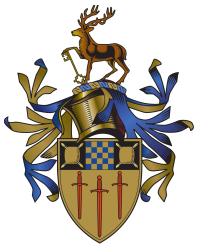

PhD Thesis

University of Surrey

Department of Mathematics

Strings and Spins in Deformed AdS/CFT

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September 21, 2023

Thesis submitted for the degree of Doctor of Philosophy in Mathematics, University of Surrey.

Abstract

In the first part of this thesis, we consider the scattering problem for the massless modes that appear in the context of $\mathrm{AdS}_3/\mathrm{CFT}_2$ from the perspective of integrable models. We start by considering the extension of the $\mathfrak{psu}(1|1)^2$ symmetry algebra by a *Modified Poincaré algebra*. Having found a consistent Hopf algebra structure, we construct suitable R-matrices, both for the undeformed and the q-deformed case of the $\mathfrak{su}(1|1)^2_{c.e.}$ algebra, and study the found coproducts in detail. We find interesting connections between the boost operator $\mathfrak{J}$ and the R-matrices. In addition, we encounter a non-coassociative structure for one of the cases we are interested in. This prompts us to make use of the notions of quasi-Hopf algebras and coassociators.

We then move from the framework of a $1+1$-dimensional short representation to the analysis of the boost operator and its coproduct in a universal, representation-independent sense. This leads us to establish a classification of boost algebras and coproducts. We arrive at six different algebraic structures, each with its own universal coproduct for the boost operator. Finally, we put particular focus on the cases that can sensibly appear within the context of AdS_3 physics.

Moving on from Hopf algebraic considerations, we stay within the context of AdS_3 string theory, and consider a particular 3-*parameter deformed background* in the *Landau-Lifshitz limit*. We construct an *effective field theory* from the Polyakov action associated to this background. This way, we obtain a Lagrangian up to next-to-leading order in the energy parameter κ, allowing us to find kinematical quantities such as the dispersion relation of these theories. We then introduce one single complex field that parametrises the relevant coordinates of the theory (corresponding to one mode stemming from the S^3 part), allowing us to canonically quantise it in the standard way. Analysing the diagrammatics of the so-obtained quantum field theory, we find interesting peculiarities related to the propagator and the ground state, and proceed to compute two-body S-matrix elements, both at leading and subleading order in the string tension λ.

In the latter part of this thesis, we then move on to a study in linear algebra, as well as particular spin chains and their Hamiltonians. We first construct an algorithmic framework that allows us to infer the generalised eigensystem of a defective complex matrix by perturbing it in such a way that it becomes diagonalisable. In this case, the eigensystem is directly accessible, and from there we can then go back to the limiting case of non-diagonalisability by again turning off the perturbation. In this construction, we find curious particularities and differences for the case of singular and non-singular geometric multiplicity of the considered eigenvalue, the latter case needing more caution in its analysis. In order to apply this recipe to the *eclectic spin chain*, we make use of the *Nested Coordinate Bethe Ansatz* and find that the analysis of the spectrum of the twisted spin chain contains sufficient information about the generalised eigensystem of the eclectic spin chain for our developed methodology to be successful.

Lay Summary

From the earth going around the sun to our smartphones working the way they do, everything in the universe happens for four reasons: The fundamental interactions. These are comprised of gravity, electromagnetism, as well as the strong and weak interaction, which can mathematically all be described as fields. Gravity, the theory dominant on large scales, can be embedded into a purely geometrical description, the general theory of relativity. The latter three forces are dominant when describing very small scales, and are described by a field theory that takes into account the quantum nature that we encounter at these scales - quantum field theory. Both of these theories are overwhelmingly successful in the description of their respective regimes, yet break down when trying to be connected. Quantum field theory relies on the idea that the interactions are mediated by point particles. The central idea of string theory is to do away with the notion of point particles and consider strings with no thickness in their stead. As simple as this conceptual replacement may be, it proves to be equally consequential: String theory features a graviton in its spectrum, making it a hopeful candidate for a quantum theory that describes gravity. String theory can be formulated on any geometrical background, but it turns out that some are more interesting and rich to analyse than others. In this PhD thesis, it will always be a particular class of geometric backgrounds that are in the focus - so-called *AdS* backgrounds. In these settings, two things will be accompanying us throughout this report, *integrability* - a particularly useful feature of some physical systems - and *deformations* - a way of continuously generating new physical models from already analysed ones, hoping that they retain crucial properties.

There will be three projects that, from different perspectives, will shine a light on string theories of this kind. In the first part, we will embark on an algebraic study related to the massless scattering in the AdS_3 context. The algebraic objects named *Hopf algebras* intrinsically feature a way that lends itself nicely to describing scattering processes, of which we will make use for both the undeformed and deformed setting. In the second project, instead of using an algebraic approach to analyse a string theoretical problem, we make use of an *effective field theory* approach in the context of a deformed AdS_3 background. These approaches allow us to analyse an a priori complicated theory in the limit of some parameter appearing in the theory, corresponding to some regime where the theory is more manageable to analyse. In this simplified theory that only describes our model under some effective assumptions, we can then use standard quantum field theoretic tools to gain insight on the scattering processes that occur. In the last part of this thesis, we move on to the topic of *spin chains*, which are quantum models that share a deep connection with AdS string theories and often feature properties of integrability. To gain a deeper insight into the energy structure of the specific spin chain we are looking at, we develop a method based on linear algebra, and successfully apply it to our problem.

Acknowledgements

First and foremost, I want to thank my principal supervisor Prof Dr Alessandro Torrielli as well as my co-supervisors Dr Juan Miguel Nieto García and Dr Martin Wolf. Alessandro and Juan Miguel both challenged me in our collaboration and supported me, especially during the trying times of the COVID-19 pandemic when I had to leave the country and work from abroad, as well as when personal family circumstances have made times tough. I was very grateful to be able to count on them for advice, both in research and elsewhere. I also thank Dr Jan B. Gutowski and Dr Andrea H. Prinsloo for having agreed to be the internal examiners in the confirmation process, and Dr Jan B. Gutowski and Dr Vidas Regelskis to be my examiners for the PhD viva. Further academics and staff involved in my PhD process I want to thank are Prof Dr Ian Roulstone, Dr Matt Turner, Dr Paul Skerrit, Gemma Park and Natasha Partt.

I am also grateful to the Doctoral College, the Department of Mathematics and the University of Surrey as a whole for making my research possible through the generous DCSA3 Award and the Turing Award, as well as awarding me "PGR Student of the Year 2022" for my achievements. I enjoyed the academic environment that Surrey has provided me with during my studies.

I was fortunate to undertake a research stay at the University of Santiago de Compostela hosted by Dr Riccardo Borsato and Dr Sibylle Driezen, generously funded by the aforementioned Turing Grant and leading to a fruitful collaboration. I also want to express my gratitude to Dr Ben Hoare and the other organisers of the IDD conference 2021 for inviting me to speak at such a prestigious event.

More broadly, I want to thank the people and friends I met and made along the way, including Daniele Farotti, Martí Berenguer, Dr Jam Sadiq, Mirco di Gioia, Dr Niels van Brempt, Praveen Kumar, Tim Meier, Silvan Schwarz, and all the people whose names escape my mind at this particular moment.

I thank Dr Sibylle Driezen, Jürg Haag, Dr Juan Miguel Nieto García, Prof Dr Alessandro Torrielli, and especially Silvan Schwarz and Linus Stillhart for providing me with useful input on my manuscript.

