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The localization calculation on supersymmetric gauge theories and its application

Naofumi HAMA
Abstract

Localization principle in supersymmetric gauge theory is one of the methods to compute an infinite dimensional path integral as a matrix model represented as finite dimensional integral. The exact observables obtained by this method can be used to characterize the gauge theory itself and can be related to other theories clued by these physical observables. In this thesis I show the explicit examples of localization calculation in various situation and one of their physical applications in particular setting. This thesis is based on [1, 2, 3]
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Chapter 1

Introduction

1.1 Background and motivation

My research interest is mainly in nonperturbative aspects of supersymmetric gauge theo-
ries to clarify which structures are essential for characterizing supersymmetric
gauge theories considering relations with other theories which have similar structures.
The physical observables which arise from nonperturbative and exact computations such
as BPS partition function and superconformal index give the information in infrared re-
gion of the gauge theories. To consider these observables therefore will give new char-
acterization and classification of supersymmetric gauge theories. In particular, this will
give us a guiding principle for analyzing the theories which have no Lagrangian. These
“non-Lagrangian” theories are very important in defining M-theory. In following in this
chapter, to explain my concerns, firstly in 1.1.1, I introduce why these “non-Lagrangian”
theories are needed in M-theory. Secondary, in 1.1.2, how the nonperturbative observ-
ables in supersymmetric gauge theories are obtained and how these quantities represent
the mathematical structure of the theories are explained. Finally, I show the example
of usage of these nonperturbative results as characterizations of geometric information of
supersymmetric gauge theories in 1.1.3, by comparing it with geometric observables which
are calculated in another theory.

1.1.1 M-theory

M-theory is the most prospective candidate for the fundamental theory in particle physics,
but the attempt to define M-theory in nonpertubative way has hard mathematical prob-
lems. One of the problems to be solved is how to describe the effective theory of multiple
M5-branes. Indeed it was found that this six dimensional gauge theory which represent \( N \)
coincident M5-branes has a certain number of supersymmetry and it is called type \( A_{N-1} \)
six dimensional \( \mathcal{N}=(2,0) \) theory, but it is also known for now that this theory has no
Lagrangian, and its property is charactarized only by geometric information of its moduli
space. In this sense, to answer the question “How can we consider the multi-body system
of M5-branes?” , we have to solve the problem “How can we treat non-Lagrangian theory?
What property does it characterize the theory without Lagrangian?”. This is the typical
standpoint that attaches importance to analyzing the theory without Lagrangian, and ac-
tually, this is the main goal of my research. In my opinion, the nonperturbative and exact calculations of physical observables in gauge theories will yield key clues to deal with this problem as I explain in the following.

1.1.2 Geometric information of supersymmetric gauge theory

In this subsection, I will show that certain geometric structures computed in nonperturbative way characterize supersymmetric gauge theories in a certain class. In particular, I will focus on the localization principle which is developed in recent years and discovers nonperturbative analytics and its application.

From decades ago, it has been very difficult to calculate physical observables in exact way because a path integral in quantum interacting gauge theory demands an infinite dimensional integral. There are some examples whose path integral can be calculated explicitly because of their mathematical structures called integrable systems, and we investigate them in detail. Localization method is also a one way to compute path integral exactly. This method uses a fermionic nilpotent symmetry of the system. Here nilpotent means that the square of this symmetric transformation is zero up to bosonic symmetries of the system such as gauge transformation and some isometries of the system. In particular, in this decade, localization calculation in supersymmetric gauge theory had a big advance which uses supersymmetry as a fermionic symmetry for localization method. The main point of localization in this case is deforming the action by adding supersymmetric exact terms. At this point, to calculate supersymmetric closed observables, path integral can be computed by this deformed action instead of the original action, and the infinite dimensional path integral localizes only on the configuration which make the deformed action to be zero. It gives us nonperturbative and exact values of supersymmetric closed observables.

This localization technique has a long history in topological field theory, and particular example is an instanton counting by using Omega deformation [4]. Recent application to supersymmetric gauge theories is started by Pestun [5]. He constructed four dimensional \( \mathcal{N} = 2 \) supersymmetric gauge theories on four-sphere \( (S^4) \), and via localization technique he obtained the explicit form of partition function of these theories. This seminal work and its generalization to various dimensions and various manifolds on which supersymmetric gauge theories are constructed had great successes in mathematical physics.

In the sense of my research purpose, the exact values of physical observables obtained by this localization technique can be applied to confirmations and developments of dualities between gauge theories. The gauge theories are represented in terms of their Lagrangians constructed from the information in the ultraviolet region. However, certain type of theories described by different Lagrangian have the same behavior in their infrared region. These theories are called IR-dual. For instance, there is a mirror symmetry which relates three dimensional supersymmetric gauge theories. From this viewpoint, the behavior of gauge theories in infrared regions are not specified by Lagrangian but some different structures. In this sense these structures, often expressed by geometrical information, are more essential in classifying gauge theories than Lagrangian. Since some of the nonperturbative values of physical observables are also prescribed by this infrared information of the theory, the easiest way to confirm the IR-dualities is comparing these physical observables.
Consequently, **nonperturbative calculation in gauge theories are useful keys to classify gauge theories.** These geometrical structures are often perceived in the relation with other theories.

### 1.1.3 The relation to quantities in another theory

Once the geometric structure is described by exact computations as noted above, same structures are found in other (not gauge) theories. The relation between these gauge theories and non-gauge theories can be a clue for an analysis of higher dimensional fundamental theories. The research from this area is also progressing. Here I take one of the most successful examples, “AGT relation” [6].

AGT relation picks up two dimensional conformal field theory. Since two dimensional conformal field theory has an infinite dimensional local conformal symmetry, certain physical observables such as correlation functions in this theory can be computed exactly. Hence the mathematical structures on which two dimensional conformal field theories are constructed can be read from their physical quantities. Conversely, two dimensional conformal field theories can be formulated in order to represent particular geometric structures. In particular, when we have a certain algebra which we want to represent by a two dimensional conformal field theory as its structure, there is a method called ”Wakimoto free field realization” [7], which adds free fields to the two dimensional conformal field theory to make its operator product algebra embodies to be the demanded algebra. For the case of AGT relation, Liouville field theory, two dimensional conformal field theory with a potential in exponential form, and Toda theory play roles of them which represent Virasolo algebra and $W_N$ algebra respectively. The correlation functions in these two dimensional conformal field theories coincide with the partition functions of the corresponding four dimensional $\mathcal{N} = 2$ $SU(N)$ supersymmetric gauge theories, called class $S$ theory [8].

This interesting relation between these well-known two theories is interpreted as two different limits of one theory, type $A_{N-1}$ six dimensional $\mathcal{N} = (2,0)$ theory, which I introduced in 1.1.1. When this theory is compactified on a twisted product of a punctured Riemann surface and a four dimensional ellipsoid ($S^4_b$), Liouville theory appears in the limit where the $S^4_b$ becomes small. On the other hand, class $S$ theory arises in the limit where the Riemann surface shrinks. Of course this relation gives some suggestions to deal with type $A_{N-1}$ six dimensional $\mathcal{N} = (2,0)$ theory. In spite of the physical importance of this theory as an M-theory, as I pointed out in 1.1.1, the type $A_{N-1}$ six dimensional $\mathcal{N} = (2,0)$ theory cannot be analyzed in a conventional way using Lagrangian. Hence AGT relation originates useful ways for considering M-theory. In this way, the essential structures are often found in the relation between different theories. In particular, in one of the most promising proof [9] of AGT relation, which treats the easiest case, the correspondence between Liouville field theory and $SU(2)$ supersymmetric gauge theory with four flavors, Liouville field theory is regarded as a representation of quantum Teichmüller space, which is moduli space of the corresponding supersymmetric gauge theory. This is a good example for treating a theory as a object which represents a geometrical structure to be considered.
1.2 Topics

1.2.1 Localization computation on four dimensional gauge theory

In the work [1] we considered four dimensional $\mathcal{N} = 2$ supersymmetric gauge theory on four dimensional ellipsoid $S^4_b$, which is deformed sphere. In particular, the result obtained by localization calculation was used for generalizations of the AGT relation. In fact, as I noted in 1.1.3, AGT relation claims the correspondences between observables in Liouville field theory and those in class $S$ theory on some manifolds which are originally not known. This work discovered that the answer was ellipsoid $S^4_b$ by confirming that the factors called “one-loop determinant” in the partition functions were actually equated to the certain factors in correlation functions of Liouville field theory. Therefore, the physical description of AGT relation in M-theory, M5 branes wrapping the product of the $S^4_b$ and the punctured Riemann surface, was established by our result, in part. The chapter 3 was devoted to introduce this work.

1.2.2 Super Rényi Entropy and their holographic dual

In the following I will show how we defined new observables in five-dimensional $\mathcal{N} = 1$ supersymmetric gauge theory in the work [3].

In the case that the part of the system cannot be observed in quantum theory, such as the system with a black hole, the physical observables called entanglement entropy is defined as an entropy which quantifies the lost information of the hidden part of the system. This entanglement entropy is a nonlocal observable and can be used as a quantum order parameter, thus it is regarded as important to compute an entanglement entropy in various situations. The easiest way to calculate it is using a replica trick, which calculates entanglement entropy from Rényi entropy, as a certain limit of the partition function of $n$ coverings surrounding an entangling surface, or the hidden part of the system. In other words, Rényi entropy is one parameter ($n$) generalization, and entanglement entropy is reproduced as Rényi entropy with $n = 1$. If the quantum field theories in $d$ dimensions have conformal symmetry and if it takes $d - 2$ dimensional hypersphere as the entangling surface, then via the coordinate transformation, the $n$ coverings of the theories becomes $n$ coverings of $d$ dimensional sphere [10]. However, the replica trick cannot be applied to the theory with supersymmetries, because the conical singularity of $n$ branching breaks supersymmetries.

To solve this problem, the authors of [11] defined new observables called Supersymmetric Rényi Entropy in three dimensional $\mathcal{N} = 2$ gauge theory constructed on $n$ coverings of $S^3$. They resolved the conical singularity and made the $n$ coverings of $S^3$ smooth and turned on background fields to preserve the supersymmetry to apply the replica trick. Henceforth, the partition function of the theory could be calculated by localization technique, and turned out to be the similar formula for those of the theory constructed on $S^3$, as I will mention in later, the results of all continuum deformation in three dimensional $\mathcal{N} = 2$ gauge theory constructed on $S^3$ converge to the result of the theory on the ellipsoid. The gravity dual of the Supersymmetric Rényi Entropy in this case was also found in [12, 13]. Similar definitions are applied in four dimensional $\mathcal{N} = 2$ gauge theory on $n$ coverings of $S^4$ in [14, 15] using four dimensional ellipsoid calculations in [1]. Additionally, [3] (which
has an overlap with [16]) we constructed five dimensional $\mathcal{N} = 1$ supersymmetric gauge theory on $n$ coverings of $S^5$, generalizing the localization formulas in one parameter from those of the theories on five sphere $S^5$. The result also matched with that obtained from the gravity dual. The only technical detail in localization computation is shown in the chapter 4.

1.2.3 AGT relation

AGT relation had not been proved for years since originated in [6]. What is more is that whether this relation was exact correspondence or not had not been verified. The main difficulty relied on that the formula for instanton corrections in the partition functions of class $\mathcal{S}$ theory was represented in the perturbative series about instantons number on the one side. On the other side, that the multi-point correlation functions in Liouville field theory constituted from the product of three-point vertices and propagators between them had to include not only primary operators but also their descendants in propagators, whose contributions were represented only in perturbative forms. To avoid this obstacle, in [2] we took the easiest case of AGT relation, the four dimensional $\mathcal{N} = 2$ supersymmetric $SU(2)$ gauge theory with four flavors on ellipsoid and Liouville field theory on sphere with four punctures. Then we considered the situation where extremely squashing limit $b \rightarrow 0$ of an ellipsoid $S^4_b$ on which supersymmetric gauge theory lived while the masses of flavors in the gauge theory and momenta in Liouville theory were taken to be satisfied certain relations. In this stage, the perturbative expansions in each side of AGT relation noted above were simplified and became exact formulas. Therefore, AGT relation could be confirmed as an exact correspondence in this case. The explicit detail is explained in the chapter 5.

1.3 Organization

The organization of this thesis is as follows. In chapter 2, I will explain the principle of localization and its history. The explicit calculations are shown in the chapter 3 for four dimensional ellipsoid case and in the chapter 4 for five dimensional ellipsoid case. They are physically applied to AGT relation in the chapter 5. I have two appendix chapters. The notations I use in this thesis are collected in the chapter A. The chapter B is complement calculation for section 5.3.
Chapter 2

Localization principle

2.1 What is localization principle

In this section, we will introduce the calculation principle of localization method.

Localization principle uses fermionic (nilpotent) transformation of the theory. Here fermionic means that bosonic transformation generated by its square constitutes the symmetries of the theory, such as gauge symmetry and translational symmetry. If this fermionic transformation $Q$ is also a symmetry of the action and the measure for the path integral, deforming the action by adding $Q$-exact term make no difference in the path integral to compute $Q$-closed physical observables. This is the localization principle.

For instance the observable $O$ to calculate in the theory whose action is $S$, bosonic fields $\phi$ and fermionic fields $\psi$ are obtained from the path integral

$$
\langle O \rangle = \int D\phi D\psi O[\phi,\psi] e^{-S[\phi,\psi]}.
$$

(2.1.1)

The value $\langle O \rangle'$ calculated by the path integral with the deformed action by $Q$-exact term $tQV[\phi,\psi]$ where $t$ is constant coefficient and $t = 0$ reduces to no deformation can be written as

$$
\langle O \rangle' = \int D\phi D\psi O[\phi,\psi] e^{-S[\phi,\psi]+tQV[\phi,\psi]}.
$$

(2.1.2)

The effect of the deformation in the action was represented by the derivative in $t$:

$$
\frac{d}{dt} \langle O \rangle' = \int D\phi D\psi O e^{-S+tQV} QV
= \int D\phi D\psi Q \{ O e^{-S+tQV} V \}.
$$

The path integral must not changed between described by the fields $\phi, \psi$ and their $Q$ transformed fields $\phi' = \phi + Q\phi, \psi' = \psi + Q\psi$. It means

$$
\langle O e^{-tQV} V \rangle = \int D\phi D\psi O[\phi,\psi] e^{-S[\phi,\psi]+tQV[\phi,\psi]} V[\phi,\psi]
= \int D\phi' D\psi' [O e^{-S+tQV} V] [\phi',\psi'],
$$

(2.1.3)
and the measure of this path integral is not transformed by $Q$, then it becomes

$$
\int D\phi'D\psi' \cdot (1 + Q) \left[ O e^{-S + tQV} \right] \cdot [\phi, \psi]
$$

$$
\equiv \int D\phi D\psi O[\phi, \psi] e^{-S[\phi, \psi] + tQV[\phi, \psi]} + \int D\phi D\psi QO e^{-S + tQV}
$$

$$
\langle O e^{-tQV} \rangle + \frac{d}{dt} \langle O \rangle' = 0.
$$

This shows that the physical observables to compute does not deformed by varying the numerical constant $t$. In particular $t = 0$ means original observable without deforming the action, and the $Q$-closed observables can be calculated with deformed action by $Q$-exact term with arbitrary $t$.

Suppose the $Q$-exact term $QV$ is taken to be positive semidefinite and $t$ is infinite. Then the path integral with this deformed action $\int D\phi D\psi O[\phi, \psi] e^{-S[\phi, \psi] + tQV[\phi, \psi]}$ is dominated by the $Q$-exact term $tQV[\phi, \psi]$, and path integral localizes onto the saddle points where $QV = 0$. It turns out that the only contributions are the classical solutions of the deforming action and the one-loop determinant around this saddle points. This method to resolve the infinite dimensional integral of the path integral into the matrix model with finite dimensional integral is called LOCALIZATION. To use it in the supersymmetric gauge theory such as the theories in this paper, we take supersymmetry which is scalar by topological twist as this fermionic transformation $Q$.

### 2.2 History

The attempts to apply localization technique to supersymmetric gauge theory were started in [4]. To evaluate the partition function contributed from instanton configurations in four dimensional $\mathcal{N} = 2$ supersymmetric gauge theories, the author deform $\mathbb{R}^4$ by twisting supersymmetries with rotational symmetries of $\mathbb{R}^4 \cong \mathbb{C}^2$. These twisting are parametrized by $\epsilon_1, \epsilon_2$, and this deformed space-time is called Omega-background. One of the instance for the explicit procedure for these twisting constructed from six dimensional theory is in following.

#### 2.2.1 Omega deformation formulated by dimensional reduction

As an example, consider $\mathcal{N} = 2$ vectormultiplet $(A_m, \lambda_A, D_{AB})$ on $\mathbb{R}^6$. $A, B = 1, 2$ are indices of $SU(2)_R$. Their supersymmetric transformation parametrized by Killing spinor $\xi_A$ is

$$
Q A_m = \xi^A \Gamma_m \lambda_A
$$

$$
Q \lambda_A = \frac{1}{2} \Gamma^{mn} \xi_A F_{mn} + D_{AB} \xi^B
$$

$$
Q D_{AB} = \xi^A \Gamma^m D_m \lambda_B + (A \leftrightarrow B).
$$

This $\xi_A$ are bosonic spinors and fundamental representations for $SU(4)$. 

9
The algebra generated by this $Q$ is turned out to be

$$Q^2 = -\xi^A \Gamma^m \xi_A D_m \tag{2.2.2}$$

which this $D_m$ is covariant derivative. This produces the translation along Killing vector $\xi^A \Gamma^m \xi$. The formula (2.2.2) can be checked by calculating $[Q^2, D_m]$. The standard susy-complemented Yang-Mills action constituted of the vectormultiplets is

$$\mathcal{L} = \text{Tr} \left[ \frac{1}{2} F_{mn} F^{mn} - \lambda^A \Gamma^m D_m \phi - \frac{1}{2} D_{AB} D^{AB} \right] \tag{2.2.3}$$

In following we consider the Kaluza-Klein dimensional reduction from $\mathbb{R}^6$ to $\mathbb{R}^4 \simeq \mathbb{C}^2(z_1, z_2)$ fibered over $T^2(x_5, x_6)$. The fibrations are parametrized by $\epsilon_1, \epsilon_2, \tilde{\epsilon}_1, \tilde{\epsilon}_2$. These explicit identifications are

$$(z_1, z_2, x_5, x_6) \sim (z_1 e^{i\epsilon_1 \tilde{\beta}_1}, z_2 e^{i\epsilon_2 \tilde{\beta}_2}, x_5 + \beta_1, x_6) \sim (z_1 e^{i\tilde{\epsilon}_1 \beta_1}, z_2 e^{i\tilde{\epsilon}_2 \beta_2}, x_5, x_6 + \beta_2). \tag{2.2.4}$$

Reparametrizing by $z_1 = w_1 e^{i\epsilon_1 \tilde{\beta}_1 x_5 + i\tilde{\epsilon}_1 \beta_1 x_6}$, $z_2 = w_2 e^{i\epsilon_2 \tilde{\beta}_2 x_5 + i\tilde{\epsilon}_2 \beta_2 x_6}$, They are formulated as

$$(w_1, w_2, x_5, x_6) \sim (w_1, w_2, x_5 + \beta_1, x_6) \sim (w_1, w_2, x_5, x_6 + \beta_2). \tag{2.2.5}$$

The metric $ds^2 = dz_1 dz_2 + dx_5^2 + dx_6^2$ also can be rewritten in this new parameters $w_1 = x_1' + i x_2'$ and $w_2 = x_3' + i x_4'$:

$$ds^2 = [dx_1' - x_5'(\epsilon_1 \beta_1 dx_5 + \tilde{\epsilon}_1 \beta_2 dx_6)]^2 + [dx_2' + x_5'(\epsilon_1 \beta_1 dx_5 + \tilde{\epsilon}_1 \beta_2 dx_6)]^2$$
$$+ [dx_3' - x_4'(\epsilon_2 \beta_1 dx_5 + \tilde{\epsilon}_2 \beta_2 dx_6)]^2 + [dx_4' + x_4'(\epsilon_2 \beta_1 dx_5 + \tilde{\epsilon}_2 \beta_2 dx_6)]^2$$
$$+ dx_5'^2 + dx_6'^2. \tag{2.2.6}$$

To make dimensional reduction, $dx_5, x_6, \beta_1, \beta_2$ are taken to be zero with proper rescaling, and vector field $A_{5,6}$ are reformulated as scalar fields in four dimension

$$-\frac{i}{2}(A_5 - iA_6) = \phi, \ -\frac{i}{2}(A_5 + iA_6) = \bar{\phi}. \tag{2.2.7}$$

When these satisfy $\tilde{\epsilon}_i \beta_2 = i \epsilon_i \beta_1$, supersymmetry tranformation rule in four dimension space-time become

$$QA^\mu = -i \xi^A \sigma^\mu \lambda_A \equiv \lambda^\mu \quad Q\phi = -i \xi^A \lambda_A \equiv -u^\mu \lambda^\mu \quad Q\tilde{\phi} = i \tilde{\xi}^A \tilde{\lambda}_A \equiv \eta$$
$$Q\lambda_A = 2\sigma^\mu \xi_A D_\mu \phi + 2i \sigma^\mu \xi_A u^\nu F_{\mu \nu}$$
$$Q\tilde{\lambda}_A = D_{AB} \xi^B + \frac{1}{2} \tilde{\sigma}^{\mu \nu} \xi_A F_{\mu \nu} - 2i \tilde{\xi}_A [\phi, \tilde{\phi}] + 2i \tilde{\xi}_A u^\mu \partial_\mu \tilde{\phi}$$

$$QD_{AB} = -i \tilde{\xi}^A \sigma^\mu D_\mu \lambda_B + 2 \xi_A (-[\phi, \bar{\lambda}_B] + u^\mu D_\mu \bar{\lambda}_B) + (A \leftrightarrow B) \tag{2.2.8}$$

where the Killing vector $u^\mu$ generates the rotations about $\mathbb{C}^2$ around the origin by

$$u^1 = -\epsilon_1 x_2, \ u^2 = \epsilon_1 x_1, \ u^3 = -\epsilon_2 x_4, \ u^4 = \epsilon_2 x_3. \tag{2.2.9}$$
The topological twisted (between $SU(2)_R$ indices and space-time indices) fields
\[ \lambda_\mu, \eta, \chi_{\mu\nu} \equiv \tilde{\xi}^A \bar{\sigma}_{\mu\nu} \tilde{\lambda}_A, \quad D_{\mu\nu} \equiv D_{AB} \xi^A \bar{\sigma}_{\mu\nu} \tilde{\xi}^B + 2F^+_{\mu\nu} \]
where $F^+_{\mu\nu}$ represents self-dual part of field strength $F_{\mu\nu}$ are transformed by
\[
\begin{align*}
Q\lambda &= 2iD_\mu \phi + 2u^\nu F_{\nu\mu} \\
Q\eta &= -2[\phi, \bar{\phi}] + 2u^\mu \partial_\mu \bar{\phi} \\
Q\chi_{\mu\nu} &= D_{AB}
\end{align*}
\]
In these terms, it is easy to confirm that the algebra generated by this supersymmetry transformation is
\[ Q^2 = -2[\phi, \cdot] + 2u^\mu D_\mu \]
where first term generates gauge transformation and in this sense by the twisted dimensional reduction (2.2.4) the four dimensional theory whose supersymmetries generates bosonic symmetries are obtained. The Yang-Mills action (2.2.3) also becomes
\[
\mathcal{L} = \text{Tr} \left[ \frac{1}{2} F^\mu_\nu F_{\mu\nu} - 4D^\mu \bar{\phi}(D_\mu \phi + iu^\nu F_{\nu\mu}) + 4(\phi, \bar{\phi}) - u^\mu D_\mu \phi \right]^2
\]
\[
- \frac{1}{4}(D_{\mu\nu} - 2F^+_{\mu\nu})(D^{\mu\nu} - 2F^{+\mu\nu})
+ 2i\eta \xi^A \bar{\sigma}^{\mu\nu} \xi_A D_\mu \lambda_\nu - \frac{1}{2} \chi_{\mu\nu} (\tilde{\xi}^A \bar{\sigma}^{\mu\nu} \bar{\sigma}^{[\lambda} \sigma^{\rho]} \xi_A) D_{[\lambda} \bar{\lambda}_\rho]
- 2\lambda_\mu (\tilde{\xi}^A \bar{\sigma}^{\mu\nu} \xi_A) [\bar{\phi}, \lambda_\nu] - 2(\eta \tilde{\xi}^A \xi_A)[\bar{\phi}, \eta] - \frac{1}{8} \chi_{\mu\nu} (\tilde{\xi}^A \bar{\sigma}^{\mu\nu} \bar{\sigma}^{\lambda\rho} \xi_A)[\phi, \chi_{\lambda\rho}]
\]
\[
= \text{Tr} \left[ Q\{\chi_{\mu\nu} F^{+\mu\nu} - \frac{1}{4} \chi_{\mu\nu} D^{\mu\nu} + \eta Q\eta + 2i\lambda_\mu D_\mu \bar{\phi}\} - \frac{1}{4} e^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \right]
\]
which is composed of supersymmetry-exact terms and an instanton term. Under the localization computation, the path integral localizes onto the saddle point of the translation along the Killing vector $u^\mu$, origin.

### 2.2.2 Fixed points theorem in instantons

Consider the supersymmetric five dimensional gauge theory on $\mathbb{C}^2(z_1, z_2) \times S^1(x_5)$ with an identification similar to that of previous subsection (2.2.4)
\[ (z_1, z_2, x_5) \sim (e^{i\beta_1} z_1, e^{i\beta_2} z_2, x_5 + \beta). \]
Take the gauge group whose rank is $N$ as $G$, and flavor symmetry whose rank is $N_f$ as $F$. The supersymmetric partition function of the theory becomes to the trace over the states with a particular instanton sector, and it is obtained from the summation of integrals over each instanton moduli space. However this moduli spaces are singular with small instantons, it can be made smooth by space-time noncommutativity which does not affect to the last result, called ADHM construction. Then the index theorem [17] about abelian
$U(1)^2 \times U(1)^N \times U(1)^{N_f}$ action where the first $U(1)^2$ rotates $\mathbb{C}^2$ and it turns out that only the contributions from the fixed points of this abelian action should be collected, or to say “localization”. With an appropriate four dimensional limit $\beta \to 0$, the instanton partition function of four dimensional $\mathcal{N} = 2$ supersymmetric gauge theory with this Omega-deformed background can be computed. The detail calculation are referred to the original paper [4] and abundant reviews for example [18, 19].

The explicit formula for the theory with the gauge group $U(N)$ and $f$ fundamental and anti-fundamental matters is as following. Let $\vec{Y} = (Y_1, \cdots, Y_N)$ be a set of Young tableaux. $k$ instanton contribution to partition function in $\mathcal{N} = 2$ $U(N)$ supersymmetric gauge theory is counted from this set of two Young tableaux $\vec{Y}$ whose total number of boxes is $k = |Y| = |Y_1| + \cdots |Y_N|$. Actually the box in these Young tableaux represents the label of fixed points. The partition function becomes

$$Z_{\text{inst}} = \sum_{\vec{Y}} q^{Y} z_{\text{vec}}(\vec{a}, \vec{Y}) \prod_{k=1}^{f} z_{\text{antifund.}}(\vec{a}, \vec{Y}_k, \mu_k) \prod_{k=1}^{f} z_{\text{fund.}}(\vec{a}, \vec{Y}_k, \mu_k)$$

(2.2.15)

where $\vec{a} = (a_1, \cdots, a_N)$ is adjoint scalar, and $\mu_i$ are masses of matter multiplets.

$$q = e^{2\pi i \tau_{UV}} = e^{-\frac{\alpha'^2}{g_{UV}^2} + i\theta_{UV}}$$

(2.2.16)

stands for coupling constant of the gauge theory. Each Young tableau $\vec{Y} = (\lambda_1 \geq \lambda_2 \geq \cdots)$ has its transpose tableau $Y^T = (\lambda'_1 \geq \lambda'_2 \geq \cdots)$. We can denote the coordinate of the box by $(i, j)$ as figure 2.1, and we also define the arm length $A_Y(s) \equiv \lambda_i - j$ and leg length $L_Y(s) = \lambda'_j - i$ to each box $s$ by using the its coordinate. Furthermore, we define the function

$$E(a, Y_i, Y_j, s) \equiv a - \epsilon_1 L_{Y_j}(s) + \epsilon_2 (A_{Y_i}(s) + 1)$$

(2.2.17)

to represent $z_{\text{vec}}$ explicitly

$$z_{\text{vec}}^{-1} = \prod_{i,j=1}^{N} \prod_{s \in Y_i} [E(a_i - a_j, Y_i, Y_j, s)] \prod_{t \in Y_j} [Q - E(a_j - a_i, Y_j, Y_i, t)].$$

(2.2.18)

where $Q = \epsilon_1 + \epsilon_2$. In addition, the function

$$\phi(a, s) \equiv a + b(i - 1) + \frac{1}{b}(j - 1)$$

(2.2.19)
is used in
\[
\begin{align*}
  z_{\text{fund}}(\vec{a}, \vec{Y}, \mu_k) &= \prod_{i=1}^{N} \prod_{s \in Y_i} (\phi(a_i, s) - i\mu_k + \frac{Q}{2}) \\
z_{\text{antifund}}(\vec{a}, \vec{Y}, \mu_k) &= \prod_{i=1}^{N} \prod_{s \in Y_i} (\phi(a_i, s) + i\mu_k + \frac{Q}{2}).
\end{align*}
\]

To derive the instanton partition function for the gauge theory with gauge group \(SU(N)\), the above results is divided by \(U(1)\) factor.

### 2.2.3 Application to gauge theory

Part of this discussion is based on [1].

The application of localization computation to derive exact formulas for the supersymmetric-closed observables such that partition function as well as expectation values of Wilson loops of the gauge theory constructed on compact manifolds is started by Pestun [5]. He considered 4D \(\mathcal{N} = 2\) supersymmetric gauge theories or Seiberg-Witten (SW) theories on round four sphere. Similar exact results have also been obtained for 3D \(\mathcal{N} \geq 2\) gauge theories on round three-sphere [20, 21, 22], its orbifold [23, 24, 25], and 2D theories on sphere [26, 27] or on squashed \(S^3\) [28]. These all served as new powerful tools to study the strong coupling behavior of the theories at low energy or other non-perturbative aspects.

At first, most of the work in this field has been focusing on theories on round spheres. A natural question would then be what other curved spaces admit rigid supersymmetry. Some systematic analysis has been made in [29, 30, 31, 32, 33] to draw conditions on the background geometry from Killing spinor equation. Also, in [34, 35, 36] another construction of supersymmetric gauge theories in three or five dimensions has been discussed in connection with contact geometry, and moreover some exact results have been worked out for theories on 3D Seifert manifolds. On the other hand, in the work [37], we constructed the gauge theory on \textit{ellipsoids} \(S^3_b\), or deformation of \(S^3\)s, preserving the supersymmetry. The definition of \(S^3_b\) when it is embedded in \(\mathbb{R}^4\) is
\[
\frac{x_1^2 + x_2^2 + x_3^2 + x_4^2}{\ell^2} = 1,
\]
then this squashing breaks the isometry \(SO(4)\) of \(S^3\) to \(U(1) \times U(1)\). The deformation is parametrized by \(b = \sqrt{\frac{\ell}{\ell'}}\), and in particular, a round sphere \(S^3\) is reproduced when \(b = 1\).

In this work we also applied the localization technique to the three dimensional \(\mathcal{N} = 2\) supersymmetric gauge theory constructed on \(S^3_b\), and the resulting physical observables were represented as nontrivial one-parameter generalizations by \(b\) from those of the theory on \(S^3\). This work stimulated the following works which studied the characterization of supersymmetric gauge theory from the viewpoint of exact values of physical observables. One of the most successful work is [38, 39, 40, 41]. They found that in the cases of continuum deformations from three dimensional \(\mathcal{N} = 2\) supersymmetric gauge theory constructed on \(S^3\) such as squashing of the \(S^3\) or changing the values of background fields, our deformation was most essential one in the sense that this was the almost only way
to deform almost integral contact structures, which determine whether the deformation gives nontrivial changes to exact values of observables or not.

For other work on supersymmetric deformations of the round $S^3$ with additional background fields, see [42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52]. Comprehensive review was recently appeared [53]. Also a part of three-sphere and of two-sphere with boundaries or two-disc which preserve supersymmetries are considered in [54, 55, 56, 57], and the theories with half-BPS vortex loop operators [58]. The above result in three dimensions implies that the correct deformation of $S^4$ should be a fibration of the ellipsoid (2.2.21) over a line segment, because the S-duality wall can then wrap the 3D fiber anywhere in four dimensions in a supersymmetric manner. In four dimensional case, $S^2 \times S^2$ is treated in [59, 60], certain theories on $T^2 \times S^2$ is computed in [61] and toric Kähler manifolds are considered in [62]. The four dimensional $\mathcal{N} = 1$ supersymmetric gauge theories with gaugino condensation is suggested to be computed in localization principle [63].

The partition functions of supersymmetric gauge theories are calculated on a round five-sphere in [64, 65, 66, 67, 68, 69] and on a squashed five-sphere in [70, 71, 72, 73]. More general five-dimensional manifolds admitting rigid supersymmetry are explored in [74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84].

The localization formula remains much room to be generalized. Especially, in the process to clarify what is the fundamental element in the partition function obtained by exact calculation in supersymmetric gauge theories, there is a recent development called factorization [85, 86, 87]. In case of certain class of three dimensional $\mathcal{N} = 2$ gauge theory constructed on $S^3_b$, the partition function is divided into a product of two parts, holomorphic block and anti-holomorphic block, each of which describes the contribution of vortices localized on north pole and south pole of $S^3_b$ respectively. The way of the gluing these two parts are specified as “s-fusion” for the partition function. This s-fusion relates the Planck constants of two blocks as $h_{NP} \leftrightarrow -\frac{1}{h_{SP}}$. There is another kind of fusion called “id-fusion”, which relates them as $h_{NP} \leftrightarrow -h_{SP}$, and the product of these parts glued by the id-fusion represents the partition function of the three dimensional gauge theory constructed on the twisted product $S^2 \times_q S^1$ where the twist is parametrized by $q(h)$. When this $S^1$ is considered as a time direction, the partition function of the theory on $S^2 \times_q S^1$ becomes superconformal index of the theory on $S^2 \times \mathbb{R}$. In other words, partition functions or superconformal indices themselves are not most fundamental structures which classify gauge theories, but factorized (anti-)holomorphic parts can characterize the geometric features. This factorization property is also found in four dimensional $\mathcal{N} = 1$ theory on $S^3 \times S^1$ [88] and certain class of five dimensional supersymmetric gauge theories [89], and opens two novel ways of analyzing the gauge theories. First one is Higgs branch localization and another one is fixing the measure of path integral.

The usual localization results noted so far are represented as an integral over Coulomb branch, the moduli space where the scalars in vector multiplets have constant values. In some cases of three dimensional theories, there are another branch in moduli space called Higgs branch, where matter multiplets have constant values configurations. The formulas for Higgs branch localization are written in integral over this branch, and turn out to be summation over vortices configurations like the results of factorization. This technique can be applied to three dimensional $\mathcal{N} = 2$ theory as found in [90, 91], and as a result, the range of localization is extended by application of Higgs branch localization, for instance
five dimensional gauge theory constructed on toric Sasaki-Einstein manifolds [92, 93]. This direction is hopeful way to generalizations of localization formula, and it is also interesting to interpret this form of Higgs branch localization in the dualities to other theories. More recently the partition function in particular four dimensional gauge theories are computed in Higgs branch localization and it is factorized [94, 95, 96].

Second usage of the factorization property is fixing the measure of a path integral. The integral measure of the partition function can absorb an overall constant and a phase as a normalization constant. However, when the vacua are discrete, the contribution from each sector must be summarized including this factor. For example, the work [97] considers the partition function of the theory built on an orbifold $S^3/Z_n$, and a resulting formula is summation over the sectors labeled by $Z_n$ holonomy. For certain theories, they determined the overall phases of each sector by comparing results with dual non-gauged theories, but in general, they are not determined. This ambiguity does not arise in superconformal index, which is defined with the measure: it counts the states one by one with particular charges. Therefore, via a transformation from superconformal indices to partition functions, the relative measures between sectors are expected to be determined by factorization technique. In particular, I am recently studying the supersymmetric gauge theory with project gauge groups, and considering the establish of the formula for this projection.
In this chapter, we show the explicit example localization calculation for the Seiberg-Witten theories constructed on four dimensional ellipsoids, which is deformed $S^4$ with isometry $U(1) \times U(1)$. Actually, to avoid IR divergence, supersymmetric gauge theories have to be constructed on compact manifold to apply localization method, and the results reflect the information of the manifolds. In this procedure, the background fields must be introduced to preserve rigid supersymmetry in spite of compactification. The most of this chapter is based on [1]. More general and recent result for the theory on warped product $T^2 \times \Sigma_2$ where $T^2$ is a flat two-torus and $\Sigma_2$ is a Riemann surface is arranged in [98].

### 3.1 Introduction

In this chapter we show derivation of the partition function of Seiberg-Witten theories on the four dimensional ellipsoids,

$$\frac{x_0^2}{r^2} + \frac{x_1^2 + x_2^2}{\ell^2} + \frac{x_3^2 + x_4^2}{\tilde{\ell}^2} = 1,$$

(3.1.1)

with some additional background fields. As can be easily guessed from the previous result [37], the additional fields include an R-symmetry gauge field which takes values on $SU(2)$ Lie algebra this time. Moreover, it turns out that the relevant off-shell 4D $\mathcal{N} = 2$ supergravity multiplet contains some more auxiliary fields, and they also have to take nonzero values to make the background supersymmetric. The paper [15] suggests that localization calculation for the theory on $n$ covering four sphere is same as this theory.

