regions of (un)circled vertices will therefore vanish. Regions of circled vertices have to be connected to one or more outgoing lines, while regions of uncircled vertices have to be connected to one or more incoming lines.

Instead of circled and uncircled vertices, we can introduce shadowed lines that cut through the diagram. All points on the shadowed side of the line are circled, while all point on the unshadowed side of the line are uncircled (see Fig. 3.3). This has as a consequence that energy can flow only across such a line from the unshadowed to the shadowed side. We can now write Eq. (3.19) in terms of different ways of cutting the diagrams as

$$F + F^* = -\sum_{\text{cuttings}} F,$$

where the functions $F$ now depend on the external momenta. This equation is called the cutting equation and it relates the real part of a diagram to the sum over all possible ways of cutting that diagram. It is essentially the optical theorem in terms of Feynman diagrams.

To see what the right-hand side of the cutting equation means, we consider the diagram in Fig. 3.4, with incoming momenta, $k_1$ and $k_2$ and loop momentum $q$. The only nonvanishing diagram on the right-hand side of the cutting equation corresponds to the third diagram in Fig. 3.3. It can be written as

$$\frac{1}{2} \frac{\lambda^2}{(2\pi)^4} \int d^4q \Theta \left( \frac{k_0}{2} - q^0 \right) \Theta \left( \frac{k_0}{2} + q^0 \right) \delta^{(4)} \left( \left( \frac{k}{2} - \tilde{q} \right)^2 - m^2 \right) \delta^{(4)} \left( \left( \frac{k}{2} + \tilde{q} \right)^2 - m^2 \right),$$

where, to be explicit, we included a factor $-i\lambda$ for each vertex (notice, however, the minus sign due to the circled vertex). We recognize that we can obtain this expression from the conventional Feynman rules for the diagram in Fig. 3.4, by replacing each Feynman propagator that gets cut, by a delta function and a theta function:

$$\frac{1}{p^2 - m^2 + i\epsilon} \to -2\pi i \delta(p^2 - m^2)\Theta(p^0).$$

(3.22)
CROSS SECTIONS AND DECAY RATES

Figure 3.4: Scalar loop diagram with incoming momenta $k_1$, $k_2$ and loop momentum $q$.

This corresponds to putting each of the cut propagators on shell. By substituting

$$\int d^4q = \int d^4p_1 \int d^4p_2 \delta^{(4)}(p_1 + p_2 - k) \quad (3.23)$$

and performing the integrals over $(p_1)^0$ and $(p_2)^0$ we may write the expression in Eq. (3.21) as

$$\frac{\lambda^2}{2} \int \frac{d^3p_1}{(2\pi)^3} \frac{1}{N^2_{p_1}} \int \frac{d^3p_2}{(2\pi)^3} \frac{1}{N^2_{p_2}} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k). \quad (3.24)$$

What we have obtained is the square of the leading-order scattering amplitude, i.e. $\lambda^2$, integrated over the two-body phase space, defined as in Eq. (2.29) for the Lorentz-violating scalar field, together with a symmetry factor $1/2$ for identical bosons in the final state. This corresponds to the right-hand side of Eq. (3.15). To summarize, it is established [17] that the real part of a diagram corresponds to the sum of all possible ways of ‘cutting’ that diagram such that the cut propagators can be put simultaneously on shell. This then corresponds to the right-hand side of Eq. (3.14). There is one extra demand here to make this identification work: the interaction Lagrangian must be Hermitian. Since the vertices are not complex conjugated in the cutting equation, to identify the shadowed side of the diagrams with $T^\dagger$, we have to demand that the vertices are either real, or that their complex conjugates exist, i.e. the interaction Lagrangian must be Hermitian. Using this one can then prove the unitarity of a theory containing only these Lorentz-violating scalar fields. We will not do this explicitly here. However the Lorentz violation for the scalar-field case, changes almost nothing about the derivation of the cutting equation, so the proof is obvious when comparing to [17], as long as one only considers concordant frames.

Just as in the case of a Lorentz-symmetric theory, fermions and vector fields introduce a few more complications when treating the unitarity of the $S$ matrix. This holds in particular for gauge bosons, because the gauge symmetry necessitates the treatment of ghosts and the use of Becchi-Rouet-Stora-Tyutin (BRST) symmetry. We will here only sketch the fermion case. The full treatment of the other subjects, although interesting, lies outside the scope of this thesis.

The general fermion propagator is given in Eq. (2.69) as

$$S_F(z) = \int \frac{d^4p}{(2\pi)^4} \frac{i \text{Adj}(\bar{\Gamma} \cdot p - \tilde{M})}{\det(\bar{\Gamma} \cdot p - M)} e^{-ipz}. \quad (3.25)$$

By performing the contour integral over $p^0$ we see that this can be written as

$$S_F(z) = \Theta(z^0)S_+(z) + \Theta(-z^0)S_-(z), \quad (3.26)$$
where $S_+(z)$ and $S_-(z)$ are given in Eqs. (2.68a) and (2.68b), respectively, and can be written as $S_\pm(z) = \text{Adj} \left( i\Gamma \cdot \partial - \hat{M} \right) \Delta_\pm(z)$, with

$$
\Delta_+(z) = \int \frac{d^3p}{(2\pi)^3} \sum_\alpha \frac{e^{-ip_\alpha z}}{(E^\alpha_u(p) - E^\alpha_v(p))(E^\alpha_u(p) + E^1_1(-p))(E^\alpha_2(p) + E^2_2(-p))},
$$

$$
\Delta_-(z) = \int \frac{d^3p}{(2\pi)^3} \sum_\alpha \frac{e^{ip_\alpha z}}{(E^\alpha_u(p) - E^\alpha_v(p))(E^\alpha_u(p) + E^1_1(-p))(E^\alpha_2(p) + E^2_2(-p))}.
$$

(3.27a)

(3.27b)

In the Lorentz-symmetric case we would now again have a suitable splitting of the Feynman propagator where $S_\pm(z)$ can be written in terms of theta functions that limit the flow of positive energy as in the above treatment of the scalar particle. For the Lorentz-violating case, matters are a little more complicated. Our first guess for an expression of $\Delta_\pm(z)$ involving such a theta function would be

$$
\Delta_\pm(z) \approx \int \frac{d^4\lambda}{(2\pi)^4} (2\pi) \delta \left( \det \left( i\Gamma \cdot \lambda - \hat{M} \right) \right) \Theta(\pm \lambda^0) e^{-i\lambda z}.
$$

(3.28)

This almost gives the correct result, i.e. the expressions in Eqs. (3.27). However, when we perform the integral over $\lambda^0$, we have to divide by the absolute value of the derivative of the argument of the delta function. Therefore, we get Eqs. (3.27), except with the absolute value of the denominators in the sum over $\alpha$. This means that one of the terms in the sum has the wrong sign, depending on whether $E^1_{u,v} > E^2_{u,v}$ or vice versa. So, we have this problem if there are nonvanishing parameters that lift degeneracy of the energies of the two spin states. Moreover, we find that $S^*_+(z) \neq S_-(z)$ for most of the Lorentz-violating parameters. In particular, this holds for CPT-violating parameters. When the theory contains nonzero CPT-violating parameters, the energy of particles and antiparticles is nondegenerate. In other words, an antiparticle cannot be interpreted as a particle moving backward in time. So the lifting of the degeneracy of the four energies of the Dirac particles severely hampers the treatment of fermions in the context of the largest-time equation and the cutting equation. The only mSME parameter that does not lift the energy degeneracy is the $c^{\mu\nu}$ parameter (the equivalent of the scalar $K^{\mu\nu}$ parameter).

This does not automatically result in the violation of unitarity by other SME parameters. At this point, it just means that the largest-time equation and the cutting equation, as described above, cannot be used to derive the unitarity of the $S$ matrix or the optical theorem in terms of Feynman diagrams. This is an interesting issue and it should be investigated further. For the moment, however, we leave it as is and move on to the Källén-Lehmann representation.

### 3.3 The Källén-Lehmann representation

The Källén-Lehmann representation of the time-ordered interacting two-point function shows that one can write this function as a weighted sum of Feynman propagators of
particles with different masses. The weight function is called the spectral density function. In Ref. [40] this representation was derived for a scalar and fermion field with Lorentz-violating parameters $c^{\mu\nu}$ (for the scalar field we called the parameter $K^{\mu\nu}$ in Section 2.4). We sketch here a different derivation, using the machinery described in the previous section (i.e. adapted from Ref. [17]). We first discuss the case of a Lorentz-violating scalar field.

Consider an arbitrary diagram which is unspecified except for the two points $x$ and $y$ (see Fig. 3.5). We distinguish two cases, namely the one where $y^0 > x^0$ and the case where $x^0 > y^0$. Using the largest-time equation, we can write for this diagram

$$
\Theta(y^0 - x^0) \sum_{\text{circlings}\{x\}} F(x, y, \cdots) = 0, \tag{3.29a}
$$

$$
\Theta(x^0 - y^0) \sum_{\text{circlings}\{y\}} F(x, y, \cdots) = 0, \tag{3.29b}
$$

where we do not include the circling of the point $x$ in the first case, or the circling of $y$ in the second case, since they are never the largest time anyway. We can sum these two expressions and take the terms with no circlings to the other side of the equality. We get

$$
-F(x, y, \cdots) = \Theta(y^0 - x^0) \sum_{\text{circlings}\{x,0\}} F(x, y, \cdots) + \Theta(x^0 - y^0) \sum_{\text{circlings}\{y,0\}} F(x, y, \cdots), \tag{3.30}
$$

since $\Theta(z) + \Theta(-z) = 1$. When we integrate this over all vertex points except $x$ and $y$, we can use the energy theta functions in the expression of $D_\pm$ of the cut propagators, to rewrite this in terms of all possible cuttings, as we did in the previous section. The result is

$$
-f(y - x) = \Theta(x^0 - y^0) F^+(y - x) + \Theta(y^0 - x^0) F^-(y - x), \tag{3.31}
$$

where

$$
F^+(y - x) = \sum_{\text{cuts}\{x,0\}} f(y - x), \tag{3.32a}
$$

$$
F^-(y - x) = \sum_{\text{cuts}\{y,0\}} f(y - x), \tag{3.32b}
$$

while $f$ is $F$ integrated over all vertex points except for $x$ and $y$. These functions depend only on $y - x$, because they are translationally invariant.
The function $F^+(y-x)$ can be represented by the diagram in Fig. 3.6. The cut in that diagram represents all possible cuts of the unknown part of the diagram. Consider next the Fourier transform of $F^+$ in the observer frame where $y=0$ (we can go to this frame because of translational invariance):

$$\tilde{F}^+(k) = \int d^4 x \ F^+(x) e^{-ikx} .$$ (3.33)

This amounts to attaching an external leg to the point $x$. Due to the fact that the $F^+$ includes all possible cuts, except one that puts $x$ on the shadowed side, we must have that $k^0 > 0$. This is because positive energy must flow from the unshadowed to the shadowed side of the cut for the diagram not to vanish. Furthermore, all cut propagators will also have a delta function that puts the corresponding particle on shell, i.e. $\tilde{p}_i^2 = m_i^2$, with $i$ labeling the particle. By momentum conservation we know that $k = \sum_i p_i$, where $p_i$ are the momenta in all the cut propagators. This then results in

$$\tilde{k} = \sum_i \tilde{p}_i ,$$ (3.34)

from which it follows that

$$\tilde{k}^2 = \sum_i \tilde{p}_i^2 + \sum_{i\neq j} \tilde{p}_i \cdot \tilde{p}_j \geq \sum_i m_i^2 + \sum_{i\neq j} |\tilde{p}_i||\tilde{p}_j|(1 - \cos \theta_{ij}) \geq 0 .$$ (3.35)

The first inequality can be derived using the dispersion relation $\tilde{p}_i^2 = m_i^2$. So we have that both $k^0$ and $\tilde{k}^2$ have to be larger than zero. We make this explicit by including theta functions in the expression for $\tilde{F}^+(k)$:

$$\tilde{F}^+(k) = \Theta(k^0)\Theta(\tilde{k}^2)\rho(\tilde{k}^2) .$$ (3.36)

As we will see in the next section, the function $\rho$ can depend on other observer Lorentz invariant structures than $\tilde{k}^2$, but we will suppress that dependence for the moment. $F^+$ can now be written as

$$F^+(-x) = \int_0^\infty ds \ \rho(s) \int \frac{d^4 k}{(2\pi)^4} \Theta(k^0)\Theta(\tilde{k}^2)\delta(s - \tilde{k}^2)e^{iks} = \int_0^\infty ds \ \frac{\rho(s)}{2\pi} D_+(-x,s) ,$$ (3.37)
where \( D_+(-x,s) \) is the expression given in Eq. (3.17) for a particle with invariant mass squared \( m^2 = s \). The function \( \rho(s) \) will of course become the spectral density function. From the definition of \( F^+ \) it can be seen that it corresponds to the right-hand side of the cutting equation in Eq. (3.20) (up to a sign and a factor \( 2\pi \)). The left-hand side of this equation is the original diagram plus its complex conjugate, which means that \( \rho(s) \) is real. Since \( F^- \) is the complex conjugate of \( F^+ \), we can, going back to Eq. (3.31), now write that
\[
-f(z) = \int_0^\infty ds \frac{\rho(s)}{2\pi} D_F(z, s),
\]
(3.38)
where \( D_F(z, s) = \Theta(z^0)D_+(z, s) + \Theta(-z^0)D_-(z, s) \) is the Feynman propagator for a particle with invariant mass squared \( m^2 = s \). If we, instead of one diagram, consider the sum of all diagrams in this equation then \( \rho(s) \) is positive (see Ref. [17]). This is then finally the Källén-Lehmann representation of the interacting two point function. It can be written as
\[
\langle \Omega | T\phi(x)\phi(y) | \Omega \rangle = \int ds \rho(s) \int \frac{d^4p}{(2\pi)^4} \frac{i}{\sqrt{p^2 - s + i\epsilon}} e^{-i(p(x-y))},
\]
(3.39)
As in the previous section, when considering SME coefficients other than \( K^{\mu\nu} \) for the scalar particle and \( c^{\mu\nu} \) for the fermion, the lifting of the degeneracy of the four energy eigenvalues causes problems for the treatment in terms of the cutting equation. Again, we will not consider these problems and their (as of yet unknown) solutions here, since they lie outside the scope of this thesis.

### 3.4 The LSZ reduction formula

As is the case when Lorentz symmetry is conserved, we now need to determine the matrix element \( \mathcal{M} \), defined in Eq. (3.10), in the interacting theory. We want to know how to calculate \( \mathcal{M} \) in terms of Feynman diagrams. The formula that establishes this relation is called the LSZ reduction formula. It relates \( S \)-matrix elements to correlation functions in the interacting theory. The latter can then be calculated in perturbation theory, using Feynman diagrams. In the SME one can derive a generalization of the LSZ reduction formula. This was done in Ref. [16], with important contributions from Ref. [40]. This derivation was limited to a the QED limit of the SME, with a subset of the possible Lorentz-violating coefficients. However, no major difficulties for the other parameters are anticipated. In particular when limiting to tree-level calculations, because in Ref [16] differences with the naïve expectations are only found at loop level. We will not rederive everything that is in Refs. [16] and [40], but we will mention some of the most important results.

In Ref. [40] the Källén-Lehmann representation of the time-ordered interacting two-point function is derived for a scalar and a fermion field. This was done including a Lorentz-violating \( c^{\mu\nu} \) parameter. The Källén-Lehmann representation is a nonperturbative result in field theory that expresses the two-point function in terms of a spectral density function as
\[
\langle \Omega | T\psi(x)\psi(y) | \Omega \rangle = \int ds \rho(s) \int \frac{d^4p}{(2\pi)^4} \frac{i}{\sqrt{p^2 - s + i\epsilon}} e^{-i(p(x-y))},
\]
(3.40)
where $\rho(s)$ is the spectral density function. Typically it can be separated into an isolated delta function for the one-particle states and a multi-particle continuum. As shown in Ref. [40], in the Lorentz-violating case the structure of the pole for the one-particle part is modified by the Lorentz-violation. It is expected to include new structures that one would not expect na"ively from looking at the Lagrangian. This is because usually the spectral density function can only depend on $p^2$, since it must be (observer) Lorentz invariant. However, now it can also depend on $c^{\mu\nu}$ for example. This allows for structures of the form

\[ (c^i)^p_p \equiv p \cdot c^i \cdot p = p^\mu g_{\mu\alpha_1} c^{\alpha_1\beta_1} \cdots g_{\beta_i\nu} p^\nu. \] (3.41)

For fermions also the analogously defined structure $(c^i)_{\gamma}^p$ (the last $p$ replaced by a gamma matrix) is allowed. It is also noted in Ref. [40] that for very high energies the one-particle part of the spectral density function could overlap with the multiparticle continuum. This turns the otherwise stable one-particle state into a resonance for some (very high) momenta. This leads to a group velocity larger than one and to Cherenkov-type decays. This is basically a result that follows from the dispersion relation, which is now modified by extra Lorentz-violating structures.

In Ref. [16] the two-point function was examined more explicitly but in a perturbative way. Explicit results where obtained up to one-loop order. This was done for a Lorentz-violating QED model, including a symmetric $c^{\mu\nu}$ parameter for the fermion and a $k^{\mu\nu}$ for the photon. The latter is related to the $k_\xi^{\mu\nu\rho\sigma}$ coefficient, which comes from the QED limit of the SME [13].

The denominator of the two-point function – usually given by $\not p - m - \Sigma(p)$ – is now

\[ \Gamma^{(2)}(p) = \Gamma^\mu p_\mu - m - \Sigma(p), \] (3.42)

where $\Sigma(p)$ is the contribution of the one-particle irreducible diagrams. It is $\Sigma(p)$ that can, and does, depend on the different structures mentioned before, due to the Lorentz-violation. These structures do not commute with $\not p$ and this complicates the explicit determination of the pole and the renormalization function of the wavefunction. Nevertheless the authors of Ref. [16] succeeded and found that close to the pole

\[ \Gamma^{(2)} = Z_R^{-1} \bar P(p) + \bar P(p) \Sigma(p) \bar P(p). \] (3.43)

Here $Z_R$ is the renormalization function of the wavefunction and $\bar P(p)$ is the Dirac operator that is usually given by $\not p - m_{\text{phys}}$. In Ref. [16] these two quantities were calculated explicitly to first order in the coupling constant and in the Lorentz violation. The renormalization function was found to be

\[ Z_R^{-1} = 1 - \frac{\alpha}{\pi} \left[ \ln \left( \frac{m}{m_\gamma} \right) - 1 + \frac{\gamma_E}{4} + \frac{1}{4} \ln \left( \frac{m^2}{4\pi\mu^2} \right) \right] - \frac{2\alpha}{3\pi m^2} \left( 2c_\mu^p - \tilde k_\mu^p \right), \] (3.44)

where $m_\gamma$ is a photon mass to regularize the IR divergences, $\gamma_E$ is the Euler-Mascheroni constant, and $\mu$ is a mass scale introduced by dimensional regularization. The loop-corrected Dirac-operator is

\[ \bar P(p) = \not p + (c_{\text{phys}})^p_\gamma - m_{\text{phys}} - \frac{\alpha}{3\pi m} \left( 2(c_{\text{phys}})^p_\gamma - (\tilde k_{\text{phys}})^p_\gamma \right), \] (3.45)
where \( m_{\text{phys}} \) is, as usual when using minimal subtraction to renormalize, given by

\[
m_{\text{phys}} = m \left[ 1 + \frac{\alpha}{\pi} \left( 1 + \frac{3}{4} \gamma_E - \frac{3}{4} \ln \left( \frac{m^2}{4\pi\mu^2} \right) \right) \right]
\]  
(3.46)

and

\[
(c_{\text{phys}})_{\mu\nu} = c_{\mu\nu} + \frac{\alpha}{\pi} \left[ -\frac{29}{36} + \frac{\gamma_E}{3} + \frac{1}{3} \ln \left( \frac{m^2}{4\pi\mu^2} \right) \right] (2c_{\mu\nu} - \tilde{k}_{\mu\nu}) .
\]  
(3.47)

The form of \( \tilde{k}_{\mu\nu} \) was not calculated in Ref. [16], but since it will differ from the tree-level value by an expression of order \( \alpha \), we can just substitute \( \tilde{k}_{\mu\nu} \) in Eq. (3.45), because we neglect terms of order \( \alpha^2 \).

For the derivation of the Lorentz-violating generalization of the LSZ reduction formula the important parts in the above are contained in Eqs. (3.44) and (3.45). We see that \( Z_R \) in Eq. (3.44) depends on momentum, which does not happen in the conventional Lorentz-symmetric case. Furthermore the operator \( \tilde{P}(p) \) in Eq. (3.45) is of a different form than the tree-level operator

\[
\tilde{P}(p)_{\text{tree}} = \not{p} + c_{\gamma} - m .
\]  
(3.48)

This also does not happen in the Lorentz-symmetric case, where no terms with a different structure are introduced by the loop corrections. Both these matters complicate the derivation of the LSZ reduction formula. It is however still possible, as is shown in Ref. [16] and the Lorentz-violating generalization of the LSZ reduction formula (for fermions) is found to be

\[
\langle f|i \rangle = \int d^4x_1 \cdots d^4y_1 \cdots d^4x' \cdots d^4y' \cdots e^{-i(p_1 \cdot x_1 + \cdots + p_l \cdot x_1 + \cdots - q_1 \cdot y_1 - \cdots - q' \cdot y'_1 - \cdots)}
\]

\[
\times \cdots (-i) Z_{R}^{-1/2}(q_1) \bar{u}^a(q_1) \tilde{P}(-q_1) \cdots i Z_{R}^{-1/2}(p'_1) \bar{v}^a(p'_1) \tilde{P}(-p'_1) \cdots
\]

\[
\times \langle \Omega | T \cdots \bar{\psi}(y'_1) \cdots \psi(y_1) \bar{\psi}(x'_1) \cdots \psi(x_1) \cdots | \Omega \rangle
\]

\[
\times \tilde{P}(p_1) u^a(p_1)(-i)Z_{R}^{-1/2}(p_1) \cdots \tilde{P}(q'_1) v^a(q'_1) i Z_{R}^{-1/2}(q'_1) \cdots
\]

\[
+ \text{disconnected terms}
\]  
(3.49)

where \( \langle f|i \rangle \) is the S-matrix element defined by

\[
\langle f|i \rangle \equiv \langle p_1 \cdots p'_1 \cdots | S | k_1 \cdots k'_1 \cdots \rangle .
\]  
(3.50)

The momenta of the incoming fermions, incoming antifermions, outgoing fermions, and outgoing antifermions are labeled by \( p, p', q, \) and \( q' \) respectively, with a subscript to label the different particles. The corresponding space-time variables are respectively denoted by \( x, x', y, \) and \( y' \). There are a few important differences with the conventional Lorentz-symmetric case, besides the presence of Lorentz-violating coefficients. Firstly, as noted before, the renormalization function \( Z_R \) depends on momentum and the Dirac operator \( \tilde{P}(p) \) contains structures that are not present in the tree-level operator in Eq. (3.48). But also the spinors that project out the correct fermion states differ from the naïve tree-level expectation, since they should obey the equations

\[
\tilde{P}(p) u^a(p) = 0 \quad (p^0 > 0),
\]  
(3.51)

\[
\tilde{P}(p) v^a(p) = 0 \quad (p^0 < 0).
\]  
(3.52)
They are thus also modified by the one-loop corrections that appear in Eq. (3.45). This does not occur in the Lorentz-symmetric case, where the spinors keep the same form at any order of the perturbative calculation. Explicit expressions for the spinors can be found in Ref. [16].

The conclusion from all this is two-fold. On the one hand, we see that the LSZ reduction formula and thus the way one calculates cross sections and decay rates is modified in ways that one might not expect from looking at tree-level results. On the other hand, all these effects occur only at loop level. Therefore all tree-level calculations are not affected and expressions for the spinors can be calculated directly from the Lagrangian.

### 3.5 Decay rates

As the final part of this chapter, we sketch the derivation of the expression for a decay rate of a particle described by a Lorentz violating scalar field. The decay rate of an unstable particle is defined as the number of decays per unit time into a specified final state, divided by the number of unstable particles present. The lifetime $\tau$ is the inverse of the sum of all decay rates into different final states. An unstable particle is usually modeled as a resonance in a scattering cross section. This is described by the relativistic Breit-Wigner formula, which says the scattering amplitude is proportional to:

$$f(p^0) \propto \frac{1}{\tilde{p}^2 - m^2 + im\Gamma} \approx \frac{1}{(1 + 2K^{00})(p_+^0 - p_0^0)(p_0^0 - p_+^0 + i\frac{m}{(1 + 2K^{00})(p_+^0 - p_0^0)\Gamma})}$$

$$= \frac{1}{N_p^2(p^0 - p_0^0 + i\frac{m}{N_p^2}\Gamma)},$$

(3.53)

where $p_+^0$ and $p_0^0$ correspond to the positive and negative energy solutions given in Eq. (2.16), while $N_p^2$ is defined in Eq. (2.21). We used the particle Lorentz violating quantity $\tilde{p}^2$ in the denominator, since this will correspond to the form of the full one particle propagator. It also makes sure that the position of the resonance in the cross section

$$\sigma \propto |f(p^0)|^2 \propto \left| \frac{1}{\tilde{p}^2 - m^2 + im\Gamma} \right|^2 \propto \frac{1}{(p^0 - p_+^0)^2 + m^2\Gamma^2}$$

is at the on-shell energy of the particle that is described by the corresponding propagator. The width of the resonance at half the maximum height is

$$\Gamma \equiv \text{FWHM} = \frac{2m}{N_p^2\Gamma}.$$  

(3.55)

The factor $2m/N_p^2$ reduces to the usual time dilatation factor $1/\gamma$ when the Lorentz violation is taken to zero. When Lorentz symmetry is obeyed, this time dilatation factor reduces to one in the restframe of the particle. In the case of the Lorentz-violating scalar field, however,

$$\frac{2m}{N_p^2} \rightarrow \frac{1}{\sqrt{1 + 2K^{00}}}$$

(3.56)
in the frame where $p = 0$.

To proceed, we compare the relativistic Breit-Wigner formula to the full interacting time-ordered two-point function for a Lorentz-violating scalar particle. When summing the geometric series of one-particle irreducible (1PI) diagram insertions in the one-particle propagator, the Fourier transform of this function can be written as

$$
\int d^4x e^{-ipx} \langle \Omega | \phi(x) \phi(0) | \Omega \rangle = \frac{i}{\tilde{p}^2 - m^2 - M^2(\tilde{p}^2)}.
$$

(3.57)

Here $-iM^2(\tilde{p}^2)$ is the sum of all 1PI insertions in the propagator. The function $M^2(\tilde{p}^2)$, actually can depend on other observer Lorentz invariants than $\tilde{p}^2$, just as the spectral density function in the Källén-Lehmann representation of the two-point function. However, we suppress this dependence here and in the following.

If the scalar field corresponding to Eq. (3.57) is unstable it can decay into other particles. Through the cutting equation, this implies that $M^2(\tilde{p}^2)$ has a nonzero imaginary part, since the right-hand side of Eq. (3.20) is nonzero. The pole in Eq. (3.57) is thus shifted from the real axis. When defining the physical mass of the particle as usual by

$$
m_{\text{phys}}^2 - m^2 - \text{Re} M^2(m_{\text{phys}}^2) = 0,
$$

we can write

$$
\int d^4x e^{-ipx} \langle \Omega | \phi(x) \phi(0) | \Omega \rangle \sim \frac{iZ}{\tilde{p}^2 - m_{\text{phys}}^2 - iZ \text{Im } M^2(\tilde{p}^2)},
$$

(3.58)

where the $\sim$ sign means that the two sides have the same pole structure in the complex plane and $Z$ is the wavefunction renormalization factor. When this propagator appears in the $s$ channel of a diagram, we can compare the the cross section given by

$$
\sigma \propto \left| \frac{1}{\tilde{p}^2 - m_{\text{phys}}^2 - iZ \text{Im } M^2(\tilde{p}^2)} \right|^2
$$

(3.59)

to the relativistic Breit-Wigner formula for the cross section in Eq. (3.54). It is not hard to see that

$$
\bar{\Gamma} = -\frac{Z}{m} \text{Im } M^2(m_{\text{phys}}^2).
$$

(3.60)

We evaluated $\text{Im } M^2(\tilde{p}^2)$ at the physical mass. This approximation is only justified if $\text{Im } M^2(\tilde{p}^2)$ is small such that the resonance is narrow. This also holds if Lorentz symmetry is obeyed.

The final step in the formula for the decay rate is a way to compute $\text{Im } M^2$. Since unstable particles never appear in asymptotic states we need the optical theorem in Eq. (3.15) for any amplitude that can be defined in terms of Feynman diagrams and not only for $S$-matrix elements. This version of the optical theorem corresponds to the cutting equation in Eq. (3.20). Using the cutting equation to relate the imaginary part of the sum of 1PI insertions in the propagator to the matrix element to all possible ways of cutting these diagrams, we can write

$$
\bar{\Gamma} = \frac{1}{2m} \sum_f d\Pi_f |\mathcal{M}(p \to f)|^2,
$$

(3.61)
where $\Pi_f$ denotes the phase space of all final particles. There is one final subtlety here, related to observations in Eqs. (3.55) and (3.56). The actual width of the resonance in a general frame is $(2m/N_p^2)\bar{\Gamma}$ and not $\bar{\Gamma}$. Therefore, we should write for the total physical decay rate, $\Gamma$, that

$$\Gamma = \frac{1}{N_p^2} \sum_f d\Pi_f |\mathcal{M}(p \to f)|^2.$$  

(3.62)

This does not reduce to Eq. (3.61) in the $p = 0$ frame or in the $v_g = 0$ frame which means that Eq. (3.61) is not the decay rate in the restframe nor the $p = 0$ frame. It is actually the decay rate in the $\tilde{p} = 0$ frame (with $\tilde{p} \approx p^\mu + K^{\mu\nu}p_\nu$).

Eq. (3.62) is the main result of this section. We used the results in the previous sections to derive it. In particular, we used the cutting equation, which followed from the largest-time equation, as described in Section 3.2. In that section we also noted that for fermions with general Lorentz-violating contributions to the Lagrangian, the proof of the cutting equation is not as straightforward as for the scalar field. Strictly speaking, Eq. (3.62) does therefore not apply to the general Lorentz-violating case. The problem lies in the derivation of diagrammatic form of the optical theorem, i.e. the cutting equation. We expect the $S$-matrix to be unitary by construction, since the Lagrangian is Hermitian. Therefore we expect the optical theorem as in Eq. (3.15) to hold. However, the $S$-matrix is defined in terms of asymptotic states, while unstable particles do never appear in asymptotic states. Nevertheless, we will take Eq. (3.62) as a working hypothesis also for other Lorentz-violating coefficients than $K^{\mu\nu}$ for the scalar field, albeit with other normalization factors than $N_p^2$ as defined in Eq. (3.56). Once again, a generalization of the diagrammatic version of the optical theorem to other Lorentz-violating coefficients would be of much interest, however, it lies outside the scope of this thesis.
Chapter 4

The $\chi^{\mu\nu}$ framework

In this chapter we will focus on a certain type of Lorentz-violating coefficients, namely the ones that modify the $W$-boson propagator, which we will parametrize by a general tensor $\chi^{\mu\nu}$. We will assume all other Lorentz-violating coefficients to be zero. This will simplify a lot of the aspects of the calculations in many cases, especially since the kinetic properties of other (external) particles are conventional. Using this framework we will calculate the allowed $\beta$-decay rate in Section 4.3, the rate of forbidden $\beta$-decay transitions in Chapter 5, and the decay rates of pions and muons in Chapter 6.

This chapter is based on:


4.1 Introduction

Most observables for Lorentz violation will get contributions from many Lorentz-violating coefficients, when we would do a calculation in the full SME. Calculating the expression, representing that observable, in terms of all the relevant parameters can become very unwieldy. The conventional way to approach this is to focus on one coefficient and to put all other parameters to zero. One then finds the expression for the desired observable in terms of this one parameter and compares with experiment to obtain bounds. Subsequently one repeats the process for other parameters, implicitly assuming that there are no cancellations between parameters.

To deal with Lorentz-violating modifications of weak-decay processes we use a slightly different approach. We summarize all Lorentz-violating contributions to the $W$-boson propagator by one tensor, since all contributions to this propagator must have the same tensor structure. The modified Lorentz-violating propagator we will use looks like

\[ \langle W^{\mu+}(p)W^{\nu-}(-p) \rangle = \frac{i(g^{\mu\nu} + \chi^{\mu\nu})}{M_W^2}. \]  

(4.1)

Notice that this is the propagator at low energy (i.e. we neglect terms of higher order in the four-momentum over the $W$ mass), since this is the relevant energy regime for all processes we want to consider. As we will see in the next section, also Lorentz-violating
contributions to a vertex to which the $W$-boson line connects give results that have exactly the same form as the Lorentz-violating tensor in Eq. (4.1) gives. Contributions of such a vertex are therefore also covered by this approach. Due to gauge invariance, however, such vertices always come with a Lorentz-violating contribution to the free-particle Lagrangian of the particles in the current that connects to the $W$ boson.

4.2 Contributions to $\chi^{\mu\nu}$

A complete list of parameters, or combinations thereof, that show up as $\chi^{\mu\nu}$ in Eq. (4.1) lies beyond the scope of this thesis, but in this section we will sketch what kind of contributions are relevant, illustrating the generality of $\chi^{\mu\nu}$ in the process. Firstly, we discuss direct contributions to the propagator of the $W$ boson coming from the kinetic gauge sector and the Higgs-gauge sector of an effective field theory (e.g. the SME). Secondly, we discuss contributions coming from vertex corrections.

To discuss contributions to the $W$-boson propagator we first consider the mSME [12, 13, 14]. Lorentz-violating contributions to the $W$-boson propagator at tree level come from the gauge sector and the Higgs sector of the mSME. Restricting ourselves to these sectors, we can obtain an expression for the $W$-boson propagator to first order in Lorentz violation and in unitarity gauge. It is given by

$$\langle W^{\mu+}(p)W^{\nu-}(-p)\rangle = \frac{-i}{p^2 - M_W^2} \left\{ g^{\mu\nu} - \frac{p^\mu p^\nu}{M_W^2} \left( k^{\mu\nu} + \frac{i}{2g} k^{\mu\phi} \phi \phi \right) - \frac{1}{p^2 - M_W^2} \left( 2k^{\mu\sigma\nu\rho} p_{\rho}p_{\sigma} + p^\mu p^\nu (k^{\rho\mu} + \frac{i}{2g} k^{\rho\phi} \phi) + p^\rho p^\mu (k^{\rho\mu} + \frac{i}{2g} k^{\rho\phi} \phi) \right) \right\}. \quad (4.2)$$

All coefficients that break Lorentz invariance are defined as in Ref. [14]. The coupling constant $g$ that appears in the denominator of the $k_{\phi\phi}$ terms is the $SU(2)$ coupling constant. From this the relevant parameters for processes at low energy can be identified. Comparing the low-energy approximation of Eq. (4.2) to Eq. (4.1) we see that

$$\chi^{\mu\nu} = -k^{\mu\nu}_{\phi\phi} - \frac{i}{2g} k^{\mu\nu}_{\phi\phi} \phi \phi + \frac{2p_{\rho}p_{\sigma}}{M_W^2} k^{\rho\sigma\mu\nu}_{W}. \quad (4.3)$$

Other terms are suppressed by powers of the four-momentum over $M_W$ relative to these.

Lorentz-violating terms of mass dimension higher than four will also give contributions to the propagator. In Ref. [29] a classification is given of dimension-five Lorentz-violating operators that are irreducible to lower-dimensional operators by the equations of motion. As an example of a dimension-five contribution to $\chi^{\mu\nu}$, we consider the Lorentz-violating operator coming from the pure gauge sector

$$\mathcal{L}_5 \supset C^{\mu\nu\rho} \text{tr} W_{\mu\lambda} D_\nu \tilde{W}_\rho^{\lambda}, \quad (4.4)$$

where $W_{\mu\nu}$ is the $SU(2)$ gauge field strength and $\tilde{W}_{\mu\nu}$ is its dual. The tensor $C^{\mu\nu\rho}$ is symmetric in all its indices and has mass dimension $-1$. (As discussed in Ref. [27] for
the photon, there are additional operators that contribute when the gauge boson is off-shell, as is the case for the processes we will consider.) When including only the term in Eq. (4.4), the W-boson propagator to first order and in unitarity gauge reads

\[
\langle W^{\mu+}(p)W^{\nu-}(-p) \rangle = \frac{-i}{p^2 - M_W^2} \left( g^{\mu\nu} - \frac{p^\mu p^\nu}{M_W^2} - \frac{4\epsilon^{\rho\mu\alpha\nu}C^{\lambda_3}g^{\rho\lambda_3}p_\lambda p_\beta p_\alpha}{(p^2 - M_W^2)} \right). \tag{4.5}
\]

Apart from the corrections in Eq. (4.5) and the preceding discussion, there are also corrections from the dimension-five Higgs-gauge sector and higher-dimensional operators. Lorentz-violating operators of dimension six and higher have not been fully classified [41, 42]. However, the general tensor \( \chi^{\mu\nu} \) in principle includes the effects of all Lorentz-violating contributions to the propagator. If all contributions to \( \chi^{\mu\nu} \) come from the W-boson propagator directly (i.e. not from the vertex), we can write

\[
\chi^{\mu\nu}_p = \sum_{n} Y_n^{\mu\nu\alpha_1\cdots\alpha_n} p_{\alpha_1} \cdots p_{\alpha_n}, \tag{4.6}
\]

where the subscript \( p \) on \( \chi^{\mu\nu}_p \) signals that we only consider contributions coming directly from the tree-level propagator. The mass dimension of \( Y_n \) is \(-n\), which implies that \( \chi^{\mu\nu}_p \) is dimensionless. This does not necessarily mean that \( Y_n \) is suppressed by \( n \) powers of the mass scale at which Lorentz symmetry is broken (e.g. the Planck mass), since it can contain powers of the W-boson mass. From Eq. (4.5), for example, we obtain \( Y_3^{\mu\lambda_3\alpha} = 4\epsilon^{\rho\mu\alpha\nu}C^{\lambda_3}g^{\rho\lambda_3}/M_W^2 \), which is suppressed by one power of the Lorentz-breaking scale, through the \( C \) coefficient of mass dimension \(-1\).

If we neglect terms that are suppressed by powers of the four momentum over the W-boson mass with respect to other terms containing the same Lorentz-violating parameter, hermiticity of the Lagrangian implies that

\[
\chi^{\mu\nu}_p(p) = \chi^{\nu\mu}_p(-p). \tag{4.7}
\]

This relation can be useful to limit the number of terms that appear in the expressions for observables when considering a particular Lorentz-violating operator. The dimension-five operator in Eq. (4.4), for example, has \( \chi^{\mu\nu}_{3\lambda_3} = -\chi^{\nu\mu}_{3\lambda_3} \). Since the \( \mu \) and \( \nu \) indices are on the epsilon tensor, \( \chi^{\mu\nu} \) is antisymmetric and real. A fact that can be used to severely simplify calculations. When the number of occurrences of \( p \) in \( \chi^{\mu\nu}_p \) is odd, CPT is violated.

