Off-shell covariantization of algebroid gauge theories

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We present a generalized method to construct field strengths and gauge symmetries that yield a Yang–Mills-type action with Lie n-algebroid gauge symmetry. The procedure makes use of off-shell covariantization in a supergeometric setting. We apply this method to the system of a 1-form gauge field and scalar fields with Lie n-algebroid gauge symmetry. We work out some characteristic examples.

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1. Introduction

Recently, many approaches for a generalization of gauge theories have been discussed. Among them, there are the so-called higher gauge theories [1], where in addition to the gauge potential higher-rank forms are introduced. Such theories are expected to appear, for instance, in the construction of the effective theory of multiple M5-branes, where a 2-form gauge potential appears.

Another approach is the promotion of the gauge algebra to an algebroid structure. This can be thought of as a generalization of the gauged nonlinear sigma model, where the structure constants of the Lie algebra become structure functions, i.e., scalar-field dependent [2–4].

A systematic way to construct higher gauge theories is to use an $L_\infty$ structure [5]. Any truncated $L_\infty$-algebra defines a gauge theory of higher-form gauge fields and the corresponding gauge symmetries are generalized to Lie n-algebras [6]. As we shall see, both generalizations, i.e., to higher gauge theory and to algebroid Yang–Mills, can be understood in a unified way using supergeometry. Actually, there is a common phenomenon in both approaches, i.e., the higher gauge theory using an $L_\infty$ structure and the approach via algebroid structure, when it comes to the formulation of the corresponding field theories. This phenomenon is the so-called fake curvature condition [7].

In a general higher gauge theory, lower-form gauge fields also exist. However, the field strength of the higher-form gauge potential is only covariant under the condition that the field strengths associated with the lower-form gauge fields vanish. This is called the fake curvature condition and results in a noninteractive theory. Therefore, it is desirable to deform the higher algebra structure to circumvent this obstruction. Such a deformation process, known as off-shell covariantization, has been analyzed in the higher gauge theory context in our previous paper [8]. There, we solved the
fake curvature condition by reducing the symmetries to Lie $n$-subalgebras, while imposing proper conditions on the auxiliary gauge fields.

In this paper, we want to address the problem of off-shell covariantization in the context of algebroid gauge theories. We apply our method to systems consisting of a 1-form gauge field and a scalar field. We formulate the corresponding higher algebroid gauge symmetries and associated gauge-invariant actions. To obtain off-shell Yang–Mills-type actions, we consider deformations of gauge transformations and field strengths. Auxiliary gauge fields are projected out and field strengths are deformed by terms proportional to the remaining curvatures.

In order to obtain proper gauge symmetries of gauge fields and field strengths, we use the supermanifold method on a so-called QP-manifold [9,10], which is a useful tool to generate a BRST–BV formalism of topological field theories [11]. Instead of starting from fields and an action, we start with a graded symplectic manifold and its Hamiltonian function corresponding to a BRST charge of the gauge algebra. Gauge fields, field strengths, and their gauge transformations are induced from the QP-manifold structure. This idea is similar to the free differential algebra method [12,13]. In our formalism, consistency is guaranteed by the underlying QP-manifold structure [11,14–16].

The advantage of the supermanifold method is that the gauge transformations and field strengths can be read off from one superfield. The starting point of our analysis is a general theory that unifies gauge theories with algebroid symmetry and those with Lie $n$-algebra symmetry. Examples are the Kotov–Strobl model [4] and the Ho–Matsuo model [17]. See also Ref. [18] for a gauge theory with a Lie 2-algebra symmetry.

The organization of this paper is as follows. In Sect. 2, we introduce QP-manifolds and describe how to use them to induce general gauge theories. In Sect. 3, we introduce our setup and explain the off-shell covariantization procedure used in this paper. In Sect. 4, we discuss the construction of $(n+1)$-dimensional higher algebroid gauge theories based on general QP-manifold structures. In Sect. 5, we construct and analyze 4D algebroid gauge theories. We derive the relations between the structure functions necessary for off-shell covariantization. Furthermore, we discuss examples including the Stückelberg formalism, non-Abelian off-shell covariantization, and an example from the Kotov–Strobl models. In Sect. 6, we examine the closure of the gauge symmetry algebra. Section 7 is devoted to the discussion.

2. QP-manifolds and gauge theories

In our approach, the underlying gauge algebra of the gauge theory is formulated by making use of so-called QP-manifolds. Therefore, we start this section by introducing the structure of a QP-manifold. Then, we show how to generate gauge transformations and field strengths that inherit the QP-manifold as underlying gauge structure. The theory is formulated in the superfield formulation of the BV formalism. There, the physical fields, ghosts, and antifields are encoded in superfields defined on a superspace. We will work out a pedagogical example of a QP-manifold that induces a Lie algebra structure and show how to use it to define an ordinary Yang–Mills gauge theory.

A QP-manifold is a convenient way to encode the algebraical structure of a BRST–BV formalism in an economical manner. To define a QP-manifold, we start with a nonnegatively graded manifold $\mathcal{M}$, which can locally be described by Grassmann-even and Grassmann-odd coordinates. The BRST operator or BRST charge of the BV formalism is associated with a vector field $Q$ of degree 1 on $\mathcal{M}$, which is nilpotent, $Q^2 = 0$. $Q$ is called the homological or Hamiltonian vector field. The BV antibracket comes from a graded symplectic structure $\omega$ of degree $n$ on $\mathcal{M}$, which induces a graded Poisson bracket $\{-, -\}$ of degree $(-n)$. A QP-manifold is said to be of degree $n$ if the graded
symplectic structure is of degree $n$. Finally, we demand the following compatibility condition between the graded symplectic structure and homological vector field: $L_Q \omega = 0$. $L_Q$ is the Lie derivative along $Q$. In general, a QP-manifold of degree $n$ induces a so-called symplectic Lie $n$-algebroid.

For any QP-manifold one can find a function $\Theta \in C^\infty(\mathcal{M})$ of degree $n+1$, which is called a Hamiltonian function, such that

$$Q = \{\Theta, -\}. \quad (2.1)$$

The nilpotency of $Q$ then translates to the classical master equation

$$\{\Theta, \Theta\} = 0. \quad (2.2)$$

Let us summarize: A QP-manifold consists of a 3-tuple $(\mathcal{M}, \omega, \Theta)$ with compatibility condition $L_Q \omega = 0$ and classical master equation $\{\Theta, \Theta\} = 0$. A QP-manifold can also be called a symplectic NQ-manifold.\footnote{The “Q” in QP-manifold refers to the Hamiltonian vector field $Q$ and the “P” refers to the induced graded Poisson structure. The “N” refers to it being nonnegatively graded.}