The organization of this chapter is as follows. in Section 3.2 we present the set of Killing spinor equations, and the action and supersymmetry of general SW theories on arbitrary curved backgrounds which support Killing spinors. Then in Section 3.3 we analyze the Killing spinor equation on ellipsoids. It will be shown that, by assuming that a Killing spinor on round $S^4$ remains after the deformation of the metric, one can solve the Killing spinor equation in favor of the background gauge and auxiliary fields and determine their form up to some arbitrariness. The square of the supersymmetry yields an isometry of
the ellipsoid which fixes two special points, i.e. the north and south poles. It is shown
that the theory looks near the two poles like the (anti-)topologically twisted theory with
Omega deformation parameter \((\epsilon_1, \epsilon_2) = (\ell^{-1}, \tilde{\ell}^{-1})\). In Section 3.4 I carry out the explicit
path integration using the SUSY localization principle. The argument here follows closely
that of Pestun [5].

3.2 Seiberg-Witten Theories on Curved Spaces

Manifolds which can support supersymmetric field theories are characterized by the exis-
tence of Killing spinors. In this chapter we consider theories which, when realized on a flat
\(\mathbb{R}^4\), have eight supercharges, i.e. 4D \(N = 2\) supersymmetric theories or Seiberg-Witten
(SW) theories. Take gauge group as \(G\). For these theories, supersymmetry is character-
ized by a pair of a chiral and an anti-chiral Killing spinors \(\xi^A = (\xi^{\alpha A}, \bar{\xi}^{\dot{\alpha} A})\), both with an
additional \(SU(2)_R\) doublet index \(A, B, \cdots\).

In the off-shell 4D \(N = 2\) standard Weyl multiplet, there are three fermionic fields,
\(\psi^I_m\) which is assigned to Poincaré supercharge, the auxiliary spinor \(\chi^I\) and \(\phi^I_m\) assigned to
conformal supercharge. To preserve supersymmetry, it is demanded that supersymmetric
transformations for these fields must be zero, and this is called Killing spinor equation.
Actually \(\phi^I_m\) is composite fields that does not needed to describe the degree of freedom
and constraints on \(\phi^I_m\) does not produce independent formula. Then in following we only
consider for gravitino \(\psi^I_m\) and fermion \(\chi^I\). They are solved simultaneously and bring two
sets of constraints on Killing spinor.

The first set is called the main equation

\[
\begin{align*}
D_m \xi^A &+ T^k l \sigma_{kl} \sigma^A m \xi^A = -i \sigma^A m \bar{\xi}^l, \\
D_m \bar{\xi}^A &+ \bar{T}^k l \bar{\sigma}_{kl} \bar{\sigma}^A m \bar{\xi}^A = -i \bar{\sigma}^A m \bar{\xi}^l
\end{align*}
\]

for some \(\xi^A, \bar{\xi}^A\). (3.2.1)

Here \(T^k l, \bar{T}^k l\) are a self-dual and an anti-self-dual real tensor background fields, and the
covariant derivatives contain a background \(SU(2)_R\) gauge field \(V_m^A B\) in addition to spin
connection \(\Omega_{m}^{a b}\).

\[
\begin{align*}
D_m \xi_A &\equiv \partial_m \xi_A + \frac{1}{4} \Omega_{m}^{a b} \sigma_{a b} \xi_A + i \xi_B V_m^B A, \\
D_m \bar{\xi}_A &\equiv \partial_m \bar{\xi}_A + \frac{1}{4} \Omega_{m}^{a b} \bar{\sigma}_{a b} \bar{\xi}_A + i \bar{\xi}_B V_m^B A.
\end{align*}
\]

(3.2.2)

The second set is called the auxiliary equation:

\[
\begin{align*}
\sigma^m \bar{\sigma}^n D_m D_n \xi_A &+ 4 D_l T_{m n} \sigma^{m n} \sigma^l \xi_A = M \xi_A, \\
\bar{\sigma}^m \sigma^n D_m D_n \bar{\xi}_A &+ 4 D_l \bar{T}_{m n} \bar{\sigma}^{m n} \bar{\sigma}^l \bar{\xi}_A = M \bar{\xi}_A.
\end{align*}
\]

(3.2.3)

where \(M\) is a scalar background field. we will later show that, if a 4D manifold with
possibly nonzero background fields \(T^k l, \bar{T}^k l, V_m^A B\) and \(M\) admits a Killing spinor satisfying
these equations, one can define SW theories on it with a rigid supersymmetry.

The above generalized Killing spinor equation was found following the suggestion of [29]
to consider the coupling to off-shell supergravity. The set of background fields and Killing
spinor equations can be compared to the auxiliary fields in the supergravity multiplet and BPS equations of [99], but there are some differences due to the change in spacetime signature. As an example, although SW theories are known to have $SU(2) \times U(1)$ R-symmetry, we do not consider background $U(1)_R$ gauge field because the $U(1)$ phase rotation is not compatible with the reality condition of SUSY parameter (A.3.3). Also, this $U(1)_R$ will be broken explicitly if the background fields $T^{kl}, T^{kl}$ take nonzero values.

### 3.2.1 Vector multiplets.

Vector multiplet consists of a gauge field $A_m$, gauginos $\lambda_{\alpha A}, \tilde{\lambda}_{\dot{\alpha} A}$, two scalar fields $\phi, \tilde{\phi}$ and an auxiliary field $D_{AB} = D_{BA}$ all taking values on the same Lie algebra. Their SUSY transformation rule is given by

\[
\begin{align*}
QA_m &= i\xi^A \sigma_m \tilde{\lambda}_A - i\tilde{\xi}^A \sigma_m \lambda_A, \\
Q\phi &= -i\xi^A \lambda_A, \\
Q\tilde{\phi} &= +i\tilde{\xi}^A \tilde{\lambda}_A, \\
Q\lambda_A &= \frac{1}{2}\sigma^{mn} \xi_A (F_{mn} + 8\tilde{\phi} T_{mn}) + 2\sigma^m \tilde{\xi}_A D_m \phi + \sigma^m D_m \xi_A \phi + 2i\xi_A [\phi, \tilde{\phi}] + D_{AB} \xi^B, \\
Q\tilde{\lambda}_A &= \frac{1}{2}\sigma^{mn} \tilde{\xi}_A (F_{mn} + 8\phi T_{mn}) + 2\sigma^m \tilde{\xi}_A D_m \tilde{\phi} + \sigma^m D_m \tilde{\xi}_A \tilde{\phi} - 2i\tilde{\xi}_A [\phi, \tilde{\phi}] + D_{AB} \tilde{\xi}^B, \\
QD_{AB} &= -i\tilde{\xi}^A \tilde{\sigma}^m D_m \lambda_B - i\xi^B \sigma^m D_m \lambda_A + i\xi_A \sigma^m D_m \tilde{\lambda}_B + i\tilde{\xi}^A \tilde{\sigma}^m D_m \lambda_A
\end{align*}
\]

Here and throughout this chapter, I take the Killing spinor $\xi$ to be Grassmann-even so that $Q$ is the supercharge which flips the statistics of the fields. The above transformation rule is compatible with the reality condition of SUSY parameter (A.3.3) if I assume

\[
(A_m)^\dagger = A_m, \quad (\lambda_{\alpha A})^\dagger = \lambda^{\dot{\alpha} A}, \quad (\tilde{\lambda}_{\dot{\alpha} A})^\dagger = \lambda^{\alpha A}, \\
\phi^\dagger = \phi, \quad (\tilde{\phi})^\dagger = \tilde{\phi}, \quad (D_{AB})^\dagger = D^{AB}.
\]

Note that $\phi, \tilde{\phi}$ are two independent real scalar fields.

The supersymmetry algebra closes off-shell, i.e. $\{Q_\xi, Q_\eta\}$ is a sum of bosonic symmetries for arbitrary pair of Killing spinors $\xi, \eta$. Here we give the formula for the square $Q^2$ of the supersymmetry for a Killing spinor $\xi$,

\[
\begin{align*}
Q^2 A_m &= i\nu^m F_{nm} + D_m \Phi, \\
Q^2 \phi &= i\nu^m D_n \phi + i[\Phi, \phi] + (w + 2\Theta) \phi, \\
Q^2 \tilde{\phi} &= i\nu^m D_n \tilde{\phi} + i[\Phi, \tilde{\phi}] + (w - 2\Theta) \tilde{\phi}, \\
Q^2 \lambda_A &= i\nu^m D_n \lambda_A + i[\Phi, \lambda_A] + (\frac{3}{2} w + \Theta) \lambda_A + \frac{1}{4} \sigma^{kl} \lambda_A D_k v_l + \Theta_{AC} D^C_B, \\
Q^2 \tilde{\lambda}_A &= i\nu^m D_n \tilde{\lambda}_A + i[\Phi, \tilde{\lambda}_A] + (\frac{3}{2} w - \Theta) \tilde{\lambda}_A + \frac{1}{4} \tilde{\sigma}^{kl} \tilde{\lambda}_A D_k v_l + \Theta_{AB} \tilde{\lambda}^B, \\
Q^2 D_{AB} &= i\nu^m D_n D_{AB} + i[\Phi, D_{AB}] + 2w D_{AB} + \Theta_{AC} D^C_B + \Theta_{BC} D^C_A.
\end{align*}
\]
where the various transformation parameters are defined as follows,

\[ u^m = 2\bar{\xi}^A \sigma^m \xi_A, \]
\[ \Phi = -2i\phi \bar{\xi}^A \xi_A + 2i\bar{\phi} \xi_A, \]
\[ w = -\frac{i}{2} (\xi^A \sigma^m D_m \bar{\xi}_A + D_m \xi^A \sigma^m \xi_A), \]
\[ \Theta = -\frac{i}{2} (\xi^A \sigma^m D_m \bar{\xi}_A - D_m \xi^A \sigma^m \xi_A), \]
\[ \Theta_{AB} = -i\xi(A\sigma^m D_m \bar{\xi}_B) + iD_m \xi(A\sigma^m \bar{\xi}_B). \]  

(3.2.7)

Note that, if \( \xi \) satisfies the main Killing spinor equation (3.2.1) only, the algebra does not close on \( D_{AB} \). The failure term

\[ \Delta_{AB} = -2i\phi(\bar{\xi}(A\bar{\sigma}^k \sigma^l D_k D_l \xi_B) + 4\xi(A\bar{\sigma}^m \sigma^k \xi_B)(D_m \bar{T}_{mn}) \]
\[ + 2i\bar{\phi}(\xi(A\sigma^m \sigma^k D_k D_l \xi_B) + 4\xi(A\sigma^m \sigma^k \xi_B)(D_k T_{mn})], \]

(3.2.8)

vanishes if \( \xi \) satisfies also the auxiliary equation.

The supersymmetric Yang-Mills Lagrangian is given by

\[ \mathcal{L}_{YM} = \text{Tr} \left[ \frac{1}{2} F_{mn} F^{mn} + 16 F_{mn}(\phi T^{mn} + \phi \bar{T}^{mn}) + 64 \phi^2 T_{mn} T^{mn} + 64 \phi^2 \bar{T}_{mn} \bar{T}^{mn} \right. \]
\[ - 4 D_m \bar{\phi} D^m \phi + 2 M \phi \bar{\phi} - 2i \lambda A \sigma^m D_m \bar{\lambda}_A - 2 \lambda A [\phi, \bar{\lambda}_A] + 2 \lambda A [\phi, \bar{\lambda}_A] \]
\[ + 4 [\phi, \bar{\phi}]^2 - \frac{1}{2} D_{AB} D_{AB} \].  

(3.2.9)

For round \( S^4 \) of radius \( \ell \) with no background \( SU(2)_R \) gauge field or auxiliary tensor fields turned on, this Lagrangian reduces to the one found in [5] with \( M = -\frac{1}{3} R = -\frac{1}{2^2} \). The action is then defined by combining \( \mathcal{L}_{YM} \) with the topological term,

\[ S_{YM} = \frac{1}{g_{YM}^2} \int d^4x \sqrt{g} \mathcal{L}_{YM} + \frac{i\theta}{8\pi^2} \int \text{Tr}(F \wedge F). \]  

(3.2.10)

Instantons and anti-instantons are topologically non-trivial configurations of gauge field satisfying \( *F = -F \) or \( *F = F \). Since the action for instantons is written in total derivative, the solutions are labeled by gauge transformation \( g : S^3 \rightarrow G \) where \( S^3 \) is boundary surface of \( \mathbb{R}^4 \). For \( G = SU(N) \) case the integer called instanton number \( \pi_3(SU(N)) = \mathbb{Z} \) labels the solution, and in other case, while the definition absorbs normalization factor, the solutions are also characterized by an integer. Take instanton number as \( n \in \mathbb{Z} \). The classical action on instanton or anti-instanton backgrounds takes values

\[ \text{instanton (} n > 0 \text{)} : -S_{YM} = 2\pi i n \tau, \quad \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2} \]
\[ \text{anti-instanton (} n < 0 \text{)} : -S_{YM} = 2\pi i n \tilde{\tau}, \quad \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2}. \]  

(3.2.11)

The Lagrangian (3.2.9) is not positive definite and path integral becomes ill-defined if the fields take values according to the reality condition (3.2.5). The actual path integral should therefore be defined with the modified contours along which

\[ \phi^\dagger = -\bar{\phi}, \quad (D_{AB})^\dagger = -D^{AB}. \]  

(3.2.12)
For $U(1)$ gauge group, there is also the Fayet-Iliopoulos type invariant. Let $w^{AB} = w^{BA}$ be a $SU(2)_R$ triplet background field satisfying

$$w^{AB} \xi_B = \frac{1}{2} \sigma^n D_n \tilde{\xi}^A + 2 T_{kl} \sigma^{kl} \tilde{\xi}^A,$$

$$w^{AB} \tilde{\xi}_B = \frac{1}{2} \bar{\sigma}^n D_n \xi^A + 2 \bar{T}_{kl} \bar{\sigma}^{kl} \xi^A. \tag{3.2.13}$$

Then one can construct the following invariant from a $U(1)$ vectormultiplet,

$$\mathcal{L}_{FI} \equiv w^{AB} D_{AB} - M (\phi + \bar{\phi}) - 64 \phi T^{kl} T_{kl} - 64 \bar{\phi} \bar{T}^{kl} \bar{T}_{kl} - 8 F^{kl} (T_{kl} + \bar{T}_{kl}). \tag{3.2.14}$$

### 3.2.2 Hypermultiplets.

The system of $r$ hypermultiplets consists of scalars $q_{AI}$ and fermions $\psi_{aI}, \bar{\psi}_{\dot{a}I}$ satisfying the reality conditions

$$(q_{AI})^\dagger = q^{AI} = \Omega^{IJ} e^{AB} q_{JB},$$

$$(\psi_{aI})^\dagger = \psi^{aI} = e^{\alpha \beta} \Omega^{IJ} \psi_{\beta J},$$

$$(\bar{\psi}_{\dot{a}I})^\dagger = \bar{\psi}^{\dot{a}I} = e^{\dot{\alpha} \dot{\beta}} \Omega^{IJ} \bar{\psi}_{\dot{\beta} J}. \tag{3.2.15}$$

Here $I, J = 1, \cdots, 2r$ are $Sp(r)$ indices and $\Omega^{IJ}$ is the real antisymmetric $Sp(r)$-invariant tensor satisfying

$$(\Omega^{IJ})^* = -\Omega_{IJ}, \quad \Omega^{IJ} \Omega_{JK} = \delta^I_K. \tag{3.2.16}$$

Pairs of $Sp(r)$ indices contracted in the order of top-left, bottom-right will be often suppressed. For example, $q^A q_A \equiv q^{AI} q_{AI}$. These matter fields can couple to vector multiplets through an embedding of the gauge group into $Sp(r)$. Namely, when vector multiplet fields such as $A_m$ are multiplied on hypermultiplet fields, they are thought of as $2r \times 2r$ matrices with elements $(A_m)_{IJ}$. The covariant derivatives of matters therefore take the form

$$D_m q_{IA} \equiv \partial_m q_{IA} - i (A_m)_{I}^{J} q_{JA} + i q_{IB} (V_m)^B_A,$$

$$D_m \psi_{aI} \equiv \partial_m \psi_{aI} - i (A_m)_{I}^{J} \psi_{aJ} + \frac{1}{4} \Omega^{\alpha \beta}_{mn} (\sigma_{\alpha \beta})_{IJ} \psi_{\beta I}, \quad \text{etc.} \tag{3.2.17}$$

It is straightforward to find on-shell SUSY transformation rule,

$$Q^{\alpha}_{\psi q_A} = -i \xi_A \psi + i \tilde{\xi}_A \bar{\psi},$$

$$Q^{\alpha}_{\bar{\psi} q_A} = 2 \sigma^m \xi_A D_m q^A + \sigma^m D_m \xi_A q^A - 4 i \xi_A \bar{\phi} q^A,$$

$$Q^{\alpha}_{\bar{\psi}} = 2 \sigma^m \xi_A D_m q^A + \sigma^m D_m \xi_A q^A - 4 i \xi_A \bar{\phi} q^A. \tag{3.2.18}$$

and the gauge covariant kinetic Lagrangian

$$\mathcal{L}^{\alpha}_{\text{mat}} = \frac{1}{2} D_m q^A D^m q_A - q^A (\phi + \bar{\phi}) q_A + \frac{i}{2} q^A D_A q_B + \frac{1}{8} (R + M) q^A q_A$$

$$- \frac{i}{2} \bar{\psi} \sigma^m D_m \psi - \frac{1}{2} \psi \phi \psi + \frac{1}{2} \bar{\psi} \bar{\phi} \bar{\psi} + \frac{i}{2} \psi \sigma^{kl} T_{kl} \psi - \frac{i}{2} \bar{\psi} \bar{\sigma}^{kl} \bar{T}_{kl} \bar{\psi}$$

$$- q^A \lambda_A \psi + \bar{\psi} \bar{\lambda}_A q^A. \tag{3.2.19}$$
It is known that one cannot make the full $\mathcal{N} = 2$ SUSY transformation law closed off-shell with finitely many auxiliary fields. For the application of localization principle, however, one focuses on the supersymmetry $Q$ corresponding to a specific choice of Killing spinor $\xi$. It is then sufficient that $Q^2$ for that specific $\xi$ is a linear sum of bosonic symmetries on all fields off-shell.

We introduce the auxiliary scalars $F_{AI}$ satisfying the reality condition

$$(F_{IA})^\dagger = F^{AI} = \Omega^{IJ} e^{AB} F_{JB},$$

and put the full Lagrangian as follows,

$$\mathcal{L}_{\text{mat}} = \mathcal{L}_{\text{mat}}^{\alpha} - \frac{1}{2} F^A F_A.$$  \hfill (3.2.20)

The supersymmetry transformation laws of fields are extended as follows,

$$Q q_A = -i\xi_A \psi + i\tilde{\xi}_A \tilde{\psi},$$

$$Q \psi = 2\sigma^m \xi_A D_m q^A + \sigma^m D_m \tilde{\xi}_A q^A - 4i\xi_A \tilde{\phi} q^A + 2\tilde{\xi}_A F^A,$$

$$Q \tilde{\psi} = 2\tilde{\sigma}^m \xi_A D_m \tilde{q}^A + \tilde{\sigma}^m D_m \xi_A \tilde{q}^A - 4i\xi_A \tilde{\phi} \tilde{q}^A + 2\tilde{\xi}_A F^A,$$

$$Q F_A = i\xi_A \sigma^m D_m \tilde{\psi} - 2\xi_A \lambda_B q^B + 2i\xi_A (\tilde{\sigma}^{kl} T_{kl}) \psi - i\xi_A \tilde{\sigma}^m D_m \psi + 2\tilde{\xi}_A \lambda_B \tilde{q}^B - 2i\tilde{\xi}_A (\tilde{\sigma}^{kl} T_{kl}) \tilde{\psi}.  \hfill (3.2.22)$$

Here the new transformation parameters $\xi, \tilde{\xi}$ are required to satisfy

$$\xi_A \xi_B - \bar{\xi}_A \bar{\xi}_B = 0,$$

$$\xi^A \xi_A + \bar{\xi}^A \bar{\xi}_A = 0,$$

$$\bar{\xi}^A \bar{\xi}_A + \xi^A \xi_A = 0,$$

$$\xi^A (\sigma^m \xi_A + \bar{\xi}^A \sigma^m \bar{\xi}_A) = 0.  \hfill (3.2.23)$$

Similar off-shell transformation rule which makes use of constrained transformation parameters like $\xi, \tilde{\xi}$ here has been written down for 4D $\mathcal{N} = 4$ gauge theories on $S^4$ in [5], and for 5D SUSY theories on $S^5$ in [65]. One can then show that $Q$ squares into a linear sum of bosonic symmetries off-shell,

$$Q^2 q_A = iv^m D_m q_A + i\Phi q_A + w q_A + \Theta_{AB} q_B,$$

$$Q^2 \psi = iv^m D_m \psi + i\Phi \psi + \frac{3}{2} w \psi - \Theta \psi + \frac{1}{2} \sigma^{kl} \tilde{\psi} D_k v_l,$$

$$Q^2 \tilde{\psi} = iv^m D_m \tilde{\psi} + i\Phi \tilde{\psi} + \frac{3}{2} w \tilde{\psi} + \Theta \psi + \frac{1}{2} \tilde{\sigma}^{kl} \tilde{\phi} D_k v_l,$$

$$Q^2 F_A = iv^m D_m F_A + i\Phi F_A + 2w F_A + \Theta_{AB} F_B. \hfill (3.2.24)$$

Here the parameters $v^m, \Phi, w, \Theta, \Theta_{AB}$ are as in (3.2.7) and

$$\Theta_{AB} = 2i\xi_A (\sigma^m D_m \tilde{\xi}_B) - 2iD_m \xi_A (\sigma^m \tilde{\xi}_B) + 4i\xi_A (\sigma^{kl} T_{kl} \tilde{\xi}_B) - 4i\tilde{\xi}_A (\sigma^{kl} T_{kl} \xi_B).  \hfill (3.2.25)$$

Note that $\xi_A, \tilde{\xi}_A$ and $F_A$ transform as doublets under a local symmetry which we call $SU(2)_{R}$, reflecting the fact that the choice of $\xi_A, \tilde{\xi}_A$ satisfying (3.2.23) is not unique. This
also means that the covariant derivative of $F_A$ contains the background $SU(2)_R$ gauge field $V_m^B$.

$$D_m F_{IA} \equiv \partial_m F_{IA} - i(A_m)_I^J F_{JA} + iF_{IB} V_m^B.$$  \hspace{1cm} (3.2.26)

For notational simplicity, we use for their doublet indices the same letters $A, B, \ldots$ as for the $SU(2)_R$ indices.

The off-shell transformation rule (3.2.22) is compatible with the reality condition of the fields (3.2.15) and (3.2.20). However, if we define the theory of hypermultiplets by the kinetic Lagrangian $L_{\text{mat}}$, we have to take the actual path integration contour in such a way that its bosonic part is positive definite. Therefore, we choose the integration contour for $F_{AI}$ differently from its real locus, so that

$$(F_{IA})^\dagger = - F^{AI}$$ \hspace{1cm} (3.2.27)

along the contour.

An important fact which will be used later is that the matter kinetic Lagrangian is supersymmetry exact. Assuming $\xi^A \xi_A - \bar{\xi}^A \bar{\xi}_A = 1$ which will be verified in the next section, one can show that

$$L_{\text{mat}} = Q V_{\text{mat}},$$

$$2V_{\text{mat}} = \psi \xi^A F_A - \bar{\psi} \bar{\xi}^A F_A + \psi \sigma^m D_m (\xi_A q^A) - \bar{\psi} \bar{\sigma}^m D_m (\bar{\xi}_A \bar{q}^A)
+ 2(i \psi \phi + \psi \sigma^{kl} T_{kl} + i q^B \lambda_B) \xi_A q^A - 2(i \bar{\psi} \bar{\phi} + \bar{\psi} \bar{\sigma}^{kl} \bar{T}_{kl} + i q^B \bar{\lambda}_B) \bar{\xi}_A \bar{q}^A.$$ \hspace{1cm} (3.2.28)

### 3.3 Supersymmetry on 4D Ellipsoids

It has been known that round spheres in various dimensions admit Killing spinors satisfying

$$D_m \zeta = \Gamma_m \zeta'$$ \hspace{1cm} for some $\zeta'$.

(3.3.1)

In [37] it was shown that the 3D ellipsoids with $U(1) \times U(1)$ isometry admit a pair of charged Killing spinors coupled to a suitably chosen background $U(1)_R$ gauge field. The ellipsoid is defined by an embedding equation in flat $\mathbb{R}^4$ with Cartesian coordinates $x_1, \cdots, x_4,$

$$\frac{x_1^2 + x_2^2}{\ell^2} + \frac{x_3^2 + x_4^2}{\tilde{\ell}^2} = 1.$$ \hspace{1cm} (3.3.2)

The goal of this section is to show that similar ellipsoids in four dimensions,

$$\frac{x_1^2}{\ell^2} + \frac{x_2^2}{\bar{\ell}^2} + \frac{x_3^2}{\tilde{\ell}^2} + \frac{x_4^2}{\tilde{\bar{\ell}}^2} = 1,$$ \hspace{1cm} (3.3.3)

admit a Killing spinor satisfying (3.2.1) and (3.2.3) if the background fields $V_m^A, T_{kl}, \bar{T}_{kl}, M$ are chosen appropriately. We will restrict to those backgrounds with $U(1) \times U(1)$ isometry, and anticipate that the square of the supersymmetry yield a linear combination of the two $U(1)$ isometries which fix the north and south poles of the ellipsoid. The ellipsoid (3.3.3) is thus parametrized by three axis-length parameters.
Introducing a polar coordinate system,
\[
\begin{align*}
  x_0 &= r \cos \rho, \\
  x_1 &= \ell \sin \rho \cos \theta \cos \varphi, \\
  x_2 &= \ell \sin \rho \cos \theta \sin \varphi, \\
  x_3 &= \tilde{\ell} \sin \rho \sin \theta \cos \chi, \\
  x_4 &= \tilde{\ell} \sin \rho \sin \theta \sin \chi,
\end{align*}
\]
(3.3.4)
the vielbein one-forms \(E^a = E^a_m dx^m\) can be chosen as
\[
\begin{align*}
  E^1 &= \sin \rho e^1, & E^2 &= \sin \rho e^2, & E^3 &= \sin \rho e^3 + h \rho, & E^4 &= g \rho, \\
(3.3.5)\end{align*}
\]
where
\[
\begin{align*}
  f &= \sqrt{\ell^2 \sin^2 \theta + \tilde{\ell}^2 \cos^2 \theta}, \\
  g &= \sqrt{r^2 \sin^2 \rho + \ell^2 \tilde{\ell}^2 f - 2 \cos^2 \rho}, \\
  h &= \frac{\tilde{\ell}^2 - \ell^2}{f} \cos \rho \sin \theta \cos \theta,
\end{align*}
\]
(3.3.6)
and \(e^a\) are vielbein of the 3D ellipsoid (3.3.2) in polar coordinates \((\varphi, \chi, \theta)\),
\[
\begin{align*}
  e^1 &= \ell \cos \theta d\varphi, & e^2 &= \tilde{\ell} \sin \theta d\chi, & e^3 &= f d\theta.
\end{align*}
\]
(3.3.7)
The spin connection \(\Omega^{ab} = \Omega^{ab}_m dx^m\) has the following components,
\[
\begin{align*}
  \Omega^{12} &= 0, & \Omega^{13} &= -\frac{\ell}{f} \sin \theta d\varphi, & \Omega^{23} &= \frac{\tilde{\ell}}{f} \cos \theta d\chi, \\
  \Omega^{14} &= \frac{\tilde{\ell}^2 \cos \rho}{gf^2} e^1, & \Omega^{24} &= \frac{\ell^2 \cos \rho}{gf^2} e^2, & \Omega^{34} &= \frac{\ell^2 \tilde{\ell}^2 \cos \rho}{gf^4} e^3.
\end{align*}
\]
(3.3.8)
Note that \(\Omega^{12}, \Omega^{13}, \Omega^{23}\) are the spin connection of the 3D ellipsoid with vielbein \(e^a\). In the polar coordinates, the ellipsoid can be seen as a warped product
\[
T^2 \times \Sigma(\theta, \rho)
\]
(3.3.9)
where two cycles of \(T^2\) are coordinated by \(\varphi\) and \(\chi\) and two radii are \(\ell \sin \rho \cos \theta\) and \(\tilde{\ell} \sin \rho \sin \theta\).

3.3.1 Killing spinors on round \(S^4\).
Killing spinor equation has solutions on the round \(S^4\) of radius \(\ell\) with no background gauge or tensor auxiliary fields turned on. The main equation (3.2.1) consists of eight equations, and we divide them into two groups. The first six equations are given by
\[
\begin{align*}
  \left( \partial_m + \frac{1}{4} \Omega^{ab}_m \tau^{ab} - \frac{i \cos \rho}{2\ell} e^a_m \tau^a \right) \xi_A &= - \sin \rho e^a_m \tau^a \xi_A, \\
  \left( \partial_m + \frac{1}{4} \Omega^{ab}_m \tau^{ab} + \frac{i \cos \rho}{2\ell} e^a_m \tau^a \right) \bar{\xi}_A &= + \sin \rho e^a_m \tau^a \bar{\xi}_A,
\end{align*}
\]
(3.3.10)
where \( a, b = 1, 2, 3 \) and the index \( m \) runs over \( \varphi, \chi, \theta \). Here \( \tau^a \) are Pauli’s matrices as before and we used \( \tau^{ab} \equiv \frac{1}{2}(\tau^a \tau^b - \tau^b \tau^a) \). The last two equations read

\[
\begin{align*}
\partial_\rho \xi_A &= -i\ell \xi'_A, \\
\partial_{\bar{\rho}} \bar{\xi}_A &= -i\ell \bar{\xi}'_A.
\end{align*}
\]

(3.3.11)

The equations (3.3.10) are solved by Killing spinors \( \kappa_{st} \) \((s, t = \pm 1)\) on round \( S^3 \) of radius \( \ell \) with coordinates \( \theta, \varphi, \chi \), which satisfy

\[
\left( \partial_m + \frac{1}{4} \Omega^{ab}_m \tau^{ab} \right) \kappa_{st} = -\frac{ist}{2\ell} e^a_m \tau^a \kappa_{st}, \quad \kappa_{st} \equiv \frac{1}{2} \begin{pmatrix} e^{\frac{s}{2}(s\chi + t\varphi - st\theta)} \\ -se^{\frac{s}{2}(s\chi + t\varphi + st\theta)} \end{pmatrix}
\]

(3.3.12)

for \( m = (\varphi, \chi, \theta) \) and \( a, b = 1, 2, 3 \). One can form a Killing vector on the \( S^3 \) as a bilinear of \( \kappa_{st} \), \( \kappa^\dagger_{st} \), \( \kappa_{st} \cdot e^{am} \partial_m = -\frac{1}{2\ell} (sd_\varphi + t\partial_\chi) \).

(3.3.13)

Recalling that \( \partial_\varphi \) and \( \partial_\chi \) are rotations in the \((x_1, x_2)\)-plane and \((x_3, x_4)\)-plane, we restrict to those with \( s = t \) so that our choice of Killing spinor corresponds to Omega deformations with \( \epsilon_1 = \epsilon_2 \) at the north pole. Assuming \( \xi_A \) and \( \bar{\xi}_A \) are all proportional to \( \kappa_{++} \) or \( \kappa_{--} \), the remaining equations (3.3.11) become

\[
\begin{align*}
-i\ell \xi'_A &= \partial_\rho \xi_A &= \cos \rho + \frac{1}{2} \sin \rho \xi_A, \\
-i\ell \bar{\xi}'_A &= \partial_{\bar{\rho}} \bar{\xi}_A &= \cos \rho - \frac{1}{2} \sin \rho \bar{\xi}_A.
\end{align*}
\]

(3.3.14)

In the following we take a particular solution of the main equation (3.2.1) which also satisfies the reality condition (A.3.3). It is also required

\[
\xi^A \xi'_A = \bar{\xi}^A \bar{\xi}'_A = 0,
\]

(3.3.15)

so that the square of the corresponding supersymmetry transformation does not give rise to dilation or \( U(1)_R \) transformation, namely \( w = \Theta = 0 \) in (3.2.7). It is unique up to the symmetries of the theory.

\[
\begin{align*}
\xi_A &= (\xi_1, \xi_2) &= \sin \rho \frac{1}{2} (\kappa_{++}, \kappa_{--}), \\
\bar{\xi}_A &= (\bar{\xi}_1, \bar{\xi}_2) &= \cos \rho \frac{1}{2} (i\kappa_{++}, -i\kappa_{--}).
\end{align*}
\]

(3.3.16)

The Killing vector which appears in the square of the supersymmetry transformation is

\[
v^m \partial_m = 2 \bar{\xi}^A \sigma^m \xi_A \partial_m = \frac{1}{\ell} (\partial_\varphi + \partial_\chi).
\]

(3.3.17)

This solution also satisfies the auxiliary equation (3.2.3) with the choice \( M = -\frac{1}{3} R = -4\ell^{-2} \).
3.3.2 A Killing spinor on ellipsoids.

Next we study the Killing spinor equation on ellipsoids (3.3.3). The strategy is to assume that, for a suitable choice of the background gauge and auxiliary fields, the Killing spinor (3.3.16) on round \( S^4 \) remains a Killing spinor also on ellipsoids. Then we will see that the Killing spinor equation can be turned into a set of linear algebraic equations on the background fields which have nontrivial solutions. A similar approach worked in the case of 3D ellipsoids [37]. Note that under this assumption the Killing vector on the ellipsoid becomes

\[
2 \bar{\xi}^A \partial_m \xi_A \partial_m = \frac{1}{\ell} \partial_\varphi + \frac{1}{\tilde{\ell}} \partial_\chi,
\]

(3.3.18)

which can be interpreted as the Omega deformation with \( \epsilon = \ell^{-1}, \epsilon_2 = \tilde{\ell}^{-1} \) near the north and south poles. This point will be explained in more detail later. In terms of (3.3.9), the Killing vector only acts on \( T^2 \) fiber and the fixed points of the translation along the Killing vector are north pole and south pole where \( T^2 \) is collapsed.

