DGLAP AND BFKL EQUATIONS IN SUPERSYMMETRIC
GAUGE THEORIES *

A.V. Kotikov\textsuperscript{1}, L.N. Lipatov\textsuperscript{2}, V.N. Velizhanin\textsuperscript{2}

\textsuperscript{1} Joint Institute for Nuclear Research, 141980 Dubna, Russia
\textsuperscript{2} St. Petersburg Nuclear Physics Institute, 188300 Gatchina, Russia

We discuss the DGLAP and BFKL equations in the next-to-leading approximation for $N = 4$ SUSY and a resummation procedure based on the AdS/CFT correspondence.

1 Introduction

The Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation \cite{BFKL} and the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equation \cite{DGLAP} are used now for a theoretical description of structure functions of the deep-inelastic $ep$ scattering (DIS) at small values of the Bjorken variable $x$. In supersymmetric gauge theories the structure of the BFKL and DGLAP equations is significantly simplified. The analyticity of the eigenvalue of the BFKL kernel as a function of the conformal spin $|n|$ gives a possibility to relate the DGLAP and BFKL equations in the case of an extended $N = 4$ SUSY.

Let us introduce the unintegrated parton distributions $\varphi_a(x, k^2_\perp)$ (hereafter $a = q, g, \varphi$ for the spinor, vector and scalar particles, respectively) and the (integrated) parton distributions $f_a(x, Q^2)$

$$f_a(x, Q^2) = \int_{k^2_\perp < Q^2} dk^2_\perp \varphi_a(x, k^2_\perp).$$

In DIS $Q^2 = -q^2$ and $x = Q^2/(2pq)$ are the Bjorken variables, $k_\perp$ is the transverse component of the parton momentum and $p$ and $q$ are the hadron and virtual photon momenta, respectively.

The DGLAP equation \cite{DGLAP} in the Lorentz spin representation has the form

$$\frac{d}{d \ln Q^2} f_a(j, Q^2) = \sum_b \gamma_{ab}(j)f_b(j, Q^2),$$

*The work is supported by grants INTAS 00-366 and RSG SS-1124.2003.2
where \( f_a(j, Q^2) \) are the Mellin momenta of parton distributions and \( \gamma_{ab}(j) \) is the anomalous dimension (AD) matrix for the twist-2 operators.

The BFKL equation \[1\] allows one to calculate the unintegrated gluon distributions at small values of \( x \)
\[
\frac{d}{d \ln (1/x)} \varphi_g(x, k^2_\perp) = \int d^2k'_\perp K(k_\perp, k'_\perp) \varphi_g(x, k^2_\perp). \tag{3}
\]
The matrix elements of the local operators \( O^a_{\mu_1, \ldots, \mu_j} \) are related to the distributions \( f_a(x, Q^2) \) for unpolarized partons as
\[
\int_0^1 dx x^{j-1} f_a(x, Q^2) = < n| \tilde{n}^{\mu_1} \ldots \tilde{n}^{\mu_j} O^a_{\mu_1, \ldots, \mu_j}| h >, \quad a = (q, g, \varphi) \tag{4}
\]and similarly for the polarized case (the vector \( \tilde{n}^\mu \sim q^\mu + x p^\mu \) is lightlike \( \tilde{n}^2 = 0 \)). Generally one can introduce mixed projections for this tensor
\[
\tilde{n}^{\mu_1} \ldots \tilde{n}^{\mu_j+\omega} O^a_{\mu_1, \ldots, \mu_j+\omega, \sigma_1, \ldots, \sigma_1} \ell^\perp \ldots \ell_1^{\perp n_1}, (l_\perp, p) = (l_\perp, q) = 0, \tag{5}
\]where \( |n| \) is conformal spin and \( \omega = j - 1 \) is an eigenvalue of the BFKL kernel \( \omega = \omega(n, |n|) \), \( m = \frac{1}{2} + i \nu + \frac{n}{2} \), \( \tilde{m} = \frac{1}{2} + i \nu - \frac{n}{2} \) with the Möbius-group invariant eigenfunction
\[
< 0| \phi(\rho_1) \phi(\rho_2) O_{|n|, \nu}(\rho_0)| 0 > = \left( \frac{\rho_{12}}{\rho_{10} \rho_{20}} \right)^m \left( \frac{\rho_{12}^*}{\rho_{10}^* \rho_{20}^*} \right)^{\tilde{m}}. \tag{6}
\]

It is important, that the anomalous dimensions \( \gamma \) do not depend on various projections of \( O_{\mu_1, \ldots, \mu_j} \).

In the leading logarithmic approximation (LLA) the BFKL dynamics has some remarkable properties in the large-\( N_c \) limit [3]: Möbius invariance, holomorphic separability of the BFKL hamiltonian leading to the holomorphic factorization of the \( n \)-gluon wave function, duality symmetry and integrability. It is equivalent to an integrable Heisenberg model [4].