Let us compute the example of a QP-manifold, which induces the structure of a Lie algebra. Let $V$ be a vector space of finite dimension. Let the basis of $V$ be denoted by $e_a$. Furthermore, let the basis of its dual $V^*$ be denoted by $e^a$. We consider the graded manifold $\mathcal{M} = T^*[n]V[1]$, where the object $[n]$ shifts the degree of the local coordinates of the cotangent fiber by $n$. When $n$ is odd, then the associated coordinate is anticommuting. The local coordinates of $V[1]$ attain degree 1. Therefore, we can describe $\mathcal{M}$ locally by coordinates $(v^a, v_a)$ of degrees $(1, n - 1)$, where $v^a$ lives in $V[1]$ and $v_a$ on the cotangent fiber. The degree of $v_a$ is $n - 1$, where the degree of $v^a$ is subtracted, since we took the cotangent fiber instead of the tangent fiber. We define the graded symplectic structure on $\mathcal{M}$ by

$$\omega = (-1)^n \delta v^a \wedge \delta v_a, \quad (2.3)$$

and the Hamiltonian function of degree $n + 1$ by

$$\Theta = \frac{1}{2} f^a_{bc} v^b v^c v_a, \quad (2.4)$$

where $f^a_{bc}$ are structure constants. The classical master equation, $\{\Theta, \Theta\} = 0$, is equivalent to $f^a_{d_1 d_2 d_3} f^b_{c_1 c_2 c_3} = 0$, which is the Jacobi identity of the structure constants. We can use derived brackets to induce the structure of a Lie algebra on $V$ as follows. First, we define an injection map $j$ by

$$j : V \oplus V^* \to V[1] \oplus T^*[n]V[1] \cong V[1] \oplus V^*[n - 1],$$

$$j : (e_a, e^a) \mapsto (v_a, v^a). \quad (2.5)$$

Then we can define a Lie bracket on $V$ by

$$[X, Y] = -f^c \{j_*(X), \Theta\}, j_*(Y)\}$$

$$= X^a Y^b f^c_{ab} e_c. \quad (2.6)$$
Here, $X = X^a e_a \mapsto j_\alpha (X) = X^a \nu_\alpha$. General derived brackets are defined by contraction of the Hamiltonian function $\Theta$ with several elements of $C^\infty (M)$ using the graded symplectic structure. We conclude that $(\mathcal{M}, \omega, \Theta)$ induces the structure of a Lie 1-algebra, i.e., a Lie algebra $(V, [-, -])$.

By making use of the BV–AKSZ formalism [11,19], one can construct a topological field theory in $n+1$ dimensions that inherits the structure of an underlying QP-manifold of degree $n$. Here, we use this method only to construct consistently the gauge transformations and field strengths. Let $\Sigma$ be the $(n+1)$-dimensional worldvolume of the topological field theory. Furthermore, let $M$ be the target space manifold in which the worldvolume shall be embedded. The first step is to promote the target space manifold $M$ to a QP-manifold $(\mathcal{M}, \omega, \Theta)$. Then, the worldvolume is promoted to the superworldvolume $T[1] \Sigma$, which captures the superfield formalism of the BV–BRST formalism to be induced in the following. We denote the local coordinates of $\Sigma$, which are of degree 0, by $\sigma^\mu$ and those of the fiber, which are of degree 1, by $\theta^\mu$. Local coordinates of even degree are Grassmann even and local coordinates of odd degree are Grassmann odd.

Let $(\mathcal{M}, \omega, Q)$ be a QP-manifold of degree $n$. Then we can define the map $a : T[1] \Sigma \rightarrow \mathcal{M}$, such that the pullback

$$ Z(\sigma, \theta) \equiv a^*(z) = \sum_{j=0}^{n+1} Z^{(j)}(\sigma, \theta) = \sum_{j=0}^{n+1} \theta^{\mu_1} \cdots \theta^{\mu_j} Z^{(j)}_{\mu_1 \cdots \mu_j}(\sigma) \quad (2.7) $$

yields a superfield [15]. Here, $z$ is a coordinate of degree $k$ on $\mathcal{M}$, and the degree of the superfield $|Z| = k$. Since the resulting object is a superfield in the BV sense, it contains associated gauge fields, ghosts, and antifields as component fields $Z_{\mu_1 \cdots \mu_j}(\sigma)$. In general, the ghost number of a field $\Psi$ is defined by degree minus form degree, $gh(\Psi) = |\Psi| - \deg(\Psi)$, where the form degree $(0, 1)$ is assigned to $(\sigma^\mu, \theta^\mu)$. Thus, by degree counting, $Z^{(j)}_{\mu_1 \cdots \mu_j}$ has ghost number $(k-j)$, $Z^{(k)}_{\mu_1 \cdots \mu_k}$ has ghost number 0, and $Z^{(k-1)}_{\mu_1 \cdots \mu_{k-1}}$ has ghost number 1. The ghost number 0 component is a physical $k$-form gauge field and the ghost number 1 component its FP ghost, i.e., the gauge parameter of the associated gauge transformation.

The super field strength corresponding to a coordinate $z$ on $\mathcal{M}$ is defined by

$$ F_z = F(z) = d \circ a^*(z) - a^* \circ Q(z), \quad (2.8) $$
and its physical component is

$$ F'_z = (d \circ a^*(z) - a^* \circ Q(z))|_{|z|+1}. \quad (2.9) $$

Here, $d = \theta^\mu \partial_\mu$ denotes the superderivative and $|z|+1$ denotes projection to the degree-$|z| + 1$ part, while setting all antifield components to zero. We get the super Bianchi identity for free:

$$ dF_z = -F \circ Q(z). \quad (2.10) $$

The associated gauge transformation is encoded in the super field strength as the degree-$|z|$ part:\footnote{This formula gives a BRST transformation.}

$$ \delta Z = (d \circ a^*(z) - a^* \circ Q(z))|_{|z|}. \quad (2.11) $$

Again, while projecting, we set all antifields to zero.
To extract the physical field directly, we define the map \( \tilde{a} : T[1] \Sigma \to \mathcal{M} \) by

\[
\tilde{a}^*(z) = \frac{1}{k!} d\sigma^{\mu_1} \wedge \cdots \wedge d\sigma^{\mu_k} Z^{(k)}_{\mu_1 \cdots \mu_k}(\sigma), \tag{2.12}
\]

where \( z \) is a coordinate of degree \( k \) on \( \mathcal{M} \). Note that we have identified \( \theta^\mu \) with \( d\sigma^\mu \). Using this map, we can rewrite the physical field strength by

\[
F_z = F(z) = d \circ \tilde{a}^*(z) - \tilde{a}^* \circ Q(z) = F_{\mid_{\theta^\mu = d\sigma^\mu}}. \tag{2.13}
\]

We conclude that, after fixing a QP-manifold \( (\mathcal{M}, \omega, Q) \), we can associate gauge fields, field strengths, and gauge transformations to each local coordinate in a consistent manner. The gauge transformations inherit the structure of the QP-manifold, which is governed by the homological vector field \( Q \).

As an example, let us construct the gauge theory of an ordinary Lie algebra using our QP-manifold example above. Corresponding to the local coordinate \( v^a \) of degree 1, we introduce a 1-form gauge field \( A^a(\sigma) \) on the target manifold as follows. As described in Eq. (2.7), the pullback of \( v^a \) along \( a^* \) yields a superfield:

\[
A^a(\sigma, \theta) \equiv a^*(v^a) = \sum_{j=0}^{n+1} A^{(j)a}(\sigma, \theta) = \sum_{j=0}^{n+1} \frac{1}{j!} \theta^{\mu_1} \cdots \theta^{\mu_j} A^{(j)a}_{\mu_1 \cdots \mu_j}(\sigma)
\]

\[
= e^a(\sigma) + \theta^\mu A^a_\mu(\sigma) + \cdots. \tag{2.14}
\]

The degree-0 part \( e^a \) corresponds to the gauge parameter with ghost number 1. According to Eq. (2.12), \( \tilde{a}^* \) extracts the \( \theta^1 \) part of the superfield \( A^a(\sigma, \theta) \), which is the physical 1-form gauge field

\[
\tilde{a}^* (v^a) = A^a = A^a_\mu(\sigma) d\sigma^\mu. \tag{2.15}
\]

From Eq. (2.8), we can compute the super field strength,

\[
F^a_A = d \circ a^*(v^a) - a^* \circ Q(v^a) \tag{2.16}
\]

\[
= \theta^\mu (\partial_\mu e^a - (-1)^n f^a_{bc} A^b_\mu e^c) + \frac{1}{2} \theta^\mu \theta^\nu (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - (-1)^n f^a_{bc} A^b_\mu A^c_\nu) + \cdots,
\]

where \( Q \) is the homological vector field associated with the Hamiltonian function by \( Q = \{\Theta, -\} \).