I also thank my mom - vielen Dank dafür, dass du mich dabei unterstützt hast, mich selbst zu sein - and my dad - plus que tu devrais être fier de moi, je devrais être fier de toi pour la force que tu as été au cours de l'année écoulée.

最后，感谢惟雨。尽管你不知道我每天都在忙些什么，但是谢谢你理解我的处境。

Statement of Originality

This thesis is a result of my own efforts. The work to which it refers is based on my PhD research projects, done in collaboration with Dr Alessandro Torrielli and Dr Juan Miguel Nieto García, which are

- arXiv:2004.02531: *Boost generator in AdS_3 integrable superstrings for general braiding*, published in JHEP.
- arXiv:2009.11171: *Boosts superalgebras based on centrally-extended* $\mathfrak{su}(1|1)^2$, published in JGeoPhys.
- arXiv:2102.06419: *Three-parameter deformation of $\mathbb{R} \times S^3$ in the Landau-Lifshitz limit*, published in JHEP.
- arXiv:2112.13883: *Jordan blocks and the Bethe ansatz I: The eclectic spin chain as a limit*, published in NuclPhysB.

In addition, I worked on a research project during my research stay in Santiago de Compostela (supported by the generous Turing Award):

- arXiv:2207.14748: *Semiclassical spectrum of a Jordanian deformation of $AdS_5 \times S^5$*, published in PhysRevD.

Any ideas, data, images or text resulting from the work of others are clearly identified as such within the work and attributed to the authors in the text or bibliography. This thesis has not been submitted for any other academic degree or professional qualification. The University of Surrey reserves the right to require an electronic version of the final document as submitted for assessment as above.

Contents

1 | String Theory and the AdS/CFT Correspondence

"... Und jedem Anfang wohnt ein Zauber inne." ["... A magic dwells in each beginning."]
– Hermann Hesse, *Stufen*

1.1 Motivation and Historical Context

The initial goal of string theory is an ambitious one: to formulate a single comprehensive quantum theory of all four forces that we (currently) consider to be governing the universe, namely gravity, the electromagnetic, weak and strong forces.

One of Sir Isaac Newton's main contribution to the scientific understanding of nature was formulating the first widely accepted description of gravity in the late 17th century. In his law of *universal gravitation*, every body of mass exerts a force on every other body of mass in an instant fashion and with infinite range. Its force is proportional to the product of the two masses m_1, m_2 and decreases with distance r:

$$F \propto \frac{m_1 m_2}{r^2} \ . \tag{1.1.1}$$

While Newton's empirical result was astounding, it needed further modifications later on. Before any meaningful amendments to the theory of the gravitational force were made, however, it was James Clerk Maxwell who in the 19th century identified the electromagnetic interaction and described it using his famous equations:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \ , \\ \nabla \cdot \mathbf{B} &= 0 \ , \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \ , \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{j} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \ . \end{aligned} \tag{1.1.2}$$

Maxwell's theory successfully unified electricity and magnetism and described them as effects of more general electromagnetic *fields*. To be able to work with them, the language of *vector calculus* needed to be developed, and apart from more involved mathematics, the (finite) speed of light in a vacuum (commonly referred to as c) also made its first prominent appearance in a physical theory. Newton's theory of mechanics and Maxwell's equations were fundamentally incompatible, and it took none other than Albert Einstein to understand this problem in his annus mirabilis at the beginning of the last century, when he formulated the *special theory of relativity (SR)*. This theory, where c is of utmost importance again,

amends Newtonian mechanics to describe physical systems with bodies moving close to the speed of light. Finally, with more advancement in astrophysical experiments and apparent discrepancies between Newton's theory and observations - such as the precession of Mercury - it was becoming more and more clear that a fundamental revolution of the understanding of mass, time and space was needed. In 1915, Einstein refined his special theory of relativity into the *general theory of relativity*, which is commonly formulated in the even more advanced mathematical language of *differential geometry*, whose central equation is Einstein's field equation

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \ . \tag{1.1.3}$$

Einstein's theory of gravity is in overwhelming agreement with experimental data, and is able to explain phenomena like gravitational red shift - where Newton's theory fell short. The scientific community also acknowledges General Relativity to be an extraordinarily beautiful theory - the left-hand side of Einstein's equations contains spacetime, whereas the right-hand side contains the matter content, inviting one to think of the famous phrase "matter tells spacetime how to bend, curvature tells matter how to move".

Einstein's theory provides us with a marvellously successful model of the very large, i.e. cosmology and gravitation, however, it seems to break down when trying to describe the very small, i.e. the (sub)atomic physical regime. It was a monumental effort of scientists in the 20th century to formulate and develop a theory of the *quantum*. At this point, we will gloss over the very beginnings of quantum mechanics (QM) involving advancements such as the proposal of particle-wave duality, the uncertainty principle and the Schrödinger and Dirac equations, and move on to quantum theories that involve *fields*. In quantum field theory (QFT), the quantum mechanical wave function is upgraded to a dynamical state. At the bottom of QFT lies the idea that particles are just quantised excitations of an underlying quantum field, that in turn depend on the coordinates of the spacetime they live in with certain dynamical constraints (as in classical Lagrangian field theory). Building on the behaviour of the operators present in QM, these fields are operator-valued and expected to satisfy certain commutation relations that do not violate the causal framework of SR (c.f. *canonical quantisation*, or for the other prominent quantisation approach, see *path integrals*). The tools of QFT allow one to (perturbatively) compute physical quantities, most notably the *amplitude*, i.e. the probability that a given initial state evolves into a specific final state over some time. For scattering processes, its square-root is proportional to the modulus of the scattering cross-section. With the instruments of QFT, the *Standard Model (SM) of particle physics* is able to describe the three other fundamental forces of nature other than gravity: The electromagnetic one (which Maxwell worked on), the weak and strong one - the former two of which can be unified. As electric and magnetic forces appear to be two independent forces in our everyday life but can be described as one single force, the weak and electromagnetic forces at low energies also appear to be two independent fundamental interactions, however, above a certain energy scale, we observe that they are two manifestations of the same force. The quantum field theoretic formulation at the bottom of this is based on work from Weinberg and Salam and involves very important descriptive tools such as the Higgs mechanism and spontaneous symmetry breaking. Beyond the electroweak unification, there have been proposals to unify the strong force together with the electromagnetic and weak forces as well into a single *Grand Unified Theory*, which is schematically summarised in 1.1.

