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Feynman Integrals and Hypergeometric Functions

Hector Luna Garcia, Luz Maria Garcia

Area de Fisica, Ciencias Basicas, UAM-Azcapotzalco. Av. San Pablo 180, 02200, Mexico, D.F.; Mexico

Ruben Mares, Enrique Ortega

Instituto Politecnico Nacional, U. P. Zacatenco, 07738 Mexico, D.F.; Mexico

Abstract. We review Davydychev method for calculating Feynman integrals for massive and no massive propagators, by employing Mellin-Barnes transformation and integration-dimensionally continuous, same that lead to hypergeometric functions. In particular an example is calculated explicitly from such a method.

1. Introduction

The need to evaluate the integrals that result from different types of Feynman diagrams has been central in relativistic quantum field theory. Methods to evaluate analytically to several classes of Feynman integrals are extremely important. In this context, it is very convenient to use the dimensional regularization scheme or integration-dimensionally continuous, although other schemes such as analytical regularization, are also used. At the present, it has been quite successful in the implementation of methods that evaluate integrals involving non-massive propagators, but depend on an external moment. Can be indicated in the literature [1-6], such methods have yielded a number of results for integrals representing propagators in multiloops. Feynman integrals with massive denominators are more complex and complicated. Currently, there are few examples to show accurate results in the case of massive integrals. An approach to massive integrals is related to the application of the operator **R** [7, 8]. The denominator is expanded in series with respect to $\frac{m^2}{k^2}$ (m is the mass and $\vec{k}$ the moment of the corresponding line in the propagator) and the inapplicability of the series to the region $|k^2| < m^2$ is compensated by the introduction of anti-terms. This method allows to obtain a finite number of terms from the asymptotic expansion of expressions for variables of type $\frac{m^2}{p^2}$ ($\vec{p}$ is external moment).

The following evaluates to an integral of Feynman by the method introduced by A.I. Davydychev and basically is based on the integral representation of Mellin-Barnes [9,10], leading to generalized hypergeometric functions. This work is divided into four parts: This introduction. The section 2, presents the mathematical basis of the method of Davydychev; in the section 3 is calculated explicitly an example to show the formal method, although this example does not correspond to a physical diagram. Finally, section 4 presents conclusions.

2. Method Davydychev. Preliminary Results

From the expression

$$ {}_1F_0 = \frac{1}{(1-z)^\nu} = \frac{1}{\Gamma(\nu)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds (-z)^s \Gamma(-s)\Gamma(\nu+s) \tag{1} $$

we have our basic formula is

$$ \frac{1}{(k^2 - m^2 + i0)^\nu} = \frac{1}{(k^2+i0)^\nu} \frac{1}{\Gamma(\nu)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \left(-\frac{m^2}{k^2+i0} \right)^s \Gamma(-s)\Gamma(\nu+s) \tag{2} $$

Where ν is the index of the line and the infinitesimal imaginary denominators define the shape causal to consider singularities in pseudo-euclidean space. Representation (2) is desirable because it can be used for both $|k^2| > m^2$ and $|k^2 < m^2$.

(a) Case $|k^2| > m^2$

In this case we have $\left(-\frac{m^2}{k^2+i0}\right)^\nu \Gamma(-s)\Gamma(\nu+s) \to 0$, on the semicircle and can be evaluatednby the residue theorem at the poles of the function $\Gamma(-s)$, see Figure 1.

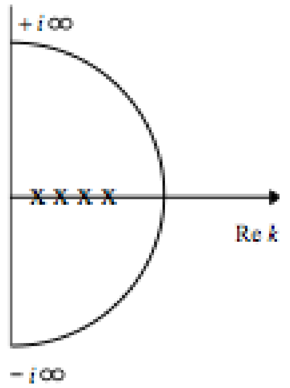

Figure 1.

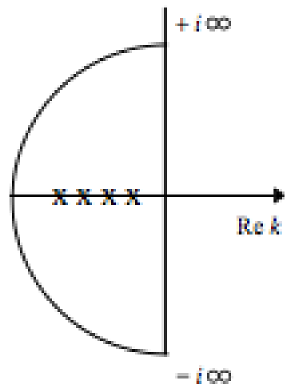

Figure 2.

(b) Case $|k^2| > m^2$. Then the integral is calculated as a sum of residues of the function $\Gamma(\nu+s)$. Once all massive denominators are represented according to the form (2), no massive integrals of the moments are evaluated, being a series that is represented in the form of functions of the type hypergeometric, see Figure 2.

3. Mellin-Barnes Contour Integral

The integral is represented by the expression

$$ \frac{1}{(k^2 - m^2 + i0)^\nu} = \frac{1}{\Gamma(\nu)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{(-m^2)^s}{(k^2+i0)^{\nu+s}} \Gamma(-s)\Gamma(\nu+s) \tag{3} $$

We can prove this equation starting with the following identity

$$ \frac{1}{(k^2 - m^2 + i0)^\nu} = \frac{1}{(k^2+i0)^\nu \left(1 - \frac{m^2}{k^2+i0}\right)^\nu} \tag{4} $$

and the representation of the hypergeometric function as a contour integral given by the expression

$$ {}_2F_1(a,b;c|z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} ds(-z)^s\frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)}\Gamma(-s) \tag{5} $$

