Conformal Field Theories in the Epsilon and $1/N$ Expansions

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Abstract

In this thesis, we study various conformal field theories in two different approximation schemes - the \( \epsilon \)-expansion in dimensional continuation, and the large \( N \) expansion. We first propose a cubic theory in \( d = 6 - \epsilon \) as the UV completion of the quartic scalar \( O(N) \) theory in \( d > 4 \). We study this theory to three-loop order and show that various operator dimensions are consistent with large-\( N \) results. This theory possesses an IR stable fixed point at real couplings for \( N > 1038 \), suggesting the existence of a perturbatively unitary interacting \( O(N) \) symmetric CFT in \( d = 5 \). Extending this model to \( Sp(N) \) symmetric theories, we find an interacting non-unitary CFT in \( d = 5 \). For the special case of \( Sp(2) \), the IR fixed point possesses an enhanced symmetry given by the supergroup \( OSp(1|2) \). We also observe that various operator dimensions of the \( Sp(2) \) theory match those from the 0-state Potts model. We provide a graph theoretic proof showing that the zero, two, and three-point functions in the \( Sp(2) \) model and the 0-state Potts model indeed match to all orders in perturbation theory, strongly suggesting their equivalence. We then study two fermionic theories in \( d = 2 + \epsilon \) - the Gross-Neveu model and the Nambu-Jona-Lasinio model, together with their UV completions in \( d = 4 - \epsilon \) given by the Gross-Neveu-Yukawa and the Nambu-Jona-Lasinio-Yukawa theories. We compute their sphere free energy and certain operator dimensions, passing all checks against large-\( N \) results. We use two sided Padé approximations with our \( \epsilon \)-expansion results to obtain estimates of various quantities in the physical dimension \( d = 3 \). Finally, we provide evidence that the \( N = 1 \) Gross-Neveu-Yukawa model which contains a 2-component Majorana fermion, and the \( N = 2 \) Nambu-Jona-Lasinio-Yukawa model which contains a 2-component Dirac fermion, both have emergent supersymmetry.
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Chapter 1

Introduction

1.1 Universality and Renormalization

Conformal Field Theories (CFT) have important applications in many areas from string theory to condensed matter physics. They are Quantum Field Theories (QFT) which are invariant under conformal transformations - transformations that preserve angles locally. In particular, scaling invariance is a part of conformal invariance. At the critical points of statistical systems, the correlation length scale $\xi$ diverges. In the continuum theory that describes the physics across distances that are large compared to the lattice spacing, the correlation functions are insensitive to the microscopic details of the model and become scale-invariant. Thus, CFTs play a primary role in the understanding of critical points in statistical mechanics. Remarkably, a wide range of seemingly unrelated statistical systems share some universal large scale properties despite their microscopic differences. In order to understand the origin of these universal properties, it is necessary to introduce the notion of renormalization.

A simple example of universality is provided by the central limit theorem. If we are summing over a large number of random variables with arbitrary distribution (provided that large deviations with respect to the mean decrease fast enough), the
asymptotic behavior is always a Gaussian, regardless of the distribution of the original random variables. One can understand this using a renormalization strategy by recursively averaging over pairs of random variables. In general, the operation of averaging over two identical random variables with distribution $\rho$ results in a random variable with a transformed distribution $T\rho$. However, the limiting distribution $\rho_*$ is necessarily a fixed point of this operation $T\rho_* = \rho_*$. One finds that the Gaussian distribution is an attractive fixed point of these operations. Thus, no matter what the initial distribution is, after repeated application of the averaging operation, all of them will tend towards the Gaussian fixed point.

Renormalization of QFTs follows a similar idea. Let’s suppose we have a QFT described by the Hamiltonian $\mathcal{H}$, and a microscopic scale $1/\Lambda$ describing, for example, the lattice spacing for a lattice model. Now we recursively integrate out short distances degrees of freedom to generate an effective Hamiltonian $\mathcal{H}_\lambda$, with $\lambda \in [0, 1]$, corresponding to increasing the scale to $1/(\lambda \Lambda)$. The various parameters in the new Hamiltonian $\mathcal{H}_\lambda$ will be different, and this leads to a transformation $T$ in the space of Hamiltonians such that:

$$\lambda \frac{d}{d\lambda} \mathcal{H}_\lambda = T[\mathcal{H}_\lambda]$$

(1.1)

Universality is related to the existence of fixed point solutions of the equation:

$$T[\mathcal{H}^*] = 0$$

(1.2)

If the above equations have an attractive fixed point, then all the Hamiltonians within the basin of attraction will eventually flow to the same fixed point - they belong to the same universality class. The Gaussian fixed point is again a stable fixed point in QFTs, but they describe free theories without interactions. However, there can also be interacting fixed points, which are more interesting.
The linear partial differential equations for the correlators of the renormalized theory, resulting from variations of the parameters under an infinitesimal change of the renormalization scale, are called renormalization group equations. In particular, using these RG equations, one can compute two functions: the $\beta$-function which describe the rate at which the couplings in $\mathcal{H}$ flows, and the anomalous dimension $\eta$, which describes the correction to the classical scaling dimensions of various operators when interactions are turned on. All of these quantities are computed order by order in perturbation theory, with higher order corrections coming from higher loop Feynman diagrams.

For example, the water-vapor phase diagram is described by the three-dimensional Ising model. Near the critical point, it is described by a Euclidean QFT of a real scalar field, $\phi$, with a $\lambda \phi^4$ interaction:

$$S = \int d^3x \left( \frac{1}{2} (\partial \phi)^2 + \frac{\lambda}{4!} \phi^4 \right).$$

The coupling constant $\lambda$ is the strength of the interaction, which will change with scale due to renormalization. It turns out, this model is weakly coupled at short distances, but it becomes strongly coupled at long distances. It is the long distance regime that is needed for describing the critical behavior, but perturbation theory in $\lambda$ cannot be applied to calculate various quantities of interest, such as the scaling dimensions of various operators. However, theorists have invented ingenious expansion schemes that have led to good approximations. Two of these include $\epsilon$-expansion in dimensional continuation, and large-$N$ expansion.

The idea behind dimensional continuation is that instead of working directly in $d = 3$, we treat $d$ as a continuous parameter. Significant simplification occurs for $d = 4 - \epsilon$ where $\epsilon \ll 1$. Then the IR stable fixed point of the Renormalization Group occurs for a weak coupling $\lambda$ of order $\epsilon$, so that a formal Wilson-Fisher expansion in $\epsilon$...
\( \epsilon \) may be developed [1]. The coefficients of the first few terms fall off rapidly, so that setting \( \epsilon = 1 \) provides a rather precise approximation that is in good agreement with the experimental and numerical results for \( d = 3 \) [2].

Another important idea has been the large-\( N \) expansion in the \( O(N) \) symmetric QFT of \( N \) real scalar fields \( \phi^i, i = 1, \ldots, N \), with interaction \( \frac{\lambda}{4}(\phi^i \phi^i)^2 \). Using a generalized Hubbard-Stratonovich transformation with an auxiliary field \( \sigma \), it is possible to develop expansion in powers of \( 1/N \) (for a comprehensive review, see [3]). The large \( N \) expansion may be developed for a range of \( d \) \([4, 5, 6, 7, 8, 9, 10, 11, 12]\) and compared with the regimes where other perturbative expansions such as dimensional continuation.

In particular, for \( d = 4 - \epsilon \), the Wilson-Fisher \( \epsilon \)-expansion may be developed for any \( N \) [2], and the results can be matched with the large \( N \) techniques. Also, for \( d = 2 + \epsilon \) the large \( N \) results match with the perturbative UV fixed point of the \( O(N) \) Non-linear Sigma Model (NL\( \sigma \)M). Using the first few terms in the large \( N \) expansion directly in \( d = 3 \) provides another approach to estimating the scaling dimensions for low values of \( N \). Thus, a combination of the large \( N \) and \( \epsilon \) expansions provides good approximations for the critical behavior in the entire range \( 2 < d < 4 \). One should note, however, that both expansions are not convergent but rather provide asymptotic series. There are continued efforts towards obtaining a more rigorous approach to the \( O(N) \) symmetric CFT’s using conformal bootstrap ideas \([13, 14, 15, 16]\), and recently it has led to more precise numerical calculations of the operator scaling dimensions in three-dimensional CFT’s \([17, 18]\). The bootstrap approach has been successfully applied not only in the range \( 2 < d < 4 \) [19], but also for \( 4 < d < 6 \) [20].
1.2 Review of Wilson-Fisher $\epsilon$-expansion

Let us consider the Euclidean field theory of $N$ real massless scalar fields with an $O(N)$ invariant quartic interaction

$$S = \int d^d x \left( \frac{1}{2} (\partial \phi^i)^2 + \frac{\lambda}{4} (\phi^i \phi^i)^2 \right). \tag{1.4}$$

As follows from dimensional analysis, the interaction term is relevant for $d < 4$ and irrelevant for $d > 4$. Hence, for $2 < d < 4$, it is expected that the quartic interaction generates a flow from the free UV fixed point to an interacting IR fixed point. In $d = 4 - \epsilon$ this fixed point can be studied perturbatively in the framework of the Wilson-Fisher $\epsilon$-expansion [1, 2]. Indeed, the one-loop beta function for the theory in $d = 4 - \epsilon$ reads

$$\beta_\lambda = -\epsilon \lambda + (N + 8) \frac{\lambda^2}{8\pi^2}, \tag{1.5}$$

and there is a weakly coupled IR fixed point at

$$\lambda^* = \frac{8\pi^2}{N + 8} \epsilon. \tag{1.6}$$

Higher order corrections in $\epsilon$ will change the value of the critical coupling, but not its existence, at least in perturbation theory. The anomalous dimensions of the fundamental field $\phi^i$ and the composite $\phi^i \phi^i$ at the fixed point can be computed to be, to leading order in $\epsilon$

$$\gamma_\phi = \frac{N + 2}{4(N + 8)^2} \epsilon^2 + O(\epsilon^3), \quad \gamma_{\phi^2} = \frac{N + 2}{N + 8} \epsilon + O(\epsilon^2), \tag{1.7}$$
corresponding to the scaling dimensions

\[
\Delta_\phi = \frac{d}{2} - 1 + \gamma_\phi = 1 - \frac{\epsilon}{2} + \frac{N + 2}{4(N + 8)^2} \epsilon^2 + \mathcal{O}(\epsilon^3), \quad (1.8)
\]

\[
\Delta_{\phi^2} = d - 2 + \gamma_{\phi^2} = 2 - \frac{6}{N + 8} \epsilon + \mathcal{O}(\epsilon^2). \quad (1.9)
\]

Note that the dimension of the $\phi^i \phi^i$ operator is $2 + \mathcal{O}(1/N)$ to leading order at large $N$, a result that follows from the large $N$ analysis reviewed below. Higher order corrections in $\epsilon$ for general $N$ may be derived by higher loop calculations in the theory (1.4), and they are known up to order $\epsilon^5$ [21, 22].

For $d > 4$, the interaction is irrelevant and so the IR fixed point is the free theory; however, one may ask about the existence of interacting UV fixed points. Working in $d = 4 + \epsilon$ for small $\epsilon$, one indeed finds a perturbative UV fixed point at (see, for example [23])

\[
\lambda_* = -\frac{8\pi^2}{N + 8} \epsilon. \quad (1.10)
\]

The anomalous dimensions at this critical point are given by the same expressions (1.7) with $\epsilon \to -\epsilon$. Note that, because $\gamma_\phi$ starts at order $\epsilon^2$, the dimension of $\phi$ stays above the unitarity bound for all $N$, at least for sufficiently small $\epsilon$. However, since the fixed point requires a negative coupling, one may worry about its stability, and it is important to study this critical point by alternative methods.

### 1.3 Review of large-$N$ expansion

A complementary approach to the $\epsilon$ expansion that can be developed at arbitrary dimension $d$ is the large $N$ expansion. The standard technique to study the theory (1.4) at large $N$ is based on introducing a Hubbard-Stratonovich auxiliary field $\sigma$ as

\[
S = \int d^d x \left( \frac{1}{2} (\partial \phi^i)^2 + \frac{1}{2} \sigma \phi^i \phi^i - \frac{\sigma^2}{4\lambda} \right). \quad (1.11)
\]
Integrating out $\sigma$ via its equation of motion $\sigma = \lambda \phi^i \phi^i$, one gets back to the original lagrangian. The quartic interaction in (1.4) may in fact be viewed as a particular example of the double trace deformations studied in [24]. One can then show that at large $N$ the dimension of $\phi^i \phi^i$ goes from $\Delta = d - 2$ at the free fixed point to $d - \Delta = 2$ at the interacting fixed point. At the conformal point, the last term in (1.11) can be dropped\footnote{This applies formally to both IR and UV fixed points.}, and the field $\sigma$ plays the role of the composite operator $\phi^i \phi^i$. One may then study the critical theory using the action

$$S_{\text{crit}} = \int d^d x \left( \frac{1}{2} (\partial \phi^i)^2 + \frac{1}{2 \sqrt{N}} \sigma \phi^i \phi^i \right)$$

(1.12)

where we have rescaled $\sigma$ by a factor of $\sqrt{N}$ for reasons that will become clear momentarily. The $1/N$ perturbation theory can be developed by integrating out the fundamental fields $\phi^i$. This generates an effective non-local kinetic term for $\sigma$

$$Z = \int D\phi D\sigma e^{-\int d^d x \left( \frac{1}{2} (\partial \phi^i)^2 + \frac{1}{2 \sqrt{N}} \sigma \phi^i \phi^i \right)}$$

$$= \int D\sigma e^{\frac{1}{2N} \int d^d x d^d y \sigma(x) \sigma(y) \langle \phi^i \phi^i(x) \phi^i \phi^i(y) \rangle_0 + O(\sigma^3)}$$

(1.13)

where we have assumed large $N$ and the subscript ‘$0$’ denotes expectation values in the free theory. We have

$$\langle \phi^i \phi^i(x) \phi^j \phi^j(y) \rangle_0 = 2N[G(x - y)]^2, \quad G(x - y) = \int \frac{d^d p}{(2\pi)^d} \frac{e^{ip(x-y)}}{p^2}.$$  

(1.14)

In momentum space, the square of the $\phi$ propagator reads

$$[G(x - y)]^2 = \int \frac{d^d p}{(2\pi)^d} e^{ip(x-y)} \tilde{G}(p)$$

$$\tilde{G}(p) = \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2(p - q)^2} = -\frac{(p^2)^{d/2-2}}{2^d(4\pi)^{d-3} \Gamma\left(\frac{d-1}{2}\right) \sin\left(\frac{\pi d}{2}\right)}$$

(1.15)
and so from (1.13) one finds the two-point function of $\sigma$ in momentum space

$$
\langle \sigma(p)\sigma(-p) \rangle = 2^{d+1}(4\pi)^{d-2}\Gamma\left(\frac{d-1}{2}\right)\sin\left(\frac{\pi d}{2}\right)(p^2)^{2-\frac{d}{2}} \equiv \tilde{C}_\sigma(p^2)^{2-\frac{d}{2}}.
$$

(1.16)

The corresponding two-point function in coordinate space can be obtained by Fourier transform and reads

$$
\langle \sigma(x)\sigma(y) \rangle = \frac{2^{d+2}\Gamma\left(\frac{d-1}{2}\right)\sin\left(\frac{\pi d}{2}\right)}{\pi^{\frac{d}{2}}\Gamma\left(\frac{d}{2} - 2\right)} \frac{1}{|x-y|^4} \equiv C_\sigma |x-y|^4.
$$

(1.17)

Indeed, this is the two-point function of a conformal scalar operator of dimension $\Delta = 2$. Note also that the coefficient $C_\sigma$ is positive in the range $2 < d < 6$.

The large $N$ perturbation theory can then be developed using the propagator (1.16)-(1.17) for $\sigma$, the canonical propagator for $\phi$ and the interaction term $\sigma\phi^i\phi^i$ in (1.12). For instance, the $1/N$ term in the anomalous dimension of $\phi^i$ can be computed from the one-loop correction to the $\phi^i$-propagator

$$
\frac{1}{N} \int \frac{d^dq}{(2\pi)^d} \frac{1}{(p-q)^2} \frac{\tilde{C}_\sigma}{(q^2)^{\frac{d}{2}-2+\delta}},
$$

(1.18)

where we have introduced a small correction $\delta$ to the power of the $\sigma$ propagator as a regulator$^2$. Doing the momentum integral by using (A.1), one obtains the result

$$
\frac{\tilde{C}_\sigma}{N} \frac{(d-4)}{(4\pi)^{\frac{d}{2}} d\Gamma\left(\frac{d}{2}\right)} (p^2)^{1-\delta} \Gamma(\delta - 1)
$$

(1.19)

where we have set $\delta = 0$ in the irrelevant factors. The $1/\delta$ pole corresponds to a logarithmic divergence and is cancelled as usual by the wave function renormalization of $\phi$. Defining the dimension of $\phi$ as

$$
\Delta_\phi = \frac{d}{2} - 1 + \frac{1}{N}\eta_1 + \frac{1}{N^2}\eta_2 + \ldots
$$

(1.20)

$^2$One may perform the calculation using a momentum cutoff, yielding the same final result.
the one-loop calculation above yields the result

\[ \eta_1 = \frac{\hat{C}_\sigma (d - 4)}{(4\pi)^{d-1}d!} = \frac{2^{d-4}(d-4)\Gamma \left( \frac{d-1}{2} \right) \sin \left( \frac{\pi d}{2} \right)}{\pi^{\frac{d}{2}}\Gamma \left( \frac{d}{2} + 1 \right)} . \]  

(1.21)

Setting \( d = 4 - \epsilon \) and expanding for small \( \epsilon \), it is straightforward to check that this agrees with the \( 1/N \) term of (1.8). The leading anomalous dimension of \( \sigma \) also takes a simple form \([4, 11, 12]\)

\[ \Delta_\sigma = 2 + \frac{4(d-1)(d-2)}{d-4} \eta_1 + \mathcal{O}(\frac{1}{N^2}). \]  

(1.22)

and can be seen to precisely agree with (1.9) in \( d = 4 - \epsilon \).

Note that the anomalous dimension of \( \phi^i \) (1.21) is positive for all \( 2 < d < 6 \). Thus, while the focus of most of the existing literature is on the range \( 2 < d < 4 \), we see no obvious problems with unitarity in continuing the large \( N \) critical \( O(N) \) theory to the range \( 4 < d < 6 \).3 A plot of \( \eta_1 \) and of the \( \sigma \) two-point function coefficient \( C_\sigma \) is given in Figure 1.1, showing that they are both positive for \( 2 < d < 6 \).

3Recall that the unitarity bound for a scalar operator is \( \Delta \geq d/2 - 1 \). One can also check that the order \( 1/N \) term in the anomalous dimension of the \( O(N) \) invariant higher spin currents, given in [10], is positive in the range \( 2 < d < 6 \), consistently with the unitarity bound \( \Delta_s \geq s + d - 2 \) for spin \( s \) operators (\( s > 1/2 \)).

Figure 1.1: The \( 1/N \) anomalous dimension of \( \phi^i \) and the coefficient of the two-point function of \( \sigma \) in the large \( N \) critical \( O(N) \) theory for \( 2 < d < 6 \).
1.4 The Gross-Neveu Model and its UV completion

In order to motivate our proposed cubic model as the UV completion of $\phi^4$ theory, we first review the relation between the Gross-Neveu (GN) model and its UV completion, the Gross-Neveu-Yukawa model.

The same $\epsilon$ and $1/N$-expansion schemes can be applied to fermionic theories as well. The GN model has a quartic interaction given by:

$$L_{\text{GN}} = \bar{\psi}_j \slashed{D} \psi^j + \frac{g}{2} (\bar{\psi}_j \psi^j)^2.$$ (1.23)

As discovered in [25], the theory is asymptotically free in $d = 2$ for $N > 2$. The $\beta$-function for the renormalized coupling constant $g$ in $d = 2 + \epsilon$ starts with:

$$\beta = \epsilon g - \frac{N-2}{2\pi} g^2 + \ldots ,$$ (1.24)

and the anomalous dimension of $\psi$ is given by [3]:

$$\Delta_\psi = \frac{1 + \epsilon}{2} + \frac{N-1}{8\pi^2} g^2.$$ (1.25)

Therefore this theory possesses a non-trivial UV fixed point given by $g_* = \frac{2\pi}{N-2} \epsilon$ in $d = 2 + \epsilon$. And the anomalous dimension at that fixed point is:

$$\Delta_\psi = \frac{1}{2} + \frac{1}{2} \epsilon + \frac{N-1}{4(N-2)^2} \epsilon + \ldots .$$ (1.26)

However, by simple power counting, this theory is not renormalizable in dimensions greater than two, and therefore in the UV, the large momentum behavior of the theory cannot be properly described via perturbation theory. It would be desirable to describe this fixed point through some other means, such as the IR fixed point of
some other theory. Indeed such a theory exists, and it is the Gross-Neveu-Yukawa theory, as we will demonstrate shortly.

First, we can still apply the Hubbard-Stratonovich analysis to the Gross-Neveu model, although it requires a bit more care. Introducing the auxiliary field $\sigma$, and dropping the quadratic term in the critical limit, we have the action

$$S_{\text{crit form}} = \int d^d x \left( - \bar{\psi}_0 \not\partial \psi_0^i + \frac{1}{\sqrt{N}} \sigma_0 \bar{\psi}_0 \psi_0^i \right), \quad (1.27)$$

where $i = 1, \ldots, \tilde{N}$ and $N = \tilde{N} \text{Tr} 1$. The propagator of the $\psi^i_0$ field reads

$$\langle \psi^i_0(p) \bar{\psi}_0(-p) \rangle_0 = \delta_{ij} \frac{i \gamma^0}{p^2}. \quad (1.28)$$

The $\sigma$ effective propagator obtained after integrating over the fundamental fields $\psi^i_0$ reads

$$\langle \sigma_0(p) \sigma_0(-p) \rangle_0 = \tilde{C}_{\sigma_0}/(p^2)^{\frac{d}{2}-1+\Delta}, \quad (1.29)$$

where

$$\tilde{C}_{\sigma_0} \equiv -2^{d+1} (4\pi)^{\frac{d-3}{2}} \Gamma \left( \frac{d-1}{2} \right) \sin \left( \frac{\pi d}{2} \right). \quad (1.30)$$

and we have introduced the regulator $\Delta$. Note that the power of $p^2$ in the propagator is $\frac{d}{2} - 1 + \Delta$ instead of $\frac{d}{2} - 2 + \Delta$ found in the scalar case. In order to cancel the divergences as $\Delta \to 0$ we have to renormalize the bare fields $\psi_0$ and $\sigma_0$:

$$\psi = Z_{\psi}^{1/2} \psi_0, \quad \sigma = Z_{\sigma}^{1/2} \sigma_0, \quad (1.31)$$
where
\[
Z_\psi = 1 + \frac{1}{N} \frac{Z_{\psi 1}}{\Delta} + O(1/N^2), \quad Z_\sigma = 1 + \frac{1}{N} \frac{Z_{\sigma 1}}{\Delta} + O(1/N^2).
\] (1.32)

The full propagators of the renormalized fields read
\[
\langle \psi^i(p) \bar{\psi}^j(-p) \rangle = \delta_j^j \tilde{C}_\psi \left( \frac{ip}{p^2} \right)^{d-\Delta_\psi + \frac{1}{2}}, \quad \langle \sigma(p)\sigma(-p) \rangle = \frac{\tilde{C}_\sigma}{(p^2)^{\frac{d}{2}-\Delta_\sigma}},
\] (1.33)

where we introduced anomalous dimensions $\Delta_\psi$ and $\Delta_\sigma$ and two-point function normalizations $\tilde{C}_\psi$ and $\tilde{C}_\sigma$ in momentum space. In particular, we have
\[
\Delta_\psi = \frac{d}{2} - \frac{1}{2} + \eta_{\text{GN}},
\] (1.34)

where $\eta_{\text{GN}} = \eta_1^\text{GN}/N + \eta_2^\text{GN}/N^2 + \ldots$, and to first order in $1/N$, we have:
\[
\eta_1^\text{GN} = \frac{\Gamma(d-1)(d-2)^2}{4\Gamma(2-d/2)\Gamma(d/2+1)\Gamma(d/2)^2},
\] (1.35)

We can expand this expression in $d = 2 + \epsilon$ and see agreement with our $\epsilon$-expansion result.

Motivated by the Hubbard-Stratonovich transformation, the Lagrangian of the GNY model [26, 27] contains a scalar field with cubic interaction with the fermions as well as a quartic self-interaction which is also marginal:
\[
\mathcal{L}_{\text{GNY}} = \frac{1}{2}(\partial_\mu \sigma)^2 + \bar{\psi}_j \not\!\partial \psi^j + g_1 \sigma \bar{\psi}_j \psi^j + \frac{1}{24} g_2 \sigma^4.
\] (1.36)
In $d = 4 - \epsilon$, the $\beta$-functions to one-loop are given by\cite{3}:

$$
\begin{align*}
\beta_{g_2} &= -\epsilon g_2 + \frac{1}{(4\pi)^2} \left(3g_2^2 + 2N g_1^2 g_2 - 12N g_1^4\right), \\
\beta_{g_1} &= -\frac{\epsilon}{2} g_1 + \frac{N + 6}{2(4\pi)^2} g_1^3,
\end{align*}
$$

where $N = N_f \text{tr} 1 = 4N_f$. The anomalous dimension for the $\psi$ field is given by:

$$
\Delta_{\psi} = \frac{3}{2} - \frac{\epsilon}{N + 6}.
$$

(1.37)

The model possesses an IR stable fixed point at the critical couplings $g_i^*$ given by

$$
\begin{align*}
\frac{(g_1^*)^2}{(4\pi)^2} &= \frac{1}{N + 6} \epsilon, \\
\frac{g_2^2}{(4\pi)^2} &= \frac{-N + 6 + \sqrt{N^2 + 132N + 36}}{6(N + 6)} \epsilon.
\end{align*}
$$

(1.39)

The anomalous dimension of $\psi$ at the fixed point is equal to:

$$
\Delta_{\psi} = \frac{3}{2} - \frac{N + 5}{2(N + 6)} \epsilon.
$$

(1.40)

Which again matches against the large $N$ result expanded near $d = 4 - \epsilon$. Therefore the UV fixed point of the GN model is described by the IR fixed point of the GNY model. The GNY model is said to be the “UV-completion” of the GN model.

Similarly, in this thesis, we are trying to show that the $O(N)$ symmetric $((\phi^i)^2)^2$ theory with a UV fixed point in $d = 4 + \epsilon$ has a UV completion given by a cubic theory with $N + 1$ scalar fields in $d = 6 - \epsilon$:

$$
\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{g_1}{2} \sigma \phi^i \phi^i + \frac{g_2}{6} \sigma^3.
$$

(1.41)
We will demonstrate this by comparing against large $N$ results of various operator dimensions.

1.5 Overview of chapters

This thesis is organized as a collection of papers. Chapters 2,3,4,6 are based on work published in [28, 29, 30, 31] respectively, with co-authors Simone Giombi, Igor Klebanov, and Grigory Tarnopolsky. Chapter 5 is based on unpublished work.

In chapter 2, we propose the alternate description of the $O(N)$ symmetric scalar field theories in $d$ dimensions with interaction $(\phi^i \phi^j)^2$ in terms of a theory of $N + 1$ massless scalars with the cubic interactions $\sigma \phi^i \phi^j$ and $\sigma^3$. Our one-loop calculation in $6 - \epsilon$ dimensions shows that this theory has an IR stable fixed point at real values of the coupling constants for $N > 1038$. We show that the $1/N$ expansions of various operator scaling dimensions match the known results for the critical $O(N)$ theory continued to $d = 6 - \epsilon$. These results suggest that, for sufficiently large $N$, there are 5-dimensional unitary $O(N)$ symmetric interacting CFT’s; they should be dual to the Vasiliev higher-spin theory in AdS$_6$ with alternate boundary conditions for the bulk scalar. Using these CFT’s we provide a new test of the 5-dimensional $F$-theorem, and also find a new counterexample for the $C_T$ theorem.

In chapter 3, we continue the study of the same theory to three loop order. We calculate the 3-loop beta functions for the two couplings and use them to determine certain operator scaling dimensions at the IR stable fixed point up to order $\epsilon^3$. We also use the beta functions to determine the corrections to the critical value of $N$ below which there is no fixed point at real couplings. We also study the theory with $N = 1$, which has a $Z_2$ symmetry under $\phi \rightarrow -\phi$. We show that it possesses an IR stable fixed point at imaginary couplings which can be reached by flow from a nearby fixed point describing a pair of $N = 0$ theories. We calculate certain operator
scaling dimensions at the IR fixed point of the $N = 1$ theory and suggest that, upon
continuation to two dimensions, it describes a non-unitary conformal minimal model.

In chapter 4, we consider a variant of our model with $Sp(N)$ symmetry composed
of $N$ anti-commuting scalars and one commuting scalar. For any even $N$ we find an
IR stable fixed point in $6 - \epsilon$ dimensions at imaginary values of coupling constants.
Borrowing our three loop calculations in 3, we develop $\epsilon$ expansions for several oper-
ator dimensions and for the sphere free energy $F$. The conjectured $F$-theorem is
obeyed in spite of the non-unitarity of the theory. Our results point to the exist-
ence of interacting non-unitary 5-dimensional CFTs with $Sp(N)$ symmetry, where
operator dimensions are real. For $N = 2$ we show that the IR fixed point possesses
an enhanced global symmetry given by the supergroup $OSp(1|2)$. This suggests the
existence of $OSp(1|2)$ symmetric CFTs in dimensions smaller than 6. We show that
the $6 - \epsilon$ expansions of the scaling dimensions and sphere free energy in our $OSp(1|2)$
model are the same as in the $q \to 0$ limit of the $q$-state Potts model.

In chapter 5, we provide further evidence of the equivalence between the $Sp(2)$
model and the 0-state Potts model. Using existing results, we find that to four-loop
order, the anomalous dimensions and beta functions of these two theories are equal
up to a sign alternating with successive loop order. Using a graph theoretic approach,
we prove this equality as well as the alternating sign. This argument holds for all
loop orders in perturbation theory. This strongly suggests that these two models are
indeed equivalent, giving an alternative description of the 0-state Potts model.

Finally, in chapter 6, we study conformal field theories with Yukawa interactions
in dimensions between 2 and 4; they provide UV completions of the Nambu-Jona-
Lasinio and Gross-Neveu models which have four-fermion interactions. We compute
the sphere free energy and certain operator scaling dimensions using dimensional
continuation. We provide new evidence that the $4 - \epsilon$ expansion of the $N = 1$ Gross-
Neveu-Yukawa model respects the supersymmetry. Our extrapolations to $d = 3$
appear to be in good agreement with the available results obtained using the numerical conformal bootstrap. We apply a similar approach to calculate the sphere free energy and operator scaling dimensions in the Nambu-Jona-Lasinio-Yukawa model, which has an additional $U(1)$ global symmetry. For $N = 2$, which corresponds to one 2-component Dirac fermion, this theory has an emergent supersymmetry with 4 supercharges, and we provide new evidence for this.
Chapter 2

Critical $O(N)$ Models in $6 - \epsilon$

Dimensions

2.1 Introduction and Summary

This chapter is based on the work published in [28], co-authored with Simone Giombi, and Igor Klebanov. We also thank David Gross, Daniel Harlow, Igor Herbut, Juan Maldacena, Giorgio Parisi, Silviu Pufu, Leonardo Rastelli, Dam Son, Grigory Tarnopolsky and Edward Witten for useful discussions.

We would like to extend the results of the $\epsilon$-expansion and large-$N$ expansion to interacting $O(N)$ models in $d > 4$. At first glance, such extensions seem impossible: the $(\phi^i \phi^i)^2$ interaction is irrelevant at the Gaussian fixed point, since it has scaling dimension $2d - 4$. Therefore, the long-distance behavior of this QFT is described by the free field theory. However, at least for large $N$, the theory possesses a UV stable fixed point whose existence may be demonstrated using the Hubbard-Stratonovich transformation. At the interacting fixed point, the scaling dimension of the operator $\phi^i \phi^i$ is $2 + O(1/N)$ for any $d$. For $d > 6$, this dimension is below the unitarity bound $d/2 - 1$. It follows that the interesting range, where the UV fixed point may be
unitary, is [32, 33, 34, 35]

\[ 4 < d < 6. \]  \tag{2.1} 

We will examine the structure of the \( O(N) \) symmetric scalar field theory in this range from various points of view.

We first note that the coefficients in the \( 1/N \) expansions derived for various quantities in [4, 5, 6, 7, 8, 9, 10, 11, 12] may be continued to the range (2.1) without any obvious difficulty. Thus, the UV fixed points make sense at least in the \( 1/N \) expansion. However, one should be concerned about the stability of the fixed points in this range of dimensions at finite \( N \). In \( d = 4 + \epsilon \), where the UV fixed point is weakly coupled, it occurs for the negative quartic coupling \( \lambda_\epsilon = -\frac{8\pi^2}{N+8} \epsilon + O(\epsilon^2) \) [36, 23]. Thus, it seems that the UV fixed point theory is unlikely to be completely stable, although it may be metastable. In order to gain a better understanding of the fixed point theory, it would be helpful to describe it via RG flow from another theory. In such a “UV complete” description, the \( O(N) \) symmetric theory we are after should appear as the conventional IR stable fixed point.

Our main result is to demonstrate that such a UV completion indeed exists: it is the \( O(N) \) symmetric theory of \( N+1 \) scalar fields with the Lagrangian

\[ \mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{g_1}{2} \sigma \phi^i \phi^i + \frac{g_2}{6} \sigma^3 . \] \tag{2.2} 

The cubic interaction terms are relevant for \( d < 6 \), so that the theory flows from the Gaussian fixed point to an interacting IR fixed point. The latter is expected to be weakly coupled for \( d = 6 - \epsilon \). The idea to study a cubic scalar theory in \( d = 6 - \epsilon \) is not new. Michael Fisher has explored such an \( \epsilon \)-expansion in the theory of a single scalar field as a possible description of the Yang-Lee edge singularity in the Ising model [37]. In that case, which corresponds to the \( N = 0 \) version of (2.2), the IR
fixed point is at an imaginary value of $g_2$.\footnote{Renormalization group calculations for similar cubic theories in $d = 6 - \epsilon$ were carried out in [38, 39, 40].} This is related to the lack of unitarity of the fixed point theory. Using the one-loop beta functions for $g_1$ and $g_2$, we will show in Section 2.3 that for large $N$ the IR stable fixed point instead occurs for real values of the couplings, thus removing conflict with unitarity. Remarkably, such unitary fixed points exist only for $N > 10^{38}$. As we show in Section 2.4, for $N \leq 10^{38}$ the IR stable fixed point becomes complex, and the theory is no longer unitary. Thus, the breakdown of the large $N$ expansion in $d = 6 - \epsilon$ occurs at a very large value, $N_{\text{crit}} = 10^{38}$. We will provide evidence, however, that for the physically interesting dimension $d = 5$, $N_{\text{crit}}$ is much smaller.

Besides its intrinsic interest, the $O(N)$ invariant scalar CFT in $d = 5$ has interesting applications to higher spin AdS/CFT dualities. There exists a class of Vasiliev theories in $\text{AdS}_{d+1}$ [41, 42, 43, 44, 45, 46] that is naturally conjectured to be dual to the $O(N)$ singlet sector of the $d$-dimensional CFT of $N$ free scalars [47, 23]. In order to extend the duality to interacting CFT’s, one adds the $O(N)$ invariant term $\lambda (\phi^i \phi^i)^2$. For $d = 3$ this leads to the well-known Wilson-Fisher fixed points [1, 2]. In the dual description of these large $N$ interacting theories, it is necessary to change the $r^{-\Delta}$ boundary conditions on the scalar field in $\text{AdS}_4$ from the $\Delta_- = 1$ to $\Delta_+ = 2 + \mathcal{O}(1/N)$ [47]. The situation is very similar for the $d = 5$ case, which should be dual to the Vasiliev theory in $\text{AdS}_6$ [23]. One can adopt the $\Delta_+ = 3$ boundary conditions on the bulk scalar, which are necessary for the duality to the free $O(N)$ theory. Alternatively, the $\Delta_- = 2 + \mathcal{O}(1/N)$ are allowed as well [48, 49]. This suggests that the dual interacting $O(N)$ CFT should exist in $d = 5$, at least for large $N$ [36, 23]. Our RG calculations lend further support to the existence of this interacting $d = 5$ CFT.

In Sections 2.3 and 2.5, using one-loop calculations for the theory (2.2) in $d = 6 - \epsilon$, we find some IR operator dimensions to order $\epsilon$, while keeping track of the dependence on $1/N$ to any desired order. We will then match our results with the $1/N$ expansions.
Figure 2.1: Interacting unitary $O(N)$ symmetric scalar CFT’s exist for dimensions $2 < d < 6$, with $d = 4$ excluded. In $6 - \epsilon$ and $4 - \epsilon$ dimensions they may be described as weakly coupled IR fixed points of the cubic and quartic scalar theories, respectively. In $4 + \epsilon$ and $2 + \epsilon$ dimensions they are weakly coupled UV fixed points of the quartic theory and of the $O(N)$ Non-linear $\sigma$ Model, respectively.

derived for the $(\phi^i\phi^i)^2$ theory in [4, 5, 6, 7, 8, 9, 10, 11, 12], evaluating them in $d = 6 - \epsilon$. The perfect match of the coefficients in these two $1/N$ expansions provides convincing evidence that the IR fixed point of the cubic $O(N)$ theory (2.2) indeed describes the same physics as the UV fixed point of the $(\phi^i\phi^i)^2$ theory. Our results thus provide evidence that, at least for large $N$, the interacting unitary $O(N)$ symmetric scalar CFT’s exist not only for $2 < d < 4$, but also for $4 < d < 6$. In Fig. 1 we sketch the entire available range $2 < d < 6$, pointing out the various perturbative descriptions of the CFT’s where $\epsilon$ expansions have been developed.

In Section 2.6 we discuss the large $N$ results for $C_T$, the coefficient of the two-point function of the stress-energy tensor [12]. We note that, as $d$ approaches 6, $C_T$ approaches that of the free theory of $N + 1$ scalar fields. This gives further evidence for our proposal that the IR fixed point of (2.2) describes the $O(N)$ symmetric CFT. We show that the RG flow from this interacting CFT to the free theory of $N$ scalars provides a counter example to the conjectured $C_T$ theorem. On the other hand, the five-dimensional version of the $F$-theorem [50] holds for this RG flow.
Our discussion of the $(\phi^i \phi^i)^2$ scalar theory in the range $4 < d < 6$ is analogous to the much earlier results [26, 27, 3] about the Gross-Neveu model [25] in the range $2 < d < 4$. The latter is a $U(\tilde{N})$ invariant theory of $\tilde{N}$ Dirac fermions with an irrelevant quartic interaction $(\bar{\psi}^i \psi^i)^2$. Using the Hubbard-Stratonovich transformation, it is not hard to show that this model has a UV fixed point, at least for large $\tilde{N}$ [3]. This CFT was conjectured [51, 52] to be dual to the type B Vasiliev theory in $\text{AdS}_4$ with appropriate boundary conditions. An alternative, UV complete description of this CFT is via the Gross-Neveu-Yukawa (GNY) model [26, 27, 3], which is a theory of $\tilde{N}$ Dirac fermions coupled to a scalar field $\sigma$ with $U(\tilde{N})$ invariant interactions $g_1 \sigma \bar{\psi}^i \psi^i + g_2 \sigma^4/24$. This description is weakly coupled in $d = 4 - \epsilon$, and it is not hard to show that the one-loop beta functions have an IR stable fixed point for any positive $\tilde{N}$. The operator dimensions at this fixed point match the large $\tilde{N}$ treatment of the Gross-Neveu model [53]. These results will be reviewed in Section 2.7, where we also discuss tests of the 3-d F-theorem [54, 50] provided by the GNY model.

2.2 Summary of large $N$ results for the critical $O(N)$ CFT

In chapter 1, we derived the anomalous dimensions of $\phi$ and $\sigma$ to first order in $1/N$. However, at higher orders in the $1/N$ expansion, a straightforward diagrammatic approach becomes rather cumbersome. However, the conformal bootstrap method developed in papers by A. N. Vasiliev and collaborators [4, 5, 6] has allowed to compute the anomalous dimension of $\phi^i$ to order $1/N^3$ and that of $\sigma$ to order $1/N^2$. These results have been successfully matched to all available orders in the $d = 2 + \epsilon$ and $d = 4 - \epsilon$ expansions, providing a strong test of their correctness. The explicit
form of $\eta_2$ in general dimensions, defined as in (1.20), reads \[4, 5]\:

$$
\eta_2 = 2\eta_2^2 (f_1 + f_2 + f_3); \\
f_1 = v'(\mu) + \mu^2 + \frac{\mu - 1}{2\mu(\mu - 1)}, \quad f_2 = \frac{\mu}{2 - \mu}v'(\mu) + \frac{\mu(3 - \mu)}{2(2 - \mu)^2}, \\
f_3 = \frac{\mu(2\mu - 3)}{2 - \mu}v'(\mu) + \frac{2\mu(\mu - 1)}{2 - \mu}; \\
v'(\mu) = \psi(2 - \mu) + \psi(2\mu - 2) - \psi(\mu - 2) - \psi(2), \\
\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad \mu = \frac{d}{2}. \quad (2.3)
$$

The expressions for the dimension of $\sigma$ at order $1/N^2$ \[5\] and for the coefficient $\eta_3$ \[6\] in arbitrary dimensions are lengthy and we do not report them explicitly here.\(^3\)

However, since they are useful to test our general picture, we write below the explicit $\epsilon$-expansions of $\Delta_\phi$ and $\Delta_\sigma$ in $d = 6 - \epsilon$, including all known terms in the $1/N$ expansion:

$$
\Delta_\phi = 2 - \frac{\epsilon}{2} + \left(\frac{1}{N} + \frac{44}{N^2} + \frac{1936}{N^3} + \ldots\right) \epsilon - \left(\frac{11}{12N} + \frac{835}{6N^2} + \frac{16352}{N^3} + \ldots\right) \epsilon^2 + O(\epsilon^3). \quad (2.4)
$$

and

$$
\Delta_\sigma = 2 + \left(\frac{40}{N} + \frac{6800}{N^2} + \ldots\right) \epsilon - \left(\frac{104}{3N} + \frac{34190}{3N^2} + \ldots\right) \epsilon^2 + O(\epsilon^3). \quad (2.5)
$$

In the next section, we will show that the order $\epsilon$ terms precisely match the one-loop anomalous dimensions at the IR fixed point of the cubic theory (2.2) in $d = 6 - \epsilon$. Higher orders in $\epsilon$ should be compared to higher loop contributions in the $d = 6 - \epsilon$ cubic theory, and it would be interesting to match them as well.

\(^2\)Ref. \[5\] contains an apparent typo: in the definition of the function $v'(\mu)$ given in Eq. (21), $\alpha = \mu - 1$ should be replaced by $\alpha = \mu - 2$. The correct formula for $\eta_2$ may be found in the earlier paper \[4\].

