Analysis of the chiral phase transition and the color superconductivity by evaluating the Wilsonian effective potential in thermal gauge theories

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Abstract

We investigate properties of the spontaneous chiral symmetry breaking and the color superconductivity in finite density QCD by using non-perturbative renormalization group. When we lower the renormalization scale, effective 4-fermi interactions are generated by the gauge interactions, which consequently bring about the spontaneous symmetry breakdown at the low energy scale. We analyze the behavior of these 4-fermi coupling constants to obtain the critical density for color superconductivity at zero temperature.

1 Introduction

In high energy region quantum chromodynamics (QCD) is verified by using the perturbation theory due to the asymptotic freedom. On the other hand, the strong gauge interactions in low energy region bring about the quark confinement and the spontaneous chiral symmetry breakdown (SxSB) as non-perturbative phenomena. Furthermore various phase structures are expected at finite temperature and finite density.

\[
\begin{align*}
\sum_{A=1}^{N^2_c-1} T^A_{d'a} T^A_{b'b} &= -\frac{N_c+1}{4N_c} (\delta_{a'a'}\delta_{b'b} - \delta_{a'b'}\delta_{a'b}) + \frac{N_c-1}{4N_c} (\delta_{a'a'}\delta_{b'b} + \delta_{a'b'}\delta_{a'b'}). \\
\end{align*}
\]

Figure 1: diquark channel exchanging one gluon.

The spontaneous chiral symmetry breakdown and the color superconductivity at zero temperature and finite density are our targets of this article. First of all we explain the color superconductivity briefly. The color superconductivity is the phenomenon that the condensation of quark-quark pair (diquark) breaks the color symmetry. The condensation is caused by the attractive interaction of the antitriplet channel of diquark. Let’s see quark-quark interactions due to one-gluon exchange (Fig. 1). To diagonalize the interaction Hamiltonian, we take irreducible representations in the color space of the quark-quark state, that is, antisymmetric state and a symmetric state. We rewrite the direct product of two color generators at the gauge vertices of the Feynman diagram in the following way,

We find that the antitriplet state is the attractive channel while the sextet state is the repulsive channel. Due to this attractive interaction, the antisymmetric state of diquark may have non-zero expectation value at low temperature and high density:

\[
\langle (\bar{\psi}^C \gamma_i \gamma_j \psi^D) \rangle \sim \epsilon_{ij} \epsilon^{abc},
\]
where $\varepsilon_{ij}$ and $\epsilon^{abc}$ are antisymmetric tensors in the flavor and the color spaces, respectively.

We need some non-perturbative methods to analyze these phenomena. The lattice QCD known as a first principle calculation is a very effective method, however, the sign problem arises at finite density. The Dyson-Schwinger (DS) equations are usually limited to the ladder approximation, where the dependence on the gauge-fixing parameter is a serious problem.

The non-perturbative renormalization group (NPRG) method [1] is a complementary approach, where we can use the approximations without these difficulty. The central object in NPRG is the effective action $S_{\text{eff}}[\phi; \Lambda]$, which is defined by integrating fields with higher momentum than the scale $\Lambda$ in the partition function:

$$Z = \int \prod_{p<\Lambda} d\phi_< (p) \prod_{\Lambda<p<\Lambda_0} d\phi_>(p) e^{-S_0[\phi_<+\phi_>; \Lambda_0]} = \int \prod_{p<\Lambda} d\phi_< (p) e^{-S_{\text{eff}}[\phi_<; \Lambda]}.$$  \hspace{1cm} (3)

The NPRG equation describes the dependence of the Wilsonian effective action $S_{\text{eff}}[\phi; \Lambda]$ on the renormalization scale (momentum cutoff) $\Lambda$ in terms of differential equations,

$$\frac{\partial}{\partial \Lambda} S_{\text{eff}}[\phi; \Lambda] = \beta[S_{\text{eff}}; \Lambda].$$ \hspace{1cm} (4)

The right side of this equation, called the $\beta$ function (actually, a functional), is evaluated by infinitesimally lowering the scale $\Lambda$ in Eq. (3).