2 NLO corrections to the BFKL and DGLAP equations in SUSY

The BFKL kernel up to NLO can be written as ([5] in QCD; [6] in SUSY)
\[
K(k_\perp, k'_\perp) = \overline{a} K_B(k_\perp, k'_\perp) + \overline{a}^2 K_{NL}(k_\perp, k'_\perp), \quad \overline{a} = g^2 N_c/(16 \pi^2), \tag{7}
\]where \( \overline{a} \) is in the \( \overline{MS} \) scheme. Eigenvalue \( \omega = \omega(n, \gamma) \) in QCD contains nonanalytic terms \( \delta_{|n|, 0} \) and \( \delta_{|n|, 2} \). \( \delta_{|n|, 2} \) is cancelled in all SUSY gauge theories, \( \delta_{|n|, 0} \) is cancelled only in the \( N = 4 \) case, where the result has the form [6]:
\[\omega = 4 \tilde{a} \chi(n, \gamma) + 4 \tilde{a}^2 \delta(n, \gamma), \quad \tilde{a} = \overline{a} + \frac{1}{3} \overline{a}^2, \quad \gamma = \frac{1}{2} + i \nu \] and \( \tilde{a} \) is in the dimension reduction scheme. \( \delta(n, \gamma) \) has the property of the hermitian separability [7].
The Regge and Bjorken representations for the cross-section of hadron production in the virtual $7^*7^*$ collisions are different

$$\delta(n, \gamma) = \phi(M) + \phi(M^*) - \frac{\omega}{2\bar{a}} \left( \rho(M) + \rho(M^*) \right), \quad M = \gamma + \frac{|n|}{2}, \quad (8)$$

but the property of the holomorphic separability is lost.

For $\rho_{11} \to \rho_{21}$ the solution of the inhomogeneous BFKL equation is

$$\langle \phi(\rho_{11}) \phi(\rho_{21}) \phi(\rho_{12}^*) \phi(\rho_{22}^*) \rangle \sim \sum_n \frac{C(\nu_\omega, |n|)}{\omega'(|n|, \nu_\omega)} |\rho_{11}'|^n |\rho_{22}'|^{2n} \left( \frac{\rho_{11}'}{\rho_{22}'} \right)^{|n|/2} E_{\nu_\omega, |n|}(\rho_{11}', \rho_{22}'),$$

where $E_{\nu_\omega, |n|}(\rho_{11}', \rho_{22}')$ is the Polyakov three-point function and $\nu_\omega$ is a solution of the algebraic equation $\omega = \omega(|n|, \nu)$. The above asymptotics has a simple interpretation in terms of the Wilson operator-product expansion

$$\lim_{\rho_{11} \to \rho_{22}} \phi(\rho_{11}) \phi(\rho_{22}) = \sum_n \frac{C(\nu_\omega, |n|)}{\omega'(|n|, \nu_\omega)} |\rho_{11}'|^n |\rho_{22}'|^{2n} \left( \frac{\rho_{11}'}{\rho_{22}'} \right)^{|n|/2} E_{\nu_\omega, |n|}(\rho_{11}'),$$

where $\Gamma_\omega = 1 + |n|^2 - \gamma(j)$ and the anomalous dimension (AD) $\gamma(j)|_{\omega \to 0} = \frac{a^2 N_c}{4\pi^2 \omega}$ belongs to a high twist operator with the conformal spin $|n| = 1, 2, \ldots$. But it was assumed [7] that after the analytic continuation of BFKL $\gamma(|n|, \omega)$ to the points $|n| = -r - 1$ ($r = 0, 1, 2, \ldots$) one can calculate singularities of AD for the twist-2 operators at negative integer $j = 1 + \omega + |n| \to -r$.

In LLA one can derive from the BFKL eigenvalue $\omega^0(n, \nu)$ in this limit $\gamma(j)|_{\omega \to 0} = \frac{a^2 N_c}{4\pi^2 \omega}$ for all $r = -1, 0, 1, \ldots$. This result is valid only for $N = 4$ SUSY where we obtain $\gamma(j) = \hat{\alpha}\gamma^{LLA}(j)$, $\gamma^{LLA}(j) = 4\left(\Psi(1) - \Psi(j - 1)\right)$ in an agreement with the direct calculation of $\gamma^{uni}(j)$ in this theory.

In the NLO approximation there is a more complicated situation. Indeed, $\gamma^{uni}(j)$ has the multiple poles $\Delta\gamma \sim \alpha^2(j + r)^{-3}$ at even $r$, which is related to an appearance of the double-logarithmic corrections $\sim (\alpha \ln^2 s)^n s^{-r}$ in the Regge limit $s \to \infty$ at upper orders $n$ of the perturbation theory.