The associated 2-form field strength is given by the ghost number zero part:

\[
F^a = F_{v^a} = d \circ \tilde{a}^*(v^a) - \tilde{a}^* \circ Q(v^a) = dA^a - (-1)^n \frac{1}{2} f^a_{bc} A^b A^c. \tag{2.17}
\]

Finally, the associated gauge transformation is given by the ghost number 1 part,

\[
\delta A^a = (d \circ a^*(z) - a^* \circ Q(z)) \big|_{z} = d\epsilon^a - (-1)^n f^a_{bc} A^b \epsilon^c, \tag{2.18}
\]

where \( \epsilon^a = a^*(v^a) \big|_{\text{deg}=0} \) is the ghost number 1 gauge parameter of the associated gauge transformation. We conclude that the QP-manifold, which induces a Lie algebra, can be utilized to construct an ordinary Yang–Mills gauge theory.

3. Setup and off-shell covariantization

In this section, we introduce our setup and method of off-shell covariantization of field strengths. Please refer to Refs. [19,20] for further details.
The method described above can be used to construct arbitrary higher gauge theories of $p$-forms. However, for our purposes in this paper, we focus on a theory containing scalar fields $X^i(\sigma)$ and a 1-form gauge field $A^a = d\sigma^\mu A^a_\mu(\sigma)$.

Our setup is as follows. We consider a QP-manifold of degree $n$, where the graded manifold is given by $\mathcal{M}_n = T^*[n]E[1], \quad n \in \mathbb{N}$, and $E \rightarrow M$ is a vector bundle. $M$ is a smooth manifold. We take the following local coordinates: $x^i$ of degree 0 on $M$ and $q^a$ of degree 1 on the fiber of the vector bundle. When we construct the associated field theory, the degree corresponds to the sum of the ghost degree and form degree. With respect to the graded cotangent bundle $T^*[n]$, we take coordinates $(\xi_i, p_a)$ of degree $(n, n - 1)$ conjugate to $(x^i, q^a)$. To summarize, the local coordinates on $\mathcal{M}_n$ are $(x^i, q^a, \xi_i, p_a)$ of degree $(0, 1, n, n - 1)$.

The symplectic form on $\mathcal{M}_n$ is defined by

$$\omega = \delta x^i \wedge \delta \xi_i + (-1)^n \delta q^a \wedge \delta p_a. \quad (3.19)$$

This induces the following graded Poisson bracket:

$$\{f, g\} = f_{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} g + f_{\alpha \beta} \frac{\partial}{\partial q^\alpha} \frac{\partial}{\partial q^\beta} g + (-1)^n f_{\alpha \beta} \frac{\partial}{\partial p_{\alpha}} \frac{\partial}{\partial p_{\beta}}, \quad (3.20)$$

where $f, g \in \mathcal{C}^\infty(\mathcal{M}_n)$.4

Now let us introduce the associated gauge fields and field strengths. For the QP-manifold $\mathcal{M}_n$ under consideration, we get a scalar field associated with the degree-0 coordinate $x^i$ and a 1-form gauge field associated with the degree-1 coordinate $q^a$, and associated field strengths:

$$\tilde{a}^a(x^i) \equiv X^i(\sigma), \quad (3.21)$$

$$\tilde{a}^a(q^a) \equiv A^a(\sigma) = A^a_\mu d\sigma^\mu, \quad (3.22)$$

$$F_X^i = d\tilde{a}^a(x^i) - \tilde{a}^a(\mathcal{Q}x^i), \quad (3.23)$$

$$F_A^a = d\tilde{a}^a(q^a) - \tilde{a}^a(\mathcal{Q}q^a). \quad (3.24)$$

In addition to that, we find $(n - 1)$- and $n$-form auxiliary gauge fields $C_a$ and $\Xi_i$ associated with the conjugate coordinates on our graded symplectic manifold:

$$\tilde{a}^a(x^i) = X^i, \quad \tilde{a}^a(q^a) = A^a, \quad (3.25)$$

$$\tilde{a}^a(\xi_i) = \Xi_i, \quad \tilde{a}^a(p_a) = C_a. \quad (3.26)$$

In this very scenario, we have gauge transformations with three independent gauge parameters corresponding to the fields $A^a$, $\Xi_i$, and $C_a$:

$$a^a(q^a)|_{\text{DEG}=0} = \epsilon^a, \quad \tilde{a}^a(\xi_i)|_{\text{DEG}=n-1} = \mu'_i, \quad \tilde{a}^a(p_a)|_{\text{DEG}=n-2} = \epsilon'_a. \quad (3.27)$$

However, in general, the field strength $F_z$ of a gauge field $\tilde{Z} = \tilde{a}^a(z)$ is transformed adjointly, $\delta F_z \sim F_z$, only on-shell since the above procedure is derived from the theory of AKSZ sigma models [11]. An action of the AKSZ sigma models is a topological field theory of BF type and the equation of motion is $F_z = 0$. If $F_z$ transforms adjointly without use of the equations of motion,
we call $F_z$ off-shell covariant. If $F_z$ is off-shell covariant, the construction of a gauge-invariant Yang–Mills-type action $S \sim F_z \wedge \ast F_z$ is possible.

The procedure to obtain off-shell covariant field strengths is as follows. We start by associating the ordinary and auxiliary gauge fields with the local coordinates of the graded manifold $\mathcal{M}$. In the next step, we project out the auxiliary gauge fields $(\mathfrak{E}_i, C_a)$ as well as the corresponding gauge degrees of freedom $\mu'_i$ and $\epsilon'_a$ by $\mathfrak{E}_i = C_a = 0$. After the projection we end up with the gauge symmetry of 1-form $A^a$ and scalar $X^i$. At this stage, the gauge symmetry is on-shell closed, in general. Then, we deform the remaining field strengths and gauge symmetries of $A^a$ and $X^i$ by adding deformation terms proportional to the field strengths. Note that the algebra underlying the gauge symmetries is not altered. As a result, we obtain a set of differential equations for the coefficient functions of the deformations. Any solution to this system leads to off-shell covariant field strengths.

4. Hamiltonian functions and field strengths

In this section, we perform a classification of Hamiltonian functions and compute the induced field strengths of the associated gauge theory.

A Hamiltonian function $\Theta$ on a general QP-manifold $M_n$ of degree $n$ is of degree $n + 1$. In this section, we examine the most general Hamiltonian function on $M_n$ by expanding it in conjugate coordinates $(\xi_i, p_a)$,

$$\Theta = \sum_k \Theta^{(k)},$$  \hspace{1cm} (4.28)

where $\Theta^{(k)}$ is a $k$th-order function in $(\xi_i, p_a)$.

The following cases occur.

A). $n \geq 4$: Since the degrees of $(\xi_i, p_a)$ are $(n, n - 1)$, the degree of $\Theta^{(k)}$ for $k \geq 2$ is larger than or equal to $2n - 2$. Therefore, if $n \geq 4$, then $\Theta^{(k)} = 0$ for $k \geq 2$ by degree counting, i.e., the general form of the Hamiltonian function is

$$\Theta = \Theta^{(0)} + \Theta^{(1)}.$$  \hspace{1cm} (4.29)

B). $n = 3$: In this case, $\Theta^{(k)} = 0$ for $k \geq 3$ by degree counting. Therefore, the expansion stops at second order:

$$\Theta = \Theta^{(0)} + \Theta^{(1)} + \Theta^{(2)}.$$  \hspace{1cm} (4.30)

This QP-manifold defines a Lie 3-algebroid structure on $E$; see Ref. [21]. Only for $n \leq 3$ does the Hamiltonian $\Theta$ provide freedom for deformations. We discuss case $n = 3$ in detail in Sect. 5.