Then using the definition of hypergeometric functions and Pochhammer's symbols, we have

$$ \frac{1}{(1-z)^a} = {}_2F_1(a,b;b|z) = \frac{1}{\Gamma(a)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} ds(-z)^s\Gamma(-s)\Gamma(a+s) \tag{6} $$

Finally, with the help of the equation (6), equation (4) can be written as the desired result

$$ \frac{1}{(k^2-m^2+i0)^\nu} = \frac{1}{(k^2+i0)^\nu}\frac{1}{\left(1-\frac{m^2}{k^2+i0}\right)^\nu} = \frac{1}{\Gamma(a)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} ds\frac{(-m^2)^s}{(k^2+i0)^{\nu+s}}\Gamma(-s)\Gamma(\nu+s) $$

4. Example: Two Points Diagram
For our example, consider the following process shown in Figure 3.

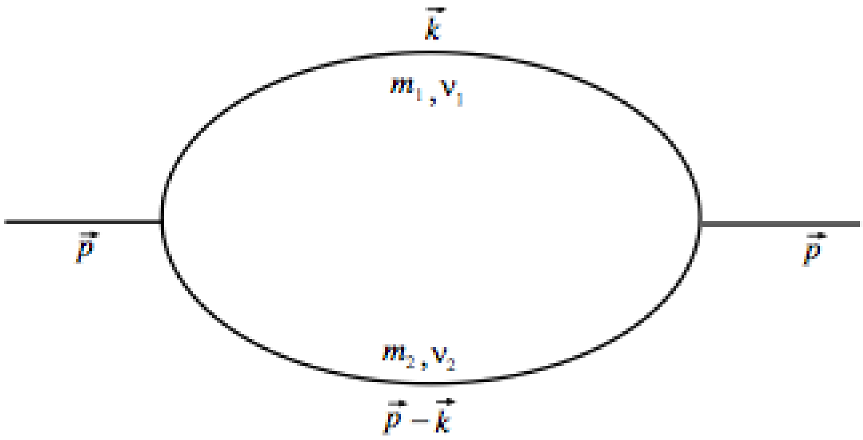

Figure 3.

In this situation, the integral of Feynman is

$$ I^{(2)}(\nu_1,\nu_2|p;m_1,m_2) = \int \frac{d^nk}{(k^2-m_1^2)^{\nu_1}((p-k)^2-m_2^2)^{\nu_2}} \tag{7} $$

where $n = 4-\epsilon$ under dimensional regularization and the continuous, the integral is dimensionally continuous and the denominators has been removed $i0$, via Wick rotation.

Finally, applying equation (3) and (7) one has

$$ \begin{aligned} I^{(2)}(\nu_1,\nu_2|p;m_1,m_2) =& \pi^{n/2}i^{1-n}(-m_2^2)^{n/2-\nu_1-\nu_2} \\ &\times \frac{1}{(2\pi i)^2}\frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty} dsdu\left(\frac{-p^2}{m_2^2}\right)^u\left(\frac{m_1^2}{m_2^2}\right)^s\Gamma(-s)\Gamma(-u) \\ &\times \Gamma(\frac{n}{2}-\nu_1-s)\frac{\Gamma(\nu_1+\nu_2-\frac{n}{2}+s+u)\Gamma(\nu_1+u+s)}{\Gamma(\frac{n}{2}+u)} \end{aligned} \tag{8} $$

The integral over s leads on the right side of the complex plane into two sets of poles corresponding to the functions $\Gamma(-s)y\ \Gamma(\frac{n}{2}-\nu_1-s)$ and applied to the integral over s, and then the integral over u; finally, $I^{(2)}$ can be written as

$$\begin{aligned} I^{(2)}(\nu_1,\nu_2|p;m_1,m_2) =& \pi^{n/2} i^{1-n} (-m_2^2)^{n/2-\nu_1-\nu_2} \frac{\Gamma\left(\nu_1+\nu_2-\frac{n}{2}\right)}{\Gamma(\nu_1+\nu_2)} \\ &\times \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{p^2}{m_2^2}\right)^j \frac{(\nu_2)_j \left(\nu_1+\nu_2-\frac{n}{2}\right)_j (\nu_1)_j}{(\nu_1+\nu_2)_{2j}} \\ &\times {}_2F_1\left(\nu_1+\nu_2-\frac{n}{2}+j, \nu_1+j; \nu_1+\nu_2+2j|1-\frac{m_1^2}{m_2^2}\right) \end{aligned} \quad (9)$$

The equation (9) can be used for the two important cases: (a) $m_1 = m_2 = m\ y\ \nu_1 = \nu_2 = 1$, and (b) $m_1 = 0,\ m_2 = 0\ y\ \nu_1 \neq \nu_2 \neq 1$. The physical case of the limit $n = 4$ for dimensionally continuous integrals is discussed in the Ref. [11].

5. Conclusions

Davydychev method based on the Barnes-Mellin transformation and the dimensionally continuous integration allowed to accurately evaluate massive Feynman integrals (mass same or different) as well no massive, in contrast to traditional methods that lead to approximations. The application of the method was applied by the authors in the Ref. [11] in the case Compton operator. Currently, the method is being applied to: (a) tensor Feynman integrals (vacuum polarization, electron self-energy and the vertex correction in the scheme of QED), (b) Kinoshita analysis on mass singularities in the context of continuous dimensionally integration, and (c) Salam analysis on the coupled divergences of the matrix S in the scheme of continuous dimensionally integration.

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