We now look at contributions to \( \chi^{\mu\nu} \) coming from Lorentz-violating corrections to a vertex connecting a left-handed fermion current and a W boson. In general, the vertex will have the form

\[
-i\gamma_v(g^{\mu\nu} + \chi^{\mu\nu}_v), \tag{4.8}
\]

where the subscript \( v \) means “vertex.” When contracting this corrected vertex with a Lorentz-symmetric propagator at low energy, this will give the same correction to a process as Eq. (4.1). In the following, we will consider a quark vertex as an illustration. Parameters from the quark sector of the mSME have been much less constrained than those that contribute to the lepton vertex. The analysis is completely analogous, however. See Section 6.4 for an application of using \( \chi^{\mu\nu} \) as coming from a Lorentz-violating vertex correction.
We consider gauge-invariant terms that can contribute to the vertex containing a few ingredients. First of all, the terms contain the left-handed quark doublet $Q_i = (u_i L, d_i L)$, where the index $i$ runs over the three quark generations. Secondly, we consider covariant derivatives of the quark doublets $D_\mu Q$. And finally, the terms under consideration may contain the $SU(2)$ gauge-field strength $W_{\mu\nu}$ and its covariant derivatives. However, the gauge-field strength can only be present once, since otherwise the term no longer describes a three-point interaction. One could also use the Higgs doublet to build terms contributing to the vertex (see Ref. [29] for examples), but since our goal is to illustrate the generality of the use of $\chi^{\mu\nu}$, and not to give an exhaustive list of all terms contributing to it, we will settle for the previously mentioned ingredients. Using these ingredients, we can build two types of terms. The first type does not contain the gauge field strength and looks like

$$L_{n+3} = T_{ij}^{\mu\alpha_1...\alpha_n} \bar{Q}_i \gamma_\mu (iD_{\alpha_1}) \cdots (iD_{\alpha_n}) Q_j , \quad (4.9)$$

where $n + 3$ is the mass dimension of the operator and $T_{ij}$ is a hermitian matrix in generation space of mass dimension $1 - n$. The gamma-matrix structure is limited to an odd number of gamma matrices, because the term is built out of a left-handed quark doublet and its conjugate. This was already mentioned in Ref. [29] and it means that the gamma-matrix structure in Eq. (4.9) is exhaustive. As an example we mention that at mass dimension four the type of term as given in Eq. (4.9) is the only gauge-invariant term that gives a contribution to the vertex correction for quarks. It is part of the mSME and was given in Eq. (2.11b). Adapted to the notation in this section, the dimension-four term looks like

$$L_4 \supset i(c_Q)_{ij} \bar{Q}_i \gamma_\mu D_\alpha Q_j . \quad (4.10)$$

The second type of terms that contribute to Lorentz violation in the vertex contains one instance of the $SU(2)$ field strength or its covariant derivatives and looks like

$$L_{m+n+5} = F_{ij}^{\mu\rho\alpha_1...\alpha_n\beta_1...\beta_m} \bar{Q}_i \gamma_\mu [D_{\alpha_1} \cdots D_{\alpha_n} W_{\rho\nu}] (iD_{\beta_1}) \cdots (iD_{\beta_m}) Q_j . \quad (4.11)$$

Here $m + n + 5$ is the mass dimension of the operator and $F_{ij}$ is again a hermitian matrix in generation space. An example of a term like this can be found in Ref. [29] and it looks like

$$L_5 \supset (c_{Q,3})_{ij} \bar{Q}_i \gamma_\mu W_\mu Q_j . \quad (4.12)$$

It is clear from Eqs. (4.9) and (4.11) that a general contribution to the parameter $\chi^{\mu\nu}$ coming from the vertex will have the form

$$\chi^{\mu\nu}_v = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} V^{\mu\rho\alpha_1...\alpha_n\beta_1...\beta_m} p_{\alpha_1} \cdots p_{\alpha_n} q_{\beta_1} \cdots q_{\beta_m} , \quad (4.13)$$

where $p$ is the $W$-boson momentum and $q$ is the momentum of one of the quarks (the momentum of the other quark can be eliminated by using momentum conservation). Including terms containing the Higgs doublet will not change this form of the vertex contribution.

There is a problem with the first type of terms, given in Eq. (4.9). These terms always contain a part that is a kinetic quark term. These kinetic terms should be taken into
account in a calculation. It is not clear, however, how such terms manifest themselves in effective parameters for the particles that are made up by the quarks. Therefore, we cannot fully treat the kind of terms given in Eq. (4.9), at least not in the way we will discuss in the next section.

A way around this problem in some cases may be found in field or coordinate redefinitions. As was shown in Ref. [31], some Lorentz-violating parameters have no observable consequences and can be removed from the Lagrangian by suitable field or coordinate redefinitions. Some parameters can be removed from a free-field theory, but as soon as interactions are included, the interaction terms prevent their removal. This means that they can be removed from the non-interacting part of the Lagrangian, but the field or coordinate redefinition used to do this will make the interaction terms Lorentz non-invariant. What can be accomplished, therefore, is that these parameters are moved from the free-field equations to the interactions. However, this is not possible for all parameters and a full analysis of this issue lies outside the scope of this thesis. See, however, Sections 6.3 and 6.4 for an example of an application of this in the context of pion decay.

From the preceding considerations, it is clear that using $\chi^{\mu\nu}$ is a general approach to Lorentz violation in weak decay processes. Moreover, the propagator in Eq. (4.1) is, to first order in Lorentz violation, compatible with the low-energy limit of the propagator of massive photons given in Ref. [43]; cf. also Ref. [44]. It has become clear that $\chi^{\mu\nu}$ can depend on the $W$-boson or quark momenta. This means that the results after integrating over momenta will differ for different momentum dependences of $\chi^{\mu\nu}$.

### 4.3 Allowed $\beta$-decay rate

In this section we consider the effect of $\chi^{\mu\nu}$ on the allowed $\beta$-decay rate of nuclei. We will consider the more general case of forbidden $\beta$ decay in Chapter 5. We place no restrictions on $\chi^{\mu\nu}$ except for tracelessness. A nonzero trace does not give any Lorentz-violation when $\chi^{\mu\nu}$ is momentum independent (corresponding to the dominant part of $\chi^{\mu\nu}$) and can be absorbed in the weak coupling constant. Results we get in terms of $\chi^{\mu\nu}$ can easily be translated to results for any Lorentz-violating parameter that shows up in the propagator of the $W$ boson.

Our calculation parallels the standard calculations of (polarized) $\beta$ decay in the conventional $V−A$ framework. We will discuss the derivation of the $\beta$-decay rate to make clear where the Lorentz violation enters and how it modifies the standard results. We follow the conventional notation for $\beta$-decay calculations [45, 46, 47, 48]. We use units such that $\hbar = c = 1$, our metric is “mostly minus,” and $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$. The effective $\beta$-decay Hamiltonian density that follows from $V−A$ theory, including the Lorentz-violating $W$-boson propagator in Eq. (4.1), is given by

$$H_\beta = (g_{\rho\sigma} + \chi_{\rho\sigma}) \left[ \bar{\psi}_p(x)\gamma^\rho(C_V + C_A\gamma^5)\psi_n(x) \right] \left[ \bar{\psi}_e(x)\gamma^\sigma(1 - \gamma^5)\psi_\nu(x) \right] + \text{h.c.} \ , \quad (4.14)$$

where $C_V$ and $C_A$ are real constants that determine the relative amplitude of the vector and axial-vector interaction, and the gamma matrices are in the Dirac basis; h.c. denotes hermitian conjugation.
We make two approximations that are customary in $\beta$-decay calculations [48]. In the first place, we evaluate the lepton current at the nuclear center, because the de Broglie wavelengths of the leptons are much larger than the nuclear radius and the current is practically constant over that range. This implies that the leptons take away zero orbital angular momentum. Secondly, since the nuclei are nonrelativistic, the “small” lower two components of the nuclear wavefunction can be neglected. These two approximations comprise allowed $\beta$ decay. Without these approximations one would have to deal also with so-called forbidden transitions. These are considered in Chapter 5. With the approximations for allowed $\beta$ decay, we can write the squared matrix element that follows from Eq. (4.14) as

$$|\mathcal{M}|^2 = \left| C_V \langle 1 \rangle J^0_\pm - C_A \langle \sigma \rangle \cdot J_\pm \right|^2,$$

(4.15)

where $J_\pm$ is the lepton current for $\beta^\pm$ decay evaluated at the nuclear center, into which we absorbed the Lorentz violation,

$$J^0_\pm = (g^\sigma + \chi^\sigma \bar{\psi}_e(0)\gamma_\sigma(1 - \gamma^5)\psi_e(0),$$

(4.16a)

$$J^\pm_\pm = (g^\sigma + \chi^\sigma \bar{\psi}_e(0)\gamma_\sigma(1 - \gamma^5)\psi_e(0),$$

(4.16b)

and $\langle 1 \rangle = \langle f | 1 | i \rangle$ and $\langle \sigma \rangle = \langle f | \sigma | i \rangle$ abbreviate the transition matrix elements of the nucleus.

The charged-lepton wavefunction is not given by the plane-wave solution of the free Dirac equation, since the $\beta$ particle feels the positive charge of the daughter nucleus after the decay. We take the spinor for allowed $\beta$ decay from Ref. [49] (our normalization differs by a factor $\sqrt{2E_e}$); it reads

$$\psi^e_{\pm}(r \to 0) = N(Z)\sqrt{E_e + m_e} \left( M \frac{\eta^e}{E_e + m_e} \right),$$

(4.17)

where

$$|N(Z)|^2 = \frac{E_e + \gamma_0 m_e}{E_e + m_e} F(E_e, Z),$$

$$\gamma_0 = \sqrt{1 - (Z\alpha)^2},$$

(4.18)

and

$$M = \frac{E_e + m_e}{E_e + \gamma_0 m_e} \left( 1 + i \frac{Z\alpha m_e}{|\mathbf{p}|} \right).$$

(4.19)

Here $E_e$, $m_e$, and $\mathbf{p}$ are the electron energy, mass, and momentum, respectively, and $\eta^e$ is a Pauli spinor. $Z$ is the atomic number of the daughter nucleus, $\alpha$ the fine-structure constant, and $F(E_e, Z)$ the usual Fermi function, which is in essence the probability to find an electron in the interior of the nucleus relative to the probability to find it at the same position without a nucleus present. The spinor of the positron is the charge conjugate of Eq. (4.17), which amounts to

$$\psi^e_{\mp}(r \to 0, Z) = i\gamma^2(\psi^e_-(r \to 0, -Z))^*.$$

(4.20)

The wavefunction of the neutrino is just the solution of the free massless Dirac equation.

Since there is no Lorentz violation in the hadronic current, the evaluation of nuclear transition matrix elements proceeds as usual through the Wigner-Eckart theorem [50], which allows one to write the matrix elements of the operators 1 and $\sigma$, given in terms of
their spherical tensor components, as a product of Clebsch-Gordan coefficients and matrix elements that are independent of the spin projection quantum number \( m \). The result is that we can write the squared matrix element Eq. (4.15) as

\[
|M|^2 = \left\{ C_F^2 |\langle 1 \rangle|^2 \delta_{jj'} |J_0|^2 + \frac{\lambda}{3} C_A^2 |\langle \sigma \rangle|^2 (|J_{+1}|^2 + |J_{-1}|^2 + |J_z|^2) \\
+ \frac{1}{2} C_A^2 |\langle \sigma \rangle|^2 \Lambda^1(|J_{+1}|^2 - |J_{-1}|^2) + \frac{1}{2} C_A^2 |\langle \sigma \rangle|^2 (|J_{+1}|^2 + |J_{-1}|^2 - 2|J_z|^2) \\
- C_V C_A \langle \sigma \rangle \delta_{jj'} \Lambda_z (J^0 J^* + J_z J^0_z) \right\},
\]

(4.21)

where the space components of the lepton current are now given in spherical coordinates as

\[
J_{\pm 1} = \pm \frac{1}{\sqrt{2}} (J^1 \pm i J^2), \quad \text{and} \quad J_z = J^3;
\]

and \( j \) and \( j' \) denote the initial and final nuclear spin, respectively, \( \langle 1 \rangle \) and \( \langle \sigma \rangle \) are the reduced matrix elements independent of the spin projection, while the \( \Lambda \) coefficients come from combinations of Clebsch-Gordan coefficients. They are given by

\[
\Lambda^{(1)} = \begin{cases} 
\langle m \rangle \left\langle \frac{m}{2} \right| \left\langle \frac{j}{2} \right| \left( j' = j - 1 \right), \\
\langle m \rangle \left\langle \frac{m}{2} \right| \left\langle \frac{j+1}{2} \right| \left( j' = j \right), \\
\langle m \rangle \left\langle \frac{m}{2} \right| \left\langle \frac{j+1}{2} \right| \left( j' = j + 1 \right)
\end{cases}, \quad \Lambda^{(2)} = \begin{cases} 
\langle m^2 \rangle - \frac{1}{2} j (j+1) \langle \frac{m}{2} \rangle \left\langle \frac{m}{2} \right| \left\langle \frac{j-1}{2} \right| \left( j' = j - 1 \right), \\
\langle m^2 \rangle + \frac{1}{2} j (j+1) \langle \frac{m}{2} \rangle \left\langle \frac{m}{2} \right| \left\langle \frac{j+1}{2} \right| \left( j' = j \right), \\
\langle m^2 \rangle - \frac{1}{2} j (j+1) \langle \frac{m}{2} \rangle \left\langle \frac{m}{2} \right| \left\langle \frac{j+3}{2} \right| \left( j' = j + 1 \right)
\end{cases},
\]

(4.23)

and

\[
\Lambda_z = \frac{\langle m \rangle}{j} \sqrt{\frac{j}{j+1}}.
\]

Here the notation \( \langle m \rangle \) and \( \langle m^2 \rangle \) means that \( m \) and \( m^2 \) must be averaged (incoherently) over the populations of the states \( m = \pm j, \pm (j-1), \ldots \). For a completely polarized source this implies that \( \langle m \rangle = j \). We denoted the \( m = 0 \) component of the current in spherical coordinates by \( J_z \) to distinguish it from the time component of the current \( J^\mu \).

The first term in Eq. (4.21) corresponds to a Fermi transition, which has \( \Delta j = j-j' = 0 \). This Fermi term is isotropic, and in particular independent of the nuclear polarization. The terms including the factor \( C_A^2 \) give Gamow-Teller transitions with \( \Delta j = 0, \pm 1 \). The first of these Gamow-Teller terms has \( \Delta j = 0 \) and is also isotropic. The second term proportional to \( C_A^2 \) is “first-order anisotropic,” characterized by the fact that it is proportional to \( \Lambda^{(1)} \propto \langle m \rangle / j \). This term gives transitions with \( \Delta j = 0, \pm 1 \). The third Gamow-Teller term, following \( \Lambda^{(2)} \), is “second-order anisotropic” and also has \( \Delta j = 0, \pm 1 \). Finally, the last term proportional to \( C_V C_A \) causes a so-called “mixed” transition. It is first-order anisotropic and gives \( \Delta j = 0 \). All anisotropic terms average to zero for randomly oriented nuclei.

Up to this point, the calculation is identical to the Lorentz-symmetric case, except for the presence of the Lorentz violation in the lepton current Eqs. (4.16a) and (4.16b). To work out Eq. (4.21) we need products of different components of the lepton current, by evaluating the general product \( J^\mu J^{\nu*} \). Since the electron and positron spinors in Eqs. (4.17) and (4.20) can be written as

\[
\psi_{e^-}^s(0) = \frac{N(Z)}{2} \left[ (1 + \gamma^0) + M (1 - \gamma^0) \right] u^s(p), \quad (4.25a)
\]

\[
\psi_{e^+}^s(0) = \frac{N^s(-Z)}{2} \left[ (1 - \gamma^0) + M (1 + \gamma^0) \right] v^s(p), \quad (4.25b)
\]
with \( u^\dagger(p) \) and \( v^\dagger(p) \) being the free Dirac spinors for the electron and positron, respectively, it is straightforward to show that for the electron

\[
\sum_{\nu \text{ spin}} J_{\mu}^\nu J_{\nu}^{\ast} = F(E_e, Z)(g^{\mu\rho} + \chi^{\mu\rho})(g^{\nu\sigma} + \chi^{\nu\sigma\ast}) \text{Tr} \left[ \slashed{P} - \gamma_\rho \gamma_\sigma (1 - \gamma^5) \right],
\]

(4.26a)

while for the positron

\[
\sum_{\nu \text{ spin}} J_{\mu}^\nu J_{\nu}^{\ast} = F(E_e, -Z)(g^{\nu\rho} + \chi^{\nu\rho})(g^{\mu\sigma} + \chi^{\mu\sigma\ast}) \text{Tr} \left[ \slashed{P} + \gamma_\rho \gamma_\sigma (1 - \gamma^5) \right],
\]

(4.26b)

where \( k^\mu \) is the neutrino four-momentum. We sum over the neutrino polarizations in Eqs. (4.26a) and (4.26b), because these are unlikely to be measured in the foreseeable future. \( P_\mu^\pm \) is given by

\[
P_{\mu}^0 = E_e \mp \hat{s}_e \cdot \hat{p},
\]

(4.27a)

\[
P_{\mu}^i = \left( 1 \mp \hat{p} \cdot \hat{s}_e (E_e - \gamma_0 m_e) \right) p^i \mp m_e \gamma_0 s^i_e \mp \sqrt{1 - \gamma_0^2 m_e (\hat{p} \times \hat{s}_e)_i},
\]

(4.27b)

with \( i \) running over the spatial indices 1, 2, 3, and \( \hat{s}_e \) the unit vector in the direction of the polarization of the \( \beta \) particle. The upper (lower) signs correspond to the electron (positron). Notice that \( P_\mu^\pm \) does not transform as a four-vector, because the Coulomb corrections are included in a boost non-invariant way in Ref. [49]. At present, this does not concern us, since we will mainly consider the effects of rotations. Neglecting the Coulomb corrections \( (Z \alpha \rightarrow 0) \) gives \( P_\mu^\pm \rightarrow p_\mu \mp m_e s^\mu_e \), with \( s^\mu \) the conventional spin four-vector.

It now becomes a simple matter of trace technology to calculate the different products of lepton currents needed in Eq. (4.21). Although this calculation is straightforward, the results are rather lengthy and not very illuminating. Their exact form is given in Appendix D.1. Substituting the results for the currents in Eq. (4.21) and remembering that the differential decay rate is given by

\[
d\Gamma = \frac{\delta(E_e + E_\nu - E_0)}{(2\pi)^5 2E_e 2E_\nu} \sum_{\nu \text{ spin}} |M|^2 d^3p d^3k,
\]

(4.28)

where \( E_0 \) is the total energy available in the decay, we find the general Lorentz-violating result for allowed \( \beta \) decay. Including Coulomb corrections and to first order in \( \chi^{\mu\nu} \), it is
4.3 ALLOWED $\beta$-DECAY RATE

given by

$$d\Gamma = \frac{1}{(2\pi)^5} d^3p \, d^3k \, \delta(E_e + E_{\nu} - E_0) \, F(E_e, \pm Z) \xi$$

$$\times \left\{ \left(1 \mp \frac{\mathbf{p} \cdot \hat{s}_e}{E_e} \right) \left[ \frac{1}{2} \left( 1 + B \frac{\mathbf{k} \cdot \hat{1}}{E_{\nu}} \right) + t + \frac{\mathbf{w}_1 \cdot \mathbf{k}}{E_{\nu}} + \mathbf{w}_2 \cdot \hat{1} \right] \right. $$

$$\left. + T_{1}^{km} \hat{k}_i \hat{m}_i + \frac{T_{2}^{kj} \hat{k}_i \hat{m}_i}{E_{\nu}} + \frac{T_{3}^{kn} \hat{k}_i \hat{m}_i}{E_{\nu}} \right\} $$

$$\times \left[ \frac{1}{2} (A - 3c \frac{\mathbf{k} \cdot \hat{1}}{E_{\nu}}) \hat{l} + \frac{1}{2} (a + c) \frac{\mathbf{k} \cdot \hat{1}}{E_{\nu}} + \hat{l} \right] + \frac{T_{1}^{lk} \hat{k}_i \hat{m}_i}{E_{\nu}} + \frac{T_{4}^{lk} \hat{k}_i \hat{m}_i}{E_{\nu}} + \frac{T_{4}^{lk} \hat{k}_i \hat{m}_i}{E_{\nu}} \right\} $$

where $\hat{1}$ is the nuclear polarization axis. The Latin indices run over the three spatial directions and summation over repeated indices is implied. The upper sign corresponds to $\beta^-$ decay, while the lower sign corresponds to $\beta^+$ decay. The different Lorentz-violating quantities are defined as

$$t = (a - \frac{1}{2}c) \chi^{00}_r,$$

$$w^i_1 = -x \chi^{0j}_r + \hat{g}(\chi^{0j}_r - \chi^{00}_r) \hat{l}, \quad w^j_2 = \tilde{K}(\chi^{0k}_r - \chi^{00}_r) - \tilde{L} \chi^j_i, \quad w^3_3 = x \chi^{0j}_r + \hat{g}(\chi^{00}_r + \chi^j_i) \hat{l},$$

$$T_1^{km} = \frac{3}{2} c \chi^{km}_r; \quad T_2^{kj} = \frac{1}{2} A \chi^{00}_r \delta^{jk} + \tilde{L}(\chi^j_r + \chi^0_i \epsilon^{ijk}) - \tilde{K}(\chi^{jk}_r + \chi^0_i \epsilon^{ijk}),$$

$$T_3^{kj} = (x + \hat{g}) \chi^{00}_r \delta^{ij} - (x \chi^{0j}_r + \hat{g} \chi^j_i) \epsilon^{ijk} - \hat{g}(\chi^{ij}_r + \chi^j_i),$$

$$T_4^{lk} = \frac{1}{2} B \chi^{00}_r \delta^{lk} - \tilde{L}(\chi^{lk}_r - \chi^{0k}_r \epsilon^{kl}) - \tilde{K}(\chi^{kl}_r - \chi^0_i \epsilon^{kl}),$$

$$S^{kmj}_1 = -\frac{3}{2} A(\chi^{00}_r \delta^{mkj} - \chi^0_i \epsilon^{slj}), \quad S^{mjk}_2 = -\frac{3}{2} B(\chi^{00}_r \delta^{mkj} - \chi^0_i \epsilon^{slj}),$$

$$S^{mjk}_3 = \tilde{L}(\chi^{00}_r \delta^{mkj} - \chi^0_i \epsilon^{slj} - \chi^{00}_r \delta^{mkj} + \chi^0_i \epsilon^{slj}),$$

$$P^{lkn}_4 = \frac{1}{2} c(\chi^{00}_r \delta^{lkn} - \chi^0_i \epsilon^{lkn}),$$

with the subscripts $r$ and $i$ denoting the real and imaginary parts of $\chi^{0\nu}$, respectively, $\chi^j = \epsilon^{mk} \chi^{mk}$, and the different constants are given by

$$\xi = 2(C^2_V \langle 1 \rangle^2 + C^2_A \langle \sigma \rangle^2), \quad x = \frac{2C^2_V \langle 1 \rangle^2}{\xi}, \quad y = \frac{2C_V C_A \langle 1 \rangle \langle \sigma \rangle}{\xi},$$

$$a = \frac{1}{3}(4x - 1), \quad c = (1 - x)\Lambda(2), \quad A = \mp(1 - x)\Lambda(1) - 2y\Lambda_2\delta_{jj'},$$

$$B = \pm(1 - x)\Lambda(1) - 2y\Lambda_2\delta_{jj'}, \quad \tilde{g} = \frac{1}{3}(1 - x)(1 + \frac{3}{2}\Lambda(2)), \quad \tilde{K} = -y\Lambda_2\delta_{jj'},$$

$$\tilde{L} = \pm\frac{1}{2}(1 - x)\Lambda(1);$$

(4.31) again, upper (lower) signs refer to $\beta^-(\beta^+)$ decay.
The nomenclature of the constants in Eq. (4.31) is such that the quantities without a “breve” occur in standard β decay, while the ones with a “breve” are Lorentz violating. Capital letters signal terms which are first-order anisotropic and the other terms are in small script. Our expressions are compatible with the existing literature for the coefficients that occur in the SM case as well (except that, for convenience, we absorbed a factor $\frac{1}{3} \Lambda^{(2)}$ in $c$). The $A_i$, for example, denotes the conventional “Wu parameter” that quantifies the correlation between the momentum of the β particle and the polarization of the parent nucleus. Furthermore, we gave some thought to the order of the terms in Eq. (4.29). First of all, the second line contains terms that can experimentally be tested without referring to the electron spin or momentum. For the other terms one does need information about the electron spin or momentum. Apart from this division, we have put conventional terms that do not violate Lorentz symmetry in the front. Finally, we ordered the terms according to their need for the knowledge of the neutrino momentum (or recoil of the daughter nucleus), nuclear spin, or both, to be experimentally accessible. See Appendix D.2 for an overview of the observability of different $\chi^{\mu\nu}$ components.

4.3.1 Sample decay rates

In this section we work out some key examples of decay rates, a pure Fermi transition, a pure Gamow-Teller transition, and neutron decay (an example of a mixed transition). For the neutron we will work out the decay rate in terms of Lorentz-violating coefficients, given in the Sun-centered inertial reference frame, to explicitly show the oscillations with time due to the rotation of the Earth. A complete overview of the observable Lorentz-violating effects in different allowed β-decay transitions (Fermi, Gamow-Teller, or mixed) is delegated to Appendix D.2, where a Table is given that contains the full set of Lorentz-violating vectors and tensors of Eq. (4.30) and their observability in different transitions.

We start with a pure Fermi transition, which has $\langle \sigma \rangle = 0$. Integrating over neutrino energy and direction and summing over electron polarization gives

$$d\Gamma_F = d\Gamma^0 \left[ 1 + 2\chi^{00}_r + \frac{2\chi^{0l}_r}{E_e} \right], \quad (4.32)$$

with

$$d\Gamma^0 = \frac{1}{8\pi^4} |p| E_e(E_e - E_0)^2 dE_e d\Omega_e F(E_e, \pm Z) \xi. \quad (4.33)$$

Therefore, measuring the β-decay rate in different directions gives access to $\chi^{0l}_r$.

For a Gamow-Teller transition in randomly oriented nuclei, after summing over electron polarization and integrating over electron momentum, we get

$$d\Gamma_{GT} = d\Gamma^0 \left[ 1 - \frac{2}{3} \chi^{00}_r + \frac{2}{3} (\chi^{10}_r + \bar{\chi}^l_i) \frac{p_f}{E_e} \right]. \quad (4.34)$$

Measuring Fermi transitions and Gamow-Teller transitions gives access to different parameters. In the former one measures $\chi^{0l}_r$, while in the latter bounds on a combination of $\chi^{10}_r$ and $\bar{\chi}^l_i$ are possible. In the case that $(\chi^{\mu\nu})^* = \chi^{\nu\mu}$, it follows that $\chi^{0l}_r = \chi^{10}_r$, so one can disentangle the bounds on $\chi^{10}_r$ and $\bar{\chi}^l_i$. 


For other parts of $\chi^{\mu\nu}$ it is necessary to obtain more directional information from the experiment. One way to accomplish this is by polarizing the parent nuclei. The relevant expression for the Gamow-Teller transition rate of polarized nuclei is
\[
d\Gamma_{GT} = d\Gamma^0 \left[ 1 - \frac{2}{3} \chi_r^{00} + \frac{5}{3} (\chi_r^{00} + \chi_i^l) \frac{p^l}{E_e} \right]
\]
\[
+ \Lambda^{(1)} \left[ (1 - \chi_r^{00}) \frac{p \cdot \hat{I}}{E_e} + (\chi_r^{00} + \chi_i^l) \frac{p^l}{E_e} - \frac{\chi_i^0 (p \times \hat{I})^l}{E_e} \right]
\]
\[
+ \Lambda^{(2)} \left[ -\chi_r^{00} (\chi_r^{00} + \chi_i^l) \frac{p^l}{E_e} + 3 \chi_r^k \hat{k} \hat{l} \frac{p^l}{E_e} - 3 \chi_r^l \hat{k} \hat{i} \frac{p \cdot \hat{I}}{E_e} \right.
\]
\[
- 3 \chi_r^{ml} \frac{\langle p \times \hat{I} \rangle^l}{E_e} \right]. \tag{4.35}
\]
By combining, for example, an asymmetry measurement for spin up and spin down, with measurements of Eqs. (4.32) and (4.34), one can extract bounds on $\chi$ (again assuming $(\chi^{\mu\nu})^* = \chi^{\nu\mu}$). In Appendix D.2 we relate $\chi$ to observables in Fermi, Gamow-Teller, and mixed transitions with respect to the decay parameters in Eqs. (4.29) and (4.30).

As a final example we look at polarized neutron $\beta$ decay. We assume that one measures the direction and energy of the outgoing electron. An experiment relevant for this example is described in Ref. [51]. For the neutron, the ratio of $C_V$ and $C_A$ has been experimentally determined to be $C_A/C_V = -1.27$. Since $\langle 1 \rangle = 1$ and $\langle \sigma \rangle = \sqrt{3}$ for neutron decay, it follows that $x = 0.17$ and $y = -0.37$. By using these values, the differential decay rate of polarized neutrons becomes
\[
d\Gamma = d\Gamma^0 \left\{ 1 - 0.21 \chi_r^{00} + (0.34 \chi_r^{0l} + 0.55 (\chi_r^{00} + \chi_i^l)) \frac{p^l}{E_e} \right.
\]
\[
+ \frac{\langle m \rangle}{j} \left[ 0.43 (\chi_r^{0k} - \chi_r^{0k}) - 0.55 \chi_i^k - (0.12 - 0.99 \chi_r^{00}) \frac{p^k}{E_e} \right.
\]
\[
- 0.99 (\chi_r^{ik} - \chi_i^0 \epsilon^{kl}) \frac{p^l}{E_e} \right\} \right) \right). \tag{4.36}
\]
There are no second-order anisotropic contributions for the neutron, because $\Lambda^{(2)} = 0$ for particles with spin $\frac{1}{2}$.

To determine the Lorentz-violating decay rate of the neutron in terms of coefficients defined in the Sun-centered inertial reference frame we use Eq. (2.5). We take the quantization axis of the nuclear spin to be in the $+z$ direction and we assume that the electrons are measured in the $\hat{x}-\hat{z}$ plane and that $p \cdot \hat{I} = v E_e \cos \theta$, with $v = |p|/E_e$ the velocity of the electron, while $\mathcal{P}$ is the polarization of the parent nuclei. The resulting expression reads
\[
d\Gamma = d\Gamma^0 \left[ 1 - 0.12 \mathcal{P} \cos \theta - \chi_i^{0T} (0.21 - 0.99 \mathcal{P} \cos \theta) \right.
\]
\[
+ Z_1 + Z_2 \cos(\Omega t) + Z_3 \sin(\Omega t) + Z_4 \cos(2\Omega t) + Z_5 \sin(2\Omega t) \right], \tag{4.37}
\]
where the quantities \( Z_i \), depending on the colatitude of the experiment, are given by

\[
Z_1 = v(0.55(X_i^{YY} - X_i^{XY} + X_i^{ZT}) + 0.34X_r^{TZ}) \cos(\theta + \zeta) + \\
+ \mathcal{P} \left[ (0.55(X_i^{YY} - X_i^{XY}) + 0.43(X_r^{XT} - X_r^{T}) \right] \cos(\theta + \zeta) \\
- 0.99vX_r^{ZT} \cos(\theta + \zeta) \cos \zeta - 0.50v(X_r^{XX} + X_r^{YY}) \sin(\theta + \zeta) \sin \zeta \quad (4.38a)
\]

\[
Z_2 = 0.55v((X_i^{YY} - X_i^{XY} + X_r^{XT}) + 0.34X_r^{TY}) \sin(\theta + \zeta) \\
+ \mathcal{P} \left[ (0.55(X_i^{YY} - X_i^{XY}) + 0.43(X_r^{XT} - X_r^{T}) \right] \sin(\theta + \zeta) \\
- 0.99v(X_r^{XX} + X_r^{ZT}) \cos \theta \sin \zeta \cos \zeta \\
- 0.99v(X_i^{YT} + X_r^{XX} \cos^2 \zeta - X_r^{ZT} \sin^2 \zeta) \sin \theta \ democratic, since they bound different combinations of \( \chi \).
\]

\[
Z_3 = 0.55v((X_i^{XX} - X_i^{XY} + X_r^{XT}) + 0.34X_r^{TY}) \sin(\theta + \zeta) \\
+ \mathcal{P} \left[ (0.55(X_i^{XX} - X_i^{XY}) + 0.43(X_r^{XT} - X_r^{T}) \right] \sin(\theta + \zeta) \\
- 0.99v(X_r^{YT} + X_r^{XY} \cos^2 \zeta + X_r^{ZT} \sin^2 \zeta) \sin \theta \cos \zeta \\
+ 0.99v(X_i^{XT} - X_r^{YT} \cos^2 \zeta - X_r^{XX} \sin^2 \zeta) \sin \theta \cos \zeta \quad (4.38b)
\]

\[
Z_4 = 0.50v\mathcal{P}(X_r^{YY} - X_r^{XX}) \sin(\theta + \zeta) \sin \zeta \quad (4.38c)
\]

\[
Z_5 = -0.50v\mathcal{P}(X_r^{XY} + X_r^{YY}) \sin(\theta + \zeta) \sin \zeta \quad (4.38d)
\]

We see indeed that Eq. (4.37) has parts \( Z_2 \) and \( Z_3 \) that oscillate with a period of one sidereal day and it has parts \( Z_4 \) and \( Z_5 \) that have half this oscillation period.

As mentioned earlier, at PSI in Switzerland, experiments on the \( \beta \) decay of polarized neutrons were performed. The data was time-stamped and an analysis looking for Lorentz-invariance violation was performed [51]. In the light of our result for neutron decay in Eq. (4.36) we believe that a more complete analysis could be performed, from which bounds on Lorentz-violating parameters in the SME could be obtained.

Finally, we mention the experiment performed at the accelerator facility “Kernfysisch versneller instituut” (KVI) in Groningen, which searched for Lorentz-violating effects in the Gamow-Teller \( \beta \) decay of \( ^{20} \)Na [18]. This experiment was designed to be sensitive to Lorentz-violating effects arising from first-order anisotropic terms in Eq. (4.29) by controlled flipping of the spin of the parent nucleus. It achieved a precision on the spin-up spin-down asymmetry of

\[
\frac{|\Gamma_{up} - \Gamma_{down}|}{\Gamma_{up} + \Gamma_{down}} < 3 \times 10^{-3} 
\]

Which resulted in bounds on \( \tilde{X}_1 = 2(k_{\phi \phi})^{2Y} + \frac{1}{2}k_{\phi W}^{2Y} \) and \( \tilde{X}_2 = 2(k_{\phi \phi})^{XZ} + \frac{1}{2}k_{\phi W}^{XZ} \), where \( (k_{\phi \phi})^{\mu \nu} \) is the imaginary antisymmetric part of \( k_{\phi \phi}^{\mu \nu} \). The obtained bounds are

\[
-9 \times 10^{-3} < \tilde{X}_1 < 2 \times 10^{-3} \\
-6 \times 10^{-3} < \tilde{X}_2 < 4 \times 10^{-3}
\]

at a 95% confidence level. These bounds are 3-5 orders of magnitude larger than the bounds we obtain from forbidden \( \beta \) decay in Eq. (5.66). However, they can be complementary, since they bound different combinations of \( \chi^{\mu \nu} \) components.
Chapter 5

Forbidden $\beta$ decay

In this chapter we generalize the results for allowed $\beta$ decay, described in Section 4.3, to forbidden $\beta$ transitions. We will calculate the general Lorentz-violating $\beta$-decay rate, that follows from the Hamiltonian:

$$H_\beta = \bar{\psi}_p(x)\gamma^\lambda(C_V + C_A\gamma^5)\psi_n(x)(g_{\lambda\tau} + \chi_{\lambda\tau})\bar{\psi}_\nu(x)\gamma^\tau(1 - \gamma^5)\psi_\nu(x) + h.c. \quad (5.1)$$

Here $C_V$ and $C_A$ are the conventional constants that give the strengths of the vector and axial-vector interaction, respectively. The Lorentz violation is parametrized by the tensor $\chi^{\lambda\tau}$, which is described in Chapter 4. Since this tensor is the only Lorentz violation in the problem, the leptons and the initial and final nuclear states can be treated as in the Lorentz-symmetric case. However, also in the present case, it is useful to sketch the main points of this treatment. It is in particular advantageous to understand how the suppression of some $\beta$-decay transitions follows from angular-momentum conservation and how this changes when rotational invariance is broken. Therefore we will go through the derivation of the $\beta$-decay rate of a general nucleus, albeit in little detail where the Lorentz violation does not change the analysis. For an extensive discussion of the general Lorentz-symmetric case see Ref. [48].

Since nuclear states are characterized by spin and parity, it is customary to expand the lepton current in the $\beta$-decay matrix element in multipoles. Compared to the multipole expansion of the photon field in the atomic case this expansion is complicated, because both vector and axial-vector currents contribute and two relativistic particles are involved, for which only the total angular momentum of each particle is a good quantum number. Moreover, the $\beta$ particle moves in the Coulomb field of the daughter nucleus.

The lowest-order terms in the multipole expansion correspond to the allowed approximation, which amounts to evaluating the lepton current at $r = 0$ and neglecting relativistic effects for the nucleus. This implies that neither of the leptons carries off orbital angular momentum.

Higher-order terms in the expansion correspond to forbidden transitions [48, 52], which are suppressed by one or more of the following small dimensionless quantities: $pR$, where $p$ is the lepton momentum and $R$ the nuclear radius ($pR$ corresponds to the ratio of the nuclear radius and the de Broglie wavelength of the lepton), $v_N$, the velocity of the decaying nucleon in units of $c$, and $\alpha Z$, the fine-structure constant times the charge of the daughter nucleus. The lowest power of these quantities that appears in the amplitude
determines the degree of forbiddenness of the transition. The transitions are classified by the nuclear-spin change $\Delta I = |I_i - I_f|$ and relative parity $\pi_i \pi_f = \pm 1$ (parity change no or yes), where $I_i$, $\pi_i$ and $I_f$, $\pi_f$ are the spins and parities of the parent and daughter nucleus, respectively. $n$-times forbidden transitions with $\Delta I = n + 1$ are called unique. Such unique forbidden transitions are advantageous, since they depend on only one nuclear matrix element, which cancels in the asymmetries that quantify Lorentz violation.

In this Chapter we derive the multipole expansion for the Lorentz-violating $\beta$-decay Hamiltonian of Eq. (5.1). Because the tensor $\chi^{\lambda\tau}$ contracts the nucleon and lepton currents in an unconventional way, the possibility arises that angular momentum is no longer conserved in the transition. In particular, it is now possible that $\Delta I = J + 1$ for $\chi^{0k}$ and $\chi^{k0}$ and that $\Delta I = J + 2$ for $\chi^{km}$, where $J$ is the total angular momentum of the leptons and Latin superscripts run over space indices. Remember that rotational invariance would imply that $\Delta I \leq J$. At the same time, however, the suppression of the transitions is, also with Lorentz violation, for the most part determined by the amount of angular momentum that is carried away by the leptons. Due to this, the parts of $\chi^{\mu\nu}$ that connect to the spin-dependent nucleon current ($\chi^{0k}$ and $\chi^{km}$) can be enhanced by a factor $\alpha_Z/pR$ with respect to the Lorentz-symmetric contributions. This enhancement factor occurs only in transitions with $\Delta I \geq 2$, i.e. starting from unique first-forbidden transitions.

A large part of our analysis (up to Eq. (5.44)) is valid for any momentum dependence of $\chi^{\lambda\tau}$. However, to do integrations over phase space we will assume $\chi^{\lambda\tau}$ to be momentum-independent, which will limit the validity of our final results to this (most important) class of Lorentz-violating parameters. The extension of these results to a momentum-dependent $\chi^{\lambda\nu}$ is straightforward, but lies outside the scope of this thesis. A $\chi^{\lambda\tau}$ that is momentum dependent can have different symmetry and CPT properties than a momentum-independent $\chi^{\lambda\tau}$ (see Eq. (4.7)).