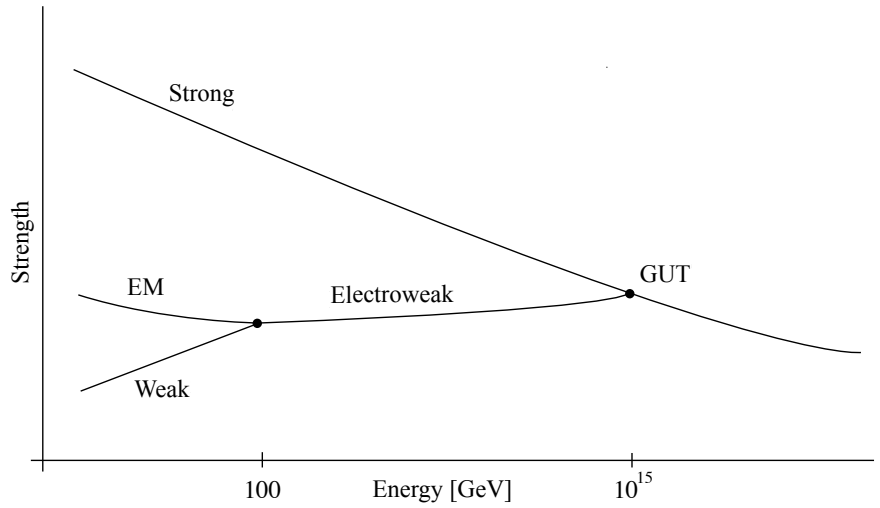


Figure 1.1: A qualitative diagram of the couplings ("force strengths") of the three SM forces at different energy scales. It is the (as of yet) unfulfilled hope of beyond SM phenomenologists that at some GUT scale, the three forces merge and can be described as one unified force.

A prominent example of such a theory is the *Pati-Salam model* with $SU(4) \times SU(2)_L \times SU(2)_R$ as the gauge group, which recently has experienced a renaissance in interest due to anomalies in flavour physics.[1] The framework of QFT is not without problems, however, and a common issue that arises is divergent behaviours and, as a consequence, the failure to make perturbative calculations work. Under certain preconditions, these pathologies can be remedied using different procedures of *renormalisation*, which is the case for the Standard Model. Unfortunately, GR seen as a (quantum) field theory is not renormalisable and therefore, while GR is a fantastic model for "all things large" and QFT together with the SM is a splendid predictor for processes related to "all things small", they are highly incompatible.

At first, this seems very contradictory - after all, stars and planets are made of the very same things as bosons and mesons. Thus, it boggles the mind why a model that describes the universe very well on small scales should not also describe it well on larger scales. This motivates the quest to find a more universal description of nature that does justice to both ends of the scale of magnitudes, but so far, this search has been largely unsuccessful.

1.2 String Theory and the Polyakov Action

The fundamental assumption of string theory is to replace the notion of (point-like) particles with (excitations of) strings, and the new action and properties that come with this (closedness/openness, length, and so on). While this might seem like a fairly inconsequential generalisation, it turns out, it changes everything.

For a point particle, the dynamical action is proportional to the length of the worldline that it sweeps out in spacetime. In the case of a string, it sweeps out what is called a *worldsheet*, and we want[2] the

[1] This is not the same kind of anomalies as the ones in QFT symmetries.
[2] Also for reasons of reparametrisation invariance.

action to be proportional to the worldsheet area. The worldsheet is a curved 2-dimensional surface embedded in spacetime - or target space, geometrically speaking. Then, the induced metric $\gamma_{\alpha\beta}$ on the worldsheet is the pullback of the target space metric $g_{\mu\nu}$ of the target space, i.e.

$$\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} g_{\mu\nu} \ , \tag{1.2.1}$$

which then lets us construct the so-called *Nambu-Goto action* for a relativistic string,

$$S = -T \int d^2\sigma \sqrt{-\det \gamma} = -T \int d^2\sigma \sqrt{-\dot{X}^2 X'^2 + (\dot{X} \cdot X')^2} \ , \tag{1.2.2}$$

where we identify T to be the tension of the string. We will always use X to refer to target space coordinates and σ to indicate worldsheet coordinates. This action is by construction proportional to the worldsheet area, but is in some regards incommodious: The equations of motion of this action are nasty, with a lot of square-root expressions that are rather complicated to deal with, especially for path integral quantisation techniques. But it describes the dynamics of a string, so whatever action we choose in its stead needs to be (classically) equivalent, i.e. have equivalent equations of motions. The so-called *Polyakov action* fits our bill:

$$S = -\frac{T}{2} \int d^2\sigma \sqrt{-\det \gamma} \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu} \ , \tag{1.2.3}$$

where $g_{\mu\nu}$ is again the target space metric (and thus can depend on the coordinates X). Apart from the manifest Poincaré-invariance of S, it is also invariant under reparametrisations (i.e. diffeomorphisms) and conformal Weyl transformations of the worldsheet metric (classically).[3] At this point, we will skip a detailed discussion on gauge fixing (that will sometimes let us choose γ in Sylvester form), mode expansions and quantisation. In this quantisation process, however, a quantity called the *central charge* arises (which in CFT is just a c-number commuting with everything), and it shares a connection with the target space dimension D. For the theory to be consistent on a quantum level, we need to set the central charge to a value that implies $D = 26$, called the critical dimension of bosonic string theory. In the second chapter of this thesis, the bosonic Polyakov action will be of great importance and interest: The background metric $g_{\mu\nu}$ will be an AdS_3 one (that is deformed in a specific way), and in addition to the symmetric metric, there will also be a term present that couples to a 2-form, called the B-field or *Kalb-Ramond* field, which can be understood as a generalisation of the electromagnetic potential.

Moving away from a purely bosonic theory and adding fermions to our model, we introduce *supersymmetry*, enhancing the previous purely bosonic symmetry of our theory to a $\mathbb{Z}_2$-graded one. To formulate a theory possessing supersymmetry, we need additional mathematical ingredients in our description, such as Lie superalgebras and Graßmannian graded variables. Skipping here again the details of the quantisation mechanism and, more importantly, if we have to introduce supersymmetry on the target space or on the worldsheet, we arrive at the critical dimension $D = 10$ for superstring theory to be consistent. An extensive explanation and review of this can be found in [GSW12b], [GSW12a], [Zwi09], [Pol05b], [Pol05a] and [Ton09], to name a few.

So far, although interesting and mathematically richer, we have not made an argument as to why we put our hope in string theory as a candidate to describe all forces and interactions of nature in a unified way. If anything, the fact that we need to work in an at least 10-dimensional spacetime to make it work makes it tempting to dismiss it right away.[4] However, string theory seems to have an enigmatic

[3] We also want this to hold on the quantum level, with the Polyakov action thus describing a 2-dimensional conformal field theory (CFT).

[4] In addition, supersymmetry is also far from established as a likely occurrence in nature.

connection to gravity. For instance, using the bosonic Polyakov action in a curved background, with the help of the partition function formulation and vertex operators, one finds that after expanding in metric coupling constants, a vanishing β functional implies the vacuum Einstein equations. And more straightforwardly, the spectrum of the closed string contains a spin-2 particle that one can identify with the *graviton*, the supposed elementary particle for the fundamental gravitational interaction, while the spectrum of the open string contains fields that we can identify with SM gauge fields. So although formulating string theory as a consistent quantum theory might seem laborious and complicated, it offers a lot of possibilities that makes one consider it to be a viable candidate for a theory of everything in a very natural way. Apart from this prospect, studying string theory has also led to interesting results in mathematics and related disciplines. We shall focus on one particular instance and type of background, where the study of strings is of particular interest.

