$n + n + \alpha \rightarrow ^{6}\text{He} + \gamma$ reaction by effective field theory approach

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The astrophysics reaction $n + n + \alpha \rightarrow ^{6}\text{He} + \gamma$ is studied by the effective field theory approach. For this purpose, as a first step, we introduce the Faddeev equation of the particle–dimer scattering amplitudes in the $2 \times 2$ cluster configuration space using the formalism based on the halo effective field theory in the channel $J^\pi = 1^{-}$. In the next step, the normalized $^{6}\text{He}$ wave function is obtained by solving the homogenous part of the particle–dimer scattering amplitudes in the channel $J^\pi = 0^{+}$. Then the electric dipole transition amplitude of reaction $n + n + \alpha \rightarrow ^{6}\text{He} + \gamma$ is calculated using the particle–dimer scattering amplitudes and the $^{6}\text{He}$ wave function. Finally, we report our results for the electric dipole strength function distribution of the $^{6}\text{He}$ halo nucleus at the leading order and compare them with experimental data and other theoretical results.

Subject Index D00, D05, D06

1. Introduction

A reliable assessment of the electromagnetic radiative recombination of two and three particles from the continuum to a bound state has been one of the major challenges of astrophysics and nuclear theory. In recent years, effective field theory (EFT) has been introduced as a well-defined scheme to construct the interactions and to evaluate the observables. In recent years, pionless effective field theory (EFT(π/)) has been applied quite successfully to the two-body reactions of the early Big Bang stages of the universe or nucleosynthesis reactions such as the $n + p \rightarrow d + \gamma$ [1–3] and $n + d \rightarrow ^{3}\text{H} + \gamma$ [4–6] processes. Study of the cross section for $n + p \rightarrow d + \gamma$ has also recently been initiated within an ab initio calculation using EFT(π) and lattice quantum chromodynamics [7]. Rapid progress has been made in the theoretical study of halo nuclei and low-energy two-body astrophysics reactions like $^7\text{Li}(n, \gamma)^8\text{Li}$ [8–10], $^7\text{Be}(p, \gamma)^8\text{B}$ [11], $^{14}\text{C}(n, \gamma)^{15}\text{C}$ [12], and $^{16}\text{O}(n, \gamma)^{17}\text{F}$ [13] using halo EFT.

A study of the $n + n + \alpha$ scattering S-matrix with three free particles in the incoming and outgoing channels has been introduced based on the Faddeev approach. This formal presentation has the capacity to numerically calculate the resonance structure of the $^6\text{He}$ system and neutron–neutron–alpha ($nna$) capture process to the $^6\text{He}$ ground state [14].

The investigation of the three-body $nna$ system by halo EFT has been the subject for describing the $^6\text{He}$ ground state using the Faddeev approach and constructing the $^6\text{He}$ wave function [15]. In contrast, the ground state of $^6\text{He}$ using interactions derived from halo EFT has been studied with the $^6\text{He}$ bound state taken from the Gamow shell model basis [16]. The continuum spectrum of $^6\text{He}$
as an \(nn\alpha\) system has also been calculated within an \textit{ab initio} approach using a no-core shell model combined with the resonating group method [17], Gamow shell model studies [18], and effective interaction hyperspherical harmonics [19].

The gravitational collapse of a star makes a higher-temperature (7 \(\sim\) 10 GK) and neutron-rich environment where rapid neutron capture (r-process) nucleosynthesis can occur. Among these processes, the two-particle radiative capture processes \(n + n + \alpha \rightarrow ^6\text{He} + \gamma\) and \(n + \alpha + \alpha \rightarrow ^9\text{Be} + \gamma\) are relevant, and play a crucial role. These reactions contribute to bridging the \( A = 5 \) and \( A = 8 \) gaps. Determination of their reaction rates is essential for the initial condition of the r-processes, namely, the amounts of seed elements and remaining neutrons [20].

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The three- and four-body recombinations such as the radiative capture reactions \(n + n + \alpha \rightarrow ^6\text{He} + \gamma\), \(n + \alpha + \alpha \rightarrow ^9\text{Be} + \gamma\), and reactions like \(\alpha + \alpha + n + n \rightarrow ^6\text{He} + \alpha\) and \(\alpha + \alpha + \alpha + n \rightarrow ^9\text{Be} + \alpha\), are the significant reactions that are competing under conditions of extreme densities and temperatures. The reaction rate and the production rate in these recombination reactions are calculated using the hyperspherical adiabatic method [21,22]. An analytical transformed harmonic oscillator basis method has also been used to study three-body systems and the rate of the \(^6\text{He}\) radiative capture reaction [23]. In addition, the Coulomb excitation of the two-neutron \(^6\text{He}\) halo nucleus has been studied with the calculation of the electric dipole (E1) strength function within a three-body model [24]. The E1 strengths are also studied in hyperspherical harmonics [19] and correlated Gaussian [25] based calculations.

Based on the application of EFT to few-nucleon systems, one can apply the EFT for the analysis of larger nuclear systems in a reliable, model-independent, and systematically order-by-order improvable manner. An ideal candidate to consider is the amplitude for the capture process of \(n + n + \alpha \rightarrow ^6\text{He} + \gamma\) in order to specify the reaction rate, cross section, and strength function by the EFT approach. The novelty of this work is that for the first time we calculate the electromagnetic radiative recombination of three particles from the continuum to a bound state by the EFT approach. In the absence of experimental data for the reaction rate and the cross section for \(n + n + \alpha \rightarrow ^6\text{He} + \gamma\), we have decided to calculate the E1 strength function and compare it with the only available experimental data [26]. Other theoretical models have also been used to present a reliable assessment [24,27–29] of this work. Therefore, the purpose of the present paper is to calculate the dominant E1 amplitude and strength function distribution by the halo EFT formalism for the \(n + n + \alpha \rightarrow ^6\text{He} + \gamma\) reaction.

### 2. Strong interactions for the \(nn\alpha\) system

The degrees of freedom for the \(nn\alpha\) system in the halo EFT are the neutrons and the alpha core. In this regime, the neutrons and the alpha core momenta are as a low-momentum scale \(Q\) and the high-momentum parameters are scaled by \(\hat{\Lambda} \sim m_\pi \sim \sqrt{m_\pi E_\alpha}\), with \(m_\pi\) as the pion mass, \(m_\alpha\) and \(E_\alpha = 20.21\) MeV denote the mass and the excitation energy of the alpha core, respectively.

In this section, we briefly explain the power counting that we use for the neutron–neutron (\(nn\)) and neutron–alpha (\(n\alpha\)) interactions that are considered for our \(nn\alpha\) calculation. The \(nn\) interaction is dominated by an S-wave virtual state. According to the values of the effective-range parameters of this state, we consider the inverse of the dineutron scattering length, \(1/(a_\pi = -18.7)\) fm\(^{-1}\), as a low-momentum scale \(Q\), and the inverse of the effective range of the \(^1\text{S}_0\ nn\) state, \(1/(r_0 = 2.75)\) fm\(^{-1}\), is related to the high-momentum scale \(\hat{\Lambda}\). The effective range expansion (ERE) for the S-wave
dineutron system at low energies is given by

\[ k \cot \delta_0 = -\frac{1}{a_0} + \frac{r_0}{2} k^2 + \cdots. \]  

(1)

As shown in Eq. (1), we deal with the effective range of the \( ^1S_0 \) \( nn \) system at the next-to-leading order (NLO). According to Eq. (1), the leading-order (LO) \( nn \) scattering T-matrix can be written as

\[ T_{nn}^{\text{ERE}}(k) = \frac{4\pi}{m_n} \frac{1}{\frac{1}{a} - i k}. \]  

(2)

where \( m_n \) is the neutron mass. Using Eq. (2), the position of the \( ^1S_0 \) \( nn \) virtual state at LO is determined by

\[ k_0 = i \gamma_s, \]  

(3)

where \( \gamma_s = \frac{1}{a} \).

For the \( n\alpha \) system at low energies, only S- and P-wave interactions are significant. Therefore, the possible partial waves in the \( n\alpha \) system are the \( ^2S_1 \), \( ^2P_1 \), and \( ^2P_3 \) states. In the following, we use the \( l\pm = 0+, 1-, \) and \( 1+ \) notations for the \( ^2S_1 \), \( ^2P_1 \), and \( ^2P_3 \) \( n\alpha \) states, respectively. Bertulani et al. introduced a power counting as [30]

\[ \frac{1}{a_{1+}} \sim Q^3, \quad \frac{r_{1+}}{2} \sim Q, \quad \frac{P_{1+}}{4} \sim \frac{1}{\Lambda}, \]  

all remaining effective-range parameters \( \sim O(\bar{\Lambda}) \),

(4)

where

\[ a_{1+} = -62.95 \text{ fm}^3, \quad r_{1+} = -0.88 \text{ fm}^{-1}, \quad P_{1+} = -3.00 \text{ fm} \]  

(5)

are the scattering length, the effective range, and the shape parameter of the partial wave 1+ in the \( n\alpha \) system [31]. Using this power counting at LO, a neutron and an alpha core interact in channel \( ^2P_3 \), which has a resonance corresponding to a shallow P-wave state [31]. As shown in Ref. [30], other effective-range parameters enter at the higher orders pertubatively; e.g., the shape parameter of the \( ^2P_3 \) state and the scattering length of the \( ^2S_1 \) partial wave have initially been considered at NLO. Finally, the non-zero contribution in the 1− partial wave appears at N3LO [30].

In contrast with Ref. [30], Bedaque et al. have suggested a different power counting [32]. The differences between these two power countings rely on the scaling of the scattering length and effective range parameters of the \( ^2P_3 \) partial wave. Bedaque et al. considered the following relations:

\[ \frac{1}{a_{1+}} \sim Q^2 \bar{\Lambda}, \quad r_{1+} \sim \bar{\Lambda}, \quad P_{1+} \sim \frac{1}{\bar{\Lambda}}, \]  

all remaining effective-range parameters \( \sim O(\bar{\Lambda}) \),

(6)

By this power counting, the effective-range parameters \( a_{1+}, r_{1+}, \) and \( a_{0+} \) (the scattering length of the partial wave 0+ in the \( n\alpha \) system) contribute in the \( n\alpha \) T-matrix at LO [32].