\(^3\)Note that \[5\] derives a result for the critical exponent $\nu$, which is related to the dimension of $\sigma$ by $\Delta_{\nu} = d - \frac{1}{\nu}$. We also note that a misprint in eq. (22) of \[6\] has been later corrected in eq. (11) of \[53\].
Using the results in [4, 5, 6, 53] we may also compute the dimension of $\phi^i$ and $\sigma$ directly in the physical dimension $d = 5$. The term of order $1/N^3$ depends on a non-trivial self-energy integral that was not evaluated for general dimension in [6]. An explicit derivation of this integral in general $d$ was later obtained in [55]. Using that result, we find in $d = 5$

$$\Delta_\phi = \frac{3}{2} + \frac{32}{15\pi^2 N} \frac{3375\pi^4 N^2}{275255197696} - \frac{89735168}{89735168} + \frac{32768 \ln 4}{9\pi^4} - \frac{229376 \zeta(3)}{3\pi^6} \frac{1}{N^3} + \ldots$$

and

$$\Delta_\sigma = 2 + \frac{512}{5\pi^2 N} + \frac{2048 (12625\pi^2 - 113552)}{1125\pi^2 N^2} + \ldots$$

(2.6)

and

$$\Delta_\sigma = 2 + \frac{512}{5\pi^2 N} + \frac{2048 (12625\pi^2 - 113552)}{1125\pi^2 N^2} + \ldots$$

(2.7)

We note that the coefficients of the $1/N$ expansion are considerably larger than in the $d = 3$ case.\footnote{In $d = 3$, one finds [5, 4, 6] (see also [18]) $\Delta_\phi = \frac{1}{2} + \frac{0.135095}{N} - \frac{0.0973367}{N^2} - \frac{0.940617}{N^3} + O(1/N^4)$ and $\Delta_\sigma = 2 - \frac{1.08076}{N} - \frac{3.0476}{N^2} + O(1/N^3)$.} Assuming the result (3.54) to order $1/N^3$, one finds that the dimension of $\phi^i$ goes below unitarity at $N_{\text{crit}} = 35$. This is much lower than the value $N_{\text{crit}} = 1038$ that we will find in $d = 6 - \epsilon$. This is just a rough estimate, since the $1/N$ expansion is only asymptotic and should be analyzed with care. Note for instance that in the $d = 3$ case, a similar estimate to order $1/N^3$ would suggest a critical value $N_{\text{crit}} = 3$, while in fact there is no lower bound: at $N = 1$ we have the 3d Ising model, where it is known [56, 17, 18, 57] that $\Delta_\phi \approx 0.518 > 1/2$. Nevertheless, the reduction from a very large value $N_{\text{crit}} = 1038$ in $d = 6 - \epsilon$ to a much smaller critical value in $d = 5$ is not unexpected. For instance, an analogous phenomenon is known to occur in the Abelian Higgs model containing $N_f$ complex scalars. A fixed point in
\[ d = 4 - \epsilon \] exists only for \( N_f \geq 183 \) [58, 3], while non-perturbative studies directly in \( d = 3 \) suggest a much lower critical value of \( N_f \). Some evidence for the reduction in the critical value of \( N_f \) as \( d \) is decreased comes from calculations of higher order corrections in \( \epsilon \) [59]. It would be nice to study such corrections for the \( O(N) \) theory in \( 6 - \epsilon \) dimensions. It would also be very interesting to explore the 5d fixed point at finite \( N \) by numerical bootstrap methods similar to what has been done in \( d = 3 \) in [18]. For the non-unitary theory with \( N = 0 \) such bootstrap studies were carried out very recently [60].

The large \( N \) critical \( O(N) \) theory in general \( d \) was further studied in a series of works by Lang and Ruhl [7, 8, 9, 10] and Petkou [11, 12]. Using conformal symmetry and self-consistency of the OPE expansion, various results about the operator spectrum of the critical theory were derived. As an example of interest to us, [10] derived an explicit formula for the anomalous dimension of the operator \( \sigma^k \) (the \( k \)-th power of the auxiliary field), which reads

\[
\Delta(\sigma^k) = 2k - \frac{2k(d-1)((k-1)d^2 + d + 4 - 3kd)}{d - 4} \frac{\eta}{N} + \mathcal{O}(\frac{1}{N^2})
\] (2.9)

where \( \eta \) is the \( 1/N \) anomalous dimension of \( \phi^i \) given in (1.21). In \( d = 6 - \epsilon \), this gives

\[
\Delta(\sigma^k) = 2k + (130k - 90k^2)\frac{\epsilon}{N} + \mathcal{O}(\epsilon^2) .
\] (2.10)

For \( k = 2, 3 \), we will be able to match this result with the one-loop operator mixing calculations in the cubic theory (2.2).

---

5We thank D. T. Son for bringing this to our attention.

6 After the original version of this paper appeared, a bootstrap study of the \( O(N) \) model in \( d = 5 \) was carried out in [61] with very encouraging results.
In [11, 12], explicit results for the 3-point function coefficients $g_{\phi\phi\sigma}$ and $g_{\sigma^3}$ were also derived. These are defined by the correlation functions

\begin{align}
\langle \phi^i(x_1)\phi^j(x_2)\sigma(x_3) \rangle &= \frac{g_{\phi\phi\sigma}}{|x_{12}|^{2\Delta_\phi-\Delta_\sigma}|x_{23}|^{\Delta_\sigma}|x_{13}|^{\Delta_\sigma}} \delta^{ij}.
\langle \sigma(x_1)\sigma(x_2)\sigma(x_3) \rangle &= \frac{g_{\sigma^3}}{(|x_{12}| |x_{23}| |x_{13}|)^{\Delta_\sigma}}.
\end{align}

The coefficient $g_{\phi\phi\sigma}$ was given in [11, 12] to order $1/N^2$ and arbitrary $d$. Expanding that result to leading order in $\epsilon = 6 - d$, we find

\begin{align}
g_{\phi\phi\sigma}^2 &= \frac{6\epsilon}{N} \left( 1 + \frac{44}{N} + O\left(\frac{1}{N^2}\right) \right).
\end{align}

This result indeed matches the value of the coupling $g_1^2$ in (2.2) at the IR fixed point, as will be shown in the next section. To leading order in $1/N$, the 3-point function coefficient $g_{\sigma^3}$ is related to $g_{\phi\phi\sigma}$ by [11]

\begin{align}
g_{\sigma^3} &= 2(d - 3)g_{\phi\phi\sigma},
\end{align}

which, as we will see, is precisely consistent with the ratio $g_2^2/g_1^2 = 6 + O(1/N)$ of the coupling constants at the $d = 6 - \epsilon$ IR fixed point.

### 2.3 The IR fixed point of the cubic theory in $d = 6 - \epsilon$

In this section, we show that the interacting scalar theory with Lagrangian (2.2) has a perturbative large $N$ IR fixed point in $d = 6 - \epsilon$, and compute the anomalous dimensions of the fundamental fields $\phi^i$ and $\sigma$ at the fixed point. The theory (2.2) has an obvious $O(N)$ symmetry, with the $N$-component vector $\phi^i$, $i = 1, \ldots, N$ transforming in the fundamental representation. Note that $g_1$ and $g_2$ are classically marginal in
\(d = 6\), and have dimension \(\frac{\epsilon}{2}\) in \(d = 6 - \epsilon\). The Feynman rules of the theory are shown in Figure 2.2.

\[
\begin{array}{ccc}
\phi & \rightarrow & j \\
\delta_{ij} & \frac{1}{p^2} & \sigma \\
- g_1 \delta_{ij} & - g_2 \\
\end{array}
\]

Figure 2.2: Feynman rules of the theory in Euclidean space.

It is not hard to compute the one-loop beta functions \(\beta_1, \beta_2\) for the couplings \(g_1, g_2\). The relevant one-loop diagrams needed to compute the counter terms \(\delta_\phi, \delta_\sigma, \delta g_1\) for the \(\sigma \phi \phi\) coupling, and \(\delta g_2\) for the \(\sigma^3\) coupling are given in Figure 2.3. Note that \(G^{m,n}\) denote the Green’s function with \(m\) \(\phi\) fields and \(n\) \(\sigma\) fields. The one-loop diagrams are labeled 1 through 7 for convenience.

Let us begin with the computation of diagram 1 in Figure 2.3.

\[
D_1 = (-g_1)^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{(p + k)^2} \frac{1}{k^2} = -g_1^2 I_1 = -\frac{p^2}{(4\pi)^3} \frac{g_1^2 \Gamma(3 - d/2)}{6 (M^2)^{3-d/2}} .
\]

Here we have used the renormalization condition \(p^2 = M^2\). This has a \(1/\epsilon\) pole in \(d = 6 - \epsilon\) which must be canceled by the counter term \(-p^2 \delta_\phi\). So we get:

\[
\delta_\phi = -\frac{1}{(4\pi)^3} \frac{g_1^2 \Gamma(3 - d/2)}{6 (M^2)^{3-d/2}} .
\]

For \(\delta_\sigma\), we have two one-loop diagrams (2 and 3). However, other than having different coupling constant factors, the integrals are identical to diagram 1. Note that

\[\text{We will state approximate expressions for the one-loop integrals } I_1 \text{ and } I_2 \text{ that are sufficient for extracting the } \log M^2 \text{ terms in } d = 6 - \epsilon. \text{ The more precise expressions are given in Appendix A.}\]
these two diagrams have a symmetry factor of 2, and there is a factor of $N$ associated with the $\phi$ loop in diagram 3. So, we have:

$$D_2 + D_3 = \frac{1}{2} (-g_2)^2 I_1 + \frac{N}{2} (-g_1)^2 I_1 = -\frac{Ng_1^2 + g_2^2}{2} I_1. \quad (2.16)$$

We arrive at the following expression for $\delta_\sigma$:

$$\delta_\sigma = -\frac{1}{(4\pi)^3} \frac{Ng_1^2 + g_2^2}{12} \frac{\Gamma(3 - d/2)}{(M^2)^{3-d/2}}. \quad (2.17)$$
Now let us compute corrections to the 3-point functions. Diagram 4 gives:

\[ D_4 = (-g_1)^2(-g_2) \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k-p)^2} \frac{1}{(k+q)^2} \frac{1}{k^2} \]

\[ = -g_1^2 g_2 I_2 = -g_1^2 g_2 \frac{\Gamma(3 - d/2)}{2(4\pi)^3 (M^2)^{3-d/2}} , \tag{2.18} \]

where \( I_2 \) is computed in Appendix A. Diagram 5 is again exactly the same as diagram 4, except for the coupling factors:

\[ D_5 = (-g_1)^3 \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k-p)^2} \frac{1}{(k+q)^2} \frac{1}{k^2} \]

\[ = -g_1^3 I_2 = -g_1^3 \frac{\Gamma(3 - d/2)}{2(4\pi)^3 (M^2)^{3-d/2}} . \tag{2.19} \]

The divergences in \( D_4 \) and \( D_5 \) must be canceled by the \(-\delta g_1\) counterterm. So we get

\[ \delta g_1 = -\frac{g_1^3 + g_1^2 g_2}{2(4\pi)^3} \frac{\Gamma(3 - d/2)}{(M^2)^{3-d/2}} . \tag{2.20} \]

Finally, we calculate \( \delta g_2 \). The diagrams have the same topology, and just differ in the coupling factors. Also, in diagram 6, we have a factor of \( N \) from the \( \phi \) loop. So we find

\[ D_6 + D_7 = N(-g_1)^3 I_2 + (-g_2)^3 I_2 \]

\[ = -(Ng_1^3 + g_2^3) I_2 = -\frac{(Ng_1^3 + g_2^3) \Gamma(3 - d/2)}{2(4\pi)^3 (M^2)^{3-d/2}} . \tag{2.21} \]

This term is canceled by \(-\delta g_2\), so we have

\[ \delta g_2 = -\frac{Ng_1^3 + g_2^3 \Gamma(3 - d/2)}{2(4\pi)^3 (M^2)^{3-d/2}} . \tag{2.22} \]
The Callan-Symanzik equation for the Green's function $G^{m,n}$ is:

$$
(M \frac{\partial}{\partial M} + \beta_1 \frac{\partial}{\partial g_1} + \beta_2 \frac{\partial}{\partial g_2} + m\gamma_\phi + n\gamma_\sigma)G^{m,n} = 0
$$

(2.23)

Let’s first apply it to $G^{2,0}$. Then, to leading order in perturbation theory, the Callan-Symanzik equation simplifies to

$$
-\frac{1}{p^2}M \frac{\partial}{\partial M} \delta_\phi + 2\gamma_\phi \frac{1}{p^2} = 0,
$$

(2.24)

which gives the anomalous dimension of $\phi^i$

$$
\gamma_\phi = \frac{1}{2} M \frac{\partial}{\partial M} \delta_\phi = \frac{1}{(4\pi)^3} \frac{g_1^2}{6}.
$$

(2.25)

Note that $\gamma_1 > 0$ as long as the coupling constant $g_1$ is real. Analogously, we obtain the $\sigma$ anomalous dimension

$$
\gamma_\sigma = \frac{1}{2} M \frac{\partial}{\partial M} \delta_\sigma = \frac{1}{(4\pi)^3} \frac{Ng_1^2 + g_2^2}{12}.
$$

(2.26)

Now we can compute the $\beta$ functions for $g_1$ and $g_2$. We have

$$
\beta_1 = -\frac{\epsilon}{2}g_1 + M \frac{\partial}{\partial M}(-\delta g_1 + \frac{1}{2}g_1(2\delta_\phi + \delta_\sigma)),
$$

(2.27)

where the first term accounts for the bare dimension of $g_1$ in $d = 6 - \epsilon$. After some simple algebra, we obtain

$$
\beta_1 = -\frac{\epsilon}{2}g_1 + \frac{(N - 8)g_1^3 - 12g_1^2g_2 + g_1g_2^2}{12(4\pi)^3}.
$$

(2.28)

Notice that when $N \gg 1$, $\beta_1$ is positive.
Finally, let us compute $\beta_2$. The Callan-Symanzik equation gives:

$$\beta_2 = -\frac{\epsilon}{2} g_2 + M \frac{\partial}{\partial M} (-\delta g_2 + \frac{1}{2} g_2 (3\delta)),$$

(2.29)

and so

$$\beta_2 = -\frac{\epsilon}{2} g_2 + \frac{-4Ng_1^3 + Ng_1^2g_2 - 3g_2^3}{4(4\pi)^3}. \tag{2.30}$$

As a check, let us note that for $N = 0$ the beta function for $g_2$ in $d = 6$ reduces to

$$\beta_2 = -\frac{3g_2^3}{4(4\pi)^3}, \tag{2.31}$$

which is the correct result for the single scalar cubic field theory in $d = 6$.

The single scalar cubic field theory in $d = 6 - \epsilon$ has no fixed points at real coupling, due to the negative sign of the beta function (2.31). It has a fixed point at imaginary coupling, which is conjectured to be related by dimensional continuation to the Yang-Lee edge singularity [37]. However, as we now show, for sufficiently large $N$, our model has a stable interacting IR fixed point. Note that for large $N$ the beta functions simplify to

$$\beta_1 = -\frac{\epsilon}{2} g_1 + \frac{Ng_1^3}{12(4\pi)^3}, \tag{2.32}$$

$$\beta_2 = -\frac{\epsilon}{2} g_2 + \frac{-4Ng_1^3 + Ng_1^2g_2}{4(4\pi)^3}. \tag{2.33}$$

This can be solved to get\(^8\)

$$g_1^* = \sqrt{\frac{6\epsilon(4\pi)^3}{N}}, \tag{2.34}$$

$$g_2^* = 6g_1^*. \tag{2.35}$$

\(^8\)There is also a physically equivalent solution with the opposite signs of $g_1, g_2$.  

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It is straightforward to compute the subleading corrections at large \( N \) by solving the exact beta function equations (4.5), (3.15) in powers of \( 1/N \). This yields

\[
g_1^* = \sqrt{\frac{6\epsilon(4\pi)^3}{N}} \left( 1 + \frac{22}{N} + \frac{726}{N^2} - \frac{326180}{N^3} + \ldots \right),
\]

\[
g_2^* = 6\sqrt{\frac{6\epsilon(4\pi)^3}{N}} \left( 1 + \frac{162}{N} + \frac{68766}{N^2} + \frac{41224420}{N^3} + \ldots \right).
\]

Note that the coefficients in this expansion appear to increase quite rapidly. This suggests that the large \( N \) expansion may break down at some finite \( N \). Indeed, we will see in Section 2.4 that this large \( N \) IR fixed point disappears at \( N \leq 1038 \) (the coupling constants go off to the complex plane). For all values of \( N \geq 1039 \), the fixed point has real couplings and is IR stable, see Section 2.4. At large \( N \), the IR stability of the fixed point can be seen from the fact that the matrix \( \frac{\partial^2 \beta}{\partial g^2_j} \) evaluated at the fixed point has two positive eigenvalues. These eigenvalues are in fact related to the dimensions of the two eigenstates coming from the operator mixing of \( \sigma^3 \) and \( \sigma \phi^i \phi^j \) operators, as will be discussed in more detail in Section 2.5.2.

We can now use the values of the fixed point couplings to compute the dimensions of the elementary fields \( \phi^i \) and \( \sigma \) in the IR. From (2.25) and (2.26) we obtain

\[
\Delta_\phi = \frac{d}{2} - 1 + \gamma_\phi = 2 - \frac{\epsilon}{2} + \frac{1}{(4\pi)^3} \frac{(g_1^*)^2}{6} = 2 - \frac{\epsilon}{2} + \frac{\epsilon}{N} + \frac{44\epsilon}{N^2} + \frac{1936\epsilon}{N^3} + \ldots
\]

and

\[
\Delta_\sigma = \frac{d}{2} - 1 + \gamma_\sigma = 2 - \frac{\epsilon}{2} + \frac{1}{(4\pi)^3} \frac{N(g_1^*)^2 + (g_2^*)^2}{12} = 2 + \frac{40\epsilon}{N} + \frac{6800\epsilon}{N^2} + \ldots
\]
Note that both dimensions are above the unitarity bound in $d = 6 - \epsilon$, namely $\Delta > \frac{d}{2} - 1$, since the anomalous dimensions are positive. Moreover, note that the order $\epsilon$ term in the dimension of $\sigma$ cancels out and we find $\Delta_\sigma = 2 + O(1/N)$. This is in perfect agreement with the large $N$ description of the critical $O(N)$ CFT. In that approach, the field $\sigma$ corresponds to the composite operator $\phi^i\phi^i$, whose dimension goes from $\Delta = d - 2$ at the free fixed point to $d - \Delta + O(1/N) = 2 + O(1/N)$ at the interacting fixed point. Furthermore, comparing (2.38), (2.39) with (2.4), (2.5), we see that the coefficients of the $1/N$ expansion precisely match the available results for the large $N$ critical $O(N)$ theory $[5, 4, 6]$ expanded at $d = 6 - \epsilon$. This is a strong test that the IR fixed point of the cubic theory in $d = 6 - \epsilon$ is identical at large $N$ to the dimensional continuation of the critical point of the $(\phi^i\phi^i)^2$ theory.

We may also match the values of the fixed point couplings (2.36), (2.37) with the large $N$ results (2.12), (2.13) for the 3-point functions coefficients in the critical $O(N)$ theory. At leading order in $\epsilon$, the 3-point functions in our cubic model simply come from a tree level calculation, and it is straightforward to verify the agreement of (2.36) with (2.12),$^9$ and that the ratio $g_1^*/g_2^* = 6$ at leading order at large $N$ agrees with (2.13).

## 2.4 Analysis of fixed points at finite $N$

In this section we analyze the one-loop fixed points for general $N$. First, let us define:

$$g_1 \equiv \sqrt{\frac{6\epsilon(4\pi)^3}{N}x},$$  \hspace{1cm} (2.40)

$$g_2 \equiv \sqrt{\frac{6\epsilon(4\pi)^3}{N}y}.$$  \hspace{1cm} (2.41)

$^9$An overall normalization factor comes from the normalization of the massless scalar propagators in $d = 6$. 

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After this, the vanishing of the $\beta$-functions (4.5), (3.15) gives

$$Nx = (N - 8)x^3 - 12x^2y + xy^2,$$  \hspace{1cm} (2.42)

$$Ny = -12Nx^3 + 3Nx^2y - 9y^3.$$  \hspace{1cm} (2.43)

These equations have 9 solutions if we allow $x$ and $y$ to be complex. One of them is the trivial solution $(0, 0)$. Two of them are purely imaginary; they occur at $(0, \pm \sqrt{N/9}i)$. There are no simple expressions for the remaining six solutions, but it is straightforward to study them numerically.

Two of them are real solutions in the second and fourth quadrant: $(-x_1, y_1)$ and $(x_1, -y_1)$, with $x_1, y_1 > 0$. They exist for any $N$, but they are not IR stable (they are saddles, i.e. there is one positive and one negative direction in the coupling space). They can be seen in both the bottom left and right graphs in Figure 2.4, corresponding to $N = 2000$ and $N = 500$ respectively.

The behavior of the remaining four solutions changes depending on the value of $N$. For $N \leq 1038$, we find that all four solutions are complex. The bottom right graph of Figure 2.4 shows that for $N = 500$ there are indeed only the two real solutions present for any $N$ discussed above.

For $N \geq 1039$, all four of these solutions become real, and they lie in the first and third quadrants: they have the form $(x_3, y_3)$, $(x_4, y_4)$, $(-x_3, -y_3)$, $(-x_4, -y_4)$, with $x_3, y_3, x_4, y_4 > 0$. To display the typical behavior of the solutions with $N \geq 1039$, in the bottom left graph of Figure 2.4 we plot the zeroes of $\beta_1$, $\beta_2$ for $N = 2000$, with regions of their signs labeled. Combining these, we can get the flow directions in each of these regions in the bottom left graph. We find that for all $N \geq 1039$, we have two stable IR fixed points that are symmetric with respect to the origin, and are labelled as red dots in the figure. These correspond to the large $N$ solution (2.36)-(2.37), and its equivalent solution with opposite signs of the couplings. For very large $N$, we see
that these solutions satisfy $g_2^* = 6g_1^*$ as in (2.35). The origin is a UV stable fixed point, and from the direction of the renormalization group flow we can see that all other fixed points are saddle points: they have one stable direction and one unstable direction.

It is interesting to treat $N$ as a continuous parameter and solve for the value $N_{\text{crit}}$ where the real IR stable fixed points disappear. First, we notice that we can factor out $x$ in (2.42), effectively making it quadratic. Moreover, if we subtract (2.43) from $\frac{y}{x}$ times (2.42), we will cancel the $Ny$ term, making the second beta function equation a homogeneous equation of order 3. After some more manipulation, we obtain:

$$N = (N - 44)x^2 + (6x - y)^2$$

$$Nx^2(6x - y) - y(4x^2 - 4y^2 + (6x - y)y) = 0.$$  \hfill (2.44)

It is convenient to rewrite these equations in terms of the following variables

$$x = \alpha, \quad y = \beta + 6\alpha.$$ \hfill (2.45)

After some algebra, we get:

$$\begin{align*}
(N - 44)\alpha^2 + \beta^2 &= N \quad \hfill (2.46) \\
840\frac{\alpha^3}{\beta^3} + (464 - N)\frac{\alpha^2}{\beta^2} + 84\frac{\alpha}{\beta} + 5 &= 0. \quad \hfill (2.47)
\end{align*}$$

$N_{\text{crit}}$ occurs when the curves defined by the two equations are tangent to each other. Notice that we have effectively decoupled the two equations, since if we write $z = \alpha/\beta$, we get

$$\begin{align*}
\alpha^2(N - 44 + z^{-2}) &= N \quad \hfill (2.48) \\
840z^3 + (464 - N)z^2 + 84z + 5 &= 0. \quad \hfill (2.49)
\end{align*}$$

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We can just first solve the second equation, then easily solve the first. To determine the critical $N$, we want the second equation to have exactly one real root, thus we require its discriminant to be zero:

$$\Delta = 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2 = 0,$$  \hspace{1cm} (2.51)

where $a, b, c, d$ are the coefficients of the cubic equation:

$$a = 840, \quad b = 464 - N, \quad c = 84, \quad d = 5.$$  \hspace{1cm} (2.52)

We then arrive at a cubic equation in terms of $N$:

$$\Delta = 20N^3 - 20784N^2 + 19392N + 1856 = 0.$$  \hspace{1cm} (2.53)

We can easily write down an analytic expression for $N$, and we find

$$N_{\text{crit}} = 1038.2660492\ldots$$  \hspace{1cm} (2.54)

Our numerical solution of the equations is consistent with this value.

For completeness, we now discuss the large $N$ behavior of the four fixed points which are not IR stable (two of them are present for any $N$, and two of them only for $N > N_{\text{crit}}$). They are obtained if we assume that, as $N \to \infty$, $x$ is $O(1)$ and $y$ is $O(\sqrt{N})$. Then, at the leading order in $N$, we get:

$$Nx = Nx^3 + xy^2, \quad Ny = 3Nx^2y - 9y^3.$$  \hspace{1cm} (2.55)

This can be solved to get four solutions: $(\sqrt{\frac{5}{6}}, \sqrt{\frac{1}{6}}\sqrt{N}), (-\sqrt{\frac{5}{6}}, \sqrt{\frac{1}{6}}\sqrt{N}), (\sqrt{\frac{5}{6}}, -\sqrt{\frac{1}{6}}\sqrt{N}), (-\sqrt{\frac{5}{6}}, -\sqrt{\frac{1}{6}}\sqrt{N})$. They correspond to the four real fixed points at large $N$ that are
The $\epsilon/\sqrt{N}$ correction does not correspond to the conventional large $N$ behavior.
2.5 Operator mixing and anomalous dimensions

2.5.1 The mixing of $\sigma^2$ and $\phi^i\phi^i$ operators

Let us compute the anomalous dimension matrix for operators $O^1 = \frac{\phi^i\phi^i}{\sqrt{N}}$ and $O^2 = \sigma^2$, where the $\sqrt{N}$ in the denominator is to ensure that the two-point functions of $O^1$ and $O^2$ are of the same order. They both have classical dimension $4 - \epsilon$ in $d = 6 - \epsilon$, so we expect them to mix.

Consider the operators $O^1$, and $O^2$ renormalized according to the convention shown in Figure 2.5.

\[
\begin{align*}
\langle \phi(p)\phi(q)\phi^2(p+q) \rangle &= \frac{1}{p^2} \frac{1}{q^2} \\
\langle \sigma(p)\sigma(q)\sigma^2(p+q) \rangle &= \frac{1}{p^2} \frac{1}{q^2}
\end{align*}
\]

Figure 2.5: Renormalization conditions for the operators $O^1$ and $O^2$.

Let $O^i_M$ denote the operator renormalized at scale $M$, and $O^i_0$ denote the bare operator. We are looking for expressions similar to (18.53) of [62], i.e. of the form:

\[
\begin{align*}
O^1_M &= O^1_0 + \delta^{11}O^1_0 + \delta^{12}O^2_0 + \delta_\phi O^1_0 \\
O^2_M &= O^2_0 + \delta^{21}O^1_0 + \delta^{22}O^2_0 + \delta_\sigma O^2_0.
\end{align*}
\]  

(2.57)

(2.58)

The $\delta^{ij}$ counterterms in the above equations are obtained by extracting the logarithmic divergence from the diagrams shown in Figure 2.6, while the $\delta_\phi O^1_0$ and $\delta_\sigma O^2_0$ terms correspond to external leg corrections, which are omitted in Figure 2.6.
Each of the terms $\delta^{ij}$ is given by canceling the divergent pieces of two of the above diagrams, as shown. For example, let us compute the first diagram of the $\delta^{11}$ counter term in Figure 2.6. We have:

$$D_1 = (-g_1)^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k-p)^2} \frac{1}{(k+q)^2} \frac{1}{k^2} \Gamma(3-d/2) \left(\frac{M^2}{2}\right)^{d/2}.$$

Next, we compute the second diagram of the $\delta^{11}$ counter term. Notice that there is a symmetry factor of 2, and also a factor of $N$ from a closed $\phi$ loop.

$$D_2 = \frac{N}{2} (-g_1)^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{(p+k)^2} \frac{1}{k^2} \frac{1}{p^2} \Gamma(3-d/2) \left(\frac{M^2}{2}\right)^{d/2}.$$

These two terms are canceled by $\delta^{11}$; therefore, to leading order in $\epsilon$ we have

$$\delta^{11} = \left( -\frac{g_1^2}{2(4\pi)^3} + \frac{Ng_1^2}{12(4\pi)^3} \right) \Gamma(3-d/2) \left(\frac{M^2}{2}\right)^{d/2}.$$

Similarly, we can calculate the other three $\delta^{ij}$. The integrals are the same, only the factors of coupling constants and $N$ (due to closed $\phi$ loops) are different. With our normalization convention of $\mathcal{O}^1$, we need to multiply $\delta^{12}$ by a factor of $\sqrt{N}$, and
divide $\delta^{21}$ by a factor of $\sqrt{N}$. Then the matrix is symmetric, $\delta^{21} = \delta^{12}$, and we find

$$\delta^{12} = \left( -\frac{\sqrt{N}g_1^2}{2(4\pi)^3} + \frac{\sqrt{N}g_1g_2}{1(4\pi)^3} \right) \frac{\Gamma(3 - d/2)}{(M^2)^{3-d/2}}, \quad (2.62)$$

$$\delta^{22} = \left( -\frac{g_2^2}{2(4\pi)^3} + \frac{g_2^2}{12(4\pi)^3} \right) \frac{\Gamma(3 - d/2)}{(M^2)^{3-d/2}}. \quad (2.63)$$

Thus, the matrix $\delta^{ij}$ is

$$\delta^{ij} = \frac{1}{12(4\pi)^3} \frac{\Gamma(3 - d/2)}{(M^2)^{3-d/2}} \begin{pmatrix}
-6g_1^2 + Ng_1^2 & -6\sqrt{N}g_1^2 + \sqrt{N}g_1g_2 \\
-6\sqrt{N}g_1^2 + \sqrt{N}g_1g_2 & -6g_2^2 + g_2^2
\end{pmatrix}. \quad (2.64)$$

The anomalous dimension matrix is given by

$$\gamma^{ij} = M \frac{\partial}{\partial M} (-\delta^{ij} + \delta_z^{ij}), \quad (2.65)$$

where we have defined

$$\delta_z^{ij} = \begin{pmatrix}
\delta_\phi & 0 \\
0 & \delta_\sigma
\end{pmatrix}. \quad (2.66)$$

Now, using expressions for $\delta_\phi$ and $\delta_\sigma$ from (2.15), (2.17), we get:

$$\gamma^{ij} = \frac{-1}{6(4\pi)^3} \begin{pmatrix}
4g_1^2 - Ng_1^2 & 6\sqrt{N}g_1^2 - \sqrt{N}g_1g_2 \\
6\sqrt{N}g_1^2 - \sqrt{N}g_1g_2 & 4g_2^2 - Ng_2^2
\end{pmatrix}. \quad (2.67)$$

The eigenvalues of this matrix will give the dimensions of the two eigenstates arising from the mixing of operators $O^1$ and $O^2$. 

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After plugging in the coupling constants at the IR fixed point from (2.36) and (2.37), and keeping its entries to order $1/N$, the matrix elements of $\gamma^{ij}$ become:

$$\gamma^{ij} = \frac{-1}{6(4\pi)^3} \begin{pmatrix} 4g_1^2 - Ng_1^2 & 6\sqrt{N}g_1^2 - \sqrt{N}g_1g_2 \\ 6\sqrt{N}g_1^2 - \sqrt{N}g_1g_2 & 4g_2^2 - Ng_1^2 \end{pmatrix}$$

$$= \epsilon \begin{pmatrix} 1 + \frac{40}{N} \frac{840}{N^{3/2}} \\ \frac{840}{N^{3/2}} & 1 - \frac{100}{N} \end{pmatrix}.$$  

To order $1/N$, the off-diagonal terms do not affect the eigenvalues, and we get the scaling dimensions

$$\Delta_- = d - 2 + \gamma_- = 4 - \frac{100\epsilon}{N} + O\left(\frac{1}{N^2}\right)$$

$$\Delta_+ = d - 2 + \gamma_+ = 4 + \frac{40\epsilon}{N} + O\left(\frac{1}{N^2}\right)$$

with the explicit eigenstates given by

$$O^+ = \frac{\sqrt{N}}{6} O^1 + O^2$$  

$$O^- = -\frac{6}{\sqrt{N}} O^1 + O^2.$$  

Satisfyingly, we see that to this order the dimension $\Delta_-$ precisely matches the dimension of the only primary field of dimension near 4 in the large $N$ UV fixed point of the quartic theory. It was determined using large $N$ methods in [10], and corresponds to $k = 2$ in (2.10). Since there are no other primaries of dimension $4 + O(1/N)$ in the critical theory, we expect that the other dimension $\Delta_+$ corresponds to a descendant. Comparing (4.26) with (2.39), we see that to this order $\Delta_+ = 2 + \Delta_\sigma$, so that the eigenstate with eigenvalue $\gamma_+$ is indeed a descendant of $\sigma$. 

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We can actually show that $\Delta_+ = 2 + \Delta_\sigma$ to all orders in $1/N$ (and for all fixed points) just using the $\beta$-function equations. To start, we redefine our variables as (2.40), (2.41). With this definition, the condition satisfied by $x$ and $y$ at the IR fixed point is given in (2.42), (2.43), or

\begin{align*}
1 &= x^2 - \frac{8}{N} x^2 - \frac{12}{N} xy + \frac{1}{N} y^2 \tag{2.73} \\
1 &= -12 \frac{x^3}{y} + 3x^2 - \frac{9}{N} y^2. \tag{2.74}
\end{align*}

The anomalous dimension matrix in this notation becomes

\begin{equation}
\gamma^{ij} = \epsilon \begin{pmatrix}
x^2 - \frac{4}{N} x^2 & \frac{1}{\sqrt{N}} (xy - 6x^2) \\
\frac{1}{\sqrt{N}} (xy - 6x^2) & x^2 - \frac{4}{N} y^2
\end{pmatrix}, \tag{2.75}
\end{equation}

which has eigenvalues

\begin{equation}
\lambda_{\pm} = \frac{(2x^2 - \frac{4}{N} x^2 - \frac{4}{N} y^2) \pm \sqrt{|\gamma|}}{2}, \tag{2.76}
\end{equation}

where $|\gamma|$ is the determinant, given by:

\begin{equation}
|\gamma| = \frac{144}{N} x^4 - \frac{48}{N} x^3 y + \frac{4}{N} x^2 y^2 + \frac{16}{N^2} x^4 - \frac{32}{N^2} x^2 y^2 + \frac{16}{N^2} y^4. \tag{2.77}
\end{equation}

Also, from the first line of (2.39), we see that in terms of $x$ and $y$ the dimension $\Delta_\sigma$ becomes

\begin{equation}
\Delta_\sigma = 2 - \frac{\epsilon}{2} + \frac{\epsilon}{2} x^2 + \frac{\epsilon}{2N^2} y^2, \tag{2.78}
\end{equation}

Thus, we have to show that:

\begin{equation}
\Delta_\sigma + 2 = 4 - \epsilon + \lambda_+ \tag{2.79}
\end{equation}
which, after some algebra, is seen to imply

\[ 1 - x^2 + \frac{4}{N} x^2 + \frac{5}{N} y^2 = \sqrt{|\gamma|}. \]  

(2.80)

Replacing \((1 - x^2)\) with \((2.73)\), we have

\[- \frac{4}{N} x^2 - \frac{12}{N} x y + \frac{6}{N} y^2 = \sqrt{|\gamma|}.\]  

(2.81)

Squaring both sides, and plugging in the expression for \(|\gamma|\), we get that the following equation is to hold for \((2.79)\) to be true

\[ 36x^4 - 12x^3y + x^2y^2 = \frac{1}{N} \left(24x^3y + 32x^2y^2 - 36xy^3 + 5y^4\right). \]  

(2.82)

To prove this equality, we again go back to \((2.73)\) and \((2.74)\). If we multiply both of these equations by \(3xy - \frac{y^2}{2}\) and subtract the former from the latter, we get exactly the expression above. Thus, \(\Delta_+ = 2 + \Delta_\sigma\) holds to all orders in \(1/N\).

### 2.5.2 The mixing of \(\sigma^3\) and \(\sigma \phi^i \phi^i\) operators

Next, we would like to calculate the mixed anomalous dimensions of the \(\Delta = 6\) operators, and show that they are all slightly irrelevant in \(d = 6 - \epsilon\). There are six operators with \(\Delta = 6\) at \(d = 6\), they are:

\[ \mathcal{O}^1 = \frac{\sigma (\phi^i)^2}{\sqrt{N}}, \quad \mathcal{O}^2 = \sigma^3, \]  

(2.83)

\[ \mathcal{O}^3 = \frac{(\partial \phi^i)^2}{\sqrt{N}}, \quad \mathcal{O}^4 = (\partial \sigma)^2, \]  

(2.84)

\[ \mathcal{O}^5 = \frac{\phi^i \Box \phi^i}{\sqrt{N}}, \quad \mathcal{O}^6 = \sigma \Box \sigma. \]  

(2.85)
However, in $d = 6 - \epsilon$, $O^1$ and $O^2$ have bare dimensions $3\frac{d-2}{2} = 6 - \frac{3}{2}\epsilon$, while $O^3$, $O^4$, $O^5$, $O^6$ have bare dimensions $2 + 2\frac{d-2}{2} = 6 - \epsilon$. Therefore, in $d = 6 - \epsilon$, $O^1$ and $O^2$ mix with each other, and $O^3$, $O^4$, $O^5$, $O^6$ mix with themselves.

We will compute the mixed anomalous dimensions of $O^1$ and $O^2$. Using the expressions for the $\beta$ functions, (4.5), (3.15), we can very simply compute the mixed anomalous dimension matrix by differentiating them with respect to appropriately normalized couplings (the rescaling is needed because two-point functions of $O^1$ and $O^2$ are off by a factor of $3N$). After some algebra, we get:

$$M^{ij} = \begin{pmatrix}
-\frac{\epsilon}{2} + \frac{3Ng_1^2 - 24g_1^2 + 2g_1g_2 + g_2^2}{12(4\pi)^4} & -\frac{12g_1^2 + 2g_1g_2}{12(4\pi)^4}\sqrt{3N} \\
-\frac{12g_1^2 + 2g_1g_2}{12(4\pi)^4}\sqrt{3N} & -\frac{\epsilon}{2} + \frac{Ng_2^2 - 9g_1^2}{4(4\pi)^4}\end{pmatrix}.$$ (2.86)

We plug in the fixed point value of $g_1$ and $g_2$ from (2.36) and (2.37) to get:

$$M^{ij} = \begin{pmatrix}
\epsilon - \frac{5040}{N^2}\epsilon + \ldots & \frac{840\sqrt{3}}{N\sqrt{N}}\epsilon + \ldots \\
\frac{840\sqrt{3}}{N\sqrt{N}}\epsilon + \ldots & \epsilon - \frac{420}{N}\epsilon - \frac{155820}{N^2}\epsilon + \ldots
\end{pmatrix}.$$ (2.87)

Notice that at this IR fixed point, both eigenvalues of $M^{ij}$ are positive, which implies that the fixed point is IR stable. This matrix has eigenvalues

$$\lambda_1 = \epsilon - \frac{3780}{N^2}\epsilon + \mathcal{O}(\frac{1}{N^3})$$ (2.88)

$$\lambda_2 = \epsilon - \frac{420}{N}\epsilon - \frac{155820}{N^2}\epsilon + \mathcal{O}(\frac{1}{N^3}).$$ (2.89)

The relation between scaling dimensions and the eigenvalues of the matrix is given by\textsuperscript{10}

$$\Delta = d + \lambda.$$ (2.90)

\textsuperscript{10}As a simple test of this formula, note that at the free UV fixed point the eigenvalues of (2.86) are $\lambda_1 = \lambda_2 = -\frac{\epsilon}{2}$, giving dimensions $\Delta_1 = \Delta_2 = 3(d/2 - 1)$ as it should be.
Thus, the scaling dimensions of the mixture of the two operators are:

\[ \Delta_1 = d + \lambda_1 = 6 + \mathcal{O}\left(\frac{1}{N^2}\right) \]
\[ \Delta_2 = d + \lambda_2 = 6 - \frac{420}{N}\epsilon + \mathcal{O}\left(\frac{1}{N^2}\right). \]

(2.91) \hspace{2cm} (2.92)

There are no \( \mathcal{O}(\epsilon) \) correction to the scaling dimensions, as we expected. As a further check, we note that \( \Delta_2 \) agrees with the dimension of the \( k = 3 \) primary operator given in (2.10). In the large \( N \) Hubbard-Stratonovich approach, this operator is \( \sigma^3 \) [10]. Our 1-loop calculation demonstrates the presence of another primary operator whose dimension is \( \Delta_1 \). Presumably, the corresponding operator in the Hubbard-Stratonovich approach is \((\partial_\mu \sigma)^2\).

We can also show that at the other fixed points discussed in Section 2.4, \( M^{ij} \) has one positive and one negative eigenvalues. If we use (2.40), (2.41) and plug them into (2.86), we get

\[ M^{ij} = \begin{pmatrix}
    -\frac{\epsilon}{2} + \frac{\epsilon}{2N}(3N x^2 - 24x^2 - 24xy + y^2) & \frac{\epsilon}{2N}(-12x^2 + 2xy)\sqrt{3N} \\
    \frac{\epsilon}{2N}(-12x^2 + 2xy)\sqrt{3N} & -\frac{\epsilon}{2} + \frac{3\epsilon}{2N}(Nx^2 - 9y^2)
\end{pmatrix}. \]

(2.93)

We have shown from solving (2.55) that

\[ x = \pm \sqrt{\frac{5}{6}} , \quad y = \pm \sqrt{\frac{1}{6}\sqrt{N}}. \]

(2.94)

Plugging these in, we find that the eigenvalues of \( M^{ij} \) at these fixed points are:

\[ \lambda_1 = \epsilon , \quad \lambda_2 = -\frac{5}{3}\epsilon , \]

(2.95)

which confirms our graphical analysis that all of these fixed points are saddle points.
2.6 Comments on $C_T$ and 5-d $F$ theorem

A quantity of interest in a CFT is the coefficient $C_T$ of the stress-tensor 2-point function, which may be defined by

$$\langle T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2) \rangle = C_T \frac{I_{\mu\nu,\rho\sigma}(x_{12})}{x_{12}^{2d}},$$

(2.96)

where $I_{\mu\nu,\rho\sigma}(x_{12})$ is a tensor structure uniquely fixed by conformal symmetry, see e.g. [63]. For $N$ free real scalar fields in dimension $d$ with canonical normalization, one has $C_T = \frac{Nd}{(d-1)S_d}$, where $S_d$ is the volume of the $d$-dimensional round sphere. In the critical $O(N)$ theory, using large $N$ methods one finds the result [11, 12]

$$C_T = \frac{Nd}{(d-1)S_d} \left( 1 + \frac{1}{N} C_{T,1} + \mathcal{O}\left( \frac{1}{N^2} \right) \right),$$

$$C_{T,1} = -\left( \frac{2C(\mu)}{\mu+1} + \frac{\mu^2 + 3\mu - 2}{\mu(\mu^2 - 1)} \right) \eta_1, \quad \mu = \frac{d}{2}$$

(2.97)

where $C(\mu) = \psi(3-\mu) + \psi(2\mu - 1) - \psi(1) - \psi(\mu)$, and $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. It is interesting to analyze the behavior of $C_{T,1}$ in $d = 6 - \epsilon$. The anomalous dimension $\eta_1$, given in (1.21), is of order $\epsilon$, but there is a pole in $C(\mu)$ from the term $\psi(3-\mu) \sim -2/\epsilon$. Hence, one finds at $d = 6 - \epsilon$

$$C_{T,1} = 1 - \frac{7}{4} \epsilon + \mathcal{O}(\epsilon^2)$$

(2.98)

and so

$$C_T = \frac{d}{(d-1)S_d} \left( N + 1 - \frac{7}{4} \epsilon \right) + \mathcal{O}(\epsilon^2).$$

(2.99)

Thus, as $d \to 6$, we find the $C_T$ coefficient of $N + 1$ free real scalars. This provides a nice check on our description of the critical $O(N)$ theory via the IR fixed point of (2.2). The $\mathcal{O}(\epsilon)$ correction may be calculated in this description as well, using the 1-loop fixed point (2.36), (2.37), but we leave this for future work.
The appearance of an extra massless scalar field as \( d \rightarrow 6 \) is also suggested by dimensional analysis. The dimension of \( \sigma \) is \( 2 + \mathcal{O}(1/N) \) in all \( d \), so as \( d \) approaches 6, it becomes the dimension of a free scalar field. Then the two-derivative kinetic term for \( \sigma \) (as well as the \( \sigma^3 \) coupling) become classically marginal. Note also that the negative sign of the order \( \epsilon \) correction in (2.99) implies that \( C_T \) decreases from the UV fixed point of \( N + 1 \) free fields to the interacting IR fixed point, consistently with the idea that \( C_T \) may be a measure of degrees of freedom \([12]\). However, expanding \( C_T \) in \( d = 4 + \epsilon \), one finds

\[
C_T = \frac{d}{(d-1)S_d^2} \left( N - \frac{5}{12} \epsilon^2 \right) + \mathcal{O}(\epsilon^3). \tag{2.100}
\]

According to our interpretation, the critical theory in \( d = 4 + \epsilon \) should be identified with the UV fixed point of the \(-\phi^4\) theory, while the IR fixed point corresponds to \( N \) free scalars. Then, (2.100) is seen to violate \( C_{T,UV}^T > C_T^{IR} \). A plot of \( C_{T,1} \) in the range \( 2 < d < 6 \) is given in Figure 2.7. Finally, let us quote the value of \( C_T \) in the interesting dimension \( d = 5 \)

\[
C_T^{d=5} = \frac{5}{4S_5^2} \left( N - \frac{1408}{1575\pi^2} \right). \tag{2.101}
\]

Again, we observe that this value is consistent with \( C_{T,UV}^T > C_T^{IR} \) for the flow from the free UV theory of \( N + 1 \) massless scalars to the interacting fixed point, but not for the flow from the interacting fixed point to the free IR theory of \( N \) massless scalars. Thus, the latter flow provides a non-supersymmetric counterexample against the possibility of a \( C_T \) theorem (for a supersymmetric counterexample, see \([64]\)). In contrast, the 5-dimensional F-theorem holds for both flows, as we discuss below.