In solving the Killing spinor equation to determine the background fields, a useful fact is that the 3D spinors \( \kappa_{st} \) on \( S^3 \) remain Killing spinors after the deformation to 3D ellipsoids if a suitable background \( U(1) \) gauge field is turned on at the same time. More explicitly, one has

\[
\left( \partial_m + \frac{1}{4} \Omega^{ab}_m \tau^{ab} + i V^{[3]}_m \right) \kappa_{\pm \pm} = - \frac{i}{2f} \epsilon_m^a \tau^a \kappa_{\pm \pm},
\]

\[
V^{[3]} = \frac{1}{2} \left( 1 - \frac{\ell}{f} \right) d\varphi + \frac{1}{2} \left( 1 - \frac{\tilde{\ell}}{f} \right) d\chi,
\]

(3.3.19)

where \( m = \varphi, \chi, \theta \) and \( a, b = 1, 2, 3 \). Another useful fact is that the following \( 2 \times 2 \) matrix,

\[
\tau^1_\theta \equiv \tau^1 \cos \theta + \tau^2 \sin \theta,
\]

(3.3.20)

satisfies \( \tau^\theta_\theta \kappa_{\pm \pm} = \mp \kappa_{\pm \pm} \) and therefore \( \tau^\theta_\theta \xi_A = -\xi_B (\tau^3)^B_A \). At this point it is convenient to regard \( \xi_A \) and \( \bar{\xi}_A \) as \( 2 \times 2 \) matrices, on which \( 2 \times 2 \) matrices with spinor indices act from the left and those with \( SU(2)_R \) indices act from the right. The latter equation can then be rewritten in the matrix form,

\[
\tau^\theta_\theta \xi = -\xi \tau^3.
\]

(3.3.21)

Hereafter all the boldface letters can be regarded as \( 2 \times 2 \) matrix quantities. By using the above equation in combination with

\[
\tau^3 \xi = \xi \{ \cos(\chi + \varphi) \tau^1 + \sin(\chi + \varphi) \tau^2 \},
\]

(3.3.22)

any \( SU(2) \) action from the right of \( \xi \) can be translated into an \( SU(2) \) action from the left, and vice versa. Note also that

\[
\tau^\theta_\theta \xi = i \tan \frac{\theta}{2} \bar{\xi}.
\]

(3.3.23)

Let us now turn to the analysis of Killing spinor equation. We introduce the notations

\[
\mathbf{V} + V^{[3]} \tau^3 \equiv \mathbf{V} = E^a \bar{\mathbf{V}}_a, \quad i \mathbf{T} \equiv \sigma_{kl} T^{kl}, \quad i \bar{\mathbf{T}} \equiv \bar{\sigma}_{kl} \bar{T}^{kl}.
\]

(3.3.24)
We also require that (3.3.15) is still satisfied on ellipsoids, and introduce a pair of anti-
ymmetric tensors $S_{kl}, \bar{S}_{kl}$ and matrices $S, \bar{S}$ by the formula
\[
\xi' = S \xi = -i\sigma_{kl} S^{kl} \xi, \quad \xi' = \bar{S} \xi = -i\sigma_{kl} \bar{S}^{kl} \xi. \tag{3.3.25}
\]
Inserting these together with (3.3.16) into the main equation (3.2.1), we obtain
\[
\xi \tilde{V}_4 + T \bar{\xi} + S \bar{\xi} = i \frac{\cos \rho + 1}{2g \sin \rho} \xi - \frac{h}{2fg \sin \rho} \tau^3 \xi - \frac{h \Omega_3^{34}}{2g} \tau^3 \xi,
\]
\[
\bar{\xi} \tilde{V}_4 + \bar{T} \xi + \bar{S} \xi = i \frac{\cos \rho - 1}{2g \sin \rho} \xi - \frac{h}{2fg \sin \rho} \tau^3 \xi + \frac{h \Omega_3^{34}}{2g} \tau^3 \xi, \tag{3.3.26}
\]
and
\[
\xi \tilde{V}_a - i T^a \xi - i \tau^a S \xi = \frac{1}{2f \sin \rho} \tau^a \xi + \frac{1}{2} \Omega_a^{b4} \tau^b \xi,
\]
\[
\bar{\xi} \tilde{V}_a + i T^a \xi + i \tau^a S \xi = \frac{1}{2f \sin \rho} \tau^a \xi - \frac{1}{2} \Omega_a^{b4} \tau^b \xi, \tag{3.3.27}
\]
where $a, b = 1, 2, 3$ and the nonzero components of $\Omega_a^{b4}$ are
\[
\Omega_1^{14} = \frac{f^2 \cos \rho}{gf^2 \sin \rho}, \quad \Omega_2^{24} = \frac{f^2 \cos \rho}{gf^2 \sin \rho}, \quad \Omega_3^{34} = \frac{f^2 f^2 \cos \rho}{gf \sin \rho}. \tag{3.3.28}
\]

The equations (3.3.26) and (3.3.27) can be regarded as a system of inhomogeneous linear algebraic equations for the unknowns $\tilde{V}, T, \bar{T}, S$ and $\bar{S}$. It is found that these equations have nontrivial solutions, and moreover the solution is not unique. A special solution for which $T, \bar{T}$ take particularly simple form is
\[
T = \frac{1}{4} \left( \frac{1}{f} - \frac{1}{g} \right) \tau_1^1 + \frac{h}{4fg} \tau_2^2,
\]
\[
\bar{T} = \frac{1}{4} \left( \frac{1}{f} - \frac{1}{g} \right) \tau_1^1 - \frac{h}{4fg} \tau_2^2,
\]
\[
S = -\frac{1}{4} \left( \frac{1}{f} + \frac{1}{g} \right) \tau_1^1 - \frac{h}{4fg} \tau_2^2,
\]
\[
\bar{S} = -\frac{1}{4} \left( \frac{1}{f} + \frac{1}{g} \right) \tau_1^1 + \frac{h}{4fg} \tau_2^2,
\]
\[
\xi \tilde{V}_1 = \left\{ \frac{\cos \theta}{2 \sin \rho} \left( \frac{1}{f} - \frac{1}{g} \right) - \frac{\sin \theta \cos \rho}{2fg} \right\} \tau_1^1 \xi + \sin \theta \cos \rho \frac{1}{2f \sin \rho} \left( 1 - \frac{f^2}{g^2} \right) \tau_2^2 \xi,
\]
\[
\xi \tilde{V}_2 = \left\{ \frac{\sin \theta}{2 \sin \rho} \left( \frac{1}{f} - \frac{1}{g} \right) + \frac{\cos \theta \cos \rho}{2fg} \right\} \tau_1^1 \xi - \cos \theta \cos \rho \frac{1}{2f \sin \rho} \left( 1 - \frac{f^2}{g^2} \right) \tau_2^2 \xi,
\]
\[
\xi \tilde{V}_3 = -\frac{\cos \rho}{2f \sin \rho} \left( 1 - \frac{f^2}{g^2} \right) \tau_3^3 \xi,
\]
\[
\xi \tilde{V}_4 = \frac{h \cos \rho}{2fg \sin \rho} \left( 1 - \frac{f^2}{g^2} \right) \tau_3^3 \xi. \tag{3.3.29}
\]
where \( \tau_0^2 \equiv i \tau_0^1 \tau_3 \). This special solution can be shifted by solutions of the homogeneous equation, namely the equations (3.3.26) and (3.3.27) with the r.h.s. set to zero. They are parametrized by three arbitrary functions \( c_1, c_2, c_3 \) as follows.

\[
\Delta T = \tan \frac{\theta}{2} (c_1 \tau_0^1 + c_2 \tau_0^2 + c_3 \tau_0^3),
\Delta T = \cot \frac{\theta}{2} (-c_1 \tau_0^1 + c_2 \tau_0^2 + c_3 \tau_0^3),
\Delta S = \cot \frac{\theta}{2} (c_1 \tau_0^1 + c_2 \tau_0^2 + c_3 \tau_0^3),
\Delta S = \tan \frac{\theta}{2} (-c_1 \tau_0^1 + c_2 \tau_0^2 + c_3 \tau_0^3),
\xi \cdot \Delta \tilde{V}_1 = -2 \sin\theta (c_2 \tau_0^2 \xi - c_1 \tau_0^1 \xi),
\xi \cdot \Delta \tilde{V}_2 = +2 \cos\theta (c_2 \tau_0^2 \xi - c_1 \tau_0^1 \xi),
\xi \cdot \Delta \tilde{V}_3 = -2c_1 \tau_0^3 \xi + 2c_3 \tau_0^1 \xi,
\xi \cdot \Delta \tilde{V}_4 = +2c_2 \tau_0^3 \xi - 2c_3 \tau_0^1 \xi. \tag{3.3.30}
\]

In \( 2 \times 2 \) matrix notations, the auxiliary equation (3.2.3) becomes

\[-4 \cot \frac{\rho}{2} \left( \sigma^m D_m S - D_m T \sigma^m \right) \tau_0^1 - 4 \sigma^m ST \sigma_m = 4 \tan \frac{\rho}{2} \left( \sigma^m D_m S - D_m T \sigma^m \right) \tau_0^1 - 4 \sigma^m ST \sigma_m = M \cdot 1. \tag{3.3.31}\]

This is satisfied by the above special solution (3.3.29) with

\[M = \frac{1}{f^2} - \frac{1}{g^2} + \frac{h^2}{f^2 g^2} - \frac{4}{fg}. \tag{3.3.32}\]

It is found that the auxiliary equation is still satisfied even after nonzero \( c_1, c_2, c_3 \) are turned on, as long as they are functions of \( \theta \) and \( \rho \) only. The shift of \( M \) is then given by

\[
\Delta M = 8 \left( \frac{1}{g} \partial_0 - \frac{h}{gf \sin \rho} \partial_\theta + \frac{f^2 \ell^2 \cos \rho}{gf^4 \sin^2 \rho} + \frac{\cos \rho (f^2 + \ell^2 - f^2)}{gf^2 \sin^2 \rho} - \frac{\cos \rho}{f \sin \rho} \right) c_1
+ 8 \left( \frac{1}{f \sin \rho} \partial_\theta + \frac{hf \ell^2 \cos \rho}{g^2 f^4 \sin \rho} + 2 \cot \theta \frac{2\theta}{f \sin \rho} - \frac{h \cos \rho}{fg \sin \rho} \right) c_2 - 16 (c_1^3 + c_2^3 + c_3^3). \tag{3.3.33}\]

We thus determined the form of all the additional background fields in order for SW theories on the ellipsoid (3.3.3) to admit a rigid supersymmetry. In the rest of this section we check two more properties of our background. The first is that the square of the supersymmetry is a sum of bosonic transformations which indeed leave all the background fields invariant. The second is that our background is regular and approaches Omega background near the two poles.

### 3.3.3 Square of SUSY.

The supersymmetry transformation \( \mathbf{Q} \) acting on fields of SW theory squares into a sum of bosonic symmetries according to (3.2.6) and (3.2.24). It can also be expressed as

\[
\mathbf{Q}^2 = i \mathcal{L}_v + \text{Gauge}(\hat{\Phi}) + \text{Lorentz}(L_{ab}) + \text{Scale}(w) + R_{U(1)}(\Theta) + R_{SU(2)}(\hat{\Theta}) + R_{SU(2)}(\hat{\Theta}_{AB}), \tag{3.3.34}
\]
where

\[
\hat{\Phi} \equiv \Phi - iv^nA_n, \\
L_{ab} \equiv D_{[a}v_{b]} + v^n\Omega_{nab}, \\
\hat{\Theta}_{AB} \equiv \Theta_{AB} + v^n\nu_{nAB}, \\
\hat{\hat{\Theta}}_{AB} \equiv \hat{\Theta}_{AB} + v^n\check{\nu}_{nAB}. 
\]

(3.3.35)

Let us compute these transformation parameters for the ellipsoid background. First of all, the condition (3.3.15) on the Killing spinor guarantees that \( w = \Theta = 0 \). Moreover one can show

\[
L_{ab} \equiv 0, \quad \hat{\Theta}_{AB} = -\frac{1}{2\ell - \frac{1}{2}}\cdot (\tau^3)^A_B
\]

(3.3.36)

using the explicit form of vielbein, spin connection and the background \( SU(2) \) gauge field obtained above. It follows that our Killing spinor is invariant under \( Q^2 \).

\[
Q^2\xi_A = i\mathcal{L}_v\xi_A - \xi_B\hat{\Theta}^B_A = 0, \\
Q^2\bar{\xi}_A = i\mathcal{L}_v\bar{\xi}_A - \bar{\xi}_B\hat{\Theta}^B_A = 0.
\]

(3.3.37)

The background fields \( V^A_mB, T_{kl}, \bar{T}_{kl}, M \) are also invariant under \( Q^2 \) since they are constructed from \( L_v \)-invariant functions and Killing spinor.

To determine the action of \( Q^2 \) on all the fields, it is still needed to determine \( \check{\xi}_A, \check{\bar{\xi}}_A \) and the background \( SU(2) \) gauge field \( \check{V}^A_mB \) which have been left somewhat ambiguous. Hereafter we take the following solution of (3.2.23).

\[
\check{\xi}_A = \cot\frac{\rho}{2}\xi_A, \quad \check{\bar{\xi}}_A = -\tan\frac{\rho}{2}\bar{\xi}_A.
\]

(3.3.38)

Note that this has an effect of gauge fixing the local \( SU(2) \) symmetry relative to \( SU(2)_R \), and the following choice of the \( SU(2)_R \) gauge field is consistent with it.

\[
\check{V}^A_mB = V^A_mB.
\]

(3.3.39)

Using (3.3.38) one can also show

\[
\hat{\hat{\Theta}}_{AB} = \hat{\Theta}_{AB}, \quad \text{therefore} \quad \hat{\hat{\Theta}}_{AB} = \hat{\Theta}_{AB},
\]

(3.3.40)

and conclude that all the background fields are invariant under \( Q^2 \).

### 3.3.4 Omega-background revisited.

Here we focus on the behavior of our ellipsoid background near the north and south poles.

Near the north pole where \( x_0 \simeq r \) in (3.3.3), the other four coordinates \( (x_1, \cdots, x_4) \) can be regarded as the Cartesian coordinates on \( \mathbb{R}^4 \). The norm of \( \xi \) approaches a constant while that of \( \bar{\xi} \) is proportional to the radial distance from the pole. In a suitable gauge, the Killing spinor should therefore take the form

\[
\check{\xi}_A^i \simeq \frac{1}{\sqrt{2}}\delta^i_A, \quad \xi_{\alpha A} \simeq -\frac{1}{2\sqrt{2\ell}}(x_1\sigma_2 - x_2\sigma_1)_{\alpha A} - \frac{1}{2\sqrt{2\ell}}(x_3\sigma_4 - x_4\sigma_3)_{\alpha A}
\]

(3.3.41)
so that
\[
2\tilde{\xi}^A \sigma^m \xi_A \cdot \frac{\partial}{\partial x_m} = \frac{1}{\ell} \left( x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} \right) + \frac{1}{\tilde{\ell}} \left( x_3 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_3} \right) = \frac{1}{\ell} \partial_\varphi + \frac{1}{\tilde{\ell}} \partial_\chi.
\] (3.3.42)

The first equation in (3.3.41) indicates that near the north pole our supersymmetry approach that of the topologically twisted theory which identifies the dotted spin SU(2) index with the SU(2) R-symmetry index. From this viewpoint, the second equation in (3.3.41) tells nothing but the fact that \(\ell^{-1}, \tilde{\ell}^{-1}\) play the role of the Omega-deformation parameters \(\epsilon_1, \epsilon_2\) [100]. Note that, for the spinor field (3.3.41) to satisfy Killing spinor equation (3.2.1) and (3.2.3) on flat \(\mathbb{R}^4\), one has to turn on the background field as follows,
\[
T^\Omega = \frac{1}{2} T_{mn} dx^m dx^n = \frac{1}{16} \left( \frac{1}{\ell} - \frac{1}{\tilde{\ell}} \right) (dx_1 dx_2 - dx_3 dx_4),
\]
\[
V^\Omega = \bar{T}^\Omega = M^\Omega = 0.
\] (3.3.43)

In other words, Omega background with \(\epsilon_1 \neq \epsilon_2\) is related to a flat \(\mathbb{R}^4\) with constant \(\bar{T}_{mn}\) to Omega background of the anti-topologically twisted theory.

In much the same way, near the south pole one can choose a gauge in which \(\xi^A_\alpha\) is proportional to the identity matrix. There the supersymmetry approaches that of the anti-topologically twisted theory with Omega deformation. One can also relate the flat \(\mathbb{R}^4\) with constant \(\bar{T}_{mn}\) to Omega background of the anti-topologically twisted theory.

It remains to check whether the ellipsoid background is regular at the two poles. To do this, we rewrite the above regular Omega background (3.3.43) with the following polar coordinates of \(\mathbb{R}^4\),
\[
x_1 = \ell \rho \cos \theta \cos \varphi, \quad x_2 = \ell \rho \cos \theta \sin \varphi, \quad x_3 = \tilde{\ell} \rho \sin \theta \cos \chi, \quad x_4 = \tilde{\ell} \rho \sin \theta \sin \chi.
\] (3.3.44)

The auxiliary field \(T\) for the Omega background then takes the form,
\[
T^\Omega = \frac{1}{16 \ell} \left( \frac{1}{\ell} - \frac{1}{\tilde{\ell}} \right) \left\{ \ell \sin \theta (E^1 E^3 + E^2 E^4) - \tilde{\ell} \cos \theta (E^1 E^4 - E^2 E^3) \right\},
\] (3.3.45)

where \(E^a\) are the natural vielbein one-forms on \(\mathbb{R}^4\) in the polar frame,
\[
E^1 = \rho \hat{e}^1, \quad E^2 = \rho \hat{e}^2, \quad E^3 = \rho \hat{e}^3 + h_{(0)} d\rho, \quad E^4 = g_{(0)} d\rho.
\] (3.3.46)

Here \(h_{(0)}\) and \(g_{(0)}\) denote the values of the functions \(h\) and \(g\) in (3.3.6) at \(\rho = 0\). Then one finds
\[
T^\Omega = -i T^\Omega_{mn} \sigma^{mn} = \frac{1}{4} \left( \frac{1}{\ell} - \frac{1}{g_{(0)}} \right) \tau^1_\theta + \frac{h_{(0)}}{4 g_{(0)}} \tau^2_\theta,
\] (3.3.47)

which agrees with the special solution (3.3.29) near the north pole. However, there is a finite mismatch between the value of \(\bar{T}\), which is zero on the Omega background (3.3.43) but nonvanishing near the north pole of (3.3.29). This indicates that our special solution has singularity at the two poles and a suitable nonzero \(c_1, c_2\) has to be chosen so as to
cancel it. A simple choice which leads to $\bar{T} = 0$ at the north pole and $T = 0$ at the south pole is given by

$$c_1 = \frac{1}{8} \left( \frac{1}{f} - \frac{1}{g} \right) \sin \rho \cos \rho, \quad c_2 = \frac{h}{8fg} \sin \rho \cos \rho. \quad (3.3.48)$$

One is still left with the freedom to shift the $c$’s by functions which vanish as $\sin^2 \rho$ or faster near the two poles.

### 3.4 Explicit Path Integration

Here we use the SUSY localization principle and evaluate partition functions of general SW theories on the ellipsoid backgrounds. The analysis follows closely that of [5]. We first focus on the theories with vector multiplets only, and introduce matter hypermultiplets later.

#### 3.4.1 Saddle points for SYM theories.

According to the SUSY localization principle explained in chapter 2, non-zero contribution to the path integral arises only from saddle points which are characterized by

$$Q \Psi = 0 \quad \text{for all the fermions } \Psi.$$

The first step in computing partition function is to find out the saddle point locus. Though we have to modify the supercharge $Q$ upon introducing BRST ghost system, the saddle point locus remain the same.

To find out the saddle point locus for vector multiplets, it is convenient to study the following quantity,

$$I_{vec} \equiv \text{Tr} \left[ (Q\lambda_{\alpha A})^\dagger (Q\lambda_{\alpha A}) + (Q\bar{\lambda}_{\dot{\alpha} A})^\dagger (Q\bar{\lambda}_{\dot{\alpha} A}) \right], \quad (3.4.1)$$

which is manifestly positive semi-definite and vanishes on saddle points. Using the transformation law and the reality condition (3.2.12), one can rewrite it as follows,

$$I_{vec} = \text{Tr} \left[ D_m \phi_1 D^m \phi_1 - [\phi_1, \phi_2]^2 - \frac{1}{2} (D_{AB} + i\phi_1 w_{AB})(D^{AB} + i\phi_1 w^{AB}) ight. \right.
\left. + \xi^A \xi_A \left( F_{mn} - 4\phi_2 T_{mn} - 4\phi_2 S_{mn} + \frac{1}{\xi^A \xi_A} v_{m} D_{n} - \phi_2 \right)^2 \right.
\left. + \xi^A \xi_A \left( F_{mn}^+ + 4\phi_2 \bar{T}_{mn} + 4\phi_2 \bar{S}_{mn} - \frac{1}{\xi^A \xi_A} v_{m} D_{n} + \phi_2 \right)^2 \right.$$
\left. \left. + \frac{1}{4\xi^A \xi_A \cdot \xi_B \xi_B} (v^m D_m \phi_2)^2 \right] \right), \quad (3.4.2)
$$

where the suffix ± for antisymmetric tensors indicates the self-dual or anti-self-dual parts, and we introduced

$$\phi_1 \equiv i(\phi + \bar{\phi}), \quad \phi_2 \equiv \phi - \bar{\phi},$$

$$w_{AB} \equiv \frac{4\xi^A \xi_A \epsilon^{mn} \xi_B (T_{mn} - S_{mn})}{\xi^C \xi_C} = - \frac{4\xi^A \xi_A \epsilon^{mn} \xi_B (\bar{T}_{mn} - \bar{S}_{mn})}{\xi^C \xi_C}. \quad (3.4.3)$$
Note that \( w_{AB} \) here satisfies the condition (3.2.13) and therefore can be used to construct FI Lagrangian.

The saddle point condition for \( \phi_2 \) and \( A_m \) is to be derived from the last three terms in the r.h.s. of (3.4.2). We argue that it is given by

\[
\phi_2 = A_m = 0 \quad \text{up to gauge choice.} \tag{3.4.4}
\]

For round sphere with \( T_{mn} = \bar{T}_{mn} = 0 \), one finds that the last three terms can be reorganized into a different “sum of squares” up to total derivatives,

\[
\mathcal{I}_{vec} = \text{Tr} \left[ \cdots + (D_m \phi_2)^2 + \xi^A \xi_A (F_{mn}^- + 4 \phi_2 S_{mn})^2 + \bar{\xi}^A \bar{\xi}_A (F_{mn}^+ - 4 \phi_2 \bar{S}_{mn})^2 \right]. \tag{3.4.5}
\]

This gives a much simpler saddle point condition which immediately leads to (3.4.4) when combined with Bianchi identity \( D_i [F_{mn}] = 0 \). However, as soon as the sphere is deformed, this reorganizing is no longer possible and one has to deal with more complicated saddle point condition which follows from (3.4.2). But if there are nontrivial solutions to the original saddle point condition on some deformed sphere, they should be continuously connected to nontrivial solutions on round sphere. Such solutions would have to be singular, since they do not minimize \( \mathcal{I}_{vec} \) of (3.4.5) which differs from (3.4.2) only by total derivatives. Thus we believe that (3.4.4) is the only solution to the saddle-point condition. It would be nice to prove this claim regorously, though we will base our subsequent analysis on this claim and obtain the most natural generalization of the result for round sphere.

Once (3.4.4) is settled, then the condition for the remaining fields are easily solved. The saddle points are thus labeled by a Lie algebra valued constant \( a_0 \), and are given by the equations

\[
A_m = 0, \quad \phi = \bar{\phi} = -\frac{i}{2} a_0, \quad D_{AB} = -i a_0 w_{AB}. \tag{3.4.6}
\]

The values of super-Yang-Mills action (3.2.9) and FI term (3.2.14) on this saddle point are

\[
\frac{1}{g_{YM}^2} \int d^4 x \sqrt{g} L_{YM} \bigg|_{\text{saddle point}} = \frac{8 \pi^2}{g_{YM}^2} \ell \ell \text{Tr}(a_0^2),
\]  
\[
\zeta \int d^4 x \sqrt{g} L_{FI} \bigg|_{\text{saddle point}} = -16 i \pi^2 \ell \ell \zeta a_0. \tag{3.4.7}
\]

They are independent of the precise choices of \( c_1, c_2, c_3 \) as long as they are smooth.

### 3.4.2 Ghosts and BRST symmetry.

For gauge fixing, we proceed in the same way as [5]. Let us introduce the Faddeev-Popov ghost field \( c \) and define the BRST transformation by

\[
Q_B A_m = D_m c, \quad Q_B \lambda_A = i \{ c, \lambda_A \}, \quad Q_B \overline{\lambda}_A = i \{ c, \overline{\lambda}_A \}, \quad Q_B D_{AB} = i \{ c, D_{AB} \}. \tag{3.4.8}
\]
We require the square of $Q_B$ to be a constant gauge rotation with parameter $a_0$, so we set
\[ Q_{Bc} = i c + a_0. \tag{3.4.9} \]
The sum of the SUSY and the BRST transformations, $\hat{Q} \equiv Q + Q_B$, will be the relevant fermionic symmetry in the application of localization principle later on. Requiring its square to act on all the fields as
\[ \hat{Q}^2 = i \mathcal{L}_v + \text{Gauge}(a_0) + R_{SU(2)}(\hat{\Theta}_{AB}), \tag{3.4.10} \]
one finds that the supersymmetry transformation of $c$ has to be,
\[ Qc = -\hat{\Phi} = -\phi_1 - i \cos \rho \phi_2 + iv^n A_n. \tag{3.4.11} \]
One also finds that the constant variable $a_0$ has to be invariant,
\[ Qa_0 = Q_B a_0 = 0. \tag{3.4.12} \]
We furthermore introduce the antighost multiplet with the transformation rules,
\[ Q_B \bar{c} = B, \quad Q_B B = i[a_0, \bar{c}], \quad Q\bar{c} = 0, \quad Q B = i \mathcal{L}_v \bar{c}. \tag{3.4.13} \]
and the multiplets of constant fields which will be used to freeze the constant modes of $c$ and $\bar{c}$.
\[
\begin{align*}
Q_B \bar{a}_0 &= \bar{c}_0, & Q_B \bar{c}_0 &= i[a_0, \bar{a}_0], & Q_B B_0 &= c_0, & Q_B c_0 &= i[a_0, B_0], \\
Q\bar{a}_0 &= 0, & Q\bar{c}_0 &= 0, & Q B_0 &= 0, & Q c_0 &= 0.
\end{align*}
\tag{3.4.14}
\]
To fix a gauge correctly, the standard way is to choose a set of conditions $G[A_m, \phi, \cdots]$ and shift the Lagrangian by the gauge-fixing term
\[ \mathcal{L}_{GF} = Q_B \mathcal{V}_{GF}, \quad \mathcal{V}_{GF} = \text{Tr} (\bar{c}G + \bar{c}B_0 + c\bar{a}_0). \tag{3.4.15} \]
We will later find it convenient to choose
\[ G = i \partial_m A^m + i \mathcal{L}_v (\cos \rho \phi_2 - v^n A_m). \tag{3.4.16} \]
For the computation of partition function using localization principle, it is more convenient to replace $Q_B$ in (3.4.15) by $Q = Q + Q_B$. As explained in [5] this replacement does not change the value of partition function.

Now that the gauge-fixed system has the fermionic symmetry $\hat{Q} \equiv Q + Q_B$, it is needed to revisit the condition for the saddle points
\[ \hat{Q} \Psi = Q \Psi + Q_B \Psi = 0 \text{ for all the fermions } \Psi. \tag{3.4.17} \]
For the fermions in vector multiplets, the added term $Q_B \Psi$ is always bilinear in fermions so that the condition for saddle points does not change. For the ghost $c$, the saddle point condition gives
\[ \hat{Q}c = i c + a_0 - \phi_1 - i \cos \rho \phi_2 + iv^n A_n = 0. \tag{3.4.18} \]
Thus $a_0$ is to be identified with the constant value of $\phi_1$ at saddle points.
3.4.3 One-loop determinant.

The value of path integral does not change under the shifts of the original Lagrangian by any \(b\)-exact quantities, \(\mathcal{L} \to \mathcal{L} + t\mathcal{Q}\mathcal{V}\). We take the regulator \(\mathcal{Q}\mathcal{V}\) so that its bosonic part is positive definite and is strictly positive anywhere away from saddle points. Since \(t\) can be taken arbitrarily large, Gaussian approximation is exact for the path integration over the fluctuations away from saddle points.

We begin by introducing some new notations for later convenience.

\[
\begin{align*}
\Psi &\equiv \mathcal{Q}\phi_2 = -i\xi^A\lambda_A - i\bar{\xi}^A\bar{\lambda}_A, \\
\Psi_m &\equiv \mathcal{Q}\lambda_m = i\xi^A\sigma_m\bar{\lambda}_A - i\bar{\xi}^A\bar{\sigma}_m\lambda_A, \\
\Xi_{AB} &\equiv 2\bar{\xi}(A\bar{\lambda}_B) - 2\xi(A\lambda_B),
\end{align*}
\]

(3.4.19)

The inverse of this relation is

\[
\begin{align*}
\lambda_A &= +i\xi_A\Psi - i\sigma^m\bar{\xi}_A\Psi_m + \xi^B\Xi_{BA}, \\
\bar{\lambda}_A &= -i\bar{\xi}_A\Psi - i\bar{\sigma}_m\xi_A\Psi_m + \bar{\xi}^B\Xi_{BA}.
\end{align*}
\]

(3.4.20)

As the regulator, we take the \(b\)-transform of the following quantity which has manifestly positive semi-definite bosonic part \(I_{\text{vec}}\),

\[
\mathcal{V} = \text{Tr}\left[(\mathcal{Q}\lambda_\alpha A)^\dagger \lambda_{\alpha A} + (\mathcal{Q}\bar{\lambda}_\dot{\alpha} A)^\dagger \bar{\lambda}_{\dot{\alpha} A}\right].
\]

(3.4.21)

Inserting (3.4.20) into this and combining with the gauge fixing term, one finds

\[
\mathcal{V} + \mathcal{V}_{\text{GF}} = \text{Tr}\left[(\hat{\mathcal{Q}}\Psi)^\dagger \Psi + (\hat{\mathcal{Q}}\Psi_m)^\dagger \Psi_m + \frac{1}{2}(\hat{\mathcal{Q}}\Xi_{AB})^\dagger \Xi_{AB} + cG + iB_0 + c\bar{a}_0\right].
\]

(3.4.22)

The integration over all the variables except for the constant \(a_0\) will be carried out under the (exact) Gaussian approximation, with the weight given by \(\mathcal{Q}(\mathcal{V} + \mathcal{V}_{\text{GF}})\) truncated up to quadratic order. In doing this, we move to a new set of path integration variables which consists of

\[
X \equiv (\phi_2, A_m; \bar{a}_0, B_0), \quad \Xi \equiv (\Xi_{AB}, \bar{c}, c)
\]

(3.4.23)

and their superpartners \(\hat{\mathcal{Q}}X, \hat{\mathcal{Q}}\Xi\). In terms of these variables one can write

\[
\mathcal{V} + \mathcal{V}_{\text{GF}}|_{\text{quad.}} = (\hat{\mathcal{Q}}X, \Xi)\begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} X \\ \hat{\mathcal{Q}}\Xi \end{pmatrix}.
\]

(3.4.24)

Then

\[
\hat{\mathcal{Q}}(\mathcal{V} + \mathcal{V}_{\text{GF}})|_{\text{quad.}} = (X \quad \hat{\mathcal{Q}}\Xi)\begin{pmatrix} H & 1 \\ 1 & \hat{\mathcal{Q}} \end{pmatrix}\begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix}\begin{pmatrix} X \\ \hat{\mathcal{Q}}\Xi \end{pmatrix}
\]

\[
- (\hat{\mathcal{Q}}X \quad \Xi)\begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} 1 \\ -H \end{pmatrix}\begin{pmatrix} \hat{\mathcal{Q}}X \\ \Xi \end{pmatrix}
\]

(3.4.25)

where \(H \equiv \hat{\mathcal{Q}}^2\) commutes with \(D_{ij}\). The Gaussian integration gives the square root of the ratio of determinants of kinetic operators for boson and fermions. The contribution from \(D_{ij}\)s to one-loop determinant of this actions are canceled in these terms, and

\[
Z^{\text{vec}}_{1\text{-loop}} \sim \sqrt{\frac{\det H_X}{\det H_{\Xi}}},
\]

(3.4.26)
One finds after some algebra that
\[
\frac{\text{det} K_{\text{fermion}}}{\text{det} K_{\text{boson}}} = \frac{\det \Xi H}{\det \chi H} = \frac{\det \text{Coker} D_{10} H}{\det \text{Ker} D_{10} H}.
\] (3.4.27)

Thus the ratio of determinants can be determined from the spectrum of the operator $H$ on the kernel and cokernel of a differential operator $D_{10}$, which is encoded in the index
\[
\text{ind} D_{10} \equiv \text{Tr}_{\text{Ker} D_{10}} \left( e^{-iHt} \right) - \text{Tr}_{\text{Coker} D_{10}} \left( e^{-iHt} \right).
\] (3.4.28)

### 3.4.4 Index of transversally elliptic operators.

In computing this index, we first drop the terms containing constant fields $B_0, \bar{a}_0$ from $\mathcal{V}_{\text{GF}}$. These constant fields are thus regarded as sitting in the kernel of $D_{10}$ and making a contribution 2 to the index. To obtain the remaining contribution, we read off the differential operator $D_{10}$ from
\[
\Xi D_{10} X + \Xi D_{11} \bar{Q} \Xi = \text{Tr} \left[ \hat{c} G - D_m c \left( \hat{Q} \Psi^m \right)^\dagger + \frac{1}{2} \Xi_{AB} \left( \hat{Q} \Xi_{AB} \right)^\dagger \right] \bigg|_{\text{quad}},
\] (3.4.29)

where we have, up to non-linear terms,
\[
\left( \hat{Q} \Psi^m \right)^\dagger = -i \mathcal{L}_\nu A_m + D_m (\hat{\Phi} - 2i \cos \rho \phi_2 + 2i v^m A_m),
\]
\[
\left( \hat{Q} \Xi_{AB} \right)^\dagger = -\xi^A \sigma^{kl} \xi^B (F_{kl} - 8\phi T_{kl} + 8\hat{\phi} S_{kl})
\]
\[
+ \bar{\xi}^A \sigma^{kl} \bar{\xi}^B (F_{kl} - 8\hat{\phi} T_{kl} + 8\phi S_{kl}) - 4\xi^A \sigma^m \xi^m (\hat{Q} \Xi_{AB}) D_n \phi_2 - D^{AB}.
\] (3.4.30)

It turns out that the operator $D_{10}$ is not elliptic but transversally elliptic with respect to the isometry $\mathcal{L}_\nu$ of the ellipsoid. Let us show this by computing explicitly its symbol.

We identify the fields $X$ and $\Xi$ with sections of bundles $\mathcal{E}_0$ and $\mathcal{E}_1$ over the ellipsoid $X$, and therefore $D_{10} : \Gamma(\mathcal{E}_0) \rightarrow \Gamma(\mathcal{E}_1)$. Its symbol $\sigma(D_{10})$ is then obtained by retaining only the terms with highest order of derivatives and making the replacement $\partial_x \rightarrow i p_x$. Thus $\sigma(D_{10})$ is a homomorphism between two vector bundles $\pi^* \mathcal{E}_0, \pi^* \mathcal{E}_1$ over the cotangent bundle $\pi : T^* X \rightarrow X$. The index of transversally elliptic operators is known to be uniquely determined by their symbols.

To write the symbol explicitly, it is convenient to introduce four unit vector fields $u^m_a \ (a = 1, \ldots, 4)$ by the formula
\[
-2i (\tau^a)^B \xi^B \sigma^m \xi_A = \sin \rho \ u^m_a \ (a = 1, 2, 3),
\]
\[
2 \xi^A \sigma^m \xi_A = \sin \rho \ u^m_4
\] (3.4.31)

and parametrize the momenta in the local orthonormal frame defined by the vielbein $u^m_a$. For example, by a slight abuse of the notation, one can write
\[
4 \xi^A \sigma^m \xi_B \partial_m = \sin \rho (\sigma^a)^B \ u^m_a \partial_m,
\]
\[
-4 \xi^A \sigma^m \xi_B \partial_m = \sin \rho (\sigma^a)^B \ u^m_a \partial_m
\] (3.4.32)

and in particular
\[
\mathcal{L}_v \equiv v^m \partial_m = \sin \rho u^m_4 \cdot i p_m = i \sin \rho \cdot p_4.
\] (3.4.33)
Using this notation together with $\Xi_a \equiv \frac{1}{2} \Xi^A B (\tau_a)^B_A$, one finds

$$\Xi \sigma(D_{10}) X = (\Xi_1, \Xi_2, \Xi_3, -\tilde{c}, ic) \begin{pmatrix} c_p p_4 & p_3 & -p_2 & -c_p p_1 & -s_p p_1 \\ -p_3 & c_p p_4 & p_1 & -c_p p_2 & -s_p p_2 \\ p_2 & -p_1 & c_p p_4 & -c_p p_3 & -s_p p_3 \\ p_1 p_4 & p_2 p_4 & p_3 p_4 & p_4^2 - 2s_p p_4^2 & 2c_p s_p p_4^2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ \phi_2 \end{pmatrix},$$

(3.4.34)

where $s_p \equiv \sin \rho$, $c_p \equiv \cos \rho$. The $5 \times 5$ matrix in the middle can be block diagonalized by a suitable change of variables within $X$ and $\Xi$.

$$\sigma(D_{10}) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ p_4 \\ 1 \end{pmatrix} \begin{pmatrix} c_p p_4 & p_3 & -p_2 & -p_1 \\ -p_3 & c_p p_4 & p_1 & -p_2 \\ p_2 & -p_1 & c_p p_4 & -p_3 \\ p_1 & p_2 & p_3 & c_p p_4 \\ -2p_4 s_p + s_p p_4^2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ c_p \end{pmatrix}.$$

(3.4.35)

The lower-right $1 \times 1$ block should give a trivial contribution to the index, since the corresponding differential operator should have just one-dimensional kernel and cokernel of constant functions. So the nontrivial contribution to the index arises from the upper-left $4 \times 4$ block of the matrix in the middle,

$$\sigma(D_{10}') = \begin{pmatrix} c_p p_4 & p_3 & -p_2 & -p_1 \\ -p_3 & c_p p_4 & p_1 & -p_2 \\ p_2 & -p_1 & c_p p_4 & -p_3 \\ p_1 & p_2 & p_3 & c_p p_4 \end{pmatrix}. $$

(3.4.36)

Near the two poles, the symbol is that of the standard self-dual or anti-self-dual complex on $\mathbb{R}^4$,

$$(\cos \rho = +1) \quad \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^+} \Omega^{2+},
 \quad \left(\cos \rho = -1 \right) \quad \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^+} \Omega^{2-}.$$

(3.4.37)

A differential operator is called elliptic if its symbol is invertible for nonzero $p_4$. The above symbol $\sigma$ is not invertible at the equator $\cos \rho = 0$ since $\sigma \sigma^T = (p_1^2 + p_2^2 + p_3^2 + \cos^2 \rho p_4^2) \cdot \text{id}$ as one can easily check. But if we restrict the momentum to be orthogonal to the vector $v$, namely $p_4 \equiv 0$, then $\sigma$ is invertible as long as $(p_1, p_2, p_3)$ are not all zero. The corresponding differential operator is then called transversally elliptic with respect to the symmetry $L_v$. The kernel and cokernel of transversally elliptic operators are generally infinite dimensional, though they are both decomposed into finite dimensional eigenspaces of $H$. Therefore, there is a bit more difficulty in the computation of index for transversally elliptic operators as compared to elliptic ones.

The operator $e^{-iHt}$ is a combination of a finite rotation of the ellipsoid, gauge rotation and $SU(2)_R$ rotation. Its action on an adjoint-valued field $O$ takes the form

$$e^{-iHt}O(x^m) = \gamma_{[O]} \cdot e^{a0t} O(x^m)e^{-a0t}, \quad (\tilde{\varphi} = \varphi + \frac{t}{2}, \tilde{\chi} = \chi + \frac{t}{2})$$

(3.4.38)
where the coefficient $\gamma_{[O]}$ encodes the action on the vector and $SU(2)_R$ indices of the field $O$. For simplicity, let us temporarily take the gauge group to be abelian.

