ON THE TWO-BEAM FFAG ACCELERATOR

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ABSTRACT

Analytical expressions for the orbits and the linear betatron oscillations of the two-beam FFAG accelerator are given and compared with the results obtained by the digital computer. Typical design parameters are discussed.

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INTRODUCTION

As pointed out by the MURA group, (1) the method of achieving very high energy collisions between elementary particles by means of colliding beams appears highly promising if the potential high intensity of the FFAG accelerators is realized. As first envisaged, a colliding beam accelerator would consist of two FFAG accelerators tangent to one another. An alternative method (2), (3) employing a pulsed accelerator and storage rings has been proposed by D. Lichtenberg and G.K. O'Neill.

This report concerns the proposed new colliding beam method, (4), (5) where both beams are in the same accelerator, circulating in opposite directions. Since this method employs a single accelerator, the troublesome problems in the other methods, such as target sections having inherently non-scaling features and beam transfer (usually not too efficient), can be avoided.

The machine is essentially a radial sector FFAG machine and has a fairly large circumference factor. However the simple structure of the magnets compared to two spiral machines, the many intersections of beams for experiments and the feasibility of changing the reaction energy make the machine an interesting possibility.

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(1) D.W. Kerst et al Phy. Rev. 102 No. 2, 1956
(2) D. Lichtenberg et al MURA-DBL/RAN/HMR-1
(3) G.K. O'Neill Phy. Rev. 102 1418 (1956)
(4) T. Ohkawa, MURA-124
(5) L.W. Jones, MURA-134
GENERAL DESCRIPTION OF THE MAGNETIC FIELD

In the magnetic fields of FFAG accelerators, there exist an infinite number of equilibrium orbits due to the nonlinearity of the field. However these orbits have extremely large circumference factors, except two orbits which are clockwise and anticlockwise respectively in a radial sector machine. These two orbits generally have different properties, such as the circumference factor and the tunes of the betatron oscillations. The one which is not used in an ordinary radial sector machine is usually unstable. We are interested here in making these two orbits cross each other at an equal energy of particles and work at the same point on the tune diagram. This can be achieved by using a magnetic field having a certain symmetry property.

We write the median plane field as

$$H_{z_0} = - H_0 \left( \frac{r}{r_0} \right)^k f(N \Theta)$$

where

$$f(N \Theta + 2 \pi) = f(N \Theta)$$

The Lagrangian for the motion in the median plane is given by

$$\mathcal{L} = \pm p \sqrt{r^2 + \dot{r}^2} + \frac{e}{c} r A_{\Phi_0}$$

where

$$\frac{e}{c} r A_{\Phi_0} = - \frac{e}{c} H_0 \frac{f}{k+2} \frac{r}{r_0} f(N \Theta)$$

primes denote derivatives with respect to $\Theta$ and $\pm$ is chosen depending on the direction of rotation.
It is obvious from the Lagrangian (1) that the condition for obtaining the identical two-way orbits is to make $f(N\theta)$ an odd function of $N\theta$, i.e.

$$f(-N\theta) = -f(N\theta)$$

(4)

because the equation of motion is identical for both directions if $f(N\theta)$ satisfies (4).

By rewriting $f(N\theta)$ in Fourier series form

$$f(N\theta) = \sum_n \left( g_n \cos nN\theta + h_n \sin nN\theta \right)$$

(5)

(4) implies

$$g_n = 0 \quad \text{for all } n$$

(6)

Customarily we put $N\theta = 0$ at the middle of the "positive magnet" and (5) and (6) become

$$f(N\theta) = f_1 \cos N\theta - f_2 \sin 2N\theta + f_3 \cos 3N\theta - \cdots$$

(7)

by putting $N\theta \rightarrow N\theta + \frac{\pi}{2}$

Especially if all even $f$'s vanish, (7) becomes

$$f(N\theta) = \sum_{j=1}^{\infty} f_{2j+1} \cos (2j+1)N\theta$$

(8)

and we have an additional symmetry around the middle of the positive magnet.

From the results of rough estimate of the orbits and the betatron oscillations around them, it is realized that the contributions from higher harmonics
of the field depend on a quantity \( \sum \frac{f_j^2}{J_j^2} \) except for the axial focusing, which depends also on \( \sum \frac{1}{J_j^2} \). In actual fields, the higher harmonic content is not very high. For example, \( f_{2j+1} \) is given by \( \frac{1}{2j+1} \) for a rectangular-shaped field. Therefore, to understand the behavior of the machine, a pure sinusoidal field is not a poor choice.