It was proposed in \([50]\) (see also \([65, 66]\)) that, for any odd dimensional Euclidean CFT, the quantity

\[
\tilde{F} = (-1)^{d+1} F = (-1)^{d-1} \log Z_{S^d}, \tag{2.102}
\]

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Figure 2.7: The $O(N^0)$ correction $C_{T,1}$ in the coefficient $C_T$ of the stress tensor two-point function in the critical $O(N)$ theory for $2 < d < 6$; see (2.97). Note that $C_{T,1}$ is negative for $2 < d \lesssim 5.22$, so that in this range of dimensions $C_T^{\text{crit}} < NC_T^{\text{free sc.}}$ for large $N$. Therefore, the $C_T$ theorem is respected in $2 < d < 4$, but violated for $4 < d \lesssim 5.22$.

should decrease under RG flows

$$\tilde{F}_{UV} > \tilde{F}_{IR}. \quad (2.103)$$

Here $F = -\log Z_{S^d}$ is the free energy of the CFT on the round sphere, which is a finite number after the power law UV divergences are regulated away (for instance, by $\zeta$-function or dimensional regularization). Note that when we put the CFT on the sphere, we have to add the conformal coupling of the scalar fields to the $S^d$ curvature. This effectively renders the vacuum metastable, both in the case of the $-\phi^4$ theory and in the cubic theory. Similarly, the CFT is metastable on $R \times S^{d-1}$, which is relevant for calculating the scaling dimensions of operators.

For $d = 3$ (where $\tilde{F} = F$), the F-theorem (2.103) was proved in [67]. However, higher dimensional versions are less well established. Using the results of this chapter we can provide a simple new test of the 5d version of the F-theorem (in this case $\tilde{F} = -F$). As we have argued, the 5d critical $O(N)$ theory can be viewed as either the IR fixed point of the cubic theory (2.2) or the UV fixed point of the quartic scalar
theory. This implies that $\tilde{F}$ should satisfy

$$N\tilde{F}_{\text{free sc.}} < \tilde{F}_{\text{crit.}} < (N + 1)\tilde{F}_{\text{free sc.}}, \quad (2.104)$$

where $\tilde{F}_{\text{free sc.}}$ is minus the free energy of a 5d free conformal scalar \[50\]

$$\tilde{F}_{\text{free sc.}} = \frac{\log 2}{128} + \frac{\zeta(3)}{128\pi^2} - \frac{15\zeta(5)}{256\pi^4} \approx 0.00574. \quad (2.105)$$

The value of $F$ at the critical point can be calculated perturbatively in $1/N$ by introducing the Hubbard-Stratonovich auxiliary field as reviewed in Section 2. The leading $O(N)$ term is the same as in the free theory, while the $O(N^0)$ term arises from the determinant of the non-local kinetic operator for the auxiliary field \[24, 68, 50\]. The result is \[23\]

$$\tilde{F}_{\text{crit.}} = N\tilde{F}_{\text{free sc.}} + \frac{3\zeta(5) + \pi^2\zeta(3)}{96\pi^2} + O\left(\frac{1}{N}\right). \quad (2.106)$$

The same answer may be obtained by computing the ratio of determinants for the bulk scalar field in the $AdS_6$ Vasiliev theory with alternate boundary conditions \[23\]. We see that the left inequality in (2.104) is satisfied, since the $O(N^0)$ correction in (2.106) is positive (this check was already made in \[23\]). More non-trivially, we observe that the right inequality also holds, because $\frac{3\zeta(5) + \pi^2\zeta(3)}{96\pi^2} \approx 0.001601$ is smaller than the value of $\tilde{F}$ for one free scalar, eq. (2.105).
2.7 Gross-Neveu-Yukawa model and a test of 3-d F-theorem

The action of the Gross-Neveu (GN) model \[25\] is given by

\[
S(\bar{\psi}, \psi) = - \int d^d x \left[ \bar{\psi} \partial \psi + \frac{1}{2} g (\bar{\psi} \psi)^2 \right].
\]  

(2.107)

Here $N = \tilde{N} \text{tr} 1$, where $\text{tr} 1$ is the trace of the identity in the Dirac matrix space, and $\tilde{N}$ is the number of Dirac fermion fields $\psi^i (i = 1, \ldots, \tilde{N})$. The parameter $N$ counts the actual number of fermion components, and it is the natural parameter to write down the $1/N$ expansion (factors of $\text{tr} 1$ never appear in the expansion coefficients if $N$ is used as the expansion parameter).

The beta functions and anomalous dimensions of this model in $d = 2 + \epsilon$ can be calculated to be (see, for instance, \[3\])

\[
\beta(g) = \epsilon g - (N - 2) \frac{g^2}{2\pi} + (N - 2) \frac{g^3}{4\pi^2} + \mathcal{O}(g^4)
\]

(2.108)

\[
\eta_\psi(g) = \frac{N - 1}{8\pi^2} g^2 - \frac{(N - 1)(N - 2)}{32\pi^3} g^3 + \mathcal{O}(g^4)
\]

(2.109)

\[
\eta_M(g) = \frac{N - 1}{2\pi} g - \frac{N - 1}{8\pi^2} g^2 + \mathcal{O}(g^3),
\]

(2.110)

where $\eta_M$ is related to the anomalous dimension of the composite field $\sigma = \bar{\psi} \psi$. We can solve the beta function for the critical value of $g$ at the fixed point:

\[
g^* = \frac{2\pi}{N - 2} \epsilon \left( 1 - \frac{\epsilon}{N - 2} \right) + \mathcal{O}(\epsilon^3).
\]

(2.111)
Plugging this value of $g^*$, we can find the dimensions of the fermion field and the composite field:

$$
\Delta_\psi = d - 1 + \eta_\psi(g^*) = \frac{1 + \epsilon}{2} + \frac{N - 1}{4(N - 2)^2} \epsilon^2 + \mathcal{O}(\epsilon^3), \quad (2.112)
$$

$$
\Delta_\sigma = d - 1 - \eta_M(g^*) = 1 - \frac{\epsilon}{N - 2} + \mathcal{O}(\epsilon^2). \quad (2.113)
$$

The UV fixed point of the GN model in $2 < d < 4$ dimensions is related to the IR fixed point of the Gross-Neveu-Yukawa (GNY) model, which has the following action [26, 27, 3]

$$
S(\bar{\psi}, \psi, \sigma) = \int d^d x \left[ -\bar{\psi}^i (\partial + g_1 \sigma) \psi^i + \frac{1}{2} (\partial_\mu \sigma)^2 + \frac{g_2}{24} \sigma^4 \right]. \quad (2.114)
$$

In $d = 4 - \epsilon$, the one-loop beta functions of the GNY model are given by:

$$
\beta_{g_1} = -\epsilon g_1 + \frac{N + 6}{16\pi^2} g_1^4 \quad (2.115)
$$

$$
\beta_{g_2} = -\epsilon g_2 + \frac{1}{8\pi^2} \left( \frac{3}{2} g_2^2 + N g_2 g_1^2 - 6N g_1^4 \right). \quad (2.116)
$$

For small $\epsilon$, one finds that there is a stable IR fixed point for any positive $N$:

$$
(g_1^*)^2 = \frac{16\pi^2 \epsilon}{N + 6} \quad (2.117)
$$

$$
g_2^* = 16\pi^2 R \epsilon, \quad (2.118)
$$

where

$$
R = \frac{24N}{(N + 6)((N - 6) + \sqrt{N^2 + 132N + 36})}. \quad (2.119)
$$

The anomalous dimensions of the elementary fields are given by

$$
\gamma_\psi = \frac{1}{32\pi^2} g_1^2, \quad \gamma_\sigma = \frac{N}{32\pi^2} g_1^2. \quad (2.120)
$$
and plugging in the value of the fixed point couplings, one obtains

\[ \Delta_\psi = \frac{d - 1}{2} + \gamma_\psi = \frac{3}{2} - \frac{N + 5}{2(N + 6)} \epsilon \]  
\[ \Delta_\sigma = \frac{d - 2}{2} + \gamma_\sigma = 1 - \frac{3}{N + 6} \epsilon . \]  

The large \( N \) expansion of the 3d critical fermion theory may be developed in arbitrary dimension following similar lines as in the scalar case reviewed in Section 2: one introduces an auxiliary field \( \sigma \) to simplify the quartic interaction, and develops the \( 1/N \) perturbation theory with the effective non-local propagator for \( \sigma \) and the \( \sigma \bar{\psi} \psi \) interaction. The anomalous dimensions of \( \psi \) and \( \sigma \) in the critical theory have been calculated respectively to order \( 1/N^3 \) and \( 1/N^2 \), and in arbitrary dimension \( d \) \cite{53, 69}. To leading order in \( 1/N \), the explicit expressions are given by

\[ \Delta_\psi = \frac{d - 1}{2} - \frac{\Gamma (d - 1) \sin \left( \frac{\pi d}{2} \right)}{\pi \Gamma \left( \frac{d}{2} - 1 \right) \Gamma \left( \frac{d}{2} + 1 \right)} \frac{1}{N} + \mathcal{O} \left( \frac{1}{N^2} \right) \]  
\[ \Delta_\sigma = 1 + \frac{4(d - 1) \Gamma (d - 1) \sin \left( \frac{\pi d}{2} \right)}{\pi (d - 2) \Gamma \left( \frac{d}{2} - 1 \right) \Gamma \left( \frac{d}{2} + 1 \right)} \frac{1}{N} + \mathcal{O} \left( \frac{1}{N^2} \right) . \]  

Expanding these expressions in \( d = 2 + \epsilon \) and \( d = 4 - \epsilon \), one can verify the agreement with (2.113) and (2.122) respectively. Note that both \( \Delta_\psi \) and \( \Delta_\sigma \) go below their respective unitarity bounds for \( d > 4 \); so, unlike in the scalar case, there is no unitary critical fermion theory above dimension four. One may check that the available subleading large \( N \) results \cite{53, 69} also precisely agree with the \( 1/N \) expansion of (2.113) and (2.122), giving strong support to the fact that the 3d critical fermionic CFT may be viewed as either the IR fixed point of the GNY model, or the UV fixed point of the GN theory.

We may use the two alternative descriptions of the critical fermion theory to provide a simple test of the 3d F-theorem, similar to the tests carried out in \cite{50}. In the UV, the GNY model (which has relevant interactions in \( d = 3 \)) is a free theory

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of $N$ fermions and one conformal scalar, while the GN model is free in the IR. Thus, the value of $F$ for the critical fermion theory should satisfy

$$NF_{\text{free fermi}} < F_{\text{crit. fermi}} < NF_{\text{free fermi}} + F_{\text{free sc.}}.$$  \hfill(2.125)

In $d = 3$, one finds

$$F_{\text{free sc.}} = \frac{\log 2}{8} - \frac{3\zeta(3)}{16\pi^2} \approx 0.0638.$$  \hfill(2.126)

The value of $F$ in the critical theory may be computed perturbatively in the $1/N$ expansion, and one obtains the result \cite{68,50}

$$F_{\text{crit. fermi}} = NF_{\text{free fermi}} + \frac{\zeta(3)}{8\pi^2} + \mathcal{O}\left(\frac{1}{N}\right).$$  \hfill(2.127)

Because $\frac{\zeta(3)}{8\pi^2} \approx 0.0152 < F_{\text{free sc.}}$, we see that indeed (2.125) is satisfied in both directions.
Chapter 3

Three-Loop Analysis of the Critical $O(N)$ Models in $6 - \epsilon$ Dimensions

3.1 Introduction and Summary

This chapter is based on the work published in [29], co-authored with Simone Giombi, Igor Klebanov, and Grigory Tarnopolsky. We also thank J. Gracey and I. Herbut for very useful correspondence, and to Y. Nakayama for valuable discussions.

We would like to extend the one-loop analysis was carried out in 2 to three loops.

We have the cubic $O(N)$ symmetric theory of $N + 1$ scalar fields $\sigma$ and $\phi^i$ in $6 - \epsilon$ dimensions with the Lagrangian:

$$
\mathcal{L} = \frac{1}{2} (\partial_\mu \phi^i)^2 + \frac{1}{2} (\partial_\mu \sigma)^2 + \frac{1}{2} g_1 \sigma (\phi^i)^2 + \frac{1}{6} g_2 \sigma^3 ,
$$

(3.1)

The one-loop beta functions showed that for $N > N_{\text{crit}}$ there exists an IR stable fixed point with real values of the two couplings. We argued that this IR fixed point of the cubic $O(N)$ theory is equivalent to the perturbatively unitary UV fixed point of the $O(N)$ model with interaction $(\phi^i \phi^i)^2$, which exists for large $N$ in $4 < d < 6$ [32, 33, 34, 35]. The $1/N$ expansions of various operator scaling dimensions we found
in chapter 2 agree with the corresponding results [4, 5, 6, 7, 8, 9, 10, 11, 12] in the quartic $O(N)$ model continued to $6 - \epsilon$ dimensions.

A surprising result of chapter 2 was that the one-loop value of $N_{\text{crit}}$ is very large: if $N_{\text{crit}}$ is treated as a continuous real parameter, then it is $\approx 1038.266$. Our interest now is in continuing the $d = 6 - \epsilon$ fixed point to $\epsilon = 1$ in the hope of finding a 5-dimensional $O(N)$ symmetric unitary CFT. In order to study the $\epsilon$ expansion of $N_{\text{crit}}$, in section 3.2 we calculate the three-loop $\beta$ functions, following the earlier work of [38, 39, 40]. In section 3.4 we find the following expansion for the critical value of $N$:

$$N_{\text{crit}} = 1038.266 - 609.840 \epsilon - 364.173 \epsilon^2 + O(\epsilon^3). \quad (3.2)$$

Neglecting further corrections, this gives $N_{\text{crit}}(\epsilon = 1) \approx 64$, but higher orders in $\epsilon$ can obviously change this value significantly. It is our hope that a conformal bootstrap approach [13, 14, 15, 16], perhaps along the lines of [61], can help determine $N_{\text{crit}}$ more precisely in $d = 5$. The bootstrap approach may also be applied in non-integer dimensions close to 6, but one should keep in mind that such theories are not strictly unitary [71].

The major reduction of $N_{\text{crit}}$ as $\epsilon$ is increased from 0 to 1 is analogous to what is known about the Abelian Higgs model in $4 - \epsilon$ dimensions. For the model containing $N_f$ complex scalars, the one-loop critical value of $N_f$ is found to be large, $N_{f,\text{crit}} \approx 183$ [58]. However, the $O(\epsilon)$ correction found from two-loop beta functions has a negative coefficient and almost exactly cancels the leading term when $\epsilon = 1$, suggesting that the $N_{f,\text{crit}}$ is small in the physically interesting three-dimensional theory [59].

---

1 These papers considered cubic field theories of $q - 1$ scalar fields that were shown in [70] to describe the $q$-state Potts model. These theories possess only discrete symmetries and generally differ from the $O(N)$ symmetric theories that we study.

2 Another possible non-perturbative approach to the theory in $4 < d < 6$ is the Exact Renormalization Group [72]. This approach does not seem to indicate the presence of a UV fixed point in the theory of $N$ scalar fields, but a search for an IR fixed point in the theory of $N + 1$ scalar fields has not been carried out yet.

3 We are grateful to Igor Herbut for pointing this out to us.
Another interesting property of the theories (4.2) is the existence of the lower critical value $N'_{\text{crit}}$ such that for $N < N'_{\text{crit}}$ there is an IR stable fixed point at imaginary values of $g_1$ and $g_2$. The simplest example of such a non-unitary theory is $N = 0$, containing only the field $\sigma$. Its $6 - \epsilon$ expansion was originally studied by Michael Fisher [37] and the continuation to $\epsilon = 4$ provides an approach to the Yang-Lee edge singularity in the two-dimensional Ising model (this is the $(2, 5)$ minimal model [73, 74] with central charge $-22/5$). From the three-loop $\beta$ functions we find the $\epsilon$ expansion

$$N'_{\text{crit}} = 1.02145 + 0.03253\epsilon - 0.00163\epsilon^2 + O(\epsilon^3).$$  

(3.3)

The smallness of the coefficients suggests that $N'_{\text{crit}} > 1$ for a range of dimensions below 6. In section 3.5 we discuss some properties of the $N = 1$ theory. We show that it possesses an unstable fixed point with $g_1^* = g_2^*$ where the lagrangian splits into that of two decoupled $N = 0$ theories. There is also an IR stable fixed point where $g_2^* = 6g_1^*/5 + O(\epsilon)$. A distinguishing feature of this non-unitary CFT is that it has a discrete $Z_2$ symmetry, and it would be interesting to search for it using the conformal bootstrap methods developed in [60]. We suggest that, when continued to two dimensions, it describes the $(3, 8)$ non-unitary conformal minimal model.

In Section 3.4.1 we also discuss unstable unitary fixed points that are present in $6 - \epsilon$ dimensions for all $N$. For $N = 1$ the fixed point has $g_1^* = -g_2^*$; it is $Z_3$ symmetric and describes the critical point of the 3-state Potts model in $6 - \epsilon$ dimensions [75].

3.2 Three-loop $\beta$-functions in $d = 6 - \epsilon$

The action of the cubic theory is

$$S = \int d^d x \left( \frac{1}{2} (\partial_\mu \phi_i^0)^2 + \frac{1}{2} (\partial_\mu \sigma_0)^2 + \frac{1}{2} g_{1,0} \sigma_0 \phi_i^0 \phi_i^0 + \frac{1}{6} g_{2,0} \sigma_0^3 \right),$$

(3.4)

\footnote{We are grateful to Yu Nakayama for valuable discussions on this issue.}
where \( \phi_0 \) and \( \sigma_0 \) are bare fields and \( g_{1,0} \) and \( g_{2,0} \) are bare coupling constants.\(^5\) As usual, we introduce renormalized fields and coupling constants by

\[
\begin{align*}
\sigma_0 &= Z_{\sigma}^{1/2} \sigma, \quad \phi_0^i = Z_{\phi}^{1/2} \phi^i, \\
g_{1,0} &= \mu \frac{Z_{g_1} Z_{\sigma}^{-1/2} Z_{\phi}^{-1}}{2} g_1, \quad g_{2,0} = \mu \frac{Z_{g_2} Z_{\sigma}^{-3/2}}{2} g_2.
\end{align*}
\]

Here \( g_1, g_2 \) are the dimensionless renormalized couplings, and \( \mu \) is the renormalization scale. We may write

\[
Z_{\sigma} = 1 + \delta_{\sigma}, \quad Z_{\phi} = 1 + \delta_{\phi}, \quad Z_{g_1} = 1 + \delta g_1, \quad Z_{g_2} = 1 + \delta g_2
\]

so that, in terms of renormalized quantities, the action reads

\[
S = \int d^d x \left( \frac{1}{2} (\partial_\mu \phi^i)^2 + \frac{1}{2} (\partial_\mu \sigma)^2 + \frac{g_1}{2} \sigma \phi^i \phi^i + \frac{g_2}{6} \sigma^3 \right)
\]

\[
+ \frac{\delta_{\phi}}{2} (\partial_\mu \phi^i)^2 + \frac{\delta_{\sigma}}{2} (\partial_\mu \sigma)^2 + \frac{\delta g_1}{2} \sigma \phi^i \phi^i + \frac{\delta g_2}{6} \sigma^3 \right).
\]

To carry out the renormalization procedure, we will use dimensional regularization \cite{76} in \( d = 6 - \epsilon \) and employ the minimal subtraction scheme \cite{77}. In this scheme, the counterterms are fixed by requiring cancellation of poles in the dimensional regulator, and have the structure

\[
\delta g_1 = \sum_{n=1}^{\infty} \frac{a_n(g_1, g_2)}{\epsilon^n}, \quad \delta g_2 = \sum_{n=1}^{\infty} \frac{b_n(g_1, g_2)}{\epsilon^n}, \quad \delta \phi = \sum_{n=1}^{\infty} \frac{z_{\phi}^n(g_1, g_2)}{\epsilon^n}, \quad \delta \sigma = \sum_{n=1}^{\infty} \frac{z_{\sigma}^n(g_1, g_2)}{\epsilon^n}.
\]

\(^{(3.8)}\)

The anomalous dimensions and \( \beta \)-functions are determined by the coefficients of the simple \( 1/\epsilon \) poles in the counterterms \cite{77}. Specifically, in our case we have that the

\(^{5}\)We do not include mass terms as we are ultimately interested in the conformal theory. In the dimensional regularization that we will be using, mass terms are not generated if we set them to zero from the start."
anomalous dimensions are given by

\[ \gamma_\phi = \frac{1}{4} \left( g_1 \frac{\partial}{\partial g_1} + g_2 \frac{\partial}{\partial g_2} \right) z_1^\phi, \quad (3.9) \]

\[ \gamma_\sigma = \frac{1}{4} \left( g_1 \frac{\partial}{\partial g_1} + g_2 \frac{\partial}{\partial g_2} \right) z_1^\sigma \quad (3.10) \]

and the \( \beta \)-functions are

\[ \beta_1(g_1, g_2) = -\frac{\epsilon}{2} g_1 + \frac{1}{2} \left( g_1 \frac{\partial}{\partial g_1} + g_2 \frac{\partial}{\partial g_2} - 1 \right) \left( a_1 - \frac{1}{2} g_1 (2z_1^\phi + z_1^\sigma) \right), \]

\[ \beta_2(g_1, g_2) = -\frac{\epsilon}{2} g_2 + \frac{1}{2} \left( g_1 \frac{\partial}{\partial g_1} + g_2 \frac{\partial}{\partial g_2} - 1 \right) \left( b_1 - \frac{3}{2} g_2 z_1^\sigma \right). \quad (3.11) \]

In other words, in order to determine the anomalous dimensions and \( \beta \)-functions, we have to calculate the coefficients of the \( 1/\epsilon \)-divergencies in the loop diagrams, from which we can read off the residues \( a_1(g_1, g_2), b_1(g_2, g_2), z_1^\phi(g_1, g_2), z_1^\sigma(g_1, g_2) \).

Working in perturbation theory, we will denote by \( a_{i1} \) the term of \( i \)-th order in the coupling constants, and similarly for the other residue functions. Then, using the results for the Feynman diagrams collected in the Appendix, we find the anomalous dimensions

\[ \gamma_\phi = -\frac{1}{2} z_1^{12} - z_1^{14} - \frac{3}{2} z_1^{16} \]

\[ = \frac{g_1^2}{6(4\pi)^3} - \frac{g_1^2}{432(4\pi)^6} \left( g_1^2 (11N - 26) - 48g_1g_2 + 11g_2^2 \right) \]

\[ - \frac{g_1^2}{31104(4\pi)^9} \left( g_1^4 (N(13N - 232) + 5184\zeta(3) - 9064) + g_1^3 g_2 (441N - 544) \right) \]

\[ - 2g_1^2 g_2^2 (193N - 2592\zeta(3) + 5881) + 942g_1g_2^3 + 327g_2^4 \]. \quad (3.12)
\[ \gamma = -\frac{1}{2}z^{12} - z^{14} - \frac{3}{2}z^{16} \]
\[ = N g_1^2 + g_2^2 + \frac{1}{12(4\pi)^3} \left( 2N g_1^4 + 48Ng_1^3g_2 - 11Ng_1^2g_2^2 + 13g_2^4 \right) \]
\[ + \frac{1}{62208(4\pi)^9} \left( 96N(12N + 11)g_1^6g_2 - 1560Ng_1^3g_2^3 + 952Ng_1^2g_2^4 \right) \]
\[ - 2Ng_1^6(1381N - 2592\zeta(3) + 4280) + g_2^6(2592\zeta(3) - 5195) \]
\[ + 3Ng_1^4g_2^2(N + 4320\zeta(3) - 8882) \]
\[ \text{(3.13)} \]

and the \( \beta \)-functions

\[ \beta_1 = -\frac{e}{2}g_1 + (a_{13} - \frac{1}{2}g_1(2z^{12} + z^{12})) + 2(a_{15} - \frac{1}{2}g_1(2z^{14} + z^{14})) + 3(a_{17} - \frac{1}{2}g_1(2z^{16} + z^{16})) \]
\[ = -\frac{e}{2}g_1 + \frac{1}{12(4\pi)^3} \left( (N - 8)g_1^2 - 12g_1g_2 + g_2^2 \right) \]
\[ - \frac{1}{432(4\pi)^6} g_1 \left( (536 + 86N)g_1^4 + 12(30 - 11N)g_1^3g_2 + (628 + 11N)g_1^2g_2^2 + 24g_1g_2^3 - 13g_2^4 \right) \]
\[ + \frac{1}{62208(4\pi)^9} g_1 \left( g_2^6(5195 - 2592\zeta(3)) + 12g_1g_2^6(-2801 + 2592\zeta(3)) \right) \]
\[ - 8g_1^2g_2^4(1245 + 119N + 7776\zeta(3)) \]
\[ + g_1^4g_2^2(-358480 + 53990N - 3N^2 - 2592(-16 + 5N)\zeta(3)) \]
\[ + 36g_1^5g_2(-500 - 3464N + N^2 + 864(5N - 6)\zeta(3)) \]
\[ - 2g_1^6(125680 - 20344N + 1831N^2 + 2592(25N + 4)\zeta(3)) \]
\[ + 48g_1^3g_2^3(95N - 3(679 + 864\zeta(3))) \]
\[
\beta_2 = -\frac{e}{2}g_2 + \left(b_{13} - \frac{3}{2}g_2 z_{12}^\sigma\right) + 2\left(b_{15} - \frac{3}{2}g_2 z_{14}^\sigma\right) + 3\left(b_{17} - \frac{3}{2}g_2 z_{16}^\sigma\right)
\]
\[
= -\frac{e}{2}g_2 + \frac{1}{4(4\pi)^3} \left(-4Ng_1^3 + Ng_1^2g_2 - 3g_2^3\right)
\]
\[
+ \frac{1}{144(4\pi)^6} \left(-24Ng_1^5 - 322Ng_1^4g_2 - 60Ng_1^3g_2^2 + 31Ng_1^2g_2^3 - 125g_2^5\right)
\]
\[
+ \frac{1}{20736(4\pi)^9} \left(-48N(713 + 577N)g_1^7 + 6272Ng_1^2g_2^5 + 48Ng_1^3g_2^4(181 + 432\zeta(3))
\]
\[
- 5g_2^7(6617 + 2592\zeta(3)) - 24Ng_1^5g_2^2(1054 + 471N + 2592\zeta(3))
\]
\[
+ 2Ng_1^6g_2(19237N - 8(3713 + 324\zeta(3))) + 3Ng_1^4g_2^3(263N - 6(7105 + 2448\zeta(3))\right).
\]

(3.15)

In the case \(N = 0\) (the single scalar cubic theory), our results are in agreement with the three-loop calculation of [39].

### 3.3 The IR fixed point

Let us introduce the notation

\[
g_1 \equiv \sqrt{\frac{6\epsilon(4\pi)^3}{N}}x, \quad g_2 \equiv \sqrt{\frac{6\epsilon(4\pi)^3}{N}}y.
\]

(3.16)
In terms of the new variables $x$ and $y$, the condition that both $\beta$-functions be zero reads

\[
0 = \frac{1}{2} x (-8x^2 + N(x^2 - 1) - 12xy + y^2) \\
- \frac{1}{12N} x \left((536 + 86N)x^4 + 12(30 - 11N)x^3y + (628 + 11N)x^2y^2 + 24xy^3 - 13y^4\right) \epsilon \\
- \frac{1}{288N^2} x \left(12xy^5(2801 - 2592\zeta(3)) + y^6(-5195 + 2592\zeta(3))\right) \\
+ 48x^3y^3(2037 - 95N + 2592\zeta(3)) \\
+ 8x^2y^4(1245 + 119N + 7776\zeta(3)) + x^4y^2(358480 - 53990N + 3N^2 + 2592(5N - 16)\zeta(3)) \\
- 36x^5y(-500 - 3464N + N^2 + 864(5N - 6)\zeta(3)) \\
+ 2x^6(125680 - 20344N + 1831N^2 + 2592(25N + 4)\zeta(3)) \epsilon^2
\] (3.17)

and

\[
0 = -\frac{1}{2} \left(9y^3 + N(12x^3 + y - 3x^2y)\right) \\
- \frac{1}{4N} \left(125y^5 + Nx^2(24x^3 + 322x^2y + 60xy^2 - 31y^3)\right) \epsilon \\
- \frac{1}{96N^2} \left(N^2x^4(27696x^3 - 38474x^2y + 11304xy^2 - 789y^3) + 5y^7(6617 + 2592\zeta(3))\right) \\
+ 34224Nx^7 - 6272Nx^6y^5 + 16Nx^6y(3713 + 324\zeta(3)) - 48Nx^3y^4(181 + 432\zeta(3)) \\
+ 48Nx^5y^2(527 + 1296\zeta(3)) + 18Nx^4y^3(7105 + 2448\zeta(3))) \epsilon^2
\] (3.18)

These equations can be solved order by order in the $\epsilon$ expansion. Using also the $1/N$ expansion, we find the fixed point values

\[
x_* = 1 + \frac{22}{N} + \frac{726}{N^2} - \frac{326180}{N^3} - \frac{349658330}{N^4} + \ldots \\
+ \left(-\frac{155}{6N} - \frac{1705}{N^2} + \frac{912545}{N^3} + \frac{3590574890}{3N^4} + \ldots\right) \epsilon \\
+ \left(\frac{1777}{144N} + \frac{29093/36 - 1170\zeta(3)}{N^2} + \ldots\right) \epsilon^2,
\] (3.19)
\[
y_* = 6(1 + \frac{162}{N} + \frac{68766}{N^2} + \frac{4124420}{N^3} + \frac{2876254870}{N^4} + \ldots \\
+ \left(\frac{-215}{2N} - \frac{86335}{N^2} - \frac{7572265}{N^3} - \frac{6963402510}{N^4} + \ldots \right) \epsilon \\
+ \left(\frac{2781}{48N} + \frac{270911}{6N^2} - \frac{157140\zeta(3)}{N^3} + \ldots \right) \epsilon^2.
\] (3.20)

This large \(N\) solution corresponds to an IR stable fixed point and generalizes the previous one-loop result. This fixed point exists and is stable to all orders in the \(1/N\) expansion.

If results beyond the \(1/N\) expansion are desired, one can determine the \(\epsilon\) expansions of \(x_*, y_*\) for finite \(N\) as follows. Plugging the expansions

\[
x_* = x_0(N) + x_1(N)\epsilon + x_2(N)\epsilon^2 + \ldots , \quad y_* = y_0(N) + y_1(N)\epsilon + y_2(N)\epsilon^2 + \ldots 
\] (3.21)

into (3.17)-(3.18), the leading order terms are found to be [28, 78]

\[
x_0(N) = \sqrt{\frac{N}{(N - 44)z(N)^2 + 1}} z(N), \quad y_0(N) = \sqrt{\frac{N}{(N - 44)z(N)^2 + 1}} (1 + 6z(N)),
\] (3.22)

where \(z(N)\) is the solution to the cubic equation

\[
840z^3 - (N - 464)z^2 + 84z + 5 = 0
\] (3.23)

with large \(N\) behavior \(z(N) = 840N + O(N^0)\). This solution is real only if \(N > 1038.27\), as can be seen from the discriminant of the above cubic equation. Once the \(x_0(N), y_0(N)\) are known, one can then determine the higher order terms in (3.21) by solving the equations (3.17)-(3.18) order by order in \(\epsilon\).

The other two roots have large \(N\) behavior \(z(N) \sim \pm \sqrt[5]{5N}\) and they are unstable IR fixed points [28].
For $N \gg 1038$ the finite $N$ exact results are close to (3.19)-(3.20), but for $N \sim 1038$ they deviate somewhat, indicating that, close to the critical $N$, the large $N$ expansion is not a good approximation (see also Figure 3.1 below).

### 3.3.1 Dimensions of $\phi$ and $\sigma$

In terms of the rescaled couplings $x$, $y$ defined in (3.16), the anomalous dimensions read

\[
\gamma_{\phi} = \frac{x^2}{N} \epsilon - \frac{x^2}{12N^2} \left( (26 - 11N)x^2 + 48xy - 11y^2 \right) \epsilon^2 \\
+ \frac{x^2}{144N^3} \left( 6(544 - 441N)x^3y - 942xy^3 - 327y^4 \right) \\
+ x^4(9064 + (232 - 13N)N - 5184\zeta(3)) \\
+ 2x^2y^2(5881 + 193N - 2592\zeta(3))) \epsilon^3, \tag{3.24}
\]

\[
\gamma_{\sigma} = \frac{Nx^2 + y^2}{2N} \epsilon - \frac{1}{12N^2} \left( 13y^4 + Nx^2(2x^2 + 48xy - 11y^2) \right) \epsilon^2 \\
+ \frac{1}{288N^3} \left( N^2x^4(2762x^2 - 1152xy - 3y^2) \\
+ 2Nx^2 \left( -528x^3y + 780xy^3 - 476y^4 + 3x^2y^2(4441 - 2160\zeta(3)) + 8x^4(535 - 324\zeta(3)) \right) \\
+ y^6(5195 - 2592\zeta(3)) \right) \epsilon^3. \tag{3.25}
\]
Plugging the fixed point values (3.19)-(3.20) into these expressions, we get the conformal dimensions of $\sigma$ and $\phi$ at the fixed point

\[ \Delta_\phi = \frac{d}{2} - 1 + \gamma_\phi \]
\[ = 2 - \frac{\epsilon}{2} + \left( \frac{1}{N} + \frac{44}{N^2} + \frac{1936}{N^3} + \ldots \right) \epsilon + \left( -\frac{11}{12N} - \frac{835}{6N^2} - \frac{16352}{N^3} + \ldots \right) \epsilon^2 \]
\[ + \left( -\frac{13}{144N} + \frac{6865}{72N^2} + \frac{54367/2 - 3672\zeta(3)}{N^3} + \ldots \right) \epsilon^3, \quad (3.26) \]
\[ \Delta_\sigma = \frac{d}{2} - 1 + \gamma_\sigma \]
\[ = 2 + \left( \frac{40}{N} + \frac{6800}{N^2} + \ldots \right) \epsilon + \left( -\frac{104}{3N} - \frac{34190}{3N^2} + \ldots \right) \epsilon^2 \]
\[ + \left( -\frac{22}{9N} + \frac{47695/18 - 2808\zeta(3)}{N^2} + \ldots \right) \epsilon^3. \quad (3.27) \]

One can verify that these results are in precise agreement with the large $N$ calculation of [4, 5, 6] for the critical $O(N)$ model in general $d$, analytically continued to $d = 6 - \epsilon$. This provides a strong check on our calculations and on our interpretation of the IR fixed point of the cubic $O(N)$ scalar theory.

The $1/N$ expansions are expected to work well for $N \gg 1038$. For any $N$ larger than the critical value, the $\epsilon$ expansions of the scaling dimensions may be determined using (3.25) and the exact analytic solutions for the fixed point location $(x_*, y_*)$. For example, in Figure 3.1 we plot the coefficient of the $O(\epsilon^3)$ term in $\Delta_\sigma$ as a function of $N$ and compare it with the corresponding $1/N$ expansion.

### 3.3.2 Dimensions of quadratic and cubic operators

In chapter 2 the mixed anomalous dimensions of quadratic operators $\sigma^2$ and $\phi^i\phi^i$ were calculated at one-loop order. These results were checked against the $O(1/N)$ term in the corresponding operator dimensions for the $O(N)$ $\phi^4$ theory [10]. In this chapter, we carry out an additional check, comparing with the $O(1/N^2)$ correction.
Figure 3.1: The $O(\epsilon^3)$ in $\Delta_{\sigma}$ as a function of $N$ for $N \geq 1039$. The $1/N$ expansion approaches the exact result as we include more terms.

found in [79], but still working to the one-loop order in $\epsilon$ (it should be straightforward to generalize the mixing calculation to higher loops, but we will not do it here).

In the quartic $O(N)$ theory with interaction $\frac{\lambda}{4} (\phi^i \phi^i)^2$, the derivative of the beta function at the fixed point coupling

$$\omega = \beta'(\lambda_*) = 4 - d + \frac{\omega_1}{N} + \frac{\omega_2}{N^2} + \ldots$$

(3.28)

is related to the dimension of the operator $(\phi^i \phi^i)^2$ by

$$\Delta_{\phi^1} = d + \omega.$$  

(3.29)

In [10, 79] the coefficient $\omega_1$ was computed as a function of dimension $d$:

$$\omega_1 = \frac{2(d-4)(d-2)(d-1)\Gamma(d)}{d\Gamma\left(2 - \frac{d}{2}\right) \Gamma\left(\frac{d}{2}\right)^3}.$$  

(3.30)
The coefficient $\omega_2$ has a more complicated structure for general $d$ which was first found in [79]. Using this result, we get that in $d = 5$, 

$$
\Delta_{\phi^4} = 4 - \frac{2048}{15\pi^2 N} \left( \frac{8192(67125\pi^2 - 589472)}{3375\pi^4 N^2} \right) + \ldots \approx 4 - \frac{13.8337}{N} - \frac{1819.66}{N^2} + \ldots \quad (3.31)
$$

Let us also quote the expansion of $\omega_2$ in $d = 4 - \epsilon$ and $d = 6 - \epsilon$:

$$
\omega_2 = 102\epsilon^2 + \left( -\frac{259}{2} + 120\zeta(3) \right) \epsilon^3 + \ldots, \quad d = 4 - \epsilon \quad (3.32)
$$

$$
\omega_2 = -49760\epsilon + \frac{237476}{3} \epsilon^2 + \left( -\frac{92480}{9} + 32616\zeta(3) \right) \epsilon^3 + \ldots, \quad d = 6 - \epsilon \quad (3.33)
$$

In $d = 4 - \epsilon$, one can check that the above results correctly reproduce the derivative of the $\beta$-function [2]

$$
\beta = -\epsilon\lambda + \frac{N + 8}{8\pi^2} \Lambda^2 - \frac{3(3N + 14)}{64\pi^4} \lambda^3
$$

$$
+ \frac{33N^2 + 480N\zeta(3) + 922N + 2112\zeta(3) + 2960}{4096\pi^6} \lambda^4 + O(\lambda^5) \quad (3.34)
$$

at the IR fixed point

$$
\lambda_* = \frac{8\pi^2}{N + 8} \epsilon + \frac{24\pi^2(3N + 14)}{(N + 8)^3} \epsilon^2
$$

$$
- \frac{\pi^2(33N^3 - 110N^2 + 96(N + 8)(5N + 22)\zeta(3) - 1760N - 4544)}{(N + 8)^5} \epsilon^3 + O(\epsilon^4). \quad (3.35)
$$

In $d = 6 - \epsilon$, the dimension of the $(\phi^i\phi^i)^2$ operator in the quartic theory should be matched to the primary operator arising from the mixing of the $\sigma^2$ and $\phi^i\phi^i$ operators in our cubic theory. In chapter 2, the mixing matrix of $\sigma^2$ and $\phi^i\phi^i$ to one-loop order was found to be

$$
\gamma^{ij} = \frac{-1}{6(4\pi)^3} \begin{pmatrix}
4g_1^2 - Ng_1^2 & 6\sqrt{N}g_1^2 - \sqrt{N}g_1g_2 \\
6\sqrt{N}g_1^2 - \sqrt{N}g_1g_2 & 4g_2^2 - Ng_2^2
\end{pmatrix}. \quad (3.36)
$$
Computing the eigenvalues $\gamma_{\pm}$ of this matrix, and inserting the values of one-loop fixed point couplings

$$g_{1*} = \sqrt{\frac{6\epsilon (4\pi)^3}{N}} \left( 1 + \frac{22}{N} + \frac{726}{N^2} - \frac{326180}{N^3} + \ldots \right), \quad (3.37)$$

$$g_{2*} = 6 \sqrt{\frac{6\epsilon (4\pi)^3}{N}} \left( 1 + \frac{162}{N} + \frac{68766}{N^2} + \frac{41224420}{N^3} + \ldots \right) \quad (3.38)$$

we find the scaling dimensions of the quadratic operators to be

$$\Delta_- = d - 2 + \gamma_- = 4 + \left( -\frac{100}{N} - \frac{49760}{N^2} - \frac{27470080}{N^3} + \ldots \right) \epsilon + \mathcal{O}(\epsilon^2), \quad (3.39)$$

$$\Delta_+ = d - 2 + \gamma_+ = 4 + \left( \frac{40}{N} + \frac{6800}{N^2} + \frac{2637760}{N^3} + \ldots \right) \epsilon + \mathcal{O}(\epsilon^2). \quad (3.40)$$

The operator with dimension $\Delta_+$ is a descendant of $\sigma$. The operator with dimension $\Delta_-$ is a primary, and comparing with (3.30), (3.33), we see that its dimension precisely agrees with the results of [79] to order $1/N^2$. The higher order terms in $\epsilon$ can be determined from mixed anomalous dimension calculations beyond one-loop, and we leave this to future work.

We now calculate the mixed anomalous dimensions of the nearly marginal operators $\mathcal{O}^1 = \sigma\phi\phi$ and $\mathcal{O}^2 = \sigma^3$. Using the beta functions written in equations (4.5)-(3.15), we can determine the anomalous dimensions of the nearly marginal operators by computing the eigenvalues of the matrix

$$M_{ij} = \frac{\partial \beta_i}{\partial g_j}. \quad (3.41)$$

Strictly speaking, this matrix is not exactly equal to the anomalous dimension mixing matrix, because it is not symmetric. However, we could make it symmetric by dividing and multiplying the off-diagonal elements by a factor $\sqrt{3N} + \mathcal{O}(\epsilon)$, which corresponds to an appropriate rescaling of the couplings. This clearly does not change
the eigenvalues of the matrix, and hence we can directly compute the eigenvalues $\lambda_{\pm}$ of (3.41), and obtain the dimensions of the eigenstate operators as

$$
\Delta_{\pm} = d + \lambda_{\pm}.
$$

(3.42)

Plugging in the fixed point values $x_*$ and $y_*$ from equations (3.19)-(3.20), we find that

$$
\Delta_+ = 6 + \left( \frac{155}{3} \epsilon^2 - \frac{1777}{36} \epsilon^3 + \ldots \right) \frac{1}{N} + O\left( \frac{1}{N^2} \right),
$$

$$
\Delta_- = 6 + \left( -420 \epsilon + 499 \epsilon^2 - \frac{1051}{12} \epsilon^3 + \ldots \right) \frac{1}{N} + O\left( \frac{1}{N^2} \right).
$$

(3.43)

The dimension of the $\sigma^k$ operator in the quartic $O(N)$ model is known to order $1/N$ as function of $d$ [10], and may be written as

$$
\Delta(\sigma^k) = 2k + \frac{k(d-2)((k-1)d^2 - d(3k-1) + 4)\Gamma(d)}{Nd\Gamma(2 - \frac{d}{2})\Gamma\left(\frac{d}{2}\right)^3} + O\left( \frac{1}{N^2} \right).
$$

(3.44)

Our result for $\Delta_-$ agrees with the $\epsilon$ expansion of this formula for $k = 3$ in $d = 6 - \epsilon$.