In the present work, we use the power counting suggested by Bertulani et al. in Ref. [30]. So, the contribution of the 1+ \( n\alpha \) state only makes the two-body interaction of the neutron and alpha
particles in our calculations. As shown in Fig. 4 of Ref. [30], when considering only the $^2P_\frac{3}{2}$ state as the dominant contribution at LO, the LO EFT parametrization of the $n\alpha$ amplitude reproduces $n\alpha$ scattering data exactly up to the neutron kinetic energy 1 MeV. Above this energy, considering the NLO corrections which relate to the $0^+$ scattering length and the $1^+$ shape parameter contributions generates good agreement with the full phase shift analysis. We also consider the values of the effective range parameters in Eq. (6) as the two-body input parameters in the calculation, and we discuss the effects of these considerations below.

As mentioned, the $n\alpha$ system has a shallow P-wave resonance in the $1^+$ channel. To obtain the position and width of this resonance in the $1^+$ $n\alpha$ state we should obtain the pole structure of the scattering T-matrix of this partial wave. Using the ERE expansion of the P-wave state,

$$k^3 \cot \delta_{1^+} = -\frac{1}{a_{1^+}} + \frac{r_{1^+}}{2} k^2 + \cdots,$$

the $n\alpha$ scattering amplitude in the $1^+$ partial wave is given by

$$T_{n\alpha}^{ERE}(k, p = k) = \frac{6\pi}{\mu} \frac{k^2 \cos \theta_{kp}}{-\frac{1}{a_{1^+}} + \frac{r_{1^+}}{2} k^2 + \cdots - ik^3},$$

where $\mu$ denotes the reduced mass of the neutron–alpha system; $k$ and $p$ indicate the incoming and outgoing center-of-mass (c.m.) momentum of the $n\alpha$ system; and $\theta_{kp}$ is the angle between them. According to the values of Eq. (5) we have three complex-conjugated poles in the $1^+$ state of the $n\alpha$ system, as

$$\kappa_1 = i\gamma_1, \quad \kappa_{\pm} = i(\bar{\gamma}_1 \pm i\bar{\gamma}_2),$$

where $\gamma_1 = 99$ MeV, $\bar{\gamma}_1 = -6$ MeV, and $\bar{\gamma}_2 = 34$ MeV [30]. We can see that the position of the resonance is at $E_0 = \frac{\bar{\gamma}_1^2 + \bar{\gamma}_2^2}{2\mu} = 0.8$ MeV. Also, $\gamma_1$ makes an unphysical bound state corresponding to a non-existent real $^3$He bound state. The binding energy with respect to $\gamma_1$ is out of range of the validity of our EFT, and so no discussion can strictly be made about it.

The Lagrangians which introduce the strong interactions of the $n\alpha\alpha$ system at LO can be written as

$$\mathcal{L} = n^+ \left(i\partial_0 + \frac{\vec{\nabla}^2}{2m_n}\right)n + \phi^+ \left(i\partial_0 + \frac{\vec{\nabla}^2}{2m_\alpha}\right)\phi + d_s^+ \left[\Delta_3 - c_{0\lambda}(i\partial_0 + \frac{\vec{\nabla}^2}{4m_n} + \frac{\gamma_\lambda^2}{m_n})\right]d_s$$

$$- \frac{\gamma}{\sqrt{8}} \frac{1}{2} \left[\eta_1 t^+ \left[\Delta_{1^+} - i\partial_0 + \frac{\vec{\nabla}^2}{2(m_n + m_\alpha)}\right]t + \frac{\gamma_1 + \gamma_2}{2} t^+ \left[n^+ \vec{S}\phi - (\vec{\nabla}n)\phi + h.c. - r t^+ \vec{S}^\dagger \cdot [\vec{\nabla}(n\phi) + h.c.]\right]\right],$$

where $n$, $\phi$, $d_s$, and $t$ are the two-component spinor field of the neutron, the bosonic alpha core field, the dineutron auxiliary field, and the $n\alpha$ auxiliary field in the partial wave channel $l^+ = 1^+$, respectively. $\eta_1$ is equal to $\pm 1$, $\gamma_1^2 = 4\pi/m_n$, and $r = (m_\alpha - m_n)/(m_\alpha + m_n)$. $\sigma_2$ is the Pauli matrix which acts on the spin space of the neutron. In Eq. (10), the $S_i$ are the $2 \times 4$ matrices which connect the states with the total angular momentums $\frac{1}{2}$ and $\frac{3}{2}$. The $S_i$ satisfy the following relations:

$$S_i S_j^\dagger = \frac{2}{3} \delta_{ij} - \frac{i}{3} \epsilon_{ijk} \sigma_k,$$

$$S_i^\dagger S_j = \frac{3}{4} \delta_{ij} - \frac{i}{6} \left[J_{ij}^{3/2}, J_j^{3/2}\right] + \frac{i}{6} \epsilon_{ijk} J_k^{3/2},$$

$$4/24$$
where the $J^{3/2}_s$ are the generators of the $J = \frac{3}{2}$ representation of the rotation group. The parameters $\Delta_s$ and $c_{0s}$ are given by matching the EFT($\pi$) $nn$ scattering amplitude to the ERE scattering amplitude of two non-relativistic nucleons around the $i\gamma_s$ [33]. Also, the $g_{1+}$ and $\Delta_{1+}$ are given as a function of the scattering length and the effective range of channel 1+ by fitting the LO EFT $n\alpha$ scattering amplitude to Eq. (8) [30],

$$g_{1+}^2 = -\frac{6\pi}{\mu^2 r_{1+}}, \quad \Delta_{1+} = \frac{1}{\mu a_{1+} r_{1+}},$$

(12)

with $\mu$ as the reduced mass of the $n\alpha$ system. According to the mentioned power counting, the LO propagators of the auxiliary field $d_s$ and $t$ are given by

$$iD_s(p_0,p) = \frac{i}{\gamma_s - \sqrt{p^2 - m_n p_0 - i\epsilon}},$$

(13)

$$iD_{1+}(p_0,p) = \frac{i\eta_{1+} \delta_{\tilde{\beta} \tilde{\alpha}}}{p_0 - \frac{p^2}{2(m_\alpha + m_\beta)} - \Delta_{1+} - i \frac{\eta_{1+} + 2 g_{1+}}{6\pi} (2\mu) \frac{2}{3} \frac{p_0}{p^2} \frac{1}{(m_\alpha + m_\beta)} + i\epsilon}.$$  (14)

where $\tilde{\alpha}$ and $\tilde{\beta}$ are the incoming and outgoing spin components of the dimeron $t$. We note that $\delta_{\tilde{\alpha} \tilde{\beta}}$ is a $4 \times 4$ unit matrix because $J = \frac{3}{2}$ for the dimeron $t$.

3. Particle–dimer scatterings in the $n\alpha$ system with $J^\pi = 1^-$

We investigate the $n\alpha$ system using the Lagrangian (1) with the dimeron auxiliary fields. Two neutrons and one alpha core can make two different dimers at the leading order: the $d_s$ (dineutron) and $t$ ($^5$He). Thus, we have four possible transitions from particle–dimer to particle–dimer states:

$$n + t \rightarrow n + t, \quad n + t \rightarrow \phi + d_s,$$

$$\phi + d_s \rightarrow n + t, \quad \phi + d_s \rightarrow \phi + d_s.$$  (15)

The Feynman diagrams giving the amplitudes of these four unphysical transitions in the channel $J^{\pi} = 1^-$ at LO are shown in Fig. 1. The dashed lozenge, square, triangle, and hexagon denote the $\phi + d_s \rightarrow \phi + d_s, n + t \rightarrow \phi + d_s, \phi + d_s \rightarrow n + t$, and $n + t \rightarrow n + t$ scattering amplitudes in the channel $J^{\pi} = 1^-$, respectively. We have two $\phi + d_s$ and $n + t$ systems, so it is convenient to write all the operators as

$$\begin{pmatrix} d_s^\dagger \phi^t \\ n_\beta^t \ t_\beta^t \end{pmatrix} \begin{pmatrix} \mathcal{O}(\phi + d_s \rightarrow \phi + d_s)_{\tilde{\alpha}}^\beta \\ \mathcal{O}(n + t \rightarrow \phi + d_s)_{\tilde{\alpha}}^\beta \\ \mathcal{O}(\phi + d_s \rightarrow n + t)_{\tilde{\alpha}}^\beta \\ \mathcal{O}(n + t \rightarrow n + t)_{\tilde{\alpha}}^\beta \end{pmatrix} \begin{pmatrix} d_s \phi \\ n_\alpha t_{\tilde{\alpha}} \end{pmatrix}. \quad \text{(16)}$$

On the other hand, each operator is presented as a $2 \times 2$ matrix which carries all the spin indices of the neutron and the dimeron $t$. So, all operators act on a tensor spin-cluster space. We emphasize that the $\alpha$ ($\tilde{\alpha}$) and $\beta$ ($\tilde{\beta}$) indices indicate the spin $\frac{1}{2}$ ($\frac{3}{2}$) components of the neutron (dimeron $t$). The contributions of the diagrams in Fig. 1 are obtained using the coupled equations. These coupled equations can be converted to one $2 \times 2$ Faddeev equation using the introduced cluster configuration space. By projecting the particle–dimer-to-particle–dimer scattering amplitude into the $n\alpha$ channel.
Fig. 1. Feynman diagrams of all possible particle–dimer scatterings in the channel $J^\pi = 1^-$ of the $nn\alpha$ system at LO. The solid and dashed lines represent the neutron and the alpha particles, respectively. The double solid line is the dineutron auxiliary field, and the double solid-dashed line indicates the auxiliary field $t$ (neutron+alpha in channel 1+). $t_{aa}^{(1^-)}$, $t_{na}^{(1^-)}$, $t_{nn}^{(1^-)}$, and $t_{nn}^{(1^-)}$ are the LO scattering amplitudes of the $\phi + d_2 \rightarrow \phi + d_1$, $n + t \rightarrow \phi + d_1$, $\phi + d_1 \rightarrow n + t$, and $n + t \rightarrow n + t$ transitions in channel $J^\pi = 1^-$, respectively.