**ORBITS**

The Lagrangian (2) gives the equation of motion in the median plane

\[
\left( \frac{r'}{\sqrt{r^2 + \gamma'^2}} \right)' - \frac{r'}{\sqrt{r^2 + \gamma'^2}} = -\frac{eH_0}{c\beta} \frac{r_{k+1}}{r_k} f(N\Theta) \quad (9)
\]

By putting \( \ln \frac{r}{r_i} = \varphi \), (9) becomes

\[
\left( \frac{\varphi'}{\sqrt{1 + \varphi'^2}} \right)' - \frac{1}{\sqrt{1 + \varphi'^2}} = -\frac{eH_0}{c\beta} \frac{r_{k+1}}{r_k} e^{(k+1)\varphi} f(N\Theta) \quad (10)
\]

where \( r_i \) is chosen as the average radius of the equilibrium orbits so that

\[
\int_{-\pi}^{\pi} \varphi'(N\Theta) = 0
\]

From (9), we get

\[
\left( \varphi'' - \varphi'^2 - 1 \right)(1 + \varphi'^2) e^{-(k+1)\varphi} = -\alpha f(N\Theta) \quad (11)
\]
where
\[ \alpha = \frac{\varepsilon H_0}{c_p} \frac{\gamma_{k+1}}{\gamma_o} \] (12)

To obtain an approximate solution for \( \varphi \), the left-hand side of (11) is expanded in power series in \( \varphi \) and the coefficients of \( \varphi \) in Fourier series evaluated by harmonic balance.

By using
\[
( \varphi'' - \varphi^2 - 1 )(1 + \varphi^2)^{-\frac{1}{2}} = -1 + \varphi'' + \frac{\varphi'^2}{2} (1 - 3 \varphi'') - \frac{3}{8} \varphi^4 (1 - 5 \varphi'') + \cdots \tag{13}
\]

\[
( - (k+1) \varphi ) = 1 - (k+1) \varphi + \frac{(k+1)^2}{2} \varphi^2 + \cdots \tag{14}
\]

and
\[
\varphi = \sum_j \varphi_j \cos j \Omega \tag{15}
\]

the left-hand side of (11) becomes
\[
( \varphi'' - \varphi'^2 - 1 )(1 + \varphi^2)^{-\frac{1}{2}} \varphi = \sum \varphi_j \cos j \Omega \tag{16}
\]
where

\[ C_0 = -1 + \frac{(k+1)N^2}{2} \left\{ q_1^2 \left\{ 1 + \frac{1}{2(k+1)} - \frac{k+1}{2N^2} \right\} + \frac{3}{2} \left\{ 1 + \frac{1}{2(k+1)} - \frac{k+1}{8N^2} \right\} \right. \]

\[ + \frac{9}{2} \left\{ \frac{k+1}{6} - \frac{3}{4} N^2 (1 + \frac{1}{2(k+1)}) \right\} + \left. \frac{(k+1)^2}{2} \left\{ \frac{k+1}{4} - \frac{7}{2} N^2 (1 + \frac{1}{2(k+1)}) \right\} + \ldots \right. \]

\[ C_1 = \frac{q_1^2}{8} \left\{ 3N^4 + (k+1)^3 - N^2 (k+1) - 3N^2 (k+1)^2 \right\} \]

\[ + \frac{q_1^2}{4} \left\{ 12N^4 + (k+1)^3 - 4N^2 (k+1) - 9N^2 (k+1)^2 \right\} \]

\[ + \frac{q_1^2}{3} \left\{ 27N^4 + \frac{(k+1)^3}{6} - 3N^2 (k+1) - \frac{5}{2} N^2 (k+1)^2 \right\} \]

\[ + \frac{q_1^2}{3} \left\{ \frac{(k+1)^3}{6} - \frac{9}{8} N^4 - \frac{5}{8} N^2 (k+1) - \frac{11}{8} N^2 (k+1)^2 \right\} \]

\[ C_2 = \frac{q_2^2}{12} \left\{ (k+1)^2 - 4N^4 + \frac{3}{8} [3N^4 + \frac{3}{8} (k+1)^3 - \frac{5}{4} N^2 (k+1)^2 + \frac{3}{4} N^2 (k+1)] \right\} \]

\[ + \frac{q_2^2}{12} \left\{ 12N^4 + \frac{(k+1)^3}{6} - N^2 (k+1) (k+2) \right\} + \frac{q_2^2}{3} \left\{ 18N^4 + \frac{(k+1)^3}{6} - N^2 (k+1)^2 \right\} \]

\[ + \frac{q_3^2}{6} \left\{ \frac{15N^4}{2} + \frac{3}{4} N^4 (4k+1) \right\} \]

\[ + \frac{q_1^2}{6} \left\{ \frac{N^2 (k+1)}{2} - \frac{N^2}{2} - \frac{(k+1)^2}{2} \right\} + \frac{q_3^2}{6} \left\{ \frac{3}{2} N^2 - \frac{(k+1)^2}{2} \right\} \]
For a pure sinusoidal field \( \phi(N\theta) = \cos N\theta \), we have

\[
C_0 = 0 \\
C_i = -\alpha \\
C_2 = 0 \\
C_3 = 0
\]