3.4 Analysis of critical $N$ as a function of $\epsilon$

We now investigate the behavior of $N_{\text{crit}}$ above which the fixed point exists at real values of the couplings. This can be defined as the value of $N$ (formally viewed as a continuous parameter) at which two real solutions of the $\beta$-function equations merge, and subsequently go off to the complex plane. Geometrically, this means that the curves on the $(g_1, g_2)$ plane defined by the zeroes of $\beta_1$ and $\beta_2$ are barely touching, i.e. they are tangent to each other. Therefore the critical $N$, as well as the corresponding
critical value of the couplings, can be determined by solving the system of equations
\[
\begin{align*}
\beta_1 &= 0, \quad \beta_2 = 0, \\
\frac{\partial \beta_1}{\partial g_1} &= \frac{\partial \beta_2}{\partial g_1}, \\
\frac{\partial \beta_1}{\partial g_2} &= \frac{\partial \beta_2}{\partial g_2}.
\end{align*}
\] (3.45)

Note that the condition in the second line is equivalent to requiring that the determinant of the anomalous dimension mixing matrix of nearly marginal operators, \(M_{ij} = \frac{\partial \beta_i}{\partial g_j}\), vanishes. This means that one of the two eigenstates becomes marginal.

Working in terms of the rescaled coupling constants defined in (3.16), we can solve the system of equations (3.45) order by order in \(\epsilon\). We assume a perturbative expansion
\[
\begin{align*}
x &= x_0 + x_1 \epsilon + x_2 \epsilon^2 + O(\epsilon^3), \\
y &= y_0 + y_1 \epsilon + y_2 \epsilon^2 + O(\epsilon^3), \\
N &= N_0 + N_1 \epsilon + N_2 \epsilon^2 + O(\epsilon^3)
\end{align*}
\] (3.46)
and plugging this into (3.45), we can solve for the undetermined coefficients uniquely. At the zeroth order, we get the equations
\[
\begin{align*}
N_0 + 8x_0^2 - N_0x_0^2 + 12x_0y_0 - y_0^2 &= 0, \\
12N_0^2 + N_0y_0 - 3N_0x_0^2y_0 + 9y_0^3 &= 0, \\
6 + \frac{(N_0 - 44)x_0}{6x_0 - y_0} &= \frac{6N_0x_0(y_0 - 6x_0)}{3N_0x_0^2 - 27y_0^2 - N_0}.
\end{align*}
\] (3.47)

The above system of equations can be solved analytically, as was done in chapter 2. We find that, up to the signs of \(x_0\) and \(y_0\), there are three inequivalent solutions
\[
\begin{align*}
x_0 &= 1.01804, \quad y_0 = 8.90305, \quad N_0 = 1038.26605, \\
x_0' &= 0.23185i, \quad y_0' = 0.25582i, \quad N_0' = 1.02145, \\
x_0'' &= 0.13175, \quad y_0'' = -0.03277, \quad N_0'' = -0.08750.
\end{align*}
\] (3.48-3.50)
Figure 3.2: The zeroes of the one-loop $\beta$ functions and the RG flow directions for $N = 2000$. The red dots correspond to the stable IR fixed points, while the black dots are unstable fixed points. As $N \to N_{\text{crit}}$, the red dot merges with the nearby black dot, and the two fixed points move into the complex plane.

The first of these solutions, with $N_{\text{crit}} = 1038.26605 + O(\epsilon)$, is of most interest to us because it is related to the large $N$ limit of the theory. For $N > N_{\text{crit}}$, we find a stable IR fixed point at real couplings $g_1$ and $g_2$.\footnote{It is stable with respect to flows of the nearly marginal couplings $g_1$ and $g_2$. As usual, there are some $O(N)$ invariant relevant operators that render this fixed point not perfectly stable.} This fixed point is shown with the red dot in Figure 3.2 (there is a second stable IR fixed point obtained by the transformation $(g_1, g_2) \to (-g_1, -g_2)$, which is a symmetry of this theory). There is also a nearby unstable fixed point, shown with a black dot, which has one stable and one unstable direction. As $N$ approaches $N_{\text{crit}}$ from above, the nearby unstable fixed
point approaches the IR stable fixed point, and they merge at $N_{\text{crit}}$. At $N < N_{\text{crit}}$, both fixed points disappear into the complex plane. As discussed in [80], this is a rather generic behavior at the lower edge of the conformal window: the conformality is lost through the annihilation of a UV fixed point and an IR fixed point. In [80] this was argued to happen at the lower (strongly coupled) edge of the conformal window for 4-dimensional $SU(N_c)$ gauge theory with $N_f$ flavors. It is interesting to observe that the same type of behavior occurs at the lower edge of the conformal window of the $O(N)$ model in $d = 6 - \epsilon$, which extends from $N_{\text{crit}}$ to infinity.

Let us identify the operator that causes the flow between the unstable fixed point and the IR stable fixed point of our primary interest. It is one of the two nearly marginal operators cubic in the fields that were studied in section 3.2. By studying the behavior of the dimensions $\Delta_1$ and $\Delta_2$ as $N \to N_{\text{crit}}$ we find that $\Delta_2 \to 6 - \epsilon$. Therefore, it is the operator corresponding to $\Delta_2$ that becomes exactly marginal for $N = N_{\text{crit}}$ and causes the flow between the IR fixed point and the nearby UV fixed point for $N$ slightly above $N_{\text{crit}}$. In bootstrap studies of the quartic $O(N)$ model this operator was denoted by $\sigma^3$ [10], i.e. it can be thought of as the "triple-trace operator" $(\phi^i \phi^i)^3$. The theory at the unstable fixed point has an unconventional large $N$ behavior where $x \sim \mathcal{O}(1)$ and $y \sim \mathcal{O}(\sqrt{N})$, so that corrections to scaling dimension proceed in powers of $N^{-1/2}$ as in chapter 2.

Let us now go back to finding the higher order corrections to $N_{\text{crit}}$ given by (3.48) (the higher order corrections to the other critical values (3.49)-(3.50) will be discussed in the next section). Once we have solved the leading order system (3.47), we can plug the solution into (3.46) and expand (3.45) up to order $\epsilon^2$. From this we obtain simple systems of linear equations from which we can determine $x_1, y_1, N_1$ and $x_2, y_2, N_2$. We find

$$
x_1 = -0.00940, \quad y_1 = -0.21024, \quad N_1 = -609.93980, \quad x_2 = 0.00690, \quad y_2 = 1.01680, \quad N_2 = -364.17333.
$$

(3.51)
Thus, to three-loop order, we conclude that

\[ N_{\text{crit}} = 1038.26605 - 609.83980 \epsilon - 364.17333 \epsilon^2 + O(\epsilon^3). \]  

(3.52)

We have also checked these expansion coefficients via a direct high-precision numerical calculation of \( N_{\text{crit}} \) for very small values of \( \epsilon \). The large and negative coefficients indicate that in the physically interesting case of \( d = 5 \), \( N_{\text{crit}} \) is likely to be much lower than the zeroth order value (this is analogous to the result [59] for the Abelian Higgs model). If we just use the first three terms and plug in \( \epsilon = 1 \), we get:

\[ N_{\text{crit}} \approx 64.253. \]  

(3.53)

For \( N < N_{\text{crit}} \) the anomalous dimensions, such as \( \gamma_{\phi} \), are no longer positive (in fact, they become complex). This loss of positivity of \( \gamma_{\phi} \) can also be seen as \( N \) is reduced in the quartic \( O(N) \) model. For example, using the \( 1/N \) expansion of \( \gamma_{\phi} \) in \( d = 5 \) [6]

\[
\gamma_{\phi} = \frac{32}{15\pi^2 N} - \frac{1427456}{3375\pi^4 N^2} + \left( \frac{275255197696}{759375\pi^6} - \frac{89735168}{2025\pi^4} + \frac{32768 \ln 4}{9\pi^4} - \frac{229376 \zeta(3)}{3\pi^6} \right) \frac{1}{N^3} + \ldots \\
= \frac{3}{2} + \frac{0.216152}{N} - \frac{4.342}{N^2} + \frac{121.673}{N^3} + \ldots
\]  

(3.54)

we find that it stops being positive for \( N < 35 \). This critical value is not too far from (3.53).
It is also instructive to study the theory using the $4 - \epsilon$ expansion. The anomalous dimensions of $\phi^i$ is [2]

$$
\gamma_{\phi} = \frac{N + 2}{4(N + 8)^2} \epsilon^2 + \frac{N + 2}{16(N + 8)^4} \left(-N^2 + 56N + 272\right) \epsilon^3 \\
+ \frac{N + 2}{64(N + 8)^6} \left(-5N^4 - 230N^3 + 1124N^2 + 17920N + 46144 - 384\zeta(3)(5N + 22)(N + 8)\right) \epsilon^4 \\
+ \mathcal{O}(\epsilon^5)
$$

(3.55)

For positive $\epsilon$ this expansion gives accurate information about the Wilson-Fisher IR fixed points [1]. For negative $\epsilon$ there exist formal UV fixed points at negative quartic coupling where we can apply this formula as well. In that case, $\gamma_{\phi}$ becomes negative for sufficiently large $|\epsilon|$ and $N < N_{\text{crit}}$, indicating that the operator $\phi^i$ violates the unitarity bound. For example for $d = 5$, corresponding to $\epsilon = -1$, we find $N_{\text{crit}} \approx 8$. Inclusion of the $\mathcal{O}(\epsilon^5)$ term (see [81]) raises this to $N_{\text{crit}} \approx 14$.

We see, therefore, that the estimates of $N_{\text{crit}}$ using the quartic $O(N)$ theory in $d = 5$ are even lower than the three-loop estimate (3.53). It seems safe to conclude that the true value is much lower than the one-loop estimate of 1038. To determine $N_{\text{crit}}$ in $d = 5$ more precisely, one needs a non-perturbative approach to the $d = 5$ theory, perhaps along the lines of the conformal bootstrap calculation in [61].

### 3.4.1 Unitary fixed points for all positive $N$

Let us note that not all real fixed points disappear for $N < N_{\text{crit}}$. The unstable real fixed points that are located in the upper left and lower right corners of Figure 3.2 exist for all positive $N$, and we would like to find their interpretation.

The fixed point with $N = 1$ has a particularly simple property that $g_1^* = -g_2^*$. This property of the solution holds for the three-loop $\beta$ functions, and we believe that it is exact. Using this, we note that the action at the fixed point is proportional to $(\sigma + i\phi)^3 + (\sigma - i\phi)^3$. Therefore, the theory at this fixed point enjoys a $Z_3$ symmetry
acting by the phase rotation on the complex combination $\sigma + i\phi$. This cubic classical action appears in the Ginzburg-Landau theory for the 3-state Potts model (see, for example, [75]). Therefore, we expect the $Z_3$ symmetric fixed point to describe the 3-state Potts model in $d = 6 - \epsilon$. The dimensions of operators at this fixed point are related by the $Z_3$ symmetry. For example, we find

$$\Delta_\phi = \Delta_\sigma = 2 - \frac{1}{3}\epsilon + \frac{2}{3}\epsilon^2 + \frac{443}{54}\epsilon^3 + O(\epsilon^4). \quad (3.56)$$

This is in agreement with the result of [39]. By calculating the eigenvalues $\lambda_\pm$ of the matrix $M_{ij} = \frac{\partial^2}{\partial \phi_i \partial \phi_j}$, we also find the dimensions (3.42) of the two cubic operators to order $\epsilon^3$:

$$\Delta_- = 6 - \frac{14}{3}\epsilon - \frac{158}{9}\epsilon^2 - \left(\frac{17380}{81} + 16\zeta(3)\right)\epsilon^3 = 6 - 4.66667\epsilon - 17.5556\epsilon^2 - 233.801\epsilon^3,$$

$$\Delta_+ = 6 - \frac{83}{18}\epsilon^2 - \left(\frac{38183}{648} + 4\zeta(3)\right)\epsilon^3 = 6 - 4.61111\epsilon^2 - 63.7326\epsilon^3. \quad (3.57)$$

The dimension $\Delta_+$ corresponds to the operator $(\sigma + i\phi)^3 + (\sigma - i\phi)^3$ which preserves the $Z_3$ symmetry and is slightly irrelevant for small $\epsilon$. The dimension $\Delta_-$ corresponds to the relevant operator $\sigma(\sigma^2 + \phi^2)$ which breaks the $Z_3$. Thus, the $Z_3$ symmetry helps stabilize the fixed point at small $\epsilon$.

Unfortunately, the $6 - \epsilon$ expansions (3.57) have growing coefficients, and it is not clear for what range of $\epsilon$ the fixed point exists. Thus, one may not be able to interpolate smoothly from the $Z_3$ symmetric fixed point in $d = 6 - \epsilon$ to $d = 2$ where the 3-state Potts model is described by the unitary $(5, 6)$ minimal model [73].

For all $N \geq 2$ we find unstable fixed points with $O(N)$ symmetry. These fixed points always have a relevant cubic operator, corresponding to a negative eigenvalue of the matrix $\frac{\partial^2}{\partial \phi_i \partial \phi_j}$. Also, they exhibit an unconventional large $N$ behavior involving half-integer powers of $N$, similarly to the unstable fixed points that appear for $N > N_{\text{crit}}$.

---

8We are grateful to Yu Nakayama for pointing this out to us.
and are shown by the black dots in the upper right and lower left corners of Figure 3.2. We leave a discussion of these fixed points for the future.

### 3.5 Non-unitary theories

In addition to the fixed points studied so far, which are perturbatively unitary and appear for $N > N_{\text{crit}}$, there exist non-unitary fixed points for $N''_{\text{crit}} < N < N'_{\text{crit}}$. The leading values of $N'_{\text{crit}}$ and $N''_{\text{crit}}$ are given in (3.49) and (3.50), respectively. Using the method developed above for finding the higher order in $\epsilon$ corrections to $N_{\text{crit}}$ we get

$$N'_{\text{crit}} = 1.02145 + 0.03253\epsilon - 0.00163\epsilon^2$$

$$x' = i \left(0.23185 + 0.08887\epsilon - 0.03956\epsilon^2\right), \quad y' = i \left(0.25582 + 0.11373\epsilon - 0.04276\epsilon^2\right)$$

(3.58)

and

$$N''_{\text{crit}} = -0.08750 + 0.34726\epsilon - 0.88274\epsilon^2$$

$$x'' = 0.13175 - 0.16716\epsilon + 0.12072\epsilon^2, \quad y'' = -0.03277 + 0.13454\epsilon - 0.35980\epsilon^2$$

(3.59)

Unfortunately, the latter expansion has growing coefficients, and we cannot extract any useful information from it. On the other hand, the higher order corrections to $N'_{\text{crit}}$ are very small, which suggests that $N'_{\text{crit}} > 1$ for range of dimensions below 6.

The theory with $N = 0$, which contains only the field $\sigma$, was originally studied by Michael Fisher as an approach to the Yang-Lee edge singularity in the Ising model [37]. Since the coupling is imaginary, it describes a non-unitary theory where some operator dimensions (e.g. $\sigma$) are below the unitarity bounds. In $d = 2$, this CFT corresponds to the $(2, 5)$ minimal model [74], which has $c = -22/5$. A conformal bootstrap approach to this model [60] has produced good results for a range of dimensions below 6.
The $N = 1$ theory, which has two fields and two coupling constants, has a more intricate structure. This theory is distinguished from the $N = 0$ case by the presence of a $Z_2$ symmetry $\phi \rightarrow -\phi$. Examining the $\beta$ functions at $N = 1$ and the eigenvalues of the matrix $\frac{\partial \beta_i}{\partial g_j}$, we observe that there exist a stable fixed point with $g_2^* = 6g_1^*/5 + O(\epsilon)$, and an unstable one with $g_1^* = g_2^*$. Introducing the field combinations

$$\sigma_1 = \sigma + \phi, \quad \sigma_2 = \sigma - \phi,$$

we note that for $g_1^* = g_2^*$ the interactions of the $N = 1$ model decouple as $\sim \sigma_1^3 + \sigma_2^3$, i.e. at this fixed point the theory is a sum of two Fisher’s $N = 0$ theories. However, one of the flow directions at this fixed point is unstable, since the corresponding operator has $\Delta O = 6 - 10\epsilon/9 + O(\epsilon^2)$ and is relevant (this value of the dimension corresponds to the negative eigenvalue of the matrix $M_{ij} = \frac{\partial \beta_i}{\partial g_j}$ at the $g_1^* = g_2^*$ fixed point). This dimension has a simple explanation as follows. The flow away from the decoupled fixed point is generated by the operator $O = \sigma_1 \sigma_2^2 + \sigma_2 \sigma_1^2$. This is allowed by the original $Z_2$ symmetry $\phi \rightarrow -\phi$, which translates into the interchange of $\sigma_1$ and $\sigma_2$. Thus,

$$\Delta O = \Delta_{\sigma}^{N=0} + \Delta_{\sigma}^{N=0} = 2 + 2\Delta_{\sigma}^{N=0},$$

where we used the fact that in the $N = 0$ theory, $\Delta_{\sigma}^{N=0} = 2 + \Delta_{\sigma}^{N=0}$ because $\sigma^2$ is a descendant. Using (3.25) for $N = 0$, we find

$$\Delta_{\sigma} = 2 - \frac{5}{9}\epsilon - \frac{43}{1458}\epsilon^2 + \left(\frac{8\zeta(3)}{243} - \frac{8375}{472392}\right)\epsilon^3 = 2 - 0.555556\epsilon - 0.0294925\epsilon^2 + 0.021845\epsilon^3$$

Substituting this into (3.61) we find the dimension of the relevant operator $O$, which indeed precisely agrees with $\Delta O = d + \lambda_-$, where $\lambda_-$ is the negative eigenvalue of $M_{ij} = \frac{\partial \beta_i}{\partial g_j}$ at the $g_1^* = g_2^*$ fixed point. Using the $\epsilon$ expansion (3.62), we find that $O$ continues to be relevant as $\epsilon$ is increased. For $\epsilon = 4$, i.e. $d = 2$, we know the exact
result in the $(2, 5)$ minimal model that $\Delta_N^{\mathcal{O}} = 0 = -2/5$, which implies $\Delta_{\mathcal{O}} = 6/5$. This strongly suggests that $\mathcal{O}$ is relevant, and the decoupled fixed point is unstable, for the entire range $2 \leq d < 6$. To describe this CFT in $d = 2$ more precisely, we note the existence of the modular invariant minimal model $M(3, 10)$, which is closely related to the product of two Yang-Lee $(2, 5)$ minimal models \cite{82, 83}.

The flow away from the unstable fixed point with $g_1^* = g_2^*$ can lead the $N = 1$ theory to the IR stable fixed point where $g_2^* = 6g_1^*/5 + O(\epsilon)$. Using our results we can deduce the $\epsilon$ expansion of various operator dimensions at this fixed point. For example,

\begin{align}
\Delta_\phi &= 2 - 0.5501\epsilon - 0.0234477\epsilon^2 + 0.0200649\epsilon^3 + \ldots \\
\Delta_\sigma &= 2 - 0.561122\epsilon - 0.0358843\epsilon^2 + 0.0236057\epsilon^3 + \ldots 
\end{align}

(3.63)

By calculating the eigenvalues $\lambda_\pm$ of the matrix $M_{ij} = \frac{\partial^2}{\partial g_i \partial g_j}$, we find the dimensions of two operators that are slightly irrelevant in $d = 6 - \epsilon$

\begin{align}
\Delta_- &= d + \lambda_- = 6 - 0.88978\epsilon + 0.0437732\epsilon^2 - 0.039585\epsilon^3, \\
\Delta_+ &= d + \lambda_+ = 6 - 0.773191\epsilon^2 + 1.59707\epsilon^3.
\end{align}

(3.64)

As $\epsilon$ is increased, these expansions suggest that the two operators become more irrelevant. It would be interesting to study this $\mathbb{Z}_2$ symmetric fixed point using a conformal bootstrap approach along the lines of \cite{60}.

Assuming that the $N = 1$ IR fixed point continues to be stable in $d = 5, 4, 3, 2$, it is interesting to look for statistical mechanical interpretations of this non-unitary CFTs. A distinguishing feature of the $N = 1$ CFT is that it has a discrete $\mathbb{Z}_2$ symmetry, while the $N = 0$ theory has no symmetries at all. As we have noted, in $d = 2$ the CFT can be obtained via deforming the $(3, 10)$ minimal model by a Virasoro primary field of dimension $6/5$ (this is the highest dimension relevant operator in that minimal
model). After analyzing the spectra of several candidate minimal models, we suggest that the end point of this RG flow is described by the \((3,8)\) minimal model with \(c = -21/4\).\(^9\) Let us note that \(M(2,5)\) and \(M(3,8)\) are members of the series of non-unitary minimal models \(M(k,3k-1)\).

In addition to the identity operator, the \(M(3,8)\) model has three Virasoro primary fields which are \(Z_2\) odd and three that are \(Z_2\) even. Comparing with the theory in \(6 - \epsilon\) dimensions, we can tentatively identify the leading \(Z_2\) odd operator as \(\phi\) and the leading \(Z_2\) even one as \(\sigma\). Obviously, further work is needed to check if the stable fixed point in \(6 - \epsilon\) dimensions with \(g^*_2 = 6g^*_1/5 + O(\epsilon)\) continued to \(\epsilon = 4\) is described by the non-unitary minimal model \(M(3,8)\).

### 3.6 Appendix A. Summary of three-loop results

The Feynman rules for our theory are depicted in Fig. B.1

\[
\begin{align*}
\begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\end{array} & = -d_{\alpha\beta\gamma} \\
\alpha & = \delta_{\alpha\beta} \frac{1}{p^2} \\
\end{array}
\]

\[
\begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\end{array} & = -(\delta g)_{\alpha\beta\gamma} \\
\alpha & = -p^2(\delta z)_{\alpha\beta}
\]

Figure 3.3: Feynman rules.

\(^9\)Note that this value is greater than the central charge of the UV theory \(M(3,10)\), which is equal to \(-44/5\). For flows between non-unitary theories the Zamolodchikov \(c\)-theorem does not hold, and it is possible that \(c_{UV} < c_{IR}\).
where we introduced symmetric tensor coupling \( d_{\alpha\beta\gamma} \) and counterterms \((\delta g)_{\alpha\beta\gamma}\), \((\delta z)_{\alpha\beta}\) with \(\alpha, \beta, \gamma = 0, 1, \ldots, N\) as

\[
\begin{align*}
  d_{000} &= g_2, \\
  d_{i0i} &= d_{i0i} = d_{0ii} = g_1, \\
  (\delta g)_{000} &= \delta g_2, \\
  (\delta g)_{i0i} &= (\delta g)_{i0i} = (\delta g)_{0ii} = \delta g_1, \\
  (\delta z)_{00} &= \delta \sigma, \\
  (\delta z)_{ii} &= \delta \phi,
\end{align*}
\]

(3.65)

where \(i = 1, \ldots, N\). The general form of a Feynman diagram in our theory could be schematically represented as

\[
\text{Feynman diagram} = \text{Integral} \times \text{Tensor structure factor.} \tag{3.66}
\]

The “Tensor structure factors” are products of the tensors \( d_{\alpha\beta\gamma} \) and \((\delta g)_{\alpha\beta\gamma}\), \((\delta z)_{\alpha\beta}\), with summation over the dummy indices. Their values for different diagrams are represented in Fig. D.1 and D.2 after parentheses \(^{10}\). The “Integrals” already include symmetry factors and are the same as in the usual \(\varphi^3\)-theory; their values are listed in Fig. D.1 and D.2 before the parentheses.

\(^{10}\)To find the “Tensor structure factor” we used the fact that it is a polynomial in \(N\), so we calculated sums of products of \(d_{\alpha\beta\gamma}, (\delta g)_{\alpha\beta\gamma}, (\delta z)_{\alpha\beta}\) explicitly for \(N = 1, 2, 3, 4, \ldots\), using Wolfram Mathematica. Having answers for \(N = 1, 2, 3, 4, \ldots\) it’s possible to restore the general \(N\) form.
3.6.1 Counterterms

\[ z_{12} = - \frac{g_1^2}{3(4\pi)^3}, \quad z_{12}^* = -\frac{Ng_1^4 + g_2^2}{6(4\pi)^3}, \quad a_{13} = -\frac{g_1^2 (g_1 + g_2)}{(4\pi)^3}, \quad b_{13} = -\frac{Ng_1^3 + g_2^3}{(4\pi)^3}, \]

(3.67)

\[ z_{14}^* = \frac{g_1^2}{432(4\pi)^6} \left( g_1^2 (11N - 26) - 48g_1 g_2 + 11g_2^2 \right), \]

\[ z_{14} = -\frac{1}{432(4\pi)^6} \left( 2Ng_1^4 + 48Ng_1^3 g_2 - 11Ng_1^2 g_2^2 + 13g_2^4 \right), \]

\[ a_{15} = -\frac{1}{144(4\pi)^6} g_1^2 \left( g_1^3 (11N + 98) - 2g_1 g_2 (7N - 38) + 101g_1 g_2^2 + 4g_2^3 \right), \]

\[ b_{15} = -\frac{1}{48(4\pi)^6} \left( 4Ng_1^5 + 54Ng_1^4 g_2 + 18Ng_1^3 g_2^3 - 7Ng_1^2 g_2^4 + 23g_2^5 \right), \]

(3.68)

\[ z_{16}^* = \frac{g_1^2}{46656(4\pi)^9} \left( g_1^4 (N(13N - 232) + 5184\zeta(3) - 9064) + 9g_1^2 g_2 (441N - 544) 
\quad - 2g_1^2 g_2 (193N - 2592\zeta(3) + 5881) + 942g_1 g_2^3 + 327g_2^4 \right), \]

\[ z_{16} = -\frac{1}{93312(4\pi)^9} \left( 2Ng_1^6 (1381N - 2592\zeta(3) + 4280) - 96N(12N + 11)g_1^5 g_2 
\quad - 3Ng_1^4 g_2^2 (N + 3432\zeta(3) - 8882) + 1560Ng_1^3 g_2^3 - 952Ng_1^2 g_2^4 - g_2^6 (2592\zeta(3) - 5195) \right), \]

\[ a_{17} = \frac{g_1^2}{15552(4\pi)^9} \left( - g_1^5 (N(531N + 10368\zeta(3) - 2600) + 23968) 
\quad + g_1^4 g_2 (99N^2 + 2592(5N - 6)\zeta(3) - 9422N - 2588) + 2g_1^3 g_2^2 (1075N + 2592\zeta(3) - 16897) 
\quad + 2g_1^2 g_2^3 (125N - 5184\zeta(3) - 3917) - g_1 g_2^4 (5184\zeta(3) + 721) + g_2^5 (2592\zeta(3) - 2801) \right), \]

\[ b_{17} = -\frac{1}{2592(4\pi)^9} \left( 2g_1^7 N (577N + 713) - 48g_1^6 g_2 N (31N - 59) 
\quad + g_1^5 g_2^2 N (423N + 2592\zeta(3) + 1010) - g_1^4 g_2^3 N (33N - 1296\zeta(3) - 6439) 
\quad - 27g_1^3 g_2^4 N (32\zeta(3) + 11) - 301Ng_1^2 g_2^5 + g_2^7 (432\zeta(3) + 1595) \right). \] (3.69)
Chapter 4

Critical $Sp(N)$ Models in $6 - \epsilon$

Dimensions and Higher Spin dS/CFT

4.1 Introduction and Summary

This chapter is based on the work published in [30], co-authored with Simone Giombi, Igor Klebanov, and Grigory Tarnopolsky. We also thank D. Anninos, D. Harlow and J. Maldacena, and especially S. Caracciolo and G. Parisi, for useful discussions.

The $O(N)$ invariant theories of $N$ massless scalar fields $\phi^i$, which interact via the potential $\frac{1}{2}(\phi^i\phi^i)^2$ possess interacting IR fixed points in dimensions $2 < d < 4$ for any positive $N$ [1]. These theories contain $O(N)$ singlet current operators with all even spin, and when $N$ is large the current anomalous dimensions are $\sim 1/N$ [2]. Since all the higher spin currents are nearly conserved, the $O(N)$ models possess a weakly broken higher spin symmetry. In the Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence [84, 85, 86], each spin $s$ conserved current in a $d$ dimensional CFT is mapped to a massless spin $s$ gauged field in $d + 1$ dimensional AdS space. For these reasons, it was conjectured [47] that the singlet sector of the critical $O(N)$ models in $d = 3$ is dual to the interacting higher spin theory in $d = 4$
containing massless gauge fields of all even positive spin [41, 42, 43, 46]. This minimal Vasiliev theory also contains a scalar field with \( m^2 = -2/\ell_{\text{AdS}}^2 \), and the two admissible boundary conditions on this field [48, 49] distinguish the interacting \( O(N) \) model from the free one (in the latter all the higher spin currents are conserved exactly). A review of the higher spin AdS/CFT dualities may be be found in [87].

A remarkable feature of the Vasiliev theories [41, 42, 43, 46] is that they are consistent not only in Anti-de Sitter, but also in de Sitter space. On general grounds one expects a CFT dual to quantum gravity in \( dS_4 \) to be a non-unitary theory defined in three dimensional Euclidean space [88]. In [89] it was proposed that the CFT dual to the minimal higher spin theory in \( dS_4 \) is the theory of an even number \( N \) of anti-commuting scalar fields \( \chi^i \) with the action

\[
S = \int d^3x \left( \frac{1}{2} \Omega_{ij} \partial_\mu \chi^i \partial^\mu \chi^j + \frac{\lambda}{4} (\Omega_{ij} \chi^i \chi^j)^2 \right).
\] (4.1)

This theory possesses \( Sp(N) \) symmetry, and \( \Omega_{ij} \) is the invariant symplectic matrix. This model was originally introduced and studied in [90, 91], where it was shown to possess an IR fixed point in \( 4 - \epsilon \) dimensions. The beta function of this model is related to that of the \( O(N) \) model via the replacement \( N \rightarrow -N \). According to the proposal of [89], the free UV fixed point of (4.1) is dual to the minimal higher spin theory in \( dS_4 \) with the Neumann future boundary conditions on the \( m^2 = 2/\ell_{dS}^2 \) bulk scalar field, and its interacting IR fixed point to the same higher spin theory but with the Dirichlet boundary conditions on the scalar field. In the latter case, the higher spin symmetry is slightly broken at large \( N \), and the de Sitter higher spin gauge fields are expected to acquire small masses through quantum effects. ¹

¹A potential difficulty with this picture is that unitarity in \( dS_{d+1} \) space requires that a massive field of spin \( s > 1 \) should satisfy \( m^2 > (s - 1)(s + d - 3)/\ell_{dS}^2 \) [92, 93, 94]. In other words, there is a finite gap between massive fields and massless ones (the latter are dual to exactly conserved currents in the CFT). However, since the masses are generated by quantum effects and are parametrically small at large \( N \), this is perhaps not fatal for bulk unitarity. It would be interesting to clarify this further.
of the de Sitter boundary conditions from the point of view of the wave function of the Universe was given in [95, 96].

In this chapter we consider an extension of the proposed higher spin dS/CFT correspondence [89] to higher dimensional de Sitter spaces, and in particular to $dS_6$. Our construction mirrors our recent work [28, 29] on the higher dimensional extensions of the higher spin AdS/CFT. It was observed long ago [32, 33, 34] that in $d > 4$ the quartic $O(N)$ models possess UV fixed points which can be studied in the large $N$ expansion. The UV completion of the $O(N)$ scalar theory in $4 < d < 6$ was proposed in [28]; it is the cubic $O(N)$ symmetric theory of $N + 1$ scalar fields $\sigma$ and $\phi^i$:

$$S = \int d^d x \left( \frac{1}{2} (\partial_{\mu}\phi^i)^2 + \frac{1}{2} (\partial_{\mu}\sigma)^2 + \frac{1}{2} g_1 \sigma (\phi^i)^2 + \frac{1}{6} g_2 \sigma^3 \right). \tag{4.2}$$

For sufficiently large $N$, this theory has an IR stable fixed point in $6 - \epsilon$ dimensions with real values of $g_1$ and $g_2$. The beta functions and anomalous dimensions were calculated to three-loop order [28, 29], and the available results agree nicely with the $1/N$ expansion of the quartic $O(N)$ model at its UV fixed point [4, 6, 7, 10, 11] when the quartic $O(N)$ model is continued to $6 - \epsilon$ dimensions. The conformal bootstrap approach to the higher dimensional $O(N)$ model was explored in [61, 20, 98].

To extend the idea of [89] to $dS_{d+1}$ with $d > 4$, we may consider non-unitary CFTs (4.1) which in $d > 4$ possess UV fixed points for large $N$. The $1/N$ expansion of operator scaling dimensions may be developed using the generalized Hubbard-Stratonovich transformation, and one finds that it is related to the $1/N$ expansion in the $O(N)$ models via the replacement $N \rightarrow -N$. In $d = 5$ this interacting fixed point should be dual to the higher spin theory in $dS_6$ with Neumann boundary conditions on the $m^2 = 6/\ell_{dS}^2$ scalar field (corresponding to the conformal dimension $\Delta = \frac{2}{5}x$).

\footnote{The one-loop beta functions of the model (4.2) in $d = 6$ were first calculated in [97].}
In search of the UV completion of these quartic CFTs in $4 < d < 6$, we introduce the cubic theory of one commuting real scalar field $\sigma$ and $N$ anti-commuting scalar fields $\chi^i$:

$$S = \int d^d x \left( \frac{1}{2} \Omega_{ij} \partial_\mu \chi^i \partial^\mu \chi^j + \frac{1}{2} (\partial_\mu \sigma)^2 + \frac{1}{2} g_1 \Omega_{ij} \chi^i \chi^j \sigma + \frac{1}{6} g_2 \sigma^3 \right). \quad (4.3)$$

Alternatively, we may combine the fields $\chi^i$ into $N/2$ complex anti-commuting scalars $\theta^\alpha$, $\alpha = 1, \ldots, N/2$ [90, 91]; then the action assumes the form

$$S = \int d^d x \left( \partial_\mu \theta^\alpha \partial^\mu \bar{\theta}^\alpha + \frac{1}{2} (\partial_\mu \sigma)^2 + g_1 \sigma \theta^\alpha \bar{\theta}^\alpha + \frac{1}{6} g_2 \sigma^3 \right). \quad (4.4)$$

We study the beta functions for this theory in $d = 6 - \epsilon$ and show that they are related to the beta functions of the theory (4.2) via the replacement $N \to -N$. For all $N$ there exists an IR fixed point of the theory (4.3) with imaginary values of $g_1$ and $g_2$. This is similar to the IR fixed point of the single scalar cubic field theory (corresponding to the $N = 0$ case of our models), which was used by Fisher [37] as an approach to the Lee-Yang edge singularity. The fact that the couplings are purely imaginary makes the integrand of the path integral oscillate rapidly at large $\sigma$; this should be contrasted with real couplings giving a potential unbounded from below.

Our results allow us to study the $6 - \epsilon$ expansion of the theory (4.3) with arbitrary $N$, and we observe that at finite $N$ there are qualitative differences between the $Sp(N)$ and $O(N)$ models which are not seen in the $1/N$ expansion. In fact, for the $Sp(N)$ model there is no analogue of the lower bound $N_{\text{crit}}$ that was found in the $O(N)$ case [28]. For the lowest value, $N = 2$, we observe some special phenomena. In this theory, which contains two real anti-commuting scalars, it is impossible to formulate the quartic interaction (4.1); thus, the cubic lagrangian (4.20) seems to be

3The $d = 5$ free $Sp(N)$ model corresponds to Dirichlet scalar boundary conditions in the $dS_6$ dual. It should be possible to extend this free $Sp(N)$ model/$dS$ higher spin duality to general dimensions, since the Vasiliev equations in $(A)dS_{d+1}$ are known for all $d$ [46].
the only possible description of the interacting theory with global \(Sp(2)\) symmetry. Furthermore, it becomes enhanced to the supergroup \(OSp(1|2)\) because at the IR fixed point the two coupling constants are related via \(g_2^* = 2g_1^*\). The enhanced symmetry implies that the scaling dimensions of \(\sigma\) and \(\chi^i\) are equal, and we check this to order \(\epsilon^3\). An example of theory with \(OSp(1|2)\) symmetry is provided by the \(q \to 0\) limit of the \(q\)-state Potts model \([99]\).\(^4\) We show that the \(6 - \epsilon\) expansions of the scaling dimensions in our \(OSp(1|2)\) symmetric theory are the same as in the \(q \to 0\) limit of the \(q\)-state Potts model.\(^5\) This provides strong evidence that the \(OSp(1|2)\) symmetric IR fixed point of the cubic theory (4.20) describes the second order transitions in the ferromagnetic \(q = 0\) Potts model, which exist in \(2 < d < 6\) \([100]\).

Using the results of \([78]\), we also compute perturbatively the sphere free energies of the models (4.3). The sphere free energy for the \(OSp(1|2)\) symmetric model is found to be the same as for the \(q = 0\) Potts model. In terms of the quantity \(\tilde{F} = \sin \left( \frac{\pi d}{2} \right) \log Z_{S^d}\), which was introduced in \([78]\) as a natural way to generalize the \(F\)-theorem \([66, 54, 50]\) to continuous dimensions, we find that the RG flow in the cubic \(Sp(N)\) models in \(d = 6 - \epsilon\) satisfies \(\tilde{F}_{UV} > \tilde{F}_{IR}\) for all \(N \geq 2\). We show that the same result holds in the model (4.1) in \(d = 4 - \epsilon\).\(^6\) This is somewhat surprising, since for non-unitary CFTs the inequality \(\tilde{F}_{UV} > \tilde{F}_{IR}\) is not always satisfied. It would be interesting to understand if this is related to the “pseudo-unitary” structure discussed in \([91]\), and to the fact that these models are presumably dual to unitary higher spin gravity theories in de Sitter space.

\(^4\)We are grateful to Giorgio Parisi for informing us about this and for important discussions.
\(^5\)We thank Sergio Caracciolo for suggesting this comparison to us and for informing us about the paper \([100]\).
\(^6\)The models (4.1) and (4.3) also satisfy the \(F\)-theorem to leading order in the large \(N\) expansion (for all \(d\)), since the leading order correction to \(\tilde{F}_{UV} - \tilde{F}_{IR}\) is of order \(N^0\), and was shown to satisfy the \(F\)-theorem in the corresponding unitary \(O(N)\) models \([50, 23, 78]\).
4.2 The IR fixed points of the cubic $Sp(N)$ theory

The beta functions and anomalous dimensions for the $Sp(N)$ symmetric model (4.3) can be obtained by replacing $N \rightarrow -N$ in the corresponding results for the cubic $O(N)$ model (4.2), which were computed in [28, 29] to three-loop order. Indeed, writing the action in the complex basis (4.4), we see that the Feynman rules and propagators are identical to those of the $O(N)$ theory written in the $U(N/2)$ basis, the only difference being that the $N/2$ complex scalars are anticommuting. Hence, for each closed loop of the $\theta^\alpha$ we get an extra minus sign, thus explaining the replacement $N \rightarrow -N$.

Using the results in [28, 29], the beta functions for the $Sp(N)$ model are then found to be:

$$\beta_1 = -\frac{\epsilon}{2}g_1 - \frac{1}{12(4\pi)^3}g_1\left((N+8)g_1^2 + 12g_1g_2 - g_2^2\right)$$

$$- \frac{1}{432(4\pi)^6}g_1\left((536-86N)g_1^4 + 12(30+11N)g_1^3g_2 + (628-11N)g_1^2g_2^2 + 24g_1g_2^3 - 13g_2^4\right) + \ldots,$$

$$\beta_2 = -\frac{\epsilon}{2}g_2 + \frac{1}{4(4\pi)^3}\left(4Ng_1^3 - Ng_1^2g_2 - 3g_2^3\right)$$

$$+ \frac{1}{144(4\pi)^6}\left(24Ng_1^5 + 322Ng_1^4g_2 + 60Ng_1^3g_2^2 - 31Ng_1^2g_2^3 - 125g_2^5\right) + \ldots \quad (4.5)$$

We have omitted the explicit three-loop terms, which can be obtained from [28, 29].

Similarly, the anomalous dimensions of the fields $\chi^i$ and $\sigma$ take the form

$$\gamma_\chi = -\frac{g_1^2}{6(4\pi)^3} + \frac{g_2^2}{432(4\pi)^6}\left(g_1^2(11N + 26) + 48g_1g_2 - 11g_2^2\right) + \ldots,$$

$$\gamma_\sigma = -\frac{Ng_1^2 - g_2^2}{12(4\pi)^3} - \frac{1}{432(4\pi)^6}\left(2Ng_1^4 + 48Ng_1^3g_2 - 11Ng_1^2g_2^2 - 13g_2^4\right) + \ldots \quad (4.6)$$

With the beta functions at hand, we can look for non-trivial fixed points of the RG flow satisfying $\beta_1(g_1^*, g_2^*) = 0$, $\beta_2(g_1^*, g_2^*) = 0$. For all positive $N$, we find two physically
equivalent fixed points with purely imaginary coupling constants, hence all operator dimensions remain real. These fixed points are IR stable for all \( N \) (the stability matrix \( M_{ij} = \frac{\partial \beta_i}{\partial g_j} \) has positive eigenvalues). Note that this is different from the \( O(N) \) versions of these models [28, 29], where one finds a critical \( N \), whose one-loop value is \( \simeq 10^{38} \), below which the IR stable fixed points with real coupling constants disappear. However, to all orders in the \( 1/N \) expansion, the fixed point couplings and conformal dimensions in the \( Sp(N) \) models are related to the ones in the \( O(N) \) models by the replacement \( N \rightarrow -N \).

Figure 4.1 shows the RG flow directions for \( N = 2 \). The arrows indicate how the coupling constants flow towards the IR. The two IR fixed points are physically equivalent because they are related by \( g_i \rightarrow -g_i \). At higher values of \( N \), the qualitative behavior of the RG flows and fixed points remain the same. We still have a UV Gaussian fixed point, and two stable IR fixed points.

A special structure emerges for \( N = 2 \). In this case we find the fixed point solution

\[
g_2^* = 2g_1^*, \quad g_1^* = i\sqrt{\frac{(4\pi)^3\epsilon}{5}} \left(1 + \frac{67}{180} \epsilon + O(\epsilon^2)\right),
\]

and the conformal dimensions of the fundamental fields are equal (the three-loop term given below can be obtained from the results in [28, 29])

\[
\Delta_\sigma = \Delta_\chi = 2 - \frac{8}{15} \epsilon - \frac{7}{450} \epsilon^2 - \frac{269 - 702\zeta(3)}{33750} \epsilon^3 + \ldots .
\]

We show in the next section that the equality of dimensions is a consequence of a symmetry enhancement from \( Sp(2) \) to the supergroup \( OSp(1|2) \).

It is natural to ask whether symmetry enhancement can occur at other values of \( N \). For instance, we can explicitly check for which \( N \) the dimensions of \( \sigma \) and \( \chi \) are equal. A direct calculation using the beta functions and anomalous dimensions up to three-loops shows that this only happens for \( N = 2 \) and \( N = -1 \). The latter
Figure 4.1: The zeroes of the one-loop $\beta$ functions and the RG flow directions for the $OSp(1|2)$ model. The coordinates are defined via $g_1 = i \sqrt{\frac{(4\pi)^3}{5}} x$, $g_2 = i \sqrt{\frac{(4\pi)^3}{5}} y$, and the red dots correspond to the stable IR fixed points.