In this article, we report the analyses of the S$\chi$SB and the color superconductivity at zero temperature and finite density in 3-flavor QCD. In NPRG language, two types of 4-fermi operators, scalar type and vector type, are automatically generated by the QCD interaction when we lower the renormalization scale. The scalar type operators are the source of S$\chi$SB just like Nambu-Jona-Lasinio model. On the other hand, the vector type operators bring about the spontaneous color symmetry breaking at low temperature and high density.

So far, the color superconductivity has been analyzed by using the ladder DS equation. In our analysis of using NPRG, diagrams beyond the ladder approximation can be taken into account, which we call the non-ladder contributions. We will evaluate the critical chemical potential for the color superconductivity by analyzing the NPRG running behaviors of the effective 4-fermi coupling constants. This is a first step analysis towards the evaluation of the diquark condensates themselves.

## 2 Non-perturbative renormalization group

Several types of the NPRG equation for quantum field theory have been derived in Refs. [2, 3, 4]. We adopt the Wegner–Houghton (WH) equation [2, 5] among them. The WH equation is given by differentiating Eq. (3) with respect to $\Lambda$. We parametrize the renormalization scale $\Lambda$ by $t \equiv \log \Lambda_0 / \Lambda$ and transform all variables into dimensionless ones by taking $\Lambda$ as a moving unit of mass. Then we have the following functional differential equation,

$$\frac{\partial}{\partial t} S_{\text{eff}}[\Phi; t] = \frac{1}{2} \int \frac{d^D p}{(2\pi)^D} \delta(1-|p|) \text{Str} \ln \left( \frac{\delta^2 S_{\text{eff}}}{\delta \Phi \delta \Phi} \right)_{p-p} - \frac{1}{2} \int \frac{d^D p d^D q}{(2\pi)^{2D}} \delta(1-|p|) \delta(1-|q|) \frac{\delta S_{\text{eff}}}{\delta \Phi_p} \left( \frac{\delta^2 S_{\text{eff}}}{\delta \Phi_p \delta \Phi_q} \right)^{-1} \frac{\delta S_{\text{eff}}}{\delta \Phi_q} - DS_{\text{eff}} - \int_0^1 \frac{d^D p}{(2\pi)^D} \Phi_p \left( \frac{d\Phi_p + \eta_p}{2} + p^\mu \frac{\partial}{\partial p^\mu} \frac{\delta}{\delta \Phi_p} S_{\text{eff}} \right),$$  \hspace{1cm} (5)

where $\Phi$ represents all types of fields (scalars, spinors, vectors, and so on), $d\Phi$ is the mass-dimension of $\Phi$ and "Str" is the super-trace for generally Grassmann-valued matrices. We introduce the wave function renormalization to normalize the kinetic term of each field, which adds the anomalous dimension term $\eta_p$ of each field.

Here we explain the physical meanings of the right-hand side of the WH equation (5) briefly (see [1] for details). The first and second terms are represented by ring-type and dumbbell-type diagrams.
of Fig. 2 respectively, whose external lines represent fields having the momentum lower than $\Lambda$. In those diagrams the momentum $l$ of every internal line is the shell mode which satisfies the condition $|l| = \Lambda$. These diagrams are nothing but the quantum corrections due to the shell mode fields, while the low momentum modes are "external" source fields. Among all the loop diagrams using the shell mode propagators, only the tree and the one-loop diagrams survive, because diagrams with two or more loops do not contribute to the derivative with respect to $t$ (or $\Lambda$). The rest of terms in Eq. (5) come out of the rescaling of fields which are necessary to make all the variables dimensionless using the moving unit $\Lambda(t)$. Change of the effective action due to these terms are called the canonical scaling. There is another origin of rescaling ($\eta_\Phi$), which is the normalization of the kinetic term of each field.

$$\Delta S_{\text{eff}} \bigg|_{\Lambda=\Lambda_0} \equiv S[\bar{\psi}, \psi, A] = \int d^4x \left\{ \bar{\psi} (\not\partial - g_s A) \psi + \frac{1}{4} (F_{\mu\nu}^a)^2 + \frac{1}{2\alpha} (\partial_\mu A_\mu)^2 \right\},$$

where we ignore the current mass of the quarks and the action has the exact chiral symmetry. Solving the NPRG equation, we get the effective action $S_{\text{eff}}(\Lambda)$ at the renormalization scale $\Lambda$, which exhibits the low-energy information including the $\chi$SB and the color superconductivity.