In LO we have one supermultiplet of operators with the same AD $\gamma^{LLA}(j)$ proportional to $\Psi(1) - \Psi(j - 1)$. In NLO the diagonal elements of the AD matrix can be expressed in terms of the universal function $\gamma(j) = -4\hat{\alpha}S_1(j - 2) + \hat{\alpha}^2\hat{Q}(j - 2)$ presented through the harmonic sums in DRED scheme [10]

$$\hat{Q}(j) = 16S_1(j)S_2(j) + 8S_3(j) - 8\tilde{S}_3(j) + 16\tilde{S}_{1,2}(j). \quad (9)$$

3 Relation between the DGLAP and BFKL equations

The Regge and Bjorken representations for the cross-section of hadron production in the virtual $\gamma^*\gamma^*$ collisions are different
\[ \sigma(s, Q^2, P^2) = \int \frac{d\omega}{2\pi i} \left( \frac{s}{|Q| |P|} \right)^\omega \varphi_\omega(Q^2, P^2) = \int \frac{d\omega}{2\pi i} \left( \frac{s}{|Q|^2} \right)^\omega \varphi_\omega(Q^2, P^2). \]

Therefore anomalous dimensions for the DGLAP and BFKL equations are related as \( \gamma = \gamma_{BFKL} + \frac{\omega}{2} \) [5]. The equation for \( \gamma \) from the BFKL approach has the symmetry \( \gamma \leftrightarrow J - \gamma \) due to the hermitian separability

\[
1 = \frac{4\hat{a}}{\omega} \left( 2\Psi(1) - \Psi(\gamma) - \Psi(J - \gamma) + \hat{a}(\phi(\gamma) + \phi(J - \gamma)) \right) + 2\hat{a}(\Psi'(\gamma) - \rho(\gamma) + \Psi'(J - \gamma) - \rho(J - \gamma)). \tag{10}
\]

Now we push \(|n| \to -r - 1\) (\( \omega = j + r \to 0 \)) and calculate the singularities of \( \gamma(j) \) at \( j \to -r \) for even \( r \) (and similar for odd \( r \))

\[
\gamma(j) = 4\hat{a} \left( \frac{1}{j + r} + K(r) + L(r)(j + r) \right) + 16\hat{a}^2 \left( \frac{1}{(j + r)^3} + \frac{T(r)}{(j + r)^2} + \frac{R(r)}{j + r} \right).
\]

It is important, that in the BFKL approach one should go beyond NLO to find \( K(r), L(r), T(r), R(r) \). On the other hand from direct calculations of the universal AD \( \gamma^{uni} \) we obtain for even \( r \) \( (\hat{S}_2(r) = \zeta(2) + S_2(r)) \)

\[
\gamma(j) = 4\hat{a} \mathcal{G}^{(0)} + 16\hat{a}^2 \left( \frac{1}{(j + r)^3} - \frac{2S_1(r + 1)}{(j + r)^2} - \frac{\hat{S}_2(r + 1)}{j + r} \right),
\]

where \( \mathcal{G}^{(0)} = \left( \frac{1}{j + r} - S_1(r + 1) - \hat{S}_2(r + 1) (j + r) \right) \). Therefore the hypothesis [7], that the BFKL equation in \( N = 4 \) SUSY contains a complete information about the DGLAP equation was qualitatively verified in the next-to-leading approximation.

4 AdS/CFT correspondence and the perturbation theory

According to the famous AdS/CFT correspondence [8] the strong-coupling limit \( \alpha_s N_c \to \infty \) is described by a classical supergravity in the anti-de Sitter space \( AdS_5 \times S^5 \). An interesting prediction was obtained for AD of twist-2 operators at \( j \to \infty \) in the strong coupling regime [9] \( \lim_{z \to \infty} a = -z^{1/2} + \frac{3\ln 2}{8\pi} + (z^{-1/2}) \), where \( \gamma(j) = a(z) \ln j \), \( z = \frac{\alpha_s N_c}{\pi} \).

On the other hand, all AD \( \gamma_i(j) \) and \( \hat{\gamma}_i(j) \) coincide at large \( j \) and our results for \( \gamma(j) \) allow one to find two first terms of the small-\( \alpha_s \) expansion \( \lim_{z \to 0} \hat{a} = -z + \pi^2/12 \ z^2 + \ldots \). To go to the strong coupling regime we perform
a resummation of the perturbative result. Namely, we present $\tilde{a}$ as a solution of the simple algebraic equation $z = -\tilde{a} + \pi^2/12 \tilde{a}^2$ and find $\tilde{a} \approx -1.103 z^{1/2} + 0.608 + O(z^{-1/2})$ in a rather good agreement with the prediction based on the AdS/CFT correspondence.

Further, we have $\gamma(j) = (j - 2) \gamma'(2)$, where $\gamma'(2) = -1.645 z + 1.353 z^2$ in the perturbation theory. From the BFKL equation one can obtain for the leading singularity of the $t$-channel partial wave in $j$-plane $\gamma = 1/2 + iv + (j - 1)/2 \rightarrow 1$ for $j \rightarrow 2, \nu \rightarrow 0$. With the use of the above summation procedure we obtain for the Pomeron intercept $j = 2 - 1.163 z^{-1/2}$ for large $z$. One can attempt also to calculate the intercept of the Pomeron at large $z$ using its perturbative expansion $j - 1 = c_1 z + c_2 z^2$ from the BFKL equation. After the resummation $j - 1 = c_1 z/(1 - c_1 z/c_2)$ we obtain in the strong coupling regime $j \simeq 2.46$ (comparing with the graviton spin $j = 2$).

References