Case corresponds to the 3-state Potts model fixed point of the theory with two commuting scalars \cite{29}; it has $g_1^*= -g_2^*$ and enhanced $Z_3$ symmetry.\footnote{For $N = -1$, there is also a (non-unitary) solution with $g_1^* = g_2^*$ which corresponds to two decoupled Fisher models \cite{37}, and hence the dimensions of the fundamental fields are trivially equal.}

The anomalous dimensions of some composite operators may be similarly obtained from the results in \cite{28, 29}. Let us quote the explicit result for the quadratic operators arising from the mixture of $\sigma^2$ and $\Omega_{ij}\chi^i\chi^j$. These operators have the same classical dimension, so we expect them to mix. The $2 \times 2$ anomalous dimension mixing matrix $\gamma^{ab}$ was given in \cite{28, 29} up to one-loop order. Extending those results to two-loop,
we find the mixing matrix

\[
\begin{pmatrix}
-\frac{g_1^2(N+4)}{6(4\pi)^3} + \frac{g_1^2(21N-134)-6g_2g_2(2N+5)+5g_2^2}{108(4\pi)^6} \\
\frac{g_1(g_2-6g_2)}{6(4\pi)^3} - \frac{g_1(6g_1(11N+20)-g_1^2g_2(11N-324)+12g_1g_2^2-13g_2^2)}{216(4\pi)^6} \\
\frac{Ng_1(6g_1g_2)}{6(4\pi)^3} + \frac{Ng_1^2(18g_1^2+16g_1g_2+15g_2^2)}{108(4\pi)^6} - \frac{Ng_1^2+4g_2^2}{6(4\pi)^3} + \frac{Ng_1^2(169g_1^2+36g_1g_2-41g_2^2)}{216(4\pi)^6} - \frac{181g_2^4}{216(4\pi)^6}
\end{pmatrix}
\]

(4.9)

where index ‘1’ corresponds to the operator \(\Omega_{ij}\chi^i\chi^j\), and index ‘2’ corresponds to \(\sigma^2\).

The diagrams contributing to the calculation are listed in Fig. 4.2.

Figure 4.2: Diagrams contributing to the mixing of \(\sigma^2\) and \(\Omega_{ij}\chi^i\chi^j\) operators to two-loop order. Notice that in these diagrams, we have not distinguished the \(\sigma\) field and \(\chi^i\) fields. Therefore, each one of these diagrams represent several diagrams where the fields are different. The solid dots represent one-loop counter terms for either the propagator or the vertex, and the crossed dots represent one-loop counter terms for the operator mixing.

The two eigenvectors of \(\gamma^{ab}\) give the two linear combinations of \(\sigma^2\) and \(\Omega_{ij}\chi^i\chi^j\), and the eigenvalues \(\gamma_{\pm}\) give their anomalous dimensions, so that \(\Delta_{\pm} = d - 2 + \gamma_{\pm}\).

We find that one of these combinations is a conformal primary, and the other one is a descendant of \(\sigma\). Indeed, after plugging in the fixed point couplings, we find \(\Delta_- = \Delta_\sigma + 2\). For instance, in the large \(N\) expansion we find the results

\[
\begin{align*}
\Delta_- &= \Delta_\sigma + 2 = 4 + \frac{-40\epsilon + \frac{104\epsilon^2}{3}}{N} + \frac{6800\epsilon - \frac{34190\epsilon^2}{3}}{N^2} + \ldots, \\
\Delta_+ &= 4 + \frac{100\epsilon - \frac{395\epsilon^2}{3}}{N} + \frac{-49760\epsilon + \frac{237476\epsilon^2}{3}}{N^2} + \ldots.
\end{align*}
\]

Upon sending \(N \to -N\), the dimension \(\Delta_\pm\) can be checked to be in agreement with the available large \(N\) results for the critical exponent \(\omega\) in the quartic \(O(N)\) theory\textsuperscript{8} [10, 79].

\textsuperscript{8}This critical exponent is related to the derivative of the beta function in the \(O(N)\) theory with quartic interaction; see [29] for more details on the comparison to the large \(N\) results.
4.2.1 Estimating operator dimensions with Padé approximants

In the previous section we calculated $\epsilon$ expansions for the operator dimensions of the cubic $Sp(N)$ theory in $d = 6 - \epsilon$. We may use the following Padé approximant to estimate the behavior of operator dimensions as we continue $d$:

$$\text{Padé}_{m,n}[d] = \frac{A_0 + A_1 d + A_2 d^2 + \ldots + A_m d^m}{1 + B_1 d + B_2 d^2 + \ldots + B_n d^n}.$$  \hspace{1cm} (4.11)

For $N = 2$, by demanding that the $\epsilon$ expansion have the same behavior as (4.8), we can fix four coefficients in the Padé approximant. Here we use Padé$_{1,2}$ and Padé$_{2,1}$, to obtain the following estimate for the operator dimension in $d = 5$:

$$\Delta_{\chi}^{N=2} = \Delta_{\sigma}^{N=2} \approx \begin{cases} 1.467 & \text{with Padé}_{1,2} \\ 1.459 & \text{with Padé}_{2,1} \end{cases}.$$

(4.12)

As expected, this is below the unitarity bound in $d = 5$, which is $\Delta = \frac{3}{2}$. The plots of different Padé approximants are shown in Fig. 4.3. For the next primary operator,
whose scaling dimension is given in (4.30), using the Padé\textsubscript{2,1} approximant we estimate $\Delta_+ \approx 3.35$ in $d = 5$. In four dimensions we estimate $\Delta_+ \approx 2.7$; this suggests that the $N = 2$ model does not have a free field description near $d = 4$.

In the $Sp(4)$ case, we may assume for $4 < d < 6$ that the IR fixed point of the cubic theory (4.3) is equivalent to the UV fixed point of the quartic theory (4.1).\footnote{This equivalence holds for $Sp(N)$ models with sufficiently large $N$, but it is not completely clear if it applies for $N = 4$.} If so, then not only do we know the $6 - \epsilon$ expansion of $\Delta_\chi$ to $O(\epsilon^3)$, we can also use the quartic theory result in $d = 4 + \epsilon$, which is known to $O(\epsilon^5)$:

$$\Delta_\chi = \begin{cases} 
2 - 0.529827\epsilon - 0.0126197\epsilon^2 + 0.0157244\epsilon^3 & \text{in } d = 6 - \epsilon \\
1 + 0.5\epsilon - 0.03125\epsilon^2 + 0.015625\epsilon^3 - 0.0730951\epsilon^4 + 0.0195503\epsilon^5 & \text{in } d = 4 + \epsilon 
\end{cases} \quad (4.13)$$

Together, these expansions allow us to determine ten coefficients of the Padé approximant. Disregarding the approximants that have a pole, we obtain the estimate $\Delta_{\chi}^{N=4} \approx 1.478$ in $d = 5$.

### 4.2.2 Sphere free energies

It is also interesting to compute the sphere free energy at the IR fixed point in $d = 6 - \epsilon$. The leading order term in the $\epsilon$ expansion for the corresponding $O(N)$ models was computed in [78]. Sending $N \to -N$, we find the following result for $\tilde{F} = \sin(\frac{\pi d}{2}) \log Z_{Sp}$ in the cubic $Sp(N)$ models

$$\tilde{F}_{\text{IR}} = \tilde{F}_{\text{UV}} - \frac{\pi}{17280} \frac{(g_2^*)^2 - 3N(g_1^*)^2}{(4\pi)^3} \epsilon + O(\epsilon^3), \quad (4.14)$$

where $\tilde{F}_{\text{UV}} = (1 - N)\tilde{F}_s$, and $\tilde{F}_s$ the value corresponding to a free conformal scalar. Plugging in the explicit solutions for the fixed point couplings $g_1^*$, $g_2^*$, it is straight-
forward to verify that, for all $N \geq 2$, we have

$$\bar{F}_{\text{UV}} > \bar{F}_{\text{IR}}. \quad (4.15)$$

For instance, for $N = 2$ we find

$$\bar{F}_{\text{IR}} = -\bar{F}_s - \frac{\pi}{43200} \epsilon^2 + \mathcal{O}(\epsilon^3). \quad (4.16)$$

Note that for the Fisher model [37] of the Lee-Yang edge singularity, corresponding to $N = 0$ and imaginary $g_2^s$, the inequality (4.15) does not hold. For general non-unitary theories the inequality does not have to hold; remarkably, it does hold for the $Sp(N)$ models with positive $N$.

Similarly, using the results of [78] for the quartic $O(N)$ theory in $d = 4 - \epsilon$, we can compute the sphere free energy at the IR fixed point of the model (4.1) in $d = 4 - \epsilon$. We find

$$\bar{F}_{\text{quartic}} = \bar{F}_{\text{free}} - \frac{\pi}{576} \frac{N(N-2)}{(N-8)^2} \epsilon^3 + \mathcal{O}(\epsilon^4), \quad (4.17)$$

where $\bar{F}_{\text{free}} = -N\bar{F}_s$. We observe that $\bar{F}_{\text{free}} > \bar{F}_{\text{quartic}}$ is satisfied for all $N > 2$. 10

The case $N = 8$ is special and needs to be treated separately. Here the one-loop term in the beta function vanishes, and we have

$$\beta_\lambda = -\epsilon \lambda + \frac{15}{32\pi^4} \lambda^3 + \mathcal{O}(\lambda^4). \quad (4.18)$$

Therefore, at the IR fixed point $\lambda_\ast^2 = \frac{32\pi^4}{15} \epsilon + \mathcal{O}(\epsilon^{3/2})$, and we get

$$\bar{F}_{\text{quartic}} = \bar{F}_{\text{free}} - \frac{\pi}{360} \epsilon^2 + \ldots \quad (4.19)$$

10For $\epsilon < 0$ the interacting theory has a UV fixed point. So, in this case $\bar{F}_{\text{free}} < \bar{F}_{\text{quartic}}$, again in agreement with the conjectured $\tilde{F}$ theorem.
4.3 Symmetry enhancement for $N = 2$

Let us write the cubic $Sp(2)$ model in terms of a real scalar $\sigma$ and a single complex anti-commuting fermion $\theta$:

$$S = \int d^4x \left( \partial_{\mu} \theta \partial^{\mu} \bar{\theta} + \frac{1}{2} (\partial_{\mu} \sigma)^2 + g_1 \sigma \theta \bar{\theta} + \frac{1}{6} g_2 \sigma^3 \right). \quad (4.20)$$

For $g_2 = 2g_1$, i.e. the fixed point relation (4.7), this action possesses a fermionic symmetry with a complex anti-commuting scalar parameter $\alpha$

$$\delta \theta = \sigma \alpha, \quad \delta \bar{\theta} = \sigma \bar{\alpha}, \quad \delta \sigma = -\alpha \bar{\theta} + \bar{\alpha} \theta. \quad (4.21)$$

As a consequence of this symmetry, the scaling dimensions of $\sigma$ and $\theta$ are equal, as seen explicitly in eq. (4.8). This complex fermionic symmetry enhances the $Sp(2)$ to $OSp(1|2)$, which is the smallest supergroup. The full set of supergroup generators can be given in the form

$$Q^+ = \frac{1}{2} \left( \sigma \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial \sigma} \right), \quad Q^- = \frac{1}{2} \left( \sigma \frac{\partial}{\partial \bar{\theta}} - \bar{\theta} \frac{\partial}{\partial \sigma} \right), \quad (4.22)$$

$$J^+ = \theta \frac{\partial}{\partial \theta}, \quad J^- = \bar{\theta} \frac{\partial}{\partial \bar{\theta}}, \quad J^3 = \frac{1}{2} \left( \theta \frac{\partial}{\partial \bar{\theta}} - \bar{\theta} \frac{\partial}{\partial \theta} \right),$$

and it is not hard to check that they satisfy the algebra of $OSp(1|2)$:

$$[J^3, J^\pm] = \pm J^\pm, \quad [J^+, J^-] = 2J^3, $$

$$[J^3, Q^\pm] = \pm \frac{1}{2} Q^\pm, \quad [J^\pm, Q^\mp] = -Q^\pm, $$

$$\{Q^\pm, Q^\mp\} = \pm \frac{1}{2} J^\pm, \quad \{Q^+, Q^-\} = \frac{1}{2} J^3, $$

We expect the operators of the theory to form representations of $OSp(1|2)$. For example, let us study the mixing of the $Sp(2)$ singlet operators $\sigma^2$ and $\theta \bar{\theta}$. Setting
$N = 2$ in (4.9), we find the scaling dimensions of the two eigenstates:

$$\Delta_+ = 4 - \frac{2}{3}\epsilon + \frac{1}{30}\epsilon^2 + \mathcal{O}(\epsilon^3), \quad \Delta_- = 4 - \frac{8}{15}\epsilon - \frac{7}{450}\epsilon^2 + \mathcal{O}(\epsilon^3). \quad (4.26)$$

The first of these dimensions corresponds to the conformal primary operator

$$O_+ = \sigma^2 + 2\theta\bar{\theta}, \quad (4.27)$$

which is invariant under all the $OSp(1|2)$ generators. The second corresponds to $O_- = \sigma^2 + \theta\bar{\theta}$, which is a conformal descendant because at the fixed point it is proportional to $\partial^\mu\partial^\nu\sigma$ by equations of motion. Indeed, we find $\Delta_- = \Delta_\sigma + 2$.

Continuation of the results for the cubic model with $OSp(1|2)$ global symmetry to finite $\epsilon$ points to the existence of such interacting CFTs in integer dimensions below 6. In [99] it was argued that the $q \to 0$ limit of the $q$-state Potts model is described by the $OSp(1|2)$ sigma model [101, 102]. There is evidence that the upper critical dimension of the spanning forest model is 6 [100], and the $6 - \epsilon$ expansions of the critical indices are [38, 100]

$$\eta = -\frac{1}{15}\epsilon - \frac{7}{225}\epsilon^2 - \frac{269 - 702\zeta(3)}{16875}\epsilon^3 + \mathcal{O}(\epsilon^4), \quad (4.28)$$
$$\nu^{-1} = 2 - \frac{1}{3}\epsilon - \frac{1}{30}\epsilon^2 - \frac{173 - 864\zeta(3)}{27000}\epsilon^3 + \mathcal{O}(\epsilon^4). \quad (4.29)$$

Remarkably, the operator dimensions we have calculated (4.8) and (4.26) agree with these expansions upon the standard identifications

$$\Delta_\chi = \frac{d}{2} - 1 + \frac{\eta}{2}, \quad \Delta_+ = d - \nu^{-1}. \quad (4.30)$$
Similarly, we can match the $6 - \epsilon$ expansions of the sphere free energies. For the $q$-state Potts model it is not hard to show that

$$
\tilde{F}_q = (q - 1)\tilde{F}_s + \frac{\pi(q - 1)(q - 2)}{8640(3q - 10)} \epsilon^2 + \mathcal{O}(\epsilon^3), \tag{4.31}
$$

and for $q = 0$ this matches the $F$ of our $OSp(1|2)$ model, (4.16). These results provide strong evidence that the $OSp(1|2)$ symmetric IR fixed point of the cubic theory (4.20) describes the $q \to 0$ limit of the $q$-state Potts model.

Numerical simulations of the spanning-forest model [100], which is equivalent to the $q \to 0$ limit of the ferromagnetic $q$-state Potts model, indicate the existence of second-order phase transitions in dimensions 3, 4 and 5. The critical exponents found in [100] are in good agreement with the Pade extrapolations of $6 - \epsilon$ expansions exhibited in section 2.1. This provides additional evidence for the existence of the critical theories with $OSp(1|2)$ symmetry. It would be of further interest to find the critical statistical models that are described by the $Sp(N)$ invariant theories with $N > 2$. 

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Chapter 5

Equivalence between the $Sp(2)$ model and the 0-state Potts model

5.1 Preliminaries

In this chapter, we will show that the 0-state Potts model and the $Sp(2)$ models are indeed equivalent at the level of tensor structures. The $N+1$-state Potts model contains a cubic interaction defined as [38]:

$$\frac{1}{6} gd_{ijk} \phi_i \phi_j \phi_k. \quad (5.1)$$

Where,

$$d_{ijk} = \sum_{\alpha=1}^{N+1} e_{\alpha}^i e_{\alpha}^j e_{\alpha}^k. \quad (5.2)$$
Where the $e_i^\alpha$ satisfies:

\begin{align}
\sum_{\alpha=1}^{N+1} e_i^\alpha &= 0, \\
\sum_{\alpha=1}^{N+1} e_i^\alpha e_j^\alpha &= (N + 1)\delta^{ij}.
\end{align}

(5.3)

\begin{align}
\sum_{\alpha=1}^{N} e_i^\alpha e_j^\alpha &= (N + 1)\delta_{\alpha\beta} - 1.
\end{align}

(5.4)

Whereas in the $O(N)$ model, the coupling constants are:

\begin{align}
d^{000} = 2, \\
d^{00i} = d^{i0i} = d^{i00} = 1.
\end{align}

(5.6)

where the index $i$ runs from 1 to $N$.

First, we note that the 4-loop results for $(N + 1)$-state Potts model as well as the $O(N)$ cubic model have been computed in [103]. For the $O(N)$ cubic model, the $Sp(2)$ model corresponds to setting $N = -2$, and $g_2 = 2g_1 = 2g$, the beta function and anomalous dimensions are:

\begin{align}
\beta_{Sp2} &= -\frac{g}{2} + \frac{5}{4}g^3 - \frac{335}{72}g^5 + \left(\frac{144889}{5184} + \frac{53\zeta(3)}{4}\right)g^7 \\
&\quad+ \left(-\frac{7502393}{31104} + \frac{53\pi^4}{144} - \frac{5297\zeta(3)}{18} - \frac{125\zeta(5)}{3}\right)g^9, \\
\gamma_{\phi Sp2} &= -\frac{1}{6}g^2 + \frac{25}{108}g^4 + \left(-\frac{14213}{7776} + \frac{5\zeta(3)}{6}\right)g^6 \\
&\quad+ \left(\frac{78667}{5184} + \frac{8\pi^4}{135} + \frac{209\zeta(3)}{108} - \frac{160\zeta(5)}{9}\right)g^8.
\end{align}

(5.7)

(5.8)

The 0-state Potts model corresponds to $N = -1$, but the coupling and other quantities are all divergent. To extract meaningful results, we set $N = -1 + \Delta$. We
get that the beta function and anomalous dimensions are:

\[
\beta_{\text{Potts}} = -\frac{g}{2} \epsilon - \left( \frac{5}{4} + \mathcal{O}(\Delta) \right) \Delta^2 g^3 + \left( -\frac{335}{72} + \mathcal{O}(\Delta) \right) \Delta^4 g^5 \\
- \left( \frac{144889}{5184} + \frac{53\zeta(3)}{4} + \mathcal{O}(\Delta) \right) \Delta^6 g^7 \\
+ \left( -\frac{7502393}{31104} + \frac{53\pi^4}{144} - \frac{5297\zeta(3)}{18} - \frac{125\zeta(5)}{3} + \mathcal{O}(\Delta) \right) \Delta^8 g^9, \\
(5.9)
\]

\[
\gamma_{\phi,\text{Potts}} = \left( \frac{1}{6} + \mathcal{O}(\Delta) \right) \Delta^2 g^2 + \left( \frac{25}{108} + \mathcal{O}(\Delta) \right) \Delta^4 g^4 + \left( \frac{14213}{7776} - \frac{5\zeta(3)}{6} + \mathcal{O}(\Delta) \right) \Delta^6 g^6 \\
+ \left( \frac{78667}{5184} + \frac{8\pi^4}{135} + \frac{209\zeta(3)}{108} - \frac{160\zeta(5)}{9} + \mathcal{O}(\Delta) \right) \Delta^8 g^8. \\
(5.10)
\]

Notice that the terms are very similar to the \( Sp(2) \) model in the \( \Delta \to 0 \) limit, if we rescale \( g \to g/\Delta \). In fact, the only difference is that the terms have alternating signs in successive loop orders. Namely, one loop contributions have opposite signs, two loop contributions have the same sign, three loop contributions have the opposite sign, etc. However, these are completely equivalent if we make the substitution \( g^2 \to -g^2 \), as one could easily verify. Therefore, they will give exactly the same answer for physical quantities such as the anomalous dimension at the fixed point.

This strongly suggests that the two models might be equivalent. We will show that the tensor structures are indeed equal for general zero, two and three-point functions, up to the alternating sign at successive loop order.

### 5.2 Review of Potts model tensor calculation

This section reviews the diagrammatic technique used in [38] by considering the following 3 point function example 5.1, which we will refer to as \( T_5 \): We have:
Figure 5.1: Diagram for what Gracey calls T5

\[ T_{5d}^{ijk} = \sum_{l_1,\ldots,l_6} d^{il_12}d^{jil_34}d^{kls_5}d^{l_1ls_6}d^{l_2ls_6} \quad (5.11) \]

\[ = \sum_{\alpha_1,\ldots,\alpha_5} \sum_{l_1,\ldots,l_6} (e_i^{\alpha_1} e_{l_1}^{\alpha_1} e_{l_2}^{\alpha_2}) (e_j^{\alpha_2} e_{l_3}^{\alpha_2} e_{l_4}^{\alpha_3}) \cdots (e_k^{\alpha_5} e_{l_5}^{\alpha_5} e_{l_6}^{\alpha_6}). \quad (5.12) \]

Now we perform the sum over the \( l \)'s, using 5.5, for example:

\[ \sum_{l_1} e_{l_1}^{\alpha_1} e_{l_4}^{\alpha_4} = (N + 1)\delta_{\alpha_1\alpha_4} - 1. \quad (5.13) \]

We will get 6 such factors, then we have:

\[ T_{5d}^{ijk} = \sum_{\alpha_1,\ldots,\alpha_5} e_i^{\alpha_1} e_j^{\alpha_2} e_k^{\alpha_3} [(N + 1)\delta_{\alpha_1\alpha_4} - 1][(N + 1)\delta_{\alpha_2\alpha_5} - 1][(N + 1)\delta_{\alpha_3\alpha_4} - 1] \]

\[ [(N + 1)\delta_{\alpha_3\alpha_5} - 1][(N + 1)\delta_{\alpha_2\alpha_5} - 1][(N + 1)\delta_{\alpha_3\alpha_4} - 1]. \quad (5.14) \]

\[ [(N + 1)\delta_{\alpha_3\alpha_5} - 1][(N + 1)\delta_{\alpha_2\alpha_5} - 1][(N + 1)\delta_{\alpha_3\alpha_4} - 1]. \quad (5.15) \]

If we multiply the 6 factors out, we will get 64 terms, carrying various powers of \((N + 1)\) and \(-1\). Each of these terms can be thought of as a coloring of the original 6 propagators in the graph. Each solid line corresponds to a factor of \((N + 1)\), while
each dashed line corresponds to a factor $-1$. We will have one diagram where all 6 lines are solid, 6 diagrams where 5 lines are solid..., until one diagram where all lines are dashed. Notice also that from 5.3, a three-point diagram will only be non-zero if all its vertices are connected by solid lines, since an isolated vertex will give zero.

Also, when there are free “clusters” of isolated solid lines in the middle of the diagram, each would contribute another factor of $(N + 1)$ from having a free index in the sum. Likewise, a vertex with three dashed lines can be considered as a trivial cluster, and also contribute a factor of $(N + 1)$ due to having a free index under the sum.

Let’s consider our example $T_5$. There are:

One diagram with no dashed lines: $(N + 1)^6$.

6 diagrams with one dashed line: $-6(N + 1)^5$.

15 diagrams with two dashed lines, but three of them don’t contribute since they have isolated points. $12(N + 1)^4$.

2 diagrams with three dashed lines. Each of them contain an isolated trivial cluster, giving an additional factor of $(N + 1)$, we get: $-2(N + 1)^4$.

No diagrams with four or more dashed lines since the outer vertices would no longer be connected.

In total, we have $T_5 = (N + 1)^6 - 6(N + 1)^5 + 10(N + 1)^4$.

In the $N \to -1$ limit, only the lowest power contributes (in this case $10(N + 1)^4$), since other terms are higher order in $\Delta$. So we only need to find the lowest term for each tensor structure.

This has a natural interpretation. It can be easily argued using induction [38], that all terms contributing to the lowest power of $(N + 1)$ have no solid line cycles, i.e. it consists of only (possibly disconnected) trees, also known as a “forest”.

Thus, in order to compute the tensor structures in the $0$-state Potts model, we need to enumerate the forests, while remembering that each one will carry $+1$ or $-1$ depending on how many dashed lines there are.
Our proof will be roughly as follows. The majority of the work will be to first show that all two-point tensor structures are the same in the Potts model and the $Sp(2)$ model. Then we can use this to deduce that it holds for three-point functions and zero point functions as well.

5.3 Prove the equivalence for two-point functions

5.3.1 Some general observations about the diagrams

First, let’s consider the Potts model. For any given two-point tensor structure, it will only be non-zero if there’s a solid line going through the middle and connecting the two outer points. There must be one and only one such line for each forest, otherwise there will be a cycle. For each forest, we call this line the “principal line”. There can be trees that are connected to the principal line, which we call “rooted trees”, and there can be trees not connected to the principal line, which we call “isolated trees”.

In the $Sp(N)$ model, we only consider the two-point function of $\theta \bar{\theta}$. For the same topology, the tensor value for $\sigma \sigma$ must be the same due to $Osp(1|2)$ invariance that was proved in chapter 4. To motivate the analogy to the Potts model case, we will denote lines of $\theta$ by solids, and $\sigma$ by dashes. It can be easily reasoned that because all vertices in $Sp(N)$ model are either dash-dash-dash or dash-solid-solid, for a Feynman diagram to be valid, it must be that there is again one and only one principal line connecting the two outer points. If this is not the case, then a solid line must either terminate (resulting in an invalid vertex of solid-dash-dash), or self-intersect (resulting in an invalid vertex of solid-solid-solid). In this case, there can be no rooted trees or isolated trees in a valid Feynman diagram. However, there can be other cycles not connected to the principal line. Further, the cycles do not share any edges.

5.2 shows an example of possible structures of a given topology in the Potts case and the $Sp(2)$ case, with the same principal line.
In the proof we will also need the following fact. We start with a given topology and a principal line. Suppose there are $P$ vertices on the principal line, $V$ vertices outside of the principal line, and $E$ edges not on the principal line. We have the following relation:

$$2E = 3V + P. \quad (5.16)$$

This can be reasoned as follows. Suppose we first add in the vertices without any edges (except those on the principal line). Since the interaction is cubic, every single vertex must have 3 edges. Each of the $P$ vertex on the principal line already has 2 edges, so they need 1 more edge. Each of the additional $V$ vertices not on the principal line needs 3 edges each. Whenever we add an edge not on the principal line, we “saturate” two edges. Hence the above relation.

Therefore, we have:

$$E - V = \frac{V + P}{2} = \frac{\text{total number of vertices}}{2} = \text{number of loops in the diagram}. \quad (5.17)$$

We will use this fact later on.
We will prove a stronger statement than the original claim. We will show that for a given topology and an arbitrary principal line, the tensor factors calculated from the 0-state Potts model agree with the \( Sp(2) \) model, with a factor of \((-1)^{\text{loop order}}\). Since this equivalence holds for any principal line, the equivalence extends to all tensor structures of that topology, and our original claim follows.

5.3.2 Defining three sets of graphs: \( G_C, G_F, G_A \)

The proof relies on the relations between three quantities, which we will define now. Given a topology and a fixed principal line, with \( V \) vertices and \( E \) edges outside of the principle line, where each edge can be either dashed or solid, we define three sets of graphs.

1. The set of “cycle graphs”, \( \{G_C\} \), consisting of all graphs on the topology where there can be zero or more solid line cycles. These cycles cannot intersect with the principal line, or themselves. These clearly have a 1-to-1 correspondence with Feynman diagrams in the \( Sp(2) \) model. Each vertex is either dash-dash-dash or dash-solid-solid. For each individual graph \( g_C \in G_C \), suppose it has \( L \) total vertices contained in \( n \) solid line cycles (hence \( V - L \) vertices not contained in them) we define its value to be \( V(g_C) = 2^{V-L} \times (-2)^n \). This corresponds to the value of the Feynman diagram in the \( Sp(2) \) model. Since there will be \( V - L \) dash-dash-dash vertices, each carrying a factor of 2, and \( n \) closed fermion loops, each carrying a factor of \(-N\), where the number of species \( N = 2 \) in our case. Therefore the sum of all graphs in \( G_C \) equals to the sum of all Feynman diagrams in the \( Sp(2) \) model with the given topology and principal line.

2. The set of “forest graphs”, \( \{G_F\} \), consisting of all graphs on the topology where there are no solid lines cycles, including those formed with the principal line. These are just the set of all forest diagrams on the given topology, which have a 1-to-1 correspondence with the diagrammatic method of calculating the tensor factor of the
0-state Potts model. For each individual graph $g_F \in G_F$, suppose it has $k$ solid edges outside of the principal line (hence it has $E - k$ dashed lines) we define its value to be $(-1)^{E-k}$. Therefore the sum of the value of all graphs in $G_F$ gives exactly the tensor factor in the 0-state Potts model with the given topology and principal line.

Therefore our goal is to show the sum of value of all graphs in $G_C$ is equal to $G_F$. To do so we need to define the following quantity which, unlike the other two quantities, does not have an obvious interpretation in terms of Feynman diagrams of either theory.

3. The set of “arrow graphs”, $\{G_A\}$, consists of all graphs on the topology where we draw arrows on each edge with the following rules. Imagine initially all edges not on the principal line are dashes. For each of the $V$ vertex not on the principle line, we make one of four choices: to color one of the three lines connected to this vertex to be solids (each of these choice carry a factor of 1, we say that such a vertex is “flowing”), or not coloring any of the lines and leaving them as dashed (this carries a factor of $-1$, we call such a vertex “empty”). The value of an individual graph $g_A \in G_A$ is just the product of all the value on its vertices, which is $V(g_A) = (-1)^{\# \text{ of empty vertices}}$.

Let’s look at the properties of $\{G_A\}$. We can make each of the 4 choices independently for each vertex, getting a total of $4^V$ possibilities. If we sum the value of all such $4^V$ diagrams, we will get $(3 - 1)^V = 2^V$, since the choice made on each vertex is independent.

Notice that we allow a line to be colored twice by its two adjacent vertices. We will keep track of which vertex colored which line with arrows. If a line connecting vertices $A$ and $B$ is colored by $A$, then an arrow is drawn on the line pointing away from $A$, likewise for $B$. If a line is colored by both vertices, we draw a double arrow. This is illustrated in figure 5.3. Since each vertex can color up to 1 of its adjacent lines, there can never be more than one arrow coming out of any vertex.

Let us now look at the properties of the diagrams in $\{G_A\}$. 103
Figure 5.3: Line 1 is colored by vertex A, line 2 is double-colored by vertices B and C. All 3 vertices are “flowing”.

Figure 5.4: Illustrating some properties of $G_A$

1. It is impossible to create a cycle with part of the principal line. Since in order to form a cycle with the principle line, you would need to color $(k+1)$ lines with only $k$ vertices, which is impossible. See 5.4 (a).

2. While it is possible to create a cycle disconnected from the principal line, it must be a simple cycle. For example, if you want to form a double cycle, with $k$ vertices, you need at least $(k+1)$ colored lines. See 5.4 (b). Thus, all cycles created must be disjoint.

Our main proof proceeds as follows. First we will show that the sum of all diagrams in $G_C$ corresponds to the sum of all diagrams in $G_A$ that does not result in any cycles, ie. all forest diagrams in $G_A$. 

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Then we will show that sum of all the forest diagrams in $G_A$ equals to the sum of diagrams in $G_F$ times the factor $(-1)^{E-V}$, which we’ve shown to be $(-1)^{\text{number of loops}}$. This explains the observation we made in the beginning of this chapter that the successive loops have alternating signs.

5.3.3 Relation between $G_A$ and $G_C$

In this section we will show that the sum of all diagrams in $G_C$ corresponds to the sum of all forest diagrams in $G_A$.

First we look at the diagram in $G_C$ where all lines outside the principal line are dashes, there is only one such diagram. That means all of the $V$ vertices outside the principal line are dash-dash-dash, with each contributing a factor of 2. This diagram has value $2^V$. On the other hand, as we’ve already noted, the sum of all diagrams in $G_A$ is exactly $2^V$.

**Conclusion 1: the only diagram with no cycle in $G_C$ is equal to the sum of all diagrams in $G_A$**

Now we look at a diagram in $G_C$ where there is exactly one solid line cycle. Suppose the cycle has length $L$. This diagram has $V - L$ triple dash vertices, each contributing a factor of 2, and $L$ solid-solid-dash vertices, each contributing a factor of 1. Finally, it has a solid line cycle, which gives an additional factor of $N = -2$. In total, it has value $-2 \times 2^{V-L}$.

Now consider all the diagrams in $G_A$ where that particular cycle is solid. We don’t care about the choice of other vertices (in particular, they can form additional cycles). So the other $V - L$ vertices contribute a factor of $2^{V-L}$. The remaining $L$ vertices in a loop must all be flowing. Further they either all flow clockwise or counterclockwise, giving 2 possibilities, and each flowing vertex gives a factor of 1. In total the sum of all such diagrams is $2 \times 2^{V-L}$.

The above argument can be easily generalized to any other cycle. So we have:
Conclusion 2: the sum of diagrams with exactly one cycle in $G_C$ is equal to the negative of the sum of all diagrams with at least one cycle in $G_A$.

Now we look at a diagram in $G_C$ with exactly 2 solid line cycles. As we argued before the two cycles must be disjoint. Suppose the length of the two cycles are $L_1$ and $L_2$. Then the value of this diagram is the product of: $2^{V-L_1-L_2}$ (for the triple-dash vertices), $(1)^{L_1+L_2}$ (for the solid-solid-dash vertices), and $N^2 = 4$.

In $G_A$, it corresponds to all diagrams with these two particular cycles, regardless of the choices of the other $V-L_1-L_2$ vertices. However, each of these cycles have 2 ways to be connected (clockwise or counterclockwise), therefore there is an additional factor of 4.

Conclusion 3: the sum of diagrams with exactly two cycles in $G_C$ is equal to the sum of all diagrams with at least two cycle in $G_A$.

We can see how this generalizes. Let $G_{C,n}$ denote the sum of all diagrams in $G_C$ with exactly $n$ cycles. When we sum up all possible diagrams in $G_C$, we get:

$$\sum_{\text{all diagrams}} G_C = G_{C,0} + G_{C,1} + G_{C,2} + \ldots.$$  \hfill (5.18)

Then let $G_{A,n+}$ be the sum of all diagrams in $G_A$ with at least $n$ cycles, we have:

$$\sum_{\text{all diagrams}} G_C = G_{C,0} + G_{C,1} + G_{C,2} + G_{C,3} + \ldots ,$$  \hfill (5.19)

$$= G_{A,0+} - G_{A,1+} + G_{A,2+} - G_{A,3+} + \ldots.$$  \hfill (5.20)

In other words, we have the sum of: all diagrams in $G_A$, minus all diagrams with at least one cycle in $G_A$, plus all diagrams with at least two cycles in $G_A$, etc. By the principle of inclusion and exclusion, the sum is exactly equal to the sum of all diagrams with no cycles in $G_A$, leaving only the forest diagrams in $G_A$. 

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Therefore, we finally conclude that, the sum of all diagrams in $G_C$ is equal to the sum of all forest diagrams in $G_A$.

### 5.3.4 Relation between $G_A$ and $G_F$

Now we will argue that each diagram in $G_F$ (which is a forest) is counted with exactly weight 1 in $G_A$, up to an overall factor depending on the number of loops in the topology. Therefore if the sum of all diagrams in $G_F$ is also equal to the sum of all forest diagrams in $G_A$, establishing our original claim.

**Case 1: a single rooted tree** If the forest diagram contains a single rooted tree with $k$ vertices and nothing else, there are $k$ vertices which need to color exactly $k$ lines. All $k$ vertices must be flowing and the direction of the arrow is fixed, as illustrated in 5.5. Thus, there is exactly one configuration which colors each rooted tree. Suppose there are $V$ vertices and $E$ edges in total. in $G_A$, $k$ of the vertices are flowing, $(V-k)$ of them are empty, and hence it has a factor of $(-1)^{V-k}$. In $G_F$, there are $E-k$ dashed lines, it carries a factor $(-1)^{E-k}$. Thus, to match the overall factor, we must multiply the Potts result by $(-1)^{E-V} = (-1)^\text{number of loops}$. 

![Principal line](image)
Case 2: a single isolated tree For a single isolated tree in $G_A$, there are $k + 1$ vertices which need to color $k$ lines. This means that either one of the vertices doesn’t color any lines, or one of the lines is double colored. There are $k + 1$ choices for where the “empty vertex” is, and $k$ choices for where the double-colored line is. However, there is a sign difference due to the fact that if a vertex doesn’t color any lines, it carries a factor $-1$. Thus, the total factor is $(-1)^{V-k}(k+1) + (-1)^{V-k-1}k = (-1)^{V-k}$. This is illustrated in Figure 5.6 Notice that the same graph in $G_F$ will carry a factor of $(-1)^{E-k}$. Again, in order to match the Potts model, we need an overall factor of $(-1)^{E-V}$. This is consistent with case 1.

Case 3: multiple rooted tree and isolated trees This is a simple generalization of the previous two cases, indeed the $G_A$ value is again consistent with the $G_F$ value, up to the same overall factor.

Combining the results from the previous section, we see that, for a given topology and a given principal line:
This proves our claim, that for a given two-point function topology and principal line, the value of the tensor structure in the $Sp(2)$ model is equal to that in the 0-state Potts model. If we sum over all possible principal lines, the equality still holds. Therefore all two-point function tensor structures in the $Sp(2)$ model are equal to that of the 0-state Potts model.

\[ \sum_{\text{all diagrams}} G_F = (-1)^{\# \text{of loops}} \sum_{\text{all forest diagrams}} G_A \]  
\[ = (-1)^{\# \text{of loops}} \sum_{\text{all diagrams}} G_C. \]  

\[ (5.21) \]
\[ (5.22) \]

5.4 three-point functions and zero point functions

The generalization to 3-point functions is rather simple. Suppose we have a three-point function $T_{3d}^{ijk}$ in the Potts model. We would like to show that it’s equal to a three-point function in the $Sp(2)$ model. Again, there are two choices of three-point functions in the $Sp(2)$ case: $\sigma^3$ or $\sigma \theta \bar{\theta}$, we can choose either one due to supersymmetry (except $\sigma^3$ is bigger by a factor of 2), so for convenience, we choose the latter.

We can convert the 3-point function to a 2-point function by merging two of its lines into one (In the $Sp(2)$ case, we must merge the $\sigma$ line with one of the two $\theta$ lines). This effectively creates a new two-point function tensor structure $T_{2new}$ related to $T_3$ via:

\[ T_{2new} = \sum_{j,k} T_3 d^{ijk} d^{jkl} = T_3 T_2 \delta^{i l}, \]  
\[ (5.23) \]

where $T_2$ is the simplest one loop tensor structure $d^{ijk} d^{jkl}$.

Thus, to show that $T_3$ is equal, we just need $T_2$ and $T_{2new}$ being equal, which is true.
The generalization to zero point functions is also simple. Notice that relation 5.20 still holds when we ignore the existence of the principal line. The consideration for diagrams in $G_F$ is also simpler because there will only be isolated trees, and no rooted trees. The equivalence will continue to hold without any issue.

Notice that the equality between all zero point functions in the two models imply that their sphere free energy $F$ will be the same.
Chapter 6

Yukawa CFTs and Emergent

Supersymmetry

6.1 Introduction and Summary

This chapter is based on the work published in [31], co-authored with Simone Giombi, Igor Klebanov, and Grigory Tarnopolsky. We also thank David Gross, David Poland, Silviu Pufu and David Simmons-Duffin for useful discussions.

Physical applications of relativistic quantum field theories with four-fermion interactions date back to Fermi’s theory of beta decay. The first application to strong interactions was the seminal Nambu and Jona-Lasinio (NJL) model. In the original paper [104] they considered the model in 3 + 1 dimensions with a single 4-component Dirac fermion and Lagrangian

\[ \mathcal{L}_{NJL} = \bar{\psi} \slashed{D} \psi + \frac{g}{2} \left( (\bar{\psi} \psi)^2 - (\bar{\psi} \gamma_5 \psi)^2 \right). \] (6.1)

In addition to the \(U(1)\) symmetry \(\psi \rightarrow e^{i\beta} \psi\), this Lagrangian possesses the \(U(1)\) chiral symmetry under \(\psi \rightarrow e^{i\alpha} \gamma_5 \psi\). Using the gap equation it was shown that the chiral \(U(1)\) can be broken spontaneously, giving rise to the massless Nambu-Goldstone
boson. This was the discovery of the crucial role of chiral symmetry breaking in the physics of strong interactions.

One of the goals of this chapter is to study a generalization of (6.1) to $N_f$ 4-component Dirac fermions $\psi_j$, $j = 1, \ldots, N_f$:

$$\mathcal{L}_{\text{NJL}} = \bar{\psi}_j \partial \psi^j + \frac{g}{2} \left( (\bar{\psi}_j \psi^j)^2 - (\bar{\psi}_j \gamma_5 \psi^j)^2 \right),$$

(6.2)

and its continuation to dimensions below 4. We define $N = 4N_f$, so that $N$ is the number of 2-component Majorana fermions in $d = 3$. In addition to the chiral $U(1)$ symmetry $\psi_j \to e^{i\alpha \gamma_5} \psi_j$ this multi-flavor NJL model possesses a $U(N_f)$ symmetry.\footnote{It is also often called the chiral Gross-Neveu model \cite{25}; in $d = 2$ it is equivalent to the $SU(2N_f)$ Thirring model \cite{105, 106}.}

When considered in $2 < d < 4$ this model gives rise to an interacting conformal field theory which describes the second-order phase transition separating the phases where the $U(1)$ chiral symmetry is broken and restored.

In addition to studying the NJL model with the $U(1)$ chiral symmetry, we will present new results for the Gross-Neveu (GN) model \cite{25}, which has a simpler quartic interaction

$$\mathcal{L}_{\text{GN}} = \bar{\psi}_j \partial \psi^j + \frac{g}{2} (\bar{\psi}_j \psi^j)^2.$$

(6.3)

Instead of the continuous chiral symmetry it possesses the discrete chiral symmetry $\psi_j \to \gamma_5 \psi_j$. As discovered in \cite{25}, in $d = 2$ this theory is asymptotically free for $N > 2$. When considered in $2 < d < 4$ this is believed to be an interacting conformal field theory which describes the second-order phase transition where the discrete chiral symmetry is broken.

In $d > 2$ the four-fermion interactions (6.2) and (6.3) are non-renormalizable. While they are renormalizable in the sense of the $1/N$ expansion \cite{107}, at finite $N$ it
is important to know the UV completion of these theories. In [26, 27] it was suggested that the UV completion in $2 < d < 4$ is provided by the appropriate Yukawa theories. The UV completion of the GN model (6.3) contains a real scalar field $\sigma$ and $N_f 4$-component Dirac fermions $\psi_j$, $j = 1, \ldots, N_f$:

$$ L_{\text{GNY}} = \frac{1}{2} (\partial_\mu \sigma)^2 + \bar{\psi}_j \gamma^\mu \psi^j + g_1 \sigma \bar{\psi}_j \psi^j + \frac{1}{24} g_2 \sigma^4. \quad (6.4) $$

This theory, known as the Gross-Neveu-Yukawa (GNY) model, will be discussed in section 6.2.