This Chapter is organized as follows. In Section 5.1 we discuss the lepton wavefunctions and in particular their spherical decomposition. In Section 5.2 we start the calculation of the general $\beta$-decay rate using the lepton wavefunctions and the Hamiltonian in Eq. (5.1). After that we will introduce the concept of $\beta$ moments in Section 5.3, which we will then use in Section 5.4 to continue our derivation of the general $\beta$-decay rate. To make the final result obtained in Chapter 5.4 more useful we will apply some approximations in Section 5.5. This will allow us to derive explicit expressions for the dominant contribution to unique and parity-forbidden transitions with a spin change $\Delta I \geq 2$, with Lorentz violation and depending on the direction of the $\beta$ particle. We will discuss these results and its implications in Section 5.6.

This chapter is based on:


### 5.1 Lepton wavefunctions

The lepton wavefunctions are solutions to the Dirac equation. For the $\beta$ particle the Dirac equation includes a Coulomb potential due to the daughter nucleus, which we treat as a point charge, with charge $eZ$. The neutrino does not feel this field and is described by the free Dirac equation. Since we assume everything to be spherically symmetric (except for
5.1 LEPTON WAVEFUNCTIONS

the Lorentz violation), it is convenient to express the solutions of the Dirac equation in terms of a superposition of spherical waves $\psi_{\kappa\mu}$. These are eigenfunctions of the operators $J^2$, $J_z$, $S^2$, and $K = \gamma^0 \sigma$ with $\sigma = -(1 + 2 \mathbf{L} \cdot \mathbf{S})$ and $\gamma^0$ the timelike Dirac matrix. The eigenvalues of these operators are $j(j + 1)$, $\mu$, $\frac{3}{4}$, and $\kappa$, respectively. The operators represent the total angular momentum ($\mathbf{J}$), its projection on the quantization axis ($J_z$), the total intrinsic spin ($\mathbf{S}$) and a quantity involving the spin-orbit interaction ($K$). The form of the spherical waves can be found for example in Ref. [48] or [50], and they are given by

$$\psi_{\kappa\mu}(\hat{r}) = \left( \begin{array}{c} g_\kappa(r) \chi_{\kappa\mu}(\hat{r}) \\ i f_{-\kappa}(r) \chi_{-\kappa\mu}(\hat{r}) \end{array} \right) e^{-iEt}.$$  \hspace{1cm} (5.2)

Here $r$ is the radial coordinate of the lepton (the origin is at the nuclear center) and $E$ the total energy of the lepton. The form of the radial wavefunctions $g_\kappa(r)$ and $f_{-\kappa}(r)$ depends on the potential that the lepton feels and is therefore different for the $\beta$ particle and the neutrino. The $\chi_{\kappa\mu}(\hat{r})$ in Eq. (5.2) is a spinor spherical harmonic and is defined as

$$\chi_{\kappa\mu}(\hat{r}) = \sum_{\xi=\pm} \chi_{\xi} Y_{l\mu} - \xi \mathcal{C}_{l^1j_1j_3;\mu-\xi,\xi,\mu},$$  \hspace{1cm} (5.3)

with $\chi_{\pm}$ the well-known spin-$\frac{1}{2}$ eigenfunctions and $Y_{lm}$ the spherical harmonics. The usual Clebsch-Gordan coefficients are denoted by $\mathcal{C}(j_1,1,j_3;\mu_1\mu_2\mu_3)$ and we will use the Condon-Shortley phase convention for them. The spinor spherical harmonics are eigenfunctions of $J^2$ and $\mathbf{L}^2$, the latter being the square of the orbital angular momentum operator. The relation between the eigenvalues of these two operators determines the sign of $\kappa$. The possible values that $\kappa$ can have are given by

$$\kappa = \begin{cases} l = 1,2,\ldots & \text{if } j = l - \frac{1}{2} \\ -(l + 1) = -1,-2,\ldots & \text{if } j = l + \frac{1}{2} \end{cases}.$$  \hspace{1cm} (5.4)

Notice in particular that $\kappa = \pm(j + \frac{1}{2})$ and that these two values of $\kappa$ are connected to two parts of the lepton wavefunction with the same $j$, but of opposite parity (since $l$ differs by one unit). A lepton state of definite angular momentum $j$ is thus described by a superposition of these two opposite-parity states and orbital angular momentum is not a good quantum number for the lepton.

The form of the radial wavefunctions, $g_\kappa(r)$ and $f_{-\kappa}(r)$, is found by solving two coupled differential equations in $r$ coming from the Dirac equation [50]. The relevant solutions for a free particle are given by

$$g_\kappa = \sqrt{\frac{p(E + m)}{\pi}} j_l(\kappa pr),$$  \hspace{1cm} (5.5a)

$$f_{-\kappa} = S(\kappa) \sqrt{\frac{p(E - m)}{\pi}} j_l(-\kappa pr),$$  \hspace{1cm} (5.5b)

where $S(\kappa)$ is the signature of $\kappa$ and $l(l) = j + \frac{1}{2} S(\kappa)$. Furthermore, $p$ and $E$ are the linear momentum and the total energy of the lepton, respectively, and $j_l(pr)$ is the spherical Bessel function. The normalization is such that the spherical waves are normalized per unit energy, as in Ref. [48].
For a negatively charged particle in a spherical potential \( V(r) = -Z e^2/r = -\alpha Z/r \) the radial wavefunctions, with the same normalization as the free spherical waves, are given by

\[
g_{\kappa} = \sqrt{\frac{p(E + m)}{4\pi}} (Q + Q^*), \tag{5.6a}
\]

\[
f_{-\kappa} = i \sqrt{\frac{p(E - m)}{4\pi}} (Q - Q^*), \tag{5.6b}
\]

with \( Q \) defined by

\[
Q = D(|\kappa|)(\gamma + i\nu)^{-\gamma} e^{-ipr + i\eta} F(\gamma + 1 + i\nu, 2\gamma + 1; 2ipr). \tag{5.7}
\]

In this \( \gamma, \nu, \) and \( \eta \) are defined as

\[
\gamma = \sqrt{\kappa^2 - (\alpha Z)^2}, \tag{5.8a}
\]

\[
\nu = \frac{\alpha Z E}{p}, \tag{5.8b}
\]

\[
e^{2i\eta} = -\frac{\kappa - im\alpha Z/p}{\gamma + i\nu}, \tag{5.8c}
\]

while \( D(|\kappa|) \) is real, constant over \( r \), and given by

\[
D(|\kappa|) = 2e^{\frac{1}{2} - \nu} |\Gamma(\gamma + i\nu)| \Gamma(2\gamma + 1), \tag{5.9}
\]

with \( \Gamma(x) \) being the well-known gamma function. The function \( F(a, b; z) \) in Eq. (5.7) is the confluent hypergeometric function defined as

\[
F(a, b; z) = \frac{\Gamma(b)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a + n) z^n}{\Gamma(b + n) n!}. \tag{5.10}
\]

The radial wavefunctions in Eqs. (5.6) are actually singular at the origin \( (r = 0) \). This singularity stems from the idealization of the nucleus as a point charge. As is conventional, we will therefore evaluate the wavefunctions at some finite nuclear radius \( R \). As mentioned in Section 4.3, the Fermi function gives the probability to find an electron in the interior of the nucleus relative to the probability to find it at the same position without a nucleus present. A useful relation between \( D(|\kappa|) \) and the Fermi function, \( F(Z, E) \), is given by

\[
D(1) = \sqrt{\frac{2F(Z, E)}{1 + \gamma_0} (2pR)^{1-\gamma_0}}, \tag{5.11}
\]

where \( \gamma_0 = \sqrt{1 - (\alpha Z)^2} \), as in Eq. (4.18), and should not be confused with the timelike gamma matrix. The difference should be clear from the context in which it is used.

For the evaluation of the decay rate of a nucleus we assume that the leptons emerge at infinity as free particles and therefore we need a plane-wave superposition of the spherical
waves given in Eq. (5.2). This spherical-wave expansion of a plane wave is well known [48] and for free particles it is given by

$$
\psi_\xi(r) = \sqrt{\frac{(2\pi)^3}{pE}} \sum_{\kappa \mu} i^l C(l_{\frac{1}{2}} j; \mu - \xi, \xi, \mu) Y^*_{\kappa \mu}(\hat{r}) \psi_{\kappa \mu}(r). \tag{5.12}
$$

For the negatively-charged leptons in a Coulomb field the spherical waves are phase shifted with respect to the free spherical waves. This can be seen by evaluating Eq. (5.6) at infinity and comparing to Eq. (5.5). The phase shift differs for different values of $\kappa$. We have to cancel this phase shift to be able to build a plane wave out of the spherical waves in a Coulomb field. For particles in a Coulomb potential the plane wave is then given by

$$
\Psi_\xi(r) = \sqrt{\frac{(2\pi)^3}{pE}} \sum_{\kappa \mu} i^l C(l_{\frac{1}{2}} j; \mu - \xi, \xi, \mu) Y^*_{\kappa \mu}(\hat{r}) \psi_{\kappa \mu}(r), \tag{5.13}
$$

where $e^{-i\delta'(\kappa)\psi_{\kappa \mu}(r)}$ are the phase-shifted spherical waves and the phase $\delta'(\kappa)$ is given by

$$
\delta'(\kappa) = \frac{1}{2} (l(\kappa) + 1)\pi - \text{arg}\Gamma(\gamma + i\nu) + \eta - \frac{1}{2} \pi \gamma. \tag{5.14}
$$

We put a apostrophe on the $\delta'(\kappa)$ in anticipation of a redefinition of this phase, given in Eq. (5.20).

Finally, we will also need the wavefunctions for antiparticles (the antineutrino and the positron). They are related to the wavefunctions of the particles by $\psi^C_{\kappa \mu}(Z) = i\gamma^2 \psi^*_{\kappa \mu}(-Z)$. Since $\sigma^C_{\kappa \mu} = S(\kappa)(-1)^{\mu+1} X_{\kappa, -\mu}$, the wavefunction $\psi^C_{\kappa \mu}$ is given by

$$
\psi^C_{\kappa \mu} = S(\kappa)(-1)^{\mu+1} \left( \frac{f_{-\kappa}(-Z)X_{-\kappa, -\mu}}{ig_{\kappa}(-Z)X_{\kappa, -\mu}} \right) e^{iEt}. \tag{5.15}
$$

## 5.2 Decay rate I

In this section we start to determine the decay rate formula. A job that we will finish in Section 5.4, after introducing $\beta$ moments as an intermezzo in Section 5.3. In the present normalization the decay rate is given by

$$
\frac{d\Gamma}{dE_\nu d\Omega_{\nu}} = 2\pi |\langle f | H_\beta | i \rangle|^2, \tag{5.16}
$$

where it is understood that the neutrino energy is determined by $E_\nu = E_0 - E_e$, with $E_0$ the energy available for the decay. In the Hamiltonian, given in Eq. (5.1), we put the expressions for the plane-wave functions of the leptons described in Section 5.1. For symmetry reasons we redefine the neutrino field by a phase given by

$$
\psi^C_{\bar{\kappa} \bar{\mu}} \rightarrow \pm S(\bar{\kappa}) \bar{e}^{-i\bar{\ell}(\bar{\kappa})}(-1)^{-\bar{\mu}-\frac{1}{2}} \psi^C_{\bar{\kappa} \bar{\mu}}, \tag{5.17}
$$

where the upper sign is for $\beta^-$ decay and the lower sign is for $\beta^+$ decay. This phase is irrelevant if we integrate over all emission directions of the neutrino and if the Lorentz-violating tensor $\chi^{\lambda \tau}$ is independent of momentum. Making use of the equality

$$
\chi^C_{\kappa \mu} X_{\kappa' \mu'} = \sum_{\xi} Y^*_{\kappa \mu - \xi} Y_{\kappa' \mu' - \xi} C(l_{\frac{1}{2}} j; \mu - \xi, \xi, \mu) C(l'_{\frac{1}{2}} j'; \mu' - \xi, \xi, \mu') \tag{5.18}
$$
for the spinor spherical harmonics and
\[ e^{i\beta'}e^{i\delta'(-\delta')} = e^{i\beta - \delta(\kappa')} , \]  
(5.19)
with
\[ \delta(\kappa) = \nu \ln(2pr) - \arg \Gamma(\gamma + i\nu) + \eta - \frac{1}{2}\pi\gamma , \]  
(5.20)
for the phases of the \( \beta \)-particle wavefunction we derive the following expression for the differential decay rate:
\[ \frac{d\Gamma}{dEd\Omega_d\Omega_{\nu}} = 2\pi \left| \sum_{\kappa\mu\bar{\kappa}\bar{\mu}} S(\kappa)(-1)^{\bar{\mu} + \frac{1}{2}} e^{i\beta(\kappa)} \chi_{\kappa\mu}(\bar{\mathbf{p}})\chi_{\bar{\kappa}\bar{\mu}}(\bar{\mathbf{q}}) \langle f | H_{\beta}(\kappa\mu\bar{\kappa}\bar{\mu})| i \rangle \right|^2 , \]  
(5.21)
where \( \mathbf{p} \) is the linear momentum of the \( \beta \) particle and \( \mathbf{q} \) is the linear momentum of the neutrino. The Hamiltonian in Eq. (5.21) is given by
\[ H_{\beta}(\kappa\mu\bar{\kappa}\bar{\mu}) = \sum_{a=1}^{A} \chi_{\lambda\tau}(\lambda V + (\gamma^5)^a C_{\Lambda}) T^a_{\pm}(g^{\lambda\tau} + \chi^{\lambda\tau} \pm i\chi^{\lambda\tau}) \mathcal{J}_a(\kappa\mu\bar{\kappa}\bar{\mu}; r^a) . \]  
(5.22)
Here, \( \chi^{\lambda\tau} = \text{Re}(\chi^{\lambda\tau}) \) and \( \chi^{\lambda\tau}_i = \text{Im}(\chi^{\lambda\tau}) \). The sum over \( a \) runs over all nucleons, \( T^a_{\pm} \) is the isospin raising (lowering) operator, and \( \mathcal{J}_a(\kappa\mu\bar{\kappa}\bar{\mu}; r^a) \) is the lepton current given by
\[ \mathcal{J}_a(\kappa\mu\bar{\kappa}\bar{\mu}; r^a) = \frac{q i^{-(\bar{\kappa})}}{\sqrt{\pi}} \left( g(\pm Z) j_{(\bar{\kappa})} \chi_{\kappa\mu}^{\dagger} \Sigma^a_{\pm} \chi_{\bar{\kappa},-\bar{\mu}} - S(\bar{\kappa}) f_{-\kappa}(\pm Z) j_{(\bar{\kappa})} \chi_{-\kappa\mu}^{\dagger} \Sigma^a_{\pm} \chi_{-\bar{\kappa},-\bar{\mu}} \right) \]  
(5.23)
with \( \Sigma^a_{\pm} = (1, \mp \sigma) \). We denote all quantum numbers related to the neutrino with a bar over them, while the quantum numbers related to the \( \beta \) particle do not have a bar. There is an implicit sum over the polarization of the \( \beta \) particle and the neutrino in Eq. (5.21), which is included in the definition of \( \chi_{\kappa\mu} \) given in Eq. (5.3). To generalize to a decay rate dependent on these polarizations one just has to put in the definition of \( \chi_{\kappa\mu} \) and omit the sum over the polarization.

To derive the selection rules that classify decay transitions, we evaluate the decay rate in terms of spherical-tensor operators. To that end, we rewrite the lepton current in terms of spherical harmonics and so called vector spherical harmonics. This will also be useful to see the effect of Lorentz violation. The vector spherical harmonics are defined by
\[ T^L_{jm} = \sum_{\rho=0,\pm1} C(L1J; m - \rho, \rho, m) Y_{L,\rho - \rho} e_{\rho} , \]  
(5.24)
with \( e_{\rho} \) the unit vector in spherical coordinates. These are related to the Cartesian coordinates by
\[ e_0 = e_z , \quad e_{\pm1} = \pm \frac{1}{\sqrt{2}} (e_x \pm ie_y) . \]  
(5.25)
To rewrite the current in terms of the (vector) spherical harmonics we use the relations

\[ \chi^\dagger_{\kappa \mu} \chi_{\bar{\kappa}, -\bar{\mu}} = \sum_L' (-1)^{\bar{\mu} - \frac{1}{2} \rho_L(j\bar{j})} C(j\bar{j}L; \mu, \bar{\mu}, \mu + \bar{\mu}) Y^*_{L,\mu+\bar{\mu}} , \tag{5.26a} \]

\[ \chi^\dagger_{\kappa \mu} \sigma \chi_{\bar{\kappa}, -\bar{\mu}} = \sum_{LJ} (-1)^{\bar{\mu} + \frac{1}{2}} \rho_J(j\bar{j}) C(j\bar{j}J; \mu, \bar{\mu}, \mu + \bar{\mu}) T_{J,-\mu-\bar{\mu}} \times (C(J1L; 000) + S(\kappa) w_J(j\bar{j}) C(J1L; 1, -1, 0)) \tag{5.26b} \]

These can be obtained by using the explicit expressions for the spinor spherical harmonics, given in Eq. (5.3). The prime on the summation means that only even values of \( l + \bar{l} + L \) are to be included, which originates from parity properties of the wavefunctions. The functions \( \rho_J(j\bar{j}) \) and \( w_J(j\bar{j}) \) are defined as

\[ \rho_J(j\bar{j}) = \sqrt{\frac{(2j+1)(2\bar{j}+1)}{4\pi(2J+1)}} C(j\bar{j}J; -\frac{1}{2}, \frac{1}{2} 0), \tag{5.27} \]

\[ w_J(j\bar{j}) = \left\{ \begin{array}{ll} \sqrt{2J+1}/ \sqrt{2J(J+1)} & , J > 0 \\ 0 & , J = 0 \end{array} \right. \tag{5.28} \]

Using the relations in Eq. (5.26) to rewrite the lepton current and putting this into the Hamiltonian in Eq. (5.22) we can write

\[ \langle f|H(\bar{\kappa})|i \rangle = \frac{q}{\sqrt{\eta}} \langle f|H(\bar{\kappa})|i \rangle \pm \langle f|H(-\bar{\kappa})|i \rangle \tag{5.29} \]

suppressing the dependence on \( \kappa, \mu \) and \( \bar{\mu} \). The matrix elements on the right-hand side of Eq. (5.29) are given by

\[ \langle f|H(\bar{\kappa})|i \rangle = (-1)^{\bar{l} - \frac{1}{2} k} \sum_{LJ} \rho_J(j\bar{j}) C(j\bar{j}J; \mu, \bar{\mu}, \mu + \bar{\mu}) \]

\[ \times \left[ \left( f|1 C_V + \gamma^5 C_A | g^{0\tau} + \chi^{0\tau}_r \pm i\chi^{0\tau}_s \right) \tilde{J}^\tau|i \rangle \right. \]

\[ + \left. \left( f|C_A + \gamma^5 C_V | \gamma^5 \gamma^0 \gamma_k (g^{k\tau} + \chi^{k\tau}_r \pm i\chi^{k\tau}_s) \tilde{J}^\tau|i \rangle \right) \right] \tag{5.30} \]

where the index \( k \) in the second term on the second line runs over spatial indices only and \( \tilde{J}^\nu \) is defined by

\[ \tilde{J}^0 = \delta_{LJ} P(\kappa\bar{\kappa}) Y^*_{L,\mu+\bar{\mu}} , \tag{5.31a} \]

\[ \tilde{J} = \mp L P(\kappa\bar{\kappa}) T^L_{J,\mu+\bar{\mu}} \tag{5.31b} \]

with

\[ P(\kappa\bar{\kappa}) = g_\kappa(pr) j_{\bar{\kappa}} - S(\bar{\kappa}) f_{-\kappa\bar{\kappa}} , \tag{5.32a} \]

\[ \bar{P}(\kappa\bar{\kappa}) = g_\kappa(pr) j_{\bar{\kappa}} + S(\bar{\kappa}) f_{-\kappa\bar{\kappa}} , \tag{5.32b} \]

\[ P_{LJ}(\kappa\bar{\kappa}) = C(J1L; 000) P(\kappa\bar{\kappa}) + S(\kappa) w_J(j\bar{j}) C(J1L; -1, 1, 0) \bar{P}(\kappa\bar{\kappa}) \tag{5.32c} \]
In Eq. (5.30) the sum over the nucleons and the isospin raising or lowering operator have been omitted and their presence is to be understood.

From Eq. (5.30) we can see the consequence of the Lorentz-violating parameter $\chi^{\lambda \tau}$: it connects 1 parts of the hadron current to $\Sigma$ parts of the lepton current and vice versa. Intuitively this can be understood by saying that the resultant intrinsic spin of the lepton pair and the spin change of the decaying nucleon are no longer directly related. In the following we will show that this causes a relative enhancement for Lorentz-violating effects in $\beta$-decay transitions with a nuclear-spin change ($\Delta I$) larger than or equal to two. To do this it is useful to introduce the concept of $\beta$ moments, which we will do as an intermezzo in the following section.

### 5.3 $\beta$ moments and Lorentz violation

In this section we introduce the concept of $\beta$ moments. These are the reduced matrix elements of the spherical tensors appearing in the $\beta$-decay amplitude. Four of these $\beta$ moments are defined by

\begin{align}
\langle f \mid \sum_a (T_\pm i^{-J} Y_{jm})^a \mid i \rangle &= C(I'JI; M'mM) \langle Y_J \rangle, \\
\langle f \mid \sum_a (T_\pm i^{-J} \gamma^5 Y_{jm})^a \mid i \rangle &= C(I'JI; M'mM) \langle \gamma^5 Y_J \rangle, \\
\langle f \mid \sum_a (T_\pm i^{-L} (\gamma^5 \gamma^0 \gamma \cdot T^L_{jm})^a \mid i \rangle &= C(I'JI; M'mM) \langle \sigma \cdot T^L_J \rangle, \\
\langle f \mid \sum_a (T_\pm i^{-L} (\gamma^0 \gamma \cdot T^L_{jm})^a \mid i \rangle &= C(I'JI; M'mM) \langle \alpha \cdot T^L_J \rangle.
\end{align}

The equality in these four equations follows from the Wigner-Eckhart theorem, that allows us to express the matrix elements of spherical tensors in terms of a Clebsch-Gordan coefficient and a reduced matrix element. These reduced matrix elements are called $\beta$ moments. In the above expressions $T_\pm$ is the isospin raising or lowering operator and the phase factors $i^{-L}$ and $i^{-J}$ are put in the definition to make sure the $\beta$ moments are real (see Ref. [48] for a derivation of this factor). The symbols that represent the $\beta$ moments in the above relations are abbreviations. One should consult the relations in Eqs. (5.33) for their exact meaning. We use $\alpha = \gamma^0 \gamma$ and $\sigma = \gamma^5 \gamma^0 \gamma$, trusting that for the latter there will be no confusion with the Pauli matrices. Furthermore, we use $m = \mu + \bar{\mu}$ as a short-hand notation.

The $\beta$ moments in Eq. (5.33) all occur for $\beta$ decay with Lorentz symmetry. If we include
Lorentz-symmetry breaking through $\chi^\lambda \tau$, there will be the following contributions:

\[
\langle f \left| \sum_a (T_{+ i} L \sigma^\rho \gamma^\nu Y_{Lm})^a \right| i \rangle = \sum_{\tilde{J} \tilde{m}} C(L1 \tilde{J}; m \rho \tilde{m}) C(I' \tilde{J} I; M' \tilde{m} M) \langle \sigma \cdot T_{\tilde{J}} \rangle ,
\]

(5.34a)

\[
\langle f \left| \sum_a (T_{+ i} L \sigma^\rho \gamma^\nu Y_{Lm})^a \right| i \rangle = \sum_{\tilde{m}} e^{\rho}_{\tilde{J} L} C(L1 \tilde{J}; \tilde{m} \rho \tilde{m}) C(I' \tilde{J} I; M' \tilde{m} M) \langle Y_{\tilde{J}} \rangle ,
\]

(5.34b)

\[
\langle f \left| \sum_a (T_{+ i} L \sigma^\rho \gamma^\nu Y_{Lm})^a \right| i \rangle = \sum_{\tilde{m}} e^{\rho}_{\tilde{J} L} C(L1 \tilde{J}; \tilde{m} \rho \tilde{m}) C(L1 \tilde{J}; \omega \sigma \tilde{m}) \times C(I' \tilde{J} I; M' \tilde{m} M) \langle \sigma \cdot T_{\tilde{J}} \rangle .
\]

(5.34c)

The first relation originates from the space-time components of $\chi^\lambda \tau$, the second one is introduced by the time-space components, and the third one comes from the space-space components. The indices $\rho$ and $\sigma$ run over spherical coordinates $0, \pm 1$, which are defined in Eq. (5.25). There are three more of these contributions that are introduced by Lorentz violation. They follow from the ones in Eqs. (5.34) by adding a factor of $\gamma^5$ to both sides of the equation.

Using that the $\beta$ moments should be invariant under space inversion, it can be shown that there are certain parity selection rules (see Ref. [48]). These entail that $\langle Y_L \rangle$ and $\langle \sigma \cdot T_{\tilde{J}} \rangle$ vanish unless the parity of the parent nucleus ($\pi_i$) and the parity of the daughter nucleus ($\pi_f$) are such that $\pi_i \pi_f = (-1)^L$. For the $\beta$ moments $\langle \gamma^5 Y_L \rangle$ and $\langle \alpha \cdot T_{\tilde{J}} \rangle$ not to vanish the relative parity must be such that $\pi_i \pi_f = (-1)^{L+1}$.

The matrix elements in Eqs. (5.34), that are related to Lorentz violation, are expressed in terms of the same $\beta$ moments that occur in the case without Lorentz violation, given in Eqs. (5.33). The difference is that $J$ (the total lepton momentum) is replaced by $\tilde{J}$, which is related to $J$ through Clebsch-Gordan coefficients given in Eqs (5.34). In the Lorentz-symmetric case the possible nuclear-spin change ($\Delta I$) is determined by the total angular momentum carried off by the lepton pair. This follows from the Clebsch-Gordan coefficients in Eqs. (5.33), which are physical only if the angular momenta obey the triangle inequality

\[
|I - I'| \leq J \leq I + I' .
\]

(5.35a)

Furthermore, the relation between the total angular momentum and the orbital angular momentum is

\[
|L - 1| \leq J \leq L + 1 ,
\]

(5.35b)

which follows from the definition of the vector spherical harmonics in Eq. (5.24). As we will see, the forbiddenness of a transition will largely be determined by $J$ (or actually $j + \bar{j}$). This relation between the forbiddenness and the total angular momenta of the leptons is unchanged in the Lorentz-violating case. However, there is a different set of triangle inequalities that determines the relation between the nuclear-spin change and
this forbiddenness:

\[
\begin{align*}
|I - I'| &\leq \tilde{J} \leq I + I', \\
|L - 1| &\leq \tilde{J} \leq L + 1, \\
|L - 1| &\leq J \leq L + 1, \\
|J - 2| &\leq \tilde{J} \leq J + 2.
\end{align*}
\]

(5.36a) (5.36b) (5.36c) (5.36d)

The last inequality is actually a consequence of the second and the third which means that \( J \) and \( \tilde{J} \) are related through \( L \). From Eq. (5.36a) we see that in the Lorentz-violating case the possible nuclear-spin changes are determined by \( \tilde{J} \), while the total angular momentum taken away by the lepton pair is related to \( \tilde{J} \) through Eq. (5.36d). Total angular momentum is therefore not conserved in all cases. However, the orbital angular momentum of the lepton pair in Eq. (5.36c) is the same as the orbital angular momentum lost by the nucleus in Eq. (5.36b). We thus say that orbital angular momentum remains conserved. Equivalently, we can say that the resultant intrinsic spin of the lepton pair is no longer directly connected to the spin change of the decaying nucleon. However, orbital angular momentum and the total intrinsic spins are not good quantum numbers, even in the Lorentz-symmetric case. Therefore the previous two statements only hold term by term in the multipole expansion and they are not statements about the physical process.

5.4 Decay rate II

In this chapter we continue to derive the general \( \beta \)-decay rate, while making use of the concepts introduced in the previous section. To this end, we write the matrix element of \( \mathcal{H}(\bar{\kappa}) \), given in Eq. (5.30) in terms of the \( \beta \) moments defined in Eqs. (5.33) and (5.34). The lepton current \( J_\mu \) depends on functions of \( pr \) and \( qr \), as can be seen from Eqs. (5.31) and (5.32). We need to pull these functions out of the matrix elements. This is done by defining some average nuclear radius \( R \) such that \( \langle f(r)S \rangle = f(R) \langle S \rangle \), with \( S \) some spherical tensor. In principle this average \( R \) is different for each \( \beta \) moment and should be labeled accordingly. However, they are all approximately equal to some root mean square nuclear radius and we will treat them as equivalent. We can then write the matrix element of \( \mathcal{H}(\bar{\kappa}) \) as

\[
\langle f|\mathcal{H}(\bar{\kappa})|i\rangle = i^{-\bar{\kappa}}(-1)^{\bar{\mu}-\frac{1}{2}} \sum_{JLm_\kappa} \left( C(I'J'I;M'm'M)C(j\bar{j}J;j\bar{j}m_J)\rho_J(j\bar{j})i^L \right.
\]

\[
\times \left( Q_0(\bar{\kappa}JLm_\kappa) \langle (C_V + \gamma^5C_A)Y_j \rangle + Q_1(\bar{\kappa}JLm_\kappa) \langle (C_A + \gamma^5C_V)\sigma \cdot T_{\bar{j}j} \rangle \right),
\]

(5.37)
with

\[ Q_0(\tilde{J}J\tilde{m}m\kappa\bar{\kappa}) = \delta_{LJ} \left( \delta_{J\ell} \delta_{m\bar{m}} P(\kappa\bar{\kappa})(1 + \chi_r^{00} \pm i\chi_i^{00}) \right. \]

\[ \pm \sum_{\rho} (-1)^\rho \left( \chi_r^{0\rho} \pm i\chi_i^{0\rho} \right) P_{LJ}(\kappa\bar{\kappa})C(L1J; \tilde{m}\rho m) \left. \right), \]

\[ Q_1(\tilde{J}J\tilde{m}m\kappa\bar{\kappa}) = \pm \sum_{\rho\sigma} \left( ((\delta^{\rho\sigma} - (-1)^\rho \chi_r^{\rho\sigma} \pm i\chi_i^{\rho\sigma})C(L1J; m - \rho, \rho, m) \times \right. \]

\[ \left. \times C(L1\tilde{J}; m - \rho, \sigma, \mu) P_{LJ}(\kappa\bar{\kappa}) \right) \right), \delta_{LJ}C(L1J; \tilde{m}\rho m)P(\kappa\bar{\kappa}) \right), \]

\[ (5.38a) \]

\[ (5.38b) \]

where now the radial wavefunctions in \( P(\kappa\bar{\kappa}) \) and \( P_{LJ}(\kappa\bar{\kappa}) \) are evaluated at some average nuclear radius \( R \), as mentioned above. The \( \rho \) and \( \sigma \) on the Lorentz-violating parameter \( \chi \) run over spherical coordinates here. The explicit zero on for example \( \chi^{0\rho} \) still represents the timelike direction, however.

We want to perform the summation over \( L \). This sum is restricted to even values of \( L + \tilde{L} \), due to parity properties of the lepton wavefunctions, as mentioned below Eq. (5.26). Since \( L + \tilde{L} = j + \bar{j} \pm 1 \) if \( S(\kappa) = S(\bar{\kappa}) \) and \( L + \tilde{L} = j + \bar{j} \) if \( S(\kappa) \neq S(\bar{\kappa}) \) we can derive the following restrictions on different possible relations between \( L \) and \( J \):

\[ \begin{align*}
\text{if } L = J \pm 1 & \quad \text{and} \quad \left\{ \begin{array}{l}
S(\kappa) = S(\bar{\kappa}) \rightarrow J + j + \bar{j} \text{ even} \\
S(\kappa) \neq S(\bar{\kappa}) \rightarrow J + j + \bar{j} \text{ odd}
\end{array} \right., \\
\text{if } L = J & \quad \text{and} \quad \left\{ \begin{array}{l}
S(\kappa) = S(\bar{\kappa}) \rightarrow J + j + \bar{j} \text{ odd} \\
S(\kappa) \neq S(\bar{\kappa}) \rightarrow J + j + \bar{j} \text{ even}
\end{array} \right.. 
\end{align*} \]

\[ (5.39a) \]

\[ (5.39b) \]

It follows that after performing the sum

\[ \langle f | \mathcal{H}(\bar{\kappa}) | i \rangle = (-1)^{\tilde{J} + \frac{1}{2}} \sum_{J\tilde{J}m} C(I^I; M\tilde{M})C(j\tilde{j}J; \mu\bar{\mu}m)\rho_{J\tilde{J}}(j\tilde{j})A_{JJ}(\kappa, \bar{\kappa}), \]

\[ (5.40) \]

where, if \( S(\kappa) = S(\bar{\kappa}) \) and \( J + j + \bar{j} \) even or \( S(\kappa) \neq S(\bar{\kappa}) \) and \( J + j + \bar{j} \) odd, \( A_{JJ}(\kappa, \bar{\kappa}) \) is given by

\[ A_{JJ}(\kappa, \bar{\kappa}) = i^{J + 1 - \tilde{I}(\kappa)} \left( g_\kappa j_{\kappa}(\kappa) B_{JJ}(S(\kappa)) - S(\bar{\kappa}) f_{-\kappa j_{\bar{\kappa}}(-\kappa)} B_{JJ}(-S(\kappa)) \right), \]

\[ (5.41a) \]

while if \( S(\kappa) \neq S(\bar{\kappa}) \) and \( J + j + \bar{j} \) even or \( S(\kappa) = S(\bar{\kappa}) \) and \( J + j + \bar{j} \) odd,

\[ A_{JJ}(\kappa, \bar{\kappa}) = i^{J - \tilde{I}(\kappa)} \left( g_\kappa j_{\kappa}(\kappa) C_{JJ}(S(\kappa)) - S(\bar{\kappa}) f_{-\kappa j_{\bar{\kappa}}(-\kappa)} C_{JJ}(-S(\kappa)) \right). \]

\[ (5.41b) \]
The quantities $B_{jj}$ and $C_{jj}$ are defined by

$$B_{jj}(S(\kappa)) = \pm \frac{1}{\sqrt{2J+1}} \left[ \left( \sqrt{J} - S(\kappa)w_j \sqrt{\frac{J+1}{2}} \right) \left( X^0_{J-1} \langle (C_V + \gamma^5C_A)Y_j \rangle \right. \right.$$  

$$+ X^1_{J-1} \left( (C_A + \gamma^5C_V)\sigma \cdot T^{J-1}_j \right) \right]$$  

$$+ \left( \sqrt{J+1} + S(\kappa)w_J \sqrt{\frac{J}{2}} \right) \left( X^0_{J+1} \langle (C_V + \gamma^5C_A)Y_j \rangle \right. \right.$$  

$$+ X^1_{J+1} \left( (C_A + \gamma^5C_V)\sigma \cdot T^{J+1}_j \right) \right], \quad (5.42a)$$

$$C_{jj}(S(\kappa)) = Z^0(S(\kappa)) \langle (C_V + \gamma^5C_A)Y_j \rangle + Z^1(S(\kappa)) \langle (C_A + \gamma^5C_V)\sigma \cdot T^{J}_j \rangle. \quad (5.42b)$$

In these definitions we used the following abbreviations:

$$X^0_{J+1} = \sum_\rho (-1)^\rho \left( \chi^0_{\rho,\rho} \pm i\chi^0_{\bar{\rho},\rho} \right) \delta_{J+1,J}C(J+1,1,J;\hat{m}\rho m), \quad (5.43a)$$

$$X^1_{J+1} = \sum_{\rho\sigma} \left( \delta^{\rho\sigma} - (-1)^\rho \left( \chi^\sigma_{\rho,\rho} \pm i\chi^\sigma_{\bar{\rho},\rho} \right) \right) C(J+1,1,J;m-\rho,\rho,m) \times$$

$$\times C(J+1,1,J;\hat{m}\rho m), \quad (5.43b)$$

$$Z^0(S(\kappa)) = (1 + \chi^0_{\rho,0} \pm i\chi^0_{\bar{\rho},0}) \delta_{jj}\delta_{m\bar{m}}$$

$$\mp S(\kappa)\frac{w_j}{\sqrt{2}} \sum_\rho (-1)^\rho \left( \chi^0_{\rho,\rho} \pm i\chi^0_{\bar{\rho},\rho} \right) \delta_{JJ}C(J1;\hat{m}\rho m), \quad (5.43c)$$

$$Z^1(S(\kappa)) = \mp S(\kappa)\frac{w_j}{\sqrt{2}} \sum_{\rho\sigma} \left( \delta^{\rho\sigma} - (-1)^\rho \left( \chi^\sigma_{\rho,\rho} \pm i\chi^\sigma_{\bar{\rho},\rho} \right) \right) C(J1;J;m-\rho,\rho,m) \times$$

$$\times C(J1;\hat{m}\rho m), \quad (5.43d)$$

The expressions for $X^0_{J-1}$ and $X^1_{J-1}$ can be obtained from the ones for $X^0_{J+1}$ and $X^1_{J+1}$ by replacing $J+1 \rightarrow J-1$ everywhere. Putting the matrix element in Eq. (5.40) into Eq. (5.29) and the result in Eq. (5.21), we can now finally write the general Lorentz-violating $\beta$-decay rate as

$$\frac{d\lambda}{dE_\nu d\Omega} = 2q^2 \sum_{\kappa\bar{\kappa}} \left| \sum_{jj\mu\bar{\mu}} e^{i\delta(\kappa)} X_{\kappa\mu} \rho_{j}(j\bar{j})C(j\bar{j}J;\mu\bar{\mu}m)C(1'\bar{J}1;M'\bar{m}M) \times$$

$$\times (A_{jj}(\kappa,\bar{\kappa}) \pm A_{jj}(\kappa,-\bar{\kappa})) \right|^2. \quad (5.44)$$

We integrated over neutrino-emission directions, which allowed us to pull the sum over $\bar{\kappa}$ and $\bar{\mu}$ out of the square. In doing this integration we assumed that $\chi^{\kappa\tau}$ is independent of momentum. We will use this assumption in the remainder of the chapter.
5.5 Approximations

The expression for the $\beta$-decay rate, given in Eq. (5.44), is still exact. In this section we will make a few approximations, to get a more useful expression. We will also derive which $\beta$ moments are dominant for which kind of $\beta$ transition.

One of the approximations we will introduce is in assuming that the Lorentz violation is small and therefore we will calculate everything to first order in $\chi$. Another approximation is that we assume the following three dimensionless quantities to be small: $R/\lambda$, $\alpha Z$ and $v_N$. These are the nuclear radius in units of the de Broglie wavelength (divided by $2\pi$), the finestructure constant times the atomic number of the daughter nucleus, and the nucleon velocity in units of the speed of light, respectively. Typical values for $R/\lambda$ and $v_N$ are cited in Ref. [48] as $R/\lambda \lesssim 1/40$ and $v_N \sim 1/10$ respectively. This approximation is conventionally called the ‘normal approximation’. Finally, we will assume that $\alpha Z/R \ll E_0$, where $E_0$ is the energy available in the decay (notice that $pR = R/\lambda$). This is usually called the ‘$\xi$ approximation’, since $\xi$ is a common abbreviation in studies of $\beta$ decay, given by $\xi = \alpha Z/2R$.