We can not exactly solve the equation (5), and we have to develop an appropriate approximation method. In general approximation is defined by projecting the original functional differential equation (5) onto some small dimensional sub-space. Here we adopt the so-called local potential approximation (LPA), where we ignore all corrections to the effective action containing derivative of fields. Then the effective action is limited to be local potential terms and the fixed (non-moving) kinetic terms. In other words our subspace here is a function space of local potentials instead of the general functional space. Actually we set all the external fields in Fig. 2 to have the vanishing momentum, that is,

$$\Phi(p) = (2\pi)^D \delta^D(p) \cdot \Phi.$$  

In this approximation, we can expand the effective action $S_{\text{eff}}$ in terms of all possible local operators respecting symmetries of QCD. Then, expressing the right-hand side of (5) by these local operators, we obtain the $\beta$ functions for the coupling constants of the operators. In fact, however, we can not evaluate an infinite number of the $\beta$ functions for infinitely higher-dimensional operators. Due to the canonical scaling, we suppose that the contributions of higher dimensional operators are relatively smaller. Therefore we truncate operators having more than a certain mass-dimension. By checking
the convergence of physical quantities with respect to the order of the truncation, we may confirm the reliability of the truncation itself.

The massless QCD with $N_f$ flavors has the following chiral and color symmetry,

$$SU(N_f)_R \times SU(N_f)_L \times SU(N_c) \times U(1)_V.$$  \hspace{1cm} (8)

The effective action generated by NPRG equation necessarily respects this symmetry since the $\beta$ function does not break any global symmetry in the initial action $S_0$. As for the working ground of NPRG equation, we adopt the complete 4-fermi operator space satisfying this symmetry[6]. Ignoring the operators which are not generated by QCD gauge interactions, the 4-fermi operators we need are the following 4 operators,

$$O_1 = (\bar{\psi} \gamma_\mu \psi)^2 - (\bar{\psi} \gamma_5 \gamma_\mu \psi)^2,$$

$$O_2 = (\bar{\psi} \gamma_\mu \psi)^2 + (\bar{\psi} \gamma_5 \gamma_\mu \psi)^2,$$

$$O_{c1} = (\bar{\psi} \gamma_\mu T^a \psi)^2 - (\bar{\psi} \gamma_5 \gamma_\mu T^a \psi)^2,$$

$$O_{c2} = (\bar{\psi} \gamma_\mu T^a \psi)^2 + (\bar{\psi} \gamma_5 \gamma_\mu T^a \psi)^2.$$  \hspace{1cm} (9)

On the other hand, we truncate higher order corrections to the gauge interactions themselves since we consider that they are not important for analysis of these symmetry breakings treated here.

Finally the effective action we analyze is

$$S_{\text{eff}}[\bar{\psi}, \psi, A; \Lambda] = \int d^4 x \left\{ \bar{\psi} (i \gamma^\mu \partial_\mu - g_A (\Lambda) A_\mu) \psi + \sum_{j}^{4\text{-fermi}} \frac{G_i (\Lambda)}{2N_f N_c} O_i (\bar{\psi}, \psi) + \frac{1}{4} (F_{\mu\nu}^a)^2 + \frac{1}{2\alpha} (\partial_\mu A_\mu)^2 \right\}.$$  \hspace{1cm} (10)

Here we use the Landau gauge ($\alpha = 0$) because it has been known to be a good gauge for LPA.

![Figure 3: All diagrams giving the $\beta$ functions for 4-fermi operators in our framework. The diagrams surrounded by the dashed line are called the ladder-type diagrams.](image)

The special diagrams (similar to the Feynman diagrams), are used in order to calculate the $\beta$ functions. In LPA the internal-line momenta of the dumbbell-type diagrams can not take the shell mode because the external-line momenta are all vanishing. Therefore the dumbbell-type diagrams do not contribute, so that we evaluate only the ring-type diagrams, which are drawn in Fig. 3.

