The current concordance model of cosmology makes definite predictions for the number density of massive gravitationally bound objects (visible as galaxy clusters) in the Universe and how this distribution evolves with redshift. It has been suggested that some recently observed objects are too massive at too early a time to exist within a standard model universe, hinting that modifications may be necessary. It is possible to consider this problem within the framework of Extreme Value Statistics (EVS): are any of these objects more extreme than the most unusual objects we expect to observe? The current answer appears to be ‘no’ and so the standard model currently passes this particular test. However, it is plausible that future observations will observe objects too extreme for the concordance model; it may then also be possible to perform model selection between different models of enhanced structure formation using these extreme clusters.

1 The Problem of Big Clusters

Cosmology in 2012 has a ‘concordance’ or standard model which, amongst other things, allows us to predict how structures should form and grow in the Universe. In particular, we can predict the co-moving number density of collapsed Cold Dark Matter haloes (visible via the galaxy clusters which reside within them) at a given mass and redshift: $n(m, z)$, the halo mass function. This distribution is expected to have a very steep cutoff at higher masses ($m \gtrsim 10^{15} M_\odot$), the steepness and evolution of which is sensitive to the initial conditions of the CDM density field, nature of gravitational collapse and background expansion rate. Hence, the abundance of massive objects in our universe has the potential to constrain primordial non-Gaussianity, modified gravity and the nature of dark energy.

In recent years new experiments such as XMM-Newton, ACT, SPT and Planck have begun to observe the Universe with sufficient sensitivity to see the most massive objects at appreciable redshifts ($z \sim 1-2$), allowing us to probe structure formation at these times. As these surveys have been ongoing, a number of analyses have calculated the expected abundance of the most
massive objects found within them. All of these analyses found galaxy clusters with masses and redshifts which meant they appeared “too-big, too-early” to exist naturally within a ΛCDM universe, indicating new physics away from the concordance model may be necessary. Furthermore, some suggested the inclusion of primordial non-Gaussianity on of $f_{NL} \sim 300-500$ could ease this tension, 1-2 orders of magnitude greater than what is observed in the CMB.

Unfortunately, these analyses have been shown to contain a bias which, when accounted for, causes the tension with the concordance model to disappear. However, we have currently only surveyed a small portion of the sky with the sensitivity to detect such clusters, and so it may be reasonable to suppose a ‘ΛCDM-killer’ may still be waiting for us to observe it. If so, it is desirable that we have statistical machinery available in order to be able to truly determine how ‘extreme’ a given cluster is without the biases present in previous methods.

2 Extreme Value Statistics of Galaxy Clusters

Rather than making inference using the mean of a sample from a probability distribution, Extreme Values Statistics (EVS) seeks to make inference using sample extrema (i.e. the greatest or least value within a sample). If $N$ i.i.d. random deviates are drawn from the underlying cumulative distribution function $F(m)$ then the probability that the largest $M_{\text{max}} \leq m$ is simply the product that each individual draw is $\leq m$, allowing us to find the probability and cumulative distribution functions:

$$\Phi(M_{\text{max}} \leq m; N) = F^N(m)$$

$$\phi(M_{\text{max}} = m; N) = NF(m) [F(m)]^{N-1}.$$  (2)

It is a seminal result in EVS that, in analogy with the central limit theorem for sample means, in the $N \rightarrow \infty$ limit the distribution of sample extremes will tend to an asymptotic distribution, the Generalised Extreme Value (GEV) distribution:

$$P_{\text{GEV}}(M_{\text{max}} \leq m; \alpha, \beta, \gamma) = \exp \left\{ - \left[ 1 + \gamma \left( \frac{m - \alpha}{\beta} \right) \right]^{-1/\gamma} \right\}$$  (3)

where $\alpha$ and $\beta$ are location and scale parameters and $\gamma$ is the shape parameter, which depends on the underlying distribution. For some distributions it is possible to analytically determine the asymptotic behaviour; for a Gaussian distribution, $\gamma = 0$ is the asymptotic value. However, if we wish to make predictions for EVS of galaxy clusters residing in CDM haloes, we can make use of the exact distribution (1) by forming the underlying distribution from the halo mass function $n(m, z)$.

3 Results

3.1 Null Tests

Figure 1 shows probability contours calculated using EVS for the mass of most massive galaxy cluster in a ΛCDM universe at each redshift, along with several of the most massive galaxy clusters so far observed. As can be seen, no currently observed cluster lies above the 99% region of this plot and so the concordance model survives this test.

If, in the future, a galaxy cluster is observed in the area above this region ruling out ΛCDM, we can also consider whether it is possible to exclude or allow different models of enhanced structure formation. To this end, we construct the EVS of galaxy clusters within two alternative cosmologies:
1. Primordial non-Gaussianity of the local type, parameterised by \( f_{NL} \) and included as an alteration to the LCDM halo mass function via a non-Gaussian correction factor \( R(f_{NL}) \).

2. ‘SUGRA003’ in which a quintessence dark energy field is coupled to the dark matter particles and has a supergravity-motivated potential. The presence of a global minimum in this potential means that the evolution of the scalar field will change sign at a particular redshift \( z_{\text{bounce}} \), with structure formation being enhanced above \( z_{\text{bounce}} \) and depleted below it.

In order to account for the full non-linear behaviour of these cosmologies we make use of the publicly available CoDECs N-body simulations\(^5\). As can be seen in figure 2, observations of ‘too massive’ clusters at different redshifts can favour or disfavour different alternative models.

**Figure 1:** 66\%, 95\% and 99\% confidence regions for the mass of the most massive cluster at a given redshift in a \( \Lambda \)CDM universe, along with the most massive high and low redshift clusters so far observed.

**Figure 2:** Shaded regions are as figure 1 for a \( \Lambda \)CDM simulation, dashed lines are the edges of these contours in the two alternative models. Lower panels show the enhancement in \( \langle M_{\text{max}} \rangle \).
3.2 Parameter Estimation

Whilst it is possible to rule out individual models using such null tests, a more efficient search technique for new physics is to parameterise away from the fiducial model and attempt to constrain those parameters. In order to place both upper and lower bounds on a parameter using extreme galaxy clusters, it is necessary to guarantee that there is no more massive cluster in the sky which has merely been missed by an observational survey. This requires a truly mass-limited survey. Fortunately, surveys which select clusters using the thermal Sunyaev-Zel'dovich (tSZ) effect are capable of approaching this ideal.

The South Pole Telescope has recently conducted a tSZ cluster survey over 2500° of the sky and present the 26 clusters with highest tSZ S/N in the region. Unfortunately, S/N does not correlate directly with mass and the survey is not expected to be truly complete. However, we make the approximation that these 26 are in fact the most massive which exist within the survey region, interpolating between a current experiment and a future one with a mass threshold low enough to ensure the contours in Figure 1 lie completely above it. We then divide the redshift intervals into bins, and assert that, if a bin is empty of clusters, the most massive cluster in that bin is not more massive than the lowest of these 26 clusters. We can then form a likelihood over e.g. the $f_{NL}$ parameter, as shown in Figure 3.

![Figure 3: Limits on $f_{NL}$ from the 26 extreme clusters in the idealised experiment described in the text. The solid line represents a sharp prior on $\sigma_8$, whilst the dashed one marginalises over the WMAP7 prior.](image)

References