The UV completion of the $U(1)$ symmetric NJL model (6.2), which contains a complex scalar field $\phi = \phi_1 + i \phi_2$, was introduced in [27]:

$$ L_{\text{NJLY}} = \frac{1}{2} (\partial_\mu \phi_1)^2 + \frac{1}{2} (\partial_\mu \phi_2)^2 + \bar{\psi}_j \gamma^\mu \psi^j + g_1 \bar{\psi}_j (\phi_1 + i \gamma_5 \phi_2) \psi^j + \frac{1}{24} g_2 (\bar{\phi} \phi)^2. \quad (6.5) $$

It has a continuous $U(1)$ chiral symmetry under\(^2\)

$$ \psi_j \rightarrow e^{i \alpha \gamma_5} \psi_j, \quad \phi \rightarrow e^{-2 i \alpha} \phi. \quad (6.6) $$

This theory, which we will call the Nambu-Jona-Lasinio-Yukawa model (NJLY), will be discussed in section 6.3.

There is a large body of literature on the GN and NJL CFTs in $d = 3$ and their applications; see, for example, [109, 26, 27, 110, 111, 112, 113, 114, 115, 3, 116, 117, 118, 119]. We will carry out further studies of these CFTs using the $4 - \epsilon$ and $2 + \epsilon$ expansions followed by Padé extrapolations. In addition to studying the scaling dimensions of some low-lying operators, we will calculate the sphere free energy $F$. The latter determines the universal entanglement entropy across a circle [66], and is

\(^2\)In $d = 3$, one may express the Lagrangian (6.2) in terms of $2N_f$ 2-component Dirac spinors $\chi^i, \chi^{i+N_f}$ by writing $\psi^i = (\chi^i, \chi^{i+N_f})$, $i = 1, \ldots, N_f$. See for instance [108] for the explicit relation between 4-component and 2-component notations in 3d. The 2-component spinors $\chi^{i,\pm} = (\chi^i \pm \chi^{i+N_f})/\sqrt{2}$ have charge $\pm 1$ under the $U(1)$ symmetry (6.6).
the quantity that enters the $F$-theorem [120, 54, 50, 67]. We will also discuss $C_T$, the normalization of the correlation function of two stress-energy tensors. For the GN model, the $1/N$ and $\epsilon$ expansions of $C_T$ were studied in [119]; in this chapter we extend these results to the NJL model and also provide the numerical estimates in $d = 3$ for various values of $N$.

When considered for $N_f = 1/2$, i.e. the single 4-component Majorana fermion (which is equivalent to one Dirac fermion in $d = 3$), the NJLY model is expected to flow to the well-known supersymmetric Wess-Zumino model with 4 supercharges. In $d < 4$ this theory defines a CFT with “emergent supersymmetry” [121, 122], in the sense that the RG flow drives the interactions to a supersymmetric IR stable fixed point, where the global $U(1)$ symmetry becomes the $U(1)_R$ symmetry (see figure 6.1). We will provide additional evidence for this using the $4 - \epsilon$ expansion of the NJLY model with $N_f = 1/2$ to two loops, both for certain scaling dimensions and for the sphere free energy. A three-loop calculation of scaling dimensions, which supports the emergent supersymmetry, was carried out recently [123].

Even more intriguingly, when the GNY model is continued to $N_f = 1/4$, which corresponds to a single 2-component Majorana fermion in $d = 3$, it appears to flow to a CFT with 2 supercharges [121, 124, 125, 118, 126]. We will show that the $O(\epsilon^2)$ corrections to scaling dimensions of operators continue to respect this emergent supersymmetry.\footnote{This is reminiscent of the symmetry enhancement from $Sp(2)$ to the supergroup $OSp(1|2)$ at the IR stable fixed point of a cubic theory in $6 - \epsilon$ dimensions in chapter 4.} This provides new support for the existence of an $\mathcal{N} = 1$ supersymmetric CFT in $d = 3$. Our Padé extrapolations of operator dimensions including the $\epsilon^2$ corrections are in good agreement with the conformal bootstrap approach to the $\mathcal{N} = 1$ supersymmetric CFT in $d = 3$ [118]. We also estimate $C_T$ and $F$ for this theory. These results will be presented in section 6.4.
Figure 6.1: RG flow and fixed point structure for the GNY model with $N = 1$ (one Majorana fermion in $d = 3$) and the NJLY model with $N = 2$ (one Dirac fermion in $d = 3$), obtained from the one-loop $\beta$-functions (6.7) and (6.68) in $d = 4 - \epsilon$. The attractive IR fixed points have “emergent” supersymmetry with 2 and 4 supercharges respectively. The red triangles denote unstable fixed points with negative quartic potential which can be seen in the one-loop analysis in $d = 4 - \epsilon$; their fate in $d = 3$ is unclear.
6.2 The Gross-Neveu-Yukawa Model

The $\beta$-functions for the GNY model with action (6.4) in $d = 4 - \epsilon$, up to two-loop order, are \[114\]

$$\beta_{g_2} = -\epsilon g_2 + \frac{1}{(4\pi)^2} \left(3g_2^2 + 2Ng_1^2g_2 - 12Ng_1^4\right) + \frac{1}{(4\pi)^4} \left(96Ng_1^6 + 7Ng_1^4g_2 - 3Ng_1^2g_2^2 - \frac{17g_2^2}{3}\right),$$

$$\beta_{g_1} = -\frac{\epsilon}{2} g_1 + \frac{N + 6}{2(4\pi)^2} g_1^3 + \frac{1}{(4\pi)^4} \left(-\frac{3}{4}(4N + 3)g_1^5 - 2g_1^3g_2^2 + \frac{9g_2^2}{12}\right), \quad (6.7)$$

where $N = N_f^\text{tr} = 4N_f$. The model possesses an IR stable fixed point at the critical couplings $g_i^*$ given by

$$\left(\frac{g_1^*}{4\pi}\right)^2 = \frac{1}{N + 6} \epsilon + \frac{(N + 66)\sqrt{N^2 + 132N + 36} - N^2 - 516N + 882}{108(N + 6)^3} \epsilon^2 + O(\epsilon^3),$$

$$\left(\frac{g_2^*}{4\pi}\right)^2 = \frac{-N + 6 + \sqrt{N^2 + 132N + 36}}{6(N + 6)} \epsilon + \frac{1}{54(N + 6)^3\sqrt{N^2 + 132N + 36}} \times \left(3N^4 + 155N^3 + 2745N^2 - 2538N + 7344\right) \epsilon^2 + O(\epsilon^3). \quad (6.8)$$

Of course, there is also a fixed point $g_1^* = 0$, $g_2^* = g_2^\text{Ising} = \frac{16\pi^2\epsilon}{3} + O(\epsilon^2)$ which corresponds to the decoupled product of the single-scalar Wilson-Fisher fixed point and $N_f$ free fermions. By looking at the derivative of the beta functions at the fixed points, one can verify that (6.8) is attractive for all $N_f$, so one can flow to it from the “Ising” fixed point along a relevant direction. Let us mention that there is formally also a third fixed point obtained from (6.8) by changing the sign of $\sqrt{N^2 + 132N + 36}$. This fixed point is unstable in $d = 4 - \epsilon$ due to the negative quartic coupling, $g_2^* < 0$, but its dimensional continuation may produce a CFT in $d = 3$.

\[116\]

\textsuperscript{4}We have reproduced them using the general two-loop results [127, 128, 129, 130, 131] for the Yukawa theories, which are reviewed in the Appendix.
The scaling dimensions of $\sigma$ and $\psi$ are found to be [114]

$$
\Delta_\sigma = 1 - \frac{\epsilon}{2} + \frac{1}{2(4\pi)^2} Ng_1^2 + \frac{1}{(4\pi)^4} \left( \frac{1}{12} g_2^2 - \frac{5}{4} N g_1^4 \right),
$$

$$
\Delta_\psi = 3 - \frac{\epsilon}{2} + \frac{1}{2(4\pi)^2} g_1^2 - \frac{1}{8(4\pi)^4} (3N + 1) g_1^4.
$$

(6.9)

At the IR stable fixed point (6.8), one gets

$$
\Delta_\sigma = 1 - \frac{3}{N + 6} \epsilon + \frac{52N^2 - 57N + 36 + (11N + 6)\sqrt{N^2 + 132N + 36}}{36(N + 6)^3} \epsilon^2 + O(\epsilon^3),
$$

$$
\Delta_\psi = \frac{3}{2} - \frac{N + 5}{2(N + 6)} \epsilon + \frac{-82N^2 + 3N + 720 + (N + 66)\sqrt{N^2 + 132N + 36}}{216(N + 6)^3} \epsilon^2 + O(\epsilon^3).
$$

(6.10)

These dimensions agree with [112] after correcting some typos in eq. (11) of that paper (in particular, the coefficient 33 should be changed to 3). Our $O(\epsilon)$ term in $\Delta_\psi$ corrects a typo in eq. (4.44) of [3].

Setting $g_1^* = 0$, $g_2^* = \frac{16\pi^2}{3} + O(\epsilon^2)$ in (6.9), we may also recover the result at the Ising fixed point $\Delta_\sigma^{\text{Ising}} = 1 - \frac{\epsilon}{2} + \frac{\epsilon^2}{108} + O(\epsilon^3)$. One can then see that the Yukawa operator $\sigma \bar{\psi} \psi$ is relevant at this decoupled fixed point, and can trigger a flow to the IR stable fixed point (6.8). We expect this to be true in $d = 3$ as well, since it is known that $\Delta_\sigma^{3d\text{ Ising}} \approx 0.518$ [57], and so $\Delta_\sigma^{3d\text{ Ising}} \approx 2.518 < 3$.

The anomalous dimension of the operator $\sigma^2$, which determines the critical exponent $\nu^{-1} = 2 - \gamma_{\sigma^2}$, may be read off from eq. (18) of [114]

$$
\gamma_{\sigma^2} = \frac{g_2}{(4\pi)^2} - \frac{1}{(4\pi)^4} (g_2^2 + Ng_2g_1^2 - 2Ng_1^4) + 2\gamma_\sigma.
$$

(6.11)

---

5 Throughout the chapter, one can obtain the corresponding $\epsilon$ expansions at the unstable fixed point by changing the sign of the square root.
At the fixed point (6.8) we find

\[ \Delta_{\sigma^2} = d - 2 + \gamma_{\sigma^2} = 2 + \frac{\sqrt{N^2 + 132N + 36} - N - 30}{6(N + 6)} \epsilon + \frac{1}{54(N + 6)^3 \sqrt{N^2 + 132N + 36}} \frac{1}{\epsilon} \]

\[ \times \left( (3N^3 + 109N^2 + 510N + 684) \sqrt{N^2 + 132N + 36} \ight. 
\[ \left. - 3N^4 - 658N^3 - 333N^2 - 15174N + 4104 \right) \epsilon^2 + O(\epsilon^3) . \tag{6.12} \]

We have also calculated the one-loop anomalous dimensions of the operators \( \bar{\psi} \psi \) and \( \sigma^3 \):

\[ \Delta_{\bar{\psi} \psi} = \frac{(N + 2)g_1^2}{(4\pi)^2} + 2\Delta_\psi, \quad \Delta_{\sigma^3} = \frac{3g_2}{(4\pi)^2} + 3\Delta_\sigma . \tag{6.13} \]

At higher orders these operators will mix, and one has to find the eigenvalues of their mixing matrix. At the fixed point (6.8), we find

\[ \Delta_{\bar{\psi} \psi} = 3 - \frac{3}{N + 6} \epsilon + O(\epsilon^2) , \quad \Delta_{\sigma^3} = 3 + \frac{\sqrt{N^2 + 132N + 36} - N - 12}{2(N + 6)} \epsilon + O(\epsilon^2) . \tag{6.14} \]

The first of these dimensions corresponds to a descendant of \( \sigma \), as can be seen from the fact that it equals \( 2 + \Delta_\sigma \).

Let us also review the known result for the \( 4 - \epsilon \) expansion of \( C_T \) in the GNY model, which was discussed in [119]; the diagrams contributing to the term \( \sim g_1^2 \) are shown in fig. 4.7 there. Evaluation of these diagrams yields

\[ C_T = NC_T,f + C_T,s - \frac{5N(g_1^2)^2}{12(4\pi)^2} = \frac{d}{S_a^2} \left( \frac{N}{2} + \frac{1}{d - 1} - \frac{5N \epsilon}{12(N + 6)} \right) . \tag{6.15} \]
where $S_d = 2\pi^{d/2}/\Gamma(d/2)$, and we used the values of $C_T$ for free scalar and fermion theories:

$$C_{T,s} = \frac{d}{(d-1)S_d^2}, \quad C_{T,f} = \frac{d}{2S_d^2}. \quad (6.16)$$

### 6.2.1 Free energy on $S^{4-\epsilon}$

In order to renormalize the theory on curved space, one should add to the action all the relevant curvature couplings that are marginal in $d = 4 - \epsilon$ \cite{132,133}

$$S_{GNY} = \int d^d x \sqrt{g} \left(\bar{\psi}_i \not{\partial} \psi^i + \frac{1}{2}(\partial_{\mu}\sigma)^2 + \frac{d-2}{8(d-1)}R\sigma^2 + g_{1,0}\bar{\psi}_i \psi^i + \frac{g_{2,0}}{24}\sigma^4 + \frac{\eta_0}{2}R\sigma^2 + a_0 W^2 + b_0 E + c_0 R^2\right), \quad (6.17)$$

where $R$ is the scalar curvature, $W^2$ is the square of the Weyl tensor, and $E$ the Euler density

$$E = R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2. \quad (6.18)$$

The parameters $\eta_0, a_0, b_0, c_0$ are bare curvature couplings whose renormalization can be fixed order by order in perturbation theory. On a sphere, the Weyl square term drops out, and to the order we work below we will only need the renormalization of the Euler coupling $b_0$ (the $R^2$ term and the renormalization of conformal coupling are expected to play a role at higher orders \cite{132,133,130}). The corresponding beta function can be extracted from the results of \cite{130,134}, and we find

$$\beta_b = \epsilon b - \frac{11N/4}{360(4\pi)^2} - \frac{1}{(4\pi)^8} \left(\frac{9}{32} Ng_1^6 + \frac{3}{8} N^2 g_1^6 + \frac{1}{4} Ng_1^4 g_2 - \frac{1}{96} Ng_1^2 g_2^2\right) + \ldots, \quad (6.19)$$
where \( b \) is the renormalized coupling, and its the relation to the bare one \( b_0 = \mu^{-\epsilon} (b - \frac{11N/4 + 1}{360(4\pi)^2 \epsilon} + \ldots) \) can be inferred from the above beta function. The coupling independent term is related to the \( \alpha \)-anomaly of the free fermions and scalar.

The calculation of the sphere free energy now proceeds as in [135, 136]. Keeping terms that contribute up to order \( \epsilon^2 \), we have

\[
F = NF_f + F_s - \frac{1}{2} g_{1,0}^2 \int d^d x d^d y \sqrt{g_x} \sqrt{g_y} \langle \sigma \bar{\psi} \psi(x) \sigma \bar{\psi} \psi(y) \rangle_0
- \frac{1}{2! (4!)} g_{2,0}^2 \int d^d x d^d y \sqrt{g_x} \sqrt{g_y} \langle \sigma^4(x) \sigma^4(y) \rangle_0
- \frac{1}{4!} g_{1,0}^4 \int d^d x d^d y d^d z d^d w \sqrt{g_x} \sqrt{g_y} \sqrt{g_z} \sqrt{g_w} \langle \sigma \bar{\psi} \psi(x) \sigma \bar{\psi} \psi(y) \sigma \bar{\psi} \psi(z) \sigma \bar{\psi} \psi(w) \rangle_0 + \delta F_b ,
\]

(6.20)

where \( \delta F_b = b_0 \int d^d x \sqrt{g} E \) is the contribution of the curvature term, and \( F_f, F_s \) are the sphere free energies of free fermion and scalar, which can be found in [78]. Starting from the flat space propagators

\[
\langle \sigma(x) \sigma(y) \rangle = C_\phi \frac{1}{|x - y|^{d-2}} , \quad \langle \psi_i(x) \bar{\psi}^j(y) \rangle = \delta^j_i C_\psi \frac{\gamma^\mu(x - y) \mu}{|x - y|^d} ,
\]

(6.21)

where \( C_\phi = \Gamma(\frac{d}{2} - 1)/(4\pi^{\frac{d}{2}}) \) and \( C_\psi = \Gamma(\frac{d}{2})/(2\pi^{\frac{d}{2}}) \), and then Weyl transforming to the sphere, we find

\[
\int d^d x d^d y \sqrt{g_x} \sqrt{g_y} \langle \sigma \bar{\psi} \psi(x) \sigma \bar{\psi} \psi(y) \rangle_0 = NC_\phi C_\psi^2 \int d^d x d^d y \sqrt{g_x} \sqrt{g_y} \frac{\gamma^\mu(x - y) \mu}{s(x, y)^{3d-4}} = NC_\phi C_\psi^2 I_2 \left( \frac{3d}{2} - 2 \right) ,
\]

\[
\int d^d x d^d y \sqrt{g_x} \sqrt{g_y} \langle \sigma^4(x) \sigma^4(y) \rangle_0 = 4! C_\phi^4 \int d^d x d^d y \sqrt{g_x} \sqrt{g_y} \frac{\gamma^\mu(x - y) \mu}{s(x, y)^{4d-2}} = 4! C_\phi^4 I_2 (2d - 4) .
\]

(6.22)
Here $I_2(\Delta)$ denotes the integrated 2-point function of an operator of dimension $\Delta$, which is given by [137, 138, 50]

$$I_2(\Delta) = \int \frac{d^d x d^d y \sqrt{g_x} \sqrt{g_y}}{s(x, y)^{2\Delta}} = (2R)^{2(d-\Delta)} \frac{2^{1-d} \pi^{d+\frac{1}{2}} \Gamma\left(\frac{d}{2} - \Delta\right)}{\Gamma\left(\frac{d+1}{2}\right) \Gamma(d-\Delta)}.$$  \hspace{1cm} (6.23)

For the 4-point function, we find

\[
\langle \bar{\sigma} \bar{\psi}(x) \bar{\psi}(y) \psi \bar{\psi}(z) \bar{\psi}(w) \rangle_0 = 6N^2 C_\phi^2 C_\psi^4 \frac{1}{s_{2d-2} s_{2} s_{2d-2} s_{2d-2}} \langle \bar{\sigma} \bar{\psi}(x) \bar{\psi}(y) \psi \bar{\psi}(z) \bar{\psi}(w) \rangle_0 - 3NC_\phi^2 C_\psi^4 \left( \frac{s_{xw}^2 s_{yz}^2 - s_{z}^2 s_{2w}^2 + s_{xy}^2 s_{2w}^2}{s_{xy} s_{yz} s_{2w} s_{2w}} \right) \times \left( \frac{2}{(s_{xy} s_{2w})^{d-2}} + \frac{1}{s_{xz} s_{yw} s_{d-2}} \right),
\]

where we used a shorthand notation for the chordal distance $s_{xy} \equiv s(x, y)$. The integral of this 4-point function over the sphere cannot be calculated explicitly, but one can find it as a series in $d = 4 - \epsilon$. For this we used the Mellin-Barnes approach, which is described in [135, 136]. The result for the integral reads

\[
\int d^d x d^d y d^d z d^d w \sqrt{g_x} \sqrt{g_y} \sqrt{g_z} \sqrt{g_w} \langle \bar{\sigma} \bar{\psi}(x) \bar{\psi}(y) \psi \bar{\psi}(z) \bar{\psi}(w) \rangle_0 = -\frac{N(N + 6)}{2(4\pi)^4 \epsilon} - \frac{N(N - 6 + 6(N + 6)(3 + \gamma + \log(4\pi R^2)))}{12(4\pi)^4} + O(\epsilon). \hspace{1cm} (6.25)
\]

Putting everything together, we find for the free energy in $d = 4 - \epsilon$

\[
F = NF_f + F_s - \frac{1}{2} g_{1,0}^2 NC_\phi C_\psi^2 I_2(3/2)(d - 2) - \frac{1}{2} g_{2,0}^2 C_\phi^4 I_2(2d - 4) + \frac{1}{4!} g_{1,0}^4 \left( \frac{N(N + 6)}{2(4\pi)^4 \epsilon} + \frac{N(N - 6 + 6(N + 6)(3 + \gamma + \log(4\pi R^2)))}{12(4\pi)^4} + O(\epsilon) \right) + \delta F_b. \hspace{1cm} (6.26)
\]
Now replacing the bare couplings with the renormalized ones

\[ g_{1,0} = \mu^2 \left( g_1 + \frac{N + 6g_1^2}{32\pi^2\epsilon} + \ldots \right), \quad g_{2,0} = \mu^\epsilon (g_2 + \ldots), \quad b_0 = \mu^{-\epsilon} \left( b - \frac{11N/4 + 1}{360(4\pi)^2\epsilon} + \ldots \right) \]

(6.27)

we find that all pole cancels, and the free energy is a finite function of the renormalized couplings \( g_1, g_2, b \).\(^6\)

As explained in [135], in order to calculate the free energy at the critical point we should now tune all couplings, including \( b \), to their fixed point values. Using (6.8), we get

\[
F = NF_f + F_s - \frac{N\epsilon}{48(N+6)} \left( (N^2 + 99N + 18)\sqrt{N^2 + 132N + 36} + (80N^2 + 2103N + 6381)N + 108 \right) \epsilon^2
\]

\[
\quad \quad \quad + \delta F_b + O(\epsilon^3).
\]

(6.28)

From the curvature beta function (6.19), we find at the critical point

\[
b_* = \frac{11N/4 + 1}{360(4\pi)^2\epsilon} + \frac{1}{(4\pi)^8 36} \left( \frac{9}{32} N(g_1^*)^6 + \frac{3}{8} N^2(g_1^*)^6 + \frac{1}{4} N(g_1^*)^4 g_2^* - \frac{1}{96} N(g_1^*)^2 (g_2^*)^2 \right) \frac{1}{\epsilon}
\]

(6.29)

and, using (6.8) and \( \int d^d x \sqrt{g} E = 64\pi^2 + O(\epsilon) \), we find that the Euler term contributes

\[
\delta F_b = \frac{N(882 + 66\sqrt{N^2 + 132N + 36} + N(516 - N + \sqrt{N^2 + 132N + 36}))}{15552(N + 6)^3} \epsilon^2 + O(\epsilon^3).
\]

(6.30)

\(^6\)Note that the coupling dependent part in the renormalization of \( b \) is necessary to cancel poles coming from diagrams at the next order. However, we still have to carefully include the Euler term, as in [135, 136], as it affects the free energy at the fixed point to order \( \epsilon^2 \).
Substituting this into (6.28), and writing the result in terms of \( \tilde{F} = -\sin(\pi d/2)F \), we find

\[
\tilde{F} = N\tilde{F}_f + \tilde{F}_s - \frac{N\pi \epsilon^2}{96(N+6)} - \frac{1}{31104(N+6)^3} \left( 161N^3 + 3690N^2 + 11880N + 216 \right.
\]

\[
+ (N^2 + 132N + 36) \sqrt{N^2 + 132N + 36} \pi \epsilon^3 + O(\epsilon^4).
\]

(6.31)

### 6.2.2 2 + \epsilon expansions

In this section we review the known results for operator dimensions at the UV fixed point of the Gross-Neveu model in \( d = 2 + \epsilon \), and then compute its sphere free energy to order \( \epsilon^3 \).

The action for the Gross-Neveu model [25] in Euclidean space in terms of bare fields and coupling reads

\[
S_{GN} = -\int d^d x \sqrt{g} \left( \bar{\psi}_i \partial \psi_i + \frac{1}{2} g_0 (\bar{\psi}_i \psi_i)^2 \right) + b_0 \int d^d x \sqrt{g} R,
\]

where \( i = 1, \ldots 2N_f \), and we have included the Euler term which is needed for the calculation of the sphere free energy below.

The \( \beta \)-function for the renormalized coupling constant \( g \) in \( d = 2 + \epsilon \) is known to be [139, 3, 140]

\[
\beta = \epsilon g - \frac{N - 2}{2\pi} g^2 + \frac{N - 2}{4\pi^2} g^3 + \frac{(N - 2)(N - 7)}{32\pi^3} g^4 + O(g^5).
\]

(6.33)

Therefore, one can see that there is a perturbative UV fixed point at a critical coupling \( g_* \) given by

\[
g_* = \frac{2\pi}{N - 2} \epsilon + \frac{2\pi}{(N - 2)^2} \epsilon^2 + \frac{\pi(N + 1)}{2(N - 2)^3} \epsilon^3 + O(\epsilon^4).
\]

(6.34)

\( ^7 \)This corrects a typo in \( g_* \) in [3] on page 59 (there it is denoted by \( u_c \)).
The scaling dimensions are found to be \([140]\)

\[
\Delta_\psi = \frac{1 + \epsilon}{2} + \frac{N - 1}{8\pi^2} g^2 - \frac{(N - 1)(N - 2)}{32\pi^3} g^3 + \mathcal{O}(g^4), \\
\Delta_\sigma = 1 + \epsilon - \frac{N - 1}{2\pi} g + \frac{N - 1}{8\pi^2} g^2 + \frac{(N - 1)(2N - 3)}{32\pi^3} g^3 + \mathcal{O}(g^4),
\]

(6.35)

where \(\sigma \sim \bar{\psi}\psi\). At the UV fixed point this gives

\[
\Delta_\psi = \frac{1}{2} + \frac{1}{2} \epsilon + \frac{N - 1}{4(N - 2)^2} \epsilon^2 - \frac{(N - 1)(N - 6)}{8(N - 2)^3} \epsilon^3 + \mathcal{O}(\epsilon^4), \\
\Delta_\sigma = 1 - \frac{1}{N - 2} \epsilon - \frac{N - 1}{2(N - 2)^2} \epsilon^2 + \frac{N(N - 1)}{4(N - 2)^3} \epsilon^3 + \mathcal{O}(\epsilon^4).
\]

(6.36, 6.37)

It is also not hard to determine the dimension

\[
\Delta_{\sigma^2} = d + \beta'(g_*) = 2 + \frac{1}{N - 2} \epsilon^2 + \frac{N - 3}{2(N - 2)^2} \epsilon^3 + \mathcal{O}(\epsilon^4).
\]

(6.38)

Let us now turn to the calculation of the free energy on \(S^{2+}\epsilon\). To order \(g^4\), we have

\[
F = NF_f - \frac{1}{2!} \left(\frac{g_0}{2}\right)^2 S_2 - \frac{1}{3!} \left(\frac{g_0}{2}\right)^3 S_3 - \frac{1}{4!} \left(\frac{g_0}{2}\right)^4 S_4 + b_0 \int d^d x \sqrt{g} R,
\]

(6.39)

where \(F_f\) is the free fermion contribution, derived as a function of \(d\) in \([78]\), and

\[
S_n = \int \prod_{i=1}^n dx_i \sqrt{g_{x_i}} \langle \psi^4(x_1)\ldots\psi^4(x_n) \rangle_{0}^{\text{conn}},
\]

(6.40)

with \(\psi^4 \equiv (\bar{\psi}_i\psi_i)^2\). Using the flat space fermion propagator in (6.21), and then performing a Weyl transformation to the sphere, we find
\[ S_2 = 2N(N-1)C_\psi^4 I_2(2d-2), \]
\[ S_3 = 8N(N-1)(N-2)C_\psi^6 I_3(2d-2), \] (6.41)

where the integral \( I_2(\Delta) \) is given in (A.12), and \( I_3(\Delta) \) denotes the integrated 3-point function [137, 138, 50]

\[
I_3(\Delta) = \int d^dx d^dy d^dz \sqrt{g_x} \sqrt{g_y} \sqrt{g_z} [s(x,y)s(y,z)s(z,x)]^{\Delta} = R^{3(d-\Delta)} \frac{8\pi^{\frac{3(1+d)}{2}} \Gamma(d - 3\Delta)}{\Gamma(d) \Gamma(\frac{1+d-\Delta}{2})^3}. \] (6.42)

For the integrated 4-point function we find

\[
S_4 = 24N(N-1)C_\psi^8 \int \prod_{i=1}^4 dx_i \sqrt{g_{x_i}} \left( \frac{2(N^2 - 3N + 4)}{s_{13}s_{14}s_{23}s_{24}} \right)^{2d-2} + \frac{2N}{s_{12}^{2d-4}s_{34}^{2d-4}s_{13}^{d}s_{14}^{d}s_{23}^{d}s_{24}^{d}}
+ \frac{2s_{13}^{d}s_{14}^{d}s_{23}^{d}s_{24}^{d}}{s_{12}^{3d-2}s_{13}^{d}s_{14}^{d}s_{23}^{d}s_{24}^{d}}
- 4(N-1) \frac{s_{12}^{2d-4}s_{13}^{d}s_{14}^{d}s_{23}^{d}s_{24}^{d}}{s_{13}^{d}s_{14}^{d}s_{23}^{d}s_{24}^{d}} \right), \] (6.43)

where \( s_{mn} \equiv s(x_m,x_n) \) is a chordal distance on a sphere. Using the methods described in [135, 136], we find

\[
S_4 = C_\psi^8 (2R)^{4(2-d)} \frac{2^{1-d} \pi^{d+1}}{\Gamma(d+1)} \frac{\Gamma(d+1)}{(\pi^{d/2})^3} 192N(N-1) \left( (N-2)^2 \left( -\frac{4}{\epsilon^2} + \frac{10}{\epsilon} - (25 + \frac{7\pi^2}{12}) \right) \right)
- 2(N-2) \left( \frac{3}{5\epsilon} - \frac{11}{5} \right) + 1 + \mathcal{O}(\epsilon) \right). \] (6.44)

After expressing the bare coupling \( g_0 \) in terms of the renormalized one

\[
g_0 = \mu^{-\epsilon} \left( g - \frac{N-2}{2\pi} \frac{g^2}{\epsilon} + \left( \frac{(N-2)^2}{4\pi^2\epsilon^2} - \frac{N-2}{8\pi^2\epsilon} \right) g^3 + \ldots \right), \] (6.45)
we find a surviving pole in $F$ of order $g^4/\epsilon$. This pole can be cancelled provided we renormalize the Euler density parameter as

$$b_0 = \mu^\epsilon \left( b - \frac{N}{48\pi \epsilon} + \frac{N(N - 1)(N - 2) g^4}{30(4\pi)^5} \right),$$

$$\beta_b = -\epsilon b + \frac{N}{48\pi} - \frac{N(N - 1)(N - 2) g^4}{6(4\pi)^5},$$  \hfill (6.46)

where the coupling independent term is due to the trace anomaly of the free fermion field, and we used $\int d^d x \sqrt{g} R = 8\pi + \mathcal{O}(\epsilon)$. In order to obtain the correct expression for $F$ at the UV fixed point in $d = 2 + \epsilon$, we should now set $g = g^*$ and $b = b^* = \frac{N}{48\pi \epsilon} - \frac{N(N - 1)(N - 2) g^4}{6(4\pi)^5}$. The contribution of the Euler term $\delta F_b = b_0 \int d^d x \sqrt{g} R$ at the fixed point (6.34) is then

$$\delta F_b = -\frac{N(N - 1)}{48(N - 2)^3} \epsilon + \mathcal{O}(\epsilon^4),$$  \hfill (6.47)

and putting this together with the contributions of $S_2$ and $S_3$, we find

$$F = NF_f + \frac{N(N - 1)}{24(N - 2)^2} \epsilon^2 - \frac{N(N - 1)(N - 3)}{16(N - 2)^3} \epsilon^3 + \mathcal{O}(\epsilon^4),$$  \hfill (6.48)

or, in terms of $\tilde{F} = -\sin(\pi d/2) F$:

$$\tilde{F} = N \tilde{F}_f + \frac{N(N - 1)\pi \epsilon^3}{48(N - 2)^2} - \frac{N(N - 1)(N - 3)\pi \epsilon^4}{32(N - 2)^3} + \mathcal{O}(\epsilon^5).$$  \hfill (6.49)

This result agrees with the one obtained in [135] using a conformal perturbation theory approach; this provides a non-trivial check on the procedure used here which involves the curvature terms.

Note that even though we had to compute $S_4$ to fix the renormalization of $b$, we cannot obtain $F$ to order $\epsilon^4$. That would require fixing $\beta_b$ to order $g^5$. 

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6.2.3 Large $N$ expansions

In this section we test the $4 - \epsilon$ and $2 + \epsilon$ expansions by comparing them with the known $1/N$ expansions \cite{109, 69, 141}. The general form of the large $N$ expansions of scaling dimensions in the GN model is\(^9\)

$$\Delta_\psi = \frac{d - 1}{2} + \frac{1}{N} \gamma_{\psi,1} + \frac{1}{N^2} \gamma_{\psi,2} + \mathcal{O}(N^{-3}),$$
$$\Delta_\sigma = 1 + \frac{1}{N} \gamma_{\sigma,1} + \frac{1}{N^2} \gamma_{\sigma,2} + \mathcal{O}(N^{-3}),$$
$$\Delta_{\sigma^2} = 2 + \frac{1}{N} \gamma_{\sigma^2,1} + \frac{1}{N^2} \gamma_{\sigma^2,2} + \mathcal{O}(N^{-3}), \quad (6.50)$$

where

$$\gamma_{\psi,1} = -\frac{\Gamma(d - 1)}{\Gamma(\frac{d}{2} - 1)\Gamma(1 - \frac{d}{2})\Gamma(\frac{d}{2} + 1)\Gamma(\frac{d}{2})}, \quad \gamma_{\psi,2} = 4\gamma_{\psi,1}^2 \left(\frac{(d - 1)\Psi(d)}{d - 2} + \frac{4d^2 - 6d + 2}{(d - 2)^2 d}\right), \quad (6.51)$$

$$\gamma_{\sigma,1} = -\frac{4d - 1}{d - 2} \gamma_{\psi,1},$$
$$\gamma_{\sigma,2} = -\gamma_{\psi,1}^2 \left(\frac{6d^2 \Theta(d)}{d - 2} + \frac{16(d - 1)^2 \Psi(d)}{(d - 2)^2} - \frac{4(d - 1)(d^4 - 2d^3 - 12d^2 + 20d - 8)}{(d - 2)^3 d}\right) \quad (6.52)$$

and

$$\gamma_{\sigma^2,1} = 4(d - 1)\gamma_{\psi,1},$$
$$\gamma_{\sigma^2,2} = -\frac{16d^2 \gamma_{\psi,1}^2}{d - 2} \left(\frac{1}{d - 4} - \frac{d^2}{2} + \frac{4}{(d - 4)^2} - \frac{6}{d - 2} - \frac{1}{d} - \frac{3}{2}\right)\Psi(d) + \frac{4}{(d - 4)^2 \gamma_{\psi,1}} \left[\frac{3d\Theta(d)}{8} \left(9 - d + \frac{12}{d - 4}\right) - \frac{(d - 3)d}{d - 4} (\Phi(d) + \Psi(d)^2) - \frac{5d}{2} + \frac{1}{2(d - 4)} - \frac{2}{(d - 4)^2}\right.$$
$$- \left.\frac{7}{d - 2} - \frac{4}{(d - 2)^2} + \frac{5}{2d} - 3 + \frac{d^2}{2} - \frac{1}{d^2}\right]. \quad (6.53)$$

\(^9\)The $1/N^3$ term in $\Delta_\psi$ may be found in \cite{142, 143}.\n
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In these equations we defined

\[
\Psi(d) = \psi(d - 1) - \psi(1) + \psi\left(2 - \frac{d}{2}\right) - \psi\left(\frac{d}{2}\right),
\]
\[
\Theta(d) = \psi'\left(\frac{d}{2}\right) - \psi'(1),
\]
\[
\Phi(d) = \psi'(d - 1) - \psi'\left(2 - \frac{d}{2}\right) - \psi'\left(\frac{d}{2}\right) + \psi'(1),
\]

(6.54)

where \(\psi(x) = \Gamma'(x)/\Gamma(x)\) denotes the digamma function.

In \(d = 3\), the above large \(N\) expansion of the scaling dimensions read

\[
\Delta_\psi = 1 + \frac{4}{3\pi^2 N} + \frac{896}{27\pi^4 N^2} = 1 + \frac{0.1351}{N} + \frac{0.3407}{N^2}, 
\]
\[
\Delta_\sigma = 1 - \frac{32}{3\pi^2 N} + \frac{9728 - 864\pi^2}{27\pi^4 N^2} = 1 - \frac{1.0808}{N} + \frac{0.4565}{N^2},
\]
\[
\Delta_{\sigma^2} = 2 + \frac{32}{3\pi^2 N} - \frac{64(632 + 27\pi^2)}{27\pi^4 N^2} = 2 + \frac{1.0808}{N} - \frac{21.864}{N^2}.
\]

(6.55, 6.56, 6.57)

For \(\tilde{F}\), the \(O(N^0)\) result can be obtained from the general formula [68, 78] for the change in \(F\) under a “double trace” deformation \(\delta S = g \int O_\Delta^2\), where \(O_\Delta\) is a scalar primary operator of dimension \(\Delta\). In terms of \(\tilde{F} = -\sin(\pi d/2)F\), the result is

\[
\delta \tilde{F} = \frac{1}{\Gamma(d + 1)} \int_0^{\Delta - \frac{d}{2}} du u \sin(\pi u) \Gamma \left(\frac{d}{2} + u\right) \Gamma \left(\frac{d}{2} - u\right).
\]

(6.58)

In the present case \(O_\Delta = \bar{\psi} \psi\), and so \(\Delta = d - 1\). Therefore,

\[
\tilde{F} = N\tilde{F}_f + \frac{1}{\Gamma(d + 1)} \int_0^{\frac{d}{2} - 1} du u \sin(\pi u) \Gamma \left(\frac{d}{2} + u\right) \Gamma \left(\frac{d}{2} - u\right) + O(1/N).
\]

(6.59)

The \(4 - \epsilon\) expansion of this agrees with the large \(N\) expansion of (6.31), i.e.

\[
\tilde{F} = N\tilde{F}_f + \tilde{F}_s - \left(\frac{\pi \epsilon^2}{96} + \frac{\pi \epsilon^3}{192}\right) + \left(\frac{\pi \epsilon^2}{16} - \frac{\pi \epsilon^3}{32}\right) \frac{1}{N} - \left(\frac{3\pi \epsilon^2}{8} - \frac{17\pi \epsilon^3}{32}\right) \frac{1}{N^2} + O(1/N^3),
\]

(6.60)
Similarly, the $2 + \epsilon$ expansion of (6.59) agrees with the large $N$ expansion of (6.49), i.e.

$$
\tilde{F} = N \tilde{F}_f + \left( \frac{\pi \epsilon^3}{48} - \frac{\pi \epsilon^4}{32} \right) + \frac{1}{N} \left( \frac{\pi \epsilon^3}{16} - \frac{\pi \epsilon^4}{16} \right) + \frac{1}{N^2} \left( \frac{\pi \epsilon^3}{6} - \frac{3\pi \epsilon^4}{32} \right) + \mathcal{O}(1/N^3) .
$$

(6.61)

For $C_T$, the relative $\mathcal{O}(1/N)$ correction to the answer for free fermions is given in [119]:

$$
C_T = NC_{T,f} \left( 1 + \frac{C_{T1}}{N} + \mathcal{O}(1/N^2) \right) ,
$$

$$
C_{T1} = -4\gamma_{\psi,1} \left( \frac{\Psi(d)}{d+2} + \frac{d-2}{(d-1)d(d+2)} \right) .
$$

(6.62)

Its expansion in $d = 4 - \epsilon$ can be seen to match (6.15), in particular reproducing the extra propagating scalar present in the GNY description.

### 6.2.4 Padé approximants

To obtain estimates for the CFT observables in $d = 3$, we will use “two-sided" Padé approximants that combine information from the $4 - \epsilon$ and $2 + \epsilon$ expansions. Namely, we consider the rational approximant $\text{Pade}_{[m,n]} = \frac{\sum_{i=1}^{m} a_i d^i}{\sum_{j=1}^{n} b_j d^j}$, where $2 < d < 4$ is the spacetime dimension, and we fix the coefficients $a_i, b_j$ so that its Taylor expansion near $d = 4$ and $d = 2$ agrees with the available perturbative results. Clearly, the “degree" $n + m$ of the approximant is bound by how many terms in the $\epsilon$-expansion are known. Such approximants may be derived for any finite $N$, and it is useful to compare their large $N$ behavior as a function of $d$ to the $1/N$-expansion results listed in the previous section. When several approximants $\text{Pade}_{[m,n]}$ are possible for the same quantity, we use such comparisons to large $N$ results to choose the one which appears to work best.
For $\Delta_\psi$, $\Delta_\sigma$ and $\Delta_{\sigma^2}$ we know the $4 - \epsilon$ expansion to order $\epsilon^2$, and the $2 + \epsilon$ expansion to order $\epsilon^3$. This allows to use Padé approximants with $m + n = 6$. For $\Delta_\psi$ and $\Delta_\sigma$, we find that $\text{Pade}_{[4,2]}$ has no poles in $2 < d < 4$ and for large $N$ is in good agreement with the results (6.51) and (6.52). For $\Delta_{\sigma^2}$, we perform the Padé on the critical exponent $\nu^{-1} = d - \Delta_{\sigma^2}$, and then translate to $\Delta_{\sigma^2}$ at the end. In this case, we find that $\text{Pade}_{[1,5]}$ is the only approximant with no poles, and it matches well to the large $N$ result. The resulting estimates in $d = 3$ for these scaling dimensions are given in Table 6.1. In Figure 6.2, we plot our 3d estimates for $\Delta_\psi$ and $\Delta_\sigma$, compared to the large $N$ curve obtained from (6.57) by eliminating $N$ to express $\Delta_\sigma$ as a function of $\Delta_\psi$. The $N = 1$ values, which correspond to the $\mathcal{N} = 1$ SUSY fixed point, are obtained in Section 6.4.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$3$</th>
<th>$4$</th>
<th>$5$</th>
<th>$6$</th>
<th>$8$</th>
<th>$20$</th>
<th>$100$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta_\psi$ (Pade$_{[4,2]}$)</td>
<td>1.066</td>
<td>1.048</td>
<td>1.037</td>
<td>1.029</td>
<td>1.021</td>
<td>1.007</td>
<td>1.0013</td>
</tr>
<tr>
<td>$\Delta_\sigma$ (Pade$_{[4,2]}$)</td>
<td>0.688</td>
<td>0.753</td>
<td>0.798</td>
<td>0.829</td>
<td>0.87</td>
<td>0.946</td>
<td>0.989</td>
</tr>
<tr>
<td>$\Delta_{\sigma^2}$ (Pade$_{[1,5]}$)</td>
<td>2.285</td>
<td>2.148</td>
<td>2.099</td>
<td>2.075</td>
<td>2.052</td>
<td>2.025</td>
<td>2.008</td>
</tr>
<tr>
<td>$F/(NF_f)$ (Pade$_{[4,4]}$)</td>
<td>1.091</td>
<td>1.060</td>
<td>1.044</td>
<td>1.034</td>
<td>1.024</td>
<td>1.008</td>
<td>1.0014</td>
</tr>
</tbody>
</table>

Table 6.1: Estimates of scaling dimensions and sphere free energy at the $d = 3$ interacting fixed point of the GN model.

For the sphere free energy of the interacting CFT, $\tilde{F}$, we find it convenient to perform the Padé approximation on the quantity

$$f(d) = \tilde{F} - N\tilde{F}_f,$$  \hspace{1cm} (6.63)

which is essentially the interacting part of the free energy in the GN description, but it includes the contribution of a free scalar from the GNY point of view. Using the results (6.31) and (6.49), we can use Padé approximants with $n + m = 8$, and we find that $\text{Pade}_{[4,4]}$ has no poles and agrees well with the large $N$ result (6.58). The resulting $d = 3$ estimates for $\tilde{F}$, normalized by the free fermion contribution $NF_f$,
Figure 6.2: Padé estimates in $d = 3$ of $\Delta_\sigma$ versus $\Delta_\psi$ for $N = 1, 2, 3, 4, 5, 6, 8, 20$, compared to the large $N$ results (6.57). The $N = 1$ value corresponds to the SUSY fixed point discussed in Section 6.4. The black dotted line is the SUSY relation $\Delta_\sigma = \Delta_\psi - 1/2$.

are given in Table 6.1 for a few values of $N$. In Figure 6.3, we also plot the result of
the constrained Padé approximants for $\tilde{F} - N\tilde{F}_f$ as a function of $2 < d < 4$ for a few
values of $N$, showing that they approach well the analytical large $N$ formula (6.58).