We start by rewriting Eq. (5.44) in a different form, where we change the summations over $\kappa$ and $\bar{\kappa}$ to summations over $\bar{j}$ and $j$, using Eq. (5.4). The decay rate is then given by

$$\frac{d\lambda}{dE d\omega_c} = 4q^2 \sum_{j\bar{j}} \sum_{j,j\bar{j}\bar{\mu}} \rho_j(j\bar{j})C(j\bar{j}J; \mu\bar{\mu})C(1'\bar{J}I; M'\bar{m}M)e^{i\delta(x)\bar{x}j}$$

$$\left[\chi_{x\mu}(g_x D_+ - f_{-x} D_-) \pm ie^{i\delta(-x) - i\delta(x)}\chi_{-x\mu}(f_x D_+ + g_{-x} D_-)\right]^2, \quad (5.45)$$

where $x = j + \frac{1}{2}$ and $\bar{x} = \bar{j} + \frac{1}{2}$ and

$$D_+ = \begin{cases} j_{l(\bar{x})} B_{jj}(+1) \pm j_{l(-\bar{x})} C_{jj}(+1) & \text{if } J + j + \bar{j} \text{ is even} \\ \pm ij_{l(-\bar{x})} B_{jj}(+1) - ij_{l(\bar{x})} C_{jj}(+1) & \text{if } J + j + \bar{j} \text{ is odd} \end{cases}, \quad (5.46a)$$

$$D_- = \begin{cases} j_{l(-\bar{x})} B_{jj}(-1) \mp j_{l(x)} C_{jj}(-1) & \text{if } J + j + \bar{j} \text{ is even} \\ \mp ij_{l(\bar{x})} B_{jj}(-1) - ij_{l(-\bar{x})} C_{jj}(-1) & \text{if } J + j + \bar{j} \text{ is odd} \end{cases}. \quad (5.46b)$$

Next we apply the approximations $pR \ll 1$ and $\alpha Z \ll 1$ to the radial wave functions of the $\beta$ particle. We get

$$g_\kappa \approx \sqrt{\frac{2pF(\pm Z, E)}{\pi(1 + \gamma_0)}} \frac{(pR)^{j-\frac{1}{2}}}{(2j)!}) \times \left\{ \begin{array}{ll} \frac{\alpha Z}{2j+1} \sqrt{E - m} & \text{if } \kappa > 0 \\ \frac{\alpha Z}{2j+1} \sqrt{E + m} & \text{if } \kappa < 0 \end{array} \right., \quad (5.47a)$$

$$f_{-\kappa} \approx \sqrt{\frac{2pF(\pm Z, E)}{\pi(1 + \gamma_0)}} \frac{(pR)^{j-\frac{1}{2}}}{(2j)!}) \times \left\{ \begin{array}{ll} \pm \sqrt{E - m} & \text{if } \kappa > 0 \\ \pm \frac{\alpha Z}{2j+1} \sqrt{E + m} & \text{if } \kappa < 0 \end{array} \right.. \quad (5.47b)$$

Using the same approximations we also get that

$$e^{i\delta(-x) - i\delta(x)} \approx i, \quad (5.48a)$$

$$e^{i\delta(x') - i\delta(x)} \approx i^{j' - j'}. \quad (5.48b)$$
Putting the results in Eqs. (5.47) and (5.48) into Eq. (5.45) we can rewrite it to

\[
\frac{d\lambda}{dE d\Omega} \approx \frac{8pq^2 F(\pm Z, E)}{\pi(1 + \gamma_0)} \sum_{j\mu} \sum_{j_f \mu_f} \rho_{j}(j\bar{j}) C(\Gamma' I I; M' \bar{m} M) C(j\bar{j} J; \mu \bar{\mu} m) (pR)^{-\frac{1}{2}} e^{i\delta(x) t^J} \\
\times \sqrt{E - m} \left[ \chi_{x\mu} + \chi_{-x\mu} (E + m) \right] \left( D_- \mp \frac{\alpha Z}{2j + 1} D_+ \right)^2 .
\]  

(5.49)

At this point we want to determine the terms that give the dominant contribution to Eq. (5.49). We need to determine the number of factors of the small quantities \( pR, qR, \alpha Z, \) or \( v_N \). From Eqs. (5.46) we see that every term in Eq. (5.49) involves a spherical Bessel function, which has the property that

\[
j_{\ell}(qR) \approx \frac{(qR)^{\ell}}{(2\ell + 1)!!} ,
\]

(5.50)

for \( qR \ll 1 \). So terms with \( \ell = \bar{j} - \frac{1}{2} \) are proportional to \( (qR)^{\frac{1}{2}} \). Combined with the factor \( (pR)^{-\frac{1}{2}} \) in Eq. (5.49), each term will be at least proportional to \( R^{i+j-1} \) (and the corresponding factors of \( q \) and \( p \)). Some terms will have a larger suppression than that, however. It follows from Eq. (5.50) that, if \( \ell = \bar{j} + \frac{1}{2} \), there will be an extra factor of \( qR \). Terms coming from \( D_+ \) are suppressed further by \( \alpha Z \), as can be seen from Eq. (5.49). And finally there is an extra factor of \( v_N \) for terms that have an extra \( \gamma^5 \) matrix, since they involve the relativistically small components of the nucleon wavefunctions. Counting the factors of these small quantities, we can determine the suppression of each term in the amplitude or decay rate, given in Eq. (5.49).

For each of these terms we then have to find the decay transitions to which it contributes and relate the suppression to the spin change \( \Delta I = |I' - I| \) and the relative parity \( \pi_i \pi_f \). Since each term is at least proportional to \( R^{i+j-1} \) its suppressions is mainly determined by the value of \( j + \bar{j} \). To determine \( j + \bar{j} \), we write

\[
\Delta I = j - j + \delta_1 + \delta_2 + \delta_3 .
\]

(5.51)

where \( \delta_1 = \bar{j} - \Delta I, \delta_2 = j + \bar{j} - J, \) and \( \delta_3 = J - \bar{J} \). With these definitions \( \delta_1 \) and \( \delta_2 \) are both nonnegative integers with \( 0 \leq \delta_1 \leq I + I' - |I - I'| \) and \( 0 \leq \delta_2 \leq j + \bar{j} - |j - \bar{j}| \). These limits for \( \delta_1 \) and \( \delta_2 \) are determined by the first and second Clebsch-Gordan coefficients in Eq. (5.49), respectively. In contrast with \( \delta_1 \) and \( \delta_2 \), \( \delta_3 \) can be negative and is limited by \( -2 \leq \delta_3 \leq 2 \). This follows from the Clebsch-Gordan coefficients in Eqs. (5.43) and the fact that \( J \) and \( \bar{J} \) should both be positive. The possible values of \( \delta_3 \) differ for terms proportional to the different quantities, defined in Eqs. (5.43). These possible values can be summarized by

\[
\begin{align*}
\text{Terms prop. to} & \quad X^0_{\pm 1} : & \delta_3 = \mp 1 , \\
& \quad X^1_{\pm 1} : & -1 \leq \delta_3 \leq 1 \pm 1 , \\
& \quad Z^0 : & \delta_3 = 0 .
\end{align*}
\]

(5.52)
5.5 APPROXIMATIONS

Terms proportional to \( X_{j+1}^0 \) and \( X_{j+1}^1 \) do not contribute if \( \delta_2 = 0 \) and \( S(\kappa) = -1 \), however. This is the case, because both \( X_{j+1}^0 \) and \( X_{j+1}^1 \) involve the factor \( \sqrt{J+1} + S(\kappa)w_J \sqrt{J} \), which vanishes if \( \delta_2 = 0 \) and \( S(\kappa) = -1 \).

Using the relation in Eq. (5.51) and including the extra factors of \( qR, \alpha Z \) and \( v_N \) mentioned earlier, we can now relate the suppression of each term in Eq. (5.49) to the \( \Delta I \) of the transitions it can contribute to.

We also have to determine the parity properties of the \( \beta \) moment in each term. As described below Eq. (5.34), \( \beta \) moments will only contribute if the parities of the parent and daughter nuclei are such that \( \pi_i \pi_f = (-1)^L \) (without an extra \( \gamma^5 \)) or \( \pi_i \pi_f = (-1)^{L+1} \) (with an extra \( \gamma^5 \)). The relation between \( L \) and \( \delta_1 \) and \( \delta_3 \) is given by

\[
\text{Terms prop. to } X_{j+1}^0 : \quad L = \Delta I + \delta_1 + \delta_3 \pm 1, \\
\text{" } \quad X_{j+1}^1 : \quad L = \Delta I + \delta_1 + \delta_3 \pm 1, \\
\text{" } \quad Z^1 : \quad L = \Delta I + \delta_1 + \delta_3, \\
\text{" } \quad Z^0 : \quad L = \Delta I + \delta_1 + \delta_3.
\]

We now have all the information to find the terms that have the least suppression by factors of \( qR, pR, \alpha Z \), and \( v_N \). From this table, we see that for a transition with \( \pi_i \pi_f = \Delta I \), to lowest order, there will be contributions from the \( \beta \) moments \( \langle Y_{\Delta I} \rangle \), \( \langle \sigma \cdot T_{\Delta I}^I \rangle \), and \( \langle \alpha \cdot T_{\Delta I}^{I-1} \rangle \). These transitions are \( \Delta I \) times forbidden and are called parity forbidden, since their relative parity differs in sign from the transitions that are one degree less forbidden, but have the same \( \Delta I \). These latter transitions (with \( \pi_i \pi_f = \Delta I - 1 \)) are called unique transitions, since their transition rate is, at lowest order, determined by only one \( \beta \) moment, namely \( \langle \sigma \cdot T_{\Delta I}^I \rangle \).

From Table D.3 we see that also with Lorentz violation the transitions are determined by these same \( \beta \) moments. However, there are now extra contributions from terms where \( j + \tilde{j} \) and \( J \) differ from their conventional values. As can be seen from the last column of the table, the order of magnitude of these terms differs from the conventional contributions by a factor \( \alpha Z/R \) (this should by multiplied by a factor of \( 1/p \) to make it dimensionless). For the parity-forbidden transitions it is the \( \beta \) moment \( \langle \alpha \cdot T_{\Delta I}^{I-1} \rangle \) that causes this enhancement, while it is \( \langle \sigma \cdot T_{\Delta I}^{I-1} \rangle \) for the unique transitions. Notice also that this enhancement occurs for transitions with \( \Delta I \geq 2 \) and that transitions with \( \Delta I \geq 3 \) will get an extra enhanced contribution, with respect to transitions with \( \Delta I = 2 \), by terms with \( j + \tilde{j} = \Delta I - 2 \).

Keeping only the contributions of the dominant \( \beta \) moments in Eq. (5.49) we can determine a usable expression for the differential decay rate of a transition with \( \Delta I \geq 3 \). In doing this we sum over initial and final nuclear polarization and use

\[
\frac{1}{2J+1} \sum_{MM'} C(I'; \tilde{J}I; M', \tilde{m}, M)C(I'; \tilde{J}I; M' \tilde{m} M) = \frac{\delta_{\tilde{j} \tilde{j}'} \delta_{\tilde{m} \tilde{m}'}}{2J+1}.
\]
Furthermore, we also sum over $\tilde{J}$, $J$, and $j$ and we use the relation

$$
(E - m) \left( \chi_{x', \mu'}^{\dagger} + \chi_{-x', \mu'} \frac{E + m}{p} \right) \left( \chi_{x, \mu} + \chi_{-x, \mu} \frac{E + m}{p} \right) = 2 \sum_{\Lambda} (-1)^{\mu' + \frac{1}{2}} \rho_{\Lambda}(j'j) C(j'j; \mu', -\mu, \mu' - \mu) Y_{\Lambda, \mu - \mu'} U(\Lambda + j + j' + 1), \tag{5.55}
$$

where the function $U(n)$ is defined by

$$
U(n) = \begin{cases} E, & \text{if } n \text{ is even} \\ p, & \text{if } n \text{ is odd} \end{cases} \tag{5.56}
$$

The resulting differential decay rate is

$$
\frac{d\lambda}{d\Omega_e dE_e} = \frac{8pq^2 F(\pm Z, E_e)}{\pi(1 + \gamma_0)} \sum_{j} \left\{ \left( \frac{(pR)^{\Delta I - j - \frac{1}{2}} (qR)^{\frac{1}{2}}}{(2(\Delta I - j))!!(2j)!!} \right)^2 \times \left[ \frac{2\Delta I + 1}{\Delta I} \frac{E}{2\pi} (\rho_{\Delta I}(\Delta I - j, j))^2 \mathcal{M}_{\Delta I - j} \right] \times \left( \rho_{\Delta I}(\Delta I, \Lambda, j, \sigma, \rho) U(\Lambda) \times \left( (-1)^{\rho} (\chi_{\sigma - \rho}^r + i\chi_{\sigma - \rho}^0) Y_{\Lambda, \mu - \mu'} + c.c. \right) \right. \\
\left. \left. \mp \frac{\alpha Z}{p R} F_2(\Delta I, \Lambda, j, \sigma, \rho) U(\Lambda + 1) \left( (-1)^{\rho} (\chi_{\sigma - \rho}^r + i\chi_{\sigma - \rho}^0) Y_{\Lambda, \mu - \mu'} + c.c. \right) \right] \right. \\
\left. \left. + \sum_{\Lambda, \rho} \frac{\alpha Z}{p R} F_3(\Delta I, \Lambda, j, \rho) U(\Lambda + 1) \left( (\chi_{\sigma, 0}^r \pm i\chi_{\sigma, 0}^0) Y_{\Lambda, \mu - \mu'} + c.c. \right) \right) \right\}, \tag{5.57}
$$

where c.c. means the complex conjugate of the preceding term between brackets. The quantities $\mathcal{M}_j$ and $\mathcal{W}$ contain the $\beta$ moments. For a transition with $\pi_i \pi_f = (-1)^{\Delta I}$ they are given by

$$
\mathcal{M}_j = C_V \langle \mathbf{\alpha} \cdot \mathbf{T}_{\Delta I}^j \rangle = \frac{\alpha Z}{2j + 1} \sqrt{\frac{\Delta I + 1}{2\Delta I + 1}} C_A \langle \mathbf{\sigma} \cdot \mathbf{T}_{\Delta I}^j \rangle \mp \frac{\alpha Z}{2j + 1} \sqrt{\frac{\Delta I}{2\Delta I + 1}} C_V \langle Y_{\Delta I} \rangle, \tag{5.58a}
$$

while for a transition with $\pi_i \pi_f = (-1)^{\Delta I + 1}$

$$
\mathcal{M}_j = \mathcal{W} = C_A \langle \mathbf{\sigma} \cdot \mathbf{T}_{\Delta I}^{j-1} \rangle, \tag{5.58b}
$$

Furthermore, in Eq. (5.57) $F_1$, $F_2$, and $F_3$ are coefficients that depend on $\Delta I, \Lambda, j, \rho$ and
\[ F_1(\Delta I, \Lambda, \tilde{j}, \sigma, \rho) = \sum_{\lambda = \pm 1} \frac{d_{\Delta I-1} \rho_{\Delta I}(\Delta I - \tilde{j}, \tilde{j}) \rho_{\Delta I-2}(\Delta I - 2 - \tilde{j}, \tilde{j}) \rho_{\Delta I-3}(\Delta I - 2 - \tilde{j}, \Delta I - \tilde{j})}{\Delta I! \Delta I! \Delta I!} \]
\[ \times \sum_{\tilde{\mu}_{\tilde{\mu}}} \left[ (-1)^{\tilde{\mu}+\frac{1}{2}} C(\Delta I - \tilde{j}, \tilde{j}, \Delta I; \tilde{j}, \tilde{j}, \Delta I; \mu, \tilde{\mu}, \mu + \tilde{\mu}) \right. \]
\[ \left. \times C(\Delta I - 1 - \tilde{j}, \tilde{j}, \Delta I - 2; \mu' + \tilde{\mu} + \tilde{\mu}, \rho, \rho, \mu' + \tilde{\mu}) C(\Delta I - 1 - \tilde{j}, \tilde{j}, \Delta I - 2; \mu' + \tilde{\mu} - \rho, \rho, \mu' + \tilde{\mu}) \right] \]
\[ (5.59a) \]

\[ F_2(\Delta I, \Lambda, \tilde{j}, \sigma, \rho) = \sum_{\lambda = \pm 1} \frac{d_{\Delta I-1} \rho_{\Delta I}(\Delta I - 1 - \tilde{j}, \tilde{j}) \rho_{\Delta I-2}(\Delta I - 2 - \tilde{j}, \tilde{j})}{\Delta I! \Delta I! \Delta I!} \]
\[ \times \sum_{\tilde{\mu}_{\tilde{\mu}}} \left[ (-1)^{\tilde{\mu}+\frac{1}{2}} C(\Delta I - \tilde{j}, \tilde{j}, \Delta I; \tilde{j}, \tilde{j}, \Delta I; \mu, \tilde{\mu}, \mu + \tilde{\mu}) \right. \]
\[ \left. \times C(\Delta I - 1 - \tilde{j}, \tilde{j}, \Delta I - 2; \mu' + \tilde{\mu} + \tilde{\mu}, \rho, \rho, \mu' + \tilde{\mu}) C(\Delta I - 1 - \tilde{j}, \tilde{j}, \Delta I - 2; \mu' + \tilde{\mu} - \rho, \rho, \mu' + \tilde{\mu}) \right] \]
\[ (5.59b) \]

\[ F_3(\Delta I, \Lambda, \tilde{j}, \rho) = \sum_{\lambda = \pm 1} \frac{d_{\Delta I-1} \rho_{\Delta I}(\Delta I - \tilde{j}, \tilde{j}) \rho_{\Delta I-2}(\Delta I - 2 - \tilde{j}, \tilde{j})}{\Delta I! \Delta I! \Delta I!} \]
\[ \times \sum_{\tilde{\mu}_{\tilde{\mu}}} \left[ (-1)^{\tilde{\mu}+\frac{1}{2}} C(\Delta I - \tilde{j}, \tilde{j}, \Delta I; \tilde{j}, \tilde{j}, \Delta I; \mu, \tilde{\mu}, \mu + \tilde{\mu}) \right. \]
\[ \left. \times C(\Delta I - 1 - \tilde{j}, \tilde{j}, \Delta I - 2; \mu' + \tilde{\mu} - \rho, \rho, \mu' + \tilde{\mu}) C(\Delta I - 1 - \tilde{j}, \tilde{j}, \Delta I - 2; \mu' + \tilde{\mu} + \rho, \rho, \mu' + \tilde{\mu}) \right] \]
\[ (5.59c) \]

Eq. (5.57) is the main result of this chapter. It contains all the dominant contributions for a general \( \beta \) decay transition with \( \Delta I \geq 3 \). To obtain the rate of a \( \Delta I = 2 \) transition we have to put \( F_1 = 0 \) in Eq. (5.57), because for \( \Delta I = 2 \) some of the Clebsch-Gordan coefficients contained in \( F_1 \) are not physical, since they do not obey the standard triangle inequalities.

### 5.6 Analysis of experiments from the 1970s

Several years before the \( W \) boson was discovered and long before searches for Lorentz violation became fashionable, two isolated experiments were performed that searched for a “preferred” direction in space in first-forbidden \( ^{90}\text{Y} \beta \) decay \([53, 54]\) and in first-forbidden
137Cs and second-forbidden 99Tc β decays [55]. The hope was that such forbidden decays would be more sensitive to violations of rotational invariance, i.e. angular-momentum conservation. In Section 5.5 we saw that in our framework for Lorentz violation, Lorentz-violating contributions to forbidden transitions are indeed enhanced. In this section we show that the experiments of Refs. [53, 54, 55] provide strong and unique bounds on Lorentz violation in the weak interaction, and in particular on previously unconstrained parameters of the SME.

In Ref. [53] the β-decay chain $^{90}$Sr($0^+, 30.2\, a$) → $^{90}$Y($2^−, 64.1\, h$) → $^{90}$Zr($0^+$) was considered, wherein the β− decay of $^{90}$Y is a $\Delta I = 2, \pi_i \pi_f = -1$, unique first-forbidden transition. A search was made for dipole and quadrupole anisotropies in the angular distribution of the electrons,

$$\Gamma(\theta) = \Gamma_0 \left(1 + \varepsilon_1 \cos \theta + \varepsilon_2 \cos^2 \theta \right),$$

(5.60)

where $\theta$ is the angle between the electron momentum and a presumed preferred direction in space. A 10 Ci $^{90}$Sr source was put in a vacuum chamber and the electron current it produced was measured on a collector plate opposite the source, giving a solid angle of nearly $2\pi$. The source was made such that only high-energy electrons could come out, assuring that only the current due to $^{90}$Y was measured. The endpoint of $^{90}$Sr is too low to contribute significantly to the current for this particular source. The chamber rotated about a vertical axis with a frequency of 0.75 Hz. An anisotropy would result in a modulation of the detected current with a frequency of 0.75 or 1.5 Hz, depending on the dipole or quadrupole nature of the anisotropy.

The data were analyzed in terms of two dipole current asymmetries,

$$\delta_{NS} = 2 \frac{i_N - i_S}{i_N + i_S}, \quad \delta_{EW} = 2 \frac{i_E - i_W}{i_E + i_W},$$

(5.61)

and one quadrupole asymmetry,

$$\delta_{2\nu} = 2 \frac{i_N + i_S - i_E - i_W}{i_N + i_S + i_E + i_W},$$

(5.62)

where $N, S, E, W$ mean north, south, east, and west, and where for instance $i_N$ denotes the mean current in the lab-fixed northern quadrant of the chamber’s rotation. These current asymmetries $\delta$ were fitted as functions of sidereal time as

$$\delta = a_0 + a_1 \sin(\omega t + \phi_1) + a_2 \sin(2\omega t + \phi_2),$$

(5.63)

where $\omega$ is the angular rotation frequency of the Earth. The extracted coefficients $a_{0,1,2}$ are given in Table 5.1. Relative phases between the different asymmetries were not considered.

<table>
<thead>
<tr>
<th>Asymmetry $\delta$</th>
<th>$10^8 \times a_0$</th>
<th>$10^8 \times a_1$</th>
<th>$10^8 \times a_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$NS$</td>
<td>$-1.9 \pm 1.0$</td>
<td>$3.2 \pm 1.9$</td>
<td>$1.7 \pm 1.9$</td>
</tr>
<tr>
<td>$EW$</td>
<td>$1.1 \pm 1.0$</td>
<td>$2.9 \pm 1.9$</td>
<td>$1.9 \pm 1.9$</td>
</tr>
<tr>
<td>$2\nu$</td>
<td>$-1.0 \pm 1.0$</td>
<td>$0.5 \pm 1.7$</td>
<td>$0.7 \pm 1.7$</td>
</tr>
</tbody>
</table>

Table 5.1: The measured values [53] for $a_{0,1,2}$ of Eq. (5.63).
and the phases $\phi_{1,2}$ between the amplitudes were not reported. Such relations would have provided stronger constraints on Lorentz violation.

By using Eq. (5.60) the expressions for $a_{0,1,2}$ were determined. Scattering inside the source, due to which the emission direction of the electrons gets partly lost when they leave the sample, had to be taken into account. With a Monte-Carlo program the probability distribution to detect an electron was determined, depending on the angle of its original direction with respect to the normal of the source. This probability distribution was then folded with Eq. (5.60). The result for the current as function of the angle $\theta_n$ between the direction of the collector plate and the presumed asymmetry axis reads

$$I(\theta) = I_0 \left[ 1 + \frac{C_1}{3 + C_2} \varepsilon_1 \cot \theta_n + \frac{C_2}{15 + 5C_2} \varepsilon_2 \cos 2\theta_n \right],$$

(5.64)

with $C_1 = 1.26$ and $C_2 = 0.39$. After transforming this equation to a standard Sun-centered reference frame, the upper limits $|\varepsilon_1| < 1.6 \times 10^{-7}$ and $|\varepsilon_2| < 2.0 \times 10^{-6}$ were determined at 90% confidence level (C.L.) [53].

We interpret the data in Table 5.1 by using Eq. (5.57), from which we obtain the relevant expression for a unique first-forbidden decay with $\Delta I = 2$ and $\pi_i \pi_f = -1$. It is given by

$$\frac{d\Gamma}{d\Omega dE} = \frac{2Epq^2 F(\pm Z, E)}{9\pi^4(1 + \gamma_0)} C_A^2 \langle \sigma \cdot T \rangle^2 R^2 \left\{ p^2 + q^2 + \frac{\alpha Z}{pR} \left[ \frac{3}{10} E \left[ \chi_r^{ij} \hat{p}^i \hat{p}^j - \frac{1}{3} \chi^0_r \right] + \frac{1}{2} p^2 \chi_r^{ij} \hat{p}^i \hat{p}^j \pm p^2 \chi_r^{10} \hat{p} \right] \right\} \tag{5.65}$$

where $F(Z, E)$ is the usual Fermi function, $p$ and $E$ are the electron momentum and energy and $q = E_0 - E$ is the neutrino momentum, with $E_0$ the energy available in the decay.

The subscripts $r$ and $i$ on the Lorentz-violating tensor indicate the real and imaginary part of $\chi^{\mu\nu}$, respectively, and $\chi^l = \epsilon^{lmk} \chi_{mk}$. The Lorentz-invariant part of the decay rate has the typical unique first-forbidden spectrum shape $\sim p^2 + q^2$. The Lorentz-violating part scales with $\alpha Z/pR$, which shows that forbidden transitions can be more sensitive to angular-momentum violation, compared to allowed ones. The enhancement is about one order of magnitude for a typical transition. With Eq. (5.65) instead of Eq. (5.60), the remaining part of the analysis parallels the analysis of Ref. [53] summarized above.

---

1Eq. (5.64) includes a correction on Eq. (4) of Ref. [53]
requires a simulation of the electron trajectories with the modified weight of the Lorentz-violating part of the expression, which, however, would entail a small modification of the original simulation. Therefore, instead, we integrate the Lorentz-violating part of Eq. (5.65) over the energy of the detected electrons, including the energy-dependent phase-space factor $\propto E p q^2 F(Z, E)$. We integrate over the top 23.4% of the energy spectrum, since the detector covered a $2\pi$ solid angle and 11.7% of the electrons escaped the source and were collected [53]. With this simplified procedure we may do the angular folding of Eq. (5.65) with the original detection-probability distribution. We checked that the limits derived below are not affected by more than 4% when changing the integrated fraction of the energy spectrum by a factor two.

We transform the tensor $\chi^{\mu\nu}$, defined in the laboratory frame, to the tensor $X^{\mu\nu}$ defined in the Sun-centered frame [19], by using $\chi^{\mu\nu} = R^\mu_\rho R^\nu_\sigma X^{\rho\sigma}$, with $R$ the appropriate rotation matrix (see Eq. (A.2)). In this way, we obtain the theoretical expressions for the coefficients $a_{0,1,2}$ listed in Table 5.2. The numerical factors in front of the coefficients in Table 5.2 are determined by the location of the experiment on Earth (the colatitude of New York is about $49^\circ$), the constants $C_1$ and $C_2$, and two phase shifts in the amplifier used in the experiment [53]. The phase correlation between $\delta_{NS}$ and $\delta_{EW}$ that our theory predicts was not measured. Therefore, $2X^{10}_r - X^1_i$ and $2X^{20}_r - X^2_i$ cannot be extracted separately and only the combined value can be determined. The phase shifts of the amplifier are the reason that $a_0 \neq 0$ for $\delta_{EW}$.

Comparing the experimental values in Table 5.1 to the theoretical predictions in Table 5.2, we derive the following limits on the Lorentz-violating coefficients at 95% C.L.:

\begin{align}
-6 \times 10^{-9} &< 2X^{30}_r - X^3_i < 2 \times 10^{-8}, \quad (5.66a) \\
-3 \times 10^{-6} &< 3X^{33}_r - X^{00}_r < 1 \times 10^{-6}, \quad (5.66b) \\
\left[ (2X^{20}_r - X^2_i)^2 + (2X^{10}_r - X^1_i)^2 \right]^{1/2} &< 4 \times 10^{-8}, \quad (5.66c) \\
\left[ (X^{13}_r + X^{31}_i)^2 + (X^{23}_r + X^{32}_i)^2 \right]^{1/2} &< 1 \times 10^{-6}, \quad (5.66d) \\
\left[ (X^{12}_r + X^{21}_i)^2 + (X^{22}_r - X^{11}_i)^2 \right]^{1/2} &< 1 \times 10^{-6}. \quad (5.66e)
\end{align}

The limit of Eq. (5.66c) was obtained from $\delta_{EW}$ only, because of the phase correlation between $\delta_{NS}$ and $\delta_{EW}$.

In Ref. [55] a similar search was made in the $\Delta I = 2$, $\pi_i \pi_f = -1$, unique first-forbidden $\beta^-$ decay $^{137}$Cs$(\frac{7}{2}^+, 30.2 a) \rightarrow ^{137m}$Ba($\frac{11}{2}^-$), and in the $\Delta I = 2$, $\pi_i \pi_f = 1$, second-forbidden $\beta^-$ decay $^{99}$Tc$(\frac{9}{2}^+, 2.1 \times 10^5 a) \rightarrow ^{99}$Ru($\frac{5}{2}^+$), by looking for a modulation of the counting rate as a function of sidereal time. An upper limit for a cos $\theta$ or cos $2\theta$ term of $3 \times 10^{-5}$ was found at 90% C.L. Although the accuracy was less than in the $^{90}$Y experiment, the setup in this experiment had a higher angular resolution. For $^{137}$Cs decay Eq. (5.65) also applies, while the expression for $^{99}$Tc decay can be obtained from Eq. (5.57) and is given
by
\[
\frac{d\Gamma}{d\Omega dE} = \frac{2Epq^2 F(\pm Z, E)}{9\pi^3(1 + \gamma_0)} R^2 \left\{ \mathcal{M}_{3/2}^2 p^2 + \mathcal{M}_{1/2}^2 q^2 \right\} + \mathcal{M}_{3/2} \mathcal{W} \frac{p^2 \alpha Z}{pR} \left[ \frac{3}{10} \frac{p}{E} \left( \chi_r^{ij} p^i p^j - \frac{1}{3} \chi_r^{00} - \frac{1}{2} \chi_r^{ij} p^i + \chi_r^{0i} p^i \right) \right],
\] (5.67)

where \(\mathcal{M}_{1/2}\) and \(\mathcal{M}_{3/2}\) depend on three \(\beta\) moments as can be seen from Eq. (5.58a). From Eq. (5.58a) it also follows that \(\mathcal{W} = 2\mathcal{M}_{3/2} - \mathcal{M}_{1/2}\). We use \(\mathcal{M}_{3/2}/\mathcal{M}_{1/2} = 0.735\), such that the spectrum shape \(\sim 0.54 p^2 + q^2\) [56, 57].

The experimental setup in Ref. [55] was made to measure electrons in two directions. In one direction, perpendicular to the Earth’s rotation axis, a count rate \(N_S\) was measured, and in the direction parallel to the Earth’s rotation axis a count rate \(N_P\). The observable \(A = N_S/N_P - 1\) was then inspected for sidereal variations. By using the expressions for the Lorentz-violating decay rates of \(^{137}\text{Cs}\) and \(^{99}\text{Tc}\), given in Eqs. (5.65) and (5.67), we obtain an expression similar to Eq. (5.63) for the observable \(A\). The amplitudes are proportional to the same combinations of Lorentz-violating coefficients as found previously for \(^{90}\text{Y}\). This time \(a_0\) is a combination of the terms found in the first column of Table 5.2. The terms for \(a_1\) can be separated to obtain the individual terms \(2X_r^{10} - \tilde{X}_r^1\) and \(2X_r^{20} - \tilde{X}_r^2\). Similarly this can be done for \(a_2\). The proportionality constants are larger than for the \(^{90}\text{Y}\) experiment. In particular, for the terms found previously in the quadrupole asymmetry the sensitivity of this setup is a factor 10 to 100 higher. However, the statistical accuracy in this experiment is much lower and the improvements on the bounds of Eq. (5.66) are insignificant. The best case is \(|3X_r^{33} - X_r^{00}| < 8 \times 10^{-6}\) at 95% C.L., instead of the bound \(3 \times 10^{-6}\) of Eq. (5.66b).

By using experiments on forbidden \(\beta\) decay, we have set strong limits on Lorentz violation in the weak interaction, in particular on the tensor \(\chi^{\mu\nu}\) that modifies the \(W\)-boson propagator. The general bounds of Eq. (5.66) can be translated into bounds on SME parameters, in terms of which
\[
\chi^{\mu\nu} = -k^{\mu\nu}_{\psi\psi} - i k^{\mu\nu}_{\psi W}/2g,
\] (5.69)
when we assume that \(\chi^{\mu\nu}\) is momentum independent (see Chapter 4); \(g\) is the SU(2) electroweak coupling constant. Since \(k_{\psi\psi}\) has a real symmetric component \(k_{\psi\psi}^S\) and an imaginary antisymmetric component \(k_{\psi\psi}^A\), while \(k_{\psi W}\) is real and antisymmetric, we derive at 95% C.L. the bounds:

\[
-5 \times 10^{-9} < (k_{\psi\psi}^S)^{ZT}, (k_{\psi\psi}^A)^{YX}, (k_{\psi W})^{YX} < 1 \times 10^{-8},
\] (5.69a)

\[
-1 \times 10^{-6} < (k_{\psi\psi}^S)^{ZZ} < 4 \times 10^{-7},
\] (5.69b)

\[
-1 \times 10^{-6} < (k_{\psi\psi}^S)^{TT} < 3 \times 10^{-6},
\] (5.69c)

\[
|M(k_{\psi\psi}^S)^{XX}|, |(k_{\psi\psi}^S)^{YY}| < 1 \times 10^{-6},
\] (5.69d)

\[
|M(k_{\psi\psi}^S)^{XT}|, |(k_{\psi\psi}^S)^{YT}|, |(k_{\psi\psi}^S)^{XT}|,
\] (5.69e)

\[
|M(k_{\psi\psi}^A)^{YZ}|, |(k_{\psi W})^{XZ}|, |(k_{\psi W})^{YZ}| < 2 \times 10^{-8},
\] (5.69f)

\[
|M(k_{\psi\psi}^S)^{XY}|, |(k_{\psi\psi}^S)^{XZ}|, |(k_{\psi\psi}^S)^{YZ}| < 5 \times 10^{-7}.
\] (5.69g)
We assumed that there are no cancellations between different parameters, i.e. when deriving a bound on one parameter, the others were set to zero. With that caveat, Eq. (5.69) provides the first strict direct bounds on these SME parameters in the electroweak sector. For the components $\chi_{\mu\nu}^r + \chi_{\nu\mu}^r$, they improve bounds from pion decay [58] by three orders of magnitude. (Indirect bounds were previously obtained for some of these parameters [59].) The validity of these indirect bounds is addressed in Ref. [58].

In order to improve on our bounds, a more sensitive $\beta$-decay experiment of the type performed in Refs. [53, 54, 55] could be designed. With theory input and by exploiting modern detector systems a number of the drawbacks of these pioneering experiments can be overcome. However, to reach their precision level will require long-running experiments with high-intensity sources.

Comparing Eqs. (5.65) and (5.67) makes it clear that a unique transition is advantageous, since it has only one dominant matrix element. This will drop out if we formulate a suitable asymmetry. We will then not need the value of this $\beta$ moment to put bounds on the Lorentz-violating parameters.

The fact that the bounds on some of the dimensionless parameters in Eq. (5.69) are about an order of magnitude better than the bounds on the relevant parameters given in Ref. [53] is mostly due to the enhancement factor that arises for forbidden transitions with $\Delta I \geq 2$. It is a relevant question to ask how this factor behaves for different isotopes. To gain insight into this, we determine what are typical values of the factors

$$\frac{3 \alpha Z}{10 E R} \frac{p^2}{p^2 + q^2} \quad \text{and} \quad \frac{\alpha Z}{p R} \frac{p^2}{p^2 + q^2}$$

for different isotopes. These factors give the enhancement of the Lorentz-violating terms in Eq. (5.65) with respect to the Lorentz-symmetric part of the decay rate. The first factor belongs to the term involving $\chi_{ij}^r \hat{p}_i \hat{p}_j - \frac{1}{3} \chi_{r0}^{00}$, while the second factor belongs to $\frac{1}{2} \chi_{i}^{ij} \hat{p}_i - \chi_{r}^{ot} \hat{p}_r$. We limit ourselves here to the case of the unique first forbidden transition. By taking the derivative of the factors in Eq. (5.70) with respect to energy, we determine for which $\beta$-particle energy the relative factors in Eq. (5.70) are the largest. We then calculate this maximum value for different isotopes, depending on the $E_0$ that is characteristic for the $\beta$ transition of the isotope. To determine the nuclear radius we use $R = 1.2 A^{1/3} \text{fm}$. The results are plotted in Fig. 5.1, where each point represents an isotope. It is clear that from the point of view of the enhancement, it is advantageous to use nuclei with a small endpoint energy. A small $E_0$ is also preferred for radiation safety reasons. On the other hand, a low $E_0$ might cause the practical problem that the $\beta$ particles are not able to leave the source or that statistics are low because of the small available phase space, which result in a long lifetime.
Figure 5.1: Maximal enhancement factors. Left: the factor in front of the $\chi^{ij}_{r} \hat{p}^i \hat{p}^j - \frac{1}{3} \chi^{00}_{r}$ term. Right: the factor in front of the $\frac{1}{2} \chi^{l}_{i} \hat{p}^l - \chi^{00}_{r} \hat{p}^l$ term.
In this chapter we discuss Lorentz violation in charged-pion decay $\pi \rightarrow \mu + \nu$ and muon decay $\mu \rightarrow e + \nu + \nu$. Pion and muon decay are two simple examples of weak-decay processes that can be calculated both with and without Lorentz violation. Additionally, they have been studied intensively in experiments. Our discussion is in terms of a $\chi^{\mu\nu}$ parameter as described in Chapter 4 and a $c^{\mu\nu}$ SME coefficient, which is the sum of $c_{R}^{\mu\nu}$ and $c_{L}^{\mu\nu}$ coefficients, described in Chapter 2. For muon decay we will also consider a CPT odd $\vartheta^{\mu}$ SME parameter, in the light of results obtained from $g - 2$ data. For a rotational invariant $\chi^{\mu\nu}$ the calculations described in this chapter have already been done in Ref. [60]. While a rotationally invariant version of the pion-decay rate with $c^{\mu\nu}$ can be found in Ref. [61].

This chapter is based on:


Section 6.7: J. P. Noordmans et al., submitted

\section{Charged-pion decay with $c^{\mu\nu}$}

In this section we calculate the Lorentz-violating pion-decay rate with a CPT even $c^{\mu\nu}$ parameter. The Lorentz-violating physics is described by the Lagrangian

$$ L = c_{\mu\nu} \left[ i \bar{\ell} \gamma^\mu \partial^\nu \ell + i \bar{\nu} \gamma^\mu \partial^\nu \nu + \bar{\ell}_L W^\nu \gamma^\mu \nu_L + \bar{\nu}_L W^\nu \gamma^\mu \ell_L \right] . \quad (6.1) $$

Here $\ell$ is the charged-lepton field and $\nu$ is the neutrino field of the corresponding flavor. Notice that the Lorentz-violating coefficients for the left-handed and right-handed fields are equal. This is a choice we make for simplicity and in terms of SME coefficients in Eq. (2.11a) it means that $(c_L)^{\mu\nu} = (c_R)^{\mu\nu}$. Gauge invariance then implies that the Lorentz-violating coefficient for the kinetic lepton term and the one for the kinetic neutrino term are equal. It also implies that these coefficients are equal to the one coupled to the interaction term of the left-handed fields with the $W$ boson. Furthermore, $c^{\mu\nu}$ needs to be real, which follows from the requirement of a Hermitian Lagrangian, and we take $c^{\mu\nu}$ to be symmetric in its spacetime indices, since the anti-symmetric part can be removed by a field redefinition [31] and is therefore not observable. We follow the procedure developed in Ref. [38] and described in Section 2.5 to calculate the pion-decay rate from Eq. (6.1).
As is described in Section 2.5, the time-derivative term in Eq. (6.1) is not conventional and we need to do a spinor redefinition for both the neutrino field and the charged-lepton field. In this case the redefinition is given by [34]
\[ \psi = A \chi = (1 - \frac{1}{2} c_{\mu 0} \gamma^0 \gamma^\mu) \chi, \]
(6.2)
where \( \chi \) is the new and physical field. Writing the Lagrangian in terms of \( \chi \) makes sure the time-derivative term is conventional and the Hamiltonian is Hermitian. Because of the redefinition the interaction term also changes. In terms of the redefined fields, which we will write again as \( \ell_L \) and \( \nu_L \), it becomes
\[ \mathcal{L} = W^{\nu} - \bar{\nu}_L (g_{\mu \nu} + C_{\mu \nu}) \gamma^\mu \ell_L = W^{\nu} - \bar{\nu}_L \tilde{\gamma}^\nu \ell_L, \]
(6.3)
where we defined
\[ \tilde{\gamma}^\mu = A (g^\mu \nu + C^\mu \nu) \gamma^\nu A = (g^\mu \nu + C^\mu \nu) \gamma^\nu, \]
(6.4)
\[ C_{\mu \nu} = c^\mu \nu - c^\mu 0 g^\nu 0 + c^\nu 0 g^\mu 0 - c^0 0 g^\mu \nu. \]
(6.5)
Notice that \( C_{\mu 0} = 0 \), which shows that extra time-derivative terms have been removed by the field redefinition. From Eq. (6.3) we see that the vertex is proportional to \( \tilde{\gamma}^\mu \), while it was proportional to \( \gamma^\mu + c^\mu \gamma^\nu \) before the redefinition. As described in Section 2.5, we can determine the dispersion relation and the solutions for the spinors from the Dirac equation. In the case where \( c_{\mu \nu} \) is the only non zero Lorentz-violating coefficient, the dispersion relation can be written as
\[ (\tilde{p}^2 - \tilde{m}^2)^2 = 0, \]
(6.6)
with
\[ \tilde{p}^\mu = p^\mu + C^\mu \nu p^\nu, \]
(6.7)
\[ \tilde{m} = m (1 - c_{00}). \]
(6.8)
This dispersion relation has two degenerate roots. After the usual reinterpretation of the negative-energy solutions, we see that the energy of both the particle and the antiparticle of either spin state is, to first order in Lorentz violation, given by
\[ E(p) = \bar{E} - \frac{c_{\mu \nu} \tilde{p}^\mu \tilde{p}^\nu}{E}, \]
(6.9)
where we introduced the convenient notation \( \tilde{p} = (\bar{E}, \bar{p}) \) with \( \bar{E} = \sqrt{\bar{p}^2 + m^2} \).