Using these $\beta$ functions, the NPRG equation is now written down as a set of ordinary differential equations as follows.

$$\frac{dg_1}{dt} = -2g_1 + \frac{1}{N_f N_c} \left\{ 3g_1^2 + \frac{3}{4} \left( 1 - \frac{1}{N_c^2} \right) g_2^2 + 2(N_f N_c + 1) g_1 g_2 + \left( N_c - \frac{1}{N_c} \right) g_1 g_2 \right\}$$

$$+ \frac{3}{4\pi} \left( 1 - \frac{1}{N_c^2} \right) g_1 \alpha_s + \frac{3N_f N_c}{16\pi^2} \left( 1 - \frac{1}{N_c^2} \right) \alpha_s^2,$$  \hspace{1cm} (11)

$$\frac{dg_2}{dt} = -2g_2 + \frac{1}{N_f N_c} \left\{ N_f N_c g_1^2 + (N_f N_c - 1) g_2^2 - \frac{3}{4} \left( 1 - \frac{1}{N_c^2} \right) g_2^2 + \left( N_c - \frac{1}{N_c} \right) g_2 g_2 \right\}$$

$$- \frac{3}{4\pi} \left( 1 - \frac{1}{N_c^2} \right) g_2 \alpha_s - \frac{3N_f N_c}{16\pi^2} \left( 1 - \frac{1}{N_c^2} \right) \alpha_s^2,$$  \hspace{1cm} (12)
Here 4-fermi coupling constants are transformed into dimensionless ones, \( g_i = \Lambda^2(t)G_i/4\pi^2 \). Note that using dimensionless variables are essential for the direct correspondence between the fixed point of the renormalization group flow and the phase boundary (in case of 2nd order phase transition). By this transformation, we can reveal the fixed-point structure of the \( \beta \) functions \([7]\). The first term, \(-2g_i\), in the above beta functions represents the canonical scaling contribution due to this transformation, and it suppresses the 4-fermi interactions. The quadratic terms coming from the ring-type diagrams generally enhance the 4-fermion interactions. Therefore, there is a point where these opposite effects are canceled, that is, a fixed point (a zero of \( \beta \) function), which corresponds to the phase boundary.

As for the gauge coupling constants, we adopt the following (N)PRG equation given by the one-loop perturbation,

\[
\frac{d}{dt} \alpha_s = \frac{\beta_0}{2\pi} \alpha_s^2, \tag{15}
\]

where \( \beta_0 = 11N_c/3 - 2N_f/3 \) and \( \alpha_s = g_i^2/4\pi \). Also we introduce the infrared cutoff effect \([10]\) that the gauge coupling constant stop increasing at a proper low energy scale, which is naturally expected by the confinement. Adopting the cutoff scheme in Ref. \([11]\), the running gauge coupling constant is given explicitly as follows (Fig. 4),

\[
\alpha_s(\Lambda) = \begin{cases} 
\frac{4\pi}{\beta_0 \log(\Lambda^2/\Lambda_{QCD}^2)} & (\Lambda > \Lambda_{\text{IF}}) \\
\frac{4\pi}{\beta_0 \log(\Lambda_{\text{IF}}^2/\Lambda_{QCD}^2)} + \frac{4\pi}{\beta_0 \log(\Lambda_1/\Lambda_{\text{IF}})} \left( \log(\Lambda_1/\Lambda_{\text{IF}}) \right)^2 & (\Lambda_1 < \Lambda < \Lambda_{\text{IF}}) \\
\frac{4\pi}{\beta_0 \log(\Lambda_{\text{IF}}^2/\Lambda_{QCD}^2)} - \frac{4\pi}{\beta_0 \log(\Lambda_1/\Lambda_{\text{IF}})} \log(\Lambda_{\text{IF}}^2/\Lambda_{QCD}^2)^2 & (\Lambda < \Lambda_1)
\end{cases}
\tag{16}
\]