with $J^\pi = 1^-$, the Faddeev equation of the diagrams in Fig. 1 can be written as

$$t^{(1^-)}(E, k_1, p_1, k_2, p_2) = K^{(1^-)}(E, k_1, p_1, k_2, p_2)$$

$$- \frac{1}{2\pi^2} \int_0^\Lambda dq \, q^2 \, K^{(1^-)}(E, q, p_1, q, p_2) \cdot D(E, q) \cdot t^{(1^-)}(E, k_1, q, k_2, q),$$

(17)

where the parameter $\Lambda$ as cutoff momentum is used to handle the ultraviolet divergences. Also, we have

$$D(E, q) \equiv \begin{pmatrix} D_s(E - \frac{q^2}{2m_\alpha}, q) & 0 \\ 0 & D_{1+}(E - \frac{q^2}{2m_\alpha}, q) \end{pmatrix},$$

$$i^{(1^-)}(E, k_1, p_1, k_2, p_2) \equiv \begin{pmatrix} i_{aa}^{(1^-)}(E, k_1, p_1) & i_{na}^{(1^-)}(E, k_2, p_1) \\ i_{an}^{(1^-)}(E, k_1, p_2) & i_{nn}^{(1^-)}(E, k_2, p_2) \end{pmatrix},$$

$$K^{(1^-)}(E, k_1, p_1, k_2, p_2) \equiv \begin{pmatrix} 0 & K_{na}^{(1^-)}(E, k_2, p_1) \\ K_{an}^{(1^-)}(E, k_1, p_2) & K_{nn}^{(1^-)}(E, k_2, p_2) \end{pmatrix}.$$  (18)
$k_1(p_1)$ and $k_2(p_2)$ indicate the incoming (outgoing) c.m. momenta of the $\phi + d_s$ and $n + t$ systems, respectively. The total c.m. energies of the particle–dimer systems are given by

$$E = \frac{k_n^2}{2} \left( \frac{1}{m_n} + \frac{1}{2m_n} \right) - \frac{\gamma_s^2}{m_n},$$

$$E = \frac{k_\alpha^2}{2} \left( \frac{1}{m_\alpha} + \frac{1}{m_\alpha + m_\alpha} \right) - \frac{\gamma_{1+}^2}{2\mu},$$

(19)

where $k_n$ and $k_\alpha$ are the on-shell c.m. momenta of the $n + t$ and $\phi + d_s$ channels, respectively, and $\gamma_{1+}$ is the momentum corresponding to the position of the $t$ resonance [30]. The detailed derivations of the $K_{nn}^{(1^-)}$, $K_{\alpha\alpha}^{(1^-)}$, and $K_{nn}^{(1)}$ kernels are presented in Appendix A.3. So, we have

$$K_{nn}^{(1^-)}(E,q,p) = -\frac{g_{1+}^2 m_n}{6} \left[ 2(1 - r) \frac{q^2 + p^2}{qp} Q_0(\varepsilon_{nn}(E,q,p)) \right],$$

$$K_{\alpha\alpha}^{(1^-)}(E,q,p) = -\frac{yg_{1+}^2 m_\alpha}{4} \left[ \frac{2}{q} Q_0(\varepsilon_{\alpha\alpha}(E,q,p)) + \frac{1 + r}{p} Q_1(\varepsilon_{\alpha\alpha}(E,q,p)) \right],$$

$$K_{nn}^{(1)}(E,q,p) = -\frac{yg_{1+}^2 m_n}{4} \left[ \frac{2}{p} Q_0(\varepsilon_{nn}(E,q,p)) + \frac{1 + r}{q} Q_1(\varepsilon_{nn}(E,q,p)) \right].$$

(20)

(21)

(22)

In the above equations, we have $Q_L(z) = \frac{1}{z} \int_1^\infty P_L(t) \frac{dt}{t}$, with $P_L(z)$ and $Q_L(z)$ the $L$th Legendre polynomials of the first and second kinds with complex arguments, respectively. Also, the functions $\varepsilon_{nn}$, $\varepsilon_{\alpha\alpha}$, and $\varepsilon_{nn}$ are given by

$$\varepsilon_{nn}(E,q,p) = \frac{m_n E - \frac{m_n}{2\mu} (q^2 + p^2)}{qp},$$

$$\varepsilon_{\alpha\alpha}(E,q,p) = \frac{m_\alpha E - \frac{m_\alpha}{2\mu} p^2}{qp},$$

$$\varepsilon_{nn}(E,q,p) = \frac{m_n E - \frac{m_n}{2\mu} q^2 - p^2}{qp}.$$  

(23)

(24)

(25)

The $J^\pi = 1^- n\alpha\alpha$ state is generated by using the $\frac{3^-}{2} n\alpha\alpha$ system with the second neutron both in the in the S- and D-wave states. According to the small contribution of the D-wave $n-t$ interaction, we have considered only the contribution of the $\frac{1^+}{2} n\alpha$ neutron in the S-wave $n-t$ components to produce the $1^- n\alpha\alpha$ system. Recently, the amplitude of the particle–dimer scattering in the general $J^\pi$ channel was obtained with respect to all possible angular momentum components in Ref. [34]. The full contribution of the $n + t \rightarrow n + t$ scattering in the $1^-$ channel has to be the same as that obtained by Braaten et al. in Ref. [34]. The difference between the combinations presented in Eq. (20) and those obtained in Ref. [34] comes from neglecting the contribution of the $\frac{5^-}{2} n\alpha$ neutrons in the $1^-$ channel.

We note that the on-shell scattering amplitude of the diagrams in Fig. 1 can be obtained by replacing the $k_1 = p_1 = k_\alpha$ and $k_2 = p_2 = k_n$ in Eq. (17). So, the on-shell $n + n + \alpha \rightarrow n + n + \alpha$ scattering amplitude in the $J^\pi = 1^-$ channel can be calculated using the relation

$$T_{nn\alpha}^{(1^-)}(E,k_\alpha,k_n) = T_{\alpha\alpha}^{(1^-)}(E,k_\alpha,k_\alpha) \cdot T_{nn\alpha}^{(1^-)}(E,k_\alpha,k_\alpha,k_n) \cdot T(E,k_\alpha,k_n).$$

(26)
where

\[ I(E, k_\alpha, k_n) = \begin{pmatrix} \mathcal{Y}(E, k_\alpha) \\ \mathcal{X}(E, k_n) \end{pmatrix}, \quad (27) \]

with

\[ \mathcal{Y}(E, k_\alpha) = -\frac{3\sqrt{2^2 - 2}}{2^3} \sqrt{\frac{1}{2}} D_\alpha (E, k_\alpha) k_n, \]

\[ \mathcal{X}(E, k_n) = -\frac{g_{1+}}{4} D_{1+} (E - \frac{k_n^2}{2m_n}, k_n) \int_{-1}^{1} d \cos \theta \left[ (1 - r) k_n + 2k' \cos \theta \right] \quad (28) \]

is a two-component vector which connects the incoming \( nn\alpha \) system to the particle–dimer states in the \( J^\pi = 1^- \) channel. In the above equation the momentum \( k' \) is a function of \( \theta \), where \( \theta \) is the angle between two incoming or outgoing neutrons:

\[ k' = -\frac{\mu}{m_\alpha} k_n \cos \theta \pm \sqrt{\left[ \frac{\mu}{m_\alpha} k_n \cos \theta \right]^2 - k_n^2 + 2\mu E}. \quad (29) \]

4. \( ^6\text{He} \) wave function

One of the major building blocks for the amplitude of the E1 \( n + n + \alpha \rightarrow ^6\text{He} + \gamma \) transition is the normalized \( ^6\text{He} \) wave function. Due to the quantum identities of the \( ^6\text{He} \) system in the ground state \( J^\pi = 0^+ \), we should introduce the Faddeev equation of the particle–dimer scatterings in the \( nn\alpha \) system with \( J^\pi = 0^+ \).

The diagrams that contribute to the amplitude of the particle–dimer scatterings in the channel \( J^\pi = 0^+ \) are shown in Fig. 2. The undashed lozenges, squares, triangles, and hexagons indicate the \( \phi + d_s \rightarrow \phi + d_s, n + t \rightarrow \phi + d_s, \phi + d_s \rightarrow n + t \), and \( n + t \rightarrow n + t \) transitions in channel \( ^6\text{He} \), respectively. Taking into account the projection operator in Eq. (A.12), and after integrating over the energy and solid angles, the Faddeev equation of the diagrams in Fig. 2 can be written as

\[ t^{(0^+)}(E, k_1, p_1, k_2, p_2) = \left[ \mathcal{K}^{(0^+)}(E, k_1, p_1, k_2, p_2) + \mathcal{H}(k_2, p_2, \Lambda) \right] \]

\[ -\frac{1}{2\pi^2} \int_0^{\Lambda} q^2 dq \left[ \mathcal{K}^{(0^+)}(E, q, p_1, q, p_2) + \mathcal{H}(q, p_2, \Lambda) \right] \cdot \mathcal{D}(E, q) \]

\[ \cdot t^{(0^+)}(E, k_1, q, k_2, q), \quad (30) \]

where

\[ t^{(0^+)}(E, k_1, p_1, k_2, p_2) \equiv \begin{pmatrix} t_{\alpha\alpha}^{(0^+)}(E, k_1, p_1) & t_{\alpha\alpha}^{(0^+)}(E, k_2, p_1) \\ t_{\alpha\alpha}^{(0^+)}(E, k_1, p_2) & t_{\alpha\alpha}^{(0^+)}(E, k_2, p_2) \end{pmatrix}. \quad (31) \]