From the Dirac equation we can in principle also determine the exact form of the (redefined) spinors. However, we will only need expressions for \( u^s(p) \bar{u}^s(p) \) and \( v^s(p) \bar{v}^s(p) \). When we normalize the physical spinors according to
\[ u^{s\dagger}(p) u^s(p) = \delta^{ss'} N^2, \]
(6.10a)
\[ v^{s\dagger}(p) v^s(p) = \delta^{ss'} N^2, \]
(6.10b)
\[ u^{s\dagger}(p) v^s(-p) = 0, \]
(6.10c)
\[ v^{s\dagger}(-p) u^s(p) = 0. \]
(6.10d)
with \( N_p = \sqrt{2(E(p) + 2c_0\mu \rho)} \), these expressions are given by

\[
\begin{align*}
  u^\nu(p) \bar{u}^\nu(p) &= \frac{N_p^2(\hat{p} + \hat{m})(1 + \gamma^5 \hat{s})}{4\hat{p}^0}, \\
  v^\nu(p) \bar{v}^\nu(p) &= \frac{N_p^2(\hat{p} - \hat{m})(1 + \gamma^5 \hat{s})}{4\hat{p}^0},
\end{align*}
\]  

(6.11a, 6.11b)

with \( \hat{s} = \left( \frac{\hat{p} \cdot \hat{s}}{m} \cdot \hat{s} + \frac{\langle \hat{p} \cdot \hat{s} \rangle \hat{p}}{m(m + p^0)} \right) \) and \( \hat{s} \) the muon spin in its restframe. That these are the correct expressions can be seen in Appendix B. We can now determine the squared matrix element for pion decay. After summing over neutrino spin and using momentum conservation, it is given by

\[
\sum_{\nu \text{ spin}} |\mathcal{M}|^2 = N_p^2 N_k^2 \tilde{m}^2 G_F^2 f_\pi^2 \frac{p^0}{k^0 p^0} (\hat{p} \pm \hat{m} \hat{s}) \cdot \hat{k},
\]  

(6.12)

where \( p \) and \( k \) are the muon and neutrino momentum, respectively, \( G_F \) is the Fermi coupling constant, \( f_\pi \simeq 92 \text{ MeV} \) is the pion-decay constant, and the upper (lower) sign applies for \( \pi^- (\pi^+) \) decay. The matrix element is proportional to the muon mass. This can be understood by the usual spin-balance argument for pion decay, which shows that, in the pion restframe, the outgoing leptons should have the same helicity, while the weak interaction only couples to the chiral component of the charged-lepton field that is of the opposite handedness. Interestingly, the Lorentz-violating spinors are eigenvectors of the operator \( \Sigma \cdot \hat{p} \) instead of the usual helicity operator \( \Sigma \cdot \hat{p} \). Also, in the pion restframe, \( \sum_{\nu \text{ spin}} |\mathcal{M}|^2 \propto (1 \pm \hat{p} \cdot \hat{s}) \), with \( \hat{p} = \hat{p}/|\hat{p}| \). This shows that the muons are polarized in the \( \pm \hat{p} \)-direction, instead of in the normal \( \pm \hat{p} \)-direction. This influences experiments that depend on pion decay for their polarized muons, such as \( g - 2 \) \[62\] or TWIST \[63\]. The first could detect the discussed effect, for example in the phase of the muon polarization, varying over the course of a sidereal day. Based on the current precision of the experiment a statistical precision of \( 10^{-6} \) seems attainable.

The differential decay rate is given by

\[
d\Gamma = \frac{1}{2m_\pi (2\pi)^3} \frac{d^3p}{4\pi^3} \frac{1}{2m_\pi (2\pi)^3} \frac{d^3k}{4\pi^3} \sum_{\nu \text{ spin}} |\mathcal{M}|^2 (2\pi)^4 \delta^4(q - p - k),
\]  

(6.13)

where \( q \) is the pion momentum. By using the dispersion relations and momentum conservation repeatedly, we find for the differential pion-decay rate in the pion restframe

\[
\frac{d\Gamma}{dQ} = \frac{G_F^2 f_\pi^2}{8\pi^2} \tilde{m}^2 \left( \tilde{M}_+ - \tilde{M}_- \right) \left( 1 + 3c_{ij}^0 + 3c_{ij}^j \hat{p}_i \hat{p}_j \right) \left( 1 \pm \hat{p} \cdot \hat{s} \right),
\]  

(6.14)

where \( \hat{p}^k = p^k/|\hat{p}| = \hat{p}^k(1 + c_{jk} \hat{p}_j \hat{p}_k) + c_{ij} \hat{p}_j \), \( \tilde{M}_+ = (m_\pi^2 + \tilde{m}^2)/2m_\pi \), \( \tilde{M}_- = (m_\pi^2 - \tilde{m}^2)/2m_\pi \), and Latin indices run over space indices only. We see that indeed the \( \pi^- (\pi^+) \) decay rate vanishes if the muon spin is antiparallel (parallel) to \( \hat{p} \). We can write Eq. (6.14) more
explicitly as
\[
\frac{d\Gamma}{d\Omega} = \frac{G_F^2 f_{\pi}^2}{8\pi^2} M^2 (M_+ - M_-) \left[ 1 + c^{00} \frac{2M_+ - M_-}{M_-} + 3c^{ij} \hat{p}_i \hat{p}_j + \pm (\hat{p} \cdot \hat{s}) \left( 1 + c^{00} \frac{2M_+ - M_-}{M_-} + 4c^{ij} \hat{p}_i \hat{p}_j \right) \mp c^{ij} \hat{s}_i \hat{p}_j \right],
\]

(6.15)

where \(M_+\) and \(M_-\) are equal to \(\tilde{M}_+\) and \(\tilde{M}_-\), with the replacement \(\tilde{m} \rightarrow m\). There are no terms proportional to \(c^{0j} \hat{p}_j\) or \(c^{ij} \hat{p}_j\) in Eq. (6.15). This implies that there will be no difference in rate for muons going in opposite directions, when the polarization of the muons is not detected. Notice also that the decay rate, integrated over muon direction, does not depend on the muon spin, i.e. \(\Gamma(\uparrow) - \Gamma(\downarrow) = 0\). We expect this to be no longer the case when the coefficients for left-handed and right-handed fields are taken to be different. The energies of the two spin states are then no longer degenerate, and the form of the operators \(u^s(p)\bar{u}^s(p)\) and \(v^s(p)\bar{v}^s(p)\) is considerably more involved (see Appendix B).

To obtain the total decay rate from Eq. (6.15), we integrate over all angles while using Eq. (6.41). The result is
\[
\frac{\Gamma}{\Gamma_0} = 1 + 2c^{00} \frac{M_+ + M_-}{M_-} \approx (1 + 7.4c^{00}),
\]

(6.16)

where \(\Gamma_0\) is the total pion-decay rate without Lorentz violation.

A derivation of the total pion to muon and neutrino decay rate similar to ours was performed in Ref. [61], starting from an isotropic and traceless \(c_{\mu\nu}\) with diagonal elements \([c^{00}, \frac{1}{3}c^{00}, \frac{1}{3}c^{00}, \frac{1}{3}c^{00}]\). The result in that paper differs from the one we get in Eq. (6.16). It is given by
\[
\frac{\Gamma}{\Gamma_0} = 1 + 2c^{00} \frac{M_+ + M_-}{M_-} \approx (1 + 9.4c^{00}).
\]

(6.17)

This can be traced back to an extra factor \((1 + c^{00})^2\) in the derivation of Ref. [61], coming from the vertex in the Feynman diagram. The cause of this extra factor is the fact that in Ref. [61] there is no field redefinition (see Eq. (6.2)) performed to make the time-derivative term conventional. In the case of Ref. [61] this is not necessary to make the Hamiltonian Hermitian, since the choice of the isotropic \(c_{\mu\nu}\) causes it to be Hermitian already. However the time derivative and the vertex (in the restframe of the pion) do have an extra factor of \(1 + c^{00}\). This is exactly the origin of the extra \((1 + c^{00})^2\) in Eq. (6.17). One of the reasons that we are convinced that our result is the correct one, is the fact that the conserved canonical energy momentum tensor also has an extra factor or \(1 + c^{00}\) in the case of Ref. [61], which causes the eigenvalue of the Hamiltonian on a one-particle state to be \((1 + c^{00})E\) instead of \(E\). This means that the one-particle states (or the fields) are not correctly normalized. Normalizing them correctly should make the extra factor of \((1 + c^{00})^2\) in Eq. (6.17) go away and reproduce our result in Eq. (6.16).
6.2 Charged-pion decay with $\chi^{\mu\nu}$

In this chapter we calculate the effects of $\chi^{\mu\nu}$ on the decay rate of two different particles: the pion and the muon. In this section we will deal with the charged pion and we will consider muon decay in Section 6.7. These are two simple examples of weak decay processes that can be calculated fairly easily both with and without Lorentz violation. For a rotational invariant $\chi^{\mu\nu}$ these calculations have already been done in Ref. [60].

In the previous section we saw that, even with a Lorentz-violating $c^{\mu\nu}$ coefficient, the matrix element for pion decay is proportional to the mass of the charged lepton. This causes the pion to decay predominantly to a muon and a muon neutrino and only very rarely to an electron and an electron neutrino (branching ratio of $10^{-4}$). For the Standard-Model this can be seen at the level of the matrix element by writing (see e.g. Ref. [64] for why this is the correct expression)

$$i\mathcal{M} \propto \bar{u}(p)\gamma(1-\gamma^5)v(k) = m_l\bar{u}(p)(1-\gamma^5)v(k)$$

(6.18)

where $q$, $p$ and $k$ are the pion, muon or electron and neutrino momentum respectively and $m_l$ is the muon or electron mass. We used momentum conservation ($q = p + k$) and the Dirac equation to derive the result that the matrix element is proportional to the charged lepton mass. Therefore the decay rate will be proportional to the square of this mass and thus

$$\frac{\Gamma(\pi \rightarrow e + \nu)}{\Gamma(\pi \rightarrow \mu + \nu)} \propto \frac{m_e^2}{m_\mu^2}.$$ 

(6.19)

For Lorentz violation parametrized by $\chi^{\mu\nu}$ we derive that

$$i\mathcal{M} \propto (g^{\mu\nu} + \chi^{\mu\nu})q^\mu\bar{u}(p)\gamma^\nu(1-\gamma^5)v(k)$$

(6.20)

and so to first order in Lorentz violation

$$\sum_{\text{spins}} |\mathcal{M}| \propto (g^{\mu\nu}g_{\rho\sigma} + \chi^{\mu\nu}g_{\rho\sigma} + g^{\mu\nu}\chi^*_{\rho\sigma})q^\mu q^\rho \text{Tr} \left[p^\nu k^\sigma(1-\gamma^5)\right].$$

(6.21)

Again using momentum conservation we find that in the restframe of the pion $p^0 = \frac{m_\pi^2+m_e^2}{2m_\pi}$ and $p = -k = \frac{m_\pi^2-m_e^2}{2m_\pi} \hat{p}$ and therefore

$$\text{Tr} \left[p^\nu k^\sigma(1-\gamma^5)\right] = \frac{m_\pi^2+m_e^2}{2m_\pi} \text{Tr} \left[\gamma^0\gamma^\nu k^\sigma(1-\gamma^5)\right] - \frac{m_\pi^2-m_e^2}{2m_\pi} \text{Tr} \left[(\hat{p} \cdot \gamma)\gamma^\nu k^\sigma(1-\gamma^5)\right].$$

(6.22)

From Eq. (6.21) we see that in the restframe of the pion, to first order in Lorentz violation, the matrix element squared is only nonvanishing if either $\nu = 0$ or $\sigma = 0$. Using this together with the masslessness of the neutrino and the fact that $p = -k$, we find that

$$\text{Tr} \left[\gamma^0\gamma^\nu k^\sigma(1-\gamma^5)\right] = \text{Tr} \left[(\hat{p} \cdot \gamma)\gamma^\nu k^\sigma(1-\gamma^5)\right]$$

$$= \begin{cases} 
\text{Tr}[k^\sigma(1-\gamma^5)] & \text{if } \nu = 0 \\
\text{Tr}[k^\nu(1-\gamma^5)] & \text{if } \sigma = 0 
\end{cases}.$$ 

(6.23)
Combining this with Eq. (6.22), we see that also in the Lorentz-violating case the decay rate will be proportional to $m_l^2$. To derive this, however, we summed over the spins of the leptons and we only looked at contributions of at most linear in Lorentz violation. Indeed, if we were to look at the contributions quadratic in Lorentz violation we would find effects that are not proportional to the lepton mass squared. We do not consider these contributions however, since we expect Lorentz violation to be very small. The spin of the charged lepton can be a relevant observable, however.

If we calculate the decay rate of the pion without summing over the spin of the charged lepton, we get that

$$
\frac{d\Gamma}{d\Omega} = \frac{G_F^2 f_\pi^2}{8\pi^2} M_2^2 (M_+ - M_-) \left[ (1 \pm \hat{p} \cdot \hat{s}) (1 + 2 \chi_0^{0j} - 2 \chi_0^{ij} \hat{p}_j) \right]
\pm \frac{m_\pi}{m} \left[ 2 \chi_0^{0j} (\hat{s}_j - (\hat{p} \cdot \hat{s}) \hat{p}_j) + 2 \chi_0^{0j} (\hat{p} \times \hat{s})_j \right],
$$

(6.24)

where the upper sign is for $\pi^-$ decay and the lower sign is for $\pi^+$ decay and $M_-$ and $M_+$ are defined as in the previous section.

Comparing Eq. (6.24) with the result for the differential pion-decay rate in terms of $c_{\mu\nu}$ in the previous section, given in Eq. (6.15), we notice that for $c_{\mu\nu}$ there are no terms proportional to $c_0 \hat{p}_j$, while for $\chi_{\mu\nu}$ there are no terms proportional to $\chi_{ij} \hat{p}_i \hat{p}_j$. In the former case there will be a nonzero dipole asymmetry in the muon direction, while one has to search for a higher-order multipole asymmetry in the latter case. Another difference between $c_{\mu\nu}$ and $\chi_{\mu\nu}$ is the enhancement factor $m_\pi/m$ for the spin-dependent terms in Eq. (6.24), which is not present in Eq. (6.15). For the dominant branching fraction $\pi \to \mu + \nu_\mu$ this is of order unity. However, if one would measure the electron spin in $\pi \to e + \nu_e$ decay, one could benefit from the sizable enhancement of $m_\pi/m_e \approx 274$. We point out that $\chi_{\mu\nu}$, in contrast with $c_{\mu\nu}$, produces a nonzero asymmetry in the spin of the muon:

$$
\frac{\Gamma(\uparrow) - \Gamma(\downarrow)}{\Gamma(\uparrow) + \Gamma(\downarrow)} = \frac{2}{3} \left( \frac{2m_\pi + m_\mu}{m} \right) \chi_r^{0z},
$$

(6.25)

where we chose the quantization axis in the $z$-direction. Finally, we notice that the decay rate in Eq. (6.15) has its maximum if $\hat{s} = \pm \hat{p}$. To first order in Lorentz violation Eq. (6.24) is proportional to $1 \pm \mathbf{V}_\ell \cdot \hat{s}$, with $\mathbf{V}_\ell$ given by $V_\ell = \hat{p}^l + 2m_\pi \left[ \chi_0^0 + \hat{p}^l (\chi^{0j}_r \hat{p}_j) - \epsilon^{ljk} \hat{p}_j (\chi^0_k) \right] / m$. Both $c_{\mu\nu}$ and $\chi_{\mu\nu}$ thus influence the polarization of the outgoing muons.

### 6.3 Coordinate choices

It is known [31, 19], that some (combinations of) SME coefficients are physically unobservable. At the level of the Lagrangian, this can be shown by using field or coordinate redefinitions to bring the Lagrangian with the apparent Lorentz violation to a conventional Lorentz-symmetric form. Since the physics does not depend on a choice of coordinates or fields, the coefficients that can be removed are unobservable in experiments. In many cases interactions between different sectors of the SME prevent the full removal of the Lorentz-violating coefficients.
As an example of this, we look at a $c_{\mu\nu}$ parameter for a fermion field of a particular species, such as in Eq. (6.1). According to Refs. [31, 19], a Lagrangian with a nonzero $c_{\mu\nu}$ parameter is equivalent to a conventional Lagrangian in a skewed coordinate system. The $c_{\mu\nu}$ can be removed by a coordinate transformation $x^\mu \rightarrow x'^\mu = x^\mu + c_{\mu\nu}x^\nu$. However, this transformation introduces $-c_{\mu\nu}$ in the other fermion sectors, while for the gauge field sector $W^{\mu\nu}W_{\mu\nu} \rightarrow W^{\mu\nu}W^{\rho\sigma}(\eta_{\mu\rho}\eta_{\nu\sigma} + 2\eta_{\mu\rho}c_{\nu\sigma} + 2\eta_{\nu\sigma}c_{\mu\rho})$. The latter has the same form as a partly nonzero $k^{\mu\nu\rho\sigma}$ parameter in the gauge field sector.

If we only consider $c_{\mu\nu}$ coefficients for fermions and the relevant parts of the $k^{\mu\nu\rho\sigma}$ coefficients for gauge fields, we can always make one sector of the SME conventional by means of a coordinate transformation. Notice that this is not a general coordinate transformation, in the usual sense. This is because we do not transform the metric, but reinterpret the coordinates with respect to the metric. This means that we make a choice which sector of the Lagrangian defines the clocks and measuring rods, and is therefore the conventional sector. The choice as to which sector is conventional, depends on the experimental setup.

### 6.4 Quark $c_{\mu\nu}$ parameters in charged-pion decay

We now turn to the quark sector of the SME. Although there are strict bounds for effective parameters from meson oscillations and measurements on the neutron and the proton [19], the best bounds on actual quark parameters are in the top quark sector and they are at the $10^{-1}-10^{-2}$ level [65]. Bounds on parameters for the other generations are lacking. Using coordinate transformations, we calculate the effects of quark parameters in leptonic pion decay.

The SM first-generation quark Lagrangian is given by

$$L_{\text{quark}} = \bar{u}(i\partial - m_u)u + \bar{d}(i\partial - m_d)d + \frac{g}{\sqrt{2}}V_{ud}\left[\bar{u}_L W^+_{\gamma\mu}d_L + \bar{d}_L W^-_{\gamma\mu}u_L\right],$$  \hspace{1cm} (6.26)

where $g$ is the $SU(2)$ coupling constant and $V_{ud}$ is the relevant entry of the CKM matrix. The corresponding Lorentz-violating part of the SME Lagrangian is

$$L_{\text{quark}}^{LV} = ic_{\mu\nu}\bar{u}\gamma^\mu\partial^\nu u + ic_{\mu\nu}\bar{d}\gamma^\mu\partial^\nu d + \frac{g}{\sqrt{2}}V_{ud}\left[c_{\mu\nu}\bar{u}_L\gamma^\mu W^+_{\nu\mu}d_L + c_{\mu\nu}\bar{d}_L\gamma^\mu W^-_{\nu\mu}u_L\right].$$  \hspace{1cm} (6.27)

This follows from Eq. (2.11b) with the assumption that Lorentz violation is equal for left-handed and right-handed quarks and that $c_{\mu\nu}$ is diagonal in flavor space. Gauge invariance then forces the parameters to be equal for up and down quarks.

As mentioned in the previous section, a coordinate transformation $x^\mu \rightarrow x'^\mu = x^\mu + c_{\mu\nu}x^\nu$ brings the quark Lagrangian to its conventional, Lorentz-symmetric, form. The coordinate transformation results in a low-energy $W$-boson propagator of the form described in Chapter 4 with $\chi^{\mu\nu} = 2c_{\mu\nu}$ and a $-c_{\mu\nu}$ coefficient for the second-generation leptons. The effect of the coordinate transformations is visualized in the diagrams in Fig. 6.1. Notice that the transformation also changes the other sectors of the SME. It depends on the experimental conditions if this is relevant. In practice observables always depend on differences between Lorentz-violating parameters of the involved particles. Pion decay
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Figure 6.1: The effect of the coordinate transformations on pion decay. Blobs represent Lorentz violation.

thus actually depends on differences between quark, lepton, and $W$-boson parameters. We will focus on the quark parameters the remainder of this section.

The calculation of the decay rate in terms of quark parameters can now be split in two parts, one dealing with Lorentz violation in the lepton kinetic terms and interaction vertex and one dealing with the Lorentz violation in the $W$-boson propagator. The former part of the calculation exactly parallels the calculation in Section 6.1, with the substitution $c_{\mu\nu} \rightarrow -c_{\mu\nu}$. The latter part, with a modified $W$-boson propagator, is treated by putting $\chi_{\mu\nu} = 2c_{\mu\nu}$ in the result of Section 6.2. Since we treat Lorentz violation to first order, we can simply combine the results in Eqs. (6.15) and (6.24), resulting in

$$
\frac{d\Gamma}{d\Omega} = \frac{G_F f_\pi^2}{8\pi^2} M_\pi^2 \left( M_+ - M_- \right) \left[ 1 + c^{00}5M_- - 2M_+ - 3c^{ij}\hat{p}_i\hat{p}_j - 4c^{0j}\hat{p}_j \right] \\
\pm (\hat{p} \cdot \hat{s}) \left[ 1 + c^{00}5M_- - 2M_+ - 4c^{ij}\hat{p}_i\hat{p}_j + 4\left( \frac{m_\pi}{m} - 1 \right) c^{0j}\hat{p}_j \right] \\
\mp 4m_\pi \frac{e^{0j}\hat{s}_j \pm e^{ij}\hat{s}_i\hat{p}_j}{m}.
$$

(6.28)

To first order in Lorentz-violating parameters, this decay rate is proportional to $1 \pm V_q \cdot \hat{s}$, with $V_q = \hat{p} \left( 1 - c^{ij}\hat{p}_i\hat{p}_k \right) - c^{ik}\hat{p}_k + 4m_\pi \left[ c^{kl} + \hat{p}^l(c^{lj}\hat{p}_j) \right] / m$, which summarizes the way the quark parameters will influence the polarization of the outgoing muons. The expression in Eq. (6.28) offers many opportunities for future experiments to constrain the $c^{\mu\nu}$ quark coefficient, by observing the muon direction or spin in pion decay.

Integrating Eq. (6.28) over muon directions and summing over spin gives the total decay rate

$$
\Gamma / \Gamma_0 = 1 + (4M_- - 2M_+)c^{00}/M_- \simeq (1 - 3.4c^{00}) .
$$

(6.29)

Since this expression holds in the restframe of the pion, the sensitivity to Lorentz-violating effects in the decay rate is enhanced by a $\gamma^2_\pi$ dependence for pions in flight ($\gamma_\pi$ is the relativistic boost factor for the pions). Due to the boost also other components of $c^{\mu\nu}$ will enter the expression, since

$$
c^{00} = \gamma^2_\pi \left[ c_{TT} + c_{(T)J}v_J + c_{JK}v_Jv_K \right] ,
$$

(6.30)

where coefficients with capital indices are defined in the Sun-centered frame, $v$ is the pion velocity in the Sun-centered frame, and $c_{(T)J} = c_{JT} + c_{JT}$. Our result can be compared with
6.5 Muon decay with $c^{\mu\nu}$

Starting from the Lagrangian for the second-generation leptons, with only a nonzero $c_{\mu\nu}$, we will calculate the total decay rate of the muon. In the calculation we will deal with the negative muon for definiteness, but the final result is identical to the result for the positive muon.

The relevant Lorentz-violating part of the Lagrangian is given in Eq (6.1). The treatment of the modified Dirac equation and its unconventional time derivative term together with the results for the dispersion relation and the Dirac spinors are as described in Section 6.1 upto Eqs. (6.11). Since we will calculate the total decay rate in this section, we will need expressions for $\sum_{s=1}^{2} u^s(p)\bar{u}^s(p)$ and $\sum_{s=1}^{2} v^s(p)\bar{v}^s(p)$. Therefore, we sum the
expressions in Eq. (6.11) over spin states and obtain
\[ \sum_{s=1}^{2} u^s(p)\bar{u}^s(p) = \frac{N_p^2(\bar{p} + \bar{m})}{2\bar{p}^0}, \]  
(6.33a)
\[ \sum_{s=1}^{2} v^s(p)\bar{v}^s(p) = \frac{N_p^2(\bar{p} - \bar{m})}{2\bar{p}^0}. \]  
(6.33b)

It is by virtue of the fact that $c^{\mu\nu}$ leaves intact the degeneracy of the roots of the dispersion relation that we can use the expressions in Eqs. (6.33). Parameters other than $c^{\mu\nu}$ or $a^\mu$ lift this degeneracy, which necessitates performing phase-space integrals before being able to sum over spin states. Since this is not the case for $c^{\mu\nu}$, we can write the muon-decay rate as
\[ d\Gamma = \frac{G_F^2}{4(2\pi)^5 N_1^2 2\bar{p}^0} \int \frac{d^3k_2}{2k_2^0} \int \frac{d^3k_1}{2k_1^0} \frac{1}{2} \sum_{\text{spins}} |\mathcal{M}|^2 \delta^4(l - k_1 - k_2 - p), \]  
(6.34)
where $l, p, k_1,$ and $k_2$ are the muon, electron, muon neutrino, and electron antineutrino momenta respectively. The matrix element in Eq. (6.34) is given by
\[ i\mathcal{M} = \left[ \bar{u}(k_1)\gamma^\mu(1 - \gamma^5)u(l) \right] \left[ \bar{u}(p)\gamma_\mu(1 - \gamma^5)v(k_2) \right], \]  
(6.35)
Using the relations for $u\bar{u}$ and $v\bar{v}$, given in Eqs. (6.33), we can determine that
\[ \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{N_p^2 N_1^2}{k_1^0 l_1^0} \text{Tr} \left[ \tilde{k}_1 \gamma^\mu \tilde{l}_1 \gamma^\nu(1 - \gamma^5) \right] \text{Tr} \left[ p\gamma_\mu \tilde{k}_2 \gamma_\nu(1 - \gamma^5) \right], \]  
(6.36)
Using Eq. (6.4) and the fact that $A\bar{A} = (1 - c_{00})$ to first order in Lorentz violation, we can write the first trace in terms of Lorentz-covariant quantities as
\[ \frac{N_p^2 N_1^2}{k_1^0 l_1^0} \text{Tr} \left[ \tilde{k}_1 \gamma^\mu \tilde{l}_1 \gamma^\nu(1 - \gamma^5) \right] = 4\text{Tr} \left[ \tilde{k}_1 \gamma^\mu \tilde{l}_1 \gamma^\nu(1 - \gamma^5) \right], \]  
(6.37)
where $\tilde{\gamma}^\mu = \gamma^\mu + c^{\mu\nu}\gamma_\nu$, $\tilde{l}_\mu = l_\mu + c^{\mu\nu}l_\nu$, and analogously for $\tilde{k}_1$. This amounts to undoing the spinor redefinition $\psi = A\chi$ and writing the trace in terms of the (original) observer Lorentz-covariant spinors. The reason we do this, is that then the integral over the neutrino momenta, given by
\[ I_1^{\alpha\beta} = \int \frac{d^3k_1}{N_1^2} \int \frac{d^3k_2}{2k_2^0} \frac{k_1^\beta k_1^\alpha}{2k_1^0} \delta^4(l - k_1 - k_2 - p), \]  
(6.38)
can be seen to be an observer Lorentz-covariant two-tensor. Unfortunately, the solution of this integral is not as easily derived as its equivalent with a $b^\mu$ parameter, given in Eq. (C.3). We will thus leave it for future work to find a solution to the integral in Eq. (6.38). Such a solution would enable us to find the muon-decay rate with $c^{\mu\nu}$, depending on electron direction and energy. For now, however, we perform the easier integral over
the momenta of the two particles without Lorentz violation, \(i.e.\) the electron/positron and the electron (anti)neutrino. For that we need the integral

\[
I_{2}^{\alpha\beta} = \int \frac{d^{3}p}{2p^{0}} \int \frac{d^{3}k_{2}}{2k_{2}^{0}} p^{\alpha} k_{2}^{\beta} \delta^{4}(l - k_{1} - k_{2} - p) = \frac{\pi}{24} \left[ q^{2} g^{\alpha\beta} + 2q^{\alpha}q^{\beta} \right] \Theta(q^{2}) \Theta(q^{0}) ,
\]

where we neglected the electron/positron mass and defined \(q = l - k_{1}\). Using this expression for \(I_{2}^{\alpha\beta}\) and standard trace technology, we determine that the decay rate is given by

\[
\Gamma = \frac{G_{F}^{2}}{12\pi^{3}} \frac{1}{N_{\nu}^{2}} \int \frac{d^{3}k_{1}}{N_{\nu_{1}}} \left[ 3l^{2}(k_{1} \cdot l) - 4(k_{1} \cdot l)^{2} + 2c_{\mu\nu} (l^{\mu}l^{\nu} + (l^{2} - 4(k_{1} \cdot l)) l^{\mu}k_{1}^{\nu} + ((k_{1} \cdot l) - 2l^{2}) k_{1}^{\mu}k_{1}^{\nu}) \right] \times \Theta(q^{2}) \Theta(q^{0}) .
\]

We also perform the integral over \(k_{1}\), since the neutrinos are not likely to be detected. We do this in the frame where the linear momentum of the muon vanishes (\(i.e.\) \(l = 0\)). In this frame the Heaviside stepfunction \(\Theta(q^{2})\) gives the requirement that \(|k_{1}| \leq \frac{m_{\mu}}{2} \left( 1 - \frac{1}{2}c_{00} + \frac{1}{2}(c_{0t} + c_{t0})\hat{k}_{1}^{i} + \frac{1}{2}c_{ij}\hat{k}_{1}^{i}\hat{k}_{1}^{j} \right)\), which becomes the upper limit of the integral over \(|k_{1}|\). The other stepfunction, \(\Theta(q^{0})\) is then automatically satisfied. To obtain the total decay rate we also use that

\[
\int d\Omega \ c_{i0}\hat{k}_{1}^{i} = 0 , \hspace{1cm} \int d\Omega \ c_{ij}\hat{k}_{1}^{i}\hat{k}_{1}^{j} = \frac{4\pi}{3} c_{ii} = \frac{4\pi}{3} c_{00} ,
\]

where the last equality follows from the requirement that \(g^{\mu\nu}c_{\mu\nu} = 0\). After performing these integrals, the final result becomes

\[
\Gamma = \frac{G_{F}^{2} m_{\mu}^{5}}{192\pi^{3}} \left( 1 - \frac{21}{5} c_{00} \right) .
\]

This result for the total muon-decay rate in the muon frame with \(l = 0\). Notice that this is not exactly the restframe of the muon, since the group velocity is \(v_{g}^{\mu} = 2c^{\mu}\) when the linear momentum of the muon is zero. However, the difference in decay rate between these two frames is quadratic in the Lorentz-violating coefficients. We will use this expression in interpreting the muon-lifetime data in Section 6.8.

### 6.6 Muon decay with \(b^{\mu}\)

In this section we calculate the muon-decay rate with a CPT-violating parameter. We start from the free Lagrangian for the charged-lepton field and the left-handed neutrino field, given by

\[
\mathcal{L} = \bar{\ell} \left( i\partial - m - \gamma^{5}\delta \right) \ell + \bar{\nu}_{L} \left( i\partial - \gamma^{5}\delta \right) \nu_{L} ,
\]

where \(a^{\mu}\) is related to the SME parameters introduced in Section 2.3 through \(a^{\mu} = (a_{L}^{\mu} + a_{R}^{\mu})/2\), while \(b^{\mu} = (a_{L}^{\mu} - a_{R}^{\mu})/2\). It is known that the \(a^{\mu}\) parameter is not observable...
[12, 31], since it can be removed from the Lagrangian by a field redefinition. We thus redefine both the neutrino and the muon field by
\[ \psi \rightarrow e^{-ia \cdot x} \psi, \]
which removes \( a^\mu \) from the theory and gives us the new Lagrangian
\[ \mathcal{L} = \bar{\ell}(i \partial - m - \gamma^5 \hat{b}) \ell + \bar{\nu}_L(i \partial - \hat{b}) \nu_L. \] (6.44)
The interaction Lagrangian, given by
\[ \mathcal{L}_{\text{int}} = \bar{\ell} W^- \nu_L + \bar{\nu}_L W^+ \ell_L, \] (6.45)
is not affected by the field redefinition. If there were no interaction terms with the \( W \) boson, we could redefine the neutrino field such that the Lorentz-violating coefficient would disappear from the neutrino part of the Lagrangian altogether. It would be physically unobservable. It can therefore not be constrained by neutrino oscillations or time of flight measurements. The coefficient is actually directly related to the coefficients discussed in Ref. [70], for which similar remarks were made.

From the Dirac equation we can determine the dispersion relation for the neutrino:
\[ \tilde{k}^2 = 0. \] (6.46)
We defined \( \tilde{k} = k^\mu \mp b^\mu \), with \( k^\mu \) the four-momentum of the neutrino. The upper (lower) sign refers to the neutrino (antineutrino), \textit{i.e.} to muon (antimuon) decay. Solving the dispersion relation for \( k^0 \) and redefining the negative solution as usual, the energies of the neutrino and antineutrino are found to be
\[ E_u = |k - b| + b^0, \] (6.47)
\[ E_v = |k + b| - b^0, \] (6.48)
with \( E_u \) referring to the neutrino energy and \( E_v \) referring to the antineutrino energy. From the Dirac equation also the spinors can be determined and we find that
\[ \sum_s u_s^\dagger(k) u_s(k) = \frac{N_{\pm k}^2(k - b)}{2(k^0 - b^0)} = \tilde{k} - \hat{b}, \] (6.49)
\[ \sum_s v_s^\dagger(k) v_s(k) = \frac{N_{\pm k}^2(k + \hat{b})}{2(k^0 + b^0)} = \tilde{k} + \hat{b}, \] (6.50)
where we normalized the spinors such that \( u^\dagger(k) u(k) = N_k^2 \) and \( v^\dagger(k) v(k) = N_{\pm k}^2 \), while \( N_{\pm k}^2 = 2|k \mp b| \). We choose this normalization such that the integrals over phase space will be observer Lorentz invariant in concordant frames (see Appendix C).

For the muon the dispersion relation and the spinor solutions are somewhat more involved (see Section 2.5 and Appendix B). The dispersion relation for example is given by
\[ (\lambda^2 - b^2 - m^2)^2 + 4b^2 \lambda^2 - 4(b \cdot \lambda)^2 = 0, \] (6.51)
which is a quartic equation in \( \lambda^0 \), giving complicated expressions for \( \lambda^0 \) in general. However, we will calculate the muon-decay rate in the muon restframe and will therefore...
restrict ourselves to the energies and spinors in that frame. The energies of muon and antimuon are given to first order in Lorentz violation by

\[ E_u^\alpha = \sqrt{m^2 + (b\theta)^2} + (-1)^\alpha |b| \approx m + (-1)^\alpha |b| , \]
\[ E_v^\alpha = \sqrt{m^2 + (b\theta)^2} + (-1)^\alpha |b| \approx m + (-1)^\alpha |b| , \]  

where \( E_u \) (\( E_v \)) denotes the muon (antimuon) energy. As expected, the degeneracy of the energy eigenvalues of the spin states is lifted.

The exact expressions for the operators \( u^\alpha(p)\bar{u}^\alpha(p) \) and \( v^\alpha(p)\bar{v}^\alpha(p) \) can be calculated using the results in Appendix B. In general such an operator can be written as

\[ u^\alpha(p)\bar{u}^\alpha(p) = \hat{S}_u^\alpha + i\gamma^5 \hat{P}_u^\alpha + (\hat{V}_u^\alpha)^\mu\gamma_\mu + (\hat{A}_u^\alpha)^\mu\gamma_\mu + (\hat{T}_u^\alpha)^\mu\nu\sigma_{\mu\nu} \]

and similarly for \( v^\alpha(p)\bar{v}^\alpha(p) \). In Appendix B the functions \( \hat{S}, \hat{P}, \hat{V}^\mu, \hat{A}, \) and \( \hat{T}^{\mu\nu} \), are determined. For our purposes we will only need the combination \( \hat{V} - \hat{A} \), as we will see later on. This quantity is, in the (anti)muon restframe, given by

\[ (\hat{V}_u^\alpha)^\mu - (\hat{A}_u^\alpha)^\mu = \frac{(E_u^\alpha)^2 - m^2 + b^2 - 2E_u^\alpha(b^\theta) - (E_u^\alpha)^2 + m^2 + b^2 - 2E_u^\alpha(b^\theta)\bar{b}^\mu}{4((E_u^\alpha)^2 - m^2 + b^2)E_u^\alpha - 2E_u^\alpha(b^\theta)^2} , \]
\[ (\hat{V}_v^\alpha)^\mu - (\hat{A}_v^\alpha)^\mu = \frac{(E_v^\alpha)^2 - m^2 + b^2 - 2E_v^\alpha(b^\theta)E_v^\alpha\bar{g}^\mu + ((E_v^\alpha)^2 + m^2 + b^2 + 2E_v^\alpha(b^\theta)\bar{b}^\mu}{4(((E_v^\alpha)^2 - m^2 + b^2)E_v^\alpha - 2E_v^\alpha(b^\theta)^2} , \]

where the expression with the subscript \( u \) relates to the muon case, while the expression for the antimuon case has a subscript \( v \). We normalized the spinors to unity.