From (17) and (18), by neglecting \( \frac{1}{k+1} \) and \( \frac{k+1}{N^2} \) compared to unity and also cubic terms of the \( b_i \), we obtain

\[
\begin{align*}
\beta_1 &\sim \sqrt{\frac{2}{k+1}} \frac{1}{N} \\
\beta_2 &\sim \frac{1}{4N^2} \\
\beta_3 &\sim \sqrt{\frac{2}{k+1}} \frac{1}{N} \left\{ \frac{k+1}{24N^2} - \frac{1}{12(k+1)} \right\} \\
\alpha &\sim \sqrt{\frac{2}{k+1}} \frac{1}{N}
\end{align*}
\]
Inserting the above values in the higher order terms in (17) and neglecting $(\frac{1}{k+1})^2$, $\frac{1}{N^2}$ and $(\frac{k+1}{N^2})^2$ compared to unity, the $q$'s are given by

$$q_1 \approx \sqrt{\frac{2}{k+1}} \frac{1}{N} \left\{ 1 - \frac{1}{4(k+1)} + \frac{3}{8} \frac{k+1}{N^2} \right\}$$

$$q_2 \approx \frac{1}{4N} \left\{ 1 + \frac{1}{k+1} + \frac{3}{8} \frac{k+1}{N^2} \right\}$$

$$q_3 \approx \sqrt{\frac{2}{k+1}} \frac{1}{N} \left\{ \frac{k+1}{24N^2} - \frac{1}{12(k+1)} \right\} \left\{ 1 - \frac{1}{4(k+1)} + \frac{3}{8} \frac{k+1}{N^2} \right\}$$

$$\alpha \approx \sqrt{\frac{2}{k+1}} N \left\{ 1 - \frac{1}{k+1} - \frac{k+1}{2N^2} \right\}$$

The circumference factor $C$ is given by

$$C \approx \alpha \left( 1 + q_1 + q_2 + q_3 \right)^{k+1}$$

$$\approx \sqrt{\frac{2}{k+1}} N \left\{ 1 + \frac{k+1}{N^2} \right\} + \mathcal{O}$$

The above estimate of the $q$'s and $\alpha$ agrees well with the results obtained from the digital computer. Since the betatron oscillations are a more sensitive check on the orbit estimates, direct comparisons of $q$'s and $\alpha$ are not shown.

The circumference factors obtained by (21) are compared with the computer results in Table 1.
TABLE I - Circumference Factor

<table>
<thead>
<tr>
<th>N</th>
<th>k+1</th>
<th>C anal</th>
<th>C digital</th>
</tr>
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<tr>
<td>64</td>
<td>200</td>
<td>8.7</td>
<td>8.6</td>
</tr>
<tr>
<td>64</td>
<td>160</td>
<td>9.4</td>
<td>9.3</td>
</tr>
<tr>
<td>64</td>
<td>120</td>
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<td>18</td>
<td>13</td>
<td>8.4</td>
<td>8.8</td>
</tr>
<tr>
<td>18</td>
<td>12</td>
<td>8.7</td>
<td>9.1</td>
</tr>
<tr>
<td>18</td>
<td>10</td>
<td>9.2</td>
<td>9.6</td>
</tr>
</tbody>
</table>

BETATRON OSCILLATION

To obtain the equations for linear betatron oscillations around the orbits, the "soft-edge equations"(6) are used, because the large scalloping of the orbits makes the method of expansion (7) used for the spiral sector machine troublesome.

In the soft-edge equations,(8)

\[
\frac{d^2x}{ds^2} + \left[ \frac{F(\delta)}{\rho_0^2} + \frac{n_0}{\rho_0^2} F(\delta) \frac{y_o}{\gamma_c} + \frac{1}{\rho_0} \frac{dF}{ds} \tan \phi \right] x = 0
\]

\[
\frac{d^2z}{ds^2} - \left[ \frac{n_0}{\rho_0^2} F(\delta) \frac{y_o}{\gamma_c} + \frac{1}{\rho_0} \frac{dF}{ds} \tan \phi \right] z = 0
\]

(6) F. T. Cole et al. R.S.I. 28, 403 (1957)
(7) F. T. Cole, MURA-F.T.C-3
(8) The same notations in reference (6) are used.
we have
\[ F(s) = \left( \frac{\gamma e}{\gamma_0} \right)^k f(N\theta) \]  
(23)

\[ \ln \frac{\gamma e}{\gamma_i} = \sum \gamma_j \cos jN\theta = h(N\theta) \]
for our case.

(22) and (23) give
\[
\frac{d^2X}{ds^2} + \frac{1}{\gamma_i^2} \left[ \alpha^2 e^{\frac{2k h(N\theta)}{f^2(N\theta)}} 
+ \alpha e^{\frac{(k-1) h(N\theta)}{f(N\theta)}} \right] X = 0
\]  
(24)

\[
\frac{d^2\zeta}{ds^2} - \frac{1}{\gamma_i^2} \left[ \alpha e^{\frac{(k-1) h(N\theta)}{f(N\theta)}} \right] \zeta = 0
\]

\[ \lambda = \gamma_i \int e^{\frac{h(N\theta)}{\sqrt{1+h^2(N\theta)}}} d\theta \]  
(25)