As shown in Appendix A.2, one can see that the kernel \( \mathcal{K}^{(0^+)} \) is finally written as

\[ \mathcal{K}^{(0^+)}(E, q_1, p_1, q_2, p_2) \equiv \begin{pmatrix} 0 & K^{(0^+)}_{\alpha\alpha}(E, q_2, p_1) \\ K^{(0^+)}_{\alpha\alpha}(E, q_1, p_2) & K^{(0^+)}_{\alpha\alpha}(E, q_2, p_2) \end{pmatrix}, \quad (32) \]
The combination of the Legendre polynomials of the second kind for the three-body interaction

\[ H \]

which connects only the incoming and outgoing \( n + t \) channels. \( \nu^{(0^+)}_{aa}, \nu^{(0^+)}_{nn}, \nu^{(0^+)}_{na} \), and \( t^{(0^+)}_{nn} \) are the LO scattering amplitudes of the \( \phi + d \rightarrow \phi + d, n + t \rightarrow \phi + d, \phi + d \rightarrow n + t \), and \( n + t \rightarrow n + t \) transitions in channel \( J^\pi = 0^+ \), respectively. All notations are the same as Fig. 1.

where

\[
K^{(0^+)}_{nn}(E, q, p) = -\frac{g^2}{6} Q_1(\varepsilon_{nn}(E, q, p)) + \frac{8}{3} Q_2(\varepsilon_{nn}(E, q, p)) + \left( \frac{4}{3} + (1 - r)^2 \right) Q_0(\varepsilon_{nn}(E, q, p)),
\]

\[
K^{(0^+)}_{na}(E, q, p) = \frac{\nu_{na}}{4\sqrt{3}} Q_1(\varepsilon_{na}(E, q, p)) + \frac{1 + r}{p} Q_0(\varepsilon_{na}(E, q, p)),
\]

\[
K^{(0^+)}_{an}(E, q, p) = \frac{\nu_{an}}{4\sqrt{3}} Q_1(\varepsilon_{an}(E, q, p)) + \frac{1 + r}{q} Q_0(\varepsilon_{an}(E, q, p)).
\]

The combination of the Legendre polynomials of the second kind for the \( n + t \rightarrow n + t \) scattering amplitude in Eq. (33) is the same as the results presented by Braaten et al. in Ref. [34], as expected. The three-body interaction \( \mathcal{H} \) in Eq. (30), which is depicted using a filled circle in Fig. 2, is given by the relation

\[
\mathcal{H}(q, p, \Lambda) \equiv \begin{pmatrix} 0 & 0 \\ 0 & -\frac{m_n}{2\Lambda} q_p \phi H_0(\Lambda) \end{pmatrix},
\]

which connects only the incoming and outgoing \( n + t \) channels as introduced in Refs. [15,16].

The normalized \( ^6\text{He} \) wave function is obtained by solving the homogeneous part of Eq. (30) by replacing \( E = -B_{2n} \), where \( B_{2n} \) is the two-neutron separation energy of the \( ^6\text{He} \) nuclei. So, the homogeneous part of Eq. (30) for calculating the \( ^6\text{He} \) wave function can be written as

\[
t_{0\text{He}}(p) = -\frac{1}{2\pi^2} \int_0^\Lambda q^2 dq \left[ K^{(0^+)}(-B_{2n}, q, p, q, p) + \mathcal{H}(q, p, \Lambda) \right] \cdot D(-B_{2n}, q) \cdot t_{0\text{He}}(q),
\]

where

\[
t_{0\text{He}}(q) \equiv \begin{pmatrix} t_{0\text{He},\phi d_s \rightarrow \phi d_s}(q) & t_{0\text{He},nt \rightarrow \phi d_s}(q) \\ t_{0\text{He},\phi d_s \rightarrow nt}(q) & t_{0\text{He},nt \rightarrow nt}(q) \end{pmatrix}.
\]
The LO three-body force counterterm $H_0$ as a function of the cutoff momentum. This result is obtained by fitting $B_{2n}$ to the experimental value 0.97 MeV for each $\Lambda$.

Generally, $t_{\text{He},xX\rightarrow yY}$ denotes the contribution of the transition $xX \rightarrow yY$ ($x, y = \phi, n$ and $X, Y = d_s, t$) to making the $^6$He. The solution of Eq. (37) is normalized for the incoming $\phi + d_s$ channel as

$$1 = -\int \frac{q^2 dq}{2\pi^2} \int \frac{q'^2 dq'}{2\pi^2} \left[ t_{\text{He}}^{(1)}(q) \right]^* \cdot D(-B_{2n}, q) \cdot \frac{\partial}{\partial E} \left[ V(E, q, q') \cdot D(E, q') \right] \biggr|_{E=-B_{2n}} \cdot t_{\text{He}}^{(1)}(q'),$$

where $t_{\text{He}}^{(1)} = t_{\text{He}} \left( \begin{array}{c} 1 \\ 0 \end{array} \right)$, and $V$ is given by

$$V(E, q, q') = K^{(0^+)}(E, q, q', q, q') + \mathcal{H}(q, q', \Lambda).$$

If we need to find the normalized contribution of the $^6$He wave function that comes from the incoming $n + t$ system, the replacement of $t_{\text{He}}^{(1)}$ by $t_{\text{He}}^{(2)} = t_{\text{He}} \left( \begin{array}{c} 0 \\ 1 \end{array} \right)$ must be done in Eq. (39).

We use the three-body force $\mathcal{H}$ to remove the cutoff dependency of the results in the $^6$He channel. The results for the LO three-body force counterterm $H_0(\Lambda)$ in Fig. 3 are obtained by adjusting $H_0(\Lambda)$ such that the two-neutron separation energy of the $^6$He nuclei $B_{2n}$, gives the experimental value $B_{2n}^{\text{exp}} = 0.97$ MeV [36] for each $\Lambda$. The manner of the three-body force $H_0$ shown in Fig. 3 differs from that obtained in Ref. [15], because Ref. [15] used a different power counting for the $1^+ \text{na}$ amplitude.

5. The $n + n + \alpha \rightarrow ^6\text{He} + \gamma$ capture process

One of the fundamental astrophysics reactions is the $n + n + \alpha \rightarrow ^6\text{He} + \gamma$ capture process. The E1 transition amplitude is the dominant contribution of this reaction. So, we have to concentrate on the transition from the initial $\text{naa}$ system with $J^\pi = 1^-$ to the final $^6$He halo nucleus bound state with $J^\pi = 0^+$. In this process, the outgoing photon only interacts with the electric charge of the alpha particle at LO. The photon–alpha interaction Lagrangian is introduced as

$$\mathcal{L}_E = \frac{eQ_\alpha}{2m_\alpha} (\vec{P} + \vec{P}') \cdot \vec{e}^*,$$

where $e$ is the electron charge and $Q_\alpha = 2$ denotes the electric charge number of the alpha particle. $\vec{P}$ ($\vec{P}'$) represents the incoming (outgoing) alpha momentum vector in the alpha–alpha–photon ($\phi\phi\gamma$)
The LO diagrams of the E1 transition amplitude. The first line depicts the diagrams that contribute to the E1 transition amplitude of $n + n + \alpha \rightarrow ^{6}\text{He} + \gamma$. The second and third lines introduce the box and oval with wavy line shown in the first line. The dotted and dashed half ovals represent $t^{(1)}_{\text{He}}$ and $t^{(2)}_{\text{He}}$, respectively. The small filled square and circle with wavy line denote the minimal substitution in the neutron–alpha–$^5\text{He}$ ($n\phi t$) vertex and the bare $^5\text{He}$ propagator, respectively. All notations are the same as Fig. 1.

vertices, and $\vec{e}_\gamma$, represents the outgoing photon polarization vector. The neutron–alpha–$^5\text{He}$–photon ($n\phi t\gamma$) and $^5\text{He}$–$^5\text{He}$–photon ($t\gamma$) vertices in Fig. 4 are given by the minimal substitution of $\vec{\nabla} \rightarrow \vec{\nabla} + ie Q_\alpha \vec{A}$ in the Lagrangian (10), with $\vec{A}$ as an external field.

The LO contribution of the E1 transition amplitude of $n + n + \alpha \rightarrow ^{6}\text{He} + \gamma$ is calculated using the diagrams in Fig. 4. In addition to the half off-shell $nn\alpha$ scattering amplitude in channel $1^-$ that is obtained from Eq. (17), we need the $^6\text{He}$ halo nucleus wave function to compute the contributions of the diagrams in Fig. 4. We calculate the normalized $^6\text{He}$ bound state wave function as presented in Sect. 4. The final E1 $n + n + \alpha \rightarrow ^{6}\text{He} + \gamma$ amplitude is obtained by the following relation:

$$M(E_i, k_\alpha, k_n) = M(E_i, k_\alpha, k_n) \vec{e}_{\gamma}^* \cdot \vec{e}, \quad (42)$$

where $\vec{e}$ denotes the polarization vector of the incoming $1^- nn\alpha$ system, and we have

$$M(E_i, k_\alpha, k_n) = \left\{ R_0(E_i, k_\alpha, k_n) + R(E_i, k_\alpha, k_n) - \frac{1}{2\pi^2} \int_0^\Lambda dq q^2 \right\} \cdot I(E_i, k_\alpha, k_n). \quad (43)$$

In Eq. (43), the contribution of the $R_0$ diagram in the first line of Fig. 4 is given by the matrix

$$R_0(E_i, k_\alpha, k_n) = \begin{pmatrix} R_0(E_i, k_\alpha) & 0 \\ 0 & R_0(E_i, k_n) \end{pmatrix},$$

where

$$R_0(E_i, k_\alpha) = \sqrt{3} \frac{e Q_\alpha}{m_\alpha} \frac{k_\alpha}{E_i - E_f} \left( 1 + \frac{D_s(E_f - \frac{k_\alpha^2}{2m_\alpha}, k_\alpha)}{D_s(E_i - \frac{k_\alpha^2}{2m_\alpha}, k_\alpha)} \right) t^{(2)}_{\text{He}}(k_\alpha). \quad (44)$$