The differential muon-decay rate is now given by

\[ d\Gamma_\alpha = \frac{G_F^2}{2(2\pi)^5} \frac{d^3p}{2\rho^5} \int \frac{d^3k_2}{2k_2^0} \int \frac{d^3k_1}{2k_1^0} \frac{1}{N_{k_1}^2} \sum_\nu \left| \mathcal{M} \right|^2 \delta^{(4)}(l - k_1 - k_2 - p) , \]

where \( l, k_1, k_2, \) and \( p \) are the momenta of the muon, muon neutrino, electron antineutrino, and the electron respectively (or of their respective antiparticles). For the negative muon the matrix element squared can be written as

\[ \sum_\nu \left| \mathcal{M} \right|^2 = \sum_\nu \left\{ \text{Tr} \left[ u(k_1)\bar{u}(k_1)\gamma^\mu(1 - \gamma^5)u(l)\bar{u}(l)\gamma^\nu(1 - \gamma^5) \right] \right. \\
\left. \times \text{Tr} \left[ u(p)\bar{u}(p)\gamma_\mu(1 - \gamma^5)v(k_2)\bar{v}(k_2)\gamma_\nu(1 - \gamma^5) \right] \right\} . \]

The first trace contains the spinors of the particles with Lorentz violation. Including the sum over neutrino spins, it can be written as

\[ 2\text{Tr} \left[ \hat{b}_1\gamma^\mu(\hat{S} + i\gamma^5\hat{P} + \hat{V} + \gamma^5\hat{A} + \hat{T}^{\mu\nu}\sigma_{\mu\nu})\gamma^\nu(1 - \gamma^5) \right] = 2\text{Tr} \left[ \hat{b}_1\gamma^\mu(\hat{V} - \hat{A})\gamma^\nu(1 - \gamma^5) \right] . \]

Where we put in the expansion of the operator \( u^\alpha\bar{u}^\alpha \) (or \( v^\alpha\bar{v}^\alpha \)) given in Eq. (6.53). We omitted the subscripts \( u, v \) and the superscript \( \alpha \). The equality shows that we indeed only
need the combination $\hat{V} - \hat{A}$, as given in Eqs. (6.54). We can now write for the differential decay rate

$$d\Gamma_\alpha = \frac{G_F^2}{12\pi^4} \frac{d^3p}{2p^0} \left( q^2 (L \cdot p) + 2(q \cdot L)(\bar{q} \cdot p) \right) \Theta(q^2) \Theta(q^0), \quad (6.58)$$

with $L^\mu = (\hat{V}_u^\alpha)^\mu - (\hat{A}_u^\alpha)^\mu$ for negative muon decay and $L^\mu = (\hat{V}_u^\alpha)^\mu - (\hat{A}_u^\alpha)^\mu$ for positive muon decay. Furthermore, $\bar{q}^\mu = q^\mu \mp b^\mu$ and we used the expression derived in Appendix C, given in Eq. (C.6). Notice that $q = p \mp b$ also depends on $\alpha$ through the $\alpha$ dependence of $t^0 = E_{u,v}^\alpha$. Neglecting the electron mass and writing $x = p^0/p^0_{\text{max}}$ with $p^0_{\text{max}} = (t^0 \mp (b^0 + b \cdot \hat{p}))$, we can bring the decay rate to the form

$$\frac{d\Gamma_\alpha}{dx d\Omega} = \frac{\Gamma_0}{2\pi} x^2 \left\{ \left[ \frac{3}{2} (1 - x) + \frac{2}{3} \bar{g}(4x - 3) \right] + \left( \frac{12b^0}{m} - \frac{3(b^0)^2 b \cdot \hat{p}}{4m|b|^2} + \frac{10b \cdot \hat{p}}{m} \right) (1 - x) \right. $$

$$\left. + \left( \frac{2b^0}{m} - \frac{3(b^0)^2 b \cdot \hat{p}}{8m|b|^2} + \frac{3b \cdot \hat{p}}{m} \right) (4x - 3) \right\} \Theta(\bar{q} \cdot b), \quad (6.59)$$

where we put in the Michel parameter $\bar{g}$ by hand [71]. This parameter $\bar{g}$ parametrizes the shape of the $\beta$-particle spectrum and in the Standard Model and the SME its value is given by $\bar{g} = 3/4$. It is used to restrict the Lorentz structure of the weak interaction.

Except for a factor $1/2$, the first line of Eq. (6.59) is the conventional Standard Model unpolarized muon-decay rate. As usual all contributions are split in a term proportional to $1 - x$ and one proportional to $4x - 3$. The latter gives the decay rate at the endpoint of the $\beta$-particle spectrum and vanishes when integrating the whole expression over $x$.

The terms on the fourth line of Eq. (6.59) are somewhat unusual. They depend on Lorentz-violating parameters, but are of zeroth order in Lorentz violation, and their size is thus expected to be comparable to the other terms. They disappear, however, when we sum the expression over $\alpha$. These terms originate from the fact that $b^\mu$ lifts the degeneracy of the energy eigenvalues of the spin states. When comparing the terms on the fourth line of Eq. (6.59) with the usual Lorentz-symmetric parity-violating terms in the muon-decay rate (see for example the first line of Eq. (6.64)), we see that these are identical when $\hat{s} = (-1)^\alpha \hat{b}$. We can say that $b^\mu$ forces the spin of the muon to be in the $\hat{b}$ direction. This causes a zeroth-order effect in the muon-decay rate, through the usual parity-violating term, where the muon spin couples to the $\beta$-particle direction.

The decay rate in Eq. (6.59), however, is not the polarized muon-decay rate. At least not in the sense that we can use it for experiments with polarized muons. In such experiments, the direction of polarization of the muon will predominantly be determined
by some polarization mechanism other than $b^{\mu}$, depending on the experimental conditions. We would then have to find the polarized spinor solutions from equations that somehow include the polarization mechanism (e.g. the Dirac equation including a magnetic field) and use them to determine the polarized muon-decay rate.

This is the reason that Eq. (6.59) and measurements of $\varrho$ cannot be used directly to put bounds on $b^{\mu}$. The most precise value of $\varrho$ was determined in an experiment using polarized muons [72] and we would need the polarized version of Eq. (6.59) to extract such bounds. We do this for $\chi^{\mu\nu}_{\mu}$ in Section 6.7, leading to the bounds in Eq. (6.68). These are in the order of $10^{-3}$-$10^{-4}$, from which we expect bounds on $b^{\mu}$ in the order of $10^{-4}$-$10^{-5}$ GeV. These are not competitive with existing bounds on the spatial components of $b^{\mu}$, given in Ref. [15], which are in the order of $10^{-22}$-$10^{-24}$ GeV. However, to our knowledge, the timelike component $b^{0}$ is not bounded by experiment. We expect that an analysis, using a precise value of $\varrho$, of the polarized muon-decay rate with $b^{\mu}$ would give bounds on $b^{0}$ in the mentioned order of $10^{-4}$-$10^{-5}$ GeV.

Integrating the expression in Eq. (6.59) over all directions, over $x$, and summing over $\alpha$ gives us a total muon-decay rate of

$$\Gamma = \Gamma^{0} \left( 1 \mp \frac{4\varrho^{0}}{m} \right).$$

We will use this expression in interpreting the muon-lifetime data in Section 6.8.

### 6.7 Muon decay with $\chi^{\mu\nu}$

Next we consider muon decay in the $\chi^{\mu\nu}$ framework. When $\chi^{\mu\nu}$ is included in the $W$-boson propagator, the matrix element for the decay $\mu^{-} \rightarrow e^{-} + \nu_{e} + \bar{\nu}_{\mu}$, corresponding to the tree-level $W$-exchange diagram, reads

$$i\mathcal{M} = \frac{G_{F}}{\sqrt{2}} (g^{\mu\nu} + \chi^{\mu\nu}) [\bar{u}(k_{1})\gamma_{\mu}(1 - \gamma_{5})u(l)] [\bar{u}(p)\gamma_{\nu}(1 - \gamma_{5})v(k_{2})],$$

where $G_{F}$ is the Fermi coupling constant, and $l$, $p$, $k_{1}$, and $k_{2}$ are the momenta of the muon, electron, muon neutrino, and electron antineutrino, respectively. From the matrix element in Eq. (6.61) and the corresponding matrix element for $\mu^{+} \rightarrow e^{+} + \nu_{e} + \bar{\nu}_{\mu}$ we derive, to first order in Lorentz violation, the muon-decay rate

$$d\Gamma = \frac{G_{F}^{2}}{24\pi^{2}} \frac{d^{3}p}{2^{10}2^{3}p^{3}} \left[ q^{2}(L \cdot Q) + 2(L \cdot q)(Q \cdot q) + 2\chi^{\nu s}_{\mu s} (2q^{2}L_{\mu}Q_{\nu} + (L \cdot Q)q_{\mu}q_{\nu} - (q \cdot Q)q_{\mu}L_{\nu} - (L \cdot q)Q_{\mu}q_{\nu}) + 2\chi^{\nu s}_{\mu s} (q^{2}L_{\mu}Q_{\nu} - (q \cdot Q)q_{\mu}L_{\nu} - (L \cdot q)Q_{\mu}q_{\nu}) + \chi^{\mu s}_{\nu s} \epsilon_{\mu \nu \rho \sigma} ((q \cdot Q)L^{\rho}q^{\sigma} - (L \cdot q)q^{\rho}Q^{\sigma}) - 2\chi^{\mu s}_{\nu s} \epsilon_{\mu \nu \rho \sigma} \lambda L^{\rho}q^{\sigma}Q^{\lambda} \right],$$

where we summed over the spins and integrated over the momenta of the (anti)neutrino. The subscripts $r$, $i$, and $s$, $a$ on the tensor $\chi^{\mu\nu}$ denote its real or imaginary and its symmetric or antisymmetric part, respectively. We defined the four-vectors $q = l - p,$
The Lorentz-violating parameters in Eq. (6.64) are defined by

\[ \text{violating terms proportional to } \frac{m}{\beta} \]

energy of the decay rate in the muon restframe four-vector \( r \mu \)on is

\[ m \]

Lorentz structure of the weak interaction. In the SM (and in our framework for Lorentz violation, cf. Eq. (6.61)) the currents have \( V - A \) structure, in which case the values

\[ L = l \mp m_\mu s \text{ and } Q = p \mp m_e r, \]

where the upper (lower) sign applies for \( \mu^- (\mu^+) \) decay; \( m_\mu \) and \( m_e \) are the muon and electron mass, respectively. The spin four-vector \( s \) of the muon is

\[ s = \left( \frac{1 \cdot \hat{s}}{m_\mu}, \hat{s} + \frac{(1 \cdot \hat{s}) l_\mu}{m_\mu} \right), \quad (6.63) \]

with \( \hat{s} \) a unit vector in the direction of the spin of the muon in its restframe; the spin four-vector \( r \) of the \( \beta^+ \) particle (electron/positron) is defined analogously.

When we sum Eq. (6.62) over the spin of the \( \beta \) particle we obtain for the differential decay rate in the muon restframe

\[
\frac{d\Gamma}{dx d\Omega} = \frac{\Gamma_0}{\pi} x^2 \left[ 3(1 - x) + \frac{2}{3} \delta(4x - 3) + (\hat{s} \cdot \hat{p}) \xi \left[ 1 - x + \frac{2\delta}{3} (4x - 3) \right] 
\right.
\]

\[
- \left( t_1 + v_1 \cdot \hat{p} \pm v_2 \cdot \hat{s} \pm v_3 \cdot (\hat{s} \times \hat{p}) 
\right)
\]

\[
+ \left( t_1^{ml} p^m p^l \pm t_2^{ml} p^m s^l + t_3^{ml} p^m (\hat{s} \times \hat{p})^l \right) (1 - x)
\]

\[
- \left( z_1 + u_1 \cdot \hat{p} \pm u_2 \cdot \hat{s} \pm u_3 \cdot (\hat{s} \times \hat{p}) 
\right)
\]

\[
+ H_1^{ml} p^m p^l \pm H_2^{ml} p^m s^l \pm H_3^{ml} p^m (\hat{s} \times \hat{p})^l \right) (4x - 3)
\]

\[
\pm (\hat{s} \cdot \hat{p}) \left[ (t_2 + v_4 \cdot \hat{p}) (1 - x) + (z_2 + u_4 \cdot \hat{p}) (4x - 3) \right] \right], \quad (6.64)
\]

where \( \Gamma_0 = G_F^2 m_\mu^5 / 192\pi^3 \) is the total decay rate in the SM, and \( x = E/E_{\text{max}} \) is the energy of the \( \beta \) particle relative to its maximum. We neglected terms proportional to \( m_e / m_\mu \), because the pertinent SM terms do not mimic Lorentz violation and the Lorentz-violating terms proportional to \( m_e / m_\mu \) will be suppressed with respect to the other ones. The Lorentz-violating parameters in Eq. (6.64) are defined by

\[
t_1 = z_1 = z_2 = \frac{1}{2} \chi_{rs}^0, \quad t_2 = \frac{5}{2} \chi_{rs}^0; \quad v_1^l = \chi_{rs}^0 + 2 \chi_{ra}^0 - 2 \chi_{ia}^l, \quad v_2^l = \frac{1}{2} \chi_{ra}^0 + \frac{7}{4} \chi_{ia}^l, \quad v_3^l = \frac{3}{2} \chi_{ia}^0 + \frac{5}{2} \chi_{is}^l, \quad v_4^l = \frac{3}{2} \chi_{ra}^0 + \frac{3}{2} \chi_{ol}^0 - \frac{3}{4} \chi_{ia}^l,
\]

\[
u_1^l = - \frac{1}{2} \chi_{ia}^l, \quad \nu_2^l = \frac{1}{2} \chi_{ra}^0 + \frac{1}{2} \chi_{ia}^l + \frac{1}{4} \chi_{is}^l, \quad \nu_3^l = \frac{1}{2} \chi_{ia}^0 + \frac{1}{2} \chi_{is}^l, \quad \nu_4^l = \frac{1}{2} \chi_{ra}^0 + \frac{1}{2} \chi_{ia}^0 - \frac{1}{4} \chi_{ia}^l,
\]

\[
u_1^l = \frac{1}{2} \chi_{rs}^0 + \frac{1}{2} \chi_{ra}^0 - \frac{1}{4} \chi_{ia}^l, \quad T_1^{ml} = \frac{3}{2} \chi_{rs}^0 + \frac{1}{2} \chi_{ra}^0 + \frac{1}{4} \chi_{ia}^l, \quad T_2^{ml} = \frac{7}{2} \chi_{rs}^0 + \frac{1}{2} \chi_{ra}^0 + \frac{1}{4} \chi_{is}^l, \quad T_3^{ml} = \frac{3}{2} \chi_{rs}^0,
\]

\[
H_1^{ml} = - \frac{1}{2} \chi_{rs}^0, \quad H_2^{ml} = \frac{1}{2} \chi_{ra}^0, \quad H_3^{ml} = \frac{1}{2} \chi_{is}^l, \quad (6.65)
\]

where \( \chi^l = e^{lnk} \chi^{mk} \). The first line of Eq. (6.64) gives the SM decay rate written in the conventional way [73]. For easy comparison, we inserted by hand the three standard Michel parameters \( \varrho, \xi, \text{ and } \delta \), which parametrize the energy and angular distribution of the \( \beta \) particles in polarized muon decay [71] and which are used to establish and test the Lorentz structure of the weak interaction. In the SM (and in our framework for Lorentz violation, cf. Eq. (6.61)) the currents have \( V - A \) structure, in which case the values
of the Michel parameters are $\varrho = 3/4$, $\xi = 1$, and $\delta = 3/4$. The TWIST collaboration
has put strong limits on non-SM contributions to the Michel parameters [72]. The next
lines of Eq. (6.64), with the parameters defined in Eq. (6.65), give Lorentz-violating,
frame-dependent contributions to muon decay.

Lorentz-violating parameters that modify the decay rate of the muon, will, in many
cases, also contribute to pion decay [74]. They affect in particular the polarization of the
muons that originate from the decay of pions. This was pointed out below Eqs. (6.12)
and (6.24). The Lorentz violation in the muon polarization will be important when the
Lorentz-symmetric $\hat{s} \cdot \hat{p}$ term is considered. Through this term, Lorentz violation in the
initial muon polarization will cause a first-order Lorentz-violating effect in the muon-decay
distribution.

Eq. (6.64) offers many possible tests of Lorentz invariance in muon decay. For example,
the dependence of the decay rate on the $\beta$ direction can be studied. In general, it is
profitable to measure over extended periods of time and record the data with “time
stamps.” One can then search for signals that oscillate with periods of one or one-half
sidereal day due to the rotation of Earth with respect to the standard Sun-centered
inertial reference frame [19]. This strategy requires reanalysis of, typically statistics-
limited, existing data [72] or new dedicated experiments. Another option is to compare
experiments performed at different velocities, i.e. with different values for the Lorentz
boost factor $\gamma$, because at higher $\gamma$ values the Lorentz-violating signals are enhanced by
a factor $\gamma^2$, so that with an equal number of events more stringent limits can be set.

The angular dependence of the SM muon-decay rate is modified in several ways in
Eq. (6.64), which is relevant for measurements of the Michel parameters. When we in-
tegrate Eq. (6.64) over the energy of the $\beta$ particle, all terms proportional to $4x - 3$
disappear. Defining the muon-polarization direction as the $z$-axis and integrating over
the azimuthal angle $\phi$ of the $\beta$ momentum, we find

$$
\frac{d\Gamma}{d\cos \theta} = \frac{\Gamma_0}{6} \left[ 3 - t_1 \mp v_z^2 \mp \cos \theta (\xi - t_2 \pm v_z^2 - T_{zz}^{zz}) \\
- \cos^2 \theta (T_{zz}^z \mp v_z^4) - \frac{1}{2} \sin^2 \theta (T_{xx}^z + T_{yy}^z) \right],
$$

(6.66)

where $\theta$ is the angle between the polarization axis and the $\beta$ momentum, with the muon
at rest in the laboratory frame. Therefore, when one determines the Michel parameter $\xi$
by fitting the $\theta$ dependence of the decay rate, one has to include a term with $\cos^2 \theta$
dependence. Because the Lorentz-violating coefficients of the $\theta$-dependent terms will vary
over the course of a sidereal day, one has to express them in the analysis in terms of $X_{\mu \nu}$,
which is $\chi^{\mu \nu}$ in the Sun-centered frame [19], and integrate over the relevant measurement
periods.

The most recent value for the Michel parameter $\varrho_{\exp} = 0.74977(26)$ [72] can already
be used to derive a bound on $\chi^{\mu \nu}$. When one measures the decay rate as a function of
positron energy, without selecting a particular direction for the positrons, Eq. (6.64) gives

$$
\frac{d\Gamma}{dx} = 4\Gamma_0 x^2 \left[ 3(1 - x)(1 + n_1 \cdot \hat{s}) + \left( \frac{2}{3} \varrho - \frac{1}{3} \chi_{00}^{00} + n_2 \cdot \hat{s} \right) (4x - 3) \right],
$$

(6.67)

with $n_1 = \chi_{0a}^{0a} + \frac{1}{2} \chi_{1a}^{1a}$ and $n_2 = \frac{1}{3} (\chi_{0a}^{0a} + \chi_{0a}^{0f} + \chi_{0a}^{1a})$. If we assign the difference between the
measured value and the SM prediction to $\chi^{\mu \nu}$, we get $\varrho_{\exp} = \varrho - \chi_{0a}^{00}/2 + (3n_2/2 - \varrho n_1) \cdot \hat{s}$,
which results in the 95% confidence limit (C.L.)

\[-5.6 \times 10^{-4} < X_{rs}^{00} - \sin \varsigma \cos \varphi \left( X_{rs}^{0Z} - \frac{1}{2} X_{ra}^{0Z} + \frac{1}{4} X_{ia}^{Z2} \right) < 1.5 \times 10^{-3} \, ,\]

(6.68)

where \( \varsigma \approx 41^\circ \) is the colatitude of Vancouver, and \( \phi \approx 52^\circ \) is the angle that the muon polarization makes with the north-south direction (anticlockwise) in the plane parallel to the surface of Earth, so \( \sin \varsigma \cos \phi = 0.404 \). We assumed that the muon polarization is parallel to the surface of Earth and we averaged over an integer number of sidereal days. \( X_{ra}^{0Z} \) does not occur in the strong bounds determined from forbidden \( \beta \) decays [75].

### 6.7.1 Total muon-decay rate with \( \chi^{\mu\nu} \)

The total decay rate obtained from Eq. (6.64) does not depend on \( \chi^{\mu\nu} \) in first order. This is a general result, already pointed out in Ref. [76], for a matrix element of the form of Eq. (6.61), i.e. two pure \( V-A \) currents contracted by \( g^{\mu\nu} + \chi^{\mu\nu} \). We rederive it here in a slightly different way.

We start by writing the square of a matrix element with a structure as in Eq. (6.61), to first order in Lorentz violation, as

\[ |\mathcal{M}|^2 \delta^4 (\Sigma p) \propto (g^{\mu\nu} g^{\rho\sigma} + g^{\rho\sigma} \chi^{\mu\nu} + g^{\mu\nu} \chi^{\rho\sigma}) \mathcal{M}_{\mu\nu\rho\sigma}^{abcd} \]  

(6.69)

Here the Latin indices \( a, b, c, d \) label the different particles, \( \delta^4 (\Sigma p) \) is the delta function ensuring overall momentum conservation, and

\[ \mathcal{M}_{\mu\nu\rho\sigma}^{abcd} = \left[ \tilde{u}_a \mathcal{O}_\mu u_b \right] \left[ \tilde{u}_c \mathcal{O}_\nu u_d \right] \left[ \tilde{u}_d \mathcal{O}_\rho u_c \right] \left[ \tilde{u}_b \mathcal{O}_\sigma u_a \right] \delta^4 (\Sigma p) \]

(6.70)

where \( \mathcal{O}_\mu = \gamma_\mu (1 - \gamma_5) \), \( P_a = p_a \mp m_a s_a \) and \( p, m, \) and \( s \) are the momentum, mass, and spin of the particle respectively. Using Fierz identities we can write

\[ \left[ \tilde{u}_a \mathcal{O}_\mu u_b \right] \left[ \tilde{u}_c \mathcal{O}_\nu u_d \right] = \frac{1}{2} \left[ \tilde{u}_a \mathcal{O}_\mu u_b \right] \left[ \tilde{u}_c \mathcal{O}_\nu u_d \right] + \frac{1}{2} \left[ \tilde{u}_d \mathcal{O}_\rho u_c \right] \left[ \tilde{u}_b \mathcal{O}_\sigma u_a \right] - \frac{1}{2} \frac{g^{\mu\nu}}{2} \left[ \tilde{u}_c \mathcal{O}_\alpha u_b \right] \left[ \tilde{u}_a \mathcal{O}_\alpha u_d \right] + \frac{i}{2} \epsilon^{\mu\nu\alpha\beta} \left[ \tilde{u}_c \mathcal{O}_\alpha u_b \right] \left[ \tilde{u}_a \mathcal{O}_\beta u_d \right] \]  

(6.71)

Contracting the left- and right-hand side with \( g^{\mu\nu} \), \( \chi^{\mu\nu}_s \), and \( \chi^{\mu\nu}_a \) gives

\[ \left[ \tilde{u}_a \mathcal{O}_\mu u_b \right] \left[ \tilde{u}_c \mathcal{O}_\nu u_d \right] = - \left[ \tilde{u}_c \mathcal{O}_\mu u_b \right] \left[ \tilde{u}_a \mathcal{O}_\nu u_d \right] \]  

(6.72a)

\[ \chi^{\mu\nu}_s \left[ \tilde{u}_a \mathcal{O}_\mu u_b \right] \left[ \tilde{u}_c \mathcal{O}_\nu u_d \right] = \chi^{\mu\nu}_s \left[ \tilde{u}_c \mathcal{O}_\mu u_b \right] \left[ \tilde{u}_a \mathcal{O}_\nu u_d \right] \]  

(6.72b)

\[ \chi^{\mu\nu}_a \left[ \tilde{u}_a \mathcal{O}_\mu u_b \right] \left[ \tilde{u}_c \mathcal{O}_\nu u_d \right] = \frac{i}{2} \chi^{\mu\nu}_a \left[ \tilde{u}_c \mathcal{O}_\mu u_b \right] \left[ \tilde{u}_a \mathcal{O}_\nu u_d \right] \]  

(6.72c)

where \( \tilde{\chi}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} \chi_{\alpha\beta} \). We therefore find that

\[ (g^{\rho\sigma} \chi^{\mu\nu}_s + g^{\mu\nu} \chi^{\rho\sigma}_s) \mathcal{M}_{\mu\nu\rho\sigma}^{abcd} = - (g^{\rho\sigma} \chi^{\mu\nu}_a + g^{\mu\nu} \chi^{\rho\sigma}_a) \mathcal{M}_{\mu\nu\rho\sigma}^{abcd} \]  

(6.73a)

\[ (g^{\rho\sigma} \chi^{\mu\nu}_a + g^{\mu\nu} \chi^{\rho\sigma}_a) \mathcal{M}_{\mu\nu\rho\sigma}^{abcd} = \frac{i}{2} (g^{\rho\sigma} \tilde{\chi}^{\mu\nu}_a - g^{\mu\nu} \tilde{\chi}^{\rho\sigma}_a) \mathcal{M}_{\mu\nu\rho\sigma}^{abcd} \]  

(6.73b)

Notice now that

\[ \widetilde{\mathcal{M}}_{\mu\nu\rho\sigma}^{abcd} \equiv \sum_{s_a,s_c} \int \frac{d^3 p_a}{2 p_a^0} \int \frac{d^3 p_c}{2 p_c^0} \mathcal{M}_{\mu\nu\rho\sigma}^{abcd} = \sum_{s_a,s_c} \int \frac{d^3 p_a}{2 p_a^0} \int \frac{d^3 p_c}{2 p_c^0} \mathcal{M}_{\mu\nu\rho\sigma}^{abcd}. \]  

(6.74)
6.8 MUON-LIFETIME DATA

<table>
<thead>
<tr>
<th></th>
<th>$\tau$ (µs)</th>
<th>$\tau'/\gamma$ (µs)</th>
<th>$10^5 \Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu^+$</td>
<td>2.1969803(21)<em>{stat}7)</em>{sys}</td>
<td>2.19730(8)<em>{stat}(10)</em>{sysu}(15)_{sysc}</td>
<td>14.6(8.9)_{tot}</td>
</tr>
<tr>
<td>$\mu^-$</td>
<td>2.196998(23)<em>{stat}(21)</em>{sys}</td>
<td>2.19767(10)<em>{stat}(4)</em>{sysu}(11)_{sysc}</td>
<td>30.4(7.0)_{tot}</td>
</tr>
<tr>
<td>$\mu$</td>
<td>2.1969804(22)_{tot}</td>
<td>2.19752(15)_{tot}</td>
<td>24.8(6.8)_{tot}</td>
</tr>
</tbody>
</table>

Table 6.2: Muon lifetimes in µs at rest, $\tau$, and in flight at $\gamma \simeq 29.3$, $\tau'$, and the relative difference $\Delta = (\tau'/\gamma - \tau)/\tau$. Listed in the rows are the values for $\mu^+$, $\mu^-$, and their average ($\mu$). The first, second, and third numbers between parentheses in the entries are total or statistical, uncorrelated systematic, and (presumed) correlated systematic errors, respectively.

Together with Eq. (6.73a) this means that, if one does not observe any properties of particles $a$ and $c$, the symmetric part of $\chi^{\mu\nu}$ does not enter the expression of the resulting decay rate. If, in addition, the properties of particle $d$ and the spin of particle $b$ are unobserved, then the tensor $\sum_{s,b,d} \int \frac{d^3p}{(2\pi)^3} M_{bd}^{\mu\nu_{pa}}$ can only be a function of $p_b$, the Minkowski metric or the Levi-Civita tensor. This causes the terms with antisymmetric $\chi^{\mu\nu}_a$ in the decay rate to vanish. Applying this to muon decay, it means that the total unpolarized decay rate of the muon, cannot depend on $\chi^{\mu\nu}$, since it can depend neither on $\chi^{\mu\nu}_a$ nor on $\chi^{\mu\nu}_s$. It also means that the total polarized decay rate of the muon can only depend on $\chi^{\mu\nu}_a -$ the antisymmetric part of $\chi^{\mu\nu}$. These conclusions are confirmed by Eq. (6.64).

6.8 Muon-lifetime data

Strong bounds on Lorentz violation can be obtained by comparing muon-lifetime measurements at different absolute muon velocities, i.e. different Lorentz boost factors $\gamma$. The available data that we used are collected in Table 6.2. The most precise measurement of the $\mu^+$ lifetime at rest, $\tau = 1/\Gamma$, comes from the MuLan experiment [77]. The $\mu^-$ lifetime at rest was derived from the MuCap experiment [78] by correcting for the muon capture rate on hydrogen, for which we used the value $\Gamma(\mu^- p \to \nu_\mu n) = 718(7)\ s^{-1}$ [79]. We also list in Table 6.2 the averaged $\mu^+$ and $\mu^-$ lifetimes as $\mu$, which is relevant when assuming CPT invariance. The most precise muon lifetime in flight, $\tau'$, was obtained from data taken for the most recent muon $g - 2$ experiment, E821 at Brookhaven National Laboratory (BNL) [80, 81], which was performed at the “magic momentum,” corresponding to $\gamma \simeq 29.3$. In the E821 experiment the muons are kept in a circular orbit (the effects of acceleration are claimed to be negligible, cf. Refs. [82, 83]). The arrival times and energies of the $\beta$ particles were recorded together with the magnetic-field strength. From these, the dilated lifetime $\tau'$ and $\gamma$ were obtained. An analysis is presented in Ref. [84], from which we take the values for the $\mu^+$ and $\mu^-$ lifetimes in flight. For $\gamma \simeq 29.3$ one finds $\tau' \simeq 64.4\ µs$, where we put approximation signs, since $\gamma$ varied slightly over the course of the experiment, but for each part of the experiment it is known to 11-20 parts per million.

The dominant systematic error that limits the precision of the final result is due to muon loss. This part of the systematic error we conservatively assume to be correlated for the $\mu^+$ and $\mu^-$ in-flight measurements. To calculate the averages we add the statistical
and uncorrelated systematic errors in quadrature to obtain a total uncorrelated error. The average $\mu^+$ and $\mu^-$ lifetime is calculated by weighing them with the inverse of the square of this error. The same weights are used to calculate the average correlated systematic error. To obtain the total errors, the correlated systematic errors were combined quadratically with the uncorrelated ones.

To determine bounds on Lorentz violation, we need to compare the lifetime for muons in flight to the one obtained for muons at rest. When Lorentz invariance holds, the muon lifetime at rest is calculated from $\tau = \tau'/\gamma$, therefore $\Delta = (\tau'/\gamma - \tau)/\tau$ is the relevant quantity. By using the values for $\tau'$ and $\gamma$ for the different parts of the $g-2$ experiment, as given in Ref. [84], together with the values for $\tau$ in Table 6.2, we calculated the values for $\Delta$ for $\mu^+$, $\mu^-$, and for the average of $\mu^+$ and $\mu^-$, respectively, as listed in the last column of Table 6.2. These results deviate from zero by 1.6 $\sigma$, 4.3 $\sigma$, and 3.7 $\sigma$, respectively. When we assume that the systematic errors on the values for $\tau'$ are completely correlated or completely uncorrelated, the deviation from zero of the value of $\Delta$ for the average of $\mu^+$ and $\mu^-$ becomes 2.8 $\sigma$ and 4.6 $\sigma$, respectively. We conclude that the available data from MuLan, MuCap, and E821 (as given in Ref. [84]) are not consistent with Lorentz invariance. To test CPT violation we consider the ratio

$$R = \frac{2(\tau_{\mu^+} - \tau_{\mu^-})}{\tau_{\mu^+} + \tau_{\mu^-}}. \quad (6.75)$$

For muon decay at rest and in flight we find $R = 0.8(1.4) \times 10^{-5}$ and $R = 1.7(1.1) \times 10^{-4}$, respectively.

### 6.8.1 Discussion

While these results are intriguing, one should keep in mind that the BNL muon $g-2$ experiment was not optimized to determine the muon lifetime. Moreover, the measurements and analyses of the muon lifetime at rest and in flight were designed to be sensitive only to the total unpolarized muon-decay rate, which as shown above does not depend on $\chi^{\mu\nu}$ in first order. In order to properly investigate the presence of Lorentz violation, details of the analysis from which the total lifetimes were extracted are required, since these analyses involved taking averages over the directions and spins of the muons. For instance, it becomes relevant that the muons in some of the measurements may be polarized. This can e.g. be due to the fact that a small residual polarization may exist in the muons used in the analyzed MuLan and $g-2$ data sets.

In the $g-2$ experiments the arrival-time distribution of the $\beta$ particles is modulated due to the muon-spin precession relative to its momentum [80]. The analysis as reported in Ref. [84] was such that the result for $\tau'$ is predominantly sensitive to the exponential decay rather than this modulation. Furthermore, the fit to the exponent of the decay curve is sensitive to the decay rate, independent of the direction or energy of the outgoing $\beta$ particles, even though the detectors are only sensitive to part of this parameter space. Also most effects of the muon polarization are removed, since it precesses around the magnetic field and the muons are unpolarized on average. (Since the Lorentz-symmetric part of the total lifetime does not depend on the spin, any Lorentz-violation in the initial spin of the muon that originates from pion decay will thus be a second-order effect.)
However, any polarization component parallel to the magnetic field does not average out and may thus cause a residual vertical polarization. Since no limit for the latter is quoted we assumed it to be zero and used the total decay rate of the muons for our analysis. When averaged over a sidereal day, a possible effect due to this residual polarization is further reduced as only the component along Earth’s axis remains.

At this moment it cannot be assessed whether such residual sensitivities would result in limits for the components of $\chi_{\mu\nu}$ that can compete with the existing bounds from measurements of forbidden $\beta$ decay [75]. Without these residual sensitivities, this measurement does not provide bounds on $\chi_{\mu\nu}$ since the total muon-decay rate does not depend on $\chi_{\mu\nu}$ (see Section 6.7.1). Also, in that case, a Lorentz-violating effect, such as the one that follows from the last column of Table 6.2, cannot be interpreted in terms of $\chi_{\mu\nu}$.

As an alternative to $\chi_{\mu\nu}$, we can interpret the effect in terms of a CPT even $c_{\mu\nu}$ parameter, or a CPT odd $b_\mu$ parameter, which both contribute to the total decay rate of the muon, as can be seen from Eqs. (6.42) and (6.60).

Eq. (6.42), when boosted to the laboratory frame, becomes

$$\Gamma' = \frac{1}{\gamma} \Gamma_0 \left[ 1 - \frac{21\gamma^2}{5} \left( \frac{c_{\mu\nu} p_\mu p_\nu}{(p^0)^2} \right) \right], \quad (6.76)$$

where $c_{\mu\nu}$ is now defined in the lab frame and $p$ is the muon momentum in the lab frame. Averaging this over a rotation of the muon around the ring and over a full sidereal day we find

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi \frac{1}{2\pi} \int_0^{2\pi/\Omega} \Omega dt \frac{2\pi}{2\pi} \Gamma' \approx \frac{1}{\gamma} \Gamma_0 \left[ 1 - 5.5\gamma^2 (c_{TT} - 0.05c_{ZZ}) \right], \quad (6.77)$$

where $\Omega$ is the angular rotation frequency of Earth, and we used that the colatitude of BNL is about 49° and that $\gamma^2 \gg 1$. We neglected effects of incomplete rotation cycles of the muon, the muon spin, and effects of incomplete sidereal days. Compared to the Lorentz-violating effect in Eq. (6.77), these effects are estimated to have a size on the order of $10^{-4}$, $10^{-2}$, and $10^{-2}$, respectively. To extract bounds on $c_{\mu\nu}$, we fit the expression in Eq. (6.77) to the four data points in Table 6.2. The four data points are the lifetimes for $\mu^+$, $\mu^-$, both at rest and in flight. The resulting fit has a probability of $p = 0.287$ and we find for $c_{TT} - 0.05c_{ZZ}$ that

$$c_{TT} - 0.05c_{ZZ} = 5.3(1.2) \times 10^{-8}, \quad (6.78)$$

which is depicted as the data point with error bar in Fig. 6.2. This value of $c_{TT}$ and $c_{ZZ}$ is three orders of magnitude more precise than the bounds found in Ref. [66]. It seems to be excluded by the bounds obtained in Ref. [85]. However, as noted in Ref. [66], these bounds apply to a spin-averaged $c_{\mu\nu}$, while our result pertain to a left-chiral $c^L_{\mu\nu}$.

In addition to $c_{\mu\nu}$, we aim to interpret the results of Table 6.2 in terms of the CPT-odd second-generation SME coefficient $b_\mu = (a^L_\mu - a^R_\mu)/2$, discussed in Section 6.6. The relevant equation is Eq. (6.60), in which the total muon-decay rate with $b_\mu$ is given. In a frame where the muons move with a boost factor $\gamma$, this expression becomes

$$\Gamma' = \frac{1}{\gamma} \Gamma_0 \left[ 1 - \frac{4\gamma b_\mu p_\alpha}{p^0 m_\mu} \right]. \quad (6.79)$$
When doing a fit to the four data points in Table 6.2, the fit has a low probability ($p = 2.2 \times 10^{-4}$). This is because the data are more compatible with CPT symmetry than with CPT violation, as was also shown below Eq. (6.75). Therefore, instead of fitting the result for $b^\mu$ separately, we fit $b^\mu$ and $c^{\mu\nu}$ simultaneously. Again, we average over the rotation of the muon around the ring and over a full sidereal day. From this we find that for a nonzero $b^\mu$ and $c^{\mu\nu}$ parameter

$$\int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^{2\pi/\Omega} \frac{\Omega dt}{2\pi} \Gamma' \simeq \frac{1}{\gamma} \Gamma_0 \left[ 1 - 5.5\gamma^2 (c^{TT} - 0.05c^{ZZ}) \mp (\gamma - 1) \frac{4b^T}{m_\mu} \right],$$

where $b^T$ is the time component of $b^\mu$ in the Sun-centered frame. When fitting this to the four data points in Table 6.2, the fit has a probability of $p = 0.567$. In Fig. 6.2 we plot the joint 95% C.L. region of $b^T$ and $c^{TT} - 0.05c^{ZZ}$. The preferred point in this parameter space with 1 $\sigma$ errors, is given by

$$c^{TT} - 0.05c^{ZZ} = 4.9(1.1) \times 10^{-8},$$
$$b^T/m_\mu = 7.3(5.0) \times 10^{-7}$$

and is represented by the dot at the center of the ellipse in Fig. 6.2.

We conclude that, when considering a CPT odd $b^\mu$ and CPT even $c^{\mu\nu}$ parameters, the data in Table 6.2 are most compatible with a CPT even parameter. The fit gets better when including both $b^\mu$ and $c^{\mu\nu}$, which point to a combination of both CPT even and CPT odd Lorentz violation.

To our knowledge, the value of the $b^T$ coefficient of the second-generation leptons, is the first of its kind. The space components of $b^\mu$ have been bounded to a level of $10^{-23}$-$10^{-24}$ by analyzing the spin precession frequency of muons [81]. The values in Fig. 6.2 are an order of magnitude larger than the bounds on the neutrino coefficients derived in Ref. [70]. However, these bounds are for first-generation coefficients.

### 6.8.2 Bounds from cosmic rays

Because large $\gamma$ factors are advantageous, cosmic-ray muons, which can have energies up to at least $10^4$ GeV, are obvious candidates to search for Lorentz violation. In Ref. [86], it was pointed out that the rate of the flavor-violating muon-decay mode $\mu \to e + \gamma$ could be enormously enhanced when Lorentz invariance is violated. Strong bounds on Lorentz violation for this decay mode were subsequently obtained in Ref. [87].