Regarding the index as the difference of the trace of $e^{-iHt}$ over $\Gamma(E_0)$ and $\Gamma(E_1)$, it should be written as a sum of contributions from the two fixed points where $\tilde{x}^m = x^m$. According to the Atiyah-Bott formula, the index is given by

$$\text{ind} D'_1|_{NP} = \int d^4x \delta^{(4)}(x - x') [\text{Tr}_{\text{boson}}(\gamma) - \text{Tr}_{\text{fermion}}(\gamma)]$$

$$= \sum_{x: \text{fixed point}} \frac{\text{Tr}_{E_0}(\gamma) - \text{Tr}_{E_1}(\gamma)}{\det(1 - \partial \tilde{x}/\partial x)},$$

(3.4.39)

where $x' = e^{tC_{\nu}}x$ and $\gamma$ is eigenvalue of $e^{-iHt}$. Near the north pole, the operator $e^{-iHt}$ acts on the local coordinates $z_1 = x_1 + ix_2, z_2 = x_3 + ix_4$ as

$$\tilde{z}_1 = e^{it}z_1 \equiv q_1z_1, \quad \tilde{z}_2 = e^{it}z_2 \equiv q_2z_2.$$  

Therefore

$$\det(1 - \partial \tilde{x}/\partial x) = (1 - q_1)(1 - \bar{q}_1)(1 - q_2)(1 - \bar{q}_2),$$

(3.4.41)

where $q_1\bar{q}_1 = q_2\bar{q}_2 = 1$. The value of $\gamma$ for various fields reads

$$\gamma[A_{z_1}] = q_1, \quad \gamma[\Xi_{11}] = \bar{q}_1 \bar{q}_2,$n
$$\gamma[A_{z_2}] = q_2, \quad \gamma[\Xi_{12}] = 1,$n
$$\gamma[A_{\bar{z}_1}] = \bar{q}_1, \quad \gamma[\Xi_{22}] = q_1q_2,$n
$$\gamma[A_{\bar{z}_2}] = \bar{q}_2, \quad \gamma[c] = 1.$$  

(3.4.42)

Then the contribution from the north pole (3.4.39) becomes

$$\text{ind} D'_1|_{NP} = \frac{(q_1 + q_2 + \bar{q}_1 + \bar{q}_2) - (\bar{q}_1 \bar{q}_2 + 1 + q_1q_2 + 1)}{(1 - q_1)(1 - q_2)(1 - \bar{q}_1)(1 - \bar{q}_2)}$$

$$= \left[-\frac{1 + q_1q_2}{(1 - q_1)(1 - q_2)}\right]_{NP}. $$

(3.4.43)

Combining it with the similar contribution from the south pole and 2 from constant modes, one obtains

$$\text{ind}(D_1) = \left[-\frac{1 + q_1q_2}{(1 - q_1)(1 - q_2)}\right] + \left[-\frac{1 + q_1q_2}{(1 - q_1)(1 - q_2)}\right] + 2.$$  

(3.4.44)

To extract the information on the multiplicity of eigenvalues of $H$, one needs to expand this expression into power series in $q_1, q_2$. The expansion does not seem to be unique, and the correct way should be found by investigating a suitable deformation of the symbol to make it non-degenerate everywhere away from the two poles. As was explained in [17] and reviewed in [5], this is the main point of difficulty in computing the index of transversally elliptic operators. At the end of the day, the correct way is to expand the first term in positive series and the second term in negative series. Thus we arrive at

$$\text{ind}(D_1) = 2 - \sum_{m,n \geq 0} \left(q_1^m q_2^n + q_1^{m+1} q_2^{n+1} + q_1^{-m} q_2^{-n} + q_1^{m-1} q_2^{-n-1}\right).$$

(3.4.45)
For non-abelian gauge group $G$, we take $a_0$ to be in the Cartan subalgebra. Then each term in the above is multiplied by
\[ \text{rk}G + \sum_{\alpha \in \Delta} \exp(ta_0 \cdot \alpha) \] (3.4.46)
where the sum runs over all roots. This finishes the computation of the index $\text{ind}(D_{10})$.

### 3.4.5 Infinite-product formula.

The one-loop determinant can be easily computed by extracting the spectrum of eigenvalues of $H$ from the index. Up to normalization factors depending only on $\ell$ and $\tilde{\ell}$, it is given by

\[ Z_{\text{vec}}^{1\text{-loop}} = \left[ \frac{\text{det}K_{\text{fermion}}}{\text{det}K_{\text{boson}}} \right]^\frac{1}{2} \]
\[ = \prod_{\alpha \in \Delta_+} \frac{1}{(\tilde{a}_0 \cdot \alpha)^2} \prod_{m,n \geq 0} (mb + nb^{-1} + Q + i\tilde{a}_0 \cdot \alpha)(mb + nb^{-1} + i\tilde{a}_0 \cdot \alpha) \]
\[ \cdot (mb + nb^{-1} + Q - i\tilde{a}_0 \cdot \alpha)(mb + nb^{-1} - i\tilde{a}_0 \cdot \alpha) \]
\[ = \prod_{\alpha \in \Delta_+} \frac{\Upsilon(i\tilde{a}_0 \cdot \alpha)\Upsilon(-i\tilde{a}_0 \cdot \alpha)}{(\tilde{a}_0 \cdot \alpha)^2}, \] (3.4.47)

where we introduced $b \equiv (\ell/\tilde{\ell})^{1/2}, Q \equiv b + \frac{1}{b}$ and $\tilde{a}_0 \equiv \sqrt{\ell\tilde{\ell}}a_0$. We also used the function $\Upsilon(x)$ as an appropriate zeta-regularized infinite product
\[ \prod_{m,n \geq 0} \left( x + mb + \frac{n}{b} \right) \left( Q - x + mb + \frac{n}{b} \right), \] (3.4.48)

which has zeroes at $x = Q + mb + \frac{n}{b}, -mb - \frac{n}{b}$ $(m, n \in \mathbb{Z}_{\geq 0})$ to express the appropriately regularized infinite products. It is characterized by
\[ \Upsilon(x) = \Upsilon(Q - x), \quad \Upsilon(Q/2) = 1, \] (3.4.49)
as well as the shift relations
\[ \Upsilon(x + b) = \Upsilon(x)\gamma(bx)b^{1-2bx}, \]
\[ \Upsilon(x + \frac{1}{b}) = \Upsilon(x)\gamma(x/b)b^{2x-1}. \quad (\gamma(x) \equiv \Gamma(x)/\Gamma(1 - x)). \] (3.4.50)

The final expression for partition function involves an integral with respect to the saddle point parameter $a_0$ over the Lie algebra, but one can restrict its integration domain to Cartan subalgebra. It gives rise to the usual Vandermonde determinant factor which cancels with the factor $(\tilde{a}_0 \cdot \alpha)^2$ in the denominator of (3.4.47).
3.4.6 Inclusion of matter.

Let us next study the case with hypermultiplet matters. The first thing to do is to solve the saddle point condition. For round $S^4$ it was shown in [5] that all the bosonic fields in hypermultiplets have to vanish at saddle points; in other words there is no Higgs branch. We claim this remains true on ellipsoids. The simplest way to see this is to consider the zero locus of the bosonic part of

$$\mathcal{L}_{\text{mat}} = \hat{Q}\nu_{\text{mat}},$$

(3.4.51)

which is the same as the bosonic part of (3.2.21). The auxiliary field $F_A$ simply has to vanish. The scalar $\phi$ has mass term which is smallest at the origin of the Coulomb branch where $\phi = \bar{\phi} = -\frac{i}{4}q_0 = 0$, and its value $\frac{1}{2}(R + M)$ is strictly positive anywhere on the ellipsoid at least when the deformation from the round sphere with $T_{mn} = T_{nm} = 0$ is not large.

The one-loop determinant can be computed in the same way as for the vector multiplets. We define new Grassmann-odd scalar fields which are doublets under $SU(2)_R$ by the formula,

$$\Psi_A = -i\xi_A\psi + i\bar{\xi}_A\bar{\psi} = Q_{QA}, \quad \Xi_A = \bar{\xi}_A\psi - \xi_A\bar{\psi}. \quad (3.4.52)$$

The inverse of this is

$$\psi = -2i\xi^A\Psi_A - 2\bar{\xi}^A\Xi_A, \quad \bar{\psi} = -2i\bar{\xi}^A\Psi_A - 2\xi^A\Xi_A. \quad (3.4.53)$$

We then rewrite the regulator Lagrangian (3.4.51) truncated up to quadratic order in terms of the variables $(q_A, Q_{QA})$ and $(\Xi_A, Q\Xi_A)$. The computation of the one-loop determinant thus reduces to that of the index of an operator $D^{\text{mat}}_{10}$ which can be read from the terms bilinear in $\Xi_A$ and $q_A$ in $\nu_{\text{mat}}$. Its symbol is given by

$$\Xi_A[\sigma(D^{\text{mat}}_{10})]^A_{BD}q^B = i\cos^2\frac{\rho}{2}\Xi_A(\sigma^a p_a)^A_{BD}q^B - i\sin^2\frac{\rho}{2}\Xi_A(\bar{\sigma}^a p_a)^A_{BD}q^B, \quad (3.4.54)$$

where we used the notation introduced in (3.4.32). The ellipticity of $D^{\text{mat}}_{10}$ is violated at $\rho = \frac{\pi}{2}$ but it is transversally elliptic with respect to the isometry generated by $L_v$.

Using Atiyah-Bott formula again, we compute the index from the action of $H$ on fields at the two poles. At the north pole it is most convenient to work with the Cartesian local coordinates $x_1, \cdots, x_4$, in terms of which the metric is flat and the Killing spinor takes the form (3.3.41). Here one can regard $q_A$ as dotted spinor and identify $\Xi_A$ as undotted spinor $\psi$. Thus one can find, for example for $r$ free hypermultiplets,

$$\gamma[\psi^{A=1}] = \frac{1}{4}q_1^{1/2}q_2^{1/2}, \quad \gamma[\psi^{A=2}] = \frac{1}{4}q_1^{1/2}q_2^{1/2},$$

$$\gamma[\psi^{A=2}] = \frac{1}{4}q_1^{1/2}q_2^{1/2}, \quad \gamma[\psi^{A=2}] = \frac{1}{4}q_1^{1/2}q_2^{1/2}. \quad (I = 1, \cdots, 2r) \quad (3.4.55)$$

Combining the contribution from the two poles one finds the index

$$\text{ind}(D^{\text{mat}}_{10}) = 2r \left[ q_1^{1/2}q_2^{1/2} \over (1 - q_1)(1 - q_2) \right] + 2r \left[ q_1^{1/2}q_2^{1/2} \over (1 - q_1)(1 - q_2) \right]$$

$$= 2r \sum_{m,n \geq 0} \left( q_1^{m+\frac{1}{2}}q_2^{n+\frac{1}{2}} + q_1^{-m-\frac{1}{2}}q_2^{-n-\frac{1}{2}} \right), \quad (3.4.56)$$

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where it is assumed that a regularization procedure similar to the case with vector multiplet determines how to expand the first line into power series. For hypermultiplets coupled to gauge symmetry, the factor $2r$ is replaced by a sum over the weight vectors of the corresponding representation. For example, the hypermultiplet is said to be in a representation $R$ of the gauge group if the index $I$ furnishes the representation $R = R \oplus \bar{R}$. Then the index is given by the replacement

$$
2r \rightarrow \sum_{\rho \in R} e^{\iota a_0 \cdot \rho} = \sum_{\rho \in R} (e^{\iota a_0 \cdot \rho} + e^{-\iota a_0 \cdot \rho}),
$$

(3.4.57)

where $\rho$ runs over all the weight vectors in a given representation $R$ or $\bar{R}$. This completes the computation of the index. It is straightforward to translate this result into the matter one-loop determinant,

$$
Z_{\text{hyp}}^{1\text{-loop}} = \prod_{\rho \in R} \prod_{m,n \geq 0} \left( mb + nb^{-1} + \frac{Q}{2} + i\hat{a}_0 \cdot \rho \right)^{-1} \left( mb + nb^{-1} + \frac{Q}{2} - i\hat{a}_0 \cdot \rho \right)^{-1}
$$

(3.4.58)

### 3.4.7 Instanton contribution.

In solving the saddle point condition for vector multiplet, the gauge field was assumed to be smooth. Relaxing this assumption, one finds from (3.4.2) that the gauge field strength can have nonzero anti-self-dual components at the north pole where $\xi_A \xi_A = \sin^2 \rho^2 = 0$, or nonzero self-dual components at the south pole where $\bar{\xi} A \bar{\xi} A = \cos^2 \rho^2 = 0$.

The system near the north pole approaches the topologically twisted theory with Omega deformation $\epsilon_1 = \ell^{-1}, \epsilon_2 = \tilde{\ell}^{-1}$, and the contribution of localized instantons is described by Nekrasov’s instanton partition function $Z_{\text{inst}}(\hat{a}_0, \epsilon_1, \epsilon_2, \tau)$. Similarly the contribution of anti-instantons localized to the south pole is evaluated by an anti-topologically twisted theory, which leads to Nekrasov’s partition function with the argument $\tilde{\tau}$.

So, the final result for the ellipsoid partition function is

$$
Z = \int d\hat{a}_0 e^{-\frac{8\pi^2}{g^2} \text{Tr}(i_0^2)} |Z_{\text{inst}}|^2 \prod_{\alpha \in \Delta_+} \Upsilon(i\hat{a}_0 \cdot \alpha) \Upsilon(-i\hat{a}_0 \cdot \alpha) \prod_{\rho \in R} \Upsilon(i\hat{a}_0 \cdot \rho + \frac{Q}{2})^{-1}.
$$

(3.4.59)

The similar deformation as one parameter generalization from the results of the theory constructed on round sphere [5] are also obtained in the theory on squashed $S^4$, which is constructed by twisting compactification from $S^4 \times S^1$ [101], but the parameter region of $b$ is different to that of this chapter.

### 3.4.8 Wilson loops.

The generalization of the above result to expectation values of supersymmetric observables is straightforward. Of particular interest are the Wilson loops. Supersymmetry requires the loops to be aligned with the direction of $v$. When $\ell, \tilde{\ell}$ are incommensurable, there are
only two classes of closed loops. One of them winds along the \( \varphi \)-direction and the other along \( \chi \)-direction, and they are both labeled by \( \rho \).

\[
S^1_{\varphi}(\rho) : (x_0, x_1, x_2, x_3, x_4) = (r \cos \rho, \ell \sin \rho \cos \varphi, \ell \sin \rho \sin \varphi, 0, 0),
\]
\[
S^1_{\chi}(\rho) : (x_0, x_1, x_2, x_3, x_4) = (r \cos \rho, 0, 0, \tilde{\ell} \sin \rho \cos \chi, \tilde{\ell} \sin \rho \sin \chi).
\]

(3.4.60)

The corresponding supersymmetric Wilson loops are given by

\[
W_{\varphi}(R) \equiv \text{Tr}_R P \exp i \int_{S^1_{\varphi}(\rho)} d\varphi \left( A_{\varphi} - 2\ell (\phi \cos^2 \frac{\rho}{2} + \bar{\phi} \sin^2 \frac{\rho}{2}) \right),
\]
\[
W_{\chi}(R) \equiv \text{Tr}_R P \exp i \int_{S^1_{\chi}(\rho)} d\chi \left( A_{\chi} - 2\tilde{\ell} (\phi \cos^2 \frac{\rho}{2} + \bar{\phi} \sin^2 \frac{\rho}{2}) \right).
\]

(3.4.61)

At saddle points they take the classical values

\[
W_{\varphi}(R) = \text{Tr}_R \exp (-2\pi b\hat{a}_0),
\]
\[
W_{\chi}(R) = \text{Tr}_R \exp (-2\pi b^{-1}\hat{a}_0).
\]

(3.4.62)

The expectation values of Wilson loops can thus be computed by inserting these expressions into the integral formula (3.4.59).
Chapter 4

$\mathcal{N} = 1$ supersymmetric theories on five dimensional ellipsoids

In this chapter, we will construct rigid $\mathcal{N} = 1$ supersymmetric gauge theories on five dimensional ellipsoids. We will construct the theories by taking the rigid limit [102] of the $\mathcal{N} = 1$ five-dimensional supergravity [103, 104]. In some aspects, this can be regarded as a theoretical challenge to define field theories on a singular manifold with supersymmetry preserved.

We will show that the Killing spinor equations and additional equations for the Killing spinors can be solved on the resolved space of the $n$-covering five-sphere. Using the supersymmetry generated by the solution of the Killing spinor equations, we perform the localization computation for the partition function on the resolved sphere that is the Hopf fibration over deformed $\mathbb{CP}^2$. There are three fixed points on $\mathbb{CP}^2$ for $U(1)^2$ actions inside $U(1) \times SO(4)$ symmetry the resolved sphere has. We notice that the resolved five-sphere can be identified with the squashed five-sphere with the squashing parameters $(\omega_1, \omega_2, \omega_3) = (1/n, 1, 1)$ near the fixed points. Translating the results for the squashed sphere [70, 71, 72, 73], we obtain the perturbative partition function on the $n$-covering five-sphere.

4.1 Rigid $\mathcal{N} = 1$ supersymmetry in five dimensions

The $\mathcal{N} = 1$ supergravity coupled to Yang-Mills and matter fields in five dimensions is constructed in [103]. This theory has an $SU(2)_R$ $R$-symmetry and the Weyl multiplet consists of the vielbein $e^a_\mu$, the graviphoton $A_\mu$, the $SU(2)_R$ gauge field $V_{\mu}^{IJ}$, the $SU(2)_R$ triplet scalar field $t^{IJ}$, the dilaton $\alpha$, the real anti-symmetric tensor $v_{ab}$, the real scalar $C$, the $SU(2)_R$-Majorana gravitino $\psi^I_\mu$ and $SU(2)_R$-Majorana fermion $\chi^I$.

Since the dilaton does not change under the supersymmetry transformation, we can fix the dilatational symmetry by $\alpha = 1$. In addition, we only consider theories without central charge which allows us to turn off the graviphoton $A_\mu = 0$. This simplifies the construction of supersymmetric field theories on a curved space from the $\mathcal{N} = 1$ supergravity as we will perform below.

Following [102], we take the rigid limit of the $\mathcal{N} = 1$ supergravity by setting the
gravitino $\psi^I_\mu$ and the fermion $\chi^I$ and their variations to be zero while letting the spacetime be curved:

$$Q\psi^I_\mu = \nabla_\mu \xi^I - i t^I_\nu \Gamma^\nu_{\mu\xi} - (V_\mu)^I_\nu \xi^\nu + \frac{i}{2} \varepsilon^\nu_\rho \Gamma^\nu_{\mu\rho} \xi^I = 0 ,$$

$$Q\chi^I = \frac{i}{2} \Gamma^\mu_{\nu} \xi^\nu v^\mu_\nu + \frac{i}{2} D_\mu t^I_\nu \Gamma^\nu_{\mu} \xi^J + v^\mu_\nu \Gamma^\nu_{\mu\nu} t^I_\nu \xi^J + \frac{1}{2} C \xi^I = 0 ,$$

where $\nabla_\mu$ is the covariant derivative with respect to the Lorentz index, and

$$\nabla_\mu v^a_b = \partial_\mu v^a_b + \omega^c_\mu v^a_c - \omega^c_\mu v^c_a ,$$

$$D_\mu t^I_\nu = \partial_\mu t^I_\nu - (V_\mu)^I_\nu t^K_J + (V_\mu)^J_K t^I_K .$$

In this limit, the supersymmetry transformations of the other background fields automatically vanish.

### 4.1.1 Supersymmetry algebra and multiplets

The $\mathcal{N} = 1$ supersymmetry algebra in five dimensions is given by

$$\{Q_{\xi^1}, Q_{\xi^2}\} = v^\mu D_\mu + \delta_M (\Theta_{ab}) + \delta_R (R^{IJ}) + \delta_G (\gamma) ,$$

where $D_\mu$ is the covariant derivative with respect to the gauge symmetry and translation, $v_\mu, \Theta_{ab}, R^{IJ}$ and $\gamma$ are parameters for the translation, Lorentz rotation, $SU(2)_R$-symmetry and gauge symmetry transformations

$$v_\mu = 2 \xi^1 \Gamma_\mu \xi^2 ,$$

$$\Theta_{ab} = 2 i (\xi^1 \Gamma_{abc} \xi^2) v^c_d - 2 i (\xi^1 \Gamma^a_{bc} \xi^2 + \xi^1 \Gamma^b_{ac} \xi^2) t^I_{IJ} ,$$

$$R^{IJ} = 6 i (\xi^1 \xi^2) t^I_{IJ} + 2 i (\xi^1 \Gamma^a_{bc} \xi^2 + \xi^1 \Gamma^b_{ac} \xi^2) v^a_b ,$$

$$\gamma = -2 i (\xi^1 \xi^2) \sigma .$$

**Vector multiplet.**

The vector multiplet contains the gauge field $A_\mu$, a real scalar $\sigma$, an $SU(2)_R$ triplet scalar $Y^{IJ}$ and an $SU(2)_R$-Majorana fermion $\lambda^I$. They transform under the supersymmetry as

$$Q A_\mu = -2 \xi^1 \Gamma_\mu \lambda ,$$

$$Q \sigma = 2 i \xi^1 \lambda ,$$

$$Q \lambda^I = \frac{1}{4} \Gamma^{\mu\nu} \xi^I F_{\mu\nu} - \frac{i}{2} \Gamma^\nu_{\mu} D_\mu \sigma - Y^{IJ} \xi^J ,$$

$$Q Y^{IJ} = -\xi^J \Gamma^\mu D_\mu \lambda^I + \frac{i}{2} \varepsilon^{\nu_\rho} \Gamma^\nu_{\mu\nu} \lambda^J + it^{IJ}_K \xi^K \lambda^I + 2 i t^{IJ}_I \xi^\lambda - i [\sigma, \xi^I \lambda^J] + (I \leftrightarrow J) .$$

---

1. We contract the $SU(2)_R$ indices from northwest to southeast direction when the indices are suppressed.
2. We can redefine the triplet scalar $Y^{IJ}$ in [103, 104, 105, 106] as $D_{IJ} = 2Y^{IJ} - 2t_{IJ}\sigma$ to make contact with [65].

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\[ F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu - [A_\mu, A_\nu]. \] The covariant derivatives are
\[ D_\mu \sigma = \partial_\mu \sigma - [A_\mu, \sigma], \]
\[ D_\mu \lambda^I = \nabla_\mu \lambda^I - (V_\mu)^I_J \lambda^J - [A_\mu, \lambda^I]. \] (4.1.6)

To put them into cohomological forms, we redefine the gaugino as [64]
\[ \lambda^I = \frac{1}{4} (-\xi^I v^\mu \Psi_\mu - \Gamma^\mu \xi^I \Psi_\mu - \Gamma^{\mu\nu} \xi^I \chi_{\mu\nu}) , \] (4.1.7)

where
\[ \Psi_\mu = -2 \xi_\mu \Gamma, \quad \chi_{\mu\nu} = \xi_{\mu\nu}. \] (4.1.8)

Then the supersymmetry transformation law (4.1.5) becomes
\[ Q A_\mu = \Psi_\mu, \]
\[ Q \Psi_\mu = v^\nu F_{\mu\nu} + i D_\mu \sigma, \]
\[ Q \sigma = -i v^\mu \Psi_\mu, \]
\[ Q \chi_{\mu\nu} \equiv H_{\mu\nu} = \frac{1}{4} (\xi^I \Gamma_{\mu\nu\rho\sigma} \xi_I) F^{\rho\sigma} - \frac{1}{2} F_{\mu\nu} - \frac{i}{2} (v_\mu D_\nu \sigma - v_\nu D_\mu \sigma) + (\xi^I \Gamma_{\mu\nu} \xi^J) Y_{IJ}, \]
\[ Q H_{\mu\nu} = v^\rho D_\rho \chi_{\mu\nu} + i [\sigma, \chi_{\mu\nu}] + \Theta_\mu^\rho \chi_{\nu \rho} - \Theta_\nu^\rho \chi_{\mu \rho}. \] (4.1.9)

The supersymmetric action of the vector multiplet is
\[ \mathcal{L}_{YM} = -\frac{2}{g^2} \text{Tr} \left[ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} D_\mu \sigma D^\mu \sigma - Y_{IJ} Y^{IJ} + 2 \sigma (2t_{IJ} Y^{IJ} - F_{\mu\nu} v^{\mu\nu}) + 2 (C - 4 t_{IJ} t^{IJ}) \sigma^2 \right. \]
\[ + 2 \lambda \Gamma^\mu D_\mu \lambda - i \lambda^I (\epsilon_{I,J} \Gamma_{\mu\nu} v^{\mu\nu} - 2 t_{IJ}) \lambda^J - 2 i \sigma [\lambda, \lambda] \left], \right. \] (4.1.10)

where \( g \) is the gauge coupling constant. The Chern-Simons term can be added in five dimensions\(^3\)
\[ \mathcal{L}_{CS}(A) = \frac{k}{24 \pi^2} \text{Tr} \left[ A \wedge dA \wedge dA + \frac{3}{2} A \wedge A \wedge dA + \frac{3}{5} A \wedge A \wedge A \wedge A \right], \] (4.1.11)

whose supersymmetric completion is [64]
\[ \mathcal{L}_{SCS} = \mathcal{L}_{CS}(A - i \sigma \kappa) - \frac{k}{8 \pi^2} \text{Tr} [\Psi \wedge \Psi \wedge \kappa \wedge F(A - i \sigma \kappa) ] , \] (4.1.12)

where \( \kappa \) and \( \Psi \) are one-forms dual to the Killing vector \( \kappa \equiv v_\mu dx^\mu \) and \( \Psi \equiv \Psi_\mu dx^\mu \), respectively.

---

\(^3\)Note that there is no imaginary unit as an overall factor because the gauge field \( A \) is anti-hermitian in our convention.
Hypermultiplet.

We can realize the supersymmetry algebra only for on-shell hypermultiplets. We, however, need one supercharge $\delta$ with unit norm $\xi = 1$ for localization which satisfies

$$Q^2 = v^\mu D_\mu + \delta_M (\Theta_{ab}) + \delta_R (R^IJ) + \delta_G (\gamma),$$

where

$$\begin{align*}
v_\mu &= \xi \Gamma_\mu \xi, \\
\Theta_{ab} &= i (\xi \Gamma_{abcd} \xi) v^{cd} - 2i (\xi I \Gamma_{ab} \xi^J) t_{IJ}, \\
R^{IJ} &= 3i t^{IJ} + 2i (\xi I \Gamma_{ab} \xi^J) v_{ab}, \\
\gamma &= -i \sigma,
\end{align*}$$

with the modified supersymmetry transformation law by introducing an auxiliary field \cite{65}. They satisfy

$$v^\mu \Gamma_\mu \xi_I = \xi_I, \quad v_\mu v_\mu = 1, \quad v_\mu \Theta_{ab} = 0, \quad \nabla_\mu v_\nu = -\Theta_{\mu\nu} .$$

The hypermultiplet consists of a scalar $q_{IA}$, a fermion $\psi_A$ and an auxiliary field $F_A^I$ with flavor indices $A = 1, 2, \cdots, 2r$ for $r$ hypermultiplets. The index $A$ is raised and lowered with a $2r \times 2r$ antisymmetric matrix $\Omega_{AB}$ as

$$q_{IA} = q^{IB} \Omega_{BA}, \quad q^{IA} = \Omega^{AB} q_{IB}, \quad \Omega^{AB} \Omega_{BC} = -\delta^A_C .$$

The reality conditions are imposed by

$$(q_{IA})^* = \epsilon_{IJ} \Omega^{AB} q_{IB}^J, \quad (\psi^A)^* = \Omega^{AB} \psi_B^{\beta} C_{\beta\alpha}, \quad (F_A^I)^* = \epsilon_{IJ} \Omega^{AB} F_B^J, \quad \Omega_{AB} = \Omega^{AB} .$$

The generators of the Lie group have one upper and one lower indices of the flavor symmetry, $\sigma_{A\dot{B}}$ for example.

Choosing the invariant tensor $\Omega$ of $Sp(2r)$ to be

$$\Omega_{AB} = \Omega^{AB} = \begin{pmatrix} 0 & 1_r \\ -1_r & 0 \end{pmatrix} ,$$

the scalar field $q_{IA}^I$ satisfying the reality condition (4.1.17) is written as a two-component vector

$$q^1 = \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix}, \quad q^2 = \begin{pmatrix} -\phi_-^* \\ \phi_+^* \end{pmatrix} ,$$

where $\phi_+$ and $\phi_-$ are in the fundamental and anti-fundamental representation of $SU(r)$ respectively. The fermion $\psi_A$ is similarly decomposed to

$$\psi^A_\dot{A} = \begin{pmatrix} \psi^{a}_{\dot{A}} \\ -\psi^{\dot{A}}_{aB} \end{pmatrix} ,$$
as a vector with spinors $\psi_A$ in the fundamental representation of $SU(r)$.

The off-shell supersymmetry transformations realizing (4.1.13) are found to be

$$
Q q^I_A = 2i\xi^I q^I_A ,
Q \psi_A = -i(D_\mu q^I_A)\Gamma^\mu \xi_I - 3t_I^J q^I_A + \xi^I \sigma_{AB} q^B_J - v_{\mu\nu} \Gamma^{\mu\nu} \xi_I q^I_A + F^I_A \xi_I ,
Q F^I_A = -2\xi^I \left( \Gamma^\mu D_\mu \psi_A + \frac{i}{2} v_{\mu\nu} \Gamma^{\mu\nu} \psi_A + i\sigma_{AB} \psi^B + 2i(\lambda^I)_{AB} q^B_J \right) ,
$$

(4.1.21)

where the checked parameter $\check{\xi}^I$ satisfies

$$
\xi_I \xi_I = \check{\xi}^I \check{\xi}_I , \quad \xi_I \check{\xi}_I = 0 , \quad \xi_I^I \Gamma_\mu \xi_I + \check{\xi}^I \Gamma_\mu \check{\xi}_I = 0 ,
$$

(4.1.22)

and the covariant derivatives are

$$
D_\mu q^I = \partial_\mu q^I - (V_\mu)^I_J q^J - A_\mu q^I ,
D_\mu \psi = \nabla_\mu \psi - A_\mu \psi .
$$

(4.1.23)

We rewrite the transformation laws (4.1.21) with the fermionic variables

$$
q_A \equiv \xi_I q^I_A , \quad \psi^+_A \equiv P_+ \psi_A ,
$$

(4.1.24)

which satisfy the “chirality” conditions

$$
P_+ q_A = \xi_I q^I_A , \quad \psi^-_A = P_- \psi_A ,
$$

(4.1.25)

$P_\pm \equiv \frac{1}{2} (1 \pm \gamma^\mu \gamma^\nu).$ For a shifted auxiliary field $\tilde{F}_A \equiv \xi_I \tilde{F}^I_A$ ($P_- \tilde{F}_A = \tilde{F}_A$), the supersymmetry transformations of the fields $(q_A, \psi^+_A, \tilde{F}_A)$ are recast in the following form:

$$
Q q_A = -i\psi^+_A ,
Q \psi^+_A = i \left( v_\mu D_\mu \delta_\mu^A + i\sigma_\mu^A \right) q_B + \frac{i}{4} \Theta_{ab} \Gamma^{ab} q_A ,
Q \psi^-_A = \tilde{F}_A ,
Q \tilde{F}_A = \left( v_\mu D_\mu \delta_\mu^A + i\sigma_\mu^A \right) \psi^-_B + \frac{1}{4} \Theta_{ab} \Gamma^{ab} \psi^-_A .
$$

The matter lagrangian reads

$$
\mathcal{L}_{\text{matter}} = D_\mu \bar{q} \Gamma^\mu q + \left( v_{\mu\nu} v^{\mu\nu} + 2t_{IJ} t^{IJ} - C - \frac{R}{4} \right) \bar{q} q - \bar{q} \tilde{q} \left( t_{IK} t^{KJ} + C_{IJ} - \sigma^2 c_{IJ} - 2Y_{IJ} \right) q^I + 2\bar{\psi} \Gamma^\mu D_\mu \psi + i\bar{\psi} \left( \Gamma_\mu \psi^{\mu\nu} + 2\sigma \right) \psi - 8i\bar{q} \lambda \psi - \tilde{F} \tilde{F} ,
$$

(4.1.26)

where the flavour indices $A, B$ are contracted from northeast to southwest.

---

4Although our off-shell supersymmetry transformation (4.1.21) differs from theirs in [104], the off-shell lagrangian still closes.
4.1.2 Resolved space

We are interested in $\mathcal{N} = 1$ supersymmetric field theories on a branched $n$-covering of five-sphere. To treat the conical singularity we replace it with a resolved space whose metric is given by

$$ds^2 = \frac{d\theta^2}{f(\theta)} + n^2 \sin^2 \theta d\tau^2 + \cos^2 \theta ds^2_{S^3},$$

(4.1.27)

where $f(\theta)$ is a smooth function behaving as

$$f(\theta) = \begin{cases} 
1/n^2, & \theta = 0, \\
1, & \epsilon < \theta < \pi/2,
\end{cases}$$

(4.1.28)

for $\epsilon \ll 1$. $ds^2_{S^3}$ is the metric of a three-sphere

$$ds^2_{S^3} = \sum_{i=1}^{3} e^i_L e^i_L.$$

(4.1.29)

The vielbein $e^i_L$ in the left invariant frame and the spin connections satisfy

$$de^i_L = \epsilon^{ijk} e^j_L \wedge e^k_L, \quad \omega^i_L = \epsilon^{ijk} e^i_L.$$

(4.1.30)

They can be parametrized by an element $g$ of $SU(2)$ group

$$ie^i_L \sigma^i = g^{-1} dg,$$

(4.1.31)

with the Pauli matrices $\sigma^i$ ($i = 1, 2, 3$). We choose the vielbein of the resolved space (4.1.27) as

$$e^1 = \frac{d\theta}{\sqrt{f(\theta)}}, \quad e^2 = n \sin \theta d\tau, \quad e^{i+2} = \cos \theta e^i_L \quad (i = 1, 2, 3),$$

(4.1.32)

and the spin connections are

$$\omega^{12} = -n \cos \theta \sqrt{f(\theta)} d\tau, \quad \omega^{i+2} = \sin \theta \sqrt{f(\theta)} e^i_L, \quad \omega^{i+2j+2} = \omega^i_{LJ}.$$ 

(4.1.33)

4.1.3 Relation between resolved space and squashed five-sphere

A five-sphere is embedded into $\mathbb{C}^3$ by complex coordinates $(z_1, z_2, z_3)$ as $|z_1|^2 + |z_2|^2 + |z_3|^3 = 1$. It has $U(1)^3$ symmetry acting on the coordinates as

$$(z_1, z_2, z_3) \rightarrow (e^{ia_1} z_1, e^{ia_2} z_2, e^{ia_3} z_3).$$

(4.1.34)

The five-sphere has the Hopf fiber representation as the $U(1)$ fibration over $\mathbb{CP}^2$. The translation along the $U(1)$ fiber is described by the overall $U(1)$ phase rotation $a_1 = a_2 = a_3$ in (4.1.34). There are three fixed points of $U(1)^2$ symmetry on the base $\mathbb{CP}^2$ at $(z_1, z_2, z_3) = (1, 0, 0), (0, 1, 0), (0, 0, 1)$. 

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Let us introduce new coordinates by
\[
\begin{align*}
    z_1 &= \sin \theta e^{i\tau}, \\
    z_2 &= \cos \theta \cos \phi \frac{e^{i(\chi+\xi)}}{2}, \\
    z_3 &= \cos \theta \sin \phi \frac{e^{i(\chi-\xi)}}{2},
\end{align*}
\]
where the ranges of the angles are taken to be \(0 \leq \theta < \pi/2\), \(0 \leq \tau < 2\pi\), \(0 \leq \phi < \pi\), \(0 \leq \chi < 4\pi\) and \(0 \leq \xi < 2\pi\). The metric is locally that of a five-sphere, but globally the \(n\)-branched cover
\[
\begin{align*}
    ds^2 &= d\theta^2 + n^2 \sin^2 \theta d\tau^2 + \cos^2 \theta ds_{S^3}^2, \\
    ds_{S^3}^2 &= \frac{1}{4} \left[ d\phi^2 + \sin^2 \phi d\xi^2 + (d\chi + \cos \phi d\xi)^2 \right].
\end{align*}
\]
In this parametrization, the translation along the Hopf fiber is given by the shifts of the angles
\[
\tau \rightarrow \tau + \frac{a}{n}, \quad \chi \rightarrow \chi + 2a,
\]
that is generated by a vector field
\[
v^\mu \partial_\mu = \frac{1}{n} \partial_\tau + 2 \partial_\chi.
\]
The three fixed points are located at
\[
\begin{align*}
    (1,0,0) : & \quad \theta = \frac{\pi}{2}, \\
    (0,1,0) : & \quad \theta = 0, \quad \phi = 0, \\
    (0,0,1) : & \quad \theta = 0, \quad \phi = \pi.
\end{align*}
\]
The vielbein for \(S^3\) are written as
\[
\begin{align*}
    e^1_L &= \frac{1}{2} \left( \sin \phi \cos \chi \ d\xi - \sin \chi \ d\phi \right), \\
    e^2_L &= \frac{1}{2} \left( \sin \chi \sin \phi \ d\xi + \cos \chi \ d\phi \right), \\
    e^3_L &= \frac{1}{2} \left( d\chi + \cos \phi \ d\xi \right).
\end{align*}
\]
Now we consider a deformation of a five-sphere satisfying \(|z_1|^2/n^2 + |z_2|^2 + |z_3|^2 = 1\). This is a special case of the squashed five-sphere defined by
\[
\omega_1^2 |z_1|^2 + \omega_2^2 |z_2|^2 + \omega_3^2 |z_3|^2 = 1,
\]
with three squashing parameters chosen as
\[
(\omega_1, \omega_2, \omega_3) = \left( \frac{1}{n}, 1, 1 \right).
\]
One can parametrize the complex coordinates with the real angles by

\[ z_1 = n \sin \theta e^{i\tau}, \]
\[ z_2 = \cos \theta \cos \frac{\phi}{2} e^{i(x+\xi)/2}, \]
\[ z_3 = \cos \theta \sin \frac{\phi}{2} e^{i(x-\xi)/2}, \]  

(4.1.43)

that gives

\[ ds^2 = \frac{d\theta^2}{f_n(\theta)} + n^2 \sin^2 \theta d\tau^2 + \cos^2 \theta ds^2_{S^3}, \]  

(4.1.44)

where \( f_n(\theta) = 1/(n^2 \cos^2 \theta + \sin^2 \theta) \). This space may be regarded as a resolved space with \( f(\theta) = f_n(\theta) \) if the partition function does not depend on the choice of \( f(\theta) \). In section 4.3, we show in detail that this identification is possible for evaluating the one-loop partition functions.