Also, the matrix kernel

$$R(E_i, k_\alpha, k_n) = \begin{pmatrix} R_2(E_i, k_n) + R_4(E_i, k_n) \\ R_1(E_i, k_n) + R_3(E_i, k_n) + \sum_{i=5}^{9} R_i(E_i, k_n) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & R_2(E_i, k_n) + R_4(E_i, k_n) \\ R_1(E_i, k_n) + R_3(E_i, k_n) + \sum_{i=5}^{9} R_i(E_i, k_n) \end{pmatrix}. \quad (45)$$
is the contribution of all the diagrams in the second and third lines of Fig. 4. The contributions of the diagrams \( R_i \) with \( i = 1, \ldots, 10 \) are given by the following relations:

\[
R_1(E_i, k_n) = -\frac{e_Q g_1^2}{12\pi^2} \frac{1}{E_f - E_i} \int_0^\Lambda dq \frac{q^2}{k_n} l_{6He}^{(2)}(q) D_1 + (E_f - \frac{q^2}{2m_n}, q) \\
\times \left[ k_n (3 - \frac{8}{3} r + r^2) + 2 (1 - r) \frac{q^2}{k_n} \left[ Q_0(\varepsilon_m(E_i, q, q)) - Q_0(\varepsilon_m(E_f, k_n, q)) \right] \right] + \left[ q (3 - r)^2 + 2 (1 - r) \frac{k_n^2}{q} \left[ Q_1(\varepsilon_m(E_i, k_n, q)) - Q_1(\varepsilon_m(E_f, k_n, q)) \right] \right] + \frac{4}{3} (3 - r) \frac{k_n}{q} \left[ Q_2(\varepsilon_m(E_i, k_n, q)) - Q_2(\varepsilon_m(E_f, k_n, q)) \right],
\]

\[ (46) \]

\[
R_2(E_i, k_n) = \frac{e_Q g_1^2}{8\sqrt{3}\pi^2} \frac{1}{E_f - E_i} \int_0^\Lambda dq \frac{q^2}{k_n} l_{6He}^{(1)}(q) \\
\times \left[ \frac{2q}{k_n} D_s(E_i - \frac{q^2}{2m_n}, q) Q_0(\varepsilon_m(E_i, k_n, q)) - D_s(E_f - \frac{q^2}{2m_n}, q) Q_0(\varepsilon_m(E_f, k_n, q)) \right] + (1 + r) \left[ D_s(E_i - \frac{q^2}{2m_n}, q) Q_1(\varepsilon_m(E_i, k_n, q)) - D_s(E_f - \frac{q^2}{2m_n}, q) Q_1(\varepsilon_m(E_f, k_n, q)) \right],
\]

\[ (47) \]

\[
R_3(E_i, k_n) = \frac{e_Q m_n g_1^2}{12\pi^2} (1 - r) \int_0^\Lambda dq \frac{q^2}{k_n} l_{6He}^{(2)}(q) D_1 + (E_f - \frac{q^2}{2m_n}, q) \\
\times \left[ \frac{2}{q} Q_1(\varepsilon_m(E_i, k_n, q)) - \frac{1}{k_n} \frac{1 - r}{q} Q_0(\varepsilon_m(E_i, k_n, q)) \right],
\]

\[ (48) \]

\[
R_4(E_i, k_n) = \frac{e_Q m_n g_1^2}{8\sqrt{3}\pi^2} (1 - r) \int_0^\Lambda dq \frac{q^2}{k_n} l_{6He}^{(1)}(q) \frac{1}{k_n q} D_s(E_f - \frac{q^2}{2m_n}, q) \\
\times Q_0(\varepsilon_m(E_i, k_n, q)),
\]

\[ (49) \]

\[
R_5(E_i, k_n) = \frac{e_Q m_n g_1^2}{12\pi^2} (1 - r) \int_0^\Lambda dq \frac{q^2}{k_n} l_{6He}^{(2)}(q) D_1 + (E_f - \frac{q^2}{2m_n}, q) \\
\times \left[ \frac{2}{k_n} Q_0(\varepsilon_m(E_i, k_n, q)) - \frac{1}{q} Q_1(\varepsilon_m(E_i, k_n, q)) \right],
\]

\[ (50) \]

\[
R_6(E_i, k_n) = -\frac{e_Q m_n g_1^2}{6\pi m_n} \frac{1}{E_f - E_i} l_{6He}^{(2)}(q) D_1 + (E_f - \frac{q^2}{2m_n}, q) \left[ I(E_i, k_n) - I(E_f, k_n) \right],
\]

\[ (51) \]
\[ R_7(E_i, k_n) = -\frac{e Q_\alpha g_{1+}^2}{12\pi} (1 - r)(1 + r - \frac{2\mu}{m_n}) k_n \]
\[ \times \sqrt{\frac{2\mu}{m_n} - \frac{\mu^2}{m_n^2}} k_n^2 - 2\mu E_{\ell} t_{\text{He}}^{(2)}(k_n) D_{1+}(E_{\ell} - \frac{k_n^2}{2m_n}, k_n), \quad (52) \]

\[ R_8(E_i, k_n) = -\frac{e Q_\alpha g_{1+}^2}{12\pi} (1 - r)(1 + r - \frac{2\mu}{m_n}) k_n \]
\[ \times \sqrt{\frac{2\mu}{m_n} - \frac{\mu^2}{m_n^2}} k_n^2 - 2\mu E_{\ell} t_{\text{He}}^{(2)}(k_n) D_{1+}(E_{\ell} - \frac{k_n^2}{2m_n}, k_n), \quad (53) \]

\[ R_9(E_i, k_n) = -\sqrt{3} \frac{e Q_\alpha m_n^+}{m_n + m_\alpha} k_n D_{1+}(E_{\ell} - \frac{k_n^2}{2m_n}, k_n) t_{\text{He}}^{(2)}(k_n), \quad (54) \]

\[ R_{10}(E_i, k_\alpha) = -\frac{e Q_\alpha m_\alpha g_{1+}}{8\pi^2} (1 - r) \int_0^\Lambda dq q^2 t_{\text{He}}^{(2)}(q) \frac{1}{k_\alpha q} D_{1+}(E_{\ell} - \frac{q^2}{2m_n}, q) \]
\[ \times Q_0(\epsilon_{\alpha \alpha}(E_i, k_\alpha, q)), \quad (55) \]

where the function \( I(E, k) \) in Eq. (51) is
\[ I(E, k) = \left[(5u - 1)u^2 k^3 + \frac{1}{15}u k \right]\left[(1 - u^2)k^2 - 2\mu E\right]^\frac{3}{2} - 2k \left(\frac{1}{3} - \frac{6u}{5}\right) \left[(1 - u^2)k^2 - 2\mu E\right]^\frac{3}{2}, \quad (56) \]

with \( u = \mu/m_\alpha \). As an example, the detailed derivation of the \( R_1 \) diagram is presented in Appendix B.

In this step, we can evaluate the strength function distribution for the halo nuclei \(^{6}\text{He} \) bound state. The E1 strength function distribution for the \(^{6}\text{He} \) halo nucleus is given by
\[ \frac{d B^{E1}(E_i)}{dE_i} = \int d\tau_{\ell} \sum_{\text{pol. ave.}} \text{Tr}[|M|^2] \delta(E_{\ell} - E_i), \quad (57) \]

where \( E_{\ell} \) is the final state energy and \( \int d\tau_{\ell} \) denotes the summation over all final states. In the above equation the trace acts on the neutron spin space. Also, the total E1 strength function of the halo nuclei \(^{6}\text{He} \) can be obtained by
\[ B^{E1}(E) = \int_0^E \frac{d B^{E1}(E_i)}{dE_i} dE_i. \quad (58) \]

6. Results and discussion

The Faddeev equation (17) is solved numerically by replacing the integrals by sums using Gaussian quadratures. After discretization of this integral equation we use the inverse matrix method. A major difficulty in doing this is the treatment of the singularities of the integral kernels. So, in order to solve the Faddeev equation (17), we try to determine all the singularities of the integral kernels. In Eq. (17), the singularities arise both from the matrix kernel \( K^{(1-)} \) and the LO propagators \( D \). The
The complex pole positions of the half off-shell kernel \( K^{(1^-)}(E, k_{\alpha}, q, k_n, p) \).

\begin{center}
\includegraphics[width=0.5\textwidth]{fig5.png}
\end{center}

Fig. 5. The complex pole positions of the half off-shell kernel \( K^{(1^-)}(E, k_{\alpha}, q, k_n, p) \).

The pole structures of the propagator of the \( ds \) auxiliary field.

\begin{center}
\includegraphics[width=0.5\textwidth]{fig6.png}
\end{center}

Fig. 6. The pole structures of the propagator of the \( ds \) auxiliary field.

former is a logarithmic singularity which comes from the vanishing of the partial fractions in the arguments of the logarithm of the Legendre polynomial of the second kind,

\[ \varepsilon_{ab}(E, q, p) \pm 1 = 0, \quad a, b = n, \alpha. \]  

(59)

In spite of the singularities in the full off-shell matrix kernel \( K^{(1^-)}(q, p, q, p) \), the ones in the half off-shell kernel \( K^{(1^-)}(k_{\alpha}, p, k_n, p) \) should be handled in the numerical calculations. Hence, we have the singularities as

\[ \varepsilon_{ab}(E, k_{\alpha}, p) \pm 1 = 0, \quad a, b = n, \alpha. \]  

(60)

The complex pole positions of the half off-shell kernels \( K^{(1^-)}(E, k_{\alpha}, p, k_n, p) \) with \( 0 < E < 6 \text{ MeV} \) have been plotted in Fig. 5.

The other singularities come from the propagators of the \( ds \) and \( t \) auxiliary fields. As shown in Fig. 6, the \( nn \) propagator has only the singularities on the real axis for \( E > 0 \). But the complicated treatments deal with the \( 1+ n\alpha \) propagator. As discussed in Sect. 2, the full propagator of the \( t \) field has an unphysical deep bound state and a shallow resonance. The position of the unphysical bound state is moving when the energy of the \( n\alpha \) system changes. The pole structures of the \( 1+ n\alpha \) propagator corresponding to the unphysical \( ^5\text{He} \) bound state and the shallow \( n\alpha \) resonance have been depicted
in Fig. 7. The curves in Figs. 7(a) and (b) have been plotted by considering $0 < E < 6$ MeV and $0 < E < 12$ MeV, respectively. These curves explicitly show that the pole positions of the $1^+ na$ propagator correspond to the values of the incoming $nna$ energy.