Figure 6.3: Padé estimates of $\tilde{F} - N\tilde{F}_f$ in $2 < d < 4$ compared to the large $N$ result (6.58).

We can use our estimates to make some tests of the $F$-theorem. In the GNY
description, we can flow to the critical theory from the free UV fixed point of $N$
fermions plus a scalar, while in the GN description one can flow to the critical theory
to the free fermions. Then, the $F$-theorem implies the inequalities

$$NF_f + F_s > F > NF_f.$$  \hspace{1cm} (6.64)

We verified that our estimates satisfy these inequalities for all values of $N$. As an example, for $N = 4$ we get $F_{GN}/(NF_f + F_s) \approx 0.93$ and $F_{GN}/(NF_f) \approx 1.06$. In the GNY description, we also see that we can flow to the critical GN point from the decoupled product of the Ising CFT and $N$ free fermions. This implies

$$NF_f + F_{\text{Ising}} > F.$$  \hspace{1cm} (6.65)

Using the estimates for $F_{\text{Ising}}$ derived in [135], we have checked that this inequality indeed holds. Using the Padé approximants as a function of $2 < d < 4$, we can also verify that both (6.64) and (6.65) are satisfied, in terms of $\tilde{F}$, in the whole range of $d$. This is in agreement with the “generalized $F$-theorem” [135].

Finally, we discuss $N = 2$, which is a special case where the $\beta$-function in $d = 2$ vanishes exactly; therefore, the theory has a line of fixed points. For $N = 2$ we cannot apply the strategy described above because the $2+\epsilon$ expansions (6.37) become singular. Directly in $d = 2$, the GN model is equivalent to the Thirring model for a single 2-component Dirac fermion which can be solved via bosonization and has a line of fixed points; the dimensions of $\psi$ and $\sigma = \bar{\psi}\psi$ depend on the interaction strength. Therefore, for these operators we can only perform the “one-sided” Padé$_{[1,1]}$ on the $4-\epsilon$ expansions. In $d = 2$ it is known that $\Delta_\sigma = 2$ (the operator is exactly marginal) and $c = 1$; we impose these boundary conditions on the Padé approximants. Then the results in $d = 3$ are

$$\Delta^{N=2}_\psi \approx 1.076, \quad \Delta^{N=2}_\sigma \approx 0.656, \quad \Delta^{N=2}_{\sigma^2} \approx 1.75.$$  \hspace{1cm} (6.66)
and for the sphere free energy

\[ F^{N=2} \approx 0.254 \approx 1.16(2F_f) \approx 0.9(2F_f + F_s) \],

in agreement with the \( F \)-theorem.

## 6.3 The Nambu-Jona-Lasinio-Yukawa Model

Using the results of [127, 128, 129, 130, 131], we have found the following \( \beta \)-functions for the NJLY model (6.5) up to two loops:

\[ \beta_1 = -\frac{\epsilon}{2}g_1 + \frac{1}{(4\pi)^2} \left( \frac{N}{2} + 2 \right) g_1^3 + \frac{1}{(4\pi)^4} \left( -\frac{8}{3}g_2^3 + \frac{1}{9}g_1 g_2^2 + (7 - 3N)g_1^5 \right), \]

\[ \beta_2 = -\epsilon g_2 + \frac{1}{(4\pi)^2} \left( \frac{10}{3}g_2^2 + 2Ng_2 g_1^2 - 12Ng_1^4 \right) - \frac{1}{(4\pi)^4} \left( \frac{20}{3}g_2^3 - 96Ng_1^6 - 2Ng_2 g_1^4 + \frac{10}{3}Ng_1^2 g_2^2 \right). \]

The one-loop terms above agree with the \( N_b = 1 \) case of the results in [113]. Solving (6.68), we find the following fixed point:

\[ \frac{(g_1^*)^2}{(4\pi)^2} = \frac{\epsilon}{N + 4} + \frac{-N^2 + 448N - 1096 + (N + 76)\sqrt{N^2 + 152N + 16}}{100(N + 4)^3} \epsilon^2 + \mathcal{O}(\epsilon^3), \]

\[ \frac{g_2^*}{(4\pi)^2} = \frac{3(\sqrt{N^2 + 152N + 16} - N + 4)}{20(N + 4)} \epsilon + \frac{9}{500(N + 4)^3 \sqrt{N^2 + 152N + 16}} \times \left( 3N^4 + 114N^3 + 764N^2 - 26192N + 1280 
\right.

\[ - (3N^3 - 114N^2 - 1932N - 320)\sqrt{N^2 + 152N + 16} ) \epsilon^2 + \mathcal{O}(\epsilon^3). \]

To get a positive solution for \( g_2^* \) we have picked the + sign for the square root. The other choice of sign gives another fixed point which is presumably unstable. In addition, there is clearly a fixed point with \( g_1^* = 0 \) and \( g_2^* \) corresponding to the \( O(2) \)
Wilson-Fisher fixed point. The solution (6.69) yields an IR stable fixed point for all values of \(N\).

The scaling dimensions of the fields are found to be

\[
\Delta_\psi = \frac{3-\epsilon}{2} + \frac{1}{(4\pi)^2} g_1^2 - \frac{1}{4(4\pi)^4} (3N+2)g_1^4,
\]
\[
\Delta_\phi = 1 - \frac{\epsilon}{2} + \frac{1}{2(4\pi)^2} Ng_1^2 + \frac{1}{(4\pi)^4} \left( \frac{1}{9}g_2^2 - \frac{3}{2}Ng_1^4 \right).
\]

(6.70)

At the fixed point (6.69) these give:

\[
\Delta_\psi = \frac{3}{2} - \frac{N+2}{2(N+4)} \epsilon + \frac{-76N^2 + 98N - 1296 + (N + 76)\sqrt{N^2 + 152N + 16}}{100(N+4)^3} \epsilon^2 + \mathcal{O}(\epsilon^3),
\]
\[
\Delta_\phi = 1 - \frac{2}{N+4} \epsilon + \frac{56N^2 - 498N + 16 + (19N + 4)\sqrt{N^2 + 152N + 16}}{50(N+4)^3} \epsilon^2 + \mathcal{O}(\epsilon^3).
\]

(6.71)

The NJL Y has two types of operators quadratic in the scalar fields: the \(U(1)\) invariant operator \(\phi \overline{\phi}\), and the charged operators \(\phi^2\) and \(\overline{\phi}^2\). The one-loop scaling dimension of \(\phi \overline{\phi}\) was determined in [113]. Using [131], we find up to two loops

\[
\Delta_{\phi \overline{\phi}} = d - 2 + \frac{4}{3(4\pi)^2} g_2 - \frac{4}{3(4\pi)^4} \left( g_2^2 + Ng_2g_1^2 \right) + 2\gamma_\phi,
\]
\[
\Delta_{\phi^2} = d - 2 + \frac{2}{3(4\pi)^2} g_2 - \frac{2}{3(4\pi)^4} \left( \frac{4}{3}g_2^2 + Ng_2g_1^2 - 6Ng_1^4 \right) + 2\gamma_\phi.
\]

(6.72)

At the fixed point this gives

\[
\Delta_{\phi \overline{\phi}} = 2 + \frac{\sqrt{N^2 + 152N + 16} - N - 16}{5(N+4)} \epsilon + \frac{1}{250(N+4)^3 \sqrt{N^2 + 152N + 16}}
\]
\[
\times \left( (17N^3 + 104N^2 + 1252N + 1120)\sqrt{N^2 + 152N + 16} - 17N^4 - 4646N^3 + 2304N^2 - 187712N + 4480 \right) \epsilon^2 + \mathcal{O}(\epsilon^3)
\]

(6.73)
and
\[ \Delta \phi^2 = 2 + \frac{\sqrt{N^2 + 152N + 16} - N - 36}{10(N + 4)} \epsilon + \frac{1}{125(N + 4)^2 \sqrt{N^2 + 152N + 16}} \times \left( (3N^3 + 571N^2 + 688N + 240)\sqrt{N^2 + 152N + 16} \\
- 3N^4 - 924N^3 + 7806N^2 - 47688N + 960 \right) \epsilon^2 + O(\epsilon^3). \tag{6.74} \]

The 4 \(-\epsilon\) expansion of \(C_T\) in the NJLY model proceeds similarly to that for the GNY model presented in [119] and reviewed in section 6.2. In the NJLY model there are two real scalar fields, and we need to replace \(T_\sigma\) by \(T_{\sigma_1} + T_{\sigma_2}\). It is not hard to check that each diagram contributing to the term \(\sim g_1^2\) picks up a factor of 2 compared to the GNY model, since each of the internal scalar lines can be either \(\phi_1\) or \(\phi_2\). Thus, we find
\[ C_T = NC_{T,f} + 2C_{T,s} - \frac{5N(g_1^*)^2}{6(4\pi)^2} = \left( \frac{N}{d} + \frac{2}{d-1} - \frac{5N \epsilon}{6(N + 4)} \right) \right). \tag{6.75} \]

### 6.3.1 Free energy on \(S^{4-\epsilon}\)

The calculation of \(F\) for the NJLY model follows the same steps as the one for the GNY model discussed earlier. The integrals are also nearly identical, except for combinatorics factors due to the fact that we have two scalar fields, one of which has \(i\gamma^5\) coupling.

The perturbative expansion of the free energy is given by
\[
F = NF_f + 2F_s - \frac{1}{2} g_{1,0}^2 \int d^d x d^d y \sqrt{g_x} \sqrt{g_y} \langle O_1(x) O_1(y) \rangle_0 \\
- \frac{1}{2! (4!)^2} g_{2,0}^2 \int d^d x d^d y \sqrt{g_x} \sqrt{g_y} \langle O_2(x) O_2 \rangle_0 \\
- \frac{1}{4!} g_{1,0}^4 \int d^d x d^d y d^d z d^d w \sqrt{g_x} \sqrt{g_y} \sqrt{g_z} \sqrt{g_w} \langle O_1(x) O_1(y) O_1(z) O_1(w) \rangle_0 + \delta F_b, \tag{6.76} \]
where we defined the operators $\mathcal{O}_1 = \bar{\psi}_j(\phi_1 + i\gamma_5\phi_2)\psi^j$ and $\mathcal{O}_2 = (\phi_1^2 + \phi_2^2)$, and $\delta F_b = b_0 \int d^4x E$ is the Euler term. Evaluating the correlation functions above, we find

$$F = NF_f + 2F_s - \frac{1}{2} \frac{g_{1,0}^2}{2} NC_\phi C_\psi^2 I_2 \left( \frac{3}{2} d - 2 \right) + \frac{1}{2} \frac{g_{2,0}^2}{(4\pi)^2} 64g_{2,0}^2 C_\phi^4 I_2 (2d - 4) + \frac{1}{4!} g_{1,0}^4 I_4 + \delta F_b,$$

(6.77)

where $I_2$ is given in (A.12) and $I_4$ corresponds to the integrated 4-point function of $\mathcal{O}_1$, for which we find

$$I_4 = \frac{N(N + 4)}{(4\pi)^4 \epsilon} + \frac{N(19N + 72 + 6(N + 4)(\gamma + \log(4\pi R^2)))}{6(4\pi)^4} + \mathcal{O}(\epsilon).$$

(6.78)

Now replacing the bare couplings with the renormalized ones

$$g_{1,0} = \mu_0^2 \left( g_1 + \frac{N + 4}{32\pi^2} \frac{g_1^3}{\epsilon} + \ldots \right), \quad g_{2,0} = \mu_0^2 \left( g_2 + \ldots \right)$$

(6.79)

we find that all poles cancel as they should. As explained in the GNY and GN calculations, in order to obtain the correct expression for $F$ at the IR fixed point in $d = 4 - \epsilon$ we need to include the effect of the Euler term. Using an improved version of the result from [130, 134], adapted to the presence of $\gamma_5$ in the vertices, we get:

$$\beta_b = \epsilon b - \frac{11N/4 + 2}{360(4\pi)^2} - \frac{1}{(4\pi)^8} \frac{1296}{1296} \left( N g_1^2 (24g_1^2 g_2 - g_2^2 + 9g_1^4 (3N - 7)) \right).$$

(6.80)

From this we can solve for the fixed point value $b_*$ with the couplings given in (6.69), and we find that the Euler term contributes

$$\delta F_b = \frac{N(-1096 + 76\sqrt{N^2 + 152N + 16} + N(448 - N + \sqrt{N^2 + 152N + 16}))}{7200(N + 4)^3} \epsilon^2.$$

(6.81)
Putting this together with the integrals in (6.77), we finally find in terms of \( \tilde{F} \):

\[
\tilde{F} = N\tilde{F}_f + 2\tilde{F}_s - \frac{N\pi\epsilon^2}{48(N+4)} - \frac{1}{14400(N+4)^3} \left( 149N^3 + 2372N^2 + 1312N + 64 \right. \\
\left. + \left( N^2 + 152N + 16 \right) \sqrt{N^2 + 152N + 16} \pi\epsilon^3 + O(\epsilon^4) \right). \quad (6.82)
\]

### 6.3.2 \( 2 + \epsilon \) expansions

In this section we consider the theory in \( 2 + \epsilon \) dimensions with 4-fermion interactions which respect the \( U(1) \) chiral symmetry. We begin with the action in Euclidean space of the form [106]

\[
S_{\text{SPV}} = -\int d^d x \left( \bar{\psi}_i \gamma^i \psi^i + \frac{1}{2} g_S (\bar{\psi}_i \psi^i)^2 + \frac{1}{2} g_P (\bar{\psi}_i \gamma_5 \psi^i)^2 + \frac{1}{2} g_V (\bar{\psi}_i \gamma_\mu \psi^i)^2 \right), \quad (6.83)
\]

where \( i = 1, \ldots 2N_f \) is the number of two-component spinors, and \( \gamma_0 = \sigma_1, \gamma_1 = \sigma_2 \) and \( \gamma_5 = -i\gamma_0 \gamma_1 = \sigma_3 \). For \( g_V = 0 \) and \( g_S = -g_P \) this reduces to the well-known chiral Gross-Neveu model [25] in \( d = 2 \). However, as we will see below, for our purposes it is not consistent to set \( g_V = 0 \) – the corresponding operator respects the \( U(1) \) chiral symmetry and gets induced.

The one-loop beta-functions and anomalous dimension of the \( \psi \) field were found in [106] using MS scheme and read

\[
\begin{align*}
\beta_S &= \epsilon g_S - \frac{1}{\pi} \left( \frac{1}{2} (N-2) g_S^2 - g_P g_S - 2 g_V (g_S + g_P) \right) + \ldots, \\
\beta_P &= \epsilon g_P + \frac{1}{\pi} \left( \frac{1}{2} (N-2) g_P^2 - g_P g_S + 2 g_V (g_S + g_P) \right) + \ldots, \\
\beta_V &= \epsilon g_V + \frac{1}{\pi} g_P g_S + \ldots, \\
\end{align*}
\]

\[ 137 \]
\[ \Delta \psi = \frac{1}{2} + \frac{1}{2} \epsilon + \frac{N}{4(2\pi)^2} \left( g_P^2 + g_S^2 + 2g_V^2 \right) - \frac{1}{4(2\pi)^2} \left( 4g_V(g_S - g_P) + (gs + g_P)^2 \right). \]  

(6.85)

We note that at the leading order in the $2 + \epsilon$ expansion the evanescent operators do not appear \cite{106, 144, 145}. One of the UV fixed points of (6.84) is$^{10}$

\[ g_S^* = -g_P^* = \frac{N}{2} g_V^* = \frac{2\pi \epsilon}{N}, \]  

(6.86)

which corresponds to the $SU(2N_f)$ Thirring model \cite{106, 146}. Indeed using the relation for the $SU(2N_f)$ generators $(T^a)^b_i (T^a)^c_i = \frac{1}{2} (\delta^b_i \delta^c_i - \frac{2}{N} \delta^b_i \delta^c_i)$ and the Fierz identity in $d = 2$ $(\gamma_\mu)_{\alpha\beta} (\gamma_\mu)_{\gamma\delta} = \delta_{\alpha\delta} \delta_{\beta\gamma} - (\gamma_5)_{\alpha\delta} (\gamma_5)_{\beta\gamma}$ one finds

\[ (\bar{\psi} \psi)^2 - (\bar{\psi} \gamma_5 \psi)^2 + \frac{2}{N} (\bar{\psi} \gamma_\mu \psi)^2 = -2 (\bar{\psi} \gamma_\mu T^a \psi)^2. \]  

(6.87)

So the action for this model is

\[ S_{SU(2N_f)Thirring} = -\int d^4x \left( \bar{\psi} i \gamma^\mu \partial_\mu \psi + \frac{1}{2} g \left( (\bar{\psi} i \gamma^\mu \psi)^2 - (\bar{\psi} i \gamma_5 \psi)^2 + \frac{2}{N} (\bar{\psi} i \gamma_\mu \psi)^2 \right) \right), \]  

(6.88)

with the beta-function and anomalous dimension

\[ \beta_g = \epsilon g - \frac{N}{2\pi} g^2 + \mathcal{O}(g^3), \quad \Delta \psi = \frac{1}{2} + \frac{1}{2} \epsilon + \frac{N^2 - 4}{8\pi^2 N} g^2 + \mathcal{O}(g^3). \]  

(6.89)

It is plausible that this model is the continuation of the NJL Y model (6.5) to $d = 2 + \epsilon$.

One finds that the UV fixed point is $g_\ast = 2\pi \epsilon / N$, and the critical anomalous dimension

$^{10}$Equations (6.84) have four different non-trivial fixed points. Two of them correspond to the GN and $SU(2N_f)$ Thirring models.
reads

\[ \Delta_\psi = \frac{1}{2} + \frac{1}{2} \epsilon + \frac{1}{2} \left( \frac{1}{N} - \frac{4}{N^3} \right) \epsilon^2 + \mathcal{O}(\epsilon^3). \] (6.90)

Also one finds that the dimension of the quartic operator is

\[ \Delta_{\phi\bar{\phi}} = 2 + \epsilon + \beta'(g_\ast) = 2 + \mathcal{O}(\epsilon^2). \] (6.91)

We can also calculate the free energy of this model. To order \( g^2 \), we have:

\[ F = NF_f - \frac{1}{2!} \left( \frac{g_0}{2} \right)^2 S_2 + \mathcal{O}(g_0^3), \] (6.92)

where \( F_f \) is the free fermion contribution and

\[ S_2 = \int dx_1 dx_2 \sqrt{g_{x_1}} \sqrt{g_{x_2}} \langle O(x_1)O(x_2) \rangle_0^{\text{conn}}, \] (6.93)

where, under the Thirring description (6.87), \( \mathcal{O} = -2(\bar{\psi}\gamma_\mu T^a \psi)^2 \). Going through the combinatorics, we find

\[ S_2 = 4(N^2 - 4)C_\psi^4 I_2(2d - 2). \] (6.94)

Evaluating this in \( d = 2 + \epsilon \) and plugging in the fixed point value (6.86) we finally find

\[ \tilde{F} = N\tilde{F}_f + \frac{(N^2 - 4)\pi\epsilon^3}{24N^2} + \mathcal{O}(\epsilon^4). \] (6.95)
6.3.3 Large $N$ expansions

For the NJL model, the $1/N$ expansions of operator dimensions again assume the general form (6.50), where now [110, 147]

$$\gamma_{\psi,1} = -\frac{2\Gamma(d-1)}{\Gamma(d/2-1)\Gamma(1-d/2)\Gamma(d/2+1)\Gamma(d/2)}, \quad \gamma_{\psi,2} = 2\gamma_{\psi,1}^2 \left( \Psi(d) + \frac{4}{d-2} + \frac{1}{d} \right),$$

(6.96)

$$\gamma_{\phi,1} = -2\gamma_{\psi,1}, \quad \gamma_{\phi,2} = -2\gamma_{\psi,2} + \frac{4d^2(d^2 - 5d + 7)}{(d-2)^3} \gamma_{\psi,1}^2,$$

(6.97)

and

$$\gamma_{\phi,1} = 4(d - 1)\gamma_{\psi,1},$$

(6.98)

$$\gamma_{\phi,2} = -8\gamma_{\psi,1}^2 \left( \frac{3d^2(3d - 8)\Theta(d)}{4(d-4)(d-2)} - \frac{(d-3)d^2(\Phi(d) + \Psi^2(d))}{(d-4)(d-2)} + \frac{4(d-2)d}{(d-4)^2\gamma_{\psi,1}} + \frac{4}{d-2} + \frac{1}{d} \right) + \frac{(d-4)d^2}{d-2} - \frac{(d-3)^2d^2}{(d-4)^2(d-2)} + \Psi(d) \left( \frac{(d-3)d^2(d^2 - 8d + 20)}{(d-4)^2(d-2)^2} - d^2 + 1 \right).$$

(6.99)

There is a considerable similarity between these results and the corresponding results for the GN model. For example, $\gamma_{\psi,1}$ in the NJL model is 2 times that in the GN model. This is because the NJL lagrangian (6.2) contains two separate double-trace operators, and hence there are two auxiliary fields, $\phi_1$ and $\phi_2$, that couple to the fermions. A similar factor of 2 appears in various other quantities.
When (6.97) and (6.99) are expanded in $d = 4 - \epsilon$ and $d = 2 + \epsilon$, they are consistent with our $\epsilon$-expansion results. Setting $d = 3$ in these equations gives \cite{110, 147}

\[
\begin{align*}
\Delta_\psi &= 1 + \frac{8}{3\pi^2 N} + \frac{1280}{27\pi^4 N^2} = 1 + \frac{0.2702}{N} + \frac{0.4867}{N^2}, \\
\Delta_\phi &= 1 - \frac{16}{3\pi^2 N} + \frac{4352}{27\pi^4 N^2} = 1 - \frac{0.5404}{N} + \frac{1.6547}{N^2}, \\
\Delta_\phi \bar{\phi} &= 2 + \frac{64}{3\pi^2 N} - \frac{128(364 + 27\pi^2)}{27\pi^4 N^2} = 2 + \frac{2.1615}{N} - \frac{30.684}{N^2}.
\end{align*}
\]

(6.100)

Turning to the sphere free energy, we can obtain the $\mathcal{O}(N^0)$ result for $\tilde{F}$ by simply doubling the $\delta \tilde{F}$ from eq. (3.8) of \cite{78}. Therefore, we have

\[
\tilde{F} = N \tilde{F}_f + \frac{2}{\Gamma(d + 1)} \int_0^{d-1} du \sin(\pi u) \Gamma \left(\frac{d}{2} + u\right) \Gamma \left(\frac{d}{2} - u\right) + \mathcal{O}(1/N). \tag{6.101}
\]

(6.101)

The $4 - \epsilon$ expansion of this agrees with the large $N$ expansion of (6.82), i.e.

\[
\tilde{F} = N \tilde{F}_f + 2 \tilde{F}_s - \left(\frac{\pi \epsilon^2}{48} + \frac{\pi \epsilon^3}{96}\right) + \left(\frac{\pi \epsilon^2}{12} - \frac{\pi \epsilon^3}{18}\right) \frac{1}{N} - \left(\frac{\pi \epsilon^2}{3} - \frac{17 \pi \epsilon^3}{36}\right) \frac{1}{N^2} + \mathcal{O}(1/N^3),
\]

(6.102)

and its $2 + \epsilon$ expansion agrees with the large $N$ expansion of (6.95), which yields

\[
\tilde{F} = N \tilde{F}_f + \frac{\pi \epsilon^3}{24} - \frac{\pi \epsilon^3}{6N^2} + \mathcal{O}(1/N^4). \tag{6.103}
\]

For $C_T$, the presence of the extra scalar field compared to the GN case \cite{119} again poses no difficulty. After some simple exercise commuting $\gamma^5$, we conclude that all the diagrams in \cite{78} contributing to $C_T$ in the GN model should receive a factor of 2

\footnote{The form of (6.91) agrees with the large $N$ result (6.99), but we haven’t compared the coefficient of $\epsilon^2$.}
due to the presence of two scalar fields. Hence, we find for the NJL model

\[
C_T = N C_{T,f} \left( 1 + \frac{C_{T1}}{N} + \mathcal{O}(1/N^2) \right),
\]

\[
C_{T1} = -4 \gamma_{\psi,1} \left( \frac{\Psi(d)}{d+2} + \frac{d-2}{(d-1)d(d+2)} \right),
\]  

(6.104)

i.e. the correction \(C_{T1}\) is just twice the corresponding term in the GN model (note that \(\gamma_{\psi,1}\) in the NJL model is twice that of the GN model). This result is, in particular, consistent with the fact that in the limit \(d \to 4\) we expect \(C_{T1}\) to reproduce the contribution of two free scalar fields. Its \(4-\epsilon\) expansion can be also seen to agree with (6.75).

### 6.3.4 Padé approximants

Following the same methods as described in Section 6.2.4, we now use the \(4-\epsilon\) and \(2+\epsilon\) expansions\(^{12}\) derived in the previous sections to obtain rational approximants of scaling dimensions and sphere free energy at the NJL fixed point in \(2 < d < 4\), for \(N > 2\) (the case \(N = 2\), which displays the emergent supersymmetry, is treated separately in Section 6.4). The results in \(d = 3\) are given in Table 6.2, indicating which approximants was chosen in each case. In figure 6.4, we also plot our 3d estimated for \(\Delta_\phi\) and \(\Delta_{\phi_0}\), compared to the large \(N\) results. From the expression for \(\Delta_{\phi_0}\) in (6.100), it appears that the expansion does not converge well already at \(\mathcal{O}(1/N^2)\), therefore we have just used (6.100) to order \(1/N\) to produce the plot below.

For the sphere free energy \(\tilde{F}\), we again find it convenient to perform the Padé approximation on the quantity \(f(d) = \tilde{F} - N \tilde{F}_f\), which corresponds to the interacting part of the NJLY free energy plus the contribution of two free scalars. In Figure 6.3, we plot the resulting Padé approximants as a function of \(2 < d < 4\) for a few values of \(N\), showing that they approach well the analytical large \(N\) formula (6.101).

\(^{12}\)For \(\Delta_{\phi_0}\), we use the boundary condition \(\Delta_{\phi_0} = 1 + \mathcal{O}(\epsilon)\) in \(d = 2 + \epsilon\).
Table 6.2: Estimates of scaling dimensions and sphere free energy at the $d = 3$ interacting fixed point of the NJL model.

<table>
<thead>
<tr>
<th>$N$</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>20</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta \varphi$ (Pade$_{[3,2]}$)</td>
<td>1.074</td>
<td>1.054</td>
<td>1.041</td>
<td>1.033</td>
<td>1.027</td>
<td>1.016</td>
<td>1.0029</td>
</tr>
<tr>
<td>$\Delta \varphi$ (Pade$_{[2,1]}$)</td>
<td>0.807</td>
<td>0.870</td>
<td>0.903</td>
<td>0.923</td>
<td>0.937</td>
<td>0.962</td>
<td>0.992</td>
</tr>
<tr>
<td>$\Delta \varphi$ (Pade$_{[3,1]}$)</td>
<td>2.018</td>
<td>2.041</td>
<td>2.055</td>
<td>2.062</td>
<td>2.064</td>
<td>2.06</td>
<td>2.022</td>
</tr>
<tr>
<td>$F/(NF_f)$ (Pade$_{[5,2]}$)</td>
<td>1.109</td>
<td>1.064</td>
<td>1.045</td>
<td>1.034</td>
<td>1.028</td>
<td>1.016</td>
<td>1.0029</td>
</tr>
</tbody>
</table>

Figure 6.4: Padé estimates in $d = 3$ of $\Delta \sigma$ versus $\Delta \psi$ for $N = 4, 6, 8, 10, 12, 20, 100$, compared to the large $N$ results (6.100).

Figure 6.5: Padé estimates of $\tilde{F} - NF_f$ in $2 < d < 4$ compared to the large $N$ result (6.101).

We can again use our estimates to test the $d = 3$ $F$-theorem and its proposed generalization in $2 < d < 4$ in terms of $\tilde{F}$. We find that in the whole range of
dimensions, the inequalities

\[ N\tilde{F}_f + 2\tilde{F}_s > \tilde{F} > N\tilde{F}_f \]  \hspace{1cm} (6.105)

hold, in accordance with the conjectured generalized \( F \)-theorem [78, 135]. Using the results in [135] for the free energy of the \( O(2) \) Wilson-Fisher model, we have also checked that

\[ N\tilde{F}_f + \tilde{F}_{O(2)} > \tilde{F} , \]  \hspace{1cm} (6.106)

which is consistent with the fact that the theory can flow from the fixed point consisting of free fermions decoupled from the \( O(2) \) model, to the IR stable NJL fixed point.

6.4 Models with Emergent Supersymmetry

6.4.1 Theory with 4 supercharges

A well-known supersymmetric theory with 4 supercharges is the Wess-Zumino model of a single chiral superfield \( \Phi \) with superpotential \( W \sim \lambda \Phi^3 \). In \( d = 4 \) the model is classically conformally invariant, but it has a non-vanishing beta function and is expected to be trivial in the IR. Continuation of this model to lower dimensions (defined such that the number of supercharges is fixed in \( 2 \leq d \leq 4 \)) was discussed in [78, 148, 149]. In \( d = 3 \), one finds the \( \mathcal{N} = 2 \) theory of a single chiral superfield (a complex scalar and a 2-component Dirac fermion) with cubic superpotential, which flows to a non-trivial CFT in the IR [150]. In \( d = 2 \), the model matches onto the \((2,2)\) supersymmetric CFT with \( c = 1 \), which is the \( k = 1 \) member of the superconformal discrete series with \( c = \frac{3k}{k+2} \).
The component Lagrangian of the WZ model in 4d reads

\[ \mathcal{L}_{\text{WZ}} = \frac{1}{2} (\partial_\mu \phi_1)^2 + \frac{1}{2} (\partial_\mu \phi_2)^2 + \frac{1}{2} \bar{\psi} \gamma \phi \psi + \frac{\lambda}{2 \sqrt{2}} \bar{\psi} (\phi_1 + i \gamma_5 \phi_2) \psi + \frac{\lambda^2}{16} (\bar{\phi} \phi)^2, \]  

(6.107)

where \( \psi \) is a 4-component Majorana fermion and \( \phi = \phi_1 + i \phi_2 \). The beta function of the model in \( d = 4 - \epsilon \) is [151]

\[ \beta_{\text{WZ}} = -\frac{\epsilon}{2} \lambda + \frac{3}{2} \frac{\lambda^3}{(4\pi)^2} - \frac{3}{2} \frac{\lambda^5}{(4\pi)^4} + \frac{15 + 36 \zeta(3)}{8} \frac{\lambda^7}{(4\pi)^6} + O(\lambda^9), \]  

(6.108)

from which one finds an IR fixed point with \( \lambda_*^2 = \frac{16 \epsilon^2}{3} + O(\epsilon^2) \). The dimension of the chiral operator \( \phi \) at the fixed point is determined by its \( R \)-charge to be

\[ \Delta_\phi = \frac{d-1}{2} R_\phi = \frac{d-1}{3}. \]  

(6.109)

One also has the exact result \( \Delta_{\phi^2} = \Delta_\phi + 1 \), since the operator \( \phi^2 \) is obtained from \( \phi \) by acting twice with the supercharges (this is because, due to cubic superpotential, one has the relation \( \Phi^2 = 0 \) in the chiral ring). It also follows from supersymmetry [121, 123] that the dimension of the operator \( \phi \bar{\phi} \) at the fixed point is given by

\[ \Delta_{\phi \bar{\phi}} = d - 2 + \beta'(\lambda_*) = 2 - \frac{1}{3} \epsilon^2 + \frac{1 + 12 \zeta(3)}{18} \epsilon^3 + O(\epsilon^4), \]  

(6.110)

which agrees with the explicit three-loop calculation of [123]. In performing Padé extrapolation of this result to \( d = 3 \), we have found that Padé\[^{1,2}\] and Padé\[^{2,1}\] give answers close to each other. Their average is \( \approx 1.909 \), which is very close to the value \( \approx 1.91 \) reported using numerical bootstrap studies [148, 149]. We can also take into account the fact that in \( d = 2 \) the dimension of \( \phi \bar{\phi} \) should approach 2. Since it also approaches 2 in \( d = 4 \), it is not a monotonic function of \( d \), which makes Padé extrapolation difficult. If we instead perform a “two-sided” extrapolation of
\[ \nu^{-1} = d - \Delta_{\phi\bar{\phi}}, \text{ and then return to } \Delta_{\phi\bar{\phi}} \text{ in } d = 3, \text{ then we find } \approx 1.94. \text{ This is somewhat further from the numerical bootstrap estimate.} \]

Since in \( d = 4 \) the WZ model includes a complex scalar and a 4-component Majorana fermion (i.e. one half of a Dirac fermion), one would expect it to correspond to \( N_f = 1/2 \) and \( N = 2 \) in the NJLY model [121, 122]. Note that the NJLY Lagrangian (6.5), specialized to the case of a single Majorana fermion, coincides with the WZ Lagrangian (6.107) provided\(^\text{13}\)

\[ 3g_1^2 = g_2 = \frac{3}{2} \lambda^2. \quad (6.111) \]

Indeed, setting \( N = 2 \) in the result for the fixed point couplings (6.8), we find:

\[
\frac{(g_1^*)^2}{(4\pi)^2} = \frac{1}{6} \epsilon + \frac{1}{18} \epsilon^2 + O(\epsilon^3),
\frac{g_2^*}{(4\pi)^2} = \frac{1}{2} \epsilon + \frac{1}{6} \epsilon^2 + O(\epsilon^3). \quad (6.112)
\]

This is precisely consistent with the relation (6.111), and gives evidence of the emergent supersymmetry in the \( N = 2 \) NJLY model. Note that for this value of \( N \), the chiral \( U(1) \) symmetry of the NJLY model becomes the \( U(1)_R \)-symmetry of the WZ model.\(^\text{14}\) Further evidence can be found by setting \( N = 2 \) in the \( 4 - \epsilon \) expansions of the operator dimensions (6.70) and (6.74), which give

\[ \Delta_\psi = \frac{3}{2} - \frac{\epsilon}{3}, \quad \Delta_\phi = 1 - \frac{\epsilon}{3}, \quad \Delta_{\phi^2} = 2 - \frac{\epsilon}{3}, \quad (6.113) \]

in agreement with the supersymmetry. In particular, the fact that the \( O(\epsilon^2) \) terms vanish is consistent with the exact result (6.109), and we also see \( \Delta_{\phi^2} = \Delta_\phi + 1 \) as

\(^\text{13}\)One should rescale \( \psi \to \psi/\sqrt{2} \) in (6.5) to get a canonical kinetic term when the fermion is Majorana.

\(^\text{14}\)For a gauge theory in \( 1 + 1 \) dimensions which exhibits emergent supersymmetry, see [152]. In that case a global \( U(1) \) symmetry turned into the \( U(1)_R \)-symmetry of the \((2,2)\) supersymmetric IR CFT.
discussed above. Furthermore, setting $N = 2$ in (6.73), we find $\Delta_{\phi^5} = 2 - \epsilon^2/3 + O(\epsilon^3)$, in agreement with the result (6.110) in the WZ model.

It is also interesting to look at the sphere free energy. Setting $N = 2$ in the $4 - \epsilon$ expansion (6.82) of $\tilde{F}$, we get

$$\tilde{F}_{N=2} = 2\tilde{F}_s + 2\tilde{F}_f - \frac{\pi\epsilon^2}{144} - \frac{\pi\epsilon^3}{162} + O(\epsilon^4).$$

(6.114)

This precisely agrees with the expansion of (5.23) in [78], which was derived using a proposal for supersymmetric localization in continuous dimension. We note that the curvature term (6.81) contributes at $O(\epsilon^3)$ order to $\tilde{F}$: $\delta \tilde{F}_b = -\frac{\pi\epsilon^4}{1296}$. This contribution is crucial for agreement with [78]. Thus, (6.114) provides a nice perturbative test of the exact formula for $\tilde{F}$ as a function of $d$ proposed in [78] for the Wess-Zumino model.

In $d = 3$, the result obtained from localization [153] yields $F_{W=\Phi^3} \approx 0.290791$. In [135], the value of $F$ for the $O(2)$ Wilson-Fisher fixed point in $d = 3$ was estimated to be $F_{O(2)} \approx 0.124$. Using also the value $F_f = \frac{1}{8} \log(2) + \frac{3\zeta(3)}{16\pi^2}$ in $d = 3$ [50], we see that

$$2F_f + F_{O(2)} \approx 0.343 > F_{W=\Phi^3},$$

(6.115)

in agreement with the RG flow depicted in Figure 6.1.

Finally, we discuss $C_T$ of the $\mathcal{N} = 2$ SCFT with superpotential $W \sim \lambda\Phi^3$. Its exact value in $d = 3$ has been determined using the supersymmetric localization [64, 154]:

$$\frac{C_T}{C_T^{\text{UV}}} = \frac{16(16\pi - 9\sqrt{3})}{243\pi} \approx 0.7268,$$

(6.116)

where $C_T^{\text{UV}} = 4C_{T,s}$ is the value for the free UV theory of two scalars and one two-component Dirac fermion. Let us compare this with a Padé extrapolation of the ratio
\( C_T / C_{T,s} \) using the boundary conditions

\[
\frac{C_T}{C_{T,s}} = \begin{cases} 
1 & \text{in } d = 2, \\
5 - \frac{11}{6} \epsilon + \mathcal{O}(\epsilon^2) & \text{in } d = 4 - \epsilon ,
\end{cases} 
\tag{6.117}
\]

where the \( 4 - \epsilon \) expansion was obtained by setting \( N = 2 \) in (6.75). The Padé approximant with these boundary conditions is

\[
\frac{C_T}{C_{T,s}} = \frac{49d - 76}{d + 20} , 
\tag{6.118}
\]

which in \( d = 3 \) gives \( C_T / C_{T,s} = 71/23 \approx 3.087 \). Comparing to the free UV CFT, this result implies \( C_{T,\text{IR}}^{d=3} / C_T^{d=4} \approx 0.77 \). This is not far from the exact result (6.116), demonstrating that the Padé approach works quite well. It would be useful to know the next order in the \( 4 - \epsilon \) expansion, which may improve the agreement.

### 6.4.2 Theory with 2 supercharges

It has been suggested that in \( d = 3 \) there exists a minimal \( \mathcal{N} = 1 \) superconformal theory containing a 2-component Majorana fermion \( \psi \) \([121, 124, 125, 118, 126]\). This theory must also contain a pseudoscalar operator \( \sigma \), whose scaling dimension is related to that of \( \psi \) by the supersymmetry,

\[
\Delta_{\sigma} = \Delta_{\psi} - \frac{1}{2} . 
\tag{6.119}
\]

Some evidence for the existence of this \( \mathcal{N} = 1 \) supersymmetric CFT was found using the conformal bootstrap \([118]\).

To describe the theory in \( d = 3 \), one can write down the Lagrangian \([121, 124, 125, 118]\)

\[
\mathcal{L}_{\mathcal{N}=1} = \frac{1}{2} (\partial_{\mu} \sigma)^2 + \frac{1}{2} \bar{\psi} \slashed{D} \psi + \frac{\lambda}{2} \sigma \bar{\psi} \psi + \frac{\lambda^2}{8} \sigma^4 . 
\tag{6.120}
\]
This model has $\mathcal{N} = 1$ supersymmetry in $d = 3$; the field content can be packaged in the real superfield $\Sigma = \sigma + \bar{\theta}\psi + \frac{1}{2}\bar{\theta}\theta f$, and the interactions follow from the cubic superpotential $W \sim \lambda \Sigma^3$. It is natural to expect that this model flows to a non-trivial $\mathcal{N} = 1$ SCFT in the IR. Note that the theory cannot be described as the UV fixed point of a lagrangian for a Majorana fermion with quartic interaction, because the term $(\bar{\psi}\psi)^2$ vanishes for a 2-component spinor.

The theory (6.120) is super-renormalizable in $d = 3$, and one may attempt its $4 - \epsilon$ expansion [121]. To formulate a Yukawa theory in $d = 4$, one strictly speaking needs a 4-component Majorana fermion, which corresponds to the GNY model with $N = 2$. However, the GNY description may be formally continued to $N = 1$. A sign of the simplification that occurs for this value is that $\sqrt{N^2 + 132N + 36}$, which appears in the $4 - \epsilon$ expansions (6.10), (6.12), equals 13 for $N = 1$. For this value of $N$, we find that the fixed point couplings in (6.8) become

$$\frac{(g_1^*)^2}{(4\pi)^2} = \frac{1}{7}\epsilon + \frac{3}{49}\epsilon^2 + \mathcal{O}(\epsilon^3),$$
$$\frac{g_2^2}{(4\pi)^2} = \frac{3}{7}\epsilon + \frac{9}{49}\epsilon^2 + \mathcal{O}(\epsilon^3).$$

(6.121)

This is consistent with the exact relation $3g_1^* = g_2$ in the SUSY model. Indeed the GNY Lagrangian (6.4), formally applied to the case of a single 2-component Majorana fermion, coincides with (6.120) when $3g_1^* = g_2 = 3\lambda^2$. The result (6.121) gives a two-loop evidence that the non-supersymmetric GNY model with $N = 1$ flows at low energies to a supersymmetric fixed point.

In [121] it was found using one-loop calculations that $\Delta_\sigma = 1 - \frac{3\epsilon}{7} - \Delta_\psi - \frac{1}{2}$. Let us check that the supersymmetry relation (6.119) continues to hold at order $\epsilon^2$. Using

\footnote{To obtain the component Lagrangian (6.120), one should eliminate the auxiliary field $f$ using its equation of motion $f \sim \lambda \sigma^2$.}
\( (6.10) \) with \( N = 1 \), we indeed find

\[
\Delta_\sigma = 1 - \frac{3}{7} \epsilon + \frac{1}{49} \epsilon^2 + \mathcal{O}(\epsilon^3), \tag{6.122}
\]

\[
\Delta_\psi = \frac{3}{2} - \frac{3}{7} \epsilon + \frac{1}{49} \epsilon^2 + \mathcal{O}(\epsilon^3). \tag{6.123}
\]

The dimensions of operators \( \sigma^2 \) and \( \sigma \psi \) should also be related by the supersymmetry,

\( \Delta_{\sigma^2} = \Delta_{\sigma \psi} - \frac{1}{2} \). Since \( \sigma \psi \) is a descendant, we also have \( \Delta_{\sigma \psi} = \Delta_\psi + 1 \). Thus, the supersymmetry relation assumes the form [125]

\[
\Delta_{\sigma^2} = \Delta_\psi + \frac{1}{2} = \Delta_\sigma + 1. \tag{6.124}
\]

Substituting \( N = 1 \) into (6.12) we find

\[
\Delta_{\sigma^2} = 2 - \frac{3}{7} \epsilon + \frac{1}{49} \epsilon^2 + \mathcal{O}(\epsilon^3), \tag{6.125}
\]

so that the supersymmetry relation (6.124) holds to order \( \epsilon^2 \). These non-trivial checks provide strong evidence that the continuation of the GNY model to \( N = 1 \) flows to a superconformal theory to all orders in the \( 4 - \epsilon \) expansion.

If we apply the standard Padé\([1,1]\) extrapolation, we find

\[
\Delta_{\sigma^2} = \frac{8d - 11}{25 - d}, \tag{6.126}
\]

which in \( d = 3 \) gives \( \Delta_{\sigma^2} = \frac{13}{22} \approx 0.59 \). \(^{16}\) This is close to the estimate of \( \Delta_\sigma \) obtained using the numerical bootstrap [118]. It is the value \( \Delta_{\sigma^2} \approx 0.582 \) where the boundary of the excluded region touches the SUSY line \( \Delta_{\sigma^2} = \Delta_\psi - 1/2 \).