### 4.1.4 Killing spinor equations

We will solve the Killing spinor equations (4.1.1) on the resolved space (4.1.27). We let spinors \( \xi^I \) be tensor products of spinors in two dimensions \( \zeta^I \) and spinors in three dimensions \( \eta^I \)

\[ \xi^I = \zeta^I \otimes \eta^I. \]  

(4.1.45)

Correspondingly, the gamma matrices that are hermitian \( (\Gamma^a)^\dagger = \Gamma^a \) can be written in tensor product forms:

\[ \Gamma^1 = \sigma^1 \otimes 1_2, \quad \Gamma^2 = \sigma^2 \otimes 1_2, \quad \Gamma^{i+2} = \sigma^3 \otimes \sigma^i, \quad (i = 1, 2, 3). \]  

(4.1.46)

The charge conjugation matrix takes the form

\[ C = \sigma_1 \otimes (i\sigma_2). \]  

(4.1.47)

With the vielbein (4.1.32) and the spin connections (4.1.33), we find the background fields

\[ t^1_1 = -t^2_2 = \frac{1}{2} \sqrt{f(\theta)}, \quad v^{12} = \mp i \frac{\sqrt{f(\theta)} - 1}{2 \cos \theta}, \]  

(4.1.48)

\[ (V_{\mu})^1_1 = -(V_{\mu})^2_2 = \mp i \frac{n \sqrt{f(\theta)}}{2} \delta_{\mu\tau}, \]

solve the first line of the Killing spinor equations (4.1.1) with the solutions

\[ \xi^1 = (e^{i\theta_1} \zeta^1) \otimes \eta_+, \quad \sigma_3 \zeta^1 = \pm \zeta^1, \]
\[ \xi^2 = (e^{-i\theta_1} \zeta^2) \otimes \eta_-, \quad \sigma_3 \zeta^2 = \mp \zeta^2, \]  

(4.1.49)

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where $\zeta^{1,2}$ are constant spinors in two dimensions and $\eta_{\pm}$ are the Killing spinors on a unit three-sphere
\[
\left( \partial_i + \frac{i}{2} \sigma_i \right) \eta_\pm = \pm \frac{i}{2} \sigma_i \eta_\pm , \quad (i = 1, 2, 3) .
\] (4.1.50)

The $SU(2)$-Majorana condition leads the relations
\[
\zeta^2 = \sigma_1 \zeta^1 , \quad \eta^- = i \sigma_2 \eta_+ ,
\] (4.1.51)
which are compatible with the solutions (4.1.49).

The second line of the Killing spinor equations (4.1.1) is satisfied by the solutions (4.1.49) if we choose the scalar field $C$ to be
\[
C = \frac{1}{4} \cot \theta f'(\theta) - \frac{f(\theta) - \sqrt{f(\theta)}}{2 \cos^2 \theta} .
\] (4.1.52)

### 4.2 Localization

We will localize the infinite-dimensional path integral of the partition function to a finite-dimensional matrix integral by adding a $Q$-exact term to the action $I \to I + t Q V$ where the localizing term $Q V$ is taken to be positive semi-definite. Since the path integral does not depend on the $Q$-exact term, we let $t$ be large so that the fixed points are given by $Q V = 0$. After determining the fixed point loci for the gauge and matter sectors, we will read off the perturbative partition function on the resolved space by identifying it with the squashed five-sphere at the three fixed points on the base $\mathbb{C}P^2$.

#### 4.2.1 Gauge sector

A localization term for the gauge sector is given similarly to [65] by
\[
V_{gauge} = -4 \text{Tr} \left[ (Q \lambda)^\dagger \lambda \right] ,
\] (4.2.1)

whose supersymmetry variation yields
\[
Q V_{gauge} \big|_{boson} = -\text{Tr} \left[ \frac{1}{4} \left( F_{\mu \nu} + \frac{1}{2} \varepsilon_{\mu \nu \rho \sigma \kappa} v^\rho F^{\sigma \kappa} \right) \left( F^{\mu \nu} + \frac{1}{2} \varepsilon^{\mu \nu \rho \sigma \kappa} v_\rho F_{\sigma \kappa} \right) 
+ \frac{1}{2} (v^\rho F_\rho \mu)(v_\mu F^{\rho \nu}) - D_\mu \sigma D^\mu \sigma + 2 Y_{IJ} Y^{IJ} \right] ,
\] (4.2.2)

where we let the hermitian conjugates of $\sigma$ and $Y_{IJ}$ be $\sigma^\dagger = -\sigma$ and $Y_{IJ}^\dagger = Y^{IJ}$. The saddle point of the localization term (4.2.2) is
\[
F_{\mu \nu} = -\frac{1}{2} \varepsilon_{\mu \nu \rho \sigma \kappa} v^\rho F^{\sigma \kappa} , \quad v^\rho F_\rho \mu = 0 , \quad D_\mu \sigma = 0 , \quad Y_{IJ} = 0 .
\] (4.2.3)

In the zero instanton sector, the gauge field is a flat connection and the saddle point becomes
\[
A_\mu = 0 , \quad \sigma = \sigma_0 = \text{const} ,
\] (4.2.4)

up to the gauge transformation.
4.2.2 Matter sector

We choose a localization term for the matter sector as

\[ V_{\text{matter}} = (Q\psi^+_A)^\dagger \psi^+_A + (Q\psi^-_A)^\dagger \psi^-_A , \]

(4.2.5)

whose supersymmetry variation yields

\[ QV_{\text{matter}}|_{\text{boson}} = \left( \frac{1}{4} \Theta_{ab} \Gamma^{ab} q_A + v^\mu D_\mu q_A \right) \dagger \left( \frac{1}{4} \Theta_{ab} \Gamma^{ab} q_A + v^\mu D_\mu q_A \right) + (\sigma q)^\dagger_A (\sigma q)^A + \tilde{F}_A^A \tilde{F}^A \]

(4.2.6)

Then the path integral localizes to the following fixed locus

\[ \left( \frac{1}{4} \Theta_{ab} \Gamma^{ab} + v^\mu D_\mu \right) q_A = 0 , \quad (\sigma q)^B q_B = 0 , \quad \tilde{F}_A = 0 . \]

(4.2.7)

Combining with (4.2.4), one finds

\[ q_A = \tilde{F}_A = 0 . \]

(4.2.8)

4.2.3 Partition function on the \( n \)-covering five-sphere

As described in section 4.1.3, the resolved five-sphere (4.1.27) can be regarded as a squashed five-sphere with \( f(\theta) = f_n(\theta) \) since they are locally equivalent near the fixed points of the four-dimensional base space in the Hopf fiber representation. Under this identification, the squashing parameters are \( (\omega_1, \omega_2, \omega_3) = (1/n, 1, 1) \) and the perturbative partition function is \([71, 72]\)

\[ Z_{\text{pert}} = \int d\sigma_0 e^{-I_0} \prod_{\alpha \in \text{positive root}} S_3 \left( \alpha(\sigma_0) \left| \frac{1}{n}, 1, 1 \right. \right) S_3 \left( \alpha(\sigma_0) + 2 + \frac{1}{n} \right| \left. \frac{1}{n}, 1, 1 \right) \]

\[ \times \prod_{\rho \in \text{weight}} S_3^{-1} \left( m + \frac{1}{2} \left( \frac{1}{n} + 2 \right) + \rho(\sigma_0) \right| \left. \frac{1}{n}, 1, 1 \right) , \]

(4.2.9)

where \( S_3 \) is the triple sine function

\[ S_3(x|\omega_1, \omega_2, \omega_3) = \prod_{p,q,r \geq 0} \left( p\omega_1 + q\omega_2 + r\omega_3 + x \right) \left( (p + 1)\omega_1 + (q + 1)\omega_2 + (r + 1)\omega_3 - x \right) . \]

(4.2.10)

Here we take the mass of hypermultiplet as \( m \), and \( I_0 \) is the classical contribution at the localization fixed point\(^5\)

\[ I_0 = -\frac{2}{g^2} \int d^5 \sqrt{g} (2C - 8t^{IJ}t_{IJ}) \text{Tr} \sigma_0^2 + \frac{i k}{24\pi^2} \int \kappa \wedge d\kappa \wedge d\kappa \text{Tr} \sigma_0^3 \]

\[ = -\frac{8\pi^3 n}{g^2} \text{Tr} \sigma_0^2 + \frac{ik\pi n}{3} \text{Tr} \sigma_0^3 , \]

(4.2.11)

where we used the volume of the \( n \)-covering five-sphere in the second equality.\(^6\) There are also instanton contributions.

\(^5\)Here, there is an imaginary unit in front of the Chern-Simons level \( k \) because of our anti-hermitian convention for Lie algebras.

\(^6\)Note that our convention leads to \( \kappa \wedge d\kappa \wedge d\kappa = 8\text{vol}(M_5) \).
4.3 Omega deformation parameters

In this section, we justify the identification of the resolved space with the squashed five-sphere with the squashing parameters (4.1.42). We relabel the phases of \( z_i \) by \( \phi_i \)

\[
\phi_1 = \tau , \quad \phi_2 = \frac{1}{2}(\chi + \xi) , \quad \phi_3 = \frac{1}{2}(\chi - \xi)
\]

(4.3.1)

then

\[
\partial_{\phi_1} = \partial_\tau , \quad \partial_{\phi_2} = \partial_\chi + \partial_\xi , \quad \partial_{\phi_3} = \partial_\chi - \partial_\xi .
\]

(4.3.2)

Since the Hopf fibration rotates all \( z_i \)'s by a same angle, the direction of this fibration can be arranged into

\[
w_\mu \partial_\mu = a \sum_i \partial_{\phi_i},
\]

and remaining part \( \partial_\nu = a_i \partial_{\phi_i} \) of Killing vector \( v = \frac{1}{n} \partial_\tau + 2 \partial_\chi = w_\mu \partial_\mu + \partial_\nu \) translates the base manifold of the Hopf fibration. Here we restrict the parameters as

\[
a_1 + a_2 + a_3 = 0 \quad \text{so that} \quad \partial_\nu \text{ acts only base manifold. These parameters are fixed to be}
\]

\[
a_1 = \frac{2}{3n}(1 - n) , \quad a_2 = a_3 = -\frac{1}{3n}(1 - n) , \quad \partial_\nu = \frac{2n+1}{3n} (\partial_\tau + 2\partial_\chi) ,
\]

(4.3.3)

and their directions are parametrized by \( \partial_\nu = w_\mu \partial_\mu \) and \( \tau'' \) by

\[
\tau' = \frac{n}{2n+1}(\tau + \chi) , \quad \tau'' = \frac{n}{2(1-n)}(2\tau - \chi) ,
\]

\[
\tau = \frac{1}{3n}((1 + 2n)\tau' + 2(1 - n)\tau'') , \quad \chi = \frac{2}{3n}((1 + 2n)\tau' - (1 - n)\tau'') .
\]

(4.3.4)

The metric of the base four-dimensional manifold \( ds^2_{M_4} \) are derived by square completion in \( d\tau' \), the direction of the Hopf fibration. In fact, by using

\[
w_\mu dx^\mu = \left( \frac{1 + 2n}{3n} \right)^2 (n^2 \sin^2 \theta + \cos^2 \theta) \left[ d\tau' + \frac{2(2n^2 \sin^2 \theta - \cos^2 \theta) d\tau'' + 3n \cos^2 \theta \cos \phi d\xi}{2(1 + 2n)(n^2 \sin^2 \theta + \cos^2 \theta)} \right],
\]

(4.3.5)

the metric of the original manifold turns out to be

\[
ds^2 = \frac{d\theta^2}{f_n(\theta)} + n^2 \sin^2 \theta d\tau^2 + \frac{1}{4} \cos^2 \theta (d\phi^2 + d\xi^2 + d\chi^2 + 2 \cos \phi d\xi d\chi) ,
\]

\[
= \left( \frac{3n}{1 + 2n} \right)^2 \frac{(w_\mu dx^\mu)^2}{n^2 \sin^2 \theta + \cos^2 \theta} ,
\]

\[
+ \frac{d\theta^2}{f_n} + \frac{1}{4} \cos^2 \theta (d\phi^2 + \sin^2 \phi d\xi^2) + \frac{\sin^2 \theta \cos^2 \theta}{n^2 \sin^2 \theta + \cos^2 \theta} \left[ (1 - n) d\tau'' - \frac{n}{2} \cos \phi d\xi \right]^2 .
\]

(4.3.6)
$ds_{M^4}^2$ is the third line of this metric. The squashing parameters in (4.1.42) are read from the one-loop determinant in instanton configurations (4.2.3). For a while, we take the translation along $\xi$ direction with charge $\delta$ into the translation of base $M^4$ generated by the Killing vector. This is expressed as $a_2 - a_3 = \delta \neq 0$, and finally $\delta$ is taken to be zero. Then the Killing vector which acts on $M^4$ is $\delta \partial_\xi + \partial_{\tau''}$. The fixed points of this translation are

$$(z_1, z_2, z_3) = (n, 0, 0), (0, 1, 0), (0, 0, 1). \quad (4.3.7)$$

In following, we determine the squashing parameters in (4.1.42) from the configurations around these fixed points.

Firstly we consider the metric around $x_1 \equiv (n, 0, 0)$. The distance from this point is parametrized by $\epsilon \ll 1$ where $\theta = \frac{\pi}{2} + \epsilon$. Hence in the first order in $\epsilon$, the metric near $(n, 0, 0)$

$$ds^2 \sim \left(3 + 2n\right)^2 (w_\mu dx^\mu)^2 + d\epsilon^2 + \frac{\epsilon^2}{4}(d\phi^2 + \sin^2 \phi d\xi^2) + \frac{1}{n^2} \left[(1 - n) d\tau'' - \frac{n}{2} \cos \phi d\xi\right]^2,$$

is interpreted as $S^1$-fibration over a two-dimensional complex plane $(\xi_1^1, \xi_2^1)$, $ds^2 = \left(\frac{3}{1+2n}\right)^2 (w_\mu dx^\mu)^2 + \sum_{i=1}^2 |d\xi_i^1|^2$, with

$$\xi_1^1 = \frac{n z_2}{z_1} \simeq -\epsilon \cos \frac{\phi}{2} e^{i\left(\frac{\xi_1^1 + 1 - n}{n} \tau''\right)} ,$$

$$\xi_2^1 = \frac{n z_3}{z_1} \simeq -\epsilon \sin \frac{\phi}{2} e^{-i\left(\frac{\xi_2^1 + 1 - n}{n} \tau''\right)} . \quad (4.3.9)$$

This two-dimensional complex plane $M^4$ is Omega-deformed by the Killing vector $\delta \partial_\xi + \partial_{\tau''}$ with Omega-deformation parameters that can be read off from the rotated angle of $\xi_i^1$ by the vector:

$$\xi_1^1 = \frac{1 - n}{n} + \frac{\delta}{2},$$

$$\xi_2^1 = \frac{1 - n}{n} - \frac{\delta}{2} \quad (4.3.10)$$

In this sense, the base manifold $M^4$ around this fixed point is interpreted as Omega-deformed four-dimensional plane $\mathbb{R}^4_{\frac{1 - n}{n} + \frac{\delta}{2}, \frac{1 - n}{n} - \frac{\delta}{2}}$.

In the same manner, we parametrize the configuration around $x_2 \equiv (0, 1, 0)$ by $\theta = \epsilon, \phi = \epsilon'$ where $\epsilon, \epsilon' \ll 1$. The metric of the original manifold is written in terms of

$$\xi_1^2 = \frac{z_1}{z_2} \simeq n e^{i\left(-\frac{\xi_1^2 + 1 - n}{n} \tau''\right)},$$

$$\xi_2^2 = \frac{z_3}{z_2} \simeq \epsilon' e^{-i\xi} . \quad (4.3.11)$$

This manipulation is expected to be justified by taking another sixth direction $S^1$ and twisting this $U(1)$ with this $S^1$, for instance.
as \( ds^2 = \left( \frac{3n}{1+2n} \right)^2 (w_\mu dx^\mu)^2 + \sum_{i=1}^{2} |dξ^2_i|^2 \). Also this base manifold is an Omega-deformed four-dimensional plane \( \mathbb{R}^4_{\frac{1}{n} - \frac{2}{n} - \delta} \). Finally, the coordinates around the third fixed point \( x_3 \equiv (0, 0, 1) \) are represented by \( \theta = \epsilon, \phi = \pi + \epsilon' \) where \( \epsilon, \epsilon' \ll 1 \). Again, the metric \( ds^2 = \left( \frac{3n}{1+2n} \right)^2 (w_\mu dx^\mu)^2 + \sum_{i=1}^{2} |dξ^2_i|^2 \) around this point is written as a four-dimensional plane \( \mathbb{R}^4_{\frac{1}{n} - \frac{2}{n} + \delta} \), coordinated by

\[
\xi_1^3 = \frac{z_1}{z_3} \simeq ne^{i(\frac{\epsilon}{2} + \frac{1-n}{n} \epsilon')} ,
\]
\[
\xi_2^3 = \frac{z_2}{z_3} \simeq -\frac{\epsilon'}{2} e^{i\xi} ,
\]

whose Omega deformation parameters are obtained from their charges.

From above results, we read a one-loop determinant \( Z_{\text{one-loop}} = \prod_j w_j^{-\frac{1}{2} c_j} \) around the saddle point configuration (4.2.3) from \( \text{ind}_{D_{10}} = \sum_j c_j e^{w_j} \). Since the indices of vector multiplets in five-dimensional \( \mathcal{N} = 1 \) theory are related to those of the self-dual complex \( \text{ind}_{D_{\text{SD}}} [107] \), which are represented by the indices \( \text{ind} \tilde{D} \) of the Dolbeault complex as

\[
\text{ind}_{D_{\text{SD}}} = \frac{1}{2} (1 + e^{i(\epsilon_1 + \epsilon_2)}) \text{ind} \tilde{D} .
\]

We calculate \( \text{ind} \tilde{D} \) in the following. The indices of Dirac complex associated with the one-loop determinant of hypermultiplets are also related to \( \text{ind} \tilde{D} \). The index of vector bundle \( V \) on a manifold \( \mathbb{M}_4 \) is calculated by the Riemann-Roch-Hirzebruch theorem as

\[
\text{ind} \tilde{D}_V = \int_{\mathbb{M}_4} \text{Td}(TM_{\mathbb{M}_4}) \text{Ch}(V) .
\]

The integral of an equivariant closed form \( \alpha \) localizes to the fixed points of the isometry \( d : \int \alpha = \sum_p \alpha(x_p) \chi(x_p) \) where \( x_p \) are fixed points of the isometry and \( \chi \) is the Euler class. The quantities in the right-hand side are evaluated in zero-form value at \( x_p \).

**Todd class**

The Todd class for \( \mathbb{C}^2 \) is \( \frac{\lambda_1}{1 - e^{i\epsilon_1}} \frac{\lambda_2}{1 - e^{i\epsilon_2}} \) where \( \lambda_{1,2} \) are Chern roots. These are deformed as \( \lambda_i \rightarrow \lambda_i + i \epsilon_i \) by Omega-deformation parameters \( \epsilon_i \). In particular, the zero-form value of the Todd class becomes

\[
\frac{-\epsilon_1^i \epsilon_2^i}{(1 - e^{i\epsilon_1})(1 - e^{i\epsilon_2})} .
\]

**Euler class**

The Euler class for an oriented four-dimensional manifold \( \mathbb{M}_4 \) is written in terms of their curvature: \( \chi(\mathbb{M}_4) = \frac{1}{12 \pi^2} e^{ijkl} R_{ij} R_{kl} \). In the case of \( \mathbb{R}^4 \), this is equal to \( -\epsilon_1^i \epsilon_2^i \).

**Kähler-Hodge manifold**

When Kähler transformation defines a line bundle, its first Chern class is equivalent to its Kähler class: \( \text{Ch}(V) = e^{c_i(V)} = e^J \) where \( J \) is the Kähler two-form. To implement
the equivariance for our case, considering that the Kähler two-form is closed including the
twist by \( \partial_{x'} \), we have to add zero-form value to \( J \). In fact,

\[
-i J_i = \frac{1}{2} \left( \frac{1}{n^2} dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3 \right) + \frac{a_1}{n^2} |z_1|^2 + a_2 |z_2|^2 + a_3 |z_3|^2 ,
\]

(4.3.15)
is closed for \( d + i \partial_{x'} \). Because of this, a contribution from the \( U(1) \) fibration is evaluated as \( \text{Ch}(V) = e^{c_1(V)} = e^{iJ_i} \). At the fixed points \( x_i \), \( i = 1, 2, 3 \), the zero-form values are \( a_i \).

### One-loop determinant

The Chern class of the base \( \mathcal{M}_4 \) is derived from the zero-form values of the Chern class
of \( \mathbb{R}^4 \), which is identified with the local geometry around fixed points \( x_i \) of \( \mathcal{M}_4 \). We assume that contributions from nonzero instanton sectors do not dominate in the large-\( N \) limit, and we are only interested in the perturbative part in the zero instanton sector. Under these assumptions, we can evaluate the contribution from the universal bundle as \( \text{Ch}(\mathcal{E}) = \text{Ch}_{\text{adj}}(e^{iA}) \), and we can use the indices on \( \mathbb{R}^4 \times S^1 \). The circumference of this \( S^1 \), which is a period of \( \tau' = \frac{n}{2n + 1} (\tau + \chi) = 2 \pi \frac{3n}{2n + 1} \), is used for the normalization of the charge \( t \) of this \( S^1 \) rotation.

Putting all together, the index for vector multiplets is

\[
\begin{align*}
I_{\text{vec}} &= \frac{1}{2} \sum_{l=-\infty}^{\infty} \left\{ \left( 1 + e^{-\frac{2i}{n} (1-n)} \right) e^{i((\frac{1+2m}{3n} - \frac{3}{3n}) (1-n))} \right. \\
&\quad + \left( 1 + e^{\frac{2i}{n} (1-n)} \right) e^{i((\frac{1+2m}{3n} - \frac{3}{3n}) (1-n))} \left( 1 - e^{-\frac{1-n}{n}} \right) + \left( 1 + e^{\frac{i}{n} (1-n)} \right) e^{i((\frac{1+2m}{3n} - \frac{3}{3n}) (1-n))} \\
&\quad \left. \left( 1 - e^{-\frac{1+n}{n}} \right) \left( 1 - e^{-i\delta} \right) \right\} \text{Ch}_{\text{adj}}(e^{i\lambda}) .
\end{align*}
\]

(4.3.16)

Each term has expansions in positive and negative power series [5], and the one-loop determinant are derived from their sum.

\[
\begin{align*}
\det_V &= \prod_{\alpha \in \text{root}} \prod_{p,q,r \geq 0} \frac{1}{i(\alpha(\lambda))^2} \left( \frac{p}{n} + q + r + i\alpha(\lambda) \right) \left( \frac{p + 1}{n} + q + r + 2 + i\alpha(\lambda) \right) , \\
&= \prod_{\alpha \in \text{positive root}} \frac{1}{i(\alpha(\lambda))^2} S_3 \left( \frac{1}{n}, 1, 1 \right) S_3 \left( i\alpha + 2 + \frac{1}{n}, 1, 1 \right) ,
\end{align*}
\]

(4.3.17)

where we took \( \delta \) to be zero. The triple sine function \( S_3(x|\omega_1, \omega_2, \omega_3) \) is defined as a
regularized infinite product \( \prod_{p,q,r \geq 0} (p\omega_1 + q\omega_2 + r\omega_3 + x)((p+1)\omega_1 + (q+1)\omega_2 + (r+1)\omega_3 - x) \).

Similarly, the contribution from a hypermultiplet in the representation \( \mathcal{R} \) and mass \( m \) is

\[
\begin{align*}
\det_H &= \prod_{\rho \in \text{weight}} \prod_{p,q,r \geq 0} \frac{1}{i(\rho(\lambda))^2} \left( \frac{p}{n} + q + r + m + \frac{1}{2} (\frac{1}{n} + 2) \pm i\rho(\lambda) \right)^{-\frac{1}{2}} \\
&\quad \times \left( \frac{p + 1}{n} + q + r + 2 - m - \frac{1}{2} (\frac{1}{n} + 2) \pm i\rho(\lambda) \right)^{-\frac{1}{2}} .
\end{align*}
\]

(4.3.18)
In case that $R$ is a real representation, it simplifies to
\[
\det_H = \prod_{\rho \in \text{weight}} \prod_{p,q,r \geq 0} \left[ \frac{p}{n} + q + r + m + \frac{1}{2} \left( \frac{1}{n} + 2 \right) + i\rho(\lambda) \right]^{-1} \\
\times \left[ \frac{p + 1}{n} + q + r + 2 - m - \frac{1}{2} \left( \frac{1}{n} + 2 \right) - i\rho(\lambda) \right]^{-1},
\] (4.3.19)
\[
= \prod_{\rho \in \text{weight}} S^{-1}_H \left( m + \frac{1}{2} \left( \frac{1}{n} + 2 \right) + i\rho \left( \frac{1}{n}, 1, 1 \right) \right).
\]
Comparing these with the one-loop partition function on the squashed five-sphere [71, 72], we confirm that the squashing parameters are given by (4.1.42) for the resolved space.

### 4.4 Adding Wilson loop

The supersymmetric Wilson loop in a representation $R$ of the gauge group is
\[
W_R = \frac{1}{\text{dim}R} \text{Tr}_R \mathcal{P} \exp \left[ - \int ds \left( A_\mu \dot{x}^\mu(s) + \text{susy completion} \right) \right].
\] (4.4.1)

In following, we particularly consider the Wilson loop which localizes on $\theta = \pi$ and wraps on $\tau$ direction. If the explicit formulas for Killing spinors (4.1.49) are substituted into $\delta[A_\tau(\theta = \pi)] = -2n\lambda^I \Gamma^2 \xi_I$, since $\Gamma^2 \xi_I = \xi_I$ for $I = 1, 2$,
\[
\delta[A_\tau(\theta = \pi)] - i n \sigma = 0,
\] (4.4.2)
and it can be computed by localization as supersymmetric invariant observable
\[
e^{- \int^{2\pi}_{\tau=0} [A_\tau(\theta = \pi) - i n \sigma] d\tau}.
\] (4.4.3)

The expectation value is almost same as partition function and only deformation is classical action part which adds $e^{2\pi i n \sigma}$
\[
(4.4.4)
\]
to those of the partition function. Similarly because of $\delta[A_\chi(\theta = 0) - i \frac{1}{2} \sigma] = 0$, the Wilson loop which localizes on $\theta = 0$
\[
\langle e^{- \int^{2\pi}_{\chi=0} [A_\chi(\theta = 0) - i \frac{1}{2} \sigma] d\chi} \rangle
\] (4.4.5)
also can be calculated, and its contribution to classical action is $e^{2\pi i n \sigma}$. Furthermore, one which localizes on $(\theta, \phi) = (0, 0)$ and extends along $\xi$ direction
\[
\langle e^{- \int^{2\pi}_{\xi=0} [A_\xi - i \frac{1}{2} \sigma] d\xi} \rangle \to e^{i \pi \sigma}
\] (4.4.6)
and which localizes on $(\theta, \phi) = (0, \pi)$
\[
\langle e^{- \int^{2\pi}_{\xi=0} [A_\xi + i \frac{1}{2} \sigma] d\xi} \rangle \to e^{-i \pi \sigma}
\] (4.4.7)
can be computed. In this sense, supersymmetric Wilson loop whose contour of the loop is the same as the orbit of the Killing vector, namely, $\dot{x}^\mu(s)/|\dot{x}(s)| = v^\mu$ can be inserted on every fixed points of $S^5$ (4.3.7).
Chapter 5

AGT relation

In this chapter, as one example for application of the exact results derived localization principle, we introduce AGT relation [6]. AGT relation relates four dimensional supersymmetric gauge theory and a particular class of two dimensional conformal field theories (CFT). These two theories looked different from each other. This relation is considered to be the two different limits of compactification of the same theory, six dimensional $\mathcal{N} = (2,0)$ theory. And with many approaches which we mention later, this correspondence turned out to be exact at least easiest case. First, we will introduce what is AGT relation in section 5.1 based on [1]. Next we will consider the particular interesting limit of AGT relation based on [2].

5.1 General AGT relation

AGT relation originally claims that partition function of four dimensional $\mathcal{N} = 2$ $SU(2)$ gauge theory constructed on a certain compact manifold is equivalent to correlators of two dimensional certain CFT, Liouville field theory (LFT). This correspondence respects the parameters of these theories. For instance, LFT has a coupling constant $b$, and this $b$ appears in gauge theory side as a parameter of the manifold on which gauge theory is constructed. This manifold is not unique, and first example is ellipsoid in section 3. There is at least one another manifold which generates same partition function [101].

Actually this is checked from previous result in simplest case. Take an example of $SU(2)$ SQCD with four fundamental hypermultiplets, which should correspond to Liouville four-point function on sphere. For $SU(2)$ SQCD with four fundamental flavors, the saddle points are labeled by a single parameter $p$, and the mass of the four hypermultiplets $\mu_1, \ldots, \mu_4$ can be introduced via suitable gauging of the $U(1)^4$ subgroup of the flavor group $SO(8)$. The one-loop part of the partition function then reads

$$Z_{1\text{-loop}} = \frac{\Upsilon(2ip)\Upsilon(-2ip)}{\prod_{i=1}^{4} \Upsilon(\frac{Q}{2} + ip + i\mu_i)\Upsilon(\frac{Q}{2} - ip + i\mu_i)}.$$  \hspace{1cm} (5.1.1)

To make correspondence with Liouville theory, we divide the four hypermultiplets into two pairs, and associate each pair with the flavor subgroup $SO(4) \simeq SU(2) \times SU(2)$. We thus get four copies of $SU(2)$ flavor groups, and denote by $p_a$ the mass parameter associated
to the $a$-th $SU(2)$. The parameters $\mu_i$ and $p_a$ are related by [8]

\[
\mu_1 = p_1 + p_2, \quad \mu_2 = p_1 - p_2, \quad \mu_3 = p_3 + p_4, \quad \mu_4 = p_3 - p_4. \tag{5.1.2}
\]

Under this identification $Z_{1\text{-loop}}$ agrees, up to some $p$-independent factors, with the product of two Liouville three point structure constants $C(p_1, p_2, p_3)$.

\[
Z_{1\text{-loop}} \sim C(p_1, p_2, p)C(p_1, p_2, -p) = C(p_1, p_2, p)C(p_1, p_2, p)R(p)^{-1}. \tag{5.1.3}
\]

Here $R(p) \equiv \Upsilon(Q + 2ip)/\Upsilon(Q - 2ip)$ is the reflection coefficient of Liouville primary operator with momentum $\alpha = \frac{Q}{2} + ip$, and $C(p_1, p_2, p_3)$ is given by [108] and derived in next section 5.2

\[
C(p_1, p_2, p_3) = \text{const} \cdot \frac{\Upsilon(Q + 2ip_1)\Upsilon(Q + 2ip_2)\Upsilon(Q + 2ip_3)}{\Upsilon(Q + 2ip_1 + 2ip_3)\Upsilon(Q + 2ip_1 + 2ip_2 - 3)\Upsilon(Q + 2ip_1 + 2ip_3 - 2)\Upsilon(Q + 2ip_1 + 2ip_2 - 3)}. \tag{5.1.4}
\]

Also the correspondence between instanton partition function and conformal block has been checked perturbatively [6]. Nekrasov instanton partition function is represented as a perturbative expansion in the instanton number. On the other hand, conformal block is expanded in descendant fields.

As we mentioned, AGT relation has a physical meaning as the two different limits of compactification of same theory, six dimensional $\mathcal{N} = (2, 0)$ theory. From this viewpoint, AGT relation is generalized to more general theories, between four dimensional type $A_{k-1}$ class $S$ gauge theory and Toda field theory with $W_k$ symmetry [109]. This class $S$ theory is defined as the theory obtained from compactifications of six dimensional $\mathcal{N} = (2, 0)$ theory on punctured Riemann surface. This six dimensional $\mathcal{N} = (2, 0)$ theory is not known well, even its Lagrangian description, but its physical importance is obvious since this theory represents the effective theory of M5-branes which are fundamental objects in M-theory. AGT relation shed light on some aspects of this mysterious theory. Many proofs for this relation have been proposed based on different ideas. For example, there is an approach using matrix models and topological string [110], the properties of conformal blocks in the series expanded form [111, 112], the action of conformal or W symmetries on instanton moduli spaces [113, 114] or the reduction of gauge theories to the theory of flat connections on Riemann surfaces [115, 9]. In particular, [9] which considers the correspondence between Liouville field theory and $SU(2)$ supersymmetric gauge theory with four flavors explains Liouville field theory as a representation of quantum Teichmüller space, which is moduli space of the corresponding supersymmetric gauge theory. When a finite group acts on the instanton moduli space and only fixed points of this action are considered, AGT relation can be variated. For example $\mathbb{Z}_2$ acts on $\mathbb{R}^4$ where gauge theory lives, AGT relation relates to supersymmetrized Liouville Field theory[116]. This relation is extended to the relationship between the theory on $\mathbb{R}^4/\mathbb{Z}_p$ and parafermionic algebra[117].

The attempts to proof shed light on different structures in 4D gauge theories that we have not been fully aware of. For other versions of the AGT-like relation, there have been a recent progress regarding the compactification of $(2, 0)$ theories on $S^3 \times S^1$ [74, 118, 77] or $S^5$ [79, 80].
5.2 Liouville Field theory

In this section, we will review Liouville field theory briefly to derive three points function (5.1.4). There are many pedagogical reviews about this topic for example [119, 120].

The Liouville Field theory is defined by the action

$$ S = \frac{1}{4\pi} \int d\bar{z} d\bar{\bar{z}} \sqrt{g} \left[ (\partial_a \phi(z, \bar{z}))^2 + 4\pi \mu e^{2b\phi(z, \bar{z})} + Q R \phi(z, \bar{z}) \right] $$

(5.2.1)

where $g$ is background metric and $R$ is its associated curvature. However, in a range of calculations for local quantities, we can use flat metric

$$ ds^2 = d\bar{z} d\bar{\bar{z}} = (d\sigma^a)^2 \quad (a = 1, 2) $$

(5.2.2)

instead of them. These choices of $\hat{g}, \hat{R}$ actually affect global quantities, for instance (5.3.6). This second term is called exponential potential. This $\mu$ is a cosmological constant, and for $\mu \geq 0$, the action is bounded from below.

To confirm the conformal symmetry of the theory, energy-momentum tensor $T^{ab}$ generated by translation should be taken. Under the translation

$$ \sigma^a \rightarrow \sigma^a + \zeta^a(\sigma^a) $$

(5.2.3)

$T^{ab}$ is defined as

$$ \delta S = -\frac{1}{2\pi} \int d^2\sqrt{g} \zeta^a \partial^b T_{ab}. $$

(5.2.4)

The measure $d^2\sqrt{g}$ does not generate any variations. From the derivate term in (5.2.1) which is same as free bosons theory

$$ T^{ab}_{\text{free}} = (\partial_a \phi)(\partial_b \phi) - \frac{1}{2} g_{ab}(\partial^c \phi)(\partial_c \phi) $$

(5.2.5)

appears. Variation on $QR\phi$ generates the term proportional to Einstein’s equation and the term originated from the part integral. It also appears $2\pi g_{ab}\mu e^{2b\phi}$ from the potential term. Taken together, it becomes

$$ T^{ab} = -(\partial^a \phi)(\partial^b \phi) + \frac{1}{2} g_{ab}(\partial^c \phi)(\partial_c \phi) - Q \phi (R^{ab} - \frac{1}{2} g^{ab} R) + Q (\partial^a \partial^b \phi - g^{ab} \partial^2 \phi) + 2\pi g^{ab}\mu e^{2b\phi} $$

(5.2.6)

Conformal invariance is confirmed by tracelessness of $T^{ab}$, but actually

$$ g_{ab} T^{ab} = -Q \partial^2 \phi + 4\pi \mu e^{2b\phi} $$

(5.2.7)

is not zero in general. In classical theory $b \rightarrow 0$ on the plane, if we substitute equation of motion (5.3.2) derived later, it becomes zero under $Q = \frac{1}{b}$. Taking account quantum effect, $Q = b + \frac{1}{b}$ which reflects the symmetry $b \leftrightarrow \frac{1}{b}$ makes it conformal theory. In following, we concentrate on mainly holomorphic part. The operator product expansion (OPE) of fields is same as free bosons theory:

$$ \phi(z)\phi(w) \sim -\frac{1}{2} \ln |z - w|^2. $$

(5.2.8)
The holomorphic part of \( T^{ab} \) of the theory on plane

\[
T(z) = T^{zz} = - (\partial \phi)^2 + Q \partial^2 \phi \quad (5.2.9)
\]
can be used to calculate conformal dimensions and primariness (the meaning is explained later) of operators. For instance, the exponential operator satisfies

\[
T(z)V_\alpha(w) \equiv T(z)e^{2\alpha} \sim - \frac{\alpha^2}{(z-w)^2}e^{2\alpha} + \frac{1}{z-w}\partial_w e^{2\alpha} + \frac{\alpha Q}{(z-w)^2} e^{2\alpha}
\]

\[
= \frac{\alpha(Q-\alpha)}{(z-w)^2} V_\alpha(w) + \frac{1}{z-w}\partial V_\alpha(w) \quad (5.2.10)
\]

which means that \( V_\alpha(w) \) is a primary field whose conformal dimension is \( \alpha(Q-\alpha) \). The actual physical parameter is only defined up to \( \alpha(Q-\alpha) \), and \( V_\alpha \) and \( V_{Q-\alpha} \) are identified. It should be represented as

\[
V_\alpha(z, \bar{z}) = R(\alpha)V_{Q-\alpha}(z, \bar{z}) \quad (5.2.11)
\]

where \( R(\alpha) \) is called a reflection coefficient which must hold \( R(\alpha)R(Q-\alpha) = 1 \). In following, we choose the normalization of \( R(Q) = -1 \). In addition, the dimension of potential term is turn out to be \( b(Q-b) \). Since the lagrangian is interpreted as free bosonic theory, which is apparently conformal, perturbed by the term whose dimension is \( b(Q-b) \), then the condition for conformal marginal is \( b(Q-b) = 1 \iff Q = b + \frac{1}{b} \).