1.3 The AdS/CFT correspondences

1.3.1 The AdS_5 / CFT_4 Case

In general, holography in physical systems describes the duality (i.e. equivalence in some defined sense) of a d-dimensional theory and a $d+1$-dimensional theory.[5] Here, we will focus our interest on the AdS/CFT duality, which, roughly speaking, relates a gravitational theory $d+1$-dimensional in Anti-de Sitter space ($\times$ compact space) with a conformal field theory (CFT) formulated on the (conformal) boundary of this space. Juan Maldacena proposed that *Type IIB* $AdS_5 \times S^5$ *String theory* is dual to an $\mathcal{N} = 4$ *supersymmetric* $SU(N)$ *gauge theory*. The original paper from Maldacena [Mal98] has sparked great interest in this subject, with close to 16'000 citations. In that same article, he also proposes a smaller sibling to this, namely AdS_3/CFT_2 - where an AdS_3 string theory denotes a string theory on a background of the form $AdS_3 \times S^3 \times M_4$, with M_4 some suitable 4-dimensional manifold that complements the $AdS_3 \times S^3$-factor to a 10-dimensional supergravity background (see e.g. the introductory part of [Ebe18]). On the CFT_2 side, we have the symmetric orbifold CFT over T^4 , which is trickier to formulate and has been, until recently, a subject of research and uncertainty. Though starting from a different dimension, the moral of this endeavour is the same: Every physical observable in AdS_3 has a partner-observable on the CFT_2 side (by considering the former on ∂AdS_3) which we expect it to coincide with, allowing us to make use of two different physical languages to describe the same physics. We shall give a brief overview of the holographic setting pertinent to our projects. At the bottom of why this concept is interesting for us lies the fact that the coupling constant of the string theory maps to the one of the (quantum) field theory, as more explicitly the weak coupling regime of the former corresponds to the strong coupling regime of the latter. We will outline this more in section 3.4. Important or pedagogical references that discuss this more comprehensively and beyond the case we are focussing on are e.g. [Wit98], [GKP02], [Nas07], [Zaf00], [DV00], [DF02], [Spi07] - all of which we are drawing inspiration from for this introduction. In our research, we shall mostly be dealing with mathematical questions relating to integrability or deformations that are motivated by a string theoretic or holographic origin, and thus we shall go over the important key points of AdS string theory, holography, and we shall do so in a very brief manner. Doing this, we also would like to comment on why approaching this topic with integrability in mind has proven to be useful.

As mentioned above, the most studied example of AdS string theory is superstring theory on a supersymmetric $AdS_5 \times S^5$ background, married to $\mathcal{N} = 4$ SYM conformal theory with gauge group

[5] With the feature that the former is formulated on the boundary of the latter.

$SU(N)$. On the CFT side, we have gauge bosons A_μ, six massless real scalars $\phi^k,\ k=1,...,6$, four chiral and anti-chiral fermions each $\Psi^a_{\alpha/\bar{\alpha}},\ a=1,..4,\ \alpha/\bar{\alpha}=1,2$ - all transforming in the adjoint representation. In broad terms, conformal symmetry can be expressed by extending the Poincaré symmetry with a dilation generator D and special conformal generator K:

$$\begin{aligned}
[D,P_\mu] &= -iP_\mu \ , \\
[D,M_{\mu\nu}] &= 0 \ , \\
[D,K_\mu] &= +iK_\mu \ , \\
[M_{\mu\nu},P_\lambda] &= -i(\eta_{\mu\lambda}P_\nu - \eta_{\lambda\nu}P_\mu) \ , \\
[M_{\mu\nu},K_\lambda] &= -i(\eta_{\mu\lambda}K_\nu - \eta_{\lambda\nu}K_\mu) \ , \\
[M_{\mu\nu},M_{\rho\sigma}] &= i(\eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\sigma}M_{\mu\rho}) \ , \\
[P_\mu,K_\nu] &= 2i(M_{\mu\nu} - \eta_{\mu\nu}D) \ .
\end{aligned} \tag{1.3.1}$$

Counting the indices while taking into account the restrictions on the generators, we can see that, in 3+1 spacetime dimensions, that we now have a 15-dimensional algebra. In addition to conformal symmetry and supersymmetry, curiously, the fields of the $\mathcal{N}=4$ SYM model exhibit a global $SU(4)$ symmetry[6], called the R-symmetry.[7] The R-symmetry, supersymmetry and conformal symmetry of $\mathcal{N}=4$ SYM are part of the larger superconformal group $PSU(2,2|4)$. Within this larger supergroup, the conformal algebra lies in the $SU(2,2)\simeq SO(2,4)$ factor of the (maximal) bosonic subgroup $SU(2,2)\times SU(4)$, while the R-symmetry is given by the latter $SU(4)\simeq SO(6)$ factor. We will go into more algebraic detail of this CFT in the discussion about its connection to spin chains later on in the next chapter. For now, one further peculiarity of this theory is that it can also be proven that the running one-loop β-function vanishes; for the one-loop case, this can be seen easily by expanding β in terms of Casimir coefficients - for any $SU(N)$ gauge theory, we have that (c.f. [GW73]),

$$\begin{aligned}
\beta(g_{YM}) = \mu\frac{\partial g_{YM}(\mu)}{\partial\mu} &= -\frac{g_{YM}^3}{96\pi^2}\left[22N - \sum_i C_i - 2\sum_j C'_j\right] \\
&= -\frac{g_{YM}^3}{96\pi^2}\underbrace{\left[22N - (6\cdot N) - 2\,(8\cdot N)\right]}_{\text{for } \mathcal{N}=4 \text{ SYM}} \\
&= 0 \ ,
\end{aligned} \tag{1.3.2}$$

where the i-sum is over real scalars, and the j-sum over Weyl fermions. To argue for the β-function to also vanish at higher orders, one needs to make use of superspace considerations and light cone arguments (see [GRcvS80], [BLE83], [Man83]).

On the string theory side, we are dealing with a background that decomposes into the following factors:

$$\text{AdS}_{n+1} = \frac{SO(n,2)}{SO(n,1)},\quad S^n = \frac{SO(n+1)}{SO(n)},\quad T^n = \underbrace{S^1\times\cdots\times S^1}_{n \text{ times}} = U(1)^n \ . \tag{1.3.3}$$

For $\text{AdS}_5\times S^5$, this translated to the spaces $SO(4,2)\times SO(6)$ (modulo some smaller homogeneous factors), which is isomorphic to $SU(2,2)\times SU(4)$. This is our bosonic ingredient, however, we want to realise a supersymmetric target space, and in a minimal fashion, we can embed this bosonic isometry

[6] With fundamental $SO(6)\simeq$ bifundamental $SU(4)$ in the boson representation.

[7] Technically, R-symmetry is part of supersymmetry, as it can be seen as the symmetry that "rotates" amongst the different copies of supersymmetry (in this case, $\mathcal{N}=4$).

group into the supergroup $PSU(2,2|4)$, giving us the target space (with quotient factors):

$$\frac{PSU(2,2|4)}{SO(4,1)\times SO(5)} \supset \mathrm{AdS}_5 \times S^5 \ . \tag{1.3.4}$$

The isometry group of this space is the (superconformal) $\mathfrak{psu}(2,2|4)$ algebra which is, consequently, according to the Wigner theorem, also the algebra that states from this theory transform under in the scattering problem. From the target space given above, one could follow the Metsaev-Tseytlin construction in [MT98] and formulate a supersymmetric coset action[8] - however, this might overload this introductory part whose main purpose is to motivate why we are dealing with certain structures that we are going to introduce later on.