When dealing with such ultrahigh energies, one has to address the question up to which energy the theoretical framework is valid. Frames that move relatively slow with respect to Earth are called concordant frames (see Section 2.2 or Ref. [20]). In these frames all Lorentz-violating parameters are expected to be small. However, in frames that are highly boosted with respect to concordant frames, the Lorentz-violating parameters can become so large that they cause problems with stability and causality in the theory (see Sections 2.4 and 2.5 or Ref. [20]). For very large boost factors the muon-decay rate could even become negative. When we denote the dimensionless Lorentz-violating effect in the muon restframe by $A$, then for large $\gamma$, for example, the tensor $\chi^{\mu\nu}$ with two Lorentz indices scales schematically as $A \propto \gamma^2 a$, where $a$ is the Lorentz-violating effect in a
Figure 6.2: Limits on CPT-even and CPT-odd Lorentz-violating couplings. The ellipse shows the joint 95% C.L. region.

concordant frame. A large $\gamma$ factor gives better bounds on $a$ when we have a bound for $A$. However, the theory can only be trusted up to some $A = A_{\text{max}}$. If we take as a guideline $A_{\text{max}} = 10^{-2}$ [20], the decay rate has to be determined with sub-percent precision to get reliable bounds. This kind of precision is very hard to achieve for ultrahigh-energy cosmic-ray muons.

The difference between the decays $\mu \to e + \nu + \nu$ and $\mu \to e + \gamma$ is that the former is the main allowed decay mode in the SM, while the latter is a forbidden process which gets enhanced with respect to the SM decay mode, even without a large boost factor. Moreover, the amplitude for $\mu \to e + \gamma$ does not interfere with a SM amplitude, because it is not a correction to an already existing SM process. Although it therefore depends quadratically on a Lorentz-violating parameter, the enhancement with $\gamma$ will also be squared, resulting in a scaling with $\gamma^4$. For $\mu \to e + \gamma$ Lorentz violation could thus become detectable for values of the boost factor $\gamma$ for which the Lorentz-violating effect is still small enough.
Chapter 7

Summary, Conclusion & Outlook

A theory that unifies general relativity and the standard model of particle physics can rightly be called the holy grail of contemporary theoretical physics. The expected energy scales where such a theory becomes relevant are so high, however, that experimental verification of theoretical ideas becomes a problem. One of the ways of surmounting this problem is the possibility that, at some high-energy scale, the unifying theory violates Lorentz and CPT invariance, possibly – and theoretically preferably – by a spontaneous symmetry-breaking mechanism. Tiny remnants of these violations might be detectable at energy scales available in present-day experiments. Any detection of such a violation of Lorentz and/or CPT invariance would be a clear signal of new physics. In many cases very tight constraints can be placed on the absence of these signals, in effect bounding quantum-gravity theories that incorporate mechanisms of Lorentz-symmetry breaking.

In this thesis – and in most of the relevant and more advanced literature – the framework that is used to study Lorentz-symmetry breaking is the Standard-Model Extension (SME). It is an effective field theory approach that in principle includes all Lorentz- and CPT-violating operators that otherwise respect the symmetries of the conventional standard model of particle physics. The operators that violate these symmetries are built out of standard-model fields, coupled to coefficients that parametrize the Lorentz and CPT violation. These coefficients are used to identify and categorize possible signals and to catalogue the experimental results in the form of constraints placed on their components.

In Chapter 2 we briefly described the Lagrangian of the minimal SME (including only operators of mass dimension $d \leq 4$) and proceeded to outline some of the QFT basics of a scalar and a Dirac field. We pointed out some known issues associated with stability and causality. These issues necessitated the introduction of a special set of observer frames in which the SME coefficients are small, the so-called concordant frames.

In Chapter 3 we then discussed general matters pertaining to cross sections and decay rates. These include the LSZ reduction formalism with a non-zero $c^{\mu\nu}$ coefficient and the derivation of a general formula for scattering cross sections and decay rates. The derivation of the latter was done using the optical theorem and the largest-time equation. We concluded that the treatment of unitarity and the optical theorem and consequently the derivation of the decay-rate formula, has not been done rigorously for all Lorentz-violating parameters. In fact, only for the scalar field $K^{\mu\nu}$ and fermion field $c^{\mu\nu}$ parameter the derivation has been completed. These parameters do not lift the degeneracy of the
energy eigenvalues and therefore the treatment in terms of the largest-time equation proceeds very similar to the Lorentz-symmetric case. For the other parameters the lifting of the degeneracy of the energy eigenvalues hampers the usual analysis. The extension of the discussion to the general Lorentz-violating case is an interesting but outstanding problem, which, unfortunately, lies outside the scope of this thesis. For decay rates we took Eq. (3.62) as a working hypothesis, also for parameters other than $K^{\mu \nu}$ and $c^{\mu \nu}$.

In Chapter 4 we introduced what we call the $\chi^{\mu \nu}$ framework. It is a general modification of the $W$-boson propagator, which entails adding a general, complex, and possibly momentum-dependent tensor $\chi^{\mu \nu}$ to it. We discuss what kind of Lorentz-violating effects will contribute in the form of such a tensor and calculate the allowed $\beta$-decay rate, including $\chi^{\mu \nu}$. We are able to derive the general polarized decay rate, dependent on the polarization and direction of the outgoing $\beta$ particle. Using this formula, bounds on $\chi^{\mu \nu}$ at the level of $10^{-2}$ were obtained from a dedicated experiment at the KVI in Groningen, the Netherlands.

Using the $\chi^{\mu \nu}$ framework we also derive the general transition rate for forbidden $\beta$ decay in Chapter 5. We identify an enhancement factor that stems from the fact that the usual selection rules for these transitions are based on angular-momentum conservation, which is violated when a general $\chi^{\mu \nu}$ coefficient is introduced into the $W$-boson propagator. Using the obtained expression for the general transition rate, together with data from experiments performed in the 1970s on the decay of $^{90}$Y, $^{137}$Cs, and $^{99}$Tc, we derive bounds in the order of $10^{-6}$-$10^{-8}$ on (combinations of) components of $\chi^{\mu \nu}$. We also translate these bounds to bounds on the SME coefficients $k_{\phi \phi}^{\mu \nu}$ and $k_{\phi W}^{\mu \nu}$.

Some of the main results of this thesis are the actual calculations of different decay rates with Lorentz violation and the bounds on Lorentz-violating coefficients we were able to derive from them. In Chapter 6 we derive the pion- and muon-decay rate with a set of different coefficients for Lorentz violation. These coefficients include a $c^{\mu \nu}$ parameter for second-generation leptons, a second-generation $b^{\mu}$ parameter and our $\chi^{\mu \nu}$ modification of the $W$-boson propagator. Using a coordinate redefinition we are also able to calculate the pion-decay rate for quark $c^{\mu \nu}$ parameters. From existing data we derive the first bounds on these quark parameters as well as some complementary bounds on the second-generation lepton $c^{\mu \nu}$ coefficients.

We also find a nonzero result for Lorentz violation. In fact, the signal differs about $4\sigma$ from the signal one would expect from Lorentz invariance. We derive this partly from data of the BNL muon $g - 2$ experiment. With this data, it is possible to determine the muon lifetime at a relativistic boost factor of about 29.3. By comparing this to the muon lifetime at rest, obtained from the MuLan experiment for $\mu^+$ and from the MuCap experiment for $\mu^-$, we were able to check the relativistic dilatation of the muon lifetime to a precision of about $10^{-4}$-$10^{-5}$. The results differ by $1.6\sigma$, $4.3\sigma$, and $3.7\sigma$ from the Lorentz-symmetric expectation, for the $\mu^+$, the $\mu^-$ and the combined results, respectively. We interpret these results in terms of a CPT-even $c^{\mu \nu}$ and a CPT-odd $b^{\mu}$ coefficient (both for second-generation leptons). It is clear that a CPT-even parameter fits the data better than a CPT-odd parameter if we fit them separately. Using both $c^{\mu \nu}$ and $b^{\mu}$ results in an even better fit.

Although these results are intriguing, one should keep in mind that the BNL muon $g - 2$ experiment was not optimized to determine the muon lifetime. Furthermore, as explained
in Chapter 6, an analysis of the data, tailored to the search for Lorentz violation, is probably necessary to draw definite conclusions. We advise a reanalysis of this effect at the next $g - 2$ experiment at FermiLab or even a dedicated experiment.

### 7.1 Summary of bounds obtained in this thesis

Using the data of two experiments from the 1970s [53, 55], we obtained direct bounds on $\chi^{\mu\nu}$, which are summarized in Eq. (5.66) and translated to bounds on the SME parameters $k^{\mu\nu}_{\phi\phi}$ and $k^{\mu\nu}_{\phi W}$ in Eq. (5.69). To give an overview of the status of limits on $\chi^{\mu\nu}$ we assume it constitutes a momentum-independent contribution to the $W$-boson propagator, i.e. we assume that it is momentum independent and that $\chi^{\mu\nu} = \chi^{\nu\mu}$.

We then distinguish two cases. In the first case we assume that there are no cancellations between different components of $X^{\mu\nu}$ in a given observable ($X^{\mu\nu}$ is $\chi^{\mu\nu}$ in the Sun-centered frame). So for example a bound of $|X^{01}_{rs}| < 10^{-8}$ would lead to a bound of about $10^{-8}$ on both $|X^{01}_{rs}|$ and $|X^{23}_{ia}|$ in this case. In the second case we allow for the maximal amount of cancellation. Then we would need an additional bound like $|\hat{X}^{1}_{ia}| < 10^{-4}$ to derive a bound of order $10^{-4}$ on both $|X^{01}_{rs}|$ and $|X^{23}_{ia}|$. We apply a logarithmic way of rounding: anything smaller than $10^{-0.5}$ is rounded to 1, while anything greater than $10^{-0.5}$ is rounded to 10. We write the results for no cancellation like:

\[
-10 \log (|X^{\mu\nu}_{rs}|) > \begin{pmatrix} 6 & 8 & 8 & 8 \\ 8 & 6 & 6 & 6 \\ 8 & 6 & 6 & 6 \end{pmatrix},
-10 \log (|X^{\mu\nu}_{ia}|) > \begin{pmatrix} \times & - & - & - \\ - & \times & 8 & 8 \\ - & 8 & \times & 8 \\ - & 8 & 8 & \times \end{pmatrix},
\]

(7.1)

where for example an entry 8 on position (0, 1) of the first matrix means that $|X^{01}_{rs}| < 10^{-8}$. The crosses in the matrix mean that the corresponding entries are identically zero (since $X^{\mu\nu}_{ia}$ is antisymmetric), while the dashes signal that there are no bounds available.

The bounds in Eq. (7.1) all come from forbidden $\beta$ decay. We can combine these limits with limits on $X^{\mu\nu}$ that are obtained elsewhere and give the order of magnitude of the bounds on $X^{\mu\nu}$, but now assuming maximal cancellation. Writing in the same symbolic way, we then get:

\[
-10 \log (|X^{\mu\nu}_{rs}|) > \begin{pmatrix} 3 & 5 & 4 & 2 \\ 5 & - & 6 & 6 \\ 4 & 6 & - & 6 \\ 2 & 6 & 6 & 3 \end{pmatrix},
-10 \log (|X^{\mu\nu}_{ia}|) > \begin{pmatrix} \times & - & - & - \\ - & \times & 2 & 4 \\ - & 2 & \times & 4 \\ - & 4 & 4 & \times \end{pmatrix}.
\]

(7.2)

Besides from forbidden $\beta$ decay, these bounds come from pion decay [58] and the result in Eq. (6.68). The bounds on $X^{TT}_{rs}$, $X^{TZ}_{rs}$, $X^{ZZ}_{rs}$, and $X^{XY}_{ia}$ in this case of maximal cancellation, in addition depend on results from neutral kaon decay obtained in Ref. [88]. These are based on the tree-level calculation of kaon decay, while QCD effects in loops should in principle be taken into account. So it would be preferable to obtain an independent measurement of a combination of these parameters. A limit better than $10^{-2}$ will result in a corresponding bound on each one of these coefficients (provided the measured combination is independent of the other measured combinations). A similar result holds for
A measurement of a different combination of these two parameters will therefore result in a bound on each of them.

It should be kept in mind, however, that the bounds Eq. (7.1) are probably the most relevant. Although it is interesting to see what happens if we allow for maximal cancellation between the different components of $\chi_{\mu\nu}$, it requires a considerable amount of finetuning to make some parameters cancel each other to a level of $10^{-8}$, while their values are around $10^{-4}$. This is even harder to imagine since this cancellation would then only happen in the Sun-centered frame. In other frames the different components of $\chi_{\mu\nu}$ mix due to the Lorentz transformations. Therefore, the values of the coefficients would be different, and the cancellation would not happen.

In addition to bounds on $\chi_{\mu\nu}$, we derived bounds on quark $c_{\mu\nu}$ coefficients, which are in the order of $10^{-4}$ and are given in Table 6.1 and Eq. (6.32). We also found that the muon-lifetime data in flight and at rest, described in Section 6.8, is incompatible with Lorentz violation. We connected this to nonzero values for second-generation $c_{\mu\nu}$ and $b_{\mu}$ parameters, described in Eq. (6.81) and Fig. 6.2.

## 7.2 Outlook

The bounds on $\chi_{\mu\nu}$, which we obtained from forbidden $\beta$ decay, are quite stringent compared to other direct bounds on the same coefficient. As mentioned in Section 5.6, they improve previous bounds from pion decay [58] by three orders of magnitude. To push these bounds further, using forbidden-$\beta$-decay experiments, would require long-running experiments with high-intensity sources. Decay experiment involving a large $\gamma$ factor could also provide the precision that is needed. However, in the latter case one would still need to obtain a precision of at least $10^{-2}$ on the decay rate to insure the validity of the theory, as is explained in Section 6.8.2.

A part of $\chi_{\mu\nu}$ for which no limits have been set, is $X_{ia}^{0l}$, with $l$ a spacelike index. From the results in this thesis we can conclude that bounds on $X_{ia}^{0l}$ can be obtained by measuring the electron/positron direction and polarization in pion or muon decay (see Eqs. (6.24) and (6.64)) or by measuring the direction of the $\beta$ particle or the neutrino in polarized allowed $\beta$ decay (see Eq. (4.29) or Table D.1).

A result that warrants further investigation is the result in Table 6.2. This is a nonzero result for Lorentz violation and should therefore be thoroughly examined. We had no access to the details of the analysis of the data from which this result was obtained. A detailed new analysis of the muon lifetime in flight is thus called for (the muon-lifetime result in rest has a better precision). This can be done either in the planned new $g - 2$ experiment [62] or in a dedicated new experiment.

Using the results in Table 6.2, we obtained nonzero values for some SME parameters in Eq. (6.81). One should keep in mind, however, that there are many other parameters in the SME that could contribute to the same effect. In first instance one should thus focus on ruling out or confirming a coefficient-independent discrepancy in the muon lifetime. This means that actual measurements of the muon lifetime are required. Should the discrepancy be confirmed, then one can start to determine what kind of coefficients are
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responsible. In that case, results in this thesis can help to define relevant observables. For example, we mentioned in Section 6.6 that an analysis of the Michel parameter $g$ in polarized-muon decay could give results for $b^T$ in the order of $10^{-4}$-$10^{-5}$ GeV. This is comparable to the value obtained in Eq. (6.81), which amounts to $b^T \sim 10^{-5}$ GeV.

Next to the experimental work that can be done to improve on the results obtained in this thesis, there are also some theoretical goals to pursue that will advance the field of testing Lorentz symmetry. One important issue in this category is the rigorous derivation of decay-rate and cross-section master formulas. We took Eq. (3.62) as a working hypothesis for decay rates in this thesis. This is motivated by other less rigorous derivations, such as in Ref. [3]. However, analyses of the LSZ formalism and the optical theorem for Feynman diagrams (the cutting equation) should also be performed in the Lorentz-violating case. These issues are to date only treated for $\epsilon_{\mu\nu}$-like parameters. As described in Chapter 3, these parameters do not break the degeneracy of the energies of the different particle and antiparticle states. The LSZ formalism and cutting equation should also be investigated for other SME parameters. In this respect one could start with $a^\mu$ and $d^{\mu\nu}$ parameters, since the former breaks CPT and not the spin degeneracy while the latter does the opposite.
Appendix A

Sun-centered inertial reference frame

When dealing with bounds on Lorentz-symmetry breaking one cannot directly compare limits obtained in different frames. This is obvious since a preferred direction might point in one direction in one frame and point in a different direction in another. The coefficients can have vastly different sizes in frames boosted relative to each other. It is therefore convenient to decide on a standard frame to report all bounds in. It is advantageous to choose a frame that has an approximately fixed orientation and velocity in space, so one does not have to attach the time of data taking to the reported limit. Furthermore, it is more likely that a Lorentz-violating tensor has constant values in a frame that is fixed with respect to the universe, than in a frame that is fixed with respect to the Earth.

The customary choice of such a frame has become the Sun-centered celestial equatorial frame (see Fig. A.1 and Refs. [25, 15]). The origin of this frame is located at the center of the Sun. The $\hat{Z}$ axis is aligned along the rotation axis of the Earth, the $\hat{X}$ axis points in the direction of the vernal equinox on the celestial sphere, while the $\hat{Y}$ axis completes the right-handed coordinate system. The orientation of the Sun-centered frame is actually not entirely fixed in space, due to precession of the Earth's rotation axis and nutation. However, for all practical purposes this frame is stationary enough.

There is potential for confusion about the direction of the $\hat{X}$ axis (and consequently the $\hat{Y}$ axis). This is because in astronomy the term vernal equinox is used in two different ways. It is a moment in time, namely the moment that the Sun passes the equator from south to north, but it also a spot on the celestial sphere, namely the spot where the Sun appears to be at the previously mentioned time. The time $T = 0$ is defined as the vernal equinox in the year 2000. Notice that even though the $\hat{X}$ axis points in the direction of the vernal equinox on the celestial sphere, the Earth is on the negative $\hat{X}$ axis at $T = 0$, i.e. the time of the vernal equinox. This is due to the two different meanings of the term vernal equinox.

To be able to express experimental results in terms of Lorentz-violating coefficients defined in the Sun-centered frame, we will have to transform quantities from the laboratory frame to the Sun-centered frame. If the coordinate system in the laboratory frame is defined with the $\hat{z}$ axis pointing perpendicular to the Earth's surface, the $\hat{x}$ axis pointing south and the $\hat{y}$ axis pointing east (see Fig. A.2), the transformation can be achieved with
the rotation matrix

\[
R(\zeta, t) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \zeta & 0 & -\sin \zeta \\
0 & 0 & 1 & 0 \\
0 & \sin \zeta & 0 & \cos \zeta
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \Omega t & \sin \Omega t & 0 \\
0 & -\sin \Omega t & \cos \Omega t & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \zeta \cos \Omega t & \cos \zeta \sin \Omega t & -\sin \zeta \\
0 & -\sin \Omega t & \cos \Omega t & 0 \\
0 & \sin \zeta \cos \Omega t & \sin \zeta \sin \Omega t & \cos \zeta
\end{pmatrix},
\]  

(A.1)

where \(\zeta\) is the colatitude of the experiment and \(\Omega \approx 2\pi/(23h 56m)\) is the Earth’s sidereal rotation frequency.

Since the Earth also moves with a nonzero velocity through the Sun-centered frame we should in principle also apply a Lorentz transformation containing the velocity of the laboratory, given by [25]

\[
\beta = \beta_\oplus \begin{pmatrix}
\sin \omega \\
-\cos \eta \cos \omega t \\
-\sin \eta \cos \omega t
\end{pmatrix} + \beta_L \begin{pmatrix}
-\sin \Omega t \\
\cos \Omega t \\
0
\end{pmatrix},
\]

(A.2)

where \(\beta_\oplus\) and \(\omega\) are, respectively, the speed and angular frequency of the Earth’s motion around the Sun. Furthermore, \(\beta_L\) is the speed of the laboratory because of the rotation of the Earth and \(\eta\) is the tilt of Earth’s axis with respect to its orbit around the Sun. Since \(\beta_\oplus \approx 1 \times 10^{-4}\) and \(\beta_L \lesssim 1.5 \times 10^{-6}\) these boosts are irrelevant in almost all cases.
Figure A.2: The lab frame with \((\hat{x}, \hat{y}, \hat{z})\)-coordinates and the Sun-centered frame with \((\hat{X}, \hat{Y}, \hat{Z})\)-coordinates. For the latter, we moved the origin to the center of the Earth, which does not matter if we neglect the velocity of the Earth and assume translational invariance.
Appendix B

$u^\alpha(p)\bar{u}^\alpha(p)$ and $v^\alpha(p)\bar{v}^\alpha(p)$

In this chapter we derive some relations that can be used to obtain explicit expressions for $u^\alpha(p)\bar{u}^\alpha(p)$ and $v^\alpha(p)\bar{v}^\alpha(p)$. As a starting point we take Eqs. (2.53) and (2.54), for the case of nondegenerate and degenerate roots of the dispersion relation respectively. We derive explicit expressions for the numerator and the denominator in these equations. Subsequently we give two examples of Lorentz-violating expressions for $u^\alpha(p)\bar{u}^\alpha(p)$ and $v^\alpha(p)\bar{v}^\alpha(p)$. One of these will explicitly show the cancellation between numerator and denominator of the factors involving degenerate eigenvalues of the Hamiltonian.

B.1 General case

We start with the numerator, i.e. the adjugate matrix of $\tilde{\Gamma} \cdot p - \tilde{M}$. We use that a general $4 \times 4$ matrix $X$ can be written in terms of the sixteen independent combinations of gamma matrices $\{1, \gamma^5, \gamma^\mu, \gamma^5\gamma^\mu, \sigma^\mu\nu\}$ like

$$X = S + i\gamma^5 P + V^\mu \gamma_\mu + A^\mu \gamma^5 \gamma_\mu + T^{\mu\nu} \sigma_{\mu\nu},$$  \hspace{1cm} (B.1)

where, comparing Eq. (B.1) to $\tilde{\Gamma} \cdot p \mp \tilde{M}$, we determine that

$$S = \mathcal{E} \cdot p \mp \mathcal{m},$$  \hspace{1cm} (B.2a)
$$P = \mathcal{F} \cdot p,$$  \hspace{1cm} (B.2b)
$$V^\mu = p^\mu + C^{\mu\nu} p_\nu \mp A^\mu,$$  \hspace{1cm} (B.2c)
$$A^\mu = D^{\mu\nu} p_\nu \mp B^\mu,$$  \hspace{1cm} (B.2d)
$$T^{\mu\nu} = \frac{1}{2} (G^{\mu\nu} p_\rho \mp \mathcal{H}^{\mu\nu}).$$  \hspace{1cm} (B.2e)

As noted in Eq. (2.55) the adjugate matrix of $X$ can be written like

$$\text{Adj}(X) = \left(\tilde{S} - \tilde{P} \gamma^5 - \tilde{V}_\mu \gamma^\mu\right) \left(S + i\gamma^5 P + V_\mu \gamma^\mu - A_\mu \gamma^5 \gamma^\mu - T_{\mu\nu} \sigma^{\mu\nu}\right),$$  \hspace{1cm} (B.3)

with

$$\tilde{S} = S^2 - P^2 + V^2 + A^2 - 2T^2,$$  \hspace{1cm} (B.4a)
$$\tilde{P} = 2 \left(iPS - V \cdot A - iT \cdot \tilde{T}\right),$$  \hspace{1cm} (B.4b)
$$\tilde{V}_\mu = 2 \left(SV_\mu + iPA_\mu - 2iT^{\mu\nu} V_\nu + 2\tilde{T}_{\mu\nu} A_\nu\right),$$  \hspace{1cm} (B.4c)
and $\tilde{T}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} T_{\rho\sigma}$. It is then tedious, but straightforward, to determine that the adjugate matrix of $X$ can be written like

$$\text{Adj} (X) = \hat{S} + i \gamma^5 \hat{P} + \hat{V}^{\mu} \gamma_\mu + \hat{A}^{\mu} \gamma^5 \gamma_\mu + \tilde{T}^{\mu\nu} \sigma_{\mu\nu},$$

(B.5)

where

$$\hat{S} = (S^2 + P^2 - V^2 + \lambda^2 - 2T^2) S - 2T \cdot \tilde{T} P + 4\tilde{T}^{\mu\nu} A_\mu V_\nu, \quad (B.6)$$

$$\hat{P} = (V^2 - S^2 - P^2 - A^2 - 2T^2) P + 2T \cdot \tilde{T} S + 4\tilde{T}^{\mu\nu} A_\mu V_\nu, \quad (B.7)$$

$$\hat{V}^{\mu} = (V^2 - S^2 - P^2 + A^2 - 2T^2) V^{\mu} - 2(V \cdot A + iT \cdot \tilde{T}) A^{\mu},$$

$$+ 4(P A_\nu - 2T_{\nu \rho} V^\rho - 2i\tilde{T}^{\nu \rho} A^{\rho}) T^{\mu\nu} - 4ST^{\mu\nu} A_\nu, \quad (B.8)$$

$$\hat{A}^{\mu} = (2T^2 - S^2 - P^2 - V^2 - A^2) A^{\mu} + 2(V \cdot A + iT \cdot \tilde{T}) V^{\mu}$$

$$- 4(SV_\nu + 2T_{\nu \rho} V^{\rho} - 2iT_{\nu \rho} V^{\rho}) T^{\mu\nu} + 4PT^{\mu\nu} V_\nu, \quad (B.9)$$

$$\tilde{T}^{\mu\nu} = (2T^2 + P^2 + A^2 - V^2 - S^2) T^{\mu\nu} + i(PS - V \cdot A - iT \cdot \tilde{T}) T^{\mu\nu}$$

$$+ P(A^{\nu} V^{\mu} - A^{\mu} V^{\nu}) + 2(T^{\mu\nu} V^{\rho} - T^{\mu\rho} V^{\nu}) V_\rho + 2i(T^{\mu\nu} V^{\rho} - T^{\mu\rho} V^{\nu}) A_\rho$$

$$- (SV_\rho - 2iT_{\rho \sigma} V^{\sigma}) A_\lambda \epsilon^{\lambda \mu \nu} - 2(T^{\rho \mu} A_\rho A^{\mu} + T^{\mu \rho} A_\rho A^{\rho} \gamma^{\nu} \gamma_\nu). \quad (B.10)$$

We can now in principle determine the adjugate matrix of $\bar{\Gamma} \cdot p \mp \bar{M}$.

To obtain expressions for $u^\alpha(p)\bar{a}^\alpha(p)$ and $v^\alpha(p)\bar{v}^\alpha(p)$ we also need the denominator in Eq. (2.53). We can obtain it by using the explicit expressions for the roots of the dispersion relation, taken from Ref. [36]. An alternative is provided by noticing that the denominator of Eq. (2.53a) is given by

$$\frac{d}{d \lambda^0} \det(\lambda^0 - H) \bigg|_{\lambda_0 = E^0_\lambda, \lambda = p} = \frac{d}{d \lambda^0} \left[ (\lambda^0 - E^1_a(\lambda))(\lambda^0 - E^2_a(\lambda))(\lambda^0 + E^1_a(-\lambda))(\lambda^0 + E^2_a(-\lambda)) \right] \bigg|_{\lambda_0 = E^0_\lambda, \lambda = p}, \quad (B.11)$$

Using Jacobi’s formula, which is given by

$$\frac{d}{dt} \det(X) = \text{Tr} \left( \text{Adj}(X) \frac{dX}{dt} \right), \quad (B.12)$$

we derive that the denominator of Eq. (2.53a) is given by

$$\text{Tr} \left( \text{Adj} \left( \bar{\Gamma} \cdot \lambda - \bar{M} \right) \gamma^0 \right) \bigg|_{\lambda_0 = E^0_\lambda, \lambda = p} = 4\hat{V}^0, \quad (B.13)$$

where it is understood that the upper signs should be used in Eqs. (B.2) to determine $\hat{V}^0$. The denominator of Eq. (2.53b) is obtained from

$$\text{Tr} \left( \text{Adj} \left( \bar{\Gamma} \cdot \lambda + \bar{M} \right) \gamma^0 \right) \bigg|_{\lambda_0 = E^0_\lambda, \lambda = p} = 4\hat{V}^0, \quad (B.14)$$

where now the lower signs in Eqs (B.2) should be used to determine $\hat{V}^0$. 

In the denominator of \( u^\alpha(p) \bar{u}^\alpha(p) \) for the case of degenerate roots of the dispersion relation (i.e. the denominator in Eqs. (2.54)) a factor \((\lambda^0 - E_u^\alpha(\lambda))\) is canceled with the numerator. However just using Eq. (B.13) and canceling the appropriate factor with the numerator, would produce an extra factor 2, as can easily be verified by calculating the derivative in Eq. (B.11) and removing a factor \((\lambda^0 - E_u^\alpha(\lambda))\) before putting \(\lambda_0 = E_u^\alpha(\lambda)\).

So in the case of two degenerate eigenvalues of \(H\), we can now get explicit expressions for \(u^\alpha(p) \bar{u}^\alpha(p)\) and \(v^\alpha(p) \bar{v}^\alpha(p)\), although the expressions can become quite unwieldy in practice. As an example we treat the case where only \(c_{\mu\nu}\) is nonzero. The dispersion relation then has two degenerate roots, so the vanishing factor \(E_u^1 - E_u^2\) has to cancel between the numerator and the denominator and we will obtain the sum over \(\alpha\), as in Eq. (2.54). Using Eqs. (B.5), (B.10), and (B.2) we find that

\[
\sum_{\alpha=1}^{2} u^\alpha(p) \bar{u}^\alpha(p) = \frac{\hat{S} + \hat{V}}{2V^0} = \frac{(\hat{p}^2 - \hat{m}^2)(\hat{p} + \hat{m})}{2(\hat{p}^2 - \hat{m}^2)p^0} = \frac{\hat{p} + \hat{m}}{2p^0},
\]

\[
\sum_{\alpha=1}^{2} v^\alpha(p) \bar{v}^\alpha(p) = \frac{\hat{S} + \hat{V}}{2V^0} = \frac{(\hat{p}^2 - \hat{m}^2)(\hat{p} - \hat{m})}{2(\hat{p}^2 - \hat{m}^2)p^0} = \frac{\hat{p} - \hat{m}}{2p^0},
\]

where \(\hat{p}^\mu = p^\mu + C^{\mu\nu}p_\nu\). In the last equality we canceled the factors in the numerator and denominator that vanish if we put \(p^0\) to its physical value. When we put the Lorentz violation to zero we obtain the conventional result for the projection operator.

We can ask what the expression for \(u^\alpha(p) \bar{u}^\alpha(p)\) is, when not summing over polarization. It is well known that for the Lorentz symmetric case (in our normalization) this is given by

\[
u^\alpha(p) \bar{u}^\alpha(p) = \frac{\hat{p} + \gamma\hat{\mu}}{4p^0}(1 + \gamma\hat{\sigma})
\]

with

\[
s_\mu = \left( \frac{p \cdot \hat{s}_\alpha}{m}, \hat{s}_\alpha + \frac{(p \cdot \hat{s}_\alpha)p}{m(m + p^0)} \right),
\]

and \(\hat{s}_\alpha\) denoting the unit spin vector in the restframe of the particle. Notice that \(s_\alpha^2 = -1\) and \(p \cdot s_\alpha = 0\). In the Lorentz-violating case where we have only a nonzero \(c^{\mu\nu}\), we can derive something similar. First we write the Dirac equation in block-matrix form, with the Dirac matrices in the Weyl basis, as

\[
\begin{pmatrix}
-\hat{m} & \hat{p} \cdot \sigma \\
\hat{p} \cdot \sigma & -\hat{m}
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix} = 0.
\]

As is the case without Lorentz violation, if we only have a \(c^{\mu\nu}\) coefficient, the Dirac equation only determines \(\phi_2\) in terms of \(\phi_1\) (or vice versa), but not the form of \(\phi_1\) itself.
In other words, there is nothing that couples to the spin of the particle\(^1\). This is different for Lorentz-violating parameters that lift the degeneracy of the energy eigenvalues of the two spin states. We will see an example of this later on in this section, when we treat the case of a nonzero \(d^{\mu \nu}\) coefficient. In the present case we are free to chose the form of \(\phi_1\). We take it to satisfy the equation

\[
V \cdot \sigma \phi_1 = 0 \, , \quad \text{with} \quad V^2 = 0 \, .
\]  

(B.19)

We take \(V\) to be

\[
V^\mu_\alpha = \frac{\tilde{p}^\mu - \tilde{m} \tilde{s}^\mu}{m} \, ,
\]

(B.20)

with

\[
\tilde{s}^\mu_\alpha = \left( \frac{\tilde{p} \cdot \tilde{s}_\alpha}{\tilde{m}} , \tilde{s}_\alpha + \frac{\tilde{p} \cdot \tilde{s}_\alpha \tilde{p}}{\tilde{m}(\tilde{m} + \tilde{p}^0)} \right) \, .
\]

(B.21)

In the frame where \(p = 0\), we have that \(\tilde{s}^2 = -1\) and \(\tilde{p} \cdot \tilde{s} = 0\). Using the fact that \(\phi_0^\dagger = V \cdot \sigma\) we can explicitly calculate \(u^\alpha(p)\bar{u}^\alpha(p)\). It is given by

\[
u^\alpha(p)\bar{u}^\alpha(p) = \frac{\tilde{p} + \tilde{m}}{4\tilde{p}^0} (1 + \gamma^5 \tilde{s}_\alpha) \, .
\]

(B.22)

Summing over \(\alpha\) will give the result in Eq. (B.15).

Finally we treat the case were both \(c^{\mu \nu}\) and \(d^{\mu \nu}\) are nonzero. We introduce the new notation \(\tilde{p}^\mu = p^\mu + (C^{\mu \nu} + D^{\mu \nu})p_\nu\) and \(\tilde{p}^\mu = p^\mu + (C^{\mu \nu} - D^{\mu \nu})p_\nu\). The expressions for \(u^\alpha(p)\bar{u}^\alpha(p)\) and \(v^\alpha(p)\bar{v}^\alpha(p)\) then become

\[
u^\alpha(p)\bar{v}^\alpha(p) = \left[ \frac{1}{2}(\tilde{p}^2 + \tilde{p}^2) + \tilde{m}^2 - m^2d^{00}d_{00} + \frac{i}{2}(\tilde{p}^2 - \tilde{p}^2)\gamma^5 + \tilde{m}(\tilde{p} + \tilde{p}) + i\epsilon_{0\lambda\mu\nu}d_{0\lambda}(\tilde{p}_\nu + \tilde{p}_\nu)\gamma_\mu \right.
\]

\[
\times \left[ \frac{1}{2}(\tilde{p} + \tilde{p}) - \frac{1}{2}\gamma^5(\tilde{p} - \tilde{p}) - \tilde{m} + \frac{\tilde{p}}{4\tilde{p}^0}(\tilde{p}^0 + \tilde{p}^0) - (\tilde{p}^2 - \tilde{p}^2)(\tilde{p}^0 - \tilde{p}^0) \right]
\]

\[
+ 4m\tilde{m}(d^{00}(\tilde{p}_\nu + \tilde{p}_\nu) - d^{00}(\tilde{p}^0 - \tilde{p}^0))^{-1} \, ,
\]

(B.23a)

\[
u^\alpha(p)\bar{v}^\alpha(p) = \left[ \frac{1}{2}(\tilde{p}^2 + \tilde{p}^2) + \tilde{m}^2 - m^2d^{00}d_{00} + \frac{i}{2}(\tilde{p}^2 - \tilde{p}^2)\gamma^5 - \tilde{m}(\tilde{p} + \tilde{p}) + i\epsilon_{0\lambda\mu\nu}d_{0\lambda}(\tilde{p}_\nu + \tilde{p}_\nu)\gamma_\mu \right.
\]

\[
\times \left[ \frac{1}{2}(\tilde{p} + \tilde{p}) - \frac{1}{2}\gamma^5(\tilde{p} - \tilde{p}) + \tilde{m} - \frac{\tilde{p}}{4\tilde{p}^0}(\tilde{p}^0 + \tilde{p}^0) - (\tilde{p}^2 - \tilde{p}^2)(\tilde{p}^0 - \tilde{p}^0) \right]
\]

\[
- 4m\tilde{m}(d^{00}(\tilde{p}_\nu - \tilde{p}_\nu) - d^{00}(\tilde{p}^0 - \tilde{p}^0))^{-1} \, .
\]

(B.23b)

In this case the expressions for the operators \(u^\alpha(p)\bar{u}^\alpha(p)\) and \(v^\alpha(p)\bar{v}^\alpha(p)\) are thus determined completely by the Dirac equation and there is no freedom left for a spin direction, as is the case when only \(c^{\mu \nu}\) is nonzero. If we put \(d^{\mu \nu}\) to zero, we have \(\tilde{p}^\mu = p^\mu + C^{\mu \nu}p_\nu\) and after summing over \(\alpha\) we obtain exactly the result in Eqs. (B.15). Notice that we do not obtain Eq. (B.22) when putting \(d^{\mu \nu}\) to zero.