By using (23) and (20), \( \lambda \) is given by
\[
\lambda - \lambda_0 \approx \gamma_i \left[ \frac{1}{2(k+1)} \right] + \frac{3}{2N^2 \sqrt{k+1}} \Delta N \theta + \ldots
\]  
(26)
Defining \( \Theta \) by
\[
\Theta = \frac{\Delta - \Delta_0}{\eta_1(1 + \frac{1}{2(k+1)})}
\]  \hspace{1cm} (27)

and assuming

\( \Theta \approx 0 \)

the equations (24) become

\[
\frac{d^2 X}{d \Theta^2} + \left(1 + \frac{1}{2(k+1)}\right)^2 \left[ \alpha^2 e^{2k h(N \Theta)} + \alpha \epsilon \left(k f(N \Theta) - h'(N \Theta) f'(N \Theta) (1 + h''(N \Theta))^{-\frac{1}{2}} \right) \right] X = 0
\]  \hspace{1cm} (28)

\[
\frac{d^2 Z}{d \Theta^2} - \left(1 + \frac{1}{2(k+1)}\right)^2 \left[ \alpha e^{2k h(N \Theta)} \right] Z = 0
\]

All terms depending on \( \Theta \) in the above equations are evaluated in

Fourier series form

\[
\epsilon^{(k-1)h(N \Theta)} (1 + h''(N \Theta))^{-\frac{1}{2}} = \sum C_j \cos jN \Theta
\]  \hspace{1cm} (29)

\[
k f(N \Theta) - h'(N \Theta) f'(N \Theta) = \sum D_j \cos jN \Theta
\]

\[
e^{2k h(N \Theta)} f'(N \Theta) = \sum E_j \cos jN \Theta
\]
where

\[ C_0 = 1 + \frac{q^2}{4} \left\{ \frac{(k-1)^2 - N^2}{2} \right\} + \frac{q^2}{4} \left\{ \frac{(k-1)^2 - 4N^2}{2} \right\} \]

\[ C_1 = \frac{g_1}{4} \left[ (k-1) + \frac{N^2}{4} (k-1) \right] \left\{ -\frac{g_1}{2} - \frac{1}{2} \frac{g_2}{2} - \frac{1}{2} g_3 (k-1)^2 \right\} \]

\[ + g_2 \left[ g_1 \{ N^2 + \frac{(k-1)}{2} \} + g_3 \{ -3N^2 + \frac{(k-1)}{2} \} - \frac{3}{2} (k-1)N^2 g_1 g_2 g_3 \right] \]

\[ C_2 = \frac{(k-1)}{4} g_2 + \frac{g_1}{4} \left\{ N^2 + (k-1)^2 \right\} + \frac{g_1}{2} g_3 \left\{ (k-1) - 3N^2 \right\} \]

\[ + (k-1)N^2 \left\{ -\frac{g_1}{4} g_2 - \frac{g_3}{2} - 2 g_1 g_2 g_3 - \frac{1}{4} g_1 g_2 g_3 \right\} \]

\[ C_3 = (k-1) g_3 + g_1 g_2 \left\{ \frac{(k-1)^2}{2} + N^2 \left\{ - (k-1)N^2 \right\} - g_1^2 g_3 \right\} \]

\[ - \frac{g_1^2 g_2}{2} \left[ - \frac{g_1}{4} g_2 - \frac{g_3}{2} + \frac{g_3}{2} - \frac{g_1}{2} g_2^2 \right] \]

\[ D_0 = - \frac{g_1}{2} N^2 \]

\[ D_1 = k - \frac{g_2}{2} N^2 \]

\[ D_2 = \frac{N^2}{2} (g_1 - g_3) \]

\[ D_3 = N^2 g_2 \]

\[ E_0 = \frac{1}{2} + \frac{k g_1}{2} + \frac{k^2}{2} \left\{ \frac{3}{2} g_1^2 + g_2^2 + g_3^2 + g_1 g_3 \right\} \]

\[ E_1 = \frac{3}{2} k g_1 + \frac{k g_3}{2} + 2 k^2 g_1 g_2 + \frac{3}{2} k^2 g_2 g_3 \]

\[ E_2 = \frac{1}{2} + k g_2 + k^2 \{ g_1 g_2 + g_1 g_3 + \frac{g_2^2}{2} + \frac{g_3^2}{2} \} \]

\[ E_3 = k (g_1 + g_3) + k^2 (2 g_1 g_2 + g_2 g_3) \]
By using (20) in (29), we obtain

\[
C_1 \cong 1 - \frac{i}{2(k+1)} + \frac{k+1}{2N^2}
\]

\[
C_2 \cong \frac{\sqrt{2(k+1)}}{N} \left[ 1 - \frac{11}{4(k+1)} + \frac{r+1}{2N^2} \right]
\]

\[
C_3 \cong \frac{\sqrt{2(k+1)}}{N} \left[ \frac{5}{18(k+1)} + \frac{k+1}{6N^2} \right] \left[ 1 - \frac{7}{4(k+1)} + \frac{3}{8} \frac{(k+1)}{N^2} \right]
\]