In the context of the AdS/CFT correspondence, one particular limit in $SU(N)$ Yang-Mills gauge theories plays a central role: the so-called 't Hooft limit (see [Hoo74]). This idea starts with classifying the resulting Feynman diagrams according to their topologies. One can see that a diagram with P propagators, V vertices and C closed lines comes with the prefactor $g_{YM}^{2P-2V}N^C$. Identifying the propagators as edges and the closed lines as faces of a 2-dimensional surface, we can make use of the Euler formula relating the genus H of a surface with the number of faces, vertices and edges and write:

$$g_{YM}^{2P-2V}N^C \equiv N^{C-P+V}\lambda^{P-V} = N^{2-2H}\lambda^{P-V}. \tag{1.3.5}$$

The 't Hooft limit is defined by the prescription $N \to \infty$ while keeping the 't Hooft coupling $\lambda = g_{YM}^2 N$ fixed. We can see that in this limit, higher-genus diagrams get suppressed while planar diagrams dominate. This is also why the 't Hooft limit sometimes is referred to as the planar limit. With the aforementioned 2-dimensional identification in mind, one can think of these planar diagrams building a tessellation of a continuous worldsheet of a string theory. Having this in mind, this observation of 't Hooft already suggests some kind of relationship with string theory.

In any concrete formulation of the AdS/CFT correspondence, one needs to make use of so-called Dp-branes (see [Ton09], [Pit16], [Val14], which we drew inspiration from). They arise in the context of open strings, which have - in contrast to closed strings - two end points, and their dynamics are also governed by the Polyakov action. If the latter is expressed in conformal gauge, we can see that for the end points of the string (X^μ at $\sigma = 0, \pi$), we demand the boundary equation

$$\partial_\sigma X \cdot \delta X = 0 \ . \tag{1.3.6}$$

This can be fulfilled by either requiring $\partial_\sigma X^\mu = 0$ (corresponding to end points move freely) or $\delta X^\mu = 0$ (corresponding to end points are fixed in space) at the end points. The former requirement is called *Neumann boundary condition*, the latter is called *Dirichlet boundary condition*. In a d-dimensional space, one can impose $X^{0,\dots,p}$ to fulfil Neumann and $X^{p+1,\dots,d-1}$ Dirichlet boundary conditions. This way, the end points of the string lie in a $p+1$-dimensional hyperplane, called the Dp-brane[9], see 1.2.

[8] This idea of constructing an action within a supertrace sTr(...) as invariant bilinear is also why the $\mathfrak{p}$ in $\mathfrak{psu}(n,n|2n)$ sometimes is omitted, as the projective factor is irrelevant.

[9] Though sometimes the p is suppressed.

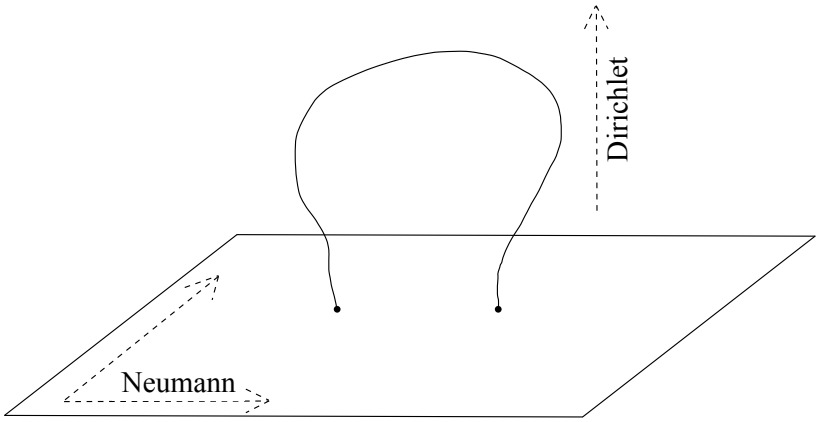


Figure 1.2: Visualisation of a D-brane.

In the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence, the stringy setup starts with N $D3$-branes, whose action can be divided into a bulk, a brane and an interaction part (that vanishes for when the Regge parameter approaches $\alpha \to 0$):

$$S_{\mathrm{Tot}} = S_{\mathrm{Brane}} + S_{\mathrm{Bulk}} + S_{\mathrm{Int}} \ . \tag{1.3.7}$$

An explicit $U(N)$ gauge symmetry is realised by the open strings with end points on the branes. However, since a $U(1)$ symmetry is associated to the stack of branes as a whole (visualised in 1.3), we are only interested in the $SU(N) = U(N)/U(1)$ factor in this description (see *Chan-Paton* index in this context).

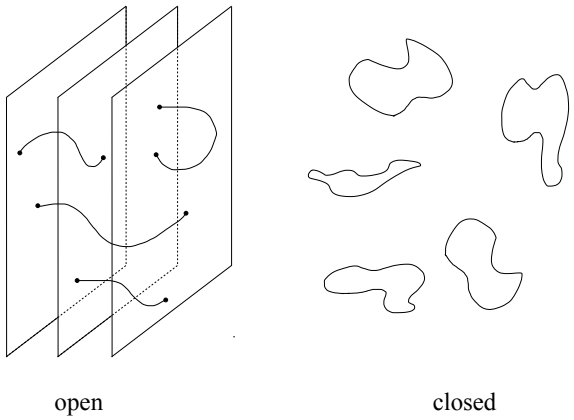


Figure 1.3: Open strings ending on a stack of D-branes, with (closed) strings propagating in the bulk.

For the (closed) string dynamics in the bulk part, we have the following metric:

$$ds^2 = \frac{1}{\sqrt{H(z)}} dx_{||}^2 + \sqrt{H(z)}(dz^2 + z^2 d\Omega_5^2,) \ , \tag{1.3.8}$$

where $z^4 H(z) = z^4 + 4\pi g_s N(\alpha')^2$, z measures the stack distance, $x_{||}^\mu$ is the spacetime coordinate parallel to the orientation of the branes. For small z compared to the parameters, we have

$$ds^2 = \frac{z^2}{2\sqrt{\pi g_s N(\alpha')^2}} dx_{||}^2 + \frac{2\sqrt{\pi g_s N(\alpha')^2}}{z^2} dz^2 + 2\sqrt{\pi g_s N(\alpha')^2}(dz^2 + z^2 d\Omega_5^2) \ . \tag{1.3.9}$$

This is exactly an $\mathrm{AdS}_5 \times S^5$ metric with radii $R_{\mathrm{AdS}_5} = R_{S^5} = \sqrt[4]{4\pi g_s N(\alpha')^2}$ - this means that we have a (type IIB) string theory in $\mathrm{AdS}_5 \times S^5$ from the bulk part, and a $\mathcal{N} = 4$ $SU(N)$ gauge theory on the boundary (branes).

Strictly speaking, in order to prove the duality, one would want to show that the coupling constants coincide $g_s = g_{YM}^2$ for all $N \in \mathbb{N}$, rather than just in some limit as e.g. the one we took above. This, as of yet, still remains unproven, however recent developments in holography - especially using the techniques of integrability - let us remain hopeful to get deeper insight into both sides of this.