Here we take a fixed scale for \( \Lambda_1, \Lambda_{\text{QCD}} \cdot e^{-1} \), and \( \Lambda_{\text{IF}} \) is left as an infrared cutoff scale parameter. Because the scale \( \Lambda_{\text{IF}} \) is rather arbitrary, we will check the dependence of our results on this parameter.\(^1\)

As for the initial condition to solve the NPRG equations, we take following values

\[ \Lambda_0 = M_Z = 91.1 \text{ GeV}, \quad \alpha_s(\Lambda_0) = 0.1176, \]

and also we set \( N_f = N_c = 3 \), and \( \Lambda_{\text{QCD}} = 241 \text{ MeV} \).

If we take the initial renormalization scale \( \Lambda_0 \) at which the perturbation theory of QCD works well, we can set the 4-fermi coupling constants \( g_i \) to be zero at \( \Lambda_0 \). When we lower the renormalization scale \( \Lambda \), the 4-fermi operators are generated by gauge interactions, that is, two diagrams in third line of Fig. 3, having only the gauge interactions. The 4-fermi operators are normally irrelevant at low energy scale because of its mass dimension larger than 4. However in case that gauge interactions are strong enough, 4-fermi interactions are generated sufficiently large so that the self interactions may overwhelms the canonical scaling, that is, they become relevant. This change of the relevance of 4-fermi operators, from irrelevant to relevant, characterizes the \( S_S \)SB at low energy in QCD.

The critical gauge coupling constant for the \( S_S \)SB obtained by NPRG method is equivalent to that by the ladder DS equation if we use the \( \beta \) function of the ladder-type diagrams (in Fig. 3) only. Note that we can easily add non-ladder diagrams to the \( \beta \) function in our NPRG framework, which may improve the dependence on the gauge-fixing parameter.

\(^1\)Physical quantities like the chiral condensates are not sensitive to the choice of the infrared cutoff scale \( \Lambda_{\text{IF}} \)[11]
4 Introducing the bare mass

As noted before, no effective operator breaking the chiral symmetry can be generated in the effective action by NPRG equation. To deal with the spontaneous breakdown of the chiral symmetry, we introduce a mass term in the effective action as follows [6]:

$$S_{\text{eff}}(\text{massless}) + \int d^4x\ m(m_0; \Lambda) \bar{\psi} \psi,$$

where $m(m_0; \Lambda) = m_0$ is a bare mass which is also regarded as an external source for chiral condensates. Due to this explicit chiral breaking mass, the 4-fermi $\beta$ functions are modified [6].

If the bare mass $m_0$ is finite, the running mass $m(m_0; \Lambda)$ is increased by the gauge interaction and the 4-fermi interactions at the infrared scale (Fig. 6). We obtain the spontaneously generated mass at the infrared scale by taking the zero bare mass limit (the chiral limit),

$$m_{\text{dyn.}} = \lim_{m_0 \to 0} \lim_{\Lambda \to 0} m(m_0; \Lambda).$$

To confirm that this method works well, we check the behaviors of the effective mass operator in detail as a function of the bare mass and the renormalization scale. In the left graph of Fig. 6, we plot the effective mass (with mass dimension, not scaled by $\Lambda$) for each bare mass at zero temperature and zero density. The dashed lines connect the same renormalization scale points. We see a “the phase transition” at some lower scale. At high-energy scale, the effective mass vanishes when the bare mass goes towards zero. Below some scale, however, behavior of the effective mass changes drastically, and there remains a finite effective mass even in the zero bare mass limit.

A different view is shown in the right graph of Fig. 6, where the renormalization flow of the 4-fermi coupling constant $g_{\text{4}}$ (dimensionless, rescaled by $\Lambda$) is added. We see that the effective mass increases rather suddenly just when the 4-fermi coupling constant is increased to have a peak. The dashed lines connect the same bare mass points. The quarks are decoupled from the interactions when the effective mass is much larger than the renormalization scale. Therefore, at the infrared limit, the effective mass (with mass-dimension) converges towards a finite certain value while the 4-fermi coupling constants (dimensionless variables) vanishes according to the canonical scaling.