In order to handle the singularities we summarize them as two classes, real and complex poles. For positive energy, the real poles come from the $nn$ propagator and the position of the unphysical $n\alpha$ bound state. The complex singularities occur due to the logarithmic singularities of the Legendre polynomials and the shallow $n\alpha$ resonance.

To integrate the Faddeev equation (17), we used the contour deformation suggested by Hetherington and Schick, namely a rotation $p \rightarrow pe^{-i\Phi}$, $\Phi > 0$ [37–39]. We should note the important practical aspects in this rotation method. In the first step we need to obtain the position of the maximum value of the real poles, $p_m$. Here, $p_m$ depends on the considered values of the energy $E$. For a total energy of the incoming $nna$ system of $0 < E < 6$ MeV, we have found $p_m \sim 140.25$ MeV. The other implementation step is choosing the rotation angle. With respect to the complex pole positions shown in Figs. 5 and 7, we considered $\Phi \sim \Phi_1 \sim \Phi_2$ a few degrees such that the rotated contour does not cross any singularities. In Fig. 7, the dashed lines denote the rotated contour. With increased total energy of the incoming $nna$ system, the use of this rotated contour may be more complicated. In this respect it is straightforward to compare Figs. 7(a) and (b), which have been plotted for the different maximum energy range of the incoming $nna$ system.
Fig. 9. (color online) Comparison of the EFT results for the E1 halo nucleus $^6$He strength function distribution (blue band) with the experimental data [26] (thick red lines as the upper and lower experimental data) and the results obtained by other theoretical methods: [24] (thin solid line), [27] (dotted line), [28] (dash-dotted line), [21] (dashed line), and [29] (long dashed line). We note that the upper and lower limits of the blue band determine the upper and lower limits due to our systematic EFT error at LO.

energy considerations. However, here we concentrate on the energy regime $0 < E < 6$ MeV so we can use this contour deformation to handle the introduced singularities. Using Eq. (26), we can compute the on-shell $1^- n + n + \alpha \rightarrow n + n + \alpha$ scattering amplitude. The EFT results for this transition have been plotted in Fig. 8. The bands denote the cutoff variation of the amplitudes between $\Lambda = 200$ MeV and 600 MeV. The small cutoff dependency shows that no three-body force is needed in this channel to renormalize the final results.

Our LO EFT results for the E1 halo nucleus $^6$He strength function distribution are compared with the experimental data [26] and other theoretical results in Fig. 9. The band in this figure represents the systematic EFT error at LO in our results. This uncertainty as a band is only plotted manually to show the upper and lower limits of changes in our LO EFT results when we consider all higher-order corrections—NLO, N2LO, and so on. The EFT results for $dB^{E1}/dE_i$ behave consistently concerning the peak position with the experimental data and other theoretical results. The higher-order EFT corrections contribute effectively in the energies beyond 1 MeV (corresponding to $k_n, k_\alpha < 50$ MeV). In this calculation, the partial wave $^2S_1$ of the $n\alpha$ system has no contribution in the $n + n + \alpha \rightarrow ^6$He + $\gamma$ amplitude [30]. Also, relative to the $^2P_1^2$ partial wave (1+), the contribution of the neutron–deuteron interaction in the $^2P_1^2$ partial wave for the $dB^{E1}/dE_i$ values is negligible [40]. So, we expect that the small discrepancy in the $dB^{E1}/dE_i$ values compensate when considering the higher-order EFT corrections, such as other $n\alpha$ partial waves. Table 1 indicates our results for the total E1 strength function $B^{E1}(E)$ in the $4$ MeV $\leq E \leq 9$ MeV energy range. We present the cutoff variations of the $B^{E1}(E)$ values from momentum cutoff $\Lambda = 200$ MeV to $\Lambda = 600$ MeV. The cutoff-independent variation indicates that apart from the $^6$He halo nucleus wave function renormalization there is no need to add the three-body force in the particle–dimer amplitudes in order to renormalize the results.
Table 1. The total E1 strength function $B^{E1}(E)$ for the $^6\text{He}$ halo nucleus bound state. The first to fifth columns denote the upper limit of the energy integration, the experimental data, the results obtained by Forssen et al., the LO EFT results, and $C = \text{Abs} \left[ 1 - \frac{B^{E1} \text{at } \Lambda=200\text{MeV}}{B^{E1} \text{at } \Lambda=600\text{MeV}} \right]$ as the cutoff variation of the LO EFT results, respectively. All $B^{E1}(E)$ values are in e² fm² units.

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Appendix A. Derivation of the kernel $\mathcal{K}^{(J^z)}$ with $J^z = 0^+$ and $J^z = 1^-$

In this appendix, we calculate in detail the exchanged-particle propagators used in Sects. 3 and 4 to obtain the LO $1^-$ particle–dimer scattering amplitudes and the $^6\text{He}$ wave function. We initially derive the unprojected kernels for all possible particle–dimer scatterings in the cluster configuration space. In the next step, we use the appropriate projection operators to construct the kernels $0^+$ and $1^-$. 

Appendix A.1. Unprojected kernels

The diagram in Fig. A.1 shows the energy and momentum of the particles in the $n+t \rightarrow n+t$ kernel. According to this diagram, before projecting, one can write the amplitude of the $n+t \rightarrow n+t$ kernel using the Feynman rules as

$$i \left( K^{\text{unproj}}_{nn} \right)_{\bar{\alpha}}(E, q, p) = \left( -i \frac{g_{1+}}{2} \right) i (S^\dagger_{1+})_{\bar{\alpha}} \left( \bar{p} + 2\bar{q} - r\bar{p} \right) i S_{\bar{\beta}}(E - \frac{q^2 + p^2}{2m_n}, \bar{p} + \bar{q}) \times \left( -i \frac{g_{1+}}{2} \right) ( -i ) (S^\dagger_{1+})_{\bar{\beta}}(\bar{q} + 2\bar{p} - r\bar{q})_i, \quad (A.1)$$

where, after summation over the spin-$\frac{1}{2}$ index, $\eta$, we have

$$i \left( K^{\text{unproj}}_{nn} \right)_{\bar{\alpha}}(E, q, p) = \frac{i}{4} g_{1+}^2 \left( S^\dagger_{1+} S_{1+} \right)_{\bar{\alpha}} \frac{[2q_j + (1 - r)p_j] \left[ 2p_i + (1 - r)q_i \right]}{E - \frac{q^2 + p^2}{2m_n} - \frac{(q + p)^2}{2m_n} + i\epsilon}. \quad (A.2)$$

All notations are as explained in the text. By simplifying Eq. (A.2), the unprojected $n+t \rightarrow n+t$ exchanged-particle kernel can be written as

$$i \left( K^{\text{unproj}}_{nn} \right)_{\bar{\alpha}}(E, q, p) = \left( -i \frac{g_{1+}^2 m_a}{4} \left( S^\dagger_{1+} S_{1+} \right)_{\bar{\alpha}} \frac{2(1 - r)q_j q_j + 4q_j p_i + (1 - r)^2 q_j p_j}{m_a E - \frac{m_a}{2}\left( q^2 + p^2 \right) - \bar{q} \cdot \bar{p} + i\epsilon} \right), \quad (A.3)$$

where $\bar{q}$ and $\bar{p}$ denote the incoming and outgoing c.m. momenta, respectively.
Fig. A.1. The propagator of the exchanged alpha particle at LO. $E$, $\vec{q}$, and $\vec{p}$ are the total energy and the incoming and outgoing momenta in the c.m. framework, respectively. All notations are as in the previous figures.

Fig. A.2. The LO propagator of the exchanged neutron with the $n + t$ and $\phi + d_s$ systems as the incoming and outgoing channels respectively. All notations are the same as Fig. A.1.

Similar to $K_{nn}^{\text{unproj.}}$, we can introduce the unprojected $K_{\alpha n}^{\text{unproj.}}$ and $K_{an}^{\text{unproj.}}$ kernels. Figure A.2 depicts the energy–momentum values of the particles in the $n + t \to \phi + d_s$ kernel. For this diagram, using the Feynman rules yields

$$i\left(K_{na}^{\text{unproj.}}\right)^{\beta}_{\alpha}(E, q, p) = \left(i \frac{\gamma}{\sqrt{8}}\right) i(\sigma_2)^{\gamma}_{\beta} iS_n(E - \frac{q^2}{2m_n} - \frac{p^2}{2m_n}, \vec{p} + \vec{q})\left(-i g_{1+} \frac{(1 + r)}{2}\right) \times \left(-i\right) S_{i}^{\eta}_{\alpha}(-\vec{q} - 2\vec{p} - r\vec{q}),$$

where this equation is written before applying the appropriate projection operators. Thus,

$$i\left(K_{na}^{\text{unproj.}}\right)^{\beta}_{\alpha}(E, q, p) = -\frac{i\gamma g_{1+} \frac{m_n}{4\sqrt{2}}}{\left(\sigma_2 S_i\right)^{\eta}_{\alpha}} \left(1 + r\right) g_{1+} \frac{2p_i}{2\mu} \frac{1}{\vec{q} \cdot \vec{p} + i\epsilon}.\quad (A.5)$$

To obtain the kernel $K_{an}^{\text{unproj.}}$, we can repeat the above steps. But we can easily write the $\phi + d_s \to n + t$ kernel at LO without additional calculation using the relation

$$i\left(K_{an}^{\text{unproj.}}\right)^{\beta}_{\alpha}(E, q, p) = i\left[\left(K_{na}^{\text{unproj.}}\right)^{\alpha}_{\beta}(E, q, p)\right]^\dagger.\quad (A.6)$$