\(^1\)However, this does not mean that \(c^{\mu \nu}\) does not influence the polarization if there would be a magnetic field present for example.
Appendix C

Observer Lorentz covariant integral with $b^\mu$

In Eq. (6.58) in Section 6.6 we used the result for an integral over neutrino momenta in the case where one of the neutrinos included Lorentz violation in the form of a nonzero $b^\mu$ parameter. In this section we derive this result. The integral is given by

$$I^{\mu\nu} = \int \frac{d^3p}{2E_p} \int \frac{d^3k}{N_{\pm k}^2} p^\mu \tilde{k}^\nu \delta(4)(\tilde{q} - p - \tilde{k}) ,$$  \hspace{1cm} (C.1)

with $\tilde{k}^\mu = k^\mu \mp b^\mu$, $E_p = |p|$, $N_{\pm k}^2 = 2|k \mp b| = \lambda_0^\pm (\pm k) - \lambda_0^\mp (\pm k)$, and $\lambda_0^\pm$ and $\lambda_0^\mp$ are the positive and negative solutions of the dispersion relation before redefinition of the negative-energy solutions. The upper sign refers to neutrinos, while the lower sign refers to antineutrinos. The integral in Eq. (C.1) is an observer Lorentz covariant two-tensor in concordant frames, as can be verified by writing for example

$$\int \frac{d^3k}{N_{\pm k}^2} = \int d^4k \delta(\tilde{k}^2)\Theta(\tilde{k}^0) . \hspace{1cm} (C.2)$$

Using the observer Lorentz covariance of $I^{\mu\nu}$ we write

$$I^{\mu\nu} = (A_1 + \epsilon_1) q^{\mu\nu} + (A_2 + \epsilon_2) q^\mu q^\nu + A_3 q^\mu b^\nu + A_4 q^\nu b^\mu , \hspace{1cm} (C.3)$$

which is its most general form, to first order in Lorentz-violating coefficients ($A_1$-$A_4$ are zeroth order in Lorentz violation, while $\epsilon_1$ and $\epsilon_2$ are first order in Lorentz violation). We could determine the values for these coefficients by contracting both sides of the equation with each of these terms. However, there is an easier way in this case. To see this we write

$$I^{\mu\nu} = \frac{d^3p}{2E_p} \int d^4k \ p^\mu \tilde{k}^\nu \delta(\tilde{k}^2) \delta(\tilde{k}^0) \delta(4)(\tilde{q} - p - \tilde{k}) , \hspace{1cm} (C.4)$$

where $\tilde{q}^\mu = q^\mu \mp b^\mu$. We can then do a linear shift on the integration variable $k$, i.e. $k \to \tilde{k}$, which, after renaming the integration variable to $k$ again and performing the integral of $k^0$, gives us

$$I^{\mu\nu} = \frac{d^3p}{2E_p} \int \frac{d^3k}{2E_k} p^\mu k^\nu \delta(4)(\tilde{q} - p - k) . \hspace{1cm} (C.5)$$
This has exactly the same form as the Lorentz-symmetric case, except \( q \rightarrow \tilde{q} \). Using known results (see e.g. [64]), we can thus immediately write

\[
I^{\mu\nu} = \frac{\pi}{24} \left( \tilde{q}^2 g^{\mu\nu} + 2\tilde{q}^\mu \tilde{q}^\nu \right) \Theta(\tilde{q}^2) \Theta(\tilde{q}^0) .
\]  
\text{(C.6)}
Appendix D

Nuclear beta decay

D.1 Lepton currents

Omitting the factor $F(E_e, \pm Z)$ and the subscript $\pm$ on $P_\pm$, the different combinations of lepton currents for allowed $\beta$ decay in Eq. (4.21) are given by

$$\frac{1}{8}|J^0|^2 = P^0 \left( \frac{1}{2} (1 + 2 \chi_0^0) k^0 - \chi_0^0 k^0 \right) - P^l \left( -\frac{1}{2} (1 + 2 \chi_0^0) k^l - \chi_0^0 k^0 + \chi^0_{ij} \epsilon_{jml} k^m \right), \tag{D.1}$$

$$\frac{1}{8}(|J^+|^2 - |J^-|^2) = \pm P^0 \left( (1 - \chi_0^0) (k \cdot \hat{I}) + \chi_i^0 (k \times \hat{I}) - k^0 (\hat{I} \cdot \hat{m}) + \chi_i^m k^l \hat{I}^m \right)$$
$$+ P^l \left( (1 - \chi_0^0) k^l \hat{I}^l - k^0 \epsilon_{jml} \chi_i^0 \hat{I}^m + \chi_i^m k^0 \hat{I}^l - k^l (\hat{I} \cdot \hat{m}) - \chi_i^0 (k \cdot \hat{I}) + \chi_i^m (k \times \hat{I}) \right) \tag{D.2}$$

$$\frac{1}{8}(|J^+|^2 + |J^-|^2) = P^0 \left( (3 \chi_i^{lm} \hat{I}^m - \chi_0^0) k^0 + (\chi_i^{lm} - 3 \chi_i^{l0} \hat{I}^m \hat{I} - \hat{I}^l) k^l + 3 \chi_i^{lm} (k \times \hat{I})^l \right)$$
$$- P^l \left( 3 (k \cdot \hat{I}) \hat{I}^l - (\chi_i^{lm} + \chi_i^{lm} \epsilon_{jkl} \hat{I}^m \hat{I}^k) k^0 \right)$$
$$- (1 + \chi_i^{lm} - 3 \chi_i^{lm} \hat{I}^m \hat{I}^l) k^l + 3 \chi_i^{lm} (k \times \hat{I})^l$$
$$+ (\chi_i^{lm} + \chi_i^{lm} \epsilon_{jlm} \hat{I}^m \hat{I}^l - 3 \chi_i^{lm} \hat{I}^m \hat{I}^l) k^m \right), \tag{D.3}$$

$$\frac{1}{8}(|J^0 J^+| + |J^0 J^-|) = P^0 \left( (1 + \chi_0^0) (k \cdot \hat{I}) + (\chi_i^{lm} - \chi_i^{0m}) \hat{I}^m k^l - \chi_i^{lm} k^m \hat{I}^l - \chi_i^{0l} (k \times \hat{I})^l \right)$$
$$- P^l \left( - (1 + \chi_0^0) \hat{I}^l k^l + \chi_i^{0l} (k \cdot \hat{I}) - (\chi_i^{lm} + \chi_i^{lm} \epsilon_{jkl} \hat{I}^m \hat{I}^k) k^0 \right)$$
$$+ \chi_i^{lm} k^l \hat{I}^m - \chi_i^{0l} \epsilon_{lml} \hat{I}^m k^l + \chi_i^{jm} \epsilon_{jkl} k^m \hat{I}^l \right), \tag{D.4}$$
where the subscripts \( r \) and \( i \) label the real and imaginary parts of \( \chi^{\mu\nu} \), while \( \chi^j = \epsilon^{imk} \chi^{mk} \), the upper (lower) signs in Eq. (D.3) refer to the case of the electron (positron), the Latin indices run over the spatial components 1, 2, 3, and summation over repeated indices is understood.

D.2 Experimental sensitivities for allowed \( \beta \) decay

In Table D.1 we give an overview of the observability of the Lorentz-violating parameter \( \chi^{\mu\nu} \) in the different types of \( \beta \)-decay transitions, Fermi, Gamow-Teller, or mixed. Although Table D.1 contains the same information as Eqs. (4.29) and (4.30), it provides a clearer overview of which Lorentz-violating parameters are accessible in which kind of experiments.

<table>
<thead>
<tr>
<th>t</th>
<th>( \chi^0_0 )</th>
<th>( \chi^0_l )</th>
<th>( \chi^l_0 )</th>
<th>( \chi^m_0 )</th>
<th>( \chi^m_l )</th>
<th>( \chi^m_i )</th>
<th>( \chi^0_0 )</th>
<th>( \chi^0_l )</th>
<th>( \chi^l_0 )</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_1 )</td>
<td>( \hat{k} )</td>
<td>F</td>
<td>GT</td>
<td>GT</td>
<td>( \chi_i^{(ml)} ) not accessible.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( w_2 )</td>
<td>( \hat{l} )</td>
<td>M</td>
<td>GT</td>
<td>M</td>
<td>If ( \chi_i^{\mu\nu} = 0 ), ( \chi^0_{ik} ) cancels ( \chi^0_{li} ).</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( w_3 )</td>
<td>( \hat{p} )</td>
<td>F</td>
<td>GT</td>
<td>GT</td>
<td>( \chi_i^{(ml)} ) not accessible.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( T_1 )</td>
<td>( \hat{l}^* )</td>
<td>GT</td>
<td></td>
<td></td>
<td>Vanishes for ( j = \frac{1}{2} ).</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( T_2 )</td>
<td>( \hat{l}, \hat{k} )</td>
<td>GT</td>
<td>GT</td>
<td>M</td>
<td>GT</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( T_3 )</td>
<td>( \hat{p}, \hat{k} )</td>
<td>X</td>
<td>GT</td>
<td>F</td>
<td>GT</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( T_4 )</td>
<td>( \hat{p}, \hat{l} )</td>
<td>GT</td>
<td>GT</td>
<td>M</td>
<td>GT</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( S_1 )</td>
<td>( \hat{l}^*, \hat{k} )</td>
<td>GT</td>
<td>GT</td>
<td></td>
<td>Vanishes for ( j = \frac{1}{2} ).</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( S_2 )</td>
<td>( \hat{l}^*, \hat{p} )</td>
<td>GT</td>
<td>GT</td>
<td></td>
<td>Vanishes for ( j = \frac{1}{2} ).</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( S_3 )</td>
<td>( \hat{p}, \hat{l}, \hat{k} )</td>
<td>M</td>
<td>GT</td>
<td>M</td>
<td>GT</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( R )</td>
<td>( \hat{p}, \hat{l}^*, \hat{k} )</td>
<td>GT</td>
<td>GT</td>
<td></td>
<td>Vanishes for ( j = \frac{1}{2} ).</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table D.1: Observability of the Lorentz-violating parameter \( \chi^{\mu\nu} \) in different \( \beta \)-decay transitions.

On the top line of Table D.1 are the different parts of \( \chi^{\mu\nu} \) that are independent under the action of rotations, in particular the rotation represented by the matrix in Eq. (2.5). If we assume that \( \chi^{\mu\nu} = \chi^{\nu\mu} \), the last three columns, corresponding to \( \chi^0_0 \), \( \chi^0_l \), and \( \chi^l_0 \), are redundant. The first of these three columns should then be removed, since \( \chi^0_0 = 0 \) if we assume \( \chi^{\mu\nu} = \chi^{\nu\mu} \). In addition, the entries in the columns of \( \chi^l_0 \) and \( \chi^0_l \) should be added to the columns of \( \chi^0_0 \) and \( \chi^0_l \), respectively.

In the left-most column the Lorentz-violating quantities which are defined in Eq. (4.30) are listed. In the second column we give the minimal directional information one needs to be able to see effects of the corresponding Lorentz-violating vector or tensor. Notice that the polarization of the \( \beta \) particle does not appear. This is because it can be seen from Eq. (4.29) that by measuring the electron direction one has access to the same Lorentz-violating observables as by measuring the polarization of the \( \beta \) particle. An asterisk on \( \hat{I} \) means that it occurs twice in the corresponding term in Eq. (4.29), and therefore it does
not change sign if one flips the nuclear spin. The entries in the Table show what is the relevant $\beta$-decay transition. F means that the component of $\chi^{\mu\nu}$ occurs in the Lorentz-violating quantity on the left in a Fermi or a mixed transition. GT has the analogous meaning for a Gamow-Teller or mixed transition. X means that the part of $\chi^{\mu\nu}$ on the top of the column shows up in the quantity in the first column in every kind of transition, while an M signals that it only is visible in a mixed transition. Finally, we used the notations $\chi^{(\mu\nu)} = \frac{1}{2}(\chi^{\mu\nu} + \chi^{\nu\mu})$ and $\chi^{[\mu\nu]} = \frac{1}{2}(\chi^{\mu\nu} - \chi^{\nu\mu})$ in the right-most Comments column.

D.3 Order of magnitude of the $\beta$ moments

The table in this appendix connects to Chapter 5 on forbidden $\beta$ decay. It summarizes which $\beta$ moments give the dominant contribution to the amplitude. As described in Section 5.5, we use the ‘normal approximation’ and the $\xi$ approximation. The first column in the table denotes the degree of forbiddenness, while the last column specifies what the order of magnitude of the contribution to the amplitude is. To make this order of magnitude a dimensionless quantity, factors of $R$ should be multiplied by factors of $q$ or $p$, depending on if $R$ originates from the wavefunction of the neutrino or of the $\beta$ particle. If there is a cross in the column labeled by LS, the corresponding term also contributes in the Lorentz-symmetric case. An asterisk in the second column means that there is no contribution for a $0 \rightarrow 0$ transition. Two asterisks signal that the term neither contributes to a $0 \rightarrow 0$ transition, nor to a $\frac{1}{2} \rightarrow \frac{1}{2}$ transition, nor to a $0 \leftrightarrow 1$ transition. To clarify from which term in Eq. (5.42) the contributions originate, we included factors of $X_{J\pm 1}^0$, $X_{J\pm 1}^1$, $Z^0$, and $Z^1$ in the $\beta$-moments column.
<table>
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<th>D.o.f.</th>
<th>$\Delta I^{n, \pi_f}$</th>
<th>$\beta$ moment</th>
<th>$\chi$-components</th>
<th>$j + j$</th>
<th>$J$</th>
<th>LS</th>
<th>O.o.m.</th>
</tr>
</thead>
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<td>$0^+$</td>
<td>$X^{0}_0 C_V (1)$</td>
<td>$\chi^{0l}$</td>
<td>1</td>
<td>1</td>
<td>$O(\alpha R^0)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$Z^{0}_0 C_V (1)$</td>
<td>$\chi^{0l}$</td>
<td>1</td>
<td>0</td>
<td>$O(\alpha R^0)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$X^{1}_1 C_A \langle \sigma \rangle$</td>
<td>$\chi^{ml}$</td>
<td>1</td>
<td>1</td>
<td>$O(\alpha R^0)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$Z^{1}_1 C_A \langle \sigma \rangle$</td>
<td>$\chi^{00}, \chi^{ml}$</td>
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<td>0</td>
<td>$O(\alpha R^0)$</td>
<td></td>
</tr>
<tr>
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<td>$0^-$</td>
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<td>$\chi^{ml}$</td>
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<td>0</td>
<td>$O(\alpha Z)$</td>
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<tr>
<td></td>
<td></td>
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<td>$\chi^{0l}, \chi^{ml}$</td>
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<td>1</td>
<td>$O(\alpha Z)$</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td>$X^{1}_{J} C_A \langle (\sigma \times f) \rangle$</td>
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<td>0</td>
<td>$O(\alpha Z)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$Z^{1}_1 C_A \langle (\sigma \cdot T^2_1) \rangle$</td>
<td>$\chi^{0l}, \chi^{ml}$</td>
<td>1</td>
<td>1</td>
<td>$O(\alpha Z)$</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td>$Z^{0}_1 C_A \langle (\sigma \cdot T^2_1) \rangle$</td>
<td>$\chi^{0l}, \chi^{ml}$</td>
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<td>1</td>
<td>$O(\alpha Z)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$X^{1}_{J} C_A \langle \alpha \rangle$</td>
<td>$\chi^{ml}$</td>
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<td>1</td>
<td>$O(\alpha Z)$</td>
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<tr>
<td></td>
<td></td>
<td>$Z^{1}_1 C_A \langle \alpha \rangle$</td>
<td>$\chi^{0l}, \chi^{ml}$</td>
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<td>0</td>
<td>$O(\alpha Z)$</td>
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<tr>
<td>2</td>
<td>$2^+$</td>
<td>$X^{1}_{J} C_A \langle (\sigma \cdot T^2_1) \rangle$</td>
<td>$\chi^{ml}$</td>
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<td>2</td>
<td>$O(\alpha Z)$</td>
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</tr>
<tr>
<td></td>
<td></td>
<td>$Z^{1}_1 C_A \langle (\sigma \cdot T^2_1) \rangle$</td>
<td>$\chi^{0l}, \chi^{ml}$</td>
<td>2</td>
<td>2</td>
<td>$O(\alpha Z)$</td>
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<tr>
<td></td>
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<td>$X^{1}_{J} C_A \langle (\sigma \cdot T^2_1) \rangle$</td>
<td>$\chi^{0l}, \chi^{ml}$</td>
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<td>2</td>
<td>$O(\alpha Z)$</td>
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<td>2</td>
<td>$O(\alpha Z)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$Z^{1}_1 C_A \langle (\sigma \cdot T^2_1) \rangle$</td>
<td>$\chi^{0l}, \chi^{ml}$</td>
<td>2</td>
<td>2</td>
<td>$O(\alpha Z)$</td>
<td></td>
</tr>
<tr>
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<td>$n^{(-1)n}$</td>
<td>$X^{1}_{J} C_A \langle (\sigma \cdot T^2_1) \rangle$</td>
<td>$\chi^{ml}$</td>
<td>3</td>
<td>3</td>
<td>$O(\alpha Z\beta)$</td>
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<tr>
<td></td>
<td></td>
<td>$X^{1}_{J} C_A \langle (\sigma \cdot T^2_1) \rangle$</td>
<td>$\chi^{ml}$</td>
<td>1</td>
<td>1</td>
<td>$O(\alpha Z\beta)$</td>
<td></td>
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<td></td>
<td></td>
<td>$X^{1}_{J} C_A \langle (\sigma \cdot T^2_1) \rangle$</td>
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<td>$Z^{1}_1 C_A \langle (\sigma \cdot T^2_1) \rangle$</td>
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<td>$Z^{0}_1 C_A \langle (\sigma \cdot T^2_1) \rangle$</td>
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<td>$O(\alpha ZR^{n-1})$</td>
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<td>$X^{1}<em>{J} C_A \langle (\alpha \cdot T^2</em>{n-1}) \rangle$</td>
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<td>$X^{1}<em>{J} C_A \langle (\sigma \cdot T^2</em>{n+1}) \rangle$</td>
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<td>$X^{1}<em>{J} C_A \langle (\sigma \cdot T^2</em>{n+1}) \rangle$</td>
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<td>$O(\alpha ZR^{n-1})$</td>
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<td>$X^{1}<em>{J} C_A \langle (\sigma \cdot T^2</em>{n+1}) \rangle$</td>
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<td>$Z^{1}<em>1 C_A \langle (\sigma \cdot T^2</em>{n+1}) \rangle$</td>
<td>$\chi^{0l}, \chi^{ml}$</td>
<td>$n$</td>
<td>$n$</td>
<td>$O(\alpha ZR^{n-1})$</td>
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</tbody>
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Table D.2: The order of magnitude of the $\beta$ moments.
Nederlandse samenvatting

Het concept symmetrie is erg belangrijk in de hedendaagse (theoretische) natuurkunde. Het is onmogelijk dit concept hier nauwgezet uit te leggen, of alle natuurkundige soorten van symmetrie te bespreken. Kort en onnauwkeurig gezegd bezit een systeem een bepaalde symmetrie wanneer dat systeem niet te onderscheiden is van het systeem dat ontstaat na een symmetrie-operatie. Een cirkel is bijvoorbeeld draaisymmetrisch, omdat na draaiing om het middelpunt (over willekeurige hoek) geen verschil te zien is met het origineel.

Het belang van symmetrie in de theoretische natuurkunde ligt in het feit dat deze beperkingen opleggen aan de theorieën die we kunnen opschrijven voor de werking van de natuur. Als we van tevoren weten dat de linker- en rechterkant van een verschijnsel uitwisselbaar zijn, dan kunnen we elke theorie die een verschil tussen deze twee voorspelt bij voorbaat afwijzen. En als we niet weten welke vorm we voor ons hebben, maar we weten wel dat deze draaisymmetrisch om zijn middelpunt is, dan hebben we al goede aanwijzingen dat we met een cirkel te maken hebben. In de basis werkt dit ook zo bij het afleiden van natuurwetten. Symmetrie geven richting bij het opstellen van modellen voor de natuur.

Dat we een bepaalde symmetrie opleggen aan een theorie volgt vaak uit experimentele ervaringen en extrapolatie, al dan niet gecombineerd met de drang naar eenvoud of schoonheid. Dit laatste lijkt misschien raar, maar symmetrieën maken modellen vaak eenvoudiger en dit wordt door veel natuurkundigen als aantrekkelijk ervaren (het rekent ook een stuk makkelijker). Hierbij moet gezegd worden dat de geschiedenis leert dat eenvoudige formules vaak de juiste zijn. In het verleden is gebleken dat symmetrieën vaak goede wegwijzers zijn bij het bouwen van modellen in de natuurkunde.

In deze context behoort Lorentzsymmetrie tot een van belangrijkste symmetrieën die er zijn. Lorentzsymmetrie is de essentie van de relativiteitstheorie van Einstein. En die relativiteitstheorie is sinds haar ontstaan uitgegroeid tot een fundament van onze theorieën voor de natuur. Met andere woorden, Lorentzsymmetrie wordt beschouwd als fundamenteel voor ons begrip van de wereld om ons heen.

Om uit te leggen wat Lorentzsymmetrie is, voeren we het begrip inertiaalstelsel in. Een inertiaalstelsel is een coördinaatenstelsel waarin een voorwerp dat met een constante snelheid beweegt, dat blijft doen (in dezelfde richting), tenzij er een kracht op werkt. In een inertiaalstelsel gaat daarnaast een voorwerp dat stillstaat niet uit zichzelf bewegen. Alle inertiaalstelsels bewegen met constante snelheden ten opzichte van elkaar of zijn onder een willekeurig hoek ten opzichte van elkaar gedraaid. Zo is het coördinaatenstelsel van iemand die ergens in de lege ruimte stilhangt een inertiaalstelsel. Maar het coördinaatenstelsel van

\footnote{We gebruiken het woord theorie en model hier als uitwisselbaar.}
iemand die met een constante snelheid langsvliegt in een ruimteschip, is dat ook. Het coördinatenstelsel van iemand in een versnellend of vertragend ruimteschip is echter geen inertiaalstelsel (zie Fig. 7.1). Voorwerpen in dit laatste ruimteschip zullen gaan bewegen ten opzichte van het ruimteschip, terwijl ze dat in de andere twee voorbeelden niet doen. Kortom, in een inertiaalstelsel maken voorwerpen, waar geen kracht op werkt, een rechtlijnige beweging met een constante snelheid, of ze volharden in een toestand van rust.

Een van de hoekstenen van de relativiteitstheorie, het relativiteitsprincipe, zegt nu dat alle inertiaalstelsels equivalent zijn. Equivalent betekent in dit geval dat de natuurwetten in elk inertiaalstelsel er precies hetzelfde uitzien. Als je in een met constante snelheid bewegende trein een experiment uitvoert, dan is de uitkomst van dat experiment hetzelfde als wanneer je het stilstaand langs het spoor zou uitvoeren. Het relativiteitsprincipe wordt ook onderschreven door het gevoel dat je wel eens in een trein hebt wanneer de trein op het spoor ernaast wegrijdt. Je kunt op dat moment soms niet onderscheiden of de trein waarin je zit rijdt, of dat het de andere trein is die beweegt. Verder betekent relativiteit dat ook de absolute oriëntatie van een systeem de natuurkunde niet beïnvloedt. Het maakt voor de uitkomst van een experiment niet uit onder welke hoek het experiment bekeken wordt of wat zijn oriëntatie in de ruimte is. Samengevat houdt het relativiteitsprincipe dus in

\[2\] Dit geldt op aarde natuurlijk niet, vanwege bijvoorbeeld de zwaartekracht. De aarde is in principe geen inertiaalstelsel.
dat de oriëntatie en de constante snelheid van een systeem geen invloed hebben op de natuurwetten in dat systeem.

Het relativiteitsprincipe stamt al uit de tijd van Galileo Galilei (1564-1642), die het beschreef in zijn “Dialogo dei due massimi sistemi del mondo”. Maar pas in 1905 was het Einstein die de verregaande gevolgen van het relativiteitsprincipe beschreef in zijn speciale relativiteitstheorie. Hij kwam tot zijn conclusies door het relativiteitsprincipe te combineren met een experimenteel resultaat uit zijn tijd. Namelijk het feit dat de snelheid van het licht constant is, onafhankelijk van het inertiaalstelsel van waaruit je het bekijkt. Als je erover nadenkt is dit raar. Een auto die met 130 kilometer per uur over de snelweg rijdt, lijkt, vanuit een tegenligger met diezelfde snelheid, een snelheid van 260 kilometer per uur te hebben. Zo werkt het niet voor licht. Een lichtbundel heeft gezien vanuit elke auto en vanuit elk inertiaalstelsel een snelheid van ongeveer een miljard kilometer per uur. Dit volgt ook uit de zogenaamde Maxwellvergelijkingen, die het gedrag van elektromagnetische velden en golven (licht is een elektromagnetische golf) beschrijven. Uit deze vergelijkingen kan de snelheid van licht afgeleid worden. Het relativiteitsprincipe zegt dat de Maxwellvergelijkingen in elk inertiaalstelsel moeten gelden en dus dat de lichtsnelheid in elk van die stelsels gelijk moet zijn.

Einstein kwam door logisch redeneren tot de conclusie dat de paradox, veroorzaakt door combinatie van de constante lichtsnelheid en het relativiteitsprincipe, opgelost kon worden door het optellen van snelheden anders te gaan doen. Met andere woorden, hij ontdekte dat om de coördinaten van het ene inertiaalstelsel af te leiden uit de coördinaten van een ander inertiaalstelsel, niet de Galileitransformaties, maar de Lorentztransformaties gebruikt moeten worden. Met formules worden deze twee coördinatentransformaties beschreven als (zie ook Fig. 7.2):

\[
\begin{align*}
\text{Galileitransformatie} & : \\
\quad t' &= t \\
\quad x' &= x - vt \\
\quad y' &= y \\
\quad z' &= z \\
\text{Lorentztransformatie} & : \\
\quad t' &= \gamma(t - \frac{v}{c^2}x) \\
\quad x' &= \gamma(x - vt) \\
\quad y' &= y \\
\quad z' &= z
\end{align*}
\]

waarbij de coördinaten met accent behoren bij een stelsel dat met een snelheid \( v \) in de \( x \)-richting beweegt, ten opzichte van het coördinatenstelsel met coördinaten zonder accent. Verder is \( \gamma \) gedefinieerd als \( \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \), waar \( c \) de lichtsnelheid is. Als de snelheid \( v \) veel kleiner is dan de lichtsnelheid, dus als \( v/c \) veel kleiner is dan 1, dan geldt dat \( \gamma \approx 1 \) en gaan de Lorentztransformaties over in de Galileitransformaties. Dit is de reden dat je in het dagelijks leven niets merkt van de gevolgen van de Lorentztransformaties.

Mochten de formules hierboven niet veel duidelijk maken, merk dan in ieder geval het volgende op: In een wereld waar de Galileitransformaties gelden is de tijd iets absoluuts (\( t = t' \)). De tijd loopt in elk inertiaalstelsel even hard. Het maakt niet uit hoe snel je beweegt en welke kant op. Alle klokken tikken met dezelfde snelheid. In een wereld waar de Lorentztransformaties gelden, hangt het verloop van de tijd wel af van hoe snel je beweegt. Klokken lopen langzamer als ze een hogere snelheid hebben. En dit geldt niet alleen voor klokken, maar voor elk natuurkundig proces. Dit zou als zeer opmerkelijk moeten overkomen. Het betekent bijvoorbeeld dat als de helft van een tweeling een ruimtereis maakt met hoge snelheid, deze bij terugkomst jonger is dan zijn of haar broer.
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Figuur 7.2: Links: de Galileitransformatie, vergelijk de $x$-as en de $x'$-as om een beeld te krijgen van de transformatie. Merk ook op dat $t = t'$. Rechts: de Lorentztransformatie, vergelijk weer de $x$-as en de $x'$-as. Merk op dat de $x'$-as gecomprimeerd is ten opzichte van de $x$-as en dat de tijd $t'$ langzamer loopt dan de tijd $t$.

of zus die op aarde is achtergebleven.

Einstein beweerde dus met zijn relativiteitstheorie dat deze laatste wereld de wereld is waarin we leven. De natuurwetten zijn hetzelfde in elk inertiaalstelsel (dit is het relativiteitsprincipe) en die inertiaalstelsels zijn aan elkaar gerelateerd via de Lorentztransformaties. Kortom de natuurwetten bezitten Lorentzsymmetrie.

Hoewel een experiment met een tweeling praktisch niet mogelijk bleek, zijn er wel equivalente experimenten gedaan met elementaire deeltjes$^3$, genaamd muonen. Muonen zijn instabiele deeltjes die na een gemiddelde tijd van ongeveer 2.2 microseconden vervallen in andere deeltjes. De tijd van ongeveer 2.2 microseconden noemen we de levensduur van het muon. Die levensduur is bepaald door middel van experimenten met muonen die stilstaan. Voor muonen die snel bewegen (met 99.9% van de lichtsnelheid) blijkt de levensduur ongeveer 60 keer langer te zijn, in overeenstemming met de Lorentztransformaties en de relativiteitstheorie. Dit experiment is niet het enige met dit resultaat. Tot op heden bevestigen alle experimenten dat de natuurwetten Lorentzsymmetrie bezitten$^4$.

Het feit dat alle experimenten de Lorentzsymmetrie van de natuur blijven bevestigen heeft ervoor gezorgd dat ze een leidraad is geworden in de zoektocht naar nieuwe natuurwetten. Zo is ze een integraal onderdeel geworden van de theorieën die op dit moment ons beste en meest fundamentele begrip van de natuur vertegenwoordigen: de “algemene relativiteitstheorie” en het “standaardmodel van de deeltjesfysica”. In het volgende bespreken we kort wat deze twee pilaren van de hedendaagse natuurkunde behelzen (zie Fig. 7.3).

De algemene relativiteitstheorie, die ook van Einstein’s hand is, beschrijft de werking van de zwaartekracht. Ze is een uitbreiding van de speciale relativiteitstheorie doordat ze ook versnellende coördinatenstelsels behandelt. Door zijn beschouwingen over versnelde coördinaatsystemen, massa en zwaartekracht, kwam Einstein tot de conclusie dat zwaartekracht iets is dat ontstaat door de interactie tussen massa of energie aan de ene

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$^3$Met elementaire deeltjes bedoelen we deeltjes die – zover we weten – niet opgebouwd zijn uit andere deeltjes.

$^4$Een uitzondering is misschien het experimentele resultaat dat beschreven staat in Sectie 6.8 van dit proefschrift, maar dat is nog verre van bevestigd.
De huidige fundamentele natuurkunde bestaat uit twee hoofdcomponenten – het standaardmodel en de algemene relativiteitstheorie – die ons in staat stellen verschillende verschijnselen met grote preciezie te beschrijven. Het verenigen van deze twee theorieën is een belangrijk doel van theoretisch natuurkundigen.

Het standaardmodel van de deeltjesfysica – kortweg het standaardmodel – is relevant als we heel kleine objecten willen behandelen, namelijk elementaire deeltjes. Het is ontstaan uit de combinatie van kwantummechanica en de speciale relativiteitstheorie (en een aantal abstractere ingrediënten, genaamd ijksymmetrieën). De speciale relativiteitstheorie is nodig omdat dergelijke kleine deeltjes vaak met hele hoge snelheden bewegen, terwijl de kwantummechanica het aspect van de zeer geringe omvang van de deeltjes behandelt. Het standaardmodel, dat ontstaan is door deze samensmelting van de kwantummechanica en speciale relativiteit, beschrijft alle ons bekende elementaire deeltjes en drie van de vier soorten interacties die de deeltjes onderling kunnen hebben. Deze drie interacties zijn de elektromagnetische, de sterke en de zwakke interactie. De interactie die ontbreekt in het standaardmodel is de zwaartekracht.

Het feit dat de zwaartekracht ontbreekt in het standaardmodel is waarschijnlijk een van de belangrijkste problemen van de tegenwoordige natuurkunde. Het komt door het feit dat algemene relativiteit en kwantummechanica niet goed samengaan. De naïeve combinatie van de twee levert onzin op. Een potentiële theorie die deze twee wel samenbrengt wordt wel een “theorie van alles” genoemd. Zo’n theorie van alles zou je de heilige graal van de hedendaagse theoretische natuurkunde kunnen noemen. Er zijn een aantal kandidaattheorieën, waarvan de snaarthetorie waarschijnlijk de bekendste is, maar voor geen van deze kandidaattheorieën is tot op heden overtuigend bewijs gevonden.

Voor een groot deel van de natuurkundige verschijnselen is het geen enkel probleem dat

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5 Deze laatste komt in de titel van dit proefschrift voor.
6 De hoop is dat zo’n theorie van alles ook andere problemen oplost, zoals die van donkere materie en energie, de massa’s van neutrino’s en de asymmetrie van materie en antimaterie.
de theorie voor objecten met veel massa en de theorie voor kleine objecten verschillend en (tot op heden) onverenigbaar zijn. In de meeste gevallen zijn de effecten van de een compleet te verwaarlozen als de ander relevant is. Zo hoeven we geen rekening te houden met de zwaartekracht als we botsingen van elementaire deeltjes bekijken en kunnen we de andere drie krachten negeren als we de draaiing van planeten om de zon beschrijven. Dit wordt moeilijker als er veel massa in een hele kleine ruimte zit, dan hebben we in principe de algemene relativiteits-theorie en het standaardmodel nodig. Zoals gezegd, gaan deze echter niet goed samen. Voorbeelden van veel massa in weinig ruimte zijn zwarte gaten en de oerknal. Om die goed te kunnen behandelen hebben we dus in principe een combinatie van het standaardmodel en algemene relativiteit nodig; ofwel een theorie van alles.

Hier komt dan Lorentzsymmetrie weer om de hoek kijken. Het blijkt namelijk dat veel kandidaattheorieën voor een theorie van alles de mogelijkheid in zich hebben om Lorentzsymmetrie te breken. Dit zou betekenen dat voor de systemen in de natuur waar die theorie nodig is – dus waar grote massa gecombineerd wordt met kleine afstanden – het toch uitmaakt met welke constante snelheid iets beweegt of welke oriëntatie het heeft. Sterker nog, ook in de meer “dagelijkse” natuurkunde zou Lorentzsymmetrie dan gebroken kunnen zijn, al zij het slechts licht gebroken. Zo licht dat de breking tot op heden aan experimentele detectie ontsnapt is. Als we deze lichte breking toch zouden kunnen detecteren of juist met steeds grotere preciezie zouden kunnen uitsluiten, dan kunnen we hiermee dus iets leren over een theorie van alles. Dit is het uiteindelijke doel van het bestuderen van de breking van Lorentzsymmetrie: het experimenteel toegang krijgen tot een eventuele theorie van alles.

Het deel waar we ons op richten in dit proefschrift is de zwakke interactie. Deze zwakke interactie is al even ter sprake gekomen toen we het standaardmodel bespraken. Het is een van de vier interacties die we tot op heden kennen. De elektromagnetische interactie en de zwaartekracht zijn de bekendste en ook het meest relevant voor ons dagelijkse leven. De andere twee, de sterke en de zwakke interactie, spelen voornamelijk een rol in het binnenste van atomen; in de kernen van die atomen om precies te zijn. De zwakke interactie is bijvoorbeeld verantwoordelijk voor bepaalde kernreacties en zonder de zwakke interactie zou de zon niet bestaan. We kunnen dus niet zeggen dat de zwakke interactie onbelangrijk is.

Een van de kernreacties, die ter sprake komt in dit proefschrift, is β-verval. Hierbij zendt een atoomkern twee elementaire deeltjes (elektron of positron en een neutrino) uit, waarbij de atoomkern zelf zodanig verandert dat een ander element ontstaat. De uitgezonden elektronen of positronen noemen we nucleaire β-straling. Het standaardmodel beschrijft precies hoe dit proces in z’n werk gaat. Het beschrijft met name met welke energie, met welke snelheid en in welke richting de verschillende deeltjes uit dit proces tevoorschijn komen. Kunnen we zonder de berekening te doen al iets zeggen over die voorspelling van het standaardmodel? Dat kunnen we inderdaad, als we ons herinneren dat Lorentzsymmetrie ingeïntegreerd zit in het standaardmodel. Het betekent bijvoorbeeld
Figuur 7.4: Het $\beta$-verval van een kern met spin (linksboven). Als alle spins dezelfde kant op wijzen hoeft een voorkeursrichting van de uitgaande deeltjes nog niet op Lorentzsymmetriebreking te wijzen (rechtsboven), als de deeltjes er maar isotroop uitkomen als de spins willekeurig georiënteerd zijn (linksonder). Als de uitgaande deeltjes ook dan een bepaalde richting kiezen betekent dit de breking van Lorentzsymmetrie (rechtsonder).

dat de elektronen gemiddeld in alle richtingen evenveel worden uitgezonden. Zou dit niet het geval zijn, dan zou er een voorkeursrichting zijn en zou het uitmaken onder welke hoek je het experiment bekijkt. Iets wat Lorentzsymmetrie verbiedt.

We moeten het bovenstaande beeld iets nuanceren. Sommige atoomkernen hebben namelijk een intrinsieke richting, iets wat we “spin” noemen. Je kunt hier een beeld van krijgen door je de kern voor te stellen alsof hij altijd in een bepaalde richting om z’n as draait. Het blijkt dat bij $\beta$-verval de elektronen wel degelijk bij voorkeur in een bepaalde richting worden uitgezonden, namelijk ten opzichte van de spin van de kern. Dit betekent echter niet het einde van Lorentzsymmetrie. Het betekent slechts dat de spin blijkbaar de uitzendrichting van de elektronen beïnvloedt. Als we een groot aantal kernen nemen waarvan de spins in alle mogelijke richtingen wijzen, dan komen de elektronen gemiddeld wel in alle richtingen evenveel uit het experiment (zie Fig. 7.4)

Als de elektronen ook in dit laatste geval niet isotroop verdeeld uit het experiment komen – en fouten van het experiment en invloeden van buitenaf zijn uitgesloten – dan betekent dit wel dat Lorentzsymmetrie gebroken is. Er is dan blijkbaar iets in “het niets” – het vacuüm – dat de richting van de elektronen beïnvloedt. Ook als het totale vervalsproces (dus gesommeerd over alle elektronrichtingen) sneller gaat wanneer de spin van de kern een bepaalde kant op wijst, houdt dit de breking van Lorentzsymmetrie in. Dit laatste voorbeeld is wat er experimenteel onderzocht is op het Kernfysisch Versneller Instituut (KVI) in Groningen (zie sectie 4.3.1).

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7Voer deze analogie niet te ver door, dan kom je in de problemen met relativiteit.
Het werk in dit proefschrift richt zich op precieze voorspellingen voor deze en andere experimenten onder de aannemer dat Lorentzsymmetrie gebroken is. Om deze voorspellingen te kunnen doen is een model voor elementaire deeltjes opgesteld, met de breking van Lorentzsymmetrie erin opgenomen. Dit model is in principe simpelweg het standaardmodel, met daaraan componenten toegevoegd die de Lorentzsymmetriebreking veroorzaken. Dit model wordt de “standaardmodel-extensie” (SME) genoemd. De toegevoegde componenten – de extensies – bevatten coëfficiënten die de breking van Lorentzsymmetrie parametriseren. Als deze parameters allemaal gelijk aan nul blijken te zijn, betekent dit dat Lorentzsymmetrie behouden is. We noemen deze parameters SME-coëfficiënten. Het doel van de experimenten is dan om bovengrenzen aan de waardes van de SME-coëfficiënten te stellen.

In dit proefschrift hebben we de invloed van de SME-coëfficiënten op vervalsreacties van deeltjes onderzocht. Meer specifiek ging het over vervalsreacties die veroorzaakt worden door de zwakke interactie. We hebben hiervoor onder andere een algemeen model opgesteld waarmee meerdere SME-coëfficiënten in een keer getest kunnen worden. Verder hebben we de vervalssnelheden van verschillende deeltjes met een set van verschillende SME-coëfficiënten uitgerekt. De vervalssnelheden die we uitgerekend hebben zijn: die van algemeen β-verval (dus voor willekeurige atoomkernen), die van pionen en die van muonen. Zoals verwacht hangen al deze vervalssnelheden nu af van bijvoorbeeld de richting van de uitgezonden deeltjes en hun spin. Het belang van de berekeningen is dat we nu voor deze vervalsreacties de precieze afhankelijkheid van de SME-coëfficiënten kennen. Daardoor kunnen we deze afhankelijkheid met experimenten op nauwkeurige wijze uitsluiten en verkrijgen we limieten op de SME-coëfficiënten en daarmee op de breking van Lorentzsymmetrie.

Ten slotte willen we kort terugkomen op de relativiteitstheorie zoals we hem in het begin bespraken. We hebben gezien dat de relativiteitstheorie voorspelt dat de tijd langzamer gaat voor deeltjes die sneller bewegen, waardoor ze langer leven (ze vervallen minder snel). Wat gebeurt er met dit gegeven als Lorentzsymmetrie gebroken is? Het antwoord is dat het in principe beide kanten op kan: het deeltje kan langer of juist korter leven dan voorspeld wordt door de relativiteitstheorie. Het punt is dat de relativiteitstheorie een hele precieze voorspelling doet: deeltjes leven γ keer langer bij hogere snelheden. Voor de eerder besproken muonen kwam dit neer op ongeveer 60 keer langer bij een snelheid van 99.9% van de lichtsnelheid. Dit is ook precies wat het standaardmodel voorspelt. De extensie van het standaardmodel geeft een andere voorspelling: afhankelijk van het teken en de grootte van de SME-coëfficiënten leeft het muon korter of langer (zie de berekeningen in hoofdstuk 6). Ook deze afhankelijkheid hebben we in dit proefschrift bekeken en formules voor afgeleid. Met deze formules en door vervalsreacties bij verschillende snelheden van de deeltjes te bekijken kunnen we nieuwe limieten op SME-coëfficiënten worden verkregen.

Samenvattend is dit dus het doel geweest van dit proefschrift: het verkrijgen van exacte formules voor allerlei vervalsreacties voor het geval dat Lorentzsymmetrie gebroken is, om vervolgens met deze formules de resultaten van experimenten te interpreteren en limieten op SME-coëfficiënten – en dus de breking van Lorentzsymmetrie – te verkrijgen. Dit

\[ \text{De vervalssnelheid van een deeltje is omgekeerd evenredig met zijn levensduur.} \]
is ons voor een heel aantal van de coëfficiënten gelukt. We hebben ook één resultaat gevonden dat Lorentzsymmetrie lijkt te schenden. Het gaat hierbij om het, in de vorige alinea besproken, muon. De snelheidsafhankelijkheid van de levensduur van het muon lijkt niet precies zo te zijn als het standaardmodel voorspelt (zie sectie 6.8). Dit zou een zeer belangrijke vondst zijn en juist daarom moeten we heel voorzichtig zijn. Dit resultaat is verre van bevestigd en moet nog veel meer onderzocht worden voordat claims gedaan kunnen worden. Tot nu toe moeten we dus concluderen dat zover wij kunnen zien Lorentzsymmetrie behouden is, in de zwakke interactie of anderszins.
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List of publications


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