C). $n = 1, 2$: The Hamiltonian $\Theta$ contains more deformation terms. In the $n = 2$ case, since $(x^i, q^a, \xi_i, p_a)$ is of degree $(0, 1, 2, 1)$, the graded manifold is $M_2 = T^*[2]\tilde{E}[1]$, and

$$\Theta = \Theta^{(0)} + \Theta^{(1)} + \Theta^{(2)} + \Theta^{(3)}.$$  \hspace{1cm} (4.31)

Then, this defines a Courant algebroid on $E$ [22,23].

For $n = 1$, $\Theta$ defines a Poisson structure on $E$. 

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4.1. Gauge fields and field strengths induced from Hamiltonian functions

First, the Hamiltonian function $\Theta^{(1)}$ reproduces a Lie algebroid for general $n$. It contains the following terms:

$$\Theta^{(1)} = \rho^i_a(x) q^a + \frac{1}{2} f^c_{ab}(x) q^a q^b p_c,$$

(4.32)

where $\rho^i_a(x), f^c_{ab}(x)$ are structure functions depending on $x$.

Lie algebroid operations are given by the following derived brackets:

$$[e_1, e_2] = -\{\{e_1, \Theta^{(1)}\}, e_2\},$$

(4.33)

$$\rho(e)f = \{\{e, \Theta^{(1)}\}, f\},$$

(4.34)

where $e, e_1, e_2 \in \Gamma(E)$ are sections of a Lie algebroid that is locally expressed by $e = e^a(x) p_a$ and $f \in C^\infty(M)$.

The classical master equation, $\{\Theta^{(1)}, \Theta^{(1)}\} = 0$, implies the following conditions on the structure constants:

$$\rho^i \frac{\partial \rho^j_b}{\partial x^i} - \rho^j_b \frac{\partial \rho^i_a}{\partial x^j} + \rho^i \rho^j a f^c_{ab} = 0,$$

(4.35)

$$\rho^i \frac{\partial f^d_{bc}}{\partial x^i} + f^d_{e} e^d f^e_{bc} = 0.$$

(4.36)

The pullback $a^*$ maps the four coordinates to superfields as follows:

$$X^i = a^*(x^i), \quad A^a = a^*(q^a),$$

(4.37)

$$\Xi_i = a^*(\xi_i), \quad C_a = a^*(p_a).$$

(4.38)

The super field strengths are given by $F_Z = d a^*(z) - a^* Q(z)$:

$$F^i_X = dX^i + (-1)^n \rho^i_a(X) A^a,$$

(4.39)

$$F^a_A = dA^a - (-1)^n \frac{1}{2} f^a_{bc}(X) A^b A^c,$$

(4.40)

$$F^{(C)}_a = dC_a - f^b_{ac}(X) C_b A^c - \rho^i_a(X) \Xi_i,$$

(4.41)

$$F^{(\Xi)}_i = d \Xi_i - \frac{1}{2} \frac{\partial f^a_{bc}}{\partial X^i}(X) C_a A^b A^c - \frac{\partial \rho^i_a}{\partial X^i}(X) \Xi_j A^a,$$

(4.42)

where $F^{(C)}$ and $F^{(\Xi)}$ are the super field strengths of $C$ and $\Xi$, respectively. When we substitute the component expansions to (4.39)–(4.42), then the corresponding degree-$(|z| + 1)$ parts are the field strengths:

$$F^i_X = dX^i + (-1)^n \rho^i_a(X) A^a,$$

(4.43)

$$F^a_A = dA^a - (-1)^n \frac{1}{2} f^a_{bc}(X) A^b A^c,$$

(4.44)

$$F^{(C)}_a = dC_a - f^b_{ac}(X) C_b A^c - \rho^i_a(X) \Xi_i,$$

(4.45)

$$F^{(\Xi)}_i = d \Xi_i - \frac{1}{2} \frac{\partial f^a_{bc}}{\partial X^i}(X) C_a A^b A^c - \frac{\partial \rho^i_a}{\partial X^i}(X) \Xi_j A^a.$$

(4.46)
The degree-$|z|$ parts of the component expansions of the super field strengths yield the gauge transformations

\[
\delta X^i = (-1)^n \rho^i_a(X) e^a, \tag{4.47}
\]

\[
\delta A^a = d\epsilon^a - (-1)^n f^{a}_{bc}(X) A^b e^c, \tag{4.48}
\]

\[
\delta C_a = d\epsilon^a - f^b_{ac}(X)(\epsilon^b \wedge A^c + C_b \wedge e^c) - \rho^i_a(X) \mu_i^a, \tag{4.49}
\]

\[
\delta \Xi_i = d\mu^i_a - \frac{1}{2} \frac{\partial f^{a}_{bc}(X)}{\partial X^i} (\epsilon^a A^b e^c + 2 C_a A^b e^c) - \frac{\partial \rho^a_j}{\partial X^i} (\mu^a_j A^a + \Xi f e^a). \tag{4.50}
\]

The gauge transformations of the gauge field strengths are

\[
\delta F^i_X = -(-1)^n \frac{\partial \rho^i_a}{\partial X^j} F^j_X e^a, \tag{4.51}
\]

\[
\delta F^a_A = -(-1)^n f^{a}_{bc} F^b_A e^c + (-1)^n \frac{\partial f^{a}_{bc}}{\partial X^i} F^j_X A^b e^c. \tag{4.52}
\]

In general, $F^a_A$ is on-shell covariant.

5. Off-shell covariantization of 4D algebroid 1-form gauge theories

In the previous sections, we discussed the structure of the Hamiltonian and canonical transformations for general $n$. To make the discussion concrete, we take a field theory for the specific case $n = 3$, i.e., $\mathcal{M}_3 = T^*[3]E[1]$. In this case, $\Theta^{(2)}$ can be included in the Hamiltonian function and we obtain interesting nontrivial examples.

First, we describe the structure of the Hamiltonians based on $\mathcal{M}_3$. Local coordinates are $(x^i, q^a, \xi_i, p_a)$ of degree $(0, 1, 3, 2)$, respectively. Since $\Theta$ is of degree 4, the Hamiltonian function is at most a second-order function in $(\xi_i, p_a)$, by degree counting, and can be expanded as $\Theta = \Theta^{(0)} + \Theta^{(1)} + \Theta^{(2)}$. Therefore, the concrete expressions are

\[
\Theta^{(0)} = \frac{1}{4!} h_{abcd}(x) q^a q^b q^c q^d, \tag{5.53}
\]

\[
\Theta^{(1)} = \frac{1}{2} f^c_{ab}(x) q^a q^b p_c + \rho^i_a(x) \xi_i q^a, \tag{5.54}
\]

\[
\Theta^{(2)} = \frac{1}{2} k^{ab}(x) p_a p_b, \tag{5.55}
\]

with additional structure functions $h_{abcd}(x), f^c_{ab}(x), \rho^i_a(x)$, and $k^{ab}(x)$.

From the classical master equation, \{\Theta, \Theta\} = 0, we obtain the following identities:

\[
\rho^i_b k^{ba} = 0, \tag{5.56}
\]

\[
\rho^k_c \frac{\partial k^{ab}}{\partial x^k} + k^{da} f^{b}_{cd} + k^{db} f^{a}_{cd} = 0, \tag{5.57}
\]

\[
\rho^k_b \frac{\partial \rho^i_a}{\partial x^k} - \rho^k_a \frac{\partial \rho^i_b}{\partial x^k} + \rho^j_c f^c_{ab} = 0, \tag{5.58}
\]
\[
2 \rho^k \frac{\partial f^a_{bc}}{\partial x^k} + k^{ae} h_{bcde} - 2 f^a_{e[b} f^e_{c]} d = 0, \quad (5.59)
\]
\[
2 \rho^k \frac{\partial h_{bcde}}{\partial x^k} + \rho^f_{[ab} h_{cdef]} = 0, \quad (5.60)
\]

which define a Lie 3-algebroid [21].