\[
D_1 \cong -(k+1) \left\{ 1 - \frac{5}{18(k+1)} \right\}
\]

\[
D_2 \cong \frac{N}{\sqrt{2(k+1)}} \left\{ 1 + \frac{1}{4} \frac{k+1}{N^2} \right\}
\]

\[
D_3 \cong \frac{1}{4} \frac{k+1}{N^2} + \frac{3}{8} \frac{k+1}{N^2} \right\}
\]

\[
E_1 \cong \frac{\sqrt{2(k+1)}}{N} \left\{ 1 - \frac{23}{18(k+1)} + \frac{13}{72} \frac{k+1}{N^2} \right\}
\]

\[
E_2 \cong \frac{1}{2} \left\{ 1 + \frac{7}{2} \frac{k+1}{N^2} \right\}
\]

\[
E_3 \cong \frac{\sqrt{2(k+1)}}{N} \left\{ 1 - \frac{4}{3(k+1)} + \frac{11}{12} \frac{k+1}{N^2} \right\}
\]
Using above values, the equations (28) become

\[
\frac{d^2 x}{d \Theta^2} + \left[ A_{x0} + A_{x1} \cos N \Theta + A_{x2} \cos 2N \Theta + A_{x3} \cos 3N \Theta \right] x = 0
\]

(31)

\[
\frac{d^2 z}{d \Theta^2} + \left[ A_{z0} + A_{z1} \cos N \Theta + A_{z2} \cos 2N \Theta + A_{z3} \cos 3N \Theta \right] z = 0
\]

where

\[
A_{x0} \geq (k+1) \left\{ 1 - \frac{7}{4(k+1)} \right\}
\]

\[
A_{x1} \geq \sqrt{2(k+1)} N \left\{ 1 + \frac{3}{8} \frac{k+1}{N^2} \right\}
\]

\[
A_{x2} \geq (k+1) \left\{ 1 - \frac{1}{3(k+1)} + \frac{k+1}{6 N^2} \right\} + \frac{2N^2}{k+1} \left\{ 1 - \frac{1}{2(k+1)} \right\}
\]

\[
A_{x3} \geq \sqrt{\frac{2}{k+1}} N \left\{ 2 - \frac{49}{6(k+1)} - \frac{25}{48} \frac{k+1}{N^2} \right\} + \frac{3}{8} \frac{\sqrt{2(k+1)}}{N} (k+1)
\]

\[
A_{z0} \geq -(k+1) \left( 1 - \frac{4}{k+1} \right) + \frac{N^2}{k+1} \left( 1 - \frac{4}{k+1} \right)
\]

\[
A_{z1} \geq -\sqrt{2(k+1)} N \left\{ 1 - \frac{2}{k+1} + \frac{3}{8} \frac{k+1}{N^2} \right\}
\]

\[
A_{z2} \geq -(k+1) \left\{ 1 - \frac{23}{6(k+1)} + \frac{k+1}{6 N^2} \right\} - \frac{N^2}{k+1} \left\{ 1 - \frac{1}{k+1} \right\}
\]

\[
A_{z3} \geq -\sqrt{\frac{2}{k+1}} N \left\{ 1 - \frac{2}{k+1} - \frac{17}{48} \frac{k+1}{N^2} \right\} - \frac{3}{8} \frac{\sqrt{2(k+1)}}{N} (k+1)
\]
To calculate the betatron oscillation frequencies from the equations (31), (9), Vogt-Nilsen's formulas are used.

\[
\cos \frac{\xi_x}{2} = \cosh \frac{2\pi \sqrt{A_{x0}}}{N} - \frac{\pi}{2N} \sinh \frac{2\pi \sqrt{A_{x0}}}{N} \left[ \frac{A_{x1}^2}{N^2 - 4A_{x0}} \right]
\]

\[
\cos \frac{\xi_z}{2} = \cosh \frac{2\pi \sqrt{-A_{z0}}}{N} - \frac{\pi}{2N} \sinh \frac{2\pi \sqrt{-A_{z0}}}{N} \left[ \frac{A_{z1}^2}{N^2 - 4A_{z0}} \right]
\]

The calculated \( \xi_x \) for various machine parameters are shown in Table 2 compared with the results obtained by digital computation.

(9) N. Vogt-Nilsen, MURA-118
In large N machines, the \( \overline{\eta} \) agree well and the apparent larger errors in \( \overline{\eta} \) are due to the fact that a small error in \( \cos \overline{\eta} \) causes a large error in \( \overline{\eta} \) because \( \overline{\eta} \) is small. For small N machine correction terms of order \( \frac{1}{k+1} \) get larger and make agreement poorer.