As shown in the right graph of Fig. 6, the peak value of the 4-fermi coupling constant increases when we lower the bare mass. In the ladder approximation without the operator expansion, we can prove that
the scalar-type 4-fermi coupling constant corresponds to the susceptibility of the chiral condensates with respect to the bare mass. Hence, taking the zero bare mass limit, this scalar-type 4-fermi coupling constant will be increased to be infinite at the transition scale, and the behavior of the effective mass with respect to $\Lambda$ is very similar to the second-order phase transition. Also in our calculation with non-ladder effects included, we may claim the correspondence between the spontaneous chiral symmetry breakdown and the divergence of the scalar type 4-fermi coupling constant in case of vanishing bare mass. It should be mentioned here that the 4-fermi $\beta$ functions in Eqs. (11) – (14) do not contain any higher dimensional operators in the LPA due to the exact chiral symmetry. This is called the perfectness of this operator coordinates. Inclusion of the bare mass term breaks this feature and there appear 6-fermi operators which contributes to 4-fermi $\beta$ functions. In this stage of calculation we ignore this sort of operator mixings, because naively they have only small effects after evaluating the chiral limit.

5 Property of 4-fermi operators

We discuss basic properties of 4-fermi operators in relation to the spontaneous breakdown of the chiral and the color symmetries. Only two 4-fermi coupling constants, $g_1$ and $g_{c1}$ appear in the $\beta$ function of the effective mass [6]. These operators are projected onto a scalar-type 4-fermi operator as follows,

$$\frac{G_1}{2N_fN_c}O_1 + \frac{G_{c1}}{2N_fN_c}O_{c1} = \frac{1}{2N_fN_c} \left[ \frac{2}{N_fN_c}G_1 + \frac{1}{N_f} \left( 1 - \frac{1}{N_c} \right) G_{c1} \right] \sum_{L=0}^{N_c-1} \left[ (\bar{\psi}\lambda_I^L\psi)^2 - (\bar{\psi}\lambda_I^{L\gamma_5}\psi)^2 \right] + \cdots ,$$

(20)

where $\lambda_I$ is the generator of $U(N_f)$. Therefore $g_1$ and $g_{c1}$ are related to the susceptibility of the chiral condensates.

Other 4-fermi operators, $O_2$ and $O_{c2}$, are vector-type operators for quark-antiquark pair. Using these 4-fermi operators, we can construct the following scalar-type operator for diquark,

$$O_d = \left( \bar{\psi}_i^C e^a \psi_j^C \right) \left( \bar{\psi}_j e^a \gamma_5 \psi_i^C \right) - \left( \bar{\psi}_i^C e^a \gamma_5 \psi_j^C \right) \left( \bar{\psi}_j e^a \gamma_5 \psi_i^C \right),$$

(21)

where $i, j$ are flavor indices and $(e^a)^{bc} \equiv e^{abc}$ is the antisymmetric tensor in the color space. This operator $O_d$ is coupled to the color antisymmetric (attractive) channel of diquark, and has been considered to induce the color superconductivity in the analysis of NJL-like effective models. Hence we suppose that $O_2$ and $O_{c2}$ trigger the diquark condensates in the framework of NPRG, which will be confirmed below.
In order to effectively take account of the symmetry breakdown of the color symmetry, we introduce a Majorana mass term explicitly breaking the color symmetry as follows,

\[ S_{\text{eff}} + \frac{1}{2} \Delta^{ij}(\Delta_0; \Lambda) \bar{\psi}_i^C e^\alpha \gamma_5 \psi_j + (\text{h.c.}) , \]

(22)

where the mass term \( \Delta^{ij}_0 \) is also regarded as the external field for the diquark condensates. The \( \beta \) function of \( \Delta^{ij}_0 \) consists only of the 4-fermi coupling constants, \( g_2 \) and \( g_{c2} \). This fact confirms that operators, \( \mathcal{O}_2 \) and \( \mathcal{O}_{c2} \), generate the diquark condensates. In addition, according to analogy of the chiral condensates, we consider that these coupling constants represent the susceptibility divergence of the diquark condensates.