Based on the general theory that we explained at the beginning, we consider the restriction of the 4D theory. The pullback \(a^i\) maps the four coordinates to superfields as follows:

\[
X^i \equiv a^i(x^i), \quad A^a \equiv a^a(q^a), \quad (5.61)
\]
\[
\Xi^i \equiv a^i(\xi_i), \quad C_a \equiv a^a(p_a), \quad (5.62)
\]

where \((X, A, \Xi, C)\) are of degree \((0, 1, 3, 2)\). The super field strengths are given by

\[
F^i_X = dX^i - \rho^i_a(X) A^a, \quad (5.63)
\]
\[
F^a_A = dA^a + \frac{1}{2} f_{abc}(X) A^b \wedge A^c + k^{ab}(X) C_b, \quad (5.64)
\]
\[
F^{(C)}_a = dC_a - f_{bac}(X) C_b A^c - \rho_j^i A_j \Xi^i + \frac{1}{3!} h_{abcd}(X) A^b A^c A^d, \quad (5.65)
\]
\[
F^{(\Xi)}_i = d\Xi_i - \frac{1}{2} \frac{\partial f_{abc}}{\partial X^i}(X) C_a A^b \wedge A^c - \frac{\partial \rho^a_j}{\partial X^i}(X) \Xi^j A^a - \frac{1}{2} \frac{\partial k^{ab}}{\partial X^i}(X) C_a C_b A^d + \frac{1}{4!} \frac{\partial h_{abcd}}{\partial X^i}(X) A^a A^b A^c A^d, \quad (5.66)
\]

where \(F^{(C)}\) and \(F^{(\Xi)}\) are the super field strengths of \(C\) and \(\Xi\), respectively. When we substitute the component expansions to Eqs. (5.63)–(5.66), then the corresponding degree-\(|z| + 1\) parts are the field strengths:

\[
F^i_X = dX^i - \rho^i_a(X) A^a, \quad (5.67)
\]
\[
F^a_A = dA^a + \frac{1}{2} f_{abc}(X) A^b \wedge A^c + k^{ab}(X) C_b, \quad (5.68)
\]
\[
F^{(C)}_a = dC_a - f_{bac}(X) C_b \wedge A^c - \rho_j^i A_j \Xi^i + \frac{1}{3!} h_{abcd}(X) A^b \wedge A^c \wedge A^d, \quad (5.69)
\]
\[
F^{(\Xi)}_i = d\Xi_i - \frac{1}{2} \frac{\partial f_{abc}}{\partial X^i}(X) C_a \wedge A^b \wedge A^c - \frac{\partial \rho^a_j}{\partial X^i}(X) \Xi^j \wedge A^a - \frac{1}{2} \frac{\partial k^{ab}}{\partial X^i}(X) C_a \wedge C_b \wedge A^d + \frac{1}{4!} \frac{\partial h_{abcd}}{\partial X^i}(X) A^a \wedge A^b \wedge A^c \wedge A^d. \quad (5.70)
\]

The degree-\(|z|\) parts of the component expansions of the super field strengths yield the gauge transformations

\[
\delta X^i = - \rho^i_a(X) \epsilon^a, \quad (5.71)
\]
\[
\delta A^a = d \epsilon^a + f_{abc}(X) A^b \epsilon^c + k^{ab}(X) \epsilon'_b, \quad (5.72)
\]
\[
\delta C_a = d \epsilon_a' - f^b_{ac} (X) (\epsilon'_b \wedge A^c + C_b \wedge \epsilon^c) - \rho^i_a (X) \mu'_i + \frac{1}{2} h_{abcd} (X) A^b \wedge A^c \epsilon^d, \tag{5.73}
\]

\[
\delta \Xi_i = d \mu'_i - \frac{1}{2} \frac{\partial f^a_{bc} (X)}{\partial X^i} (\epsilon'_a \wedge A^b \wedge A^c + 2 C_a \wedge A^b \epsilon^c) - \frac{\partial \rho^i_a (X)}{\partial X^i} (\mu'_i \wedge A^a + \Xi_j \epsilon^a) - \frac{\partial k_{ab}}{\partial X^i} (X) C_a \wedge \epsilon_b' + \frac{1}{3!} \frac{\partial h_{abcd}}{\partial X^i} (X) A^a \wedge A^b \wedge A^c \epsilon^d. \tag{5.74}
\]

The gauge transformations of the field strengths are

\[
\delta F^i_X = \frac{\partial \rho^i_a}{\partial X^j} F^j_X \epsilon^a, \tag{5.75}
\]

\[
\delta F^a_A = -f^a_{bc} F^b_X \epsilon^c - \frac{\partial k_{ab}}{\partial X^j} F^j_X \wedge \epsilon'_b - \frac{\partial f^a_{bc}}{\partial X^j} F^j_X \wedge A^b \epsilon^c. \tag{5.76}
\]

One recognizes from Eq. (5.76) that \( F^a_A \) does not transform off-shell covariantly unless \( k_{ab} (X) \) and \( f^a_{bc} (X) \) are constants.

We seek nontrivial deformations of gauge transformations and field strengths that lead to off-shell covariant gauge structures. This is done by adding terms to the field strengths and gauge transformations using the fundamental fields and lower-form field strengths. Before introducing deformation terms, the auxiliary gauge fields are projected out by imposing \( \Xi_i = C_a = 0 \).

By form degree counting, we assume the following structure of deformations of the field strengths in terms of \( X^i \) and \( A^a \):

\[
\hat{F}^i_X = F^i_X = dX^i - \rho^i_a (X) A^a, \tag{5.77}
\]

\[
\hat{F}^a_A = F^a_A \big|_{C_a = 0} + K^a_{ij} (X) F^j_X \wedge A^c + L^a_{ij} (X) F^j_X \wedge F_X^j \]

\[
= dA^a + \frac{1}{2} f^a_{bc} (X) A^b \wedge A^c + K^a_{ij} (X) F^j_X \wedge A^c + L^a_{ij} (X) F^j_X \wedge F^j_X, \tag{5.78}
\]

where \( K^a_{ij} (X) \) and \( L^a_{ij} (X) \) are functions. The gauge transformations of \( (X^i, A^a) \) should be of the following form:

\[
\hat{\delta} X^i = \delta X^i = -\rho^i_a (X) \hat{\epsilon}^a, \tag{5.79}
\]

\[
\hat{\delta} A^a = \delta A^a + N^a_{ci} (X) F^i_X \hat{\epsilon}^c
\]

\[
= d \hat{\epsilon}^a + f^a_{bc} (X) A^b \hat{\epsilon}^c + N^a_{ci} (X) F^i_X \hat{\epsilon}^c, \tag{5.80}
\]

where \( N^a_{ci} (X) \) is a function.