\(^{16}\)We note that our estimate \( \Delta_{\sigma^2} \approx 1.59 \) in the \( N = 1 \) theory is below 2, just like the estimate (6.66) in the \( N = 2 \) theory. This is in contrast with the large \( N \) behavior (6.57) where \( \Delta_{\sigma^2} > 2 \). Thus, perhaps not surprisingly, the theories with \( N = 1, 2 \) are, in some respects, rather far from the large \( N \) limit.
It is also important to know how the theory behaves when continued to \( d = 2 \). It is plausible that the \( d = 2 \) theory has \( \mathcal{N} = 1 \) superconformal symmetry, and the obvious candidate is the tri-critical Ising model [124, 126], which is the simplest supersymmetric minimal model [73, 155]. The Padé extrapolation (6.126) gives \( \Delta_\sigma \approx 0.217 \), which is quite close to the dimension \( 1/5 \) of the energy operator in the tri-critical Ising model. This provides new evidence that the GNY model with \( \mathcal{N} = 1 \) extrapolates to the tri-critical Ising model in \( d = 2 \); in figure 6.6 we show how the operator spectrum matches with the exact results in \( d = 2 \). Imposing the boundary condition that \( \Delta_\sigma = 1/5 \) in \( d = 2 \) enables us to perform a “two-sided” Padé estimate. The resulting value in \( d = 3 \) is \( \approx 0.588 \), which is very close to that following from (6.126). The agreement with the bootstrap result \( \Delta_\sigma \approx 0.582 \) is excellent.

Figure 6.6: Qualitative picture of the interpolation of operator dimensions from the \( d = 4 \) free theory to the tri-critical Ising model \( \mathcal{M}(4,5) \) in \( d = 2 \), indicating our estimated values for the \( \mathcal{N} = 1 \) SCFT in \( d = 3 \). In \( 2 \leq d < 4 \), the operators \( \sigma \psi \), and a linear combination of \( \sigma^3 \) and \( \bar{\psi} \psi \), are expected to be conformal descendants of \( \psi \) and \( \sigma \) respectively.

From the \( \beta \)-functions (6.7) for the GNY model we may also deduce the dimensions of two primary operators that are mixtures of \( \sigma^4 \) and \( \sigma \bar{\psi} \psi \). They are determined by
the eigenvalues $\lambda_1, \lambda_2$ of the matrix $\partial \beta_i/\partial g_j$. At the fixed point for $N = 1$ we find

\[
\begin{align*}
\Delta_1 &= d + \lambda_1 = 4 - \frac{3}{7} \epsilon^2 + O(\epsilon^3), \\
\Delta_2 &= d + \lambda_2 = 4 + \frac{6}{7} \epsilon - \frac{95}{49} \epsilon^2 + O(\epsilon^3),
\end{align*}
\]  
(6.127)

Since $\Delta_1$ corresponds to the $\theta \bar{\theta}$ component of the superfield $\Sigma^3$, we know that

\[
\Delta_{\sigma^3} = \Delta_1 - 1 = 3 - \frac{3}{7} \epsilon^2 + O(\epsilon^3).
\]  
(6.128)

As a check, we may set $N = 1$ in (6.14) and see that the order $\epsilon$ term in $\Delta_{\sigma^3}$ indeed vanishes. It is further possible to argue that this primary operator, when continued to $d = 2$, matches onto the dimension 3 operator in the tri-critical Ising model, i.e. the product of left and right supercurrents. This implies that $\Delta_{\sigma^3}$ is not monotonic as a function of $d$ and is likely to be somewhat smaller than 3 in $d = 3$.\footnote{This is in line with a statement in \cite{118} that a kink lies on the SUSY line when $\Delta_{\sigma^3}$ is slightly smaller than 3.}

Simply setting $\epsilon = 1$ in (6.128) gives $\Delta_{\sigma^3} \approx 2.57$, but this probably underestimates it. One way to implement the exact boundary condition $\Delta_{\sigma^3} = 3$ in $d = 2$ is to extrapolate the quantity $\tilde{\Delta} = (\Delta_{\sigma^3} - 3)/(d - 2)$, instead of $\Delta_{\sigma^3}$ itself [22]. This approach would yield the estimate $\Delta_{\sigma^3} \approx 2.79$ in $d = 3$. This slightly relevant operator is parity odd in $d = 3$, while the dimension $\Delta_1$ corresponds to an irrelevant parity even operator.

Similarly, we can perform extrapolation of $\tilde{F}$ and check if the $d = 2$ value is close to $\pi c/6$ where $c = 7/10$ is the central charge of the tri-critical Ising model. Setting $N = 1$ in (6.31), we have

\[
\tilde{F} = \tilde{F}_s + \tilde{F}_f - \frac{\pi}{6} \left( \frac{\epsilon^2}{112} + \frac{\epsilon^3}{98} + O(\epsilon^4) \right).
\]  
(6.129)

Performing a Padé approximation of the quantity $f(d) = \tilde{F} - \tilde{F}_f$, we find that the average of the standard Padé[2,1] and Padé[1,2] approximants yields $\tilde{F}/\tilde{F}_s \approx 0.68$,\footnote{This is in line with a statement in \cite{118} that a kink lies on the SUSY line when $\Delta_{\sigma^3}$ is slightly smaller than 3.}
which is quite close to \( c = 7/10 \). Therefore, in order to get a better estimate in \( d = 3 \), it makes sense to impose it as an exact boundary condition in \( d = 2 \). Following this procedure, and taking an average of the Padé approximants with \( n + m = 4 \), we find the \( d = 3 \) estimate

\[
F \approx 0.158. \tag{6.130}
\]

In the UV, we have the free CFT of one scalar and one Majorana fermion, which has \( F_{\text{UV}} = \frac{1}{4} \log 2 \approx 0.173 \), and therefore we find \( F_{\text{IR}}/F_{\text{UV}} \approx 0.91 \). This is a check of the \( F \)-theorem for the flow from the free to the interacting \( \mathcal{N} = 1 \) SCFT. It is also interesting to compare the value of \( F \) at the SUSY fixed point to the decoupled Ising fixed point in Figure 6.1. A plot comparing \( \tilde{F} - \tilde{F}_f \) with \( \tilde{F}_{\text{Ising}} \), which was obtained in [135], is given in Figure 6.7. It shows that \( \tilde{F} < \tilde{F}_f + \tilde{F}_{\text{Ising}} \) in the whole range \( 2 < d < 4 \), in agreement with the generalized \( F \)-theorem [78, 135] and the expectation that the SUSY fixed point is IR stable.

![Figure 6.7: Comparison of \( \tilde{F} - \tilde{F}_f \) at the SUSY fixed point and \( \tilde{F}_{\text{Ising}} \) (from [135]) in \( 2 \leq d \leq 4 \).](image-url)

Finally we consider \( C_T \). Its \( 4 - \epsilon \)-expansion for general \( N \) is given in eq. (6.15), and we can use it to estimate the \( C_T \) value at the \( \mathcal{N} = 1 \) SUSY fixed point in \( d = 3 \). We can perform a Padé approximation on the ratio \( C_T/C_{T,s} \), where \( C_{T,s} \) is the free
scalar value, using the boundary conditions

\[
\frac{C_T}{C_{T,s}} = \begin{cases} 
\frac{7}{10} & \text{in } d = 2, \\
\frac{5}{2} - \frac{19}{28} \epsilon + \mathcal{O}(\epsilon^2) & \text{in } d = 4 - \epsilon, 
\end{cases} 
\]  

(6.131)

where we have imposed the \(d = 2\) value corresponding to the tri-critical Ising model. A Padé\([1,1]\) approximant with these boundary conditions is

\[
\frac{C_T}{C_{T,s}} = \frac{497d - 728}{62d + 256}, 
\]  

(6.132)

which in \(d = 3\) gives \(C_T/C_{T,s} \approx 1.73\). This value lies well within the region allowed by the bootstrap in fig. 8 of [118] (note that \(\Delta_\psi \approx 1.09\)). Comparing to the free UV CFT of one scalar and one Majorana fermion, we have found \(C_T^{\text{IR}}/C_T^{\text{UV}} \approx 0.86\). It would be useful to know the next order in the \(4 - \epsilon\) expansion in order to obtain a more precise estimate. Based on the comparison of the Padé for the \(N = 2\) model with the exact result, we may expect the \(\epsilon^2\) correction to reduce \(C_T\) somewhat in \(d = 3\).

Let us also include a brief discussion of the non-supersymmetric fixed point of the \(N = 1\) GNY model, marked by the red triangle in figure 6.1. While it has \(g_2^* < 0\) in \(d = 4 - \epsilon\), it may become stable for sufficiently small \(d\). Changing the sign of the square root in (6.31), we find the \(4 - \epsilon\) expansion of the sphere free energy at this fixed point:

\[
\tilde{F}_{\text{non-SUSY}} = \tilde{F}_s + \tilde{F}_f - \frac{\pi}{6} \left( \frac{\epsilon^2}{112} + \frac{6875\epsilon^3}{889056} + \mathcal{O}(\epsilon^4) \right). 
\]  

(6.133)

Extrapolating this to \(d = 2\), we estimate \(c_{\text{non-SUSY}} \approx 0.78\). This is close to central charge 4/5 of the (5,6) minimal model. This model includes primary fields of conformal weight 1/40 and 21/40, so that one may form a spin 1/2 field with weights
\[(h, \tilde{h}) = (1/40, 21/40)\] that could correspond to \(\psi\) in the Yukawa theory (such a field is not present in the standard modular invariants that retain fields of integer spin only).\footnote{Another possibility is that the continuation of the non-supersymmetric fixed point to \(d = 2\) gives the \((6, 7)\) minimal model (we are grateful to the referee for suggesting this). Its central charge \(6/7 \approx 0.857\) is also not far from the extrapolation of \(\Delta_{\sigma^2}\) to \(d = 2\). The spectrum of the \((6, 7)\) model contains a spin \(1/2\) operator with weights \((h, \tilde{h}) = (5/56, 33/56)\).}

In \(d = 3\) our extrapolation gives \(F_{\text{non-SUSY}} \approx 0.16\). The latter quantity is bigger than (6.130); therefore, the non-SUSY fixed point, if it is stable, can flow to the SUSY one in \(d = 3\).

It is also interesting to look at the scaling dimensions of \(\sigma, \psi\) and \(\sigma^2\) at the non-supersymmetric fixed point. Changing the sign of the square roots in (6.10) and (6.11), we find for \(N = 1\):

\[
\begin{align*}
\Delta_{\sigma}^{\text{non-SUSY}} &= 1 - \frac{3}{7} \epsilon - \frac{95}{6174} \epsilon^2 + \mathcal{O}(\epsilon^3), \\
\Delta_{\psi}^{\text{non-SUSY}} &= \frac{3}{2} - \frac{3}{7} \epsilon - \frac{115}{37044} \epsilon^2 + \mathcal{O}(\epsilon^3), \\
\Delta_{\sigma^2}^{\text{non-SUSY}} &= -\frac{22}{21} \epsilon + \frac{1117}{9261} \epsilon^2 + \mathcal{O}(\epsilon^3)
\end{align*}
\] (6.134)

Using Pade\([1,1]\) extrapolations to \(d = 2\) we get

\[
\begin{align*}
\Delta_{\sigma}^{\text{non-SUSY}} &\approx 0.077 , & \Delta_{\psi}^{\text{non-SUSY}} &\approx 0.63 , & \Delta_{\sigma^2}^{\text{non-SUSY}} &\approx 0.297 .
\end{align*}
\] (6.135)

These numbers are not far from the corresponding operator dimensions in either the \((5, 6)\) or the \((6, 7)\) minimal models. For example, in the \((5, 6)\) interpretation the exact scaling dimension of \(\psi\) in \(d = 2\) should be \(11/20\), while in the \((6, 7)\) it should be \(19/28\). The Padé value we find lies between the two, but the accuracy of the extrapolation all the way to \(d = 2\) is hard to assess.

The Pade\([1,1]\) extrapolations of (6.134) to \(d = 3\) yield \(\Delta_{\sigma}^{\text{non-SUSY}} \approx 0.555\) and \(\Delta_{\psi}^{\text{non-SUSY}} \approx 1.068\). Interestingly, these values are not far from the feature at \((\Delta_{\sigma}, \Delta_{\psi}) \approx (0.565, 1.078)\), which lies below the supersymmetry line in figure 7 of the
Changing the sign of the square root in (6.14), we also find

\[ \Delta_{\sigma_3}^{\text{non-SUSY}} = 3 - \frac{13}{7} \epsilon + \mathcal{O}(\epsilon^2) , \]  

(6.136)

which suggests that in \( d = 3 \) the dimension of this parity-odd operator is less than 3. A higher loop analysis of the operator dimensions is, of course, desirable.
Appendix A

Useful integrals for one-loop calculation

Let us evaluate the following useful one-loop integral in dimensional regularization

\[ I(\alpha, \beta) = \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^{2\alpha}(p+q)^{2\beta}} \]

\[ = \int_0^1 dx \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \Gamma(\alpha + \beta) \int \frac{d^d q}{(2\pi)^d} \frac{1}{[q^2 + x(1-x)p^2]^{\alpha + \beta}} \]

\[ = \int_0^1 dx \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty dt \frac{d^d q}{(2\pi)^d} e^{-t(q^2 + x(1-x)p^2)} \]

\[ = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(\frac{d}{2} - \alpha)\Gamma(\frac{d}{2} - \beta)\Gamma(\alpha + \beta - \frac{d}{2})}{\Gamma(\alpha)\Gamma(\beta)\Gamma(d - \alpha - \beta)} \left( \frac{1}{p^2} \right)^{\alpha + \beta - \frac{d}{2}} \quad (A.1) \]

In the calculations in \( d = 6 - \epsilon \), we often encounter the case \( \alpha = 1, \beta = 1 \), which gives

\[ I_1 \equiv I(1, 1) = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(3 - \frac{d}{2})\Gamma(\frac{d}{2} - 1)^2}{(2 - \frac{d}{2})\Gamma(d - 2)} \left( \frac{1}{p^2} \right)^{2 - \frac{d}{2}} \quad (A.2) \]

For the purpose of extracting the logarithmic terms in \( d = 6 - \epsilon \), it is sufficient to use the approximation

\[ I_1 \to -\frac{p^2}{6(4\pi)^3} \frac{\Gamma(3 - d/2)}{(p^2)^{3-d/2}}. \quad (A.3) \]
We will also need the following three-propagator integral

\[ I_2 = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k-p)^2} \frac{1}{(k+q)^2} \frac{1}{k^2} \]  \hspace{1cm} (A.4)

\[ = 2 \int_0^1 dx dy dz \delta(1-x-y-z) \int \frac{d^d k}{(2\pi)^d} \frac{1}{[x(k-p)^2 + y(k+q)^2 + zk^2]^3} \]  \hspace{1cm} (A.5)

\[ = 2 \int_0^1 dx dy dz \delta(1-x-y-z) \int \frac{d^d k}{(2\pi)^d} \frac{1}{[k^2 - 2xp + 2yq + xp^2 + yq^2]^3} \]  \hspace{1cm} (A.6)

Defining \( l \equiv k - xp + yq \), we get:

\[ I_2 = 2 \int_0^1 dx dy dz \delta(1-x-y-z) \int \frac{d^d l}{(2\pi)^d} \frac{1}{[l^2 + \Delta]^3} \]  \hspace{1cm} (A.7)

with \( \Delta = x(1-x)p^2 + y(1-y)q^2 + 2xyq \). Evaluating the standard momentum integral, we obtain

\[ I_2 = \int_0^1 dx dy dz \delta(1-x-y-z) \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(3-d/2)}{\Delta^{3-d/2}} \]. \hspace{1cm} (A.8)

In the RG calculations of Section 3, we use the renormalization conditions \[62\] \( p^2 = q^2 = (p+q)^2 = M^2 \), which imply \( 2p \cdot q = -M^2 \), and hence \( \Delta = M^2(x(1-x) + y(1-y) - xy) \). So we can write

\[ I_2 = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(3-d/2)}{(M^2)^{3-d/2}} \int_0^1 dx dy dz \delta(1-x-y-z) \frac{1}{(x(1-x) + y(1-y) - xy)^{3-d/2}} \]. \hspace{1cm} (A.9)

In \( d = 6 - \epsilon \) the Feynman parameter integral becomes

\[ \int_0^1 dx dy dz \delta(1-x-y-z) \left( 1 - \frac{\epsilon}{2} \log(x(1-x) + y(1-y) - xy) + O(\epsilon^2) \right) = \frac{1}{2} + O(\epsilon) \]. \hspace{1cm} (A.10)

Thus,

\[ I_2 = \frac{1}{2(4\pi)^3} \left( \frac{2}{\epsilon} - \log M^2 + A + O(\epsilon) \right) , \] \hspace{1cm} (A.11)
where $A$ is an unimportant constant. For the purpose of extracting the log $M^2$ terms in $d = 6 - \epsilon$, it is sufficient to use the approximation

$$I_2 \to \frac{1}{2(4\pi)^3} \frac{\Gamma(3 - d/2)}{(M^2)^{3-d/2}}.$$  \hspace{1cm} (A.12)
Appendix B

Summary of three-loop results

The Feynman rules for our theory are depicted in Fig. B.1

\[ \begin{align*}
\text{Figure B.1: Feynman rules.}
\end{align*} \]

where we introduced symmetric tensor coupling \( d_{\alpha\beta\gamma} \) and counterterms \( (\delta g)_{\alpha\beta\gamma} \), \( (\delta z)_{\alpha\beta} \) with \( \alpha, \beta, \gamma = 0, 1, ..., N \) as

\[
\begin{align*}
d_{000} &= g_2, & d_{ii0} &= d_{i0i} &= d_{0ii} &= g_1, \\
(\delta g)_{000} &= \delta g_2, & (\delta g)_{ii0} &= (\delta g)_{i0i} &= (\delta g)_{0ii} &= \delta g_1, \\
(\delta z)_{00} &= \delta \sigma, & (\delta z)_{ii} &= \delta \phi,
\end{align*}
\]

(B.1)
where \( i = 1, \ldots, N \). The general form of a Feynman diagram in our theory could be schematically represented as

\[
\text{Feynman diagram} = \text{Integral} \times \text{Tensor structure factor.} \tag{B.2}
\]

The “Tensor structure factors” are products of the tensors \( d_{\alpha\beta\gamma} \) and \( (\delta g)_{\alpha\beta\gamma} \), \( (\delta z)_{\alpha\beta} \), with summation over the dummy indices. Their values for different diagrams are represented in Fig. D.1 and D.2 after parentheses \(^1\). The “Integrals” already include symmetry factors and are the same as in the usual \( \varphi^3 \)-theory; their values are listed in Fig. D.1 and D.2 before the parentheses.

\(^1\)To find the “Tensor structure factor” we used the fact that it is a polynomial in \( N \), so we calculated sums of products of \( d_{\alpha\beta\gamma}, (\delta g)_{\alpha\beta\gamma}, (\delta z)_{\alpha\beta} \) explicitly for \( N = 1, 2, 3, 4, \ldots \), using Wolfram Mathematica. Having answers for \( N = 1, 2, 3, 4, \ldots \) it’s possible to restore the general \( N \) form.
B.1 Counterterms

\[ z_{i2}^\phi = -\frac{g_1^2}{3(4\pi)^3}, \quad z_{i2}^\sigma = -\frac{Ng_1^2 + g_2^2}{6(4\pi)^3}, \quad a_{13} = -\frac{g_1^2 (g_1 + g_2)}{(4\pi)^3}, \quad b_{13} = -\frac{Ng_1^3 + g_2^3}{(4\pi)^3}, \]

(B.3)

\[ z_{i4}^\phi = \frac{g_1^2}{432(4\pi)^6} (g_1^2(11N - 26) - 48g_1g_2 + 11g_2^2), \]

\[ z_{i4}^\sigma = -\frac{1}{432(4\pi)^6} (2Ng_1^4 + 48Ng_1^2g_2 - 11Ng_1^2g_2^2 + 13g_2^4), \]

\[ a_{15} = -\frac{1}{144(4\pi)^6}g_1^2(g_1^2(11N + 98) - 2g_1^2g_2(7N - 38) + 101g_1g_2^2 + 4g_2^3), \]

\[ b_{15} = -\frac{1}{48(4\pi)^6}(4Ng_1^5 + 54Ng_1^4g_2 + 18Ng_1^3g_2^2 - 7Ng_1^2g_2^3 + 23g_2^5), \]

(B.4)

\[ z_{i6}^\phi = \frac{g_1^2}{46656(4\pi)^9}(g_1^4(N(13N - 232) + 5184\zeta(3) - 9064) + g_1^3g_2(441N - 544)
- 2g_1^2g_2(193N - 2592\zeta(3) + 5881) + 942g_1g_2^3 + 327g_2^4), \]

\[ z_{i6}^\sigma = -\frac{1}{93312(4\pi)^9}(2Ng_1^6(1381N - 2592\zeta(3) + 4280) - 96N(12N + 11)g_1^5g_2
- 3Ng_1^4g_2^2(N + 4320\zeta(3) - 8882) + 1560Ng_1^3g_2^3 - 952Ng_1^2g_2^4 - g_2^6(2592\zeta(3) - 5195)), \]

\[ a_{17} = \frac{g_1^2}{15552(4\pi)^9}( - g_1^5(N(531N + 10368\zeta(3) - 2600) + 23968) + g_1^4g_2(99N^2 + 2592(5N - 6)\zeta(3) - 9422N - 2588) + 2g_1^3g_2^2(1075N + 2592\zeta(3) - 16897)
+ 2g_1^2g_2^3(125N - 5184\zeta(3) - 3917) - g_1g_2^4(5184\zeta(3) + 721) + g_2^5(2592\zeta(3) - 2801)), \]

\[ b_{17} = -\frac{1}{2592(4\pi)^9}(2g_1^7N(577N + 713) - 48g_1^6g_2N(31N - 59)
+ g_1^5g_2^2N(423N + 2592\zeta(3) + 1010) - g_1^4g_2^3N(33N - 1296\zeta(3) - 6439)
- 27g_1^3g_2^4N(32\zeta(3) + 11) - 301Ng_1^2g_2^5 + g_2^7(432\zeta(3) + 1595)). \]  

(B.5)
Appendix C

Sample diagram calculations

C.1 Some useful integrals

Many of the diagrams listed in figure D.1 are recursively primitive, so they can be easily evaluated using the integral:

\[
I(\alpha, \beta) = \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^{2\alpha}(p-k)^{2\beta}} = \frac{L_d(\alpha, \beta)}{(k^2)^{\alpha + \beta - d/2}}, \tag{C.1}
\]

where

\[
L_d(\alpha, \beta) = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma\left(\frac{d}{2} - \alpha\right)\Gamma\left(\frac{d}{2} - \beta\right)\Gamma(\alpha + \beta - \frac{d}{2})}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\frac{d}{2} - \alpha - \beta)}. \tag{C.2}
\]

For the more complicated integrals, we use the mathematica program FIRE\cite{156}, which uses integration-by-parts (IBP) relations to turn them into simpler “master integrals”, which we then evaluate by hand.

There are two categories of diagrams which show up quite frequently as subdiagrams, the “special KITE” diagrams and the “ChT” diagrams shown in Figure C.1.
Figure C.1: The Special \textit{KITE} and \textit{ChT} diagrams, the numbers labeling each propagator denote its index.

The special \textit{KITE} diagram is a two-loop diagram corresponding to the following integral:

\[
SK(\alpha) = \frac{1}{(2\pi)^d (2\pi)^d} \frac{1}{p^2(q+k)^2(q+k)^2(p-q)^2} \cdot (C.3)
\]

Notice that the power of the middle propagator is arbitrary. Via the Gegenbauer Polynomial technique as described in [157], this integral can be expressed as an infinite sum of gamma functions.

\[
SK(\alpha) = -\frac{2}{(4\pi)^d (k^2)^{4+\alpha-d}} \frac{1}{\Gamma(2\alpha)} \Gamma(\lambda-\alpha) \Gamma(\lambda+1-2\alpha) \Gamma(2\lambda+1/2-\alpha) \Gamma(1/2-2\lambda+\alpha) \\
\times \sum_{n=0}^{\infty} \frac{\Gamma(n+2\lambda) \Gamma(n+1-\lambda+\alpha)}{\Gamma(n+\alpha+1) (n+1-\lambda+\alpha)} \\
+ \Gamma(2\lambda) \Gamma(\lambda-\alpha-1) \Gamma(2\lambda-\alpha) \Gamma(\alpha+1-2\lambda) \\
\times \frac{\Gamma(1/2) \Gamma(3\lambda-\alpha-1) \Gamma(2\lambda-\alpha) \Gamma(\alpha+1-2\lambda)}{\Gamma(\lambda) \Gamma(2\lambda+1/2-\alpha) \Gamma(1/2-2\lambda+\alpha)} \\
+ \frac{\Gamma(\lambda) \Gamma(3\lambda-\alpha) \Gamma(2\lambda-\alpha) \Gamma(\alpha+1-2\lambda) \Gamma(\lambda-\alpha-1) \Gamma(2\lambda-\alpha) \Gamma(\alpha+1-2\lambda)}{\Gamma(2\lambda+1/2-\alpha) \Gamma(1/2-2\lambda+\alpha)} (C.4)
\]

where \( \lambda = d/2 - 1 \). In the case of \( d = 6 - \epsilon \) we have found that, for example:

\[
SK\left(2\frac{d}{2}\right) = \frac{1}{(4\pi)^d (k^2)^{6-3d/2}} \left( \frac{-54}{\epsilon^2} + \frac{-71 + 24\gamma}{1296\epsilon} + \frac{-14641 + 8520\gamma - 1440\gamma^2 + 120\pi^2}{15520} + \ldots \right) (C.5)
\]

The \( \epsilon \)-expansion of the above result can also be verified indirectly with the mathematica packages MBTools implementing the Mellin-Barnes representation [158].

The \textit{ChT} diagram is another variation of the \textit{KITE} diagram. It correspond to the integral:

\[
ChT(\alpha, \beta) = \frac{1}{(2\pi)^d (2\pi)^d} \frac{1}{p^2(p+k)^2q^2(q+k)^2(p-q)^2} \cdot (C.6)
\]
In this diagram, one triangle of the \( KITE \) diagram all have indices 1, and the other two lines have arbitrary indices \( \alpha \) and \( \beta \). This diagram was evaluated in position space by Vasiliev et. al. in [5]. Their answer is:

\[
ChT(\alpha, \beta) = \frac{\pi^d v(d - 2)}{\Gamma\left(\frac{d}{2} - 1\right)} \frac{1}{(x^2)^{d/2-3+\alpha+\beta}} \times \left( \frac{v(\alpha)v(2 - \alpha)}{(1 - \beta)(\alpha + \beta - 2)} + \frac{v(\beta)v(2 - \beta)}{(1 - \alpha)(\alpha + \beta - 2)} + \frac{v(\alpha + \beta - 1)v(3 - \alpha - \beta)}{(\alpha - 1)(\beta - 1)} \right),
\]

where \( v(\alpha) = \frac{\Gamma(d/2-\alpha)}{\Gamma(\alpha)} \). For our purpose, we just need to fourier transform this expression to momentum space.

We also need variations of the \( SK \) and \( ChT \) diagrams, with a particular index raised by 1, for example. However, we can use FIRE to relate them to the original version of these diagrams.

### C.2 Example of a two point function diagram

We will evaluate the three-loop ladder diagram which is the first diagram in Figure D.1(e). It corresponds to the integral:

\[
LADDER = \int \frac{d^dp d^dq d^dr}{(2\pi)^{3d}} \frac{1}{p^2(p + k)^2(p - r)^2(r + k)^2(r - q)^2(q + k)^2}, \quad (C.8)
\]

where the loop momenta are \( p, q, \) and \( r \). The external momentum is \( k \). Using FIRE, it can be reduced to a sum of five master integrals, denoted as \( M_A, \ldots, M_E \):

\[
LADDER = \frac{4(2d - 5)(3d - 8)(9d^2 - 65d + 118)}{(d - 4)^4k^8} M_A - \frac{12(d - 3)(3d - 10)(3d - 8)}{(d - 4)^3k^6} M_C
\[
+ \frac{32(d - 3)^2(2d - 7)}{(d - 4)^3k^6} M_B + \frac{4(d - 3)^2M_E}{(d - 4)^2k^4} + \frac{3(d - 3)(3d - 10)}{(d - 4)^2k^4} M_D.
\]

(C.9)
The diagrams corresponding to the master integrals are listed in Figure C.2. Among these master integrals, only $M_D$ is non-primitive, the rest can be calculated easily. However, if we integrate over the middle loop, we see that $M_D$ is in fact related to the special $KITE$ diagram $SK(2 - d/2)$. We have:

$$
M_A = \frac{L_d(1,1)L_d(1,2-d/2)L_d(1,3-d)}{(k^2)^{4-3d/2}}, \quad M_B = \frac{(L_d(1,1))^2L_d(1,4-d)}{(k^2)^{5-3d/2}},
$$

$$
M_C = \frac{(L_d(1,1))^2L_d(1,2-d/2)}{(k^2)^{5-3d/2}},
$$

$$
M_D = L_d(1,1)SK(2-d/2), \quad M_E = \frac{(L_d(1,1))^3}{(k^2)^{6-3d/2}}.
$$

(C.10)

Plugging in $d = 6 - \epsilon$ and expanding in $\epsilon$, we find that:

$$
LADDER = \frac{k^2}{(4\pi)^{3d/2}} \left( - \frac{2}{9\epsilon^3} + \frac{-115 + 36\gamma + \log k^2}{108\epsilon^2} 
+ \frac{-4043 + 18(115 - 18\gamma)\gamma + 18\pi^2 - 18\log k^2(-115 + 36\gamma + 18\log k^2)}{1296\epsilon} + \ldots \right).
$$

(C.11)

C.3 Example of a three point function diagram

We will evaluate the three-loop diagram found in Figure D.2(f). In order to employ the same techniques used for the two-point functions, we impose that the momentum running through the three points are $p$, $-p$, and $0$, as a three-point function with three arbitrary momenta are much more difficult to compute.
Figure C.3: The three orientations of the same diagram topology correspond to different integrals.

However, since the momenta are asymmetric, it is necessary to consider all three “orientations” of each topology of the diagram. Notice that the tensor factors mentioned in the previous section will also be different. As an illustration, let’s denote the three orientations of the diagram we are considering by $I_1$, $I_2$, and $I_3$. After taking into account that one of the external momenta is zero, they are each equivalent to a two-point function as shown in Figure C.3. All lines have indices 1, except those with black dots, which have indices 2.

$I_1$ contains a subdiagram that is equivalent to $ChT(1,2)$, which can be evaluated easily using our formula before; after that, the diagram is primitive. The other two diagrams be reduced via FIRE into the master integrals $M_A$, $M_B$, $M_C$, and $M_D$ as in the $LADDER$ diagram. Again, in $d = 6 - \epsilon$, we find that:
\[ I_1 = \frac{1}{(4\pi)^{3d/2}} \left( \frac{1}{6\epsilon^3} + \frac{5 - 2\gamma - 2 \log k^2}{8\epsilon^2} \frac{173 + 18(\gamma - 5) - \pi^2 + 18 \log k^2 (-5 + 2\gamma + \log k^2)}{96\epsilon} + \ldots \right) \]  \hspace{1cm} \text{(C.12)}

\[ I_2 = \frac{1}{(4\pi)^{3d/2}} \left( \frac{1}{6\epsilon^3} + \frac{5 - 2\gamma - 2 \log k^2}{8\epsilon^2} \frac{125 + 18(\gamma - 5) - \pi^2 + 18 \log k^2 (-5 + 2\gamma + \log k^2)}{96\epsilon} + \ldots \right) \]  \hspace{1cm} \text{(C.13)}

\[ I_3 = \frac{1}{(4\pi)^{3d/2}} \left( \frac{1}{6\epsilon^3} + \frac{5 - 2\gamma - 2 \log k^2}{8\epsilon^2} \frac{125 + 18(\gamma - 5) - \pi^2 + 18 \log k^2 (-5 + 2\gamma + \log k^2)}{96\epsilon} + \ldots \right) \]  \hspace{1cm} \text{(C.14)}
Appendix D

Beta functions and anomalous dimensions for general Yukawa theories

In this appendix we list known general results for $\beta$- and $\gamma$-functions for general Yukawa theories in $d = 4$ with the Lagrangian

\[
L = \frac{1}{2} (\partial_\mu \phi_i)^2 + \bar{\psi} \Gamma_i \psi \phi_i + \frac{1}{4!} g_{ijkl} \phi_i \phi_j \phi_k \phi_l ,
\]

where $\phi_i$, with $i = 1, \ldots, N_b$ are real scalar fields, $\psi_\alpha$, with $\alpha = 1, \ldots, N_f$ are four-component Dirac spinors, and the matrices $\Gamma_i$ have the following general form

\[
\Gamma_i = S_i \otimes 1 + iP_i \otimes \gamma_5, \quad \Gamma_i^\dagger = S_i^\dagger \otimes 1 - iP_i^\dagger \otimes \gamma_5
\]

and act in the flavor and spinor spaces and are not necessarily Hermitian. We also assume that $\gamma_5^2 = 1$.

Using the papers [127, 128, 129] and [130] one can find the $\beta$- and $\gamma$-functions of the general Yukawa theory. We note that one can use results of [127, 128, 129]
for the four-component spinors, see sec. 4 in [128]. For the $\gamma$-functions the result reads (see formulas (3.6), (4.4) in [127] and (7.2) in [130])

$$\gamma_\psi = \frac{1}{(4\pi)^2} \frac{1}{2} \Gamma_i \Gamma_i^\dagger - \frac{1}{(4\pi)^4} \left( \frac{1}{8} \Gamma_i \Gamma_j \Gamma_i \Gamma_j + \frac{3}{8} tr(\Gamma_i \Gamma_j) \right),$$

$$\gamma_{\phi,ij} = \frac{1}{2(4\pi)^2} tr(\Gamma_i \Gamma_j^\dagger) + \frac{1}{(4\pi)^4} \left( \frac{1}{12} g_{iklm} g_{jklm} - \frac{3}{4} tr(\Gamma_j \Gamma_k \Gamma_i \Gamma_k) - \frac{1}{2} tr(\Gamma_j \Gamma_k \Gamma_i \Gamma_k) \right).$$

(D.3)

For the anomalous mixing matrix of the $N_b(N_b + 1)/2$ quadratic operators $O_{ij} = \phi_i \phi_j$ with $i \leq j$ we have [131]

$$\gamma_{ij,kl} = \gamma_{\phi,ml} \delta_{ij,ml} + \gamma_{\phi,ml} \delta_{ij,km} + \begin{cases} M_{ij,kl} + M_{ij,ik}, & k \neq l, \\ M_{ij,ik}, & k = l, \end{cases}$$

(i.e. $\delta_{11,11} = 1, \delta_{11,12} = 0, \delta_{12,12} = 1, \delta_{22,22} = 1, \ldots$) and

$$M_{ij,kl} = \frac{1}{(4\pi)^2} g_{ijkl} - \frac{1}{(4\pi)^4} \left( g_{ikmn} g_{jlmn} + tr(\Gamma_i \Gamma_j) g_{ijkl} - 2 tr(\Gamma_i \Gamma_k \Gamma_j \Gamma_l) \right).$$

(D.4)

For the $\beta$-functions we have (see (3.3) in [128] and (7.2) in [130])

$$\beta_i = \frac{1}{(4\pi)^2} \left( \frac{1}{2} (\Gamma_i^2 \Gamma_i + \Gamma_i \Gamma_i^2) + 2 \Gamma_j \Gamma_j^\dagger \Gamma_j \right) + \frac{1}{(4\pi)^4} \left( 2 \Gamma_j \Gamma_j^\dagger \Gamma_i (\Gamma_k \Gamma_j \Gamma_i - \Gamma_i^\dagger \Gamma_k^\dagger) - \Gamma_j (\Gamma_i \Gamma_i^\dagger + \Gamma_i \Gamma_i^2) \Gamma_j - \frac{1}{8} (\Gamma_j \Gamma_i \Gamma_j \Gamma_i + \Gamma_i \Gamma_i \Gamma_j \Gamma_j) - tr(\Gamma_i \Gamma_k) \Gamma_j \Gamma_j \Gamma_j \right) - \frac{3}{8} tr(\Gamma_i \Gamma_k)(\Gamma_j \Gamma_i \Gamma_k \Gamma_k) - \Gamma_j tr(\frac{3}{8} (\Gamma_j \Gamma_j \Gamma_i + \Gamma_i \Gamma_i \Gamma_j) + \frac{1}{2} \Gamma_j \Gamma_k \Gamma_i \Gamma_k) - 2 g_{ijkl} \Gamma_j \Gamma_i \Gamma_k \Gamma_l + \frac{1}{12} g_{iklm} g_{jklm} \Gamma_j \right)$$

(D.6)
and (see (4.3) in [129])

\[
\beta_{ijkl} = \frac{1}{(4\pi)^2} \left( \frac{1}{8} \sum_{\text{perm}} g_{ijmn} g_{mnkl} - \frac{1}{2} \sum_{\text{perm}} \text{tr}(\Gamma_i \Gamma_j \Gamma_k \Gamma_l) + \frac{1}{2} \sum_{a=i,j,k,l} \text{tr}(\Gamma_a^\dagger \Gamma_a) g_{ijkl} \right) \\
+ \frac{1}{(4\pi)^4} \left( \frac{1}{12} \sum_{a=ijkl} g_{amnp} g_{amnp} - \frac{1}{4} \sum_{\text{perm}} g_{ijmn} g_{kmpq} g_{lmpr} - \frac{1}{8} \sum_{\text{perm}} \text{Tr}(\Gamma_a^\dagger \Gamma_a \Gamma_a^\dagger \Gamma_a) \right) \\
+ \frac{1}{2} \sum_{\text{perm}} g_{ijkl} \text{tr}(\Gamma_k \Gamma_m^\dagger \Gamma_i \Gamma_l^\dagger) - g_{ijkl} \sum_{a=ijkl} \left( \frac{3}{4} \text{tr}(\Gamma_a \Gamma_a \Gamma_m \Gamma_m^\dagger) + \frac{1}{2} \text{tr}(\Gamma_a \Gamma_a^\dagger \Gamma_a \Gamma_a^\dagger) \right) \\
+ \sum_{\text{perm}} \left( \text{tr}(\Gamma_m \Gamma_i \Gamma_j \Gamma_k^\dagger) + 2\text{tr}(\Gamma_m \Gamma_i^\dagger \Gamma_m \Gamma_j \Gamma_k^\dagger) + \text{tr}(\Gamma_i \Gamma_j \Gamma_m^\dagger \Gamma_k \Gamma_l^\dagger) \right),
\]

(D.7)

where \(\sum_{\text{perm}}\) denotes the sum over all permutation of the indices \(i, j, k, l\) and \(\Gamma^2 = \Gamma_i^\dagger \Gamma_i\) and \(\Gamma^{2\dagger} = \Gamma_i \Gamma_i^\dagger\), also \(\sum_{a=ijkl} f(a) \equiv f(i) + f(j) + f(k) + f(l)\). Traces \(\text{tr}(\Gamma \ldots)\) are over the flavor and spinor indices and \(\text{tr}1 = 4, \text{tr}\gamma_5 = 0\).

Looking at the previous expressions for the \(\beta\)- and \(\gamma\)-functions, one can easily generalize the result of [130] for \(\beta_b\), when the matrices \(\Gamma_i\) are not Hermitian:

\[
\beta_b = -\frac{1}{(4\pi)^8} \frac{1}{144} \left( \frac{1}{8} \text{tr}(\Gamma_i^\dagger \Gamma_i \Gamma_j^\dagger \Gamma_j \Gamma^2) + \text{tr}(\Gamma_i^\dagger \Gamma_i \Gamma_j^\dagger \Gamma_j \Gamma_j^\dagger \Gamma_k^\dagger \Gamma_k) - \text{tr}(\Gamma_i^\dagger \Gamma_i \Gamma_j^\dagger \Gamma_j \Gamma_j^\dagger \Gamma_k^\dagger \Gamma_k) \right) \\
+ \frac{3}{4} \text{tr}(\Gamma_i^\dagger \Gamma_j) \text{tr}(\Gamma_i^\dagger \Gamma_j \Gamma^2 + \Gamma_i^\dagger \Gamma_k \Gamma_j \Gamma_k) + g_{ijkl} \text{tr}(\Gamma_i^\dagger \Gamma_j \Gamma_k \Gamma_l^\dagger) - \frac{1}{24} g_{iklm} g_{jklm} \text{tr}(\Gamma_i^\dagger \Gamma_j) \right).
\]

(D.8)

Now to use these formulas for the GNY model one simply takes

\[
\Gamma_1 = g_1 1_{N_f \times N_f} \otimes 1, \quad g_{1111} = g_2.
\]

(D.9)
and finds the results (6.7), (6.9), (6.11) and (6.19). In the case of the NJL model, we have

\[ \Gamma_1 = g_1 1_{N_f \times N_f} \otimes 1, \quad \Gamma_2 = ig_1 1_{N_f \times N_f} \otimes \gamma_5, \]

\[ g_{1111} = g_{2222} = g_2, \quad g_{1122} = g_{1221} = \cdots = \frac{1}{3} g_2, \quad (D.10) \]

and we obtain the results (6.68), (6.70), (6.72) and (6.80).
Figure D.1: Values of derivatives of two-point diagrams. The upper row in parenthesis is for $\langle \sigma \sigma \rangle$ and the lower is for $\langle \phi \phi \rangle$. 
Figure D.2: Values of three-point diagrams. The upper row in parenthesis is for \(\langle \sigma \sigma \sigma \rangle\) and the lower is for \(\langle \sigma \phi \phi \rangle\).
\[
\frac{1}{(4\pi)^3} 158^c \left( 1 - \frac{17}{12} + \frac{10k^2}{9} \right) \times \left\{ 3 \left( 2Ng_1^7 + N_2 N_1g_1g_2^3 + N_2g_1^2g_2^3 + g_1^7 \right) \right.
\left. g_1^1 \left( (3N + 4)g_1^1 + (N + 3)g_1^1g_2^1 + (N + 3)g_1^1g_2^3 + g_1g_2^3 + g_1^5 \right) \right.
\left. - g_1^1 \left( (3N + 4)g_1^1 + (2N + 6)g_1g_2 + (2N + 1)g_1g_2^3 + (N + 2)g_1g_2^3 + 2g_1g_2^3 + g_1^5 \right) \right.
\left. - g_1^1 \left( (N + 2)g_1^2 + N_2g_1^2g_2^3 + Ng_1^2g_2^3 + g_1^2g_2^3 \right) \right.
\left. g_1^1 \left( (4N + 4)g_1^2 + (5N + 2)g_1g_2 + (N + 2)g_1g_2^3 + (2N + 1)g_1g_2^3 + g_1g_2^3 + 2g_1^2 \right) \right.
\left. - g_1^1 \left( (N^2 + 8)g_1^2 + (2N^2 + 4)g_1g_2 + 2N_1g_1g_2^3 + 4Ng_1g_2^3 + g_1g_2^3 + 2g_1^2 \right) \right.
\left. - g_1^1 \left( (4N + 4)g_1^2 + N(N + 4)g_1g_2 + 4g_1g_2^3 + (2N + 4)g_1g_2^3 + g_2^3 \right) \right.
\left. - g_1^1 \left( (N + 3)g_1^2g_2^3 + N_2g_1^2g_2^3 + g_1^2g_2^3 \right) \right.
\left. g_1^1 \left( 3g_1^2 + (N + 2)g_1g_2 + g_1g_2^3 + 2g_1g_2^3 \right) \right.
\left. - g_1^1 \left( 3 \left( 2Ng_1^7 + N_2 N_1g_1g_2^3 + N_2g_1^2g_2^3 + g_1^7 \right) \right) \right.
\left. g_1^1 \left( (2N + 5)g_1^2 + 7g_1^2g_2 + 3g_1^2g_2 + 5g_1g_2 + 2g_1^2 \right) \right. \right. \]
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