1.3.2 The AdS_3 / CFT_2 Case

In AdS_3, the motivation is much the same. However, here we have several different candidates for backgrounds: $\mathrm{AdS}_3 \times S^3 \times M^4$, where M^4 is some suitable 4-dimensional manifold, restricted to either of the cases: $M^4 \in \{K3, S^3 \times S^1, T^4\}$. Only these 4-dimensional backgrounds are actually supersymmetric in the sense that they solve the supergravity equations. For us, the $M^4 = K3$ case is not of interest and will not be considered at any point in this thesis,[10] and the super-cosets we will look into are[11]

$$\mathrm{AdS}_3 \times S^3 \times S^3 \times S^1 \cong \frac{D(2,1;\alpha)^2}{SL(2) \times SU(2)^2} \times U(1) \ , \tag{1.3.10}$$

$$\mathrm{AdS}_3 \times S^3 \times T^4 \cong \frac{PSU(1,1|2)^2}{SL(2) \times SU(2)} \times U(1)^4 \ . \tag{1.3.11}$$

We are not going into excessive detail about the underlying brane configurations and arising metrics (see [CT98], [Mal98], [CT98] for specific derivations). However, the latter - being the slightly simpler of the two siblings - is achieved by compactifying four spacetime directions on T^4, and afterwards putting Q_1 $D1$-branes along one of the remaining non-compact directions and then Q_5 $D5$-branes along this direction and wrapping the former four compactified ones.[12] The metric associated with the T^4-coset case is given by

$$ds^2 = \frac{dx_{||}^2}{\sqrt{H_1 H_5}} + \sqrt{H_1 H_5}\left(dz^2 + z^2 d\Omega_3^2\right) \ , \tag{1.3.12}$$

where the functions H_i are again related to the separation of the Di-branes and given by $H_5 = 1 + c_5 \frac{Q_5}{z^2}$ and $H_1 = 1 + c_1 \frac{Q_1}{z^2}$, with c_i being coefficients depending on coupling constants,α' and manifold volumes. We chose the x-coordinate to go along the $D1$-brane and $dx_{||}^2 = dx^2 - dt^2$. In the near-horizon limit, we end up with radii $R_{\mathrm{AdS}_3}^2 = R_{S^3}^2 = \sqrt{Q_5 Q_1}$[13] and superisometry algebra

$$\mathfrak{psu}(1,1|2)_L \oplus \mathfrak{psu}(1,1|2)_R \ . \tag{1.3.13}$$

[10] The integrability analysis of the T^4 background is assumed to give rise to similar results as the $K3$, as the $K3$ manifold can be seen as a kind of discrete quotient of T^4.

[11] Here, we already present the super-cosets, i.e. the below "$\cong$" signs are to be understood as some kind of super-embedding rather than isomorphisms.

[12] There is also another realisation based on a $F1 - NS5$ setup, for which mixed fluxes can be more naturally formulated.

[13] To make sense of the units here, we need to remember that in our framework units are distinct *up to standard quantities* such as $c, \hbar$, the Planck mass/length, and so on. See also the convention used in [Sfo15].

For the setup of $\mathrm{AdS}_3 \times S^3 \times S^3 \times S^1$, rather than a $D1-D5$ brane stacks setup, we now deal with a $D1-D5'-D5$ brane stacks setup, with the metric in this case looking like

$$ds^2 = \frac{ds^2\left(\mathbb{M}_{(1,1)}\right)}{H_1} + H_5' dz'^2 + H_5 dz^2 \ , \tag{1.3.14}$$

where $\mathbb{M}_{(1,1)}$ is the $1+1$-dimensional Minkowskian metric, and the harmonics $H_5 = 1 + \frac{1}{z^2}$ and $H_5' = 1 + \frac{1}{z'^2}$, and $H_1 = H_5' H_5$. The separations of the two branes are measured by z and z'. Similar to the above, we then get as superisometry algebra of the background

$$\mathfrak{d}(2,1;\alpha)_L \oplus \mathfrak{d}(2,1;\alpha)_R \ , \tag{1.3.15}$$

and the radii R_1, R_2 of the two spheres satisfy

$$\frac{1}{R^2_{\mathrm{AdS}_3}} = \frac{1}{R_1^2} + \frac{1}{R_2^2}$$

$$\Leftrightarrow \tag{1.3.16}$$

$$\frac{R^2_{\mathrm{AdS}_3}}{R_1^2} + \frac{R^2_{\mathrm{AdS}_3}}{R_2^2} = 1 \ , \tag{1.3.17}$$

where R_{AdS_3} is the AdS_3 radius, as before. This condition on the radii follows from the fact that we want our background to satisfy the supergravity equations. We can see the α parameter of the superisometry group $D(2,1;\alpha)^2$ appearing through

$$\frac{R^2_{\mathrm{AdS}_3}}{R_1^2} = \alpha, \quad \frac{R^2_{\mathrm{AdS}_3}}{R_2^2} = 1-\alpha \ , \tag{1.3.18}$$

which satisfies the above-mentioned constraint on the radii. For $\alpha \in \{0,1\}$, the AdS radius R_{AdS_3} and either R_1 or R_2 coincide, with the other one decompactifying - in this limit, we get the other super-background of $AdS_3 \times S^3 \times T^4$ which has the superisometry group $PSU(1,1|2)^2$, as we previously stated. Up to some subtleties, the dual CFT to the string theory on one kind of this background is given by the large N limit of the (free) symmetric product orbifold $\mathrm{Sym}^N(T^4)$, which is to this day not as fully understood as its AdS_5/SYM counterpart. We will go into more details about different regimes, limits and aspects of the correspondences later on when motivating the utility of integrability.

1.4 Outline of the Thesis

This thesis is based on a collection of papers I published with my collaborators ([GW21], [NGTW20], [GTW20], [NGW22]). These projects were, in their topics and subject matter, quite different: Whereas [GTW20] was a natural continuation of [NGTW20], the topics of [GW21] and [NGW22] are of a distinct nature. However, in all of the efforts presented in this thesis, we are always revolving around the topics of integrability and deformation in AdS/CFT, and we shall always try to motivate this bigger picture in our endeavours.

In chapter 1, we motivated the historical context of string theory, different AdS/CFT correspondences, and explained the first appearances of algebraic structures within this.

In chapter 2, we outline the algebraic background necessary to understand chapters 4 and 6, as well as 3 to some extent. In this chapter, we will first go over the fundamentals of spectral theory in linear algebra, the canonical Jordan Normal Form for endomorphisms on finite dimensional vector spaces over an algebraically closed field. We then enter the field of Lie theory, and comment on the canonical classifications that exist to categorise (some) Lie algebras and Lie superalgebras. After this, we will review general notions appearing in algebra such as the ones of coalgebras and bialgebras. Lastly, we will discuss in more detail the theory of Hopf algebras, as they share an intimate connection with quantum integrability.

Moving on from mathematical preliminaries, we start to delve into the topic of integrability in chapter 3. To this end, we start by motivating the concept of integrability and explaining important concepts of classical integrability and their appearances in physics - we will not make use of integrability in the strictly classical sense, but a brief motivation of this is sensible before moving on to its quantum analogue. Here, we briefly talk about the appearance and properties of R-matrices, and how they connect to the algebraic exposition we made one chapter prior. We then make things more concrete by introducing the concept of spin chains - which often constitute prime examples of quantum integrable systems - and then illustrate many features by the example of different spin chains. In this context, we also present one example of the different Bethe Ansätze - the Coordinate Bethe Ansatz - and apply it to the spin chain systems that we introduced. We end the chapter about integrability by talking about its applications, appearances and successes within the physical context that we are interested in.