Let us compute the gauge transformations of Eqs. (5.77) and (5.78) using Eqs. (5.79) and (5.80). Employing the Bianchi identities derived from Eq. (2.10),

\[
d F^i_X = \frac{\partial \rho^i_a}{\partial X^j} F^j_X A^a + \rho^i_a F^a_A, \tag{5.81}
\]

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we can compute $\hat{\delta} F_A^a$. The requirement that the coefficients of $F_X^i d e^h$, $F_X^i \wedge A^a e^b$, $F_X^i \wedge A^a e^b$, and $\delta F_A^i \wedge F_X^j e^b$ in $\hat{\delta} F_A^a$ vanish gives relations among $K$, $L$, and $N$:

$$N_{bi}^a = K_{bi}^a,$$

$$\frac{\partial f_{bc}^a}{\partial X^i} + f_{db}^a K_{ci}^d + K_{bi}^d d_{bc}^d - \frac{\partial N_{ci}^a}{\partial X^j} \rho_{ib}^j + \frac{\partial K_{bi}^d}{\partial X^j} \rho_{ib}^j - N_{ci}^a \frac{\partial \rho_{ib}^j}{\partial X^i} + K_{bi}^d \frac{\partial \rho_{ib}^j}{\partial X^i} = f_{de}^a K_{bi}^d,$$

$$\frac{1}{2} \frac{\partial N_{cj}^a}{\partial X^i} + \frac{1}{2} K_{bi}^d N_{cj}^b + L_{bi}^c \frac{\partial \rho_{cj}^b}{\partial X^j} - (i \leftrightarrow j) = f_{bc}^a t_{ij}^b - \frac{\partial L_{ij}^a}{\partial X^k} \rho_{cj}^b.$$  

Under these conditions the field strength is off-shell covariant:

$$\hat{\delta} F_A^a = -(f_{bc}^a + N_{ci}^a \rho_{ib}^j) \hat{F}_A^b e^c.$$

The final step of our strategy is to solve the above system of equations. Although it might be a cumbersome task to find the general solution, we can find a nice subset of solutions using a suitable ansatz, as we will see in the next section. Based on this ansatz, concrete examples can be computed. In particular, the second example provides a new result.

5.1. Examples

5.1.1. Stückelberg formalism

First, we consider a trivial example to show that this formalism is a generalization of a known formalism. The starting point is the QP-manifold $M_3 = T^* [3] E[1]$, where $E = TM$ is a tangent bundle. Recall that the local coordinates $(x^i, q^a, \xi_i, p_a)$ are of degree $(0, 1, 3, 2)$. We take

$$f_{bc}^a = k^{ab}(x) = h_{abcd} = 0,$$

$$\rho_{ij}^a = m \delta_{ij}^a = \text{constant},$$

where $i$ and $a$ run over the same index range. Then, the Hamiltonian function is

$$\Theta = m \xi_a q^a,$$

which trivially satisfies the classical master equation, $\{\Theta, \Theta\} = 0$. The resulting field strengths are

$$F_X^a = dX^a - mA^a,$$

$$F_A^a = dA^a,$$

$$F_{a(C)}^a = dC_a + m \Xi_a,$$

$$F_{a(\Xi)}^a = d\Xi_a.$$  

The gauge transformations of the gauge fields are

$$\delta X^a = - m e^a,$$

$$\delta A^a = d e^a,$$

$$\delta C_a = d e_a^i + m \mu_a^i,$$

$$\delta \Xi_a = d \mu_a^i.$$  

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From these equations, the gauge transformations of the field strengths are trivially covariant:

\[
\delta F^a_X = 0, \quad (5.95)
\]
\[
\delta F^a_A = 0. \quad (5.96)
\]

The gauge-invariant action,

\[
S = \int \text{tr}((F_A \wedge *F_A) + (F_X \wedge *F_X))
\]
\[
= \int F_{\mu\nu}^A F_A^{\mu\nu} + (\partial_\mu X^a - m A^a_\mu)(\partial^\mu X^a - m A^{\mu a}), \quad (5.97)
\]

is the so-called Stückelberg formalism of the massive vector field \( A^a_\mu \). We conclude that our formalism provides a nonlinear generalization of the Stückelberg formalism.

### 5.1.2. Non-Abelian gauged nonlinear sigma models

We list a simple but nontrivial example, taking again \( M_3 \) with local coordinates \((x^i, q^a, \xi_i, p_a)\) of degree \((0, 1, 3, 2)\) as a starting point. Let the structure constants be

\[
f^{abc} = \text{constant}, \quad \rho^i_{\cdot a} = h_{abcd} = 0, \quad k^{ab}(x) = \text{arbitrary}.
\]

Then, the Hamiltonian function is

\[
\Theta = \frac{1}{2} f^{abc} q^b q^c p_a + \frac{1}{2} k^{ab}(x)p_a p_b. \quad (5.98)
\]

The resulting field strengths are

\[
F^i_X = dx^i, \quad (5.99)
\]
\[
F^a_A = dA^a + \frac{1}{2} f^{abc} A^b A^c + k^{ab} C_b, \quad (5.100)
\]
\[
F^{(C)}_a = dC_a - f^{bc} A^b \wedge C_c, \quad (5.101)
\]
\[
F^i(\Xi) = d\Xi_i - \frac{1}{2} \frac{\partial k^{ab}}{\partial X^i} (X) C_a \wedge C_b. \quad (5.102)
\]

The gauge transformations of the gauge fields are

\[
\delta X^i = 0, \quad (5.103)
\]
\[
\delta A^a = d\epsilon^a + f^{abc} A^b \epsilon^c + k^{ab}(X) \epsilon^b, \quad (5.104)
\]
\[
\delta C_a = d\epsilon'_a - f^{bc} A^c \wedge \epsilon'_b + \epsilon^c C_b, \quad (5.105)
\]
\[
\delta \Xi_i = d\mu'_i - \frac{\partial k^{ab}}{\partial X^i} (X) C_a \wedge \epsilon'_b. \quad (5.106)
\]

Using these equations, we compute the gauge transformations of the field strengths as

\[
\delta F^i_X = 0, \quad (5.107)
\]
\[
\delta F^a_A = -f^{abc} F^b_A \epsilon^c - \frac{\partial k^{ab}}{\partial X^i} F^i_X \wedge \epsilon'_b. \quad (5.108)
\]

The gauge transformation of \( F^a_A \) is not off-shell covariant.
Let us apply our formalism to this system. A solution of Eqs. (5.82)–(5.84) in this example is

\[ K^a_{bi} = N^a_{bi} = \delta^a_b \frac{\partial w}{\partial X^i}(X), \quad L^a_{ij} = 0, \quad (5.109) \]

where \( w(X) \) is an arbitrary function. The covariantized field strengths and gauge transformations are computed as

\[ \hat{F}^i_X = F^i_X = dX^i, \quad (5.110) \]
\[ \hat{F}^a_A = dA^a + \frac{1}{2} f^{ab}c A^b A^c + \frac{\partial w}{\partial X^i} F^i_X \wedge A^a, \quad (5.111) \]
\[ \hat{\delta} A^a = d\hat{\epsilon}^a + f^{ab}c A^b \hat{\epsilon}^c + \frac{\partial w}{\partial X^i} F^i_X \hat{\epsilon}^a, \quad (5.112) \]
\[ \hat{\delta} X^i = 0. \quad (5.113) \]

Finally, we obtain

\[ \hat{\delta} F^i_X = 0, \quad (5.114) \]
\[ \hat{\delta} F^a_A = -f^{ab}c \hat{F}^b_A \hat{\epsilon}^c. \quad (5.115) \]

Assume that \( M \) is 1D. Then, we drop the index \( i \) and take

\[ K^a_{b} = N^a_{b} = \frac{\delta^a_b}{X}, \quad (5.116) \]

which yields

\[ F_X = dX, \quad (5.117) \]
\[ \hat{F}^a_A = dA^a + \frac{1}{2} f^{ab}c A^b A^c + \frac{1}{X} F_X \wedge A^a, \quad (5.118) \]
\[ \hat{\delta} A^a = d\hat{\epsilon}^a + f^{ab}c A^b \hat{\epsilon}^c + \frac{1}{X} F_X \hat{\epsilon}^a, \quad (5.119) \]
\[ \hat{\delta} X = 0. \quad (5.120) \]