The smooth approximation may be used for estimating \( \overline{\eta} \) roughly. The equations (31) are approximately

\[
\frac{d^2X}{d(\overline{\xi})^2} + \left[ (k+1) + \sqrt{2(k+1)} \right] N \cos N \theta + \left[ k+1 + \frac{2N^2}{k+1} \cos 2N \theta \right] \overline{X} \geq 0
\]

\[
\frac{d^2\overline{\xi}}{d(\overline{\xi})^2} + \left[ -k+1 + \frac{N^2}{k+1} - \sqrt{2(k+1)} N \cos N \theta - \left[ k+1 + \frac{N^2}{k+1} \right] \cos 2N \theta \right] \overline{\xi} \geq 0
\]
The \( \frac{\varphi_j}{\pi} \) are given by

\[
\frac{\varphi_k}{\pi} \sim \frac{2\sqrt{2(k+1)}}{N} \tag{34}
\]

\[
\frac{\varphi_k}{\pi} \sim \frac{2}{\sqrt{k+1}}
\]

By multiplying (34) each other, we obtain

\[
\left( \frac{\varphi_k}{\pi} \right) \left( \frac{\varphi_k}{\pi} \right) \sim \frac{4\sqrt{2}}{N} \tag{35}
\]

For a working point in the first stable region, (35) should be less than unity and it indicates that there is a lower limit on \( N \) to obtain stable motions.

In Fig. 1 \( \varphi_x \) is plotted against \( \varphi_k \).

From (21) and (34), the circumference factor \( C \) is roughly given by

\[
C \sim \frac{4}{\left( \frac{\varphi_k}{\pi} \right)} + 2 \tag{36}
\]

and this shows the minimum circumference factor for a sinusoidal field is \( \sim 6 \)

For a rectangular field this figure would be reduced by a factor \( \sqrt{2} \).

Since we would rather use \( \varphi_x \) between \( (2/3)\pi \) and \( \pi \) to make the circumference factor as small as possible, the smooth approximation always underestimates \( \varphi_x \). If we plot \( \varphi_x \) against \( \frac{k+1}{N^2} \), it looks more like a straight line than the parabola expected by the smooth approximation.

Furthermore, if \( \varphi_x \) is plotted against \( \frac{k+1}{N^2} \) all points lie approximately on a straight line independent of \( N \), as shown in Fig. 2. So we have a handy empirical formula for \( \varphi_x \).
\[ \frac{\sqrt{x}}{11.9 \left( \frac{N^2}{N^2+3.5} \right)} + 0.175 \]  

(37)

Similarly, \( \sqrt{x} \) is plotted against \( \frac{2}{\sqrt{N+1}} \), which the smooth approximation predicts in Fig. 3 and we obtain a handy formula for \( \sqrt{x} \):

\[ \frac{\sqrt{x}}{11.90} \approx 1.105 \left( \frac{2}{\sqrt{N+1}} - 0.03 \right) \]  

(38)

The \( \sqrt{x} \) obtained by (37) and (38) are compared with the computer results in Table 3 and Table 4.

\[
\begin{array}{|c|c|c|c|}
\hline
N & k+1 & \frac{\sqrt{x}}{11} \text{ emp. form.} & \frac{\sqrt{x}}{11} \text{ digit} \\
\hline
64 & 200 & 0.763 & 0.764 \\
 & 160 & 0.646 & 0.641 \\
 & 120 & 0.531 & 0.531 \\
36 & 40 & 0.565 & 0.563 \\
24 & 25 & 0.743 & 0.743 \\
 & 18 & 0.599 & 0.595 \\
18 & 13 & 0.744 & 0.750 \\
 & 12 & 0.708 & 0.710 \\
 & 10 & 0.634 & 0.636 \\
16 & 9.5 & 0.733 & 0.741 \\
\hline
\end{array}
\]
### HIGHER HARMONICS

The magnetic fields in actual machines contain higher harmonics which are usually smaller than those of rectangular fields. The effects of the $j$-th harmonic on the orbits and the gradient focusing are reduced by a factor $\frac{1}{j^2}$ and very little change in $\bar{U_X}$ is expected by adding harmonics. Table 5 shows the $\bar{U_X}$ hardly change by adding third harmonics.
The circumference factor is reduced by adding harmonics, since the orbits are almost unaffected and the peak field is reduced. When the harmonic content is smaller than that of a rectangular field the circumference factor is approximately given by

\[ C \sim C_{\text{sin}} \times \frac{\sum f_{j+1}}{f} \]  

(39)

where \( C_{\text{sin}} \) is the circumference factor of sinusoidal field and the field is assumed to have the form (8).

According to the smooth approximation, \( \bar{\gamma} \) is given by

\[ \left( \frac{\bar{\gamma}}{\pi} \right)^2 \sim \left( \frac{\bar{\gamma}}{\pi} \right)_{s_{+n}} \times F^2 \]  

(40)

where

\[ F^2 = \sum \frac{f_{j+1}}{f_j^2} \]

for sinusoidal field
\( \left( \frac{J_x}{\pi} \right)^2 / \left( \frac{J_z}{\pi} \right)^2 \) is plotted against \( F^2 \) in Fig. 40. It is clear in the figure that the increment of \( \bar{\nu}_z \) is more than that given by the smooth approximation. \( \bar{\nu}_z \) is given very roughly by

\[ \left( \frac{J_x}{\pi} \right)^2 \sim \left( \frac{J_z}{\pi} \right)^2 \left[ F^2 + \sqrt{F^2} / \alpha + 0.16 \right] \]  

**SPIRALLING**

In a large \( N \) machine, say \( N > 100 \), spiralling might be necessary to obtain a comfortable \( \bar{\nu}_z \). The orbits and the radial tunes change very little. The smooth approximation formula for the axial tune is given by

\[ \left( \frac{\nu_x}{\pi} \right)^2 \sim \frac{4 F^2}{k_{\text{ax}}^2} \left[ 1 + \frac{2}{N^2 W^2} \right] \]  

Stability limits would be decreased by spiralling due to the additional non-linearity of the fields.