In chapter 4, we then move on to our projects that we studied in [NGTW20] and [GTW20]. Here, our setting was the algebraic study of the massless $\mathrm{AdS}_3/\mathrm{CFT}_2$ scattering problem from an integrability perspective. Proceeding analogously, first with a chosen representation, and then moving on to the representation independent case, we endow the generators of our algebra with a Hopf algebraic structure, including the additional boost generator $\mathfrak{J}$ that we introduce. In the representation-dependent case, we find suitable R-matrices for each of the particular Hopf algebraic structures that we chose initially, while for the representation-independent case we focus on a purely algebraic study that will give rise to a categorisation of boost relations and algebras.

Leaving the realm of algebraic study of integrability, we move on to present the work conducted in our publication [GW21] in chapter 5. Here, we are remaining within the context of AdS_3 and deformations, but choose an effective field theory approach by considering a limit where it reduces to a deformed LL model. We canonically quantise our effective Lagrangians and compute 2-body S-matrix elements by means of Feynman diagrammatics. For this chapter, we try to make all calculations explicit in a detailed and illustrative manner in the main part of the text and make an exposition of some standard tools of QFT within the presentation of our work, rather than dedicating a separate section or appendix to the calculations or methods.

In the last chapter of the main part, chapter 6, we move on to studying spectral and analytical properties of operators, and apply the results we find and methodology we develop to the case of the eclectic spin chain. In the first part of this chapter, we analyse operators with singular geometric multiplicity. We then proceed with relaxing this condition. This requires us to refine the algorithmic framework we developed in the first part of the chapter. The strongly twisted Hamiltonian appearing in the context of the eclectic spin chain then proves to be an ideal candidate for our methods to be applied to. We then successfully cross-check our results with known results within the literature available to us.

In chapter 7, we present conclusions on the chapters 4, 5 and 6, going over the contextual questions we posed ourselves initially as well as the important results. We then propose possible directions for future research for each of the projects we presented.

Lastly, in appendix A, we provide details on the calculations and extend some considerations of chapters 4 and 6. For the former chapter, since most algebraic computations in the representation-dependent case (quasi-cocommutativity relations, supercommutators, et cetera) have been performed using computer algebra, we limit ourselves to illustrate in the appendix the computations regarding the non-coassociative curiosity we found. For the representation-independent case, we try to place the necessary calculations already in the main part of this thesis. For the project of the latter chapter, we placed the more general cases as well as some more involved calculations in the appendix for the interested to study.

2 | Algebraic Footing

"Wer fremde Sprachen nicht kennt, weiß nichts von seiner eigenen." ["He who knows no foreign languages knows nothing of his own."]

– Johann Wolfgang von Goethe, *Maximen und Reflexionen*

In this chapter, we want to give a brief introduction to the pertinent algebraic theory for this thesis. Although it might seem an excessive excursion to the mathematical fundamentals, we shall delve into some aspects of linear algebra in the beginning of this chapter. The reason for this is that in the penultimate project that we present in this thesis, we will heavily rely on (and, in some part, extend) results from spectral theory of linear algebra. As our methodology will involve a lot of technical intricacies and subtleties, a more comprehensive recapitulation is reasonable. The standard literature of linear algebra that we will rely on and refer the reader to encompasses the works [MK08], [Jän03], and [Fis13]. Going beyond linear algebra and delving into the field of quantum algebra, for further reading on this, we refer the reader to [Kas95], [FH04], [Hum72], [Maj03] and - for a more geometric approach [Fec11]. With regard to a more pedagogical introduction, we recommend looking at the lecture notes of [Bei15]. We will to some extent follow these references when giving this introduction.

Hereinafter, whenever we refer to a field $\mathbb{K}$, we mean $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and likewise, we will omit the word "linear" when referring to maps. Particularly for the latter part of this introduction, it is also true that whenever we mention vector spaces (which we generally assume to be finite-dimensional), we can define analogous algebraic structures on modules over unital rings.

2.1 Linear Algebra

Naturally, we will assume familiarity of the reader with concepts such as vector spaces, homomorphisms, matrices and determinants and start our exposition with the most basic definitions of eigentheory. Let $M \in M_{d\times d}(\mathbb{C})$ be a complex square matrix of dimension $d \times d$. For any such matrix, we can define the associated *characteristic polynomial*[1] as[2]

$$\mathrm{char}_M(x) = \det(x\mathbb{1} - M) \ . \tag{2.1.1}$$

[1] Many authors define the characteristic polynomial as $\det(M - x\mathbb{1}_d)$, thus differing from our definition of a (monic) characteristic polynomial by a factor of $(-1)^d$. This does not affect the important properties of it, such as the roots and polynomial divisors.

[2] In the context of linear algebra, we denote the identity matrix as $\mathbb{1}$. If we additionally want to specify the dimension of the identity matrix, we refer to the $r \times r$ identity matrix as $\mathbb{1}_r$, but its dimensions can in most instances in this thesis be directly inferred.

The fundamental d'Alembert-Gauss theorem tells us that this polynomial has n roots, if multiplicity is factored in.[3] The roots $x = \lambda$ of this polynomial are called *eigenvalues* of M and have the defining property that a non-zero vector $v \in \mathbb{C}^d$ exists such that

$$v \in \ker(M - \lambda \mathbb{1}), \text{ or equivalenty } Mv = \lambda v \ , \tag{2.1.2}$$

in which case we call v an *eigenvector* of M associated with the eigenvalue λ, and $\ker(M - \lambda\mathbb{1})$ the eigenspace associated with λ. Any eigenvalue of a matrix carries two types of multiplicities: the algebraic and the geometric multiplicity. The *algebraic multiplicity* n of an eigenvalue λ is the degree of the root of λ in $\text{char}_M(x)$, which is another way of saying that $n \in \mathbb{N}_0$ is maximal such that $(x-\lambda)^n \big| \text{char}_M(x)$. The *geometric multiplicity* α is the dimension of the associated eigenspace, i.e. $\alpha = \dim\left(\ker(M - \lambda\mathbb{1})\right) = \text{nul}(M - \lambda\mathbb{1})$. For any of the eigenvalues $\{\lambda_1, \cdots, \lambda_r\}$ of M, we always have $\alpha_i \leq n_i$.[4] For the case where we have $\alpha_i = n_i \ \forall i \in \{1, \cdots, r\}$, we say that M is *diagonalisable* (or *semi-simple*). In this case, there exist d linearly independent eigenvectors of M, which spans out the whole space using this *eigenbasis*. Equivalently, this means that there exists a diagonal matrix that M is similar to via basis change to the eigenbasis, i.e.

$$M \sim \begin{pmatrix} D_1 & 0 & 0 & \dots \\ 0 & D_2 & 0 & \dots \\ 0 & 0 & D_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \ , \text{ with each block being of the form } D_i = \begin{pmatrix} \lambda_i & 0 & 0 & 0 & \dots \\ 0 & \lambda_i & 0 & 0 & \dots \\ 0 & 0 & \lambda_i & 0 & \dots \\ 0 & 0 & 0 & \lambda_i & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \ . \tag{2.1.3}$$