So, the $\phi + d_s \to n + t$ kernel at LO, $K_{an}^{(LO)}$, is given by

$$i\left(K_{an}^{\text{unproj.}}\right)^{\beta}_{\alpha}(E, q, p) = -\frac{i\gamma g_{1+} \frac{m_n}{4\sqrt{2}}}{\left(\sigma_2 S_i\right)^{\eta}_{\alpha}} \left(1 + r\right) g_{1+} \frac{2q_j}{2\mu} \frac{1}{\vec{q} \cdot \vec{p} + i\epsilon}.\quad (A.7)$$
In the cluster configuration space, the unprojected kernels (A.3), (A.5), and (A.7) can be rewritten as

\[
\begin{pmatrix}
\hat{\kappa}_{\alpha,\alpha}^{\text{unproj.}}(E, q_1, q_2, p_1, p_2) \\
0 \\
A_i(E, q_2, p_1) \left( \sigma_2 \hat{S}_i \right)^{\hat{\sigma}}_\alpha \\
B_j(E, q_1, p_2) \left( \sigma_2 \hat{S}_j \right)^{\hat{\sigma}}_\alpha \\
C_{ij}(E, q_2, p_2) \left( \sigma_2 \hat{S}_i \right)^{\hat{\sigma}}_\alpha
\end{pmatrix}.
\]  
(A.8)

Let us remember that if the total c.m. energies of the \( n + t \) and \( \phi + d_s \) systems are equal, the c.m. on-shell momentum values for the \( n + t \) and \( \phi + d_s \) channels are different. On the other hand, with respect to Eq. (19) one can see that \( k_n \neq k_\alpha \), so we have to consider the different incoming and outgoing momenta for each transition when we use the cluster configuration space. In Eq. (A.8), \( q_1 \) (\( p_1 \)) and \( q_2 \) (\( p_2 \)) represent the incoming (outgoing) c.m. momenta in the \( \phi + d_s \) and \( n + t \) states, respectively. Also, the \( A_i, B_j, \) and \( C_{ij} \) functions are given by

\[
A_i(E, q, p) = -\frac{y g_{1+} m_n}{4\sqrt{2}} \left[ \frac{(1 + r) q_i + 2 p_i}{m_n E - \vec{q}^2 - \frac{m_n}{2\mu} \vec{p}^2 - \vec{q} \cdot \vec{p} + i\epsilon} \right],
\]  
(A.9)

\[
B_j(E, q, p) = -\frac{y g_{1+} m_n}{4\sqrt{2}} \left[ \frac{(1 + r) q_j + 2 q_j}{m_n E - \frac{m_n}{2\mu} q^2 - p^2 - \vec{q} \cdot \vec{p} + i\epsilon} \right],
\]  
(A.10)

and

\[
C_{ij}(E, q, p) = -i \frac{g_{1+}^2 m_n}{4} \left[ \frac{4 q_j p_i + 2 (1 - r) (q_j q_j + p_j p_j) + (1 - r)^2 q_j p_j}{m_n E - \frac{m_n}{2\mu} (q^2 + p^2) - \vec{q} \cdot \vec{p}} \right].
\]  
(A.11)

### Appendix A.2. Projecting to the \( ^6 \text{He} \) channel

The projection operator of the \( ^6 \text{He} \) channel in the cluster configuration space is given by

\[
\begin{pmatrix}
\hat{P}_{0+}^{\beta_1, \beta_2} \\
0
\end{pmatrix}_{\alpha, \alpha} = \begin{pmatrix}
\mathbf{1}_{\hat{\alpha}}^{\beta_1} \\
0 \\
\sqrt{\frac{3}{2}} \left( \sigma_2 \hat{e} \cdot \hat{S} \right)^{\beta_2}_{\hat{\alpha}}
\end{pmatrix},
\]  
(A.12)

where \( \hat{e} \) denotes the unit vector of the c.m. momentum of the \( n + t \) system and \( \mathbf{1} \) is the \( 2 \times 2 \) unit matrix. The 11 and 22 elements of \( \hat{P}_{0+} \) are \( 2 \times 2 \) and \( 2 \times 4 \) spin matrices, and project the \( \phi + d_s \) and \( n + t \) systems into the \( ^6 \text{He} \) channel, respectively. The projection operator in Eq. (A.12) justifies the unitary relations

\[
\frac{1}{4\pi} \int d\Omega_e \left( \hat{P}_{0+}^{\beta_1, \beta_2} \left( \hat{P}_{0+}^{\dag} \right)_{\alpha_1, \alpha_2}^{\gamma_1, \gamma_2} \right) = \delta_{\alpha_1}^{\beta_1} \delta_{\alpha_2}^{\beta_2},
\]

\[
\frac{1}{4\pi} \int d\Omega_e \left( \hat{P}_{0+}^{\beta_1, \beta_2} \left( \hat{P}_{0+}^{\dag} \right)_{\alpha_1, \alpha_2}^{\gamma_1, \gamma_2} \right) = \delta_{\alpha}^{\beta} \delta_{\alpha}^{\beta}.
\]  
(A.13)

The relations in Eq. (A.13) can be proved by using

\[
\frac{1}{4\pi} \int d\Omega_e \left[ \frac{3}{2} \sigma_2 \hat{e} \cdot \hat{S} \hat{S}^{\dag} \cdot \hat{e} \sigma_2 \right] = \mathbf{1}_{2 \times 2}.
\]  
(A.14)

Applying the projection operator \( \hat{P}_{0+} \), one can project the incoming and outgoing systems into the \( ^6 \text{He} \) channel. The projected \( 2 \times 2 \) kernel is achieved by using the relation

\[
\left( \hat{K}_{0+}^{\beta_1, \beta_2} \right)_{\alpha_1, \alpha_2}^{\gamma_1, \gamma_2} (E, q_1, q_2, p_1, p_2) = \left( \hat{P}_{0+}^{\gamma_1, \gamma_2} \left( \hat{K}_{\text{unproj.}}^{\beta_1, \beta_2} \right)_{\alpha_1, \alpha_2}^{\gamma_1, \gamma_2} (E, q_1, q_2, p_1, p_2) \left( \hat{P}_{0+}^{\dag} \right)_{\alpha_1, \alpha_2}^{\gamma_1, \gamma_2} \right).
\]  
(A.15)
where the summation over all repeated indices should be done. So, one can find in the $^6$He channel,

$$
P_{0^+} \left( \begin{array}{cc}
0 & A_i(E, q_2, P_1) \sigma_2 S_i \\
B_j(E, q_1, p_2) S^\dag_j \sigma_2 & C\alpha_j(E, q_2, p_2) S^\dag_j S_i \\
\end{array} \right) P^\dag_{0^+}
$$

$$
= 2\sqrt{\frac{2}{3}} \left( \begin{array}{cc}
0 & A_i(E, q_2, P_1) e_i \\
B_j(E, q_1, p_2) e_i e_j & \sqrt{\frac{2}{3}} C\alpha_j(E, q_2, p_2) e_i e_j \\
\end{array} \right).
$$

(A.16)

According to Eqs. (A.15) and (A.16), the projected exchanged-particle propagators in the cluster configuration space are given by

$$
K^{(0^+)}(E, q_1, q_2, P_1, P_2) = \left( \begin{array}{cc}
0 & K^{(0^+)}_{na}(E, q_2, P_1) \\
K^{(0^+)}_{an}(E, q_1, P_2) & K^{(0^+)}_{nn}(E, q_2, P_2) \\
\end{array} \right),
$$

(A.17)

where

$$
K^{(0^+)}_{na}(E, q, P) = -\frac{y g_1 + m_n}{4\sqrt{3}} \frac{1}{2} \int_{-1}^{1} d \cos \theta \left[ \frac{(1 + r)q + 2p \cos \theta}{m_n E - q^2 - \frac{m_n}{2\mu} p^2 - qp \cos \theta + i\epsilon} \right],
$$

(A.18)

$$
K^{(0^+)}_{an}(E, q, P) = -\frac{y g_1 + m_n}{4\sqrt{3}} \frac{1}{2} \int_{-1}^{1} d \cos \theta \left[ \frac{(1 + r)p + 2q \cos \theta}{m_n E - \frac{m_n}{2\mu} q^2 - p^2 - qp \cos \theta + i\epsilon} \right],
$$

(A.19)

and

$$
K^{(0^+)}_{nn}(E, q, P) = -\frac{g_1^2 + m_n}{6} \frac{1}{2} \int_{-1}^{1} d \cos \theta \left[ \frac{4pq \cos^2 \theta + 2(1 - r)(q^2 + p^2) \cos \theta + (1 - r)^2 qp}{m_n E - \frac{m_n}{2\mu} (q^2 + p^2) - qp \cos \theta} \right].
$$

(A.20)

The $\theta$ angle in the above equations indicates the angle between the initial ($\vec{q}$) and final ($\vec{p}$) c.m. momenta. With respect to the relation

$$
Q_L(z) = \frac{1}{2} \int_{-1}^{1} dt P_L(t) \frac{z - t}{z - t},
$$

(A.21)

where $P_L(z)$ and $Q_L(z)$ are the $L$th Legendre polynomials of the first and second kinds with complex arguments, respectively, and using the relations

$$
1 = P_0(\cos \theta), \quad \cos \theta = P_1(\cos \theta), \quad \cos^2 \theta = \frac{1}{3} \left( 2P_2(\cos \theta) + P_0(\cos \theta) \right),
$$

(A.22)

Eqs. (A.18)–(A.20) convert to

$$
K^{(0^+)}_{na}(E, q, P) = -\frac{y g_1 + m_N}{4\sqrt{3}} \left[ \frac{2}{q} Q_1(\epsilon_{na}(E, q, P)) + \frac{1 + r}{p} Q_0(\epsilon_{na}(E, q, P)) \right],
$$

(A.23)

$$
K^{(0^+)}_{an}(E, q, P) = -\frac{y g_1 + m_N}{4\sqrt{3}} \left[ \frac{2}{q} Q_1(\epsilon_{an}(E, q, P)) + \frac{1 + r}{p} Q_0(\epsilon_{an}(E, q, P)) \right],
$$

(A.24)
and
\[
K^{(0^+)}_{mn}(E, q, p) = -\frac{g_1^2 + m_\alpha}{6} \left[ 2(1 - r) \frac{q^2 + p^2}{qp} Q_1(\epsilon_{nn}(E, q, p)) + \frac{8}{3} Q_2(\epsilon_{nn}(E, q, p)) + \left( \frac{4}{3} + (1 - r)^2 \right) Q_0(\epsilon_{nn}(E, q, p)) \right].
\tag{A.25}
\]