By the redefinition of \( X \) via

\[ \varphi = \log |X|, \quad (5.121) \]

the equations can be rewritten in a nonsingular form:

\[ F_X = e^\varphi d\varphi, \quad (5.122) \]
\[ \hat{F}^a_A = dA^a + \frac{1}{2} f^{ab}c A^b A^c + d\varphi \wedge A^a, \quad (5.123) \]
\[ \hat{\delta} A^a = d\hat{\epsilon}^a + f^{ab}c A^b \hat{\epsilon}^c + d\varphi \hat{\epsilon}^a, \quad (5.124) \]
\[ \hat{\delta} \varphi = 0. \quad (5.125) \]
5.1.3. Kotov–Strobl model

As third example we formulate the model proposed in Ref. [4]. Here, we consider a QP-manifold of degree two, $\mathcal{M}_2 = T^*\mathbb{R}^2\mathbb{E}$, in order to demonstrate the covariantization procedure for the Kotov–Strobl model. Note that the resulting gauge theory is not restricted to any dimension. The local coordinates of $\mathcal{M}_2$ are denoted by $(x^i, \xi, q^a)$ of degree $(0, 2, 1)$. The fiber coordinates of $\mathbb{E}$ are identified by introducing a fiber metric $\lambda_{ab}$. The graded symplectic form is defined by

$$\omega = \delta x^i \wedge \delta \xi_i + \frac{1}{2} \lambda_{ab}(x) \delta q^a \wedge \delta q^b.$$  \hfill (5.126)

The most general form of the Hamiltonian is given by

$$\Theta = \rho^i_a(x) \xi_i q^a + \frac{1}{3!} h_{abc}(x) q^a q^b q^c.$$  \hfill (5.127)

In order to construct the Kotov–Strobl model, we take $\mathcal{M}$ as a 2D manifold and $\mathbb{E}$ a vector bundle over $\mathcal{M}$ with 1D fiber. Let us denote the local coordinates of $\mathcal{M}_2$ by $(x, y) := (x^1, x^2)$, $(\xi, \eta) := (\xi_1, \xi_2)$, and $q := q^1$ and take the following Hamiltonian function:

$$\Theta = -e^{-\frac{1}{2} \xi y} \eta q,$$  \hfill (5.128)

where $\lambda$ is a constant. That corresponds to choosing

$$h_{abc} = 0, \quad \lambda_{11} = 1, \quad \rho^1 = 0, \quad \rho^2 = e^{-\frac{1}{2} \xi y}.$$  \hfill (5.129)

The associated superfields are defined as

$$X \equiv a^a(x), \quad Y \equiv a^a(y), \quad A \equiv a^a(q), \quad \Xi \equiv a^a(\xi), \quad H \equiv a^a(\eta).$$  \hfill (5.130)

Using Eqs. (2.13) and (2.11), we obtain the following field strengths:

$$F_X = dX,$$  \hfill (5.131)

$$F_Y = dY - e^{-\frac{1}{2} \xi y} A,$$  \hfill (5.132)

$$F_A = dA + e^{-\frac{1}{2} \xi y} H,$$  \hfill (5.133)

$$F_\Xi = d\Xi - \frac{\lambda}{2} Ye^{-\frac{1}{2} \xi y} HA,$$  \hfill (5.134)

$$F_H = dH - \frac{\lambda}{2} Xe^{-\frac{1}{2} \xi y} HA.$$  \hfill (5.135)

---

\textsuperscript{5} We can use the fiber coordinates $(q^a, p_a)$ of $E[1]$ and $E^*[1]$. 

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and gauge transformations of the gauge fields:

\[ \delta X = 0, \quad (5.136) \]
\[ \delta Y = -e^{-\frac{1}{2}XY} \epsilon, \quad (5.137) \]
\[ \delta A = d\epsilon + e^{-\frac{1}{2}XY} \mu'_2, \quad (5.138) \]
\[ \delta \Xi = d\mu'_1 - \frac{\lambda}{2} Ye^{-\frac{1}{2}XY} (\mu'_2 A + H \epsilon), \quad (5.139) \]
\[ \delta H = d\mu'_2 - \frac{\lambda}{2} Xe^{-\frac{1}{2}XY} (\mu'_2 A + H \epsilon). \quad (5.140) \]

Here, \( \epsilon \) is the 0-form gauge parameter corresponding to \( A \), and \( \mu'_1 \) and \( \mu'_2 \) are the 1-form gauge parameters corresponding to \( \Xi \) and \( H \), respectively. We are only interested in the gauge transformations and field strengths of the fields \((X, Y, A)\). The gauge transformations of the field strengths are computed as

\[ \delta F_X = 0, \quad (5.141) \]
\[ \delta F_Y = -\frac{\lambda}{2} Ye^{-\frac{1}{2}XY} F_X \epsilon - \frac{\lambda}{2} Xe^{-\frac{1}{2}XY} F_Y \epsilon - e^{-\frac{1}{2}XY} \mu'_2, \quad (5.142) \]
\[ \delta F_A^a = \frac{\lambda}{2} Ye^{-\frac{1}{2}XY} F_X \mu'_2 + \frac{\lambda}{2} Xe^{-\frac{1}{2}XY} F_Y \mu'_2. \quad (5.143) \]

The gauge transformations of \( F_Y \) and \( F_A \) are not off-shell covariant.

We apply the off-shell covariantization procedure to this theory. The possible deformations of the field strengths and gauge transformations are

\[ \hat{F}_A = F_A + J(X, Y) F_X \wedge A + K(X, Y) F_Y \wedge A + L(X, Y) F_X \wedge F_Y, \quad (5.144) \]
\[ \hat{\delta} A = d\hat{\epsilon} + M(X, Y) F_X \wedge \hat{\epsilon} + N(X, Y) F_Y \wedge \hat{\epsilon}, \quad (5.145) \]

where we determine the functions \( J, K, L, M, N \) of the scalar fields \( X, Y \). Deformations of the other field strengths and gauge transformations need not be considered.

One solution is \( M = -\frac{1}{2} Y \) and \( N = 0 \). In this case, \( \hat{\delta} F_Y \) is covariantized as

\[ \hat{\delta} F_Y = -\frac{\lambda}{2} e^{-\frac{1}{2}XY} XF_Y \hat{\epsilon}. \quad (5.146) \]

In the next step, we require off-shell covariance of \( \hat{\delta} F_A \). This determines \( J = -\frac{1}{2} Y, K = 0, \) and \( L = -\frac{1}{2} Ye^{\frac{1}{2}XY} \). The resulting field strengths and gauge transformations are

\[ \hat{F}_A = dA - \frac{\lambda}{2} Ye^{\frac{1}{2}XY} F_X \wedge F_Y \]
\[ = dA - \frac{\lambda}{2} Ye^{\frac{1}{2}XY} dX \wedge dY, \quad (5.147) \]
\[ \hat{\delta} A = d\hat{\epsilon} - \frac{\lambda}{2} Ye^{\frac{1}{2}XY} \hat{\epsilon}. \quad (5.148) \]

The gauge transformation of \( \hat{F}_A \) is computed as

\[ \hat{\delta} F_A = 0, \quad (5.149) \]

which is off-shell covariant.
Invariant action  Since the scalar field strength $F^i_X = dX^i - \rho^i_a(X)A^a$ transforms off-shell covariantly,

$$\hat{\delta} F^i_X = \frac{\partial \rho^i_a}{\partial X^j} F^j_X \hat{\epsilon}^a,$$  \hfill (5.150)

the action

$$\int g_{ij}(X) F^i_X \wedge * F^j_X$$  \hfill (5.151)

is invariant if $g_{ij}(X)$ satisfies

$$\hat{\delta} g_{ij}(X) = - \left( g_{kj} \frac{\partial \rho^k_a}{\partial X^i} + g_{ik} \frac{\partial \rho^k_a}{\partial X^j} \right) \hat{\epsilon}^a. \hfill (5.152)$$