At first, it might seem that amongst the totality of complex $d \times d$ matrices, diagonalisable ones might constitute a minority[5] - after all, the above stated feature of diagonalisable is immensely powerful. However, within the space of $M_{d\times d}(\mathbb{C})$, the set of diagonalisable matrices lies dense. Nonetheless, for the cases where $\alpha_i < n_i$, we have no such relation at our disposal, and these matrices cannot be brought into diagonal form, in which case we call it *defective*. This is most obvious when considering that, since at least for some $j \in \{1, \cdots, r\}$, we have $\alpha_i < n_i$ and thus

$$\sum_i \dim\ker(M - \lambda_i \mathbb{1}) = \sum_i \alpha_i < \sum_i n_i = d = \dim(\mathbb{C}^d) \ , \tag{2.1.4}$$

which means we cannot construct a basis solely consisting out of eigenvectors. However, we can generalise the concept of eigenvectors in order to rectify this. We can relax the condition (2.1.2), and define a *generalised eigenvector* (or *hauptvector*) of rank n of M associated to the eigenvalue λ_i as a vector $v_{i,\alpha}^{(n)}$ satisfying

$$(M - \lambda_i \mathbb{1}) v_{i,\alpha}^{(n)} = v_{i,\alpha}^{(n-1)} \ , \tag{2.1.5}$$

where α_i is the geometric multiplicity associated with λ_i with $v^{(1)}$ being the eigenvector. Recall that a true eigenvector (for which we shall drop the (1) superindex) is by definition non-zero, which means that the above condition defining a *vector space flag* does not trivialise.[6] The property that $(M - \lambda_i\mathbb{1})^n v_{i,\alpha}^{(n)} = 0$

[3] This statement hinges on the fact that we are considering a polynomial over the complex numbers. Over $\mathbb{R}$, $\text{char}_M(x)$ does not necessarily split into linear factors, as some polynomials of degree 2 (such as $x^2 + 1$) are irreducible in $\mathbb{R}[x]$.

[4] We will suppress the index of α wherever it clutters the notation (producing indices of depth 2) and its reference is clear.

[5] Of course, only in a figurative sense, as both sets are infinite.

[6] This only defines a vector space flag if, for the next higher space in the flag chain, one considers the generalised eigenvectors *up to that rank* to be part of it, too.

follows immediately, as

$$\begin{aligned}(M-\lambda_i\mathbb{1})^n v_{i,\alpha}^{(n)} &= (M-\lambda_i\mathbb{1})^{n-1} v_{i,\alpha}^{(n-1)} \\ &= \cdots = (M-\lambda_i\mathbb{1}) v_{i,\alpha} \\ &= 0\ . \end{aligned} \tag{2.1.6}$$

Furthermore, the vectors defined by (2.1.5) form a chain that spans out a *proper* flag, meaning that they are linearly independent (but not necessarily orthogonal).[7]

As we have stated before, non-diagonalisable $d \times d$ matrices cannot generate d linearly independent eigenvectors that can then form a basis of the vector space. However, from these chains, we can indeed define n linearly independent generalised eigenvectors that in turn can form a basis of our underlying vector space, called the *generalised eigenbasis*. While we can never bring a non-diagonalisable matrix M into diagonal form using automorphisms (i.e. basis changes), we can bring it into a canonical block-diagonal form called the *Jordan Normal Form (JNF)*, first stated by Camille Jordan in 1870.[8] In its generalised eigenbasis, any given complex matrix M takes the form

$$M = \begin{pmatrix} J_1 & 0 & 0 & \dots \\ 0 & J_2 & 0 & \dots \\ 0 & 0 & J_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}\ , \text{ with each block being of the form } J_i = \begin{pmatrix} \lambda_i & 1 & 0 & 0 & \dots \\ 0 & \lambda_i & 1 & 0 & \dots \\ 0 & 0 & \lambda_i & 1 & \dots \\ 0 & 0 & 0 & \lambda_i & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\ , \tag{2.1.7}$$

where we refer to J_i as a *Jordan block*. The following main properties hold for the JNF of M:

- The geometric multiplicity of a given eigenvalue λ_i indicates how many Jordan Blocks associated to it we find in the JNF.
- The total sum of the sizes of all Jordan blocks associated to λ corresponds to its algebraic multiplicity, as this is simply the number of times λ_i appears on the diagonal of the JNF.
- The JNF of a diagonalisable matrix is purely diagonal, i.e. all Jordan blocks are of size 1. In this case, the notion of eigenbasis and generalised eigenbasis coincide.

There are several other illustrative properties of the JNF related to the minimal polynomial, but for the sake of brevity, we will not elaborate on this aspect of eigentheory.[9] Linear combinations of generalised eigenvectors of rank p again constitute a generalised eigenvector of the same rank, however, the generalised eigenvectors of rank $p-1$ that they are associated with may change. Even more so, any linear combination of generalised eigenvectors featuring just one generalised eigenvector of higher rank p than

[7] There exists the equivalent definition of generalised eigenvectors of rank n, demanding them to simultaneously satisfy

$$\begin{aligned}(M-\lambda_i\mathbb{1})^n v_{i,\alpha}^{(n)} &= 0 \\ (M-\lambda_i\mathbb{1})^{n-1} v_{i,\alpha}^{(n)} &\neq 0\ .\end{aligned}$$

This characterisation seems more straightforward, as there is no necessity to define any generalised eigenvector of lower rank that is necessary to define generalised eigenvectors of higher rank. However, as we shall see later on, we will be very interested in the eigenvectors of different rank associated to a given generalised eigenvector, as they form a chain that permits us to know more about the Jordan structure of the matrix that they are stemming from. More concisely, it allows us to define a *strict filtration*.

[8] Not to be confused with Pascual Jordan or Wilhelm Jordan.

[9] For the interested reader, we recommend the study of [Yan14], where Cayley theory and minimal polynomials are discussed in some detail.

the others constitutes a generalised eigenvector of rank p with the same generalised eigenvector of rank $p-1$ that was associated to originally. With this in mind, we are led to conclude the following:

Theorem 2.1.1. *The dimension of the vector space spanned out by generalised eigenvectors of rank l is smaller or equal than the dimension of the true eigenspace. This means that we have* $\dim\left\{\ker\left(M-\lambda\mathbb{1}\right)^{l}\right\}-\dim\left\{\ker\left(M-\lambda\mathbb{1}\right)^{l-1}\right\}\leq\dim\left\{\ker\left(M-\lambda\mathbb{1}\right)\right\}$.

More generally, we have that $\dim\left\{\ker\left(M-\lambda\mathbb{1}\right)^{l}\right\}-\dim\left\{\ker\left(M-\lambda\mathbb{1}\right)^{l-1}\right\}\leq\dim\left\{\ker\left(M-\lambda\mathbb{1}\right)^{n}\right\}-\dim\left\{\ker\left(M-\lambda\mathbb{1}\right)^{n-1}\right\}$ *holds for any two integers $n<l$.*

Proof. We want to use the first isomorphism theorem to prove the former claim. Let us therefore establish the following first: Given any vector $w\in\ker\left(M-\lambda\mathbb{1}\right)^{l}$, we have that $\left(M-\lambda\mathbb{1}\right)^{l-1}w\in\ker\left(M-\lambda\mathbb{1}\right)$ by definition. Therefore, we have the domain and codomain of the following set of maps:

$$\left(M-\lambda\right)^{l-1}:\ker\left(M-\lambda\mathbb{1}\right)^{l}\to\ker\left(M-\lambda\mathbb{1}\right)\,, \tag{2.1.8}$$

which implies together with the first isomorphism theorem that

$$\ker\left(M-\lambda\mathbb{1}\right)^{l}\Big/\ker\left(M-\lambda\mathbb{1}\right)^{l-1}\simeq\mathrm{Im}(M-\lambda\mathbb{1})^{l-1}\subseteq\ker\left(M-\lambda\mathbb{1}\right)\