In Eqs. (A.23)–(A.25), \(\epsilon_{nn}, \epsilon_{an},\) and \(\epsilon_{nn}\) are obtained using the relations
\[
\epsilon_{na}(E, q, p) = \frac{m_n E - q^2 - \frac{m_n q^2}{2p}}{qp},
\tag{A.26}
\]
\[
\epsilon_{an}(E, q, p) = \frac{m_n E - \frac{m_n q^2}{2p} - p^2}{qp},
\tag{A.27}
\]
\[
\epsilon_{nn}(E, q, p) = \frac{m_n E - \frac{m_n q^2}{2p} (q^2 + p^2)}{qp}.
\tag{A.28}
\]

**Appendix A.3. Projecting to the \(J^\pi = 1^-\) channel**

As the \(^{6}\text{He}\) channel we can project the particle–dimer states of the \(nn\alpha\) system to the \(J^\pi = 1^-\) channel. For this purpose, we should use the matrix operator
\[
(P_1^-)^{\beta_1, \beta_2}_{\alpha, \alpha} = \begin{pmatrix}
\sqrt{3} \hat{e} \cdot \hat{e}^{1} & 0 \\
0 & \sqrt{\frac{3}{2}} (\sigma_2 \hat{e} \cdot \hat{S})^{\beta_2}_{\alpha}
\end{pmatrix}
\tag{A.29}
\]
to project the \(\phi + d_s\) and \(n + t\) systems into the \(J^\pi = 1^-\) channel. As in the previous section, we can project the unprojected kernel (A.8) to the \(J^\pi = 1^-\) system using Eq. (A.29). Therefore, we have
\[
(P_1^-) \begin{pmatrix}
0 & A_1(E, q_2, p_1) \sigma_2 S_i \\
B_1(E, q_1, p_2) S^i_2 & C_{ij}(E, q_2, p_2) S^i_2 S_j
\end{pmatrix} (P_1^+) = \begin{pmatrix}
0 & A_1(E, q_1, p_1) e^{i}_j \\
B_1(E, q_1, p_2) e^{i}_{p_2} & \frac{1}{2} C_{ij}(E, q_2, p_2) \delta_{ij}
\end{pmatrix}.
\tag{A.30}
\]

Taking into account Eqs. (A.15) and (A.30), in the cluster configuration space the kernel projected to the \(J^\pi = 1^-\) channel is given by
\[
K^{(1^-)}_{E, q_1, q_2, p_1, p_2} = \begin{pmatrix}
0 & K^{(1^-)}_{na}(E, q_2, p_1) \\
K^{(1^-)}_{an}(E, q_1, p_2) & K^{(1^-)}_{nn}(E, q_2, p_2)
\end{pmatrix},
\tag{A.31}
\]
where
\[
K^{(1^-)}_{na}(E, q, p) = -\frac{g_1^2 + m_\alpha}{4} \frac{1}{2} \int_{-1}^{1} d \cos \theta \left[ \frac{(1 + r)q \cos \theta + 2p}{m_n E - q^2 - \frac{m_n q^2}{2p} - qp \cos \theta + i\epsilon} \right],
\tag{A.32}
\]
\[
K^{(1^-)}_{an}(E, q, p) = -\frac{g_1^2 + m_\alpha}{4} \frac{1}{2} \int_{-1}^{1} d \cos \theta \left[ \frac{(1 + r)p \cos \theta + 2q}{m_n E - \frac{m_n q^2}{2p} - p^2 - q \cos \theta + i\epsilon} \right].
\tag{A.33}
\]
and

\[
K_{nn}^{(1)}(E, q, p) = -\frac{g^2_{\alpha} m_1}{6} \left[ \frac{1}{2} \int_{-1}^{1} d\cos \theta \left[ \frac{2(1-r)(q^2 + p^2) + (4 + (1-r)^2)q \cos \theta}{m_\alpha E - \frac{m_\alpha}{2M}(q^2 + p^2) - q \cos \theta} \right] \right]. \tag{A.34}
\]

Using Eqs. (A.21) and (A.22), the projected 1− kernels are finally obtained by

\[
K_{nn}^{(1)}(E, q, p) = -\frac{g^2_{\alpha} m_1}{4} \left[ \frac{2}{q} Q_0(\epsilon_{\alpha n}(E, q, p)) + \frac{1+r}{p} Q_1(\epsilon_{\alpha n}(E, q, p)) \right], \tag{A.35}
\]

\[
K_{\alpha n}^{(1)}(E, q, p) = -\frac{g^2_{\alpha} m_1}{4} \left[ \frac{2}{q} Q_0(\epsilon_{n\alpha}(E, q, p)) + \frac{1+r}{q} Q_1(\epsilon_{n\alpha}(E, q, p)) \right], \tag{A.36}
\]

and

\[
K_{n\alpha}^{(1)}(E, q, p) = -\frac{g^2_{\alpha} m_1}{6} \left[ \frac{2}{q} Q_0(\epsilon_{\alpha n}(E, q, p)) + \frac{1+r}{q} Q_1(\epsilon_{\alpha n}(E, q, p)) \right]. \tag{A.37}
\]

**Appendix B. The calculation for the contribution of the R1 diagram**

In the following, we calculate in detail one of the diagrams in Fig. 4 which contribute to the amplitude of the E1 \(nn\alpha\) capture transition. We focus on the first diagram of the second line in Fig. 4, i.e. the diagram \(R_1\). Figure B.1 shows the diagram \(R_1\) with all the energy–momentum values of the particles. After integrating over the energy of the loop, \(q_0\), the contribution of the diagram \(R_1\) is given by

\[
\tilde{R}_1(E_i, k) = -\frac{e Q_0 g^2_{\alpha}}{4} \frac{1}{E_t - E_i} \int_{0}^{\Lambda} \frac{q^2 \ dq \ 2\pi^2}{2\pi^2} t_{0,\alpha}(q) D_1 + \left( E_t - \frac{q^2}{2m_\alpha} \right) \times \int \frac{d\Omega_q}{4\pi} \left[ \frac{1}{m_\alpha E_i - \frac{m_\alpha}{2\mu}(k^2 + q^2) - k \cdot \hat{q}} - \frac{1}{m_\alpha E_\alpha - \frac{m_\alpha}{2\mu}(k^2 + q^2) - k \cdot \hat{q}} \right] \times \left[ 2(1-r)(k_{ik} q_i + (1-r)^2k_{i} q_i) + 4k_{j} q_i + (1-r)^2k_{i} q_{ij} \right] S_j^\dagger S_i (\tilde{k} + \hat{q}) \cdot \tilde{\sigma}_y. \tag{B.1}
\]

In the above equation, for convenience the \(k_\alpha\) momentum is constituted by \(k\). The dominated contribution of the E1 \(n + n + \alpha \rightarrow ^6\text{He} + \gamma\) process is given by the transition from the initial \(J^\pi = 1^+\) channel to the final \(0^+\) \(^6\text{He}\) state. Taking into account the projection operators (A.12) and (A.29), the projection of the diagram \(R_1\) to the E1 \(1^\to 0^+\) transition is done using the following relation:

\[
\left( \sqrt{\frac{3}{2}} \sigma_2 \tilde{S} \cdot \hat{e}_i \right) S_j^\dagger S_i \left( \sqrt{\frac{3}{2}} \tilde{S}^\dagger \cdot \hat{e}_j \sigma_2 \right) = \frac{2}{3} \epsilon_j^i \epsilon_i. \tag{B.2}
\]

With respect to Eq. (B.2), we have

\[
\tilde{R}_1(E_i, k) = -\frac{e Q_0 g^2_{\alpha}}{6} \frac{1}{E_t - E_i} \int_{0}^{\Lambda} \frac{q^2 \ dq \ 2\pi^2}{2\pi^2} t_{0,\alpha}(q) D_1 + \left( E_t - \frac{q^2}{2m_\alpha} \right) \frac{1}{2} \int_{-1}^{1} d\cos \theta \times \left[ \frac{1}{m_\alpha E_i - \frac{m_\alpha}{2\mu}(k^2 + q^2) - kq \cos \theta} - \frac{1}{m_\alpha E_\alpha - \frac{m_\alpha}{2\mu}(k^2 + q^2) - kq \cos \theta} \right] \times \left[ \left( (1-r)(2q^3 + (1-r)k^2 q) \right) + \left( (3-r)k^2 q + 2(1-r)k^3 \right) \cos \theta \right. \\
+ \left. 2(3-r)k^2 q \cos \theta \right] \tilde{\sigma}_y \cdot \hat{e}.
\tag{B.3}
\]
Fig. B.1. The diagram $R_1$ of Fig. 4.

As in the previous appendix, using Eqs. (A.21) and (A.22) the relation (B.2) is replaced by

$$
\bar{R}_1(E_i, k) = -\frac{e Q_0 g_1^2}{6} \frac{1}{E_f - E_i} \int_0^\Lambda q^2 dq \left[ \frac{1}{2\pi^2} t_{He}(q) D_1(E_f - q^2/E) \right]
$$

$$
\times \left\{ \left[ (3 - \frac{8}{3} r + r^2) k + 2(1 - r) \frac{q^2}{k} \right] \left[ Q_0(\epsilon_{nn}(E_i, q, k) - Q_0(\epsilon_{nn}(E_f, q, k)) \right]
$$

$$
+ \left[ (3 - r)^2 q + 2(1 - r) \frac{k^2}{q} \right] \left[ Q_1(\epsilon_{nn}(E_i, q, k) - Q_1(\epsilon_{nn}(E_f, q, k)) \right]
$$

$$
+ \frac{4}{3} (3 - r) k \left[ Q_2(\epsilon_{nn}(E_i, q, k) - Q_2(\epsilon_{nn}(E_f, q, k)) \right]\right\} \vec{\epsilon}_y^* \cdot \hat{e}.
$$

Finally, we can write the contribution of the diagram $R_1$ without the $\vec{\epsilon}_y^* \cdot \hat{e}$ factor as

$$
\tilde{R}_1(E_i, k) = R_1(E_i, k) \vec{\epsilon}_y^* \cdot \hat{e}.
$$

References