**STABILITY LIMITS**

According to G. Parzen's \(^{(10)}\) formula for stability limit due to non-linear resonance lines, there are no remarkable differences from the ordinary radial sector machine, except that the Walkinshaw's line \( \nu_x - 2 \bar{\nu}_z = 0 \) is almost forbidden in this machine. \(^{(10)}\)

\(^{(10)}\) G. Parzen, MURA-300
The stability limit amplitude of the radial motion due to the resonance line is given by

\[ A = \frac{2}{MN^2} \left| \left( \frac{\Omega_x}{2\pi} \right)^2 - \left( \frac{1}{3} \right)^2 \right| \tag{43} \]

where

\[ M = \frac{1}{4} \frac{(k+1)^2}{N^4} \left( 1 + \frac{28(k+1)}{N^2} \right) \]

The above formula is compared with the results from the computer in the following table.

<table>
<thead>
<tr>
<th>N</th>
<th>k+1</th>
<th>A anal</th>
<th>A digit</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>16.1</td>
<td>1.9 \times 10^{-2}</td>
<td>1.3 \times 10^{-2}</td>
</tr>
<tr>
<td>36</td>
<td>58</td>
<td>5.0 \times 10^{-3}</td>
<td>3.9 \times 10^{-3}</td>
</tr>
</tbody>
</table>

For the sum resonance line, \( \Omega_x + 2\Omega_z = 2\pi \), the stability limit amplitudes are given by

\[ A = \frac{1}{M N^2} \left| \left( \frac{\Omega_x}{2\pi} \right)^2 - \left( \frac{\Omega_z}{2\pi} \right)^2 \right| \]

\[ B = \frac{1}{M N^2} \left\{ 2 \left| \left( \frac{\Omega_x}{2\pi} \right)^2 - \left( \frac{\Omega_z}{2\pi} \right)^2 \right| \left( \frac{\Omega_z}{2\pi} \right)^2 - \left( \frac{\Omega_x}{2\pi} \right)^2 \right\}^{\frac{1}{2}} \tag{44} \]

where

\[ M = \frac{\alpha}{4} \frac{(k+1)^2}{N^4} \]

A and B depend on the tunes \( \Omega_x \) and \( \Omega_z \) at the stability boundary to which \( \Omega_{x0} \) and \( \Omega_{z0} \) are driven. It takes long series of computer runs to obtain
A and B as a function of initial values of \( x \) and \( z \). This survey has not been done. However, few values of A and B obtained so far indicate fair agreements with (44).

**TWO-BEAM - ONE-BEAM TUNING**

Since the symmetric machine is a special case of the radial sector FFAG machine, the machine can be used as a one-beam radial sector machine by changing the magnitude of the magnetic fields in the positive and the negative magnets.

If \( f(\pi \Theta) \) is given by

\[
f(\pi \Theta) = \varepsilon + \cos \pi \Theta \quad (45)
\]

the smooth approximation gives

\[
\left( \frac{J_x}{\pi} \right)^2 \sim \frac{4(k+1)}{N^2} \left\{ 1 + \frac{\alpha^2(k+1)}{2N^2} \right\} \quad (46)
\]

\[
\left( \frac{J_z}{\pi} \right)^2 \sim 4 \left\{ \frac{\alpha^2}{2N^2} + \frac{\alpha^2(k+1)^2}{2N^4} - \frac{k+1}{N^2} \right\}
\]

where

\[
\alpha' = \frac{N^2}{k+1} \left[ \frac{\varepsilon^2}{4} + \frac{2(k+1)}{N^2} - \frac{\varepsilon}{2} \right]
\]

\[\text{(ii) } \alpha' \text{ is } \frac{1}{2} \ln \frac{a}{b} \text{ in G. Parzen's notation in his report MURA-273}\]
By eliminating $x'$ in (46), we obtain

$$\left(\frac{v_c}{\pi}\right)^2 \left[1 + \frac{N^2}{(k+1)^2} \right] \sim \frac{4}{k+1} + \frac{\delta(k+1)}{N^2} \quad (47)$$

This shows the working points move along a hyperbola by changing the magnets excitation. It must be noted that this is the smooth approximation result and shows only rough behavior of the working points, especially for high $\sigma, \omega$.

**EXAMPLES OF DESIGN PARAMETERS**

In the following, typical sets of parameters for rather small size machines are discussed.