We will not proceed to direct evaluation of the diquark condensates in this report. In the next section, however, we will study the change of RG flows of the 4-fermi coupling constants due to finite density, which will give us a clue about the color superconductivity phase transition at zero temperature and finite density.

6 Results

Fig. 7 shows RG of 4-fermi coupling constants with various values of the chemical potential. Here we ignore finite-density effects to the running of the gauge coupling constant.

We should mention here that the scalar-type 4-fermi coupling constants, \( g_1 \) and \( g_{c1} \), diverge at infrared region obeying the behavior of the susceptibility of the chiral condensates if we make the bare mass smaller to be zero. The vector-type 4-fermi coupling constants, \( g_2 \) and \( g_{c2} \), corresponding to the susceptibility of the diquark condensates, also diverge because the \( \beta \) functions of \( g_1 \) and \( g_{c1} \) include not only the self coupling constants but also \( g_2 \) and \( g_{c2} \). This mixing between the scalar-type and the vector-type 4-fermi operators comes from the non-ladder effects. Of course we do not conclude the color superconductivity from this divergence of \( g_2 \) and \( g_{c2} \).

Let us look at the chemical potential dependence in detail. The absolute value of peaks of the scalar-type 4-fermi coupling constants (\( g_1 \) and \( g_{c1} \)) are larger than the vector-type ones at low density (\( \mu = 0.01 \) GeV). Increasing the chemical potential, the vector type 4-fermi operators are enhanced while the scalar type ones remain rather stable. Finally the vector-type 4-fermi coupling constants diverge at a density (\( \mu \sim 0.26 \) GeV). This behavior implies the color superconductivity phase transition occurs due to high density.

Next we evaluate the critical density \( \mu_c \) for the color superconducting phase transition. In Fig. 8 we plot the inverse of the peak value of \( g_{c2} \) with respect to density \( \mu \), where points connected by a line are those with the same bare mass \( m_0 \). The critical density \( \mu_c \) is given by a point at which the inverse of the peak value goes to zero. We show the dependence of \( \mu_c \) on the bare mass in the left graph of Fig. 9, where we extrapolate \( \mu_c \) linearly to get the zero bare mass limit.

The dependence of the extrapolated critical densities \( \mu_c \) on the infrared cutoff scale \( \Lambda_{\text{IR}} \) is small as shown in the right graph of Fig. 9.

We conclude that the critical chemical potential for the color superconductivity at zero temperature reads

\[ \mu_c = 0.16 \text{ GeV} \ (\Lambda_{\text{IR}} = 0.393 \text{ GeV}) \]

Note that our result above is approximately a half of that which has been obtained by other methods until now. We expect that the critical density might become larger if we use the two-loop \( \beta \) function for the gauge coupling constant.

7 Summary

We analyzed the color superconductivity at zero temperature and finite density by using the WH equation, one of the methods of NPRG. In this report, we worked with the one-loop \( \beta \) function for the gauge coupling constant and the complete \( \beta \) functions in the 4-fermi operator space respecting the chiral symmetry of QCD. We have taken into account the diagrams beyond the ladder approximation.
We introduced two types of mass operators explicitly breaking the chiral and the color symmetries in order to evaluate the size and the criticality of the spontaneous breakdown of these symmetries. These masses at the initial renormalization scale are nothing but the external fields giving the chiral and the diquark condensates, respectively.

We have classified the 4-fermi operators into two categories, which are the scalar type operator generating the chiral condensates and the vector type ones generating the diquark condensates. According to this observation, we investigated the running behavior of the 4-fermi coupling constants at zero temperature and finite density. We obtained the critical density (chemical potential) by evaluating the divergence point of the 4-fermi operators responsible for the color superconductivity. This is the first result of the critical density for the color superconductivity in the NPRG (LPA) framework with non-ladder effects included.

References

Figure 8: $1/(\text{the peak value of } g_{t2})$ vs. density

Figure 9: The left figure shows the zero bare mass extrapolation of the critical density $\mu_c$, and the right figure plots the extrapolated critical density for the different values of the infrared cutoff scale $\Lambda_{1F}$.