The central charge \( c \) of the theory can be read from the most singular term in OPE of \( T(z)T(w) \), which is

\[
2(\partial_z \partial_w (\frac{1}{2} \ln |z-w|^2))^2 + Q^2 \partial^2_z \partial^2_w (\frac{1}{2} \ln |z-w|^2) = \frac{1 + 3! Q^2}{2(z-w)^4} \quad (5.2.12)
\]

and which means \( c = 1 + 6Q^2 \). The generators of the algebra are defined in

\[
L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z) \iff \sum_n L_n z^{-n-2}. \quad (5.2.13)
\]

The contour is taken to be circle encircling origin. If it is considered as time-slice in radial quantization, comparing to (5.2.4), \( L_n \) are interpreted as generators of conformal transformations

\[
\zeta = z^{n+1}. \quad (5.2.14)
\]

The action on primary operator (5.2.10) can be rewritten as

\[
T(z)V_\alpha(w) = \frac{\alpha(Q-\alpha)}{(z-w)^2} V_\alpha(w) + \frac{1}{z-w}\partial V_\alpha(w) + \sum_{n=0}^{\infty} (z-w)^n L_{n+2} V_\alpha(w) \quad (5.2.15)
\]

and implies that \( L_{-1} \) acts as differential. The algebra \([L_m, L_n] \) whose product is defined by time-ordering in radial quantization is computed as the transformation rules under (5.2.14) of \( L_m \) themselves, read from \( TT \) OPE, and it turns out to be to be

\[
[L_m, L_n] = \oint \frac{dz}{2\pi i} \left[ \frac{c}{12} (m^3 - m) z^{m+n-1} + (m-n) z^{m+n+1} T(z) \right]
\]

\[
= (m-n)L_{m+n} + \frac{c}{12} m(m^2 - 1) \delta_{m+n,0}, \quad (5.2.16)
\]
called Virasoro algebra.

Take eigenstate $|\psi\rangle$ of $L_0$ whose eigenvalue is $h$: $L_0|\psi\rangle = h|\psi\rangle$. The state $L_\alpha|\psi\rangle$ is also an eigenstate of $L_0$ and its eigenvalue is $h - \alpha$. It means that $L_{\alpha > 0}$ raises its eigenvalue for $L_0$ and $L_{\alpha > 0}$ lowers it. If we interpret $L_0$ as Hamiltonian and its eigenvalue as energy, it must be bounded from below. In other words, there should be a primary state $|\psi\rangle$

\[ L_{\alpha > 0}|\psi\rangle = 0, \quad L_0|\psi\rangle = h|\psi\rangle. \tag{5.2.17} \]

When we consider (5.2.13), the condition for primariness is that OPE with $T(z)$ does not have more singular than $-2$th order. In this sense, (5.2.10) reveals that $V_\alpha$ is primary field whose eigenvalue is conformal dimension $\alpha(Q - \alpha)$.

The eigenstate $|\psi\rangle$ called a highest weight state and its descendant fields generated by acting Virasoro operators $L_\alpha$ on the primary field

\[ L_{-l}|\psi\rangle \equiv L_{-n_1} \cdots L_{-n_r}|\psi\rangle \quad (n_1 \geq n_2 \geq \cdots \geq n_r > 0) \tag{5.2.18} \]

generate a one of representations of Virasoro algebra, labeled by a highest weight, or conformal dimension of $|\psi\rangle$. $|I\rangle = \sum_{i=1}^r n_i$ is a parameter called level. Under the assumption $L_0^\dagger = \Lambda_0$,

\[ \langle \psi|\psi'\rangle \propto \delta_{\psi\psi'}, \tag{5.2.19} \]

orthogonal. In terms of reflection coefficient (5.2.11), two points function between primary operators should be

\[ \langle V_{\alpha_1}(z)V_{\alpha_2}(w)\rangle = f(z - w)[\delta(Q - \alpha_1 - \alpha_2) + R(\alpha)\delta(\alpha_1 - \alpha_2)] \tag{5.2.20} \]

where coordinate dependency is determined by conformal symmetry.

Some representations satisfy

\[ \exists I, \forall J \quad \text{s.t.} \quad L_J(L_{-I}|h\rangle) = 0 \tag{5.2.21} \]

where $|h\rangle$ is a primary state whose conformal dimension is $h$, or a descendant field $L_{-I}|h\rangle$ is also a primary field. This $|\Delta\rangle = L_{-I}|h\rangle$ is called a null vector, and $|h\rangle$ is degenerate state. For instance, a nontrivial condition for a null vector $|\Delta\rangle = L_{-l}|h\rangle$ whose level is one is only $L_1|\Delta\rangle = 0$.

\[ L_1|\Delta\rangle = [L_1, L_{-l}]|h\rangle = 2h|h\rangle = 0 \tag{5.2.22} \]

and the condition is only $h = 0$. The null vectors with level two should be written as $|\Delta\rangle = (L_{-2} + aL_{-1}^2)|h\rangle$ where $a$ is numerical constant. The nontrivial conditions $L_1|\Delta\rangle = L_2|\Delta\rangle = 0$ mean that

\[ 3|\Delta\rangle + a\{L_1, L_{-1}\}|l\rangle = (3 + 2a(1 + 2h))|\Delta\rangle = 0 \quad \Rightarrow \quad a = \frac{-3}{2(1 + 2h)} \]

\[ (4h + \frac{c}{2})|h\rangle + 6ah|h\rangle = 0 \quad \Rightarrow \quad (2 + 3a)h = \frac{c}{4}. \tag{5.2.23} \]

They are solved simultaneously:

\[ h = \frac{-(c - 5) \pm \sqrt{(c - 1)(c - 25)}}{16}. \tag{5.2.24} \]
and by submitting $c = 1 + 6Q^2$, it becomes

$$h = \frac{-3(b^2 + \frac{1}{b^2}) \pm 3(b^2 - \frac{1}{b^2}) - 4}{8} = \frac{-3b^2 - 2}{4},$$

and degenerate state satisfies

$$(L_{-2} + \frac{1}{b^2}L_{-1}) |\Delta\rangle = 0.$$  

(5.2.25)

In terms of momentum $\alpha$ which is $h = \alpha(Q - \alpha)$, this formula is $\alpha = -\frac{1}{2}b^{\pm 1}$. The general results are obtained from Kac determinant whose matrix elements are $\langle h | L_I L_J | h \rangle$, and it turns out that null vectors are labeled by $(r, s)$ where $r, s \in \mathbb{N}$ with level $rs$ whose momenta are

$$\alpha = \frac{1}{2} \left( (-r + 1)b + (-s + 1)\frac{1}{b} \right) \leq 0.$$  

(5.2.26)

In following, we will derive the three points function of primary fields. We concentrated on only the holomorphic part $z$, but anti-holomorphic part $\bar{z}$ is also explained in same way. It is assumed that the state generated by OPE between different representations belongs to other representations. For example, the product of general operators $W(I, h, \bar{I}, \bar{h})(z, \bar{z})$ and $W(I', h', \bar{I}', \bar{h}')(w, \bar{w})$ is factorized into

$$W(I, h, \bar{I}, \bar{h})(z, \bar{z})W(I', h', \bar{I}', \bar{h}')(w, \bar{w}) \sim \sum_{\tilde{I}, \tilde{h}} f_{\tilde{I}, \tilde{h}, I, h, I', h'} (z_1 - z_2) |\tilde{I} \rangle \langle I| - |\tilde{I} \rangle \langle L_{-I}| V_{\tilde{h}}(z_2).$$  

(5.2.27)

and symbolically represented as

$$[h, \bar{h}] \times [h', \bar{h}'] \sim \sum_{\tilde{h}, \tilde{h}} [\tilde{h}, \tilde{h}].$$  

(5.2.28)

Here we focus on holomorphic part again. By dimensional analysis, coordinate dependency can be factorized as

$$V^{I, h}(z_1)V^{I', h'}(z_2) \sim \int d\alpha F_{\alpha, 0}(\alpha, Q - \alpha) f_{I, \alpha, h} V^{I, h}(z_1) \times (z_1 - z_2)^{|I| - |\alpha|} L_{-I} V^{\bar{h}}(z_2).$$  

(5.2.29)

The summation over $h$ becomes integral. Schematically, OPE between primary operators as product of representations can be written as

$$V_{\alpha_1} V_{\alpha_2} \sim \int d\alpha F_{\alpha_1, \alpha_2}(\alpha) V_{\alpha}.$$  

(5.2.30)

Taking a OPE between a primary state and a degenerate state $(r, s) = (2, 1)$ particularly, it becomes

$$V_{\alpha}(z_1)V_{(2, 1)}(z_2) \sim \int d\alpha f_{0, 0, \alpha}(Q - \alpha, h) (z_1 - z_2) \times (z_1 - z_2)^{|I|} L_{-I} V^{h}(z_2).$$  

(5.2.31)

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where \( h' \) is given in (5.2.24). Since the holomorphy of \( T(z) \) at \( z \to \infty \) defined by \( w = \frac{1}{z} \) patch requires its coordinate dependence \( T(z) = O(z^{-4}) \), the correlation function in Liouville theory satisfies Ward identity

\[
\oint_{\infty} \mathrm{d}z \zeta(z) \left( T(z) \prod_{i=1}^{N} V_{\alpha_i}(z_i) \right) = 0 \quad (5.2.33)
\]

where \( \zeta(z) = O(z^2) \). Consider three-point function with one degenerate operator \((r, s) = (2, 1)\). By submitting (5.2.15) and the facts that \( L_{-1} \) acts as differential, it is

\[
\langle T(z)V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{(2,1)}(z_3) \rangle = \sum_{i=1}^{3} \left( \frac{h_i}{(z - z_i)^2} + \frac{1}{z - z_i} \partial_{z_i} \right) \langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{(2,1)}(z_3) \rangle + \sum_{n=0}^{\infty} [(z - z_1)^n \langle (L_{-n+2}V_{\alpha_1}(z_1))V_{\alpha_2}(z_2)V_{(2,1)}(z_3) \rangle \\
+ (z - z_2)^n \langle V_{\alpha_1}(z_1)(L_{-n+2}V_{\alpha_2}(z_2))V_{(2,1)}(z_3) \rangle \\
+ (z - z_3)^n \langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)(L_{-n+2}V_{(2,1)}(z_3)) \rangle \rangle \quad (5.2.34)
\]

where \( h_i = \alpha_i(Q - \alpha_i) \) are conformal dimensions. When \( \zeta(z) = 1 \), the poles with first order are collected and the Ward identity is

\[
\sum_{i=1}^{3} \partial_{z_i} \langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{(2,1)}(z_3) \rangle = 0. \quad (5.2.35)
\]

In case for \( \zeta(z) = \frac{1}{z - z_3} = \sum_{k=0}^{\infty} (-1)^k \frac{(z - z_3)^k}{(z_i - z_3)^{k+1}} \), this identity shows

\[
\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)(L_{-2}V_{(2,1)}(z_3)) \rangle = \sum_{i=1,2} \frac{h_i}{(z_i - z_3)^2} \langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{(2,1)}(z_3) \rangle - \left( \frac{1}{z_1 - z_3} \partial_{z_1} - \frac{1}{z_2 - z_3} \partial_{z_2} \right) \langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{(2,1)}(z_3) \rangle = 0. \quad (5.2.36)
\]

Meanwhile \( V_{(2,1)} \) satisfies the formula for degenerate operator (5.2.26) which is substituted to (5.2.36) to make it called BPZ equation

\[
\left[ \sum_{i=1,2} \frac{h_i}{(z_i - z_3)^2} - \left( \frac{1}{z_1 - z_3} \partial_{z_1} - \frac{1}{z_2 - z_3} \partial_{z_2} \right) + \frac{1}{b^2} \partial_{z_3} \right] \langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{(2,1)}(z_3) \rangle = 0. \quad (5.2.37)
\]

The coordinate dependence of correlation function can be factorized so as to satisfy conformal transformation rule such that [121]

\[
\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{(2,1)}(z_3) \rangle = C_{\alpha_1\alpha_2\alpha_3}|z_{12}|^{2(h_3 - h_1 - h_2)}|z_{23}|^{2(h_1 - h_2 - h_3)}|z_{31}|^{2(h_2 - h_3 - h_1)} \quad (5.2.38)
\]
where \( z_{ij} = z_i - z_j \) and \( h_3 = \frac{1}{2} - \frac{3}{4b^2} \). This is confirmed by Ward identity with \( \zeta(z) = (z - z_2)(z - z_3) \) which provides

\[
(h_1(2z_1 - z_2 - z_3) + z_{12}z_{13}\partial_{z_1} + z_{23}(h_2 - h_3)) \langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{(2,1)}(z_3) \rangle = 0 \quad (5.2.39)
\]

and its cyclic equations. This \( C_{\alpha_1\alpha_2\alpha_3} \) relates with OPE coefficient in (5.2.31) using two points function (5.2.20) as

\[
C_{\alpha_1\alpha_2\alpha_3} = \int d\alpha \langle F_{\alpha_1\alpha_2,\alpha}V_{\alpha}V_{\alpha_3} \rangle = F_{\alpha_1\alpha_2,\alpha}(\delta(Q - \alpha - \alpha_3) + R(\alpha)\delta(\alpha - \alpha_3))
\]

which is solved under the assumption \( F_{\alpha_1\alpha_2,Q - \alpha_3} = R(\alpha_3)F_{\alpha_1\alpha_2,\alpha_3} \). It means

\[
F_{\alpha_1\alpha_2,Q - \alpha_3} = \frac{1}{2}C_{\alpha_1\alpha_2\alpha_3}.
\]

By using (5.2.38) and take the terms with lowest order of \( z_{23} \), the Ward identity implies

\[
C_{\alpha_1\alpha_2\alpha_3} \left[ h_2 - (h_1 - h_2 - h_3) + \frac{1}{b^2} (h_1 - h_2 - h_3)(h_1 - h_2 - h_3 - 1) \right] = 0, \quad (5.2.42)
\]

and \( h_2 \) can be determined in terms of \( h_1 \) and \( b \). Actually, if we take \( t = h_1 - h_3 = h_1 + \frac{1}{2}(1 + \frac{3}{2}b^2) \) and \( x = t - h_2 \), this equation is

\[
\frac{x^2}{b^2} - \left( \frac{1}{b^2} + 2 \right) x + t = 0 \quad (5.2.43)
\]

and solved as

\[
h_2 = h_1 - \frac{b^2}{4} \pm b\sqrt{\frac{Q^2}{4} - h_1} = \left( \alpha_1 \mp \frac{b}{2} \right) \left( Q - \alpha_1 \pm \frac{b}{2} \right). \quad (5.2.44)
\]

which means \( \alpha_2 = \alpha_1 \pm \frac{b}{2} \). Note on (5.2.19), in terms of (5.2.29), this result can be written as

\[
[\alpha] \times \left[ \frac{b}{2} \right] \sim [\alpha + \frac{b}{2}] + [\alpha - \frac{b}{2}], \quad (5.2.45)
\]

or in terms of (5.2.31),

\[
V_{(2,1)}V_{\alpha} \sim C_+(\alpha)V_{\alpha + \frac{b}{2}} + C_-(\alpha)V_{\alpha - \frac{b}{2}} \quad (5.2.46)
\]

where \( C_\pm(\alpha) = F_{(2,1),\alpha,\alpha \pm \frac{b}{2}} = \frac{1}{2}C_{(2,1),\alpha,\alpha \pm \frac{b}{2}} \). In particular, since

\[
R(\alpha + \frac{b}{2})C_+(\alpha) = C_{(2,1),\alpha,\alpha + \frac{b}{2}} \]
\[
R(\alpha - \frac{b}{2})C_-(\alpha) = \frac{1}{2}C_{(2,1),\alpha,\alpha - \frac{b}{2}}, \quad (5.2.47)
\]

\( C_+ \) can be related to \( C_- \) as

\[
C_+(\alpha) = \frac{R(\alpha)}{R(\alpha + \frac{b}{2})}C_-(\alpha + \frac{b}{2}). \quad (5.2.48)
\]

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In following, we take a normalization where \( C_-(\alpha) = 1 \).

In general, Liouville theory does not hold a momentum conservation law. In particular, OPE between degenerate operators is

\[
[(r, s)] \times [(2, 1)] \sim [(r + 1, s)] + [(r - 1, s)].
\]  

(5.2.49)

Almost same discussion can be applied for the case with degenerate state with \((r, s) = (1, 2)\) which states

\[
\alpha \times \left[ \frac{1}{2b} \right] \sim \left[ \frac{1}{2b} \right] + \left\langle \alpha \right\rangle
\]

(5.2.50)

\[
((r, s)] \times [(1, 2)] \sim [(r, s + 1)] + [(r, s - 1)].
\]  

(5.2.51)

When these formula are applied to \([(2, 1)] \times [(r, s)] \times [\alpha],

\[
[(r, s)] \times \left( \alpha + \frac{b}{2} \right) \sim \left( [(r + 1, s)] + [(r - 1, s)] \right) \times [\alpha].
\]  

(5.2.52)

Once start from the trivial case with \([(r, s) = (1, 1)]\) (if \(r - 1 = 0\), such representation does not appear) and attempt it recursively and with \([(1, 2)] \times [(r, s)] \times [\alpha],\) it turns out that

\[
[(r, s)] \times [\alpha] \sim \sum_{i=0}^{r} \sum_{j=0}^{s-1} \left[ \alpha + \frac{1}{2} \left( (-r + 1)b + (-s + 1) \frac{1}{b} \right) \right] + ib \frac{1}{b}.
\]

(5.2.53)

To derive three points function, here we consider four points function with a degenerate operator \(V_{(2,1)}\). The Ward identity for \(\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{\alpha_3}(z_3)V_{(2,1)}(z_4)\rangle\) with

\[
\zeta(z) = \frac{(z - z_1)(z - z_2)(z - z_3)}{z - z_4}
\]

\[
= \sum_{k=0}^{\infty} (-k)^k \frac{(z - z_1)^k}{z_{14}^{k+1}} \left( (z - z_1)^2 + (z_{12} + z_{13})(z - z_1)^2 + z_{12}z_{13}(z - z_1) \right)
\]

\[
= (z - z_4)^2 - (z_{14} + z_{24} + z_{34})(z - z_4) + (z_{14}z_{24} + z_{14}z_{34} + z_{24}z_{34}) - \frac{z_{14}z_{24}z_{34}}{z - z_4}
\]

is

\[
- z_{14}z_{24}z_{34} \langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{\alpha_3}(z_3)L_{-2}V_{(2,1)}(z_4) \rangle + \left[ \frac{z_{12}z_{13}}{z_{14}}h_1 + \frac{z_{21}z_{23}}{z_{24}}h_2 + \frac{z_{31}z_{32}}{z_{34}}h_3 - (z_{14} + z_{24} + z_{34})h_4 \right] \langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{\alpha_3}(z_3)V_{(2,1)}(z_4) \rangle
\]

\[
+ (z_{14}z_{24} + z_{14}z_{34} + z_{24}z_{34}) \partial_{z_4} \langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{\alpha_3}(z_3)V_{(2,1)}(z_4) \rangle = 0.
\]  

(5.2.55)

Inserting the formula for degenerate operator, it becomes

\[
\frac{z_{14}z_{24}z_{34}}{b^2} \partial_{z_4} \langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{\alpha_3}(z_3)V_{(2,1)}(z_4) \rangle + \left[ \frac{z_{12}z_{13}}{z_{14}}h_1 + \frac{z_{21}z_{23}}{z_{24}}h_2 + \frac{z_{31}z_{32}}{z_{34}}h_3 - (z_{14} + z_{24} + z_{34})h_4 \right] \langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{\alpha_3}(z_3)V_{(2,1)}(z_4) \rangle
\]

\[
+ (z_{14}z_{24} + z_{14}z_{34} + z_{24}z_{34}) \partial_{z_4} \langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{\alpha_3}(z_3)V_{(2,1)}(z_4) \rangle = 0.
\]  

(5.2.56)
In general, three of four coordinates of primary operators can be taken arbitrarily without loss of generality, so we take \((z_1, z_2, z_3, z_4) = (0, \infty, 1, z)\) and leave the highest order of \(z_2\) to translate it as

\[
-\frac{z(1-z)}{b^2} \partial_z^2 - \frac{h_1}{z} + h_2 - \frac{h_3}{1-z} - h_4 + (1-2z) \partial_z \biggl( V_{\alpha_1}(0)V_{\alpha_2}(\infty)V_{\alpha_3}(1)V_{(2,1)}(z) \biggr) = 0.
\]  

(5.2.57)

This second order differential equation has two linear independent solutions as

\[
F_1(z) = z^{b\alpha_1}(1-z)^{b\alpha_3} F \left( \frac{1}{2} + b(\alpha_1 + \alpha_2 + \alpha_3 - \frac{3}{2}Q), \frac{1}{2} + b(\alpha_1 - \alpha_2 + \alpha_3 - \frac{Q}{2}), 1 + b(2\alpha_1 - Q), z \right)
\]

and

\[
F_2(z) = z^{bQ-\alpha_1}(1-z)^{b\alpha_3} F \left( \frac{1}{2} + b(-\alpha_1 + \alpha_2 + \alpha_3 - \frac{Q}{2}), \frac{1}{2} + b(-\alpha_1 - \alpha_2 + \alpha_3 + \frac{Q}{2}), 1 - b(2\alpha_1 - Q), z \right)
\]

(5.2.58)

(5.2.59)

where \(F(a, b, c, x)\) are hypergeometric function which are solutions of

\[
(x(1-x)\partial_x^2 + [c - (a + b + 1)x] \partial_x - ab) F(a, b, c, x) = 0
\]

(5.2.60)

used for \(|z| < 1\) and they are related by \(F_2(z) \xrightarrow{\alpha_1 \to Q-\alpha_1} F_1(z)\). Also hypergeometric functions satisfy the Kummer’s transformation

\[
F(a, b, c, x) = F(b, a, c, x)
\]

(5.2.61)

\[
F(a, b, c, x) = (1-x)^{c-a-b} F(c-a, c-b, c, x)
\]

(5.2.62)

by which reflection symmetry about \(\alpha_2\) and \(\alpha_3\) are also guaranteed. The hypergeometric functions which are defined as the solutions of (5.2.60) have different representation used for other parametric region. For example, in \(|1-z| < 1\),

\[
F_1^r(z) = z^{b\alpha_1}(1-z)^{b\alpha_3} \times F \left( \frac{1}{2} + b(\alpha_1 + \alpha_2 + \alpha_3 - \frac{3}{2}Q), \frac{1}{2} + b(\alpha_1 - \alpha_2 + \alpha_3 - \frac{Q}{2}), 1 + b(2\alpha_3 - Q), 1 - z \right)
\]

\[
= z^{b(Q-\alpha_1)}(1-z)^{b\alpha_3} F \left( \frac{1}{2} + b(\alpha_{-1-2+3} + \frac{Q}{2}), \frac{1}{2} + b(\alpha_{-1+2+3} - \frac{Q}{2}), 1 + b(2\alpha_3 - Q), 1 - z \right)
\]

(5.2.63)

and

\[
F_2^r(z) = z^{b\alpha_1}(1-z)^{b(Q-\alpha_3)} F \left( \frac{1}{2} + b(\alpha_{-1-2+3} + \frac{Q}{2}), \frac{1}{2} + b(\alpha_{1+2-3} - \frac{Q}{2}), 1 + b(-2\alpha_3 + Q), 1 - z \right)
\]

\[
= z^{b(Q-\alpha_1)}(1-z)^{b(Q-\alpha_3)} \times F \left( \frac{1}{2} + b(\alpha_{-1+2+3} + \frac{Q}{2}), \frac{1}{2} - b(\alpha_{1+2+3} - \frac{3}{2}Q), 1 + b(Q - 2\alpha_3), 1 - z \right)
\]

(5.2.64)
which are related to previous results as

\[
\frac{F(a, b, c, z)}{Γ(c)} = \frac{Γ(a + b - c)}{Γ(a)Γ(b)}(1 - z)^{c-a-b}F(c - a, c - b, c - a - b + 1, 1 - z) + \frac{Γ(c - a - b)}{Γ(c - a)Γ(c - b)}F(a, b, a + b - c + 1, 1 - z)
\]

(5.2.65)

to

\[
\frac{F_1(z)}{Γ(1 + b(2α_1 - Q))} = \frac{Γ(b(2α_3 - Q))}{Γ(\frac{1}{2} + b(α_{1+2+3} - \frac{3}{2}Q))Γ(\frac{1}{2} + b(α_{1-2+3} - \frac{Q}{2})}F^t_1(z)
\]

\[
+ \frac{Γ(b(Q - 2α_3))}{Γ(\frac{1}{2} + b(α_{1-2-3} + \frac{Q}{2}))Γ(\frac{1}{2} + b(α_{1+2-3} - \frac{Q}{2})}F^s_1(z)
\]

(5.2.66)

and

\[
\frac{F_2(z)}{Γ(1 - b(2α_1 - Q))} = \frac{Γ(b(2α_3 - Q))}{Γ(\frac{1}{2} + b(α_{1-1+2+3} - \frac{3}{2}Q))Γ(\frac{1}{2} + b(α_{1-2+3} + \frac{Q}{2})}F^t_2(z)
\]

\[
+ \frac{Γ(b(Q - 2α_3))}{Γ(\frac{1}{2} - b(α_{1+2+3} - \frac{3}{2}Q))Γ(\frac{1}{2} - b(α_{1-2-3} - \frac{Q}{2})}F^s_1(z)
\]

(5.2.67)

where \(α_{1+2-3} = α_1 + α_2 - α_3\) and so on. The index \(t\) in (5.2.63) and (5.2.64) means that in \(|1 - z| < 1\) the OPE has taken firstly between \(V_{α_1}(1)\) and \(V_{(2,1)}(z)\) and that comparing with (5.2.58) which takes \(V_{α_1}(0)\) and \(V_{(2,1)}(z)\) OPE first, the four points function is calculated in \(t\)-channel. In this sense, \(F_1(z)\) and \(F_2(z)\) should be noted as \(F^t_1(z)\) and \(F^s_2(z)\) to clarify them being used for \(s\)-channel. Actually, the dimensional analysis on OPE \(V_{(2,1)}(z)V_{α_1}(0) \sim \sum LV_h(0)\) starts with conformal dimension \(z^{h-h_{(2,1)}-α(Q-α)}\) where \(h = (α \pm \frac{b}{2})(Q - α \mp \frac{b}{2})\) and for \(α + \frac{b}{2}\) the exponent \(b(Q - α_1)\) corresponds to the exponent of \(F^s_2(z)\) and in those of \(α - \frac{b}{2}\) case the exponent \(bα_1\) is related to \(F^t_1(z)\). For \(t\)-channel, comparing the exponent of OPE \((1 - z)^{h-h_{(2,1)}-α(Q-α)}\), the intermediate momentum \(α_3 - \frac{b}{2}\) corresponds to \(F^t_1(z)\) and that of \(α_3 + \frac{b}{2}\) is \(F^s_2(z)\).

Since almost same discussion can be applied for anti-holomorphic part, the four points function is represented in bilinear combinations of these \(F^t_{1,2}(z)\) and their complex conjugates with \(s\)-channel, and if we take \(t\)-channel, it must be a bilinear combination of \(F^t_{1,2}(z)\) and their complex conjugates. By demanding the invariance around the monodromy at the singularity \(z = 0, 1\) of the solution, they are diagonal in terms of \(F_{1,2}\), and

\[
(V_{α_1}(0)V_{α_2}(∞)V_{α_3}(1)V_{(2,1)}(z)) = \sum_{i=1,2} c^*_i |F_i^t(z)|^2 = \sum_{i=1,2} c_i |F_i^s(z)|^2.
\]

(5.2.68)

The ratios of these coefficients are obtained by substitution of (5.2.66) (5.2.67). First, the fact that cross terms does not appear in (5.2.68) implies

\[
c^*_i = -b^2(2α_1 - Q)^2 c_i
\]

\[
× γ(\frac{1}{2} - b(α_{1+2+3} - \frac{3}{2}Q))γ(\frac{1}{2} - b(α_{1-2+3} - \frac{Q}{2}))
\]

\[
× γ(\frac{1}{2} - b(α_{1-2-3} + \frac{Q}{2}))γ(\frac{1}{2} - b(α_{1+2-3} - \frac{Q}{2}))γ^2(b(2α_1 - Q))
\]

(5.2.69)
where $\gamma(x) \equiv \frac{\Gamma(x)}{\Gamma(1-x)}$. Since gamma function satisfies
\[
\Gamma(x + \frac{1}{2})\Gamma(-x + \frac{1}{2}) = \frac{\pi}{\cos \pi x} \Rightarrow \Gamma^2(x)\sin \pi x = \pi \gamma(x),
\]  
(5.2.70)

they are turned out to be
\[
c'_2 = c'_4 \left( \frac{\Gamma(1+b(2\alpha_1-Q))\Gamma(b(2\alpha_3-Q))}{\Gamma(\frac{1}{2} + b(\alpha_{1+2+3} - \frac{3}{2}Q)))\Gamma(\frac{1}{2} + b(\alpha_{1-2+3} - \frac{3}{2}Q))} \right)^2
+ c'_2 \left( \frac{\Gamma(1-b(2\alpha_1-Q))\Gamma(b(2\alpha_3-Q))}{\Gamma(\frac{1}{2} + b(\alpha_{1-1+2+3} - \frac{3}{2}Q)))\Gamma(\frac{1}{2} + b(\alpha_{-1-2+3} + \frac{3}{2}Q))} \right)^2
= c'_4 \frac{\gamma(b(2\alpha_3-Q))\gamma(1+b(2\alpha_1-Q))}{\gamma(\frac{1}{2} + b(\alpha_{1+2+3} - \frac{3}{2}Q))\gamma(\frac{1}{2} + b(\alpha_{1-2+3} - \frac{3}{2}Q))},
\]  
(5.2.71)

and
\[
c'_4 = c'_4 \frac{\gamma(b(Q-2\alpha_3))\gamma(1+b(2\alpha_1-Q))}{\gamma(\frac{1}{2} + b(\alpha_{1-1+2+3} - \frac{3}{2}Q))\gamma(\frac{1}{2} + b(\alpha_{1+2+3} - \frac{3}{2}Q))}.
\]  
(5.2.72)

In following, we have to determine three points function $C_{\alpha_1,\alpha_2,\alpha_3}$. So far we already know the equations (5.2.68)(5.2.69)(5.2.72) which the $C_{\alpha_1,\alpha_2,\alpha_3}$ have to satisfies in terms of four points function, and all we have to do is determine it in particular situation. Since also we know the relation (5.2.48), we can consider the case with four points function $\langle V_{(2,1)}(z)V_\alpha(0)V_{Q-\alpha}(\infty)V_{(2,1)}(1) \rangle$ which degenerates to two points function, and solve it for reflection coefficient $R(\alpha)$.

In s-channel where $|z| < 1$, OPE between $V_{(2,1)}(z)$ and $V_\alpha(0)$ raises $C_{\pm}(\alpha)$. For case with intermediate momentum $\alpha - \frac{b}{2}$ figured in Fig.5.1 which picks up $C_{-}(\alpha)$, another vertex is represented with
\[
V_{(2,1)}V_{\alpha}V_Q \sim \left( C_+(\alpha - \frac{b}{2}V_\alpha + C_-(\alpha - \frac{b}{2}V_{\alpha-b}) \right) V_{Q-\alpha}.
\]  
(5.2.73)

Comparing with (5.2.20), this vertex is $C_+(\alpha - \frac{b}{2})$. The discussion in paragraph below (5.2.67) shows that this diagram corresponds to $F^s_1$. Substituting into (5.2.48) with normalization $C_-(\alpha) = 1$,
\[
c'_4 = C_+(\alpha - \frac{b}{2})C_-(\alpha) = \frac{R(\alpha(\alpha - \frac{b}{2}))}{R(\alpha)}.
\]  
(5.2.74)
Similarly from $C_+(\alpha)(C_+(\alpha + \frac{b}{2})V_{\alpha+b} + C_-(\alpha + \frac{b}{2})V_{\alpha})V_{Q-\alpha}$,

$$c_2^b = \frac{R(\alpha)}{R(\alpha + \frac{b}{2})}$$  \hspace{1cm} (5.2.75)

and we already know their ratio (5.2.69).

The $t$-channel corresponds to the diagram Fig. 5.2. On OPE $V_{(2,1)}V_{(2,1)} \sim C_+(-\frac{b}{2})V_0 + C_-(\frac{b}{2})V_0$ where $V_0$ is identity operator consider the former term. This intermediate momentum corresponds to $F^2_z$. Since (5.2.20), it becomes $V_{\alpha}V_{Q-\alpha}V_0 \sim 1$ and

$$c_2^t = C_+(-\frac{b}{2}) = \frac{R(-\frac{b}{2})}{R(0)}.$$  \hspace{1cm} (5.2.76)

Taken together, the ratio of the coefficients generates the equation about reflection coefficient such as

$$\frac{c_2^t}{c_2^b} = \frac{R(-\frac{b}{2})R(\alpha)}{R(0)R(\alpha - \frac{b}{2})} = \frac{\gamma(-b(b + Q))\gamma(1 + b(2\alpha - Q))}{\gamma(\frac{b}{2} + b(Q - \frac{b}{2} - \frac{3}{2}Q))\gamma(\frac{b}{2} + b(2\alpha - Q - \frac{b}{2} - \frac{3}{2}Q))}$$

$$= \frac{\gamma(bQ)\gamma(2b(Q - \alpha))}{\gamma(2bQ)\gamma(b(Q - 2\alpha))}. \hspace{1cm} (5.2.77)$$

Same deriviation on $\langle V_{(1,2)}(z)V_{\alpha}(0)V_{Q-\alpha}(\infty)V_{(1,2)}(1) \rangle$ gives the equation which reflects $b \leftrightarrow \frac{1}{b}$ symmetry:

$$\frac{c_2^t}{c_2^b} = \frac{R(-\frac{1}{2b})R(\alpha)}{R(0)R(\alpha - \frac{1}{2b})} = \frac{\gamma(Q)^2\gamma(\frac{2}{b}(Q - 2\alpha))}{\gamma(Q)^2\gamma(\frac{2}{b}(Q - \alpha))}.$$  \hspace{1cm} (5.2.78)

We can confirm that

$$R(\alpha) = -\mu^Q-2\alpha(2\alpha - Q)^2\gamma(b(2\alpha - Q))\gamma(\frac{1}{b}(2\alpha - Q))$$  \hspace{1cm} (5.2.79)

which also has $b \leftrightarrow \frac{1}{b}$ symmetry satisfies

$$\frac{R(-\frac{b}{2})R(\alpha)}{R(0)R(\alpha - \frac{b}{2})} = \frac{(b + Q)^2(2\alpha - Q)^2\gamma(-b(b + Q))\gamma(-\frac{b+Q}{b}b(2\alpha - Q))\gamma(\frac{1}{b}(2\alpha - Q))}{Q^2(2\alpha - b - Q)^2\gamma(-\frac{b+Q}{b}b(2\alpha - b - Q))\gamma(\frac{1}{b}(2\alpha - b - Q))}$$

$$= \frac{\gamma(bQ)\gamma(2b(Q - \alpha))}{\gamma(2bQ)\gamma(b(Q - 2\alpha))}. \hspace{1cm} (5.2.80)$$
If Liouville theory is placed on a manifold with nontrivial topology, it turns out that the cosmological constant $\mu$ here is related to $\mu$ in Lagrangian (5.2.1). The result is used to determine $C_{+}^{(\alpha)} = \frac{R(\alpha)}{R(\alpha + \frac{b}{2})} = \mu^b b^{-\frac{b}{2}}(2\alpha - Q) \gamma(2\alpha b)$. (5.2.81)

To determine $c_1^s$, we have to consider more general momenta assignment $\langle V_{(2,1)}(z)V_{\alpha_1}(0)V_{\alpha_2}(\infty)V_{\alpha_3}(1) \rangle$ again. Corresponding diagram to $s$-channel taking a particular intermediate momentum is Fig.5.3 and the coefficient $c_1^s$ is $C_{-}^{(\alpha)}C_{\alpha_1+\frac{b}{2},\alpha_2,\alpha_3}$ and $c_2^s = C_{+}^{(\alpha)}C_{\alpha_1+\frac{b}{2},\alpha_2,\alpha_3}$. We already know their ratio (5.2.69).

$$\frac{C_{\alpha_1+\frac{b}{2},\alpha_2,\alpha_3}}{C_{\alpha_1-\frac{b}{2},\alpha_2,\alpha_3}} = \frac{c_2^s}{c_1^sC_{+}^{(\alpha)}}$$

$$= \mu^{-b}b^{-4}\gamma(1 + b(2\alpha_1 - Q))\gamma\left(\frac{1}{2} + b(\alpha_1 + \alpha_2 + \alpha_3 - \frac{3}{2}Q)\right)\gamma\left(\frac{1}{2} - b(\alpha_1 - \alpha_2 - \alpha_3 + \frac{Q}{2})\right)\gamma\left(\frac{1}{2} - b(\alpha_1 + \alpha_2 - 3\alpha_3 - Q)\right).$$

(5.2.82)

Note the shift relation (3.4.50), and we can confirm that three points function (5.1.4) where $\alpha_i = Q \frac{2}{2} + ip_i$ satisfies this recursive formula and can determine the constant in previous equation such as

$$C_{\alpha_1\alpha_2\alpha_3} = \frac{[\frac{\gamma}{(\alpha_1+\alpha_2+\alpha_3)\gamma}(0)\gamma(2\alpha_1)\gamma(2\alpha_2)\gamma(2\alpha_3)]}{[\gamma(\alpha_1+\alpha_2+\alpha_3 - Q)\gamma(\alpha_1+\alpha_2+\alpha_3 - Q)\gamma(\alpha_1+\alpha_2+\alpha_3 - Q)\gamma(\alpha_1+\alpha_2+\alpha_3 - Q)]}. (5.2.83)$$

This formula is called DOZZ formula.