In this example, the action is given by

$$S = \int F_X \wedge * F_X + V(X) + e^{\lambda XY} F_Y \wedge * F_Y + \hat{F}_A \wedge * \hat{F}_A$$  \hfill (5.153)

and is invariant under gauge transformations. The gauge transformation of the third term is given by

$$\hat{\delta} (e^{\lambda XY} F_Y \wedge * F_Y) = 2e^{\lambda XY} (\hat{\delta} F_Y) \wedge * F_Y - (\hat{\delta} e^{\lambda XY}) F_Y \wedge * F_Y$$

$$= e^{\lambda XY} \lambda X \epsilon F_Y \wedge * F_Y - e^{\lambda XY} \lambda X \epsilon F_Y \wedge * F_Y = 0. \hfill (5.154)$$

Correspondence to the Kotov–Strobl model.  By the off-shell covariantization procedure, we obtain the field strengths

$$F_X = dX,$$  \hfill (5.155)

$$F_Y = dY - e^{-\frac{\lambda}{2}XY} A,$$  \hfill (5.156)

$$\hat{F}_A = dA + \frac{\lambda}{2} e^{\frac{\lambda}{2}XY} YdY \wedge dX,$$  \hfill (5.157)

and the gauge transformations of the gauge fields

$$\hat{\delta} X = 0,$$  \hfill (5.158)

$$\hat{\delta} Y = - e^{-\frac{\lambda}{2}XY} \hat{\epsilon},$$  \hfill (5.159)

$$\hat{\delta} A = d\hat{\epsilon} - \frac{\lambda}{2} dXY \hat{\epsilon}. \hfill (5.160)$$

From these equations, the gauge transformations of the field strengths are computed as

$$\hat{\delta} F_X = 0,$$  \hfill (5.161)

$$\hat{\delta} F_Y = - \frac{\lambda}{2} e^{-\frac{\lambda}{2}XY} XF_Y \hat{\epsilon},$$  \hfill (5.162)

$$\hat{\delta} F_A = 0.$$  \hfill (5.163)
Redefining the gauge field $A$, the gauge parameter $\epsilon$, and the field strength $F_A$ as

\begin{align}
\tilde{\epsilon} &\equiv e^{-\frac{1}{2}XY} \epsilon, \\
\tilde{A} &\equiv e^{-\frac{1}{2}XY} A, \\
G_A &\equiv e^{-\frac{1}{2}XY} F_A,
\end{align}

we obtain the following field strengths from Eqs. (5.155)–(5.157):

\begin{align}
F_X &= dX, \\
F_Y &= dY - \tilde{A}, \\
G_A &= d\tilde{A} + \frac{\lambda}{2} (XdY \wedge \tilde{A} + YF_Y \wedge dX).
\end{align}

These are the field strengths discussed in Ref. [4]. We can rewrite the gauge transformations of the gauge fields using $\tilde{\epsilon}$ by

\begin{align}
\hat{\delta} X &= 0, \\
\hat{\delta} Y &= -\tilde{\epsilon}, \\
\hat{\delta} \tilde{A} &= d\tilde{\epsilon} - \frac{\lambda}{2} XF_Y \tilde{\epsilon}.
\end{align}

Then, the gauge transformations of the field strengths are given by

\begin{align}
\hat{\delta} F_X &= 0, \\
\hat{\delta} F_Y &= \frac{\lambda}{2} XF_Y \tilde{\epsilon}, \\
\hat{\delta} G_A &= \frac{\lambda}{2} XG_A \tilde{\epsilon} - \frac{\lambda}{2} XF_Y \wedge F_X \tilde{\epsilon} + \left(\frac{\lambda}{2}\right)^2 X(1 - Y)F_Y \wedge F_X \tilde{\epsilon},
\end{align}

which are the same expressions as in Ref. [4].

6. Gauge algebras

Finally, we discuss the closure of the gauge symmetry algebra. For this, we write the gauge transformations as

\begin{align}
\hat{\delta} X^i &= -\rho^i_a(X) \tilde{\epsilon}^a, \\
\hat{\delta} A^a &= d\tilde{\epsilon}^a + f^{a}{}_{bc} A^b \tilde{\epsilon}^c + N^a_{\alpha} F^i_\alpha \tilde{\epsilon}^c,
\end{align}

where the gauge parameter $\tilde{\epsilon}^a$ is an ordinary function. We find that two gauge transformations $\tilde{\delta}_1$ and $\tilde{\delta}_2$ close to $\tilde{\delta}_3$ by $[\tilde{\delta}_1, \tilde{\delta}_2] = \tilde{\delta}_3$ with $\tilde{\epsilon}^a = f^{a}{}_{bc} \tilde{\epsilon}^b \tilde{\epsilon}^c$, where $\tilde{\delta}_i$ denotes the gauge transformation with respective gauge parameters $\tilde{\epsilon}_i$:

\begin{align}
\left[\tilde{\delta}_1, \tilde{\delta}_2\right] X^i &= \tilde{\delta}_3 X^i, \\
\left[\tilde{\delta}_1, \tilde{\delta}_2\right] A^a &= \tilde{\delta}_3 A^a + A_{\alpha}^{a} F^i_{\alpha} \tilde{\epsilon}^b \tilde{\epsilon}^c,
\end{align}
where

\[
\Lambda^a_{ibc} = -\frac{1}{2} \mathcal{N}^a_{dib} f^d_{bc} + f^a_{dcb} \mathcal{N}^d_{bi} + \frac{\partial \mathcal{N}^a_{hi}}{\partial X^j} \rho^j_c + \mathcal{N}^a_{cij} \frac{\partial \rho^j_h}{\partial X^i} - (b \leftrightarrow c). \quad (6.180)
\]

The gauge transformation of \( X^i \) is off-shell closed. Off-shell closure of the gauge transformation of \( A^a \) requires

\[
\Lambda^a_{ibc} = 0, \quad (6.181)
\]

which is satisfied in our examples.

7. Discussion

In this paper, we have investigated the method to obtain off-shell covariant gauge transformations and field strengths of higher gauge theories in Ref. [8] and applied it to a system of algebroid gauge theory with 1-form gauge field and scalars, where the structure constants become scalar-field-dependent functions. We have demonstrated off-shell covariantization of a gauge theory based on a Lie 2-algebroid and a Lie 3-algebroid. Recall that the resulting gauge theory is not restricted to any dimension. For covariantization, we deform field strengths and gauge transformations. The starting point of this procedure is an on-shell (i.e., \( F_z = 0 \)) covariant theory. Since the gauge transformations and field strengths are deformed by terms proportional to the lower field strengths, they are consistent if the theory is kept on-shell.

There are several directions to develop the approach presented in this paper. The extension of the method to gauge theories with Lie \( n \)-algebroid gauge symmetry induced from a QP-manifold of degree \( n \) is straightforward. Similar conditions corresponding to Eqs. (5.82)–(5.84) can be computed for arbitrary \( n \).

Here, we have formulated the Kotov–Strobl model using a QP-manifold of degree two. However, we can also construct the Kotov–Strobl model from a QP-manifold of degree three. For this, a further generalization of the procedure is necessary. Another possible application of our method is to investigate multiple M5-brane systems [24,25]. We can add scalar fields to the analysis conducted in Ref. [8]. The procedure in this paper can also be applied to investigate the properties of supergravity in connection with tensor hierarchy. Furthermore, gauge theoretical formulations of gravity such as the vielbein formalism or the gauge theory of the Poincaré group can also be treated in this formalism. It would also be interesting to compare the present formalism with the approach taken in Ref. [26]. We expect that our approach will shed new light on the analysis of such systems.

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