Unfortunately direct Forocyl-Formesh calculation cannot be used with vanishing average value of the magnetic field. The calculations are done as follows. In the magnet configurations of Forocyl agenda, a finite value of potential on the positive (negative) magnet and zero on the negative (positive) magnet are given. The configurations have the symmetry around the centers of the positive magnets and also around that of the negative magnets. By picking up only odd Fourier coefficients of the median plane fields in the output of Forocyl, we have the median plane field when both positive and negative magnets are energized. This field can be used to calculate dynamics by the Well Tempered Five program. (The alternative is to feed this field in Tempermesh and use Formesh for dynamics)\(^{(12)}\) It must be noted that all field coefficients are normalized to $f_0 = 1$ and then multiplied by $\alpha'$ to run dynamics to keep $\alpha'$ within the range of the program.

\(^{(12)}\) This procedure is being used for New Model by A. Sessler and F.T. Cole.
N for small size (electron) machine might range 14～24 considering the magnet gaps and straight sections. In this range N = 18 must be discarded because the working points stay always too close to the line $\sqrt{x} + 2\sqrt{y} = 2\pi$ which is an essential and dangerous line.

**EXAMPLE 1.**

\[ N = 20 \quad \text{k = 15.1} \]

The configuration of magnets is shown in Fig. 5.

Fourier components of median plane fields

<table>
<thead>
<tr>
<th>Order</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>1.000 000</td>
</tr>
<tr>
<td>3rd</td>
<td>-0.228407</td>
</tr>
<tr>
<td>5th</td>
<td>0.054433</td>
</tr>
<tr>
<td>7th</td>
<td>0.004272</td>
</tr>
<tr>
<td>9th</td>
<td>-0.017548</td>
</tr>
<tr>
<td>11th</td>
<td>0.014469</td>
</tr>
<tr>
<td>13th</td>
<td>-0.008186</td>
</tr>
</tbody>
</table>

Circumference factor 7.2

\[ \sqrt{\frac{x}{\pi}} \approx 0.727 \quad \sqrt{\frac{y}{\pi}} \approx 0.547 \]

Phase plots of x and y motion are shown in Fig. 8 and Fig. 9.

Radius at injection 200 cm

at output 275 cm

Energy at injection 100 Kev

at output 50 Mev

$H_{max}$ 4300 gauss

**EXAMPLE 2.**

\[ N = 16 \quad \text{k = 8.5} \]

The magnet configuration is shown in Fig. 6.

Fourier components of the median plane fields
EXAMPLE 3.

\[ N = 36 \quad k = 57 \]

The magnet configuration is shown in Fig. 7.

Fourier components of the median plane fields

\[
\begin{align*}
1\text{st} &: 1.000000 \\
3\text{rd} &: -0.228394 \\
5\text{th} &: 0.054391 \\
7\text{th} &: 0.004341 \\
9\text{th} &: -0.017617 \\
11\text{th} &: 0.014520 \\
13\text{th} &: -0.008215 \\
\end{align*}
\]

\[
\frac{\xi}{\pi} \sim 0.004341 \quad \frac{\eta}{\pi} \sim 0.014520
\]

Circumference factor \(7.07\)

\[
\frac{\xi}{\pi} \sim 0.718 \quad \frac{\eta}{\pi} \sim 0.275
\]

\[ H_{\text{max}} \quad 3500 \text{ gausses} \]
Radius at injection 890 cm  
      at output 1000 cm  
Energy at injection 100 Kev  
      at output 300 Mev  
$H_{\text{max}}$ 7000 gauss

Other Problems

Since the machine is different from the ordinary FFAG accelerator only in the particle dynamics, the problems such as, acceleration, injection, stacking, space charge effects and so on, are the same as in the ordinary FFAG machine. The discussions of these problems can be found in numerous MURA reports.
\[ \frac{\sigma^2}{\pi} \]

- \( N = 18 \)
- \( N = 24 \)
- \( N = 64 \)

Fig. 1
$\frac{\sigma^2}{\pi}$

- $N = 18$
- $N = 24$
- $N = 36$
- $N = 64$

Fig. 2

$\frac{k + 3.5}{N^2}$
Fig. 3

\[ \frac{\bar{y}}{\pi} \]

- \( \star \) \( N = 18 \)
- \( \times \) \( N = 24 \)
- \( \bullet \) \( N = 64 \)

\[ \frac{2}{\sqrt{k+1}} \]

\[ \bar{y} \approx 1.05 \left( \frac{2}{\sqrt{k+1}} - 0.03 \right) \]
$N = 18$

$N = 24$

$N = 64$

$\left( \frac{y}{\sin \frac{\pi}{N}} \right)^2 = \left( \frac{y_0}{\sin \frac{\pi}{N}} \right)^2 + F^2$

$k+1 = 18 \quad \sigma_y = 0.487 \pi$

$k+1 = 120 \quad \sigma_y = 0.178 \pi$

$\sigma_y = 0.987 \pi$
median plane

Fig. 5

median plane

Fig. 6

median plane

Fig. 7