### 5.3 AGT relation in the light asymptotic limit

#### 5.3.1 Introduction

Beside the suggestive physical meaning for six dimensional $\mathcal{N} = (2, 0)$ theory, AGT relation itself is also interesting subject. Since the representations of two equivalent observables, partition functions of class $\mathcal{S}$ theories and correlation functions of two dimensional theories,
their hidden symmetric structures in one side can be extracted from other side repre-
ation. From the point of this view, the manifest symmetries in correlation functions of
LFT, such as permutation symmetries or reflection symmetries, have to be interpreted in
supersymmetric gauge theory side. In fact, in the explicit correspondence mentioned in
previous section, the factor called one-loop determinant part in partition functions of the
gauge theory was written in product of Liouville three-points structure constants, which
enjoy reflection symmetry, but this symmetry is not manifest in gauge theory side, and
its physical meaning is not clear. In addition, permutation symmetry is not also manifest.
The parameters mapping between inserted momenta to LFT and masses of hypermulti-
plets in supersymmetric gauge theory side does not have permutation symmetry, in the
first place. As seen above, making a comparison between the structure of two sides of
AGT relation.

Here we consider “light asymptotic limit” [122, 123] of AGT relation. In this limit, the
central charge of LFT become infinity, and at the same time, the weights of inserted fields
in correlation functions go to zero. Then n-points functions are turn out to be classical, or
the integral over elements of $H^3_3$ and since the integrand is product of n inserted functions,
this has manifest permutation symmetry. The partition function of supersymmetric gauge
theory which corresponds to it in AGT relation must respect this factorized form in this
light asymptotic limit. In previous example of $SU(2)$ SQCD theory with four fundamental
hypermultiplets on ellipsoids, this limit corresponds to completely squashed limit and two
of four masses of hypermultiplets become very heavier. In this section, we confirm that the
partition function in this example explicitly reproduce the Liouville correlation function
in this limit. This means that AGT relation can be proven in this example explicitly.

The organization of this section is as follows. In section 5.3.2, we review the light
asymptotic limit in LFT to see that the correlation functions have manifest permutation
symmetry. Then in section 5.3.3 we interpret this limit in supersymmetric gauge the-
ory side, and shows that partition functions become the Liouville correlation function in
this limit. Finally in section 5.3.4, we present some physical meaning of this limit in
gauge theory side. We have one appendix for section B, which shows orthonormality of
wavefunctions in Liouville theory.

5.3.2 Light asymptotic limit

In this subsection, we review the Liouville field theory in light asymptotic limit, and
derive the explicit form of four-points function on sphere. Correlation functions of primary
operators are

$$Z \equiv \int \mathcal{D}\phi e^{-S_{\phi} + \sum_i 2\alpha_i \phi(z_i)}$$

(5.3.1)

where the conformal dimensions of the primary operators are $\Delta_{\alpha_i} = \alpha_i (Q - \alpha_i)$. Now we
consider “classical limit”($c \to \infty, \ b \to 0$). at the same time, in particular, we concentrate on
“light asymptotic limit”[122, 123] rest of the paper. In this limit, we consider the field
whose weight behave as $\alpha_i = b \eta_i \to 0$ where $\eta_i$s are fixed.

When the correlation functions (5.3.1) are computed in this limit, inserted operators
have no effects on the solution of classical equation of motion, Liouville equation. Substi-
tuting the solution of Liouville equation for normalized Liouville field $\varphi = b\phi$

$$\partial \bar{\partial} \varphi = \pi \mu b^2 e^{2\varphi}$$  \hspace{1cm} (5.3.2)

into $Z$, it is shown that the remaining integration is integral over the $\mathcal{H}^+_3$ symmetry which is contained in LFT as following. Now we define energy-momentum tensor $t$ by

$$t = -\left(\partial \varphi\right)^2 + \partial^2 \varphi$$  \hspace{1cm} (5.3.3)

where $\phi$ stands for the solution of Liouville equation. This $t$, which satisfies $\bar{\partial} t = 0$, is holomorphic.

$$\bar{\partial} t = -2\left(\partial \varphi\right) \cdot \bar{\partial} \partial \varphi + \partial \partial^2 \varphi = \left[-2(\partial \varphi) + \partial\right] \left(\pi \mu b^2 e^{2\varphi}\right) \to 0,$$  \hspace{1cm} (5.3.4)

Since in the light asymptotic limit there is no pole which is generated by inserted operators, its weight imply the behavior under light asymptotic limit: $\lim_{b \to 0} t = 0$. Now also we introduce $\psi = e^{-\varphi}$. As this satisfies $(\partial^2 - t)\psi = 0$, it suffices $\partial^2 \psi = 0$ in light asymptotic limit, and it follows that $\psi$ is linear function of $z$. Almost same discussion about $\bar{z}$ is substantiated, and it is denoted in

$$\psi = A'(\alpha z \bar{z} + \beta z + \gamma \bar{z} + \delta) \equiv A'(z \ 1) g \left(\begin{array}{c} \bar{z} \\ 1 \end{array}\right).$$  \hspace{1cm} (5.3.5)

where $g$ is element of two by two matrix. These are solutions for Liouville equation on flat space (5.3.2), but if we integrate this equation over the sphere, an effect of the curvature term appears so that

$$\mu b^2 \int d^2 z e^{2\varphi} + 1 = 0.$$  \hspace{1cm} (5.3.6)

Physical field $\phi(z)$ cannot satisfy this equation, so it turns out that correlation functions for LFT in light asymptotic limit on sphere does not have classical solutions.

In this case, we can define another quantities to which classical analysis can be applied [124]. Laplace transformation is applied to $Z$, then

$$Z_A = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\mu Z e^{\mu A}$$

is introduced, and original $Z$ is denoted by $Z = \int_{0}^{\infty} dA Z_A e^{-\mu A}$. In this integral, we shift the integral variable as $\varphi \to \varphi + \frac{1}{2} \log A$, they become

$$Z_A = A^{-\frac{1}{3b^2}} - 1 + \sum_i \eta_i Z_{A=1}$$  \hspace{1cm} (5.3.8)

and

$$Z = \mu^\frac{1}{2b^2} - \sum \eta_i \Gamma \left(\sum \eta_i - \frac{1}{b^2}\right) Z_{A=1}.$$  \hspace{1cm} (5.3.9)

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This Gamma function is diverging, and $Z_{A=1}$ remains finite. From now on, we consider these $Z_A$ are universal quantities which characterize behaviors of correlation functions. Equations of motion for $Z_A$ are previous

$$\partial \bar{\partial} \varphi = \pi \mu b^2 e^{2\varphi}$$

(5.3.10)

and for constraint conditions

$$A = \int dz^2 e^{2\varphi}.$$  

(5.3.11)

Integrating (5.3.10) leads to (5.3.6), and they are solved by $\mu = -\frac{1}{\sqrt{A}}$. Inserting the solutions (5.3.5) to this, we can take $A' = e^{\sqrt{\pi}}$ and $g \in \mathcal{H}_3^+$ as the classical solutions for $Z_A$. Because of reality of $\phi(z)$, which represents physical field, it is obvious that $g$ is hermite and has positive trace. Now it is written as

$$Z_A = \frac{A^{-(1+\frac{1}{2\pi})}}{2\pi i} \int_{\mathcal{H}_3^+} dg \prod_i e^{2n_i \varphi} = \frac{A^{-(1+\frac{1}{2\pi})}}{2\pi i} \int_{\mathcal{H}_3^+} dg \prod_i \left( \sqrt{\frac{\pi}{A}} (z_i - 1) g\left(\frac{z_i}{1}\right) \right)^{-2n_i}$$

(5.3.12)

in terms of the $n$ points function $\prod_i \Phi_{z_i}^{n_i}(g)$. These $\Phi_{z_i}^{n_i}(g)$ generate orthonormal basis of $\mathcal{H}_3^+$ symmetry

$$-2\pi^3 \delta(g - g') = \int d\eta dz^2 (1 - 2\eta)^2 \Phi_{z}^{\eta}(g) \Phi_{z}^{1-\eta}(g').$$

(5.3.13)

The derivation of this orthonormality is discussed in [125] in detail, and we review it in Appendix B. The original correlation functions are

$$Z = \mu^{\frac{1}{2\pi}} - \sum n_i \Gamma \left( -\frac{1}{b^2} + \sum n_i \right) \frac{1}{2\pi i + \sum n_i} \int_{\mathcal{H}_3^+} dg \prod_i \Phi_{z_i}^{n_i}(g).$$

(5.3.14)

In following, we consider $\int_{\mathcal{H}_3^+} dg \prod_i \Phi_{z_i}^{n_i}(g)$ as correlation functions, and compare them to partition functions of gauge theory.

**Four points function**

Here we derive the four points function in terms of integral over the intermediate momentum between three points function

$$\langle \Phi_{z_1}^{n_1} \Phi_{z_2}^{n_2} \Phi_{z_3}^{n_3} \rangle = \int dg \Phi_{z_1}^{n_1} \Phi_{z_2}^{n_2} \Phi_{z_3}^{n_3}.$$  

(5.3.15)

The four points functions are expanded in these such as

$$\langle \Phi_{z_1}^{n_1} \Phi_{z_2}^{n_2} \Phi_{z_3}^{n_3} \Phi_{z_4}^{n_4} \rangle = -\frac{1}{2\pi^3} \int d\eta d^2 z (1 - 2\eta)^2 \langle \Phi_{z_1}^{n_1} \Phi_{z_2}^{n_2} \Phi_{z_4}^{n_4} \rangle \langle \Phi_{z_3}^{n_3} \Phi_{z_4}^{1-\eta} \rangle.$$  

(5.3.16)
This three points function which is constrained to $SL(2, \mathbb{C})$ invariant is written explicitly by affine coordinates and their conformal dimensions as in previous section,

$$
\langle \Phi_{z_1}^{\eta_1} \Phi_{z_2}^{\eta_2} \Phi_{z_3}^{\eta_3} \rangle = \frac{\pi \Gamma(\eta_1 + \eta_2 + \eta_3 - 1) \Gamma(-\eta_3 + \eta_2 + \eta_1) \Gamma(-\eta_2 + \eta_1 + \eta_3) \Gamma(-\eta_1 + \eta_2 + \eta_3)}{2 \Gamma(2\eta_1) \Gamma(2\eta_2) \Gamma(2\eta_3) \Gamma(2\eta)} \times \sqrt{|z_{12}|^{2(\eta_1 - \eta_2 - \eta_3)} |z_{23}|^{2(\eta_2 - \eta_3 - \eta_1)} |z_{31}|^{2(\eta_3 - \eta_1 - \eta_2)}}. \quad (5.3.17)
$$

In Appendix B, we derive this correlation function. Substituting this into (5.3.16), the explicit form of four points function is given as

$$
\langle \Phi_{z_1}^{\eta_1} \Phi_{z_2}^{\eta_2} \Phi_{z_3}^{\eta_3} \Phi_{z_4}^{\eta_4} \rangle = \frac{1}{8\pi} \int d\eta \eta \int \Gamma(\eta + 2\eta_1 + \eta_2 + 1) \Gamma(-\eta + \eta_1 + 2) \Gamma(\eta + \eta_2 + 1) \Gamma(-\eta_3 + \eta_4 + 1) \Gamma(1 - \eta - \eta_3 - 4) \Gamma(1 - \eta - \eta_2 - 4) \Gamma(1 - \eta - \eta_1 - 4) \Gamma(1 - \eta - \eta_4 - 4)
$$

$$
\times \sqrt{|z_{12}|^{2(\eta_2 - \eta_1 + 2)} |z_{23}|^{2(\eta_3 - \eta_2 - \eta_1)} |z_{34}|^{2(1 - \eta - \eta_3 - 4)} |z_{41}|^{2(\eta_4 - \eta_3 - 1 + \eta)} |z - z_1|^{2(\eta_1 - \eta - 2)} |z - z_2|^{2(\eta_2 - \eta - 1)} |z - z_3|^{2(\eta_3 - \eta - 1)} |z - z_4|^{2(\eta_4 - \eta - 1)} \Gamma(2\eta_3) \Gamma(2\eta_4) \Gamma(2\eta) \Gamma(2\eta_1) \Gamma(2\eta_2) \Gamma(2\eta_1 + \eta_2 + \eta_3 + \eta_4)
$$

where $\eta_{i+} = \eta_i \pm \eta_j$. Before integrating $d^2 z$, we consider the holomorphic part

$$
z_{12}^{\eta - \eta_1 + 2} \int dz (z_2 - z)^{\eta_1 - \eta} (z - z_1)^{\eta_2 - \eta} (z - z_3)^{\eta_3 - \eta} (z - z_4)^{\eta_4 - \eta - 1} \eta \eta_1 \eta_2 \eta_3 \eta_4
$$

$$
\times \sqrt{|z_{12}|^{2(\eta_2 - \eta_1 + 2)} |z_{23}|^{2(\eta_3 - \eta_2 - \eta_1)} |z_{34}|^{2(1 - \eta - \eta_3 - 4)} |z_{41}|^{2(\eta_4 - \eta_3 - 1 + \eta)} |z - z_1|^{2(\eta_1 - \eta - 2)} |z - z_2|^{2(\eta_2 - \eta - 1)} |z - z_3|^{2(\eta_3 - \eta - 1)} |z - z_4|^{2(\eta_4 - \eta - 1)} \Gamma(2\eta_3) \Gamma(2\eta_4) \Gamma(2\eta) \Gamma(2\eta_1) \Gamma(2\eta_2) \Gamma(2\eta_1 + \eta_2 + \eta_3 + \eta_4)
$$

where $z$ is redifined to $z = \frac{z_2}{z_{12}}$. This integral can be written as a linear combination of two hypergeometric functions

$$
F_1(\eta, \eta_1 = 1, \ldots, 4, z_2^\prime) = z_2^{-1} F(1 - \eta - \eta_1 - 1 - \eta + \eta_3 - 2, 1 - \eta; z_2^\prime)
$$

and

$$
F_2(\eta, \eta_1, z_2^\prime) = F(1 - \eta - \eta_1 - 1 - \eta + \eta_3 - 2, 2; z_2).
$$

The integral over anti-holomorphic coordenates is also almost same, then the result of integrations over $d^2 z$ in (5.3.18) is denoted in bilinear combinations of these $F_{1,2}(\eta, \eta_1, z_2^\prime)$ and their complex conjugates. The coefficients of them are determined by demanding the invariance around the monodromy at the singularity $z_2^\prime = 0, 1$ of the solution. Then this integral turns out to be

$$
\int d^2 z |z_2^\prime - z|^{2(\eta_1 - \eta - 2)} |z_2^{\prime} - 2(\eta_1 - \eta - 2)| z - 1 |^{2(\eta_1 - \eta - 2)}
$$

$$
\times \frac{\pi \gamma(2\eta - 1) |z_2^\prime|^{2(1 - 2\eta)}}{\gamma(\eta - \eta_1 - 1) \gamma(\eta + \eta_1 - 1)} F(1 - \eta - \eta_1 - 2, 1 - \eta + \eta_3 - 2, 2(1 - \eta); z_2^\prime)
$$

and

$$
\frac{\pi \gamma(\eta + \eta_3 - 1)}{\gamma(2\eta) \gamma(1 - \eta + \eta_3 - 1)} F(1 - \eta - \eta_1 - 1 - \eta + \eta_3 - 2, 2; z_2).
$$

$$
\quad (5.3.23)
$$

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Substituting this to the four points function (5.3.18), explicit formula of the four points function whose inserted coordinates are fixed to \((z_1, z_2, z_3, z_4) = (0, z'_2, 1, \infty)\) come out as

\[
\frac{1}{8} \int \frac{d\eta}{\Gamma(\eta + 1)} \frac{\Gamma(\eta + 1 + \eta_{1+2})}{\Gamma(2\eta_1)\Gamma(2\eta_2)\Gamma(2\eta - 1)}
\times \Gamma(\eta_{1+4} - \eta) \frac{\Gamma(1 - \eta + \eta_{1+4})}{\Gamma(2\eta_3)\Gamma(2\eta_4)\Gamma(1 - \eta - \eta_{1+4})}
\times \left[ \frac{\gamma(2\eta - 1)}{\gamma(\eta - 1)} \frac{|z_2'|^{2(1 - \eta_{1+2})}}{\gamma(\eta - \eta_{1+2})}\right] F(1 - \eta + \eta_{1+2}, 1 - \eta + \eta_{1+4}, 2(1 - \eta); z_2')^2
\]

\[
+ \frac{\gamma(\eta + \eta_{1+4})}{\gamma(\eta - \eta_{1+4})} \frac{|z_2'|^{2(n_{1+2})}}{F(\eta - \eta_{1+2}, \eta + \eta_{1+4}, 2\eta; z_2')}^2 \right].
\]

Each term in third and fourth lines looks symmetric under the interchange of weights between conjugate pair \(\eta \to 1 - \eta\), and particularly, since both \(\eta\) and \(1 - \eta\) belong to \(1/2 + i\mathbb{R}\), the contours of integrals about \(\eta\) and \(1 - \eta\) are same. Because of that, we can reset the integral variable of first term to \(1 - \eta \to \eta\), and compute in following

\[
\langle \Phi_{\eta_{1+2}}^{\eta_1} \Phi_{\eta_{1+3}}^{\eta_2} \Phi_{\eta_{1+4}}^{\eta_3} \rangle = \frac{1}{4} \int \frac{d\eta}{\Gamma(\eta + 1)} \left| z_2' \right|^{2(\eta - \eta_{1+2})} \frac{\Gamma(\eta - \eta_{1+2})}{\Gamma(2\eta_1)\Gamma(2\eta_3)}
\times \frac{\Gamma(1 - \eta + \eta_{1+4})}{\Gamma(2\eta_4)}
\times \frac{\Gamma(1 - \eta - \eta_{1+4})}{\gamma(\eta - \eta_{1+4})} \frac{\Gamma(1 - \eta + \eta_{1+4})}{\gamma(\eta - \eta_{1+4})} \frac{|z_2'|^{2(n_{1+2})}}{F(\eta - \eta_{1+2}, \eta + \eta_{1+4}, 2\eta; z_2')}^2
\]

\[
\times \Gamma(\eta + \eta_{1+2}) \Gamma(\eta - \eta_{1+2}) \Gamma(-\eta + \eta_{1+2}) \Gamma(1 - \eta + \eta_{1+4}) \Gamma(\eta + \eta_{1+4}) \Gamma(1 - \eta - \eta_{1+4}).
\]

In addition, the four points function inserted at original coordinates can be related this four points function of fixed coordinates by multiplying the dependence of coordinates as follows

\[
\langle \Phi_{\eta_{1+2}}^{\eta_1} \Phi_{\eta_{1+3}}^{\eta_2} \Phi_{\eta_{1+4}}^{\eta_3} \rangle = \left| z_{42}' \right|^{-4n_2} \left| z_{31}' \right|^{-2n_{1+2} - 2n_{1+4}} \left| z_{43}' \right|^{-2n_{1+2} + 2n_{1+4}} \left| z_{41}' \right|^{-2n_{1+2} + 2n_{1+4}} \langle \Phi_{\eta_{1+2}}^{\eta_1} \Phi_{\eta_{1+3}}^{\eta_2} \Phi_{\eta_{1+4}}^{\eta_3} \rangle
\]

where the constant factors ignored in (5.3.20) are included. \(z_2'\) is \(SL(2, \mathbb{C})\)-invariant cross ratio \(z_2' = \frac{z_{12}^4 z_{34}^4}{z_{13} z_{24}}\).

### 5.3.3 Parameter coincidences in AGT relation

In this subsection, we proved AGT relation in this light asymptotic limit explicitly. First we introduce the parameters which are used in gauge theory. These reparametrize the weights by real parameters such as \(\alpha_i = b\eta_i = \frac{Q}{2} + ip_i\), \(\alpha = b\eta = \frac{Q}{2} + ip\). Furthermore, to interpret these parameters in AGT relation, as seen in (5.1.2) the momenta and masses are related as

\[
\mu_1 = p_1 + p_2, \quad \mu_2 = -p_1 + p_2, \quad \mu_3 = -p_3 + p_4, \quad \mu_4 = -p_3 - p_4
\]
where these parameters \( \mu_a \) represent masses of hypermultiplets in gauge theory. Two of these four hypermultiplets are heavy and rest two are light in this limit. In AGT relation, the four point function of LFT dealt so far is conjectured to coincide with partition function of \( SU(2) \) SQCD with four fundamental flavours. This partition function is calculated by localization method in [1] as explained in previous chapters as follows

\[
Z_{\text{ell}} = \int dp \, e^{-S_{YM}} |Z_{\text{inst}}(p, \ell^{-1}, \tilde{\ell}^{-1}, \tau)|^2 \prod_{i=1}^{4} \prod_{\pm} \frac{\Upsilon(2ip)\Upsilon(-2ip)}{\Upsilon(\frac{Q}{2} \pm ip + i\mu_i)}
\]

(5.3.27)

where \( S_{YM} \) is classical contribution from Yang-Mills term, \( Z_{\text{inst}} \) is instanton contribution, and remaining part arises from one-loop determinant of vector multiplet and four fundamental matter multiplet. In rest of this subsection, we analyze the behavior of this partition function in light asymptotic limit, and show this function coincides with above result of four point function (5.3.25) in this limit.

**One-loop determinant part**

First, we consider one-loop determinant part. Let it be rewritten in LFT parameter

\[
\frac{\Upsilon(2ip)\Upsilon(-2ip)}{\prod_{\pm} \prod_{i=1}^{4} \Upsilon(\frac{Q}{2} \pm ip + i\mu_i)} = \frac{\Upsilon(-Q + 2b\eta)\Upsilon(2b\eta)}{\Upsilon(b\eta - Q + b\eta_{1+2})\Upsilon(b\eta + b\eta_{-1-2})\Upsilon(b\eta - Q + b\eta_{3+4})\Upsilon(b\eta + b\eta_{3-4})}
\]

\[
\times \frac{1}{\Upsilon(-b\eta + b\eta_{1+2})\Upsilon(b\eta - b\eta_{-1-2})\Upsilon(-b\eta + b\eta_{3+4})\Upsilon(b\eta - b\eta_{3-4})}.
\]

(5.3.28)

However, the definition of \( \Upsilon(x) \) (3.4.48) has information about only vanishing points and constraints which are imposed by hand. Then we can replace \( \Upsilon(x) \) by the function whose vanishing points place same as those of \( \Upsilon(x) \) in this limit. Vanishing points of \( \Upsilon \) function is

\[
x = Q + mb + \frac{n}{b}, \quad -mb - \frac{n}{b} \quad (m, n \in \mathbb{Z}_{\geq 0}).
\]

(5.3.29)

Here we consider the vanishing points of the following function in this limit

\[
\hat{\Upsilon}(x) \equiv \Upsilon(bx), \quad \check{\Upsilon}(x) \equiv \Upsilon(-Q + bx)
\]

(5.3.30)

which appear in (5.3.28). The former satisfies

\[
\hat{\Upsilon}(x) = 0 \iff x = m + 1 + \frac{n + 1}{b^2}, -m - \frac{n}{b^2} \quad (m, n \in \mathbb{Z}_{\geq 0}),
\]

(5.3.31)

and the vanishing points which remain finite in the \( b \to 0 \) limit are \( n = 0 \) and \( x \in \mathbb{Z}_{\leq 0} \). This behavior is mimicked by \( \Gamma \) function whose first order poles are placed at \( \mathbb{Z}_{\leq 0} \):

\[
\hat{\Upsilon}(x) \to \frac{1}{\Gamma(x)}.
\]

(5.3.32)

This argument can be applied to \( \check{\Upsilon}(x) \). Its vanishing points are

\[
\check{\Upsilon}(x) = 0 \iff x = m + 2 + \frac{n + 2}{b^2}, \quad -(m - 1) - \frac{n - 1}{b^2} \quad (m, n \in \mathbb{Z}_{\geq 0}),
\]

(5.3.33)
and those out of them which stay finite are \( n = 1 \) and \( x \in \mathbb{Z}_{\leq 1} \). This function is imitated by

\[
\tilde{Y}(x) \to \frac{1}{\Gamma(x - 1)}.
\]

(5.3.34)

Consequently, the one-loop determinant part (5.3.28) is written in \( \Gamma \) function in light asymptotic limit as

\[
Z_{1-\text{loop}}^{\tilde{b} \to 0}(\eta, \eta_k) = \frac{\Gamma(\eta + \eta_{1+2} - 1)\Gamma(\eta + \eta_{1-2})\Gamma(-\eta + \eta_{1+2})\Gamma(\eta - \eta_{1-2})}{\Gamma(2\eta)} \times \frac{\Gamma(\eta + \eta_{3+4} - 1)\Gamma(\eta + \eta_{3-4})\Gamma(-\eta + \eta_{3+4})\Gamma(\eta - \eta_{3-4})}{\Gamma(2\eta - 1)}.
\]

(5.3.35)

This function can be submitted to the four points function (5.3.25) in LFT.

**Instanton contribution part**

In following, we consider the instanton contribution part \( Z_{\text{inst}} \). Let \( \tilde{Y} = (Y_1, Y_2) \) be two Young tableaux. Nekrasov partition function formula (2.2.15) in this case \( k \) instanton contribution is counted from this pair of two Young tableaux \( \tilde{Y} \) whose total number of boxes is \( k = |\tilde{Y}| = |Y_1| + |Y_2| \). Take that \( \tilde{a} = (a_1, a_2) = (a, -a) = (ip, -ip) \) is adjoint scalar, and \( \mu_i \) are masses of fundamental multiplets. Explicitly

\[
z_{\text{vec}}^{-1} = \prod_{i,j=1}^{2} \prod_{s \in Y_1}^{s} \prod_{t \in Y_1}^{t} [E(a_i - a_j, Y_i, Y_j, s)] \prod_{t \in Y_1}^{t} (Q - E(a_j - a_i, Y_j, Y_i, t))
\]

and

\[
z_{\text{fund}}(\tilde{a}, \tilde{Y}, \mu_k) = \prod_{s \in Y_1}^{s} (\phi(a_1, s) - i\mu_k + \frac{Q}{2}) \prod_{t \in Y_2}^{t} (\phi(a_2, t) - i\mu_k + \frac{Q}{2}),
\]

\[
z_{\text{antifund}}(\tilde{a}, \tilde{Y}, \mu_k) = \prod_{s \in Y_1}^{s} (\phi(a_1, s) + i\mu_k + \frac{Q}{2}) \prod_{t \in Y_2}^{t} (\phi(a_2, t) + i\mu_k + \frac{Q}{2}).
\]

(5.3.37)

Above results are instanton contributions for gauge theory whose gauge group is \( U(2) \), and to apply them to our \( SU(2) \) partition function, it must be divided by \( U(1) \) factor \(^1\)

\[
(1 - q)^{\frac{1}{2}(Q + i\mu_{1+2})(Q - i\mu_{3+4})} = (1 - q)^{2b^2 \eta_2 \eta_3}.
\]

(5.3.38)

\(^1\)The difference between this \( U(1) \) factor and that of in [6] is accounted for by relation of the parametrization (5.3.27) and this power of \( U(1) \) factor [126].
Now let us consider the behavior of these formulas in light asymptotic limit. Concluding remark about this behavior is summarized in [127], and we review this argument rest of this subsection so that the formulas turn out to be hypergeometric function.

First, in this limit, since the $U(1)$ factor (5.3.38) goes to 1, the division by this factor does not affect $Z_{\text{inst}}$. Second, the contributions from each multiplet are dominated by the terms which has $\frac{1}{b}$ in each factors. The numerator part of $Z_{\text{inst}}$ is written in parameters of (5.3.27) as product of  

\[ z_{\text{fund}}(\vec{a}, \vec{Y}, \mu_3) = \prod_{s \in Y_1} \left[ \frac{j-1}{b} + b(\eta_{0+3-4} + i - 1) \right] \prod_{t \in Y_2} \left[ \frac{j'}{b} + b(-\eta_{0-3+4} + i') \right] \]

\[ z_{\text{fund}}(\vec{a}, \vec{Y}, \mu_4) = \prod_{s \in Y_1} \left[ \frac{j-2}{b} + b(\eta_{0+3+1+2} + i - 2) \right] \prod_{t \in Y_2} \left[ \frac{j'-1}{b} + b(-\eta_{0-3-1+2} + i' - 1) \right] \]

\[ z_{\text{antifund}}(\vec{a}, \vec{Y}, \mu_1) = \prod_{s \in Y_1} \left[ \frac{j-2}{b} + b(\eta_{0+1+2} + i - 2) \right] \prod_{t \in Y_2} \left[ \frac{j'-1}{b} + b(-\eta_{0-1+2} + i' - 1) \right] \]

\[ z_{\text{antifund}}(\vec{a}, \vec{Y}, \mu_2) = \prod_{s \in Y_1} \left[ \frac{j-1}{b} + b(\eta_{0-1+2} + i - 1) \right] \prod_{t \in Y_2} \left[ \frac{j'}{b} + b(-\eta_{0+1+2} + i') \right] \]  \hspace{1cm} (5.3.39)

where $\eta = \eta_0$, $\eta_{i+j+k} = \eta_i + \eta_j + \eta_k$ and the coordinates of $s \in Y_1$ are $(i, j)$ and those of $t \in Y_2$ are $(i', j')$. The denominator part is

\[ z_{\text{vec}}^{-1}(\vec{a}, \vec{Y}) = \prod_{s \in Y_1} \left[ \frac{1}{b} (A_{Y_1}(s) + 1) - bL_{Y_1}(s) \right] \prod_{t \in Y_2} \left[ \frac{1}{b} A_{Y_1}(s) + b(L_{Y_1}(s) + 1) \right] \]

\[ \times \prod_{s \in Y_1} \left[ \frac{1}{b} A_{Y_1}(s) + b(2\eta - L_{Y_2}(s) - 1) \right] \prod_{t \in Y_2} \left[ \frac{1}{b} (A_{Y_2}(t) - 1) + b(-2\eta + L_{Y_2}(s)) \right] \]

\[ \times \prod_{t \in Y_2} \left[ \frac{1}{b} (A_{Y_2}(t) + 2) - b(L_{Y_1}(t) + 2\eta - 1) \right] \prod_{t \in Y_2} \left[ \frac{1}{b} (A_{Y_2}(t) + 1) + b(2\eta + L_{Y_1}(t)) \right] \]

\[ \times \prod_{s \in Y_1} \left[ \frac{1}{b} (A_{Y_2}(t) + 1) - bL_{Y_2}(t) \right] \prod_{t \in Y_2} \left[ \frac{1}{b} A_{Y_2}(t) + b(L_{Y_2}(t) + 1) \right] . \]  \hspace{1cm} (5.3.40)

Thus when $|Y|$ is fixed, to determine which Young tableau $\vec{Y}$ has dominant behavior in light asymptotic limit, what we all have to do is comparing the powers of $b$ of the denominator and those of the numerator. They look naively both $(\frac{1}{b})^8$ and seem to cancel each other, but as it is they do not. Note that the coordinates in coefficients of $\frac{1}{b}$ are those of boxes over which the product is taken such as $j, j' \geq 1$ and $A_{Y_{1,2}} \geq 0$, and that the number of boxes which satisfy $A_{Y_1} = 0$ equals to the number of $\lambda_{1,1} = (4, 1)$. Furthermore, we obtain $(2\eta A_{Y_1} = 1) = (2\eta = 2)$. Consequently, the powers of $b$ which arises from the terms whose coefficients of $\frac{1}{b}$ are zero are

\[ (\text{denominator}) \propto b^{2(\eta_{j=1} + (\eta_{j=2} + (\eta_{j'=1})} \]

\[ (\text{numerator}) \propto b^{2(\eta_{j=1} + 2(\eta_{j=2} + 2(\eta_{j'=1}))} \]  \hspace{1cm} (5.3.41)

Taking them into account, we conclude that

\[ Z_{\text{inst}}^{b\to0} \propto b^{2(\eta_{j=2} + (\eta_{j'=1}))} . \]  \hspace{1cm} (5.3.42)
Figure 5.4: Young tableau which dominates in light asymptotic limit

Each fixed $|Y|$, there is a Young tableau whose power of $b$ (5.3.42) is least, and it dominates in the summation which is taken over all $\vec{Y}$s. This tableau is that whose $Y_2$ is empty and $Y_1$ is aligned in first row (as figured in Fig.5.4). Since all boxes in this tableau satisfy $j = 1$, it turns out that $A_{Y_1}(s) = 0$. Furthermore, because of $L_{Y_1}(s) = |Y| - i$ and $L_{Y_2}(s) = -i$, they are calculated as

$$(\text{denominator}) = \prod_{i=1}^{|Y|} (|Y| - i + 1)(2\eta + i - 1) + O(b)$$

and we obtain

$$Z_{\text{inst}}^{b \to 0} = \prod_{|Y| = 1}^{\infty} \frac{|Y|^{|Y|-1}}{|Y|!} \prod_{i=0}^{\eta_{0+3-4} - i + 1}(\eta_{0-1+2} + i - 1) + O(b)$$

(5.3.43)

(5.3.44)

Let coupling constant $q$ introduced in (2.2.16) be identified with $z'_2$ which is used to parametrize the sphere on which LFT lives, then (5.3.25) become

$$\langle \Phi_{\eta_1} \Phi_{\eta_2} \Phi_{\eta_3} \Phi_{\eta_4} \rangle = -\frac{1}{4\prod_i \Gamma(2\eta_i - 1)} \int d\eta |z'_2|^{2(\eta - \eta_1 + \eta_2)} |Z_{\text{inst}}^{b \to 0} Z_{1-\text{loop}}^{b \to 0}(\eta, \eta_i) \rangle. \quad (5.3.46)$$

In fact, with the above identification between coupling constant $q$ and $z'_2$, classical Yang-Mills action can be parametrized by expectation value of scalar fields $\eta$ such as

$$e^{-S_{YM}} = e^{-\frac{s\eta^2}{g_{YM}} 2p^2} \to |z'_2|^{2\eta}$$

(5.3.47)

where some unrelated factors are ignored. Taken together, aside from $\eta$-unrelated factors, it is shown that

$$\langle \Phi_{\eta_1} \Phi_{\eta_2} \Phi_{\eta_3} \Phi_{\eta_4} \rangle \propto Z_{\text{inst}}^{b \to 0} = \int d\eta e^{-S_{YM}} |Z_{\text{inst}}^{b \to 0} Z_{1-\text{loop}}^{b \to 0}(\eta, \eta_i) \rangle. \quad (5.3.48)$$

AGT relation is explicitly proven under the light asymptotic limit.
5.3.4 Short summary

The simple formula for the Liouville four-point correlators in the light asymptotic limit, and the corresponding simplification of the 4D ellipsoid partition function, both suggest that a certain 2D gauge theory may describe the limit. In particular, the investigation shows that the sum over Young tableaux simplifies to a sum over linear arrays of boxes, which may imply that the instanton sum turns into a vortex sum in this limit.

The light asymptotic limit corresponds to the limit of an extremely squashed 4-sphere, i.e. $\ell \ll \hat{\ell}$ in (3.3.3). In this limit, one has various choices regarding the behavior of the other axis-length $r$. In one typical choice $r = \ell \ll \ell$, the ellipsoid degenerates to a small $S^2$ fibered over a large disk with a small Omega deformation about the origin, and becomes $S^2 \times \mathbb{R}^2$ in the limit. In another typical choice $r = \hat{\ell} \gg \ell$, the role of the base and fiber is interchanged, and one has the $(x_1, x_2)$-plane $\mathbb{R}^2$ with a large Omega deformation fibered over a large $S^2$. The result of this section could therefore be compared with the recent results on SUSY gauge theories on $S^2$ [26, 27] or squashed $S^2$ [28], especially the theories appearing on 2D surface defects corresponding to Liouville degenerate operator insertions [128, 129, 130, 27]. At present, it is not clear to us which is the more suitable picture to understand the simplification of 4D partition function. In either picture, since $\epsilon_1/\epsilon_2 \to 0$ in the limit, the physics near the north and south poles of the ellipsoid is a 2D $\mathcal{N} = (2, 2)$ SUSY theory in the Nekrasov-Shatashvili limit [131]. Furthermore, in addition to this “dimensional reduction”, one also has to send some of the mass parameters to infinity to reproduce the Liouville light asymptotic limit. It would be interesting to understand fully the mechanism of this dimensional reduction.
Appendix A

Notation

A.1 Six dimension

Here \( m = 1, \cdots, 6 \) represent directions of \( \mathbb{R}^6 \).

A.1.1 Clifford algebra

\( \Gamma_m \) which satisfies

\[
\{ \Gamma_m, \Gamma_n \} = 2\delta_{mn}
\]

represents Clifford algebra in six dimensional space. The chirality in this space is projected by

\[
\Gamma^{123456} = \text{diag} \left( i1_4, -i1_4 \right) \equiv i\Gamma^7.
\]

There is rotational symmetry \( SO(6) \simeq SU(4) \) on \( \mathbb{R}^6 \). Spinors whose eigenvalues of \( \Gamma^7 \) are +1 are fundamental representations of \( SU(4) \), and -1 are anti-fundamental representations.

A.1.2 Charge conjugate

Charge conjugation matrix \( C \) is defined by

\[
C\Gamma^aC^{-1} = -(\Gamma^a)^T.
\]

A.2 Five dimension

- The Greek indices \( \mu, \nu, \cdots \) are for the space-time indices.

- The Roman indices \( a, b, \cdots \) are for the local Lorentz frame. To translate them into the space-time indices, we use vielbein \( e^\mu_a \). For instance, \( v^{\mu\nu} = v^{ab} e^\mu_a e^\nu_b \).

- The capital Roman indices \( I, J, \cdots \) stand for the \( SU(2)_R \) symmetry, \( I, J = 1, 2 \).
• The capital Roman indices $A, B, \cdots$ stand for the $Sp(2r)$ flavor symmetry, $A, B = 1, 2, \cdots, 2r$.

• The spinor indices are denoted by the Greek indices $\alpha, \beta, \cdots$.

• The generators of the Lie algebra $T^A$ of a Lie group is anti-hermitian: $T^A = -T^A$ satisfying $[T_A, T_B] = -f^{C}_{AB}T_C$ with the structure constant $f^{C}_{AB}$ and the Killing form $\text{Tr}(T_AT_B) = -\frac{1}{2}\delta_{AB}$.

• All spinors are Grassmann odd except Killing spinors that are treated as Grassmann even variables.

• The gauge transformation $\delta_G$ for a gauge field $A$ defined by

$\delta_G(\epsilon)A = d\epsilon - [A, \epsilon]$ , \hspace{1cm} (A.2.1)

and for a matter field $\Phi$ by

$\delta_G(\epsilon)\Phi = \epsilon\Phi$ . \hspace{1cm} (A.2.2)

A.2.1 Gamma matrices and spinors

In the Euclidean five dimensions, the gamma matrices satisfying the Clifford algebra

$\{\Gamma_a, \Gamma_b\} = 2\delta_{ab} , \hspace{1cm} (a, b = 1, 2, \cdots, 5)$ , \hspace{1cm} (A.2.3)

are $4 \times 4$ matrices. We use the gamma matrices with multiple indices

$\Gamma^{a_1a_2\cdots a_n} \equiv \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{\sigma} \Gamma^{a_{\sigma(1)}}a_{\sigma(2)} \cdots \Gamma^{a_{\sigma(n)}}$ , \hspace{1cm} (A.2.4)

where $|\sigma|$ is the order of an element $\sigma$ of the permutation group $S_n$. With this notation, the product of the gamma matrices obeys the relation

$\Gamma_{1\cdots 5} = -1_4$ . \hspace{1cm} (A.2.5)

The charge conjugation matrix $C$ satisfies

$C^T = -C , \hspace{1cm} C^T C = 1 , \hspace{1cm} \Gamma_a^T = C\Gamma_a C^{-1}$ . \hspace{1cm} (A.2.6)

We introduce a spinor $\psi^\alpha$ with upper index $\alpha$. Then the positions of indices of the matrices are

$(\Gamma_a)^{\alpha}_{\beta} , \hspace{1cm} C_{\alpha\beta} , \hspace{1cm} C^{\alpha\beta}C_{\beta\gamma} = -\delta^\alpha_\gamma , \hspace{1cm} (C^\alpha_{\beta})^* = C^{\alpha\beta} = C_{\alpha\beta}$ . \hspace{1cm} (A.2.7)

We raise and lower the spinor index by the charge conjugation matrix

$\psi_\alpha = \psi^\beta C_{\beta\alpha} , \hspace{1cm} \psi^{\alpha} = C^{\alpha\beta}\psi_\beta$ , \hspace{1cm} (A.2.8)
and the contraction is always defined by
\[ \chi \psi \equiv \chi_\alpha \psi^\alpha. \] (A.2.9)

Spinors have an SU(2) index \( I \) (\( I = 1, 2 \)) which can be raised or lowered by the skew-symmetric matrix \( \epsilon_{IJ} \):
\[ \psi^I = \epsilon^{IJ} \psi_J, \quad \psi_J = \psi^J \epsilon_{IJ}, \quad \epsilon_{IJ} = \epsilon^{IJ} = (i\sigma_2)_{IJ}. \] (A.2.10)

We can impose the SU(2)-Majorana condition on spinors with the charge conjugation matrix
\[ (\psi^\alpha_I)^* = \epsilon_{IJ} \psi^\beta_J C_{\beta\alpha}. \] (A.2.11)

If we denote spinors as \( \epsilon = (\epsilon^1, \epsilon^2)^T \), the condition yields
\[ \epsilon^2 = C^* (\epsilon^1)^*. \] (A.2.12)

The bilinear terms of spinors with SU(2) indices behave under the exchange of the two spinors as
\[ \psi^I \Gamma^a_a a_1 a_2 \cdots a_n \chi^J = t_n \chi^J \Gamma^a_a a_1 a_2 \cdots a_n \psi^I, \quad t_n = \begin{cases} +1, & n = 0, 1, 4, 5 \\ -1, & n = 2, 3 \end{cases}. \] (A.2.13)

When the indices of SU(2) are contracted, there appears a minus sign in the right hand side.

For triple spinors \( (\xi, \eta, \psi) \), the Fierz identities hold
\[ (\Gamma^\mu \xi)^\alpha (\eta \Gamma^\mu \psi) = -2\psi^\alpha (\xi \eta) - 4\eta^\alpha (\psi \xi), \]
\[ (\Gamma^\mu \eta)^\alpha (\eta \Gamma^\mu \psi) = 4\psi^\alpha (\xi \eta) - 4\eta^\alpha (\psi \xi), \]
where \( s = 1 \) for Grassmann odd spinors and \( s = 0 \) for Grassmann even spinors.

### A.2.2 Spin connection and Lie derivatives

The spin connection for a given vielbein is defined by
\[ \omega^a_\mu = e^a_b \nabla_\mu e^b, \] (A.2.15)
and the Riemann tensor is given using the spin connection by
\[ R^{ab}_{\mu\nu} = \partial_\mu \omega^a_\nu - \partial_\nu \omega^a_\mu + \omega^a_\mu \omega^b_\nu - \omega^a_\nu \omega^b_\mu. \] (A.2.16)

The Ricci scalar is then
\[ R = e^a_c e^c_b R_{ab}. \] (A.2.17)

This convention yields a negative Ricci curvature for a round sphere.

The covariant derivatives for spinors are defined by
\[ \nabla_\mu \zeta = \partial_\mu \zeta + \frac{1}{4} \omega^a_\mu \Gamma_{ab} \zeta. \] (A.2.18)
A.3 Four dimension

Here \( \mu, \nu, \cdots \) represent directions of four dimensional space-time.

A.3.1 Spinor indices

Under the 4D rotation group \( SO(4) \simeq SU(2) \times SU(2) \), chiral and anti-chiral spinors transform as doublets of the first and the second \( SU(2) \), respectively. We use the indices \( \alpha, \beta, \cdots \) and \( \dot{\alpha}, \dot{\beta}, \cdots \) for chiral and anti-chiral spinors. These indices are raised and lowered by the antisymmetric invariant tensors \( \epsilon^{\alpha\beta}, \epsilon^{\dot{\alpha}\dot{\beta}}, \epsilon_{\alpha\beta}, \epsilon_{\dot{\alpha}\dot{\beta}} \) with nonzero elements

\[
\epsilon^{12} = -\epsilon^{21} = -\epsilon_{12} = \epsilon_{21} = 1. \tag{A.3.1}
\]

Pairs of undotted indices are suppressed when contracted in the up-left, down-right order, and similarly for contracted dotted indices in the down-left, up-right order.

We introduce a set of \( 2 \times 2 \) matrices \( (\sigma^a)_{\alpha\dot{\beta}} \) and \( (\bar{\sigma}^a)_{\dot{\alpha}\beta} \) with \( a = 1, \cdots, 4 \) satisfying standard algebras. In terms of Pauli’s matrices \( \tau^a \) they are given by

\[
\begin{align*}
\sigma^a &= -i\tau^a, & \bar{\sigma}^a &= i\tau^a, \\
\sigma^4 &= 1, & \bar{\sigma}^4 &= 1. \tag{A.3.2}
\end{align*}
\]

We also use \( \sigma_{ab} = \frac{1}{2}(\sigma_a\sigma_b - \sigma_b\sigma_a) \) and \( \bar{\sigma}_{ab} = \frac{1}{2}(\bar{\sigma}_a\bar{\sigma}_b - \bar{\sigma}_b\bar{\sigma}_a) \). Note that \( \sigma_{ab} \) is anti self-dual, namely \( \sigma_{ab} = -\frac{1}{2}\epsilon_{abcd}\sigma^{cd} \), while \( \bar{\sigma}_{ab} \) is self-dual.

A.3.2 \( SU(2)_R \) index

We use the tensors \( \epsilon^{AB}, \epsilon_{AB} \) with nonzero elements \( (A.3.1) \) to raise or lower \( SU(2)_R \) indices \( A, B \cdots \). We also require the Killing spinors to satisfy the reality condition

\[
(\xi_{\alpha A})^\dagger = \epsilon^{A\alpha} \xi_{A\beta} \epsilon^{\beta B} \xi_{\beta B}, \quad (\bar{\xi}_{\dot{\alpha} A})^\dagger = \epsilon^{A\dot{\alpha}} \bar{\xi}_{A\dot{\beta}} \epsilon^{\dot{\beta} B} \bar{\xi}_{\dot{\beta} B}. \tag{A.3.3}
\]

A.3.3 Clifford algebra

Gamma matrices \( \gamma^\mu \) which satisfy \( \{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu} \) are explicitly taken to be

\[
\gamma^\mu = \begin{pmatrix}
(\bar{\sigma}^{\dot{\alpha}\beta}) & (\sigma^\mu)_{\alpha\dot{\beta}}
\end{pmatrix},
\gamma^5 = \begin{pmatrix}
1 \\
-1
\end{pmatrix},
\gamma^6 \equiv 1_4. \tag{A.3.4}
\]

When they are embedded in six dimensional Clifford algebra, the gamma matrices in six dimension can be written as

\[
\Gamma^m = \begin{pmatrix}
\epsilon^{m\nu}\tilde{\gamma}^{m} & \epsilon^{m\nu} \epsilon^{-1}
\end{pmatrix} \tag{A.3.5}
\]

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where $\epsilon$ is written in antisymmetric invariant tensors $\epsilon^{\alpha\beta}$ as

$$\epsilon = -\begin{pmatrix} \epsilon^{\alpha\beta} \\ \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}.$$  \hspace{1cm} (A.3.6)

Then six dimensional spinor which belongs to fundamental representation of $SU(4)$ is projected to four dimensional spinors such that

$$\lambda^{(6d)}_A = \begin{pmatrix} \lambda_A^\alpha \\ \bar{\lambda}^\dot{\alpha}_A \\ 0_{(4)} \end{pmatrix}$$  \hspace{1cm} (A.3.7)
Appendix B

Appendix for section 5.3

B.1 Orthonormality of wavefunctions

Here we review the orthonormality of wavefunctions (5.3.13) in $SL(2, \mathbb{C})$

$$\Phi_{\eta i}^{(\eta)}(g) = \left( z_i, 1 \, g \left( \frac{z_i}{1} \right) \right)^{-2\eta_i}$$  \hspace{1cm} (B.1.1)

satisfy

$$-2\pi^3 \delta(g - g') = \int d\eta d\bar{z}^2 (1 - 2\eta)^2 \Phi_{\eta i}^{(\eta)}(g) \Phi_{1 - \eta}^{1 - \eta}(g').$$  \hspace{1cm} (B.1.2)

The discussion in more detail is summarized in [125] and the references in it.

$g$, the element of $SL(2, \mathbb{C})$ is in fact constrained to be hermite and an element of $H_3^+$ because of reality of $\phi(z)$. Let it be parametrized by three parameters such as

$$g = \left( e^\phi \sqrt{1 + \gamma \bar{\gamma}} \right) \frac{\partial}{\partial \gamma} e^{-\phi} \sqrt{1 + \gamma \bar{\gamma}} \partial \phi$$ \hspace{1cm} (B.1.3)

where $\phi$ is real parameter and differs from previous $\phi(z)$, which represents Liouville field. We rewrite this wavefunction in other form using the fact that they satisfy same differential equations generated by $SL(2, \mathbb{C})$ generators. Three generators are denoted in

$$L_{-1} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad L_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$ \hspace{1cm} (B.1.4)

They generate

$$J^a f(g) \equiv \frac{\partial}{\partial \tau} f(e^{-\tau L_a} e^{-\tau L_0^1})|_{\tau=0}, \quad \bar{J}^a f(g) \equiv \frac{\partial}{\partial \bar{\tau}} f(e^{-\tau L_a} e^{-\tau L_0^1})|_{\bar{\tau}=0}.$$ \hspace{1cm} (B.1.5)

In explicit form, they are written as

$$J^- = e^{-\phi} \sqrt{1 + \gamma \bar{\gamma}} \frac{\partial}{\partial \gamma} + \frac{1}{2} e^{-\phi} \frac{\bar{\gamma}}{\sqrt{1 + \gamma \bar{\gamma}}} \frac{\partial}{\partial \phi}$$

$$J^+ = -e^\phi \sqrt{1 + \gamma \bar{\gamma}} \frac{\partial}{\partial \gamma} + \frac{1}{2} e^\phi \frac{\gamma}{\sqrt{1 + \gamma \bar{\gamma}}} \frac{\partial}{\partial \phi}$$

$$J^0 = \frac{1}{2} \left( -\bar{\gamma} \frac{\partial}{\partial \gamma} + \gamma \frac{\partial}{\partial \bar{\gamma}} - \frac{\partial}{\partial \phi} \right)$$ \hspace{1cm} (B.1.6)

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Their complex conjugates are related by $(J^a)^\dagger = -\bar{J}^a$. Let us obtain the plane wave basis $\Psi(j, z, \bar{z})$ of $\mathcal{H}_J^a$ by diagonalizing three (maximum number of commuting differential operators) differential operators which are composed of these $\bar{J}^a, J^a$, and identify $\Psi(j, z, \bar{z})$ as $\Phi_\eta^2(g)$ where $\eta = -j$. The differential operators which are diagonalized by $\Psi(j, z, \bar{z})$ are taken as

\[
K^0 = (J^0 - \bar{J}^0) = -\gamma \frac{\partial}{\partial \gamma} + \bar{\gamma} \frac{\partial}{\partial \bar{\gamma}} = i \frac{\partial}{\partial \varphi}, \quad F^0 = i(J^0 + \bar{J}^0) = -i \frac{\partial}{\partial \varphi}
\]

\[
Q = (J^0)^2 - \frac{1}{2} J^+ J^- - \frac{1}{2} J^- J^+ = (\bar{J}^0)^2 - \frac{1}{2} \bar{J}^+ \bar{J}^- - \frac{1}{2} \bar{J}^- \bar{J}^+
\]

\[
= \frac{1}{4} \left( \gamma \frac{\partial}{\partial \gamma} - \bar{\gamma} \frac{\partial}{\partial \bar{\gamma}} \right)^2 + \frac{1}{4(1 + \gamma \bar{\gamma})} \frac{\partial^2}{\partial \varphi^2} + \frac{1}{2} \left( \gamma \frac{\partial}{\partial \gamma} + \bar{\gamma} \frac{\partial}{\partial \bar{\gamma}} \right) + (1 + \gamma \bar{\gamma}) \frac{\partial^2}{\partial \gamma^2} \tag{B.1.7}
\]

where $\gamma = e^{i\varphi} \sqrt{y}$ where $\varphi \in [-\pi, \pi], y \in \mathbb{R}$. The eigenvalue of $K^0$ is integer $n$, and of $F^0$ is real number $p$. We consider the eigenfunction $\Psi_{np}$ of $K^0, F^0$, and impose it the condition that diagonalize $Q$. In this basis, $Q$ is written in

\[
Q = -\frac{n^2}{4y} - \frac{p^2}{4(1 + y)} + \frac{\partial}{\partial y} y(1 + y) \frac{\partial}{\partial y}. \tag{B.1.8}
\]

We introduce the eigenfunction $f(y, j)$ of $Q$ whose eigenvalue is $j(j + 1)$ and $g(y, j)$ by ansatz that $f(y, j) = y^{\frac{1}{2}} (1 + y)^{\frac{n}{2}} g(y, j)$ satisfy $Qf(y, j) = j(j + 1)f(y, j)$. Substituting (B.1.8), it turns out

\[
x(1-x) \frac{\partial^2}{\partial x^2} g(x, j) + (1 + ip - x(2 + n + ip)) \frac{\partial}{\partial x} g(x, j) - \left( \frac{n + ip}{2} - j \right) \left( \frac{n + ip}{2} + j + 1 \right) g(x, j) = 0 \tag{B.1.9}
\]

where $x = 1 + y$. This equation is hypergeometric differential equation. Using one of the solution of this equation, $f(y, j)$ is $f(y, j) = y^{\frac{1}{2}} (1 + y)^{\frac{n}{2}} F\left(\frac{1}{2} |n| + ip - j, \frac{1}{2} |n| + ip + j + 1, 1 + |n|; -y \right)$ where $F(\alpha, \beta, \gamma; x)$ is hypergeometric function. However, since $n$ appears as $n^2$ in $Q$, requiring the regularity when $y \to 0$, this $f(y, j)$ should be

\[
f_{np}(y, j) = y^{\frac{|n|}{2}} (1 + y)^{\frac{n}{2}} F\left(\frac{1}{2} |n| + ip - j, \frac{1}{2} |n| + ip + j + 1, 1 + |n|; -y \right). \tag{B.1.10}
\]

This solution is written in

\[
f_{np}(y, j) = B_{np}(y) g^\dagger \left( 1 + \frac{1}{y} \right)^{\frac{i}{2}} F\left(\frac{1}{2} |n| + ip - j, -\frac{1}{2} (|n| - ip) - j, -2j; -\frac{1}{y} \right) \tag{B.1.11}
\]

where their coefficient functions are

\[
B_{np}(y) = \frac{\Gamma(1 + |n|) \Gamma(1 + 2j)}{\Gamma(\frac{1}{2} |n| + ip + 1 + j) \Gamma(\frac{1}{2} (|n| - ip) + 1 + j)}. \tag{B.1.12}
\]

which is useful to see the behavior in infinity

\[
f_{np}(y, j) = f_{np}(y, j) + f_{np}(y, -1 - j) \tag{B.1.13}
\]
where each term satisfy $Qf^{np} = j(j + 1)f^{np}$. Fourier transformation which changes this $(n, p, j)$ basis to $(z, \bar{z}, j)$ leads to the explicit form of wavefunction

$$\Psi(j, z, \bar{z}|g) = \frac{1}{(2\pi)^2} \sum_n \int \frac{dp}{\sqrt{B_{np}(-j - 1)}} e^{i\arg(z)}|z|^{2j - ip} \frac{e^{in\varphi - ip\varphi}}{B_{np}(-j - 1)} f^{np}(y, j).$$  \hspace{1cm} (B.1.14)

To identify this as (B.1.1)

$$\Phi(j, z, \bar{z}|g) = \frac{2j + 1}{\pi} \left[ (z^1)^p (\bar{z})^q \right]^{2j}$$

$$= \frac{2j + 1}{\pi} [p^2 e^{\phi} \sqrt{1 + \gamma^2} + r(\gamma e^{i\xi} + \gamma e^{-i\xi}) + e^{-\phi} \sqrt{1 + \gamma^2}]^{2j} \hspace{1cm} (B.1.15)$$

where $z = re^{i\xi}$, first let us confirm their proportionality. To do so, it is sufficient to check that they satisfy these three differential equations:

$$\begin{cases}
\left( \frac{\partial}{\partial \bar{z}} - \frac{\partial}{\partial \varphi} \right) \Phi(j, z, \bar{z}|g) = \frac{\partial}{\partial \varphi} \Psi(j, z, \bar{z}|g) = 0 \\
\left( -\frac{\partial}{\partial \phi} + r \frac{\partial}{\partial r} - 2j \right) \Psi(j, z, \bar{z}|g) = \left( -\frac{\partial}{\partial \varphi} + r \frac{\partial}{\partial r} - 2j \right) \Phi(j, z, \bar{z}|g) = 0 \\
Q \Phi(j, z, \bar{z}|g) = j(j + 1) \Phi(j, z, \bar{z}|g), \hspace{1cm} Q \Psi(j, z, \bar{z}|g) = j(j + 1) \Psi(j, z, \bar{z}|g) \hspace{1cm} (B.1.16)
\end{cases}$$

These are verified by direct calculation, and since both of them are regular at $y = 0$, we can conclude that $\Phi(j, z, \bar{z}|g) \propto \Psi(j, z, \bar{z}|g)$. The proportional constant does not depend on $g$, hence we evaluate it in $y \to \infty, \phi = \varphi = 0$. In this limit,

$$\Phi(j, z, \bar{z}|y = \infty, \phi = \varphi = 0) = \frac{2j + 1}{\pi} y^j |re^{i\xi} + 1|^{4j} \hspace{1cm} (B.1.17)$$

and

$$\Psi(j, z, \bar{z}|y = \infty, \phi = \varphi = 0) = \frac{y^j}{(2\pi)^2} \sum_n \int \frac{dp}{\sqrt{B_{np}(-j - 1)}} e^{2j - ip} e^{in\xi}. \hspace{1cm} (B.1.18)$$

Latter $\Psi(j, z, \bar{z}|y = \infty, \phi = \varphi = 0)$ looks as Fourier transformation of $\frac{B_{np}(j)}{B_{np}(-j - 1)}$. Actually, their inverse Fourier transformations to $(n, p)$ basis are

$$\Psi_{np} = \frac{1}{(2\pi)^2} \sum_{n'} \int \frac{dp'}{\sqrt{B_{np'}(-j - 1)}} e^{-in\xi + in'\xi'} e^{-ip' + ip - 2} = \frac{B_{np}(j)}{B_{np}(-j - 1)} \hspace{1cm} (B.1.19)$$

and

$$\Phi_{np} = \frac{2j + 1}{\pi} \int d^2 z \ z^n |z|^{-2j - 2 + ip - n} |z + 1|^{4j}. \hspace{1cm} (B.1.20)$$

This integral (B.1.20) is directly performed, and equated to (B.1.19) so that it can be concluded to $\Phi(j, z, \bar{z}|g) = -\Psi(j, z, \bar{z}|g)$. Submitting $z$ which is separated into real and imaginary part $z = z_1 + iz_2$,

$$\Phi_{np} = \frac{2j + 1}{\pi} \int dz_1(z_1 + iz_2)^n (z_1^2 + z_2^2)^{-j - 1 + \frac{3}{2} - n} ((z_1 + 1)^2 + z_2^2)^{2j}. \hspace{1cm} (B.21)$$
This integral is computed straightforwardly as in [132]. Since \( n \in \mathbb{Z} \), singular points are on the imaginary axis of \( z_2 \), and we can deform the integral contour of \( z_2 \) to almost on the imaginary axis. Using \( 0 < \epsilon \ll 1 \), this deformation is established by \( z_2 \mapsto i\epsilon^{-2i\epsilon}z_2 \), and submitting \( z_\pm = z_1 \pm z_2 \), the integral in the first order of \( \epsilon \) become

\[
\Phi_{np} = -\frac{2j + 1}{2\pi} \int dz_+ dz_- (z_+ + i\epsilon(z_+ - z_-))^n [(z_+ - i\epsilon(z_+ - z_-))(z_- + i\epsilon((z_+ - z_-))]^{-j-1+\frac{j}{2}p-rac{n}{2}} \times (z_+ + 1 - i\epsilon(z_+ - z_-))^{2j}(z_- + 1 + i\epsilon(z_+ - z_-))^{2j} \\
= -\frac{2j + 1}{2\pi} \int_{-\infty}^{\infty} dz_+ (z_+ - i\epsilon(z_+ - z_-))^{-j-1+\frac{j}{2}p-rac{n}{2}} (z_- + 1 - i\epsilon(z_+ - z_-))^{2j} \\
\times \int_{-\infty}^{\infty} dz_- (z_+ + i\epsilon(z_+ - z_-))^{-j-1+\frac{j}{2}p+\frac{n}{2}} (z_- + 1 + i\epsilon(z_+ - z_-))^{2j}.
\]

(B.1.22)

In \( z_- \) integral, each factor has a singularity at \( z_- = 0 \) and \( z_- = -1 \) respectively. According to the value of \( z_+ \), relative position of integral contour and these singularities are different as follows.

- 1. When \( z_+ \in (-\infty, -1) \), regarding the singularity at \( z_- = 0 \), \( \epsilon(z_+ - z_-) = \epsilon z_+ < 0 \) thus contour passes under the singularity, and regarding the singularity at \( z_- = -1 \), \( \epsilon(z_+ - z_-) = \epsilon(z_+ + 1) < 0 \) thus contour passes under the singularity. The contour go through underneath both of two singularities, we can add contour at infinity to enclose the contour without taking singularities inside so as to see that this region does not contribute to the integral.

- 2. When \( z_+ \in (0, \infty) \), regarding the singularity at \( z_- = 0 \), \( \epsilon(z_+ - z_-) > 0 \) thus contour passes above the singularity, and regarding the singularity at \( z_- = -1 \), \( \epsilon(z_+ - z_-) > 0 \) thus contour passes above the singularity. The contour go through over both of two singularities, we can add contour at infinity to enclose the contour without taking singularities inside so as to see that this region does not contribute to the integral.

- 3. When \( z_+ \in (-1,0) \), regarding the singularity at \( z_- = 0 \), \( \epsilon(z_+ - z_-) > 0 \) thus contour passes above the singularity, and regarding the singularity at \( z_- = -1 \), \( \epsilon(z_+ - z_-) < 0 \) thus contour passes under the singularity. This contour is enclosed by circulating at infinity in upper half plane, such that the contour can be deformed to pick a singularity only at \( z_- = 0 \). This contour is illustrated in Fig.B.1.
Then
\[
\Phi_{np} = \frac{2j + 1}{\pi} B(2j + 1, -j + \frac{i}{2}p - \frac{n}{2}) B(-j - \frac{i}{2}p - \frac{n}{2}, -j + \frac{i}{2}p - \frac{n}{2}) \sin \pi(-j + \frac{i}{2}p - \frac{n}{2})
\]  
\begin{equation}
(B.1.23)
\end{equation}

where \(B(\alpha, \beta)\) is beta function which satisfies \(B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}\). By substituting this, it turns out that
\[
\Phi_{np} = -\frac{B_{np}(j)}{B_{np}(-j - 1)} = -\Psi_{np}
\]  
\begin{equation}
(B.1.24)
\end{equation}

where \(B_{np}(j)\) is defined in previous (B.1.12). This result imply \(\Phi(j, z, \bar{z}|g) = -\Psi(j, z, \bar{z}|g)\), and to justify (B.1.2) we can use \(\Psi(j, z, \bar{z}|g)\) as wavefunction of \(\mathcal{H}_3^+\). Using reality of \(\Phi(j, z, \bar{z}|g)\) and explicit form (B.1.14),
\[
\int dz^2 dz\Psi(j, z, \bar{z}|g)\Psi(-1 - j, z, \bar{z}|g') = \frac{1}{(2\pi)^2} \sum_n \int dp dq e^{i\nu(\nu'-\nu) + ip(\nu'-\nu)} B_{np}(-1 - j) B_{np}(j) f^{np}(y, j) f^{np}(y', -1 - j)
\]  
\begin{equation}
(B.1.25)
\end{equation}

In following, we interpret this formula as Fourier transformation of orthonormal relation in \((n, p)\) basis. The eigenvalue \(q = j(j + 1)\) of \(Q\) is solved to
\[
q = \begin{cases} 
-\frac{1}{2} + \sqrt{\frac{1}{4} + q} & (q \in \mathbb{C} \setminus [-\infty, -\frac{1}{4}]) \\
-\frac{1}{2} + i\sqrt{-\frac{1}{4} - q} & (q \in [-\infty, -\frac{1}{4}])
\end{cases}
\]

To formulate the spectral projection of this branch cut, we first compose a resolvent of \(Q\). Denoting polynomials \(p(y) = y(1 + y)\) and \(r(y) = \frac{y^2}{4y} + \frac{p^2}{4(1+y)}\), it can be written \(Q = \partial_y p(y)\partial_y - r(y)\). Let us take
\[
R(y, y', q) = N^{-1}[\theta(y - y') f^{np}(y, q) f^{np}(y', q) + \theta(y' - y) f^{np}(y, q) f^{np}(y', q)]
\]  
\begin{equation}
(B.1.26)
\end{equation}

where \(N\) is normalization factor which is evaluated in following and \(f^{np}(y, j)\) is introduced in (B.1.11). Direct calculation shows
\[
(Q - q)R(y, y', q) = N^{-1} p(y)[f^{np}(y, q)\partial_y f^{np}(y, q) - f^{np}(y, q)\partial_y f^{np}(y, q)]\delta(y - y').
\]  
\begin{equation}
(B.1.27)
\end{equation}

Taking into account \((Q - q)f^{np}(y, q) = (Q - q)f^{np}(y, q) = 0, \quad \partial_y (p(y)[f^{np}(y, q)\partial_y f^{np}(y, q) - f^{np}(y, q)\partial_y f^{np}(y, q)]) = 0.\) \begin{equation}
(B.1.28)
\end{equation}

This means the coefficient of right hand side of (B.1.27) does not depend on \(y\), and when \(N\) is evaluated as this coefficient at specific \(y\), \((Q - q)R(y, y', q) = \delta(y - y').\) For instance, we evaluate \(N\) in \(y \to \infty\) as
\[
N = \lim_{y \to \infty} p(y)[f^{np}(y, q)\partial_y f^{np}(y, q) - f^{np}(y, q)\partial_y f^{np}(y, q)] = (2j + 1) B_{np}(j) B_{np}(-j - 1).\]  
\begin{equation}
(B.1.29)
\end{equation}

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By substituting above, spectral projection

\[ P_{[-\infty, -\frac{1}{2}]} = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \left[ \int_{-\infty}^{-\frac{1}{2}} dq R(q + i\epsilon) - \int_{-\infty}^{-\frac{1}{2}} dq R(q - i\epsilon) \right] \]  

(B.1.30)

of branch cut of \( j \) that \( j(q + i\epsilon) = j(q), j(q - i\epsilon) = -1 - j(q) \) turns out to be

\[
\delta(y - y') = \frac{1}{2\pi i} \int_{-\infty}^{-\frac{1}{2}} dq \frac{1}{2j + 1} B_{np}(j) B_{np}(-j - 1) \times \\
\left[ \theta(y - y')(f_{np}(y, j(q)) + f_{np}(y, -j(q) - 1)) f_{np}(y, j(q)) + \theta(y' - y)(f_{np}(y', j(q)) + f_{np}(y', -j(q) - 1)) f_{np}(y, j(q)) \right] \\
= \frac{1}{2\pi i} \int_{P_+} dq f_{np}(y, j)f_{np}(y', j) B_{np}(j) B_{np}(-j - 1). 
\]  

(B.1.31)

where \( P_+ = -\frac{1}{2} + i\mathbb{R} \). This is an orthonormal relation in \((n, p)\) basis. Regarding this result, (B.1.25) become

\[
\int dz^2 dz d\Psi(j, z, \bar{z}|g)\Psi(1 - j, z, \bar{z}|g') = -2\pi i \delta(y - y')\delta(\varphi - \varphi')\delta(\phi - \phi').
\]  

(B.1.32)

The integral measure of \( g \in SL(2, \mathbb{C}) \) where

\[
g = \begin{pmatrix}
\alpha \\ \beta \\
\gamma \\ \delta
\end{pmatrix}
\]  

is

\[
dg = \delta(\alpha\delta - \beta\gamma - 1)d\alpha d\beta d\gamma d\delta = \frac{1}{\alpha}d\alpha d\beta d\gamma.
\]  

(B.1.33)

Parameters in (B.1.3) are also useful to write this measure: \( dg = d\phi d\gamma d\bar{\gamma} \). Using \((y, \varphi)\), which is defined in \( \gamma = \sqrt{\varphi} e^{i\varphi} \), it becomes \( dg = -id\phi dy d\varphi \). Thus we conclude that

\[
\Phi(z, \eta, g) \equiv \frac{1}{\sqrt{2\pi} \frac{\sqrt{\varphi} e^{i\varphi}}{2}} \Phi(j, z, \bar{z}|g) = -\frac{1 + 2j}{\sqrt{2\pi} \frac{\sqrt{\varphi} e^{i\varphi}}{2}} \left( \begin{array}{c} z \\ 1 \end{array} \right) \left( \begin{array}{c} j \\ 1 \end{array} \right)^{2j} 
\]  

(B.1.34)

where we identify \( \eta = -j \) satisfy

\[
\delta(g - g') = \int d\eta dz^2 \Phi(z, \eta, g)\Phi(z, 1 - \eta, g').
\]  

(B.1.35)

In (5.3.12), we defined wavefunctions by

\[
\Phi^\eta_z(g) = \left( \begin{array}{c} z \\ 1 \end{array} \right) \left( \begin{array}{c} j \bar{z} \\ 1 \end{array} \right)^{-2\eta}
\]  

(B.1.36)

This definition is different from the wavefunction defined at (B.1.35) by coefficient which depends only on \( \eta \). This difference is included in normalizations of correlation functions. For example, above orthonormality become

\[
-2\pi^2 \delta(g - g') = \int d\eta dz^2 (1 - 2\eta)^2 \Phi^\eta_z(g)\Phi^1_{-\eta}(g').
\]  

(B.1.37)
We can use above formulas to derive three points function (5.3.17). Let us parametrize

\[ g = \begin{pmatrix} e^{\phi} & -e^{\phi} \bar{\eta} \\
-e^{\phi} \gamma & e^{\phi} \bar{\eta} + e^{-\phi} \end{pmatrix} \]  

(B.1.39)

to write

\[ \langle \Phi_{z_1} \Phi_{z_2} \Phi_{z_3} \rangle = \int d\phi d\gamma d\bar{\gamma} e^{2\phi} \prod_{i=1}^{3} \left[ e^{\phi}|z_i - \gamma|^2 + e^{-\phi} \right]^{-2\eta_i}. \]  

(B.1.40)

It is known that correlation functions in CFT are split into the coordinates dependences part and conformal block part. Here let \( C(\eta_1, \eta_2, \eta_3) \) stands for conformal block part. Then

\[ \langle \Phi_{z_1} \Phi_{z_2} \Phi_{z_3} \rangle = |z_{12}|^{-2(\eta_1 + \eta_2 - \eta_3)} \langle z_{23} |^{-2(\eta_2 + \eta_3 - \eta_1)} |z_{31}|^{-2(\eta_3 + \eta_1 - \eta_2)} C(\eta_1, \eta_2, \eta_3) \]  

(B.1.41)

where \( z_{ij} = z_i - z_j \), and in explicit form,

\[ C(\eta_1, \eta_2, \eta_3) = \int d\phi d\gamma d\bar{\gamma} e^{2(1-m-\eta_2-\eta_3)} \left[ |\gamma|^2 + e^{-2\phi} \right]^{-2\eta_1} \left[ (1-\gamma)^2 + e^{-2\phi} \right]^{-2\eta_2}. \]  

(B.1.42)

Substituting \( e^{2\phi} = u, \gamma = x + iy \) and using the formula \( x^{-2\eta_i} = \frac{1}{\Gamma(2\eta_i)} \int ds e^{-sx}s^{2\eta_i-1} \), it becomes

\[ C(\eta_1, \eta_2, \eta_3) = \frac{1}{\Gamma(2\eta_1)\Gamma(2\eta_2)} \int du dx dy ds dt u^{-(2-\eta_1-\eta_2-\eta_3)} s^{2\eta_1-1} t^{2\eta_2-1} e^{-(s+t)y^2 -(s+t)(x-\frac{u}{s+t})^2 -(s+t)u + \frac{ux}{s+t}} \]
\[ = \frac{\pi}{2} \int du ds dt \frac{1}{s+t} u^{-(2-\eta_1-\eta_2-\eta_3)} s^{2\eta_1-1} t^{2\eta_2-1} e^{-\frac{xt}{s+t}} (s+t)u. \]  

(B.1.43)

Furthermore, we take \( s = \ell \theta, t = \ell(1-\theta) \), then integral about \( u \) are represented by \( \Gamma \) function such that

\[ C(\eta_1, \eta_2, \eta_3) = \frac{\pi \Gamma(\eta_1 + \eta_2 + \eta_3 - 1)}{2\Gamma(2\eta_1)\Gamma(2\eta_2)} \int d\ell d\theta \ell^{\eta_1+\eta_2-\eta_3-1} \theta^{2\eta_2 - 1} (1-\theta)^{2\eta_2 - 1} e^{-\ell\theta(1-\theta)} \]
\[ = \frac{\pi \Gamma(\eta_1 + \eta_2 + \eta_3 - 1)}{2\Gamma(2\eta_1)\Gamma(2\eta_2)} \Gamma(\eta_1 + \eta_2 - \eta_3) B(\eta_1 - \eta_2 + \eta_3, -\eta_1 + \eta_2 + \eta_3) \]
\[ = \frac{\pi \Gamma(\eta_1 + \eta_2 + \eta_3 - 1) \Gamma(-\eta_3 + \eta_2 + \eta_1) \Gamma(-\eta_2 + \eta_1 + \eta_3) \Gamma(-\eta_1 + \eta_2 + \eta_3)}{2\Gamma(2\eta_1)\Gamma(2\eta_2)\Gamma(2\eta_3)}. \]  

(B.1.44)

This is conformal block of the three points function (5.3.17).
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—君のやっているのは物理じゃなくて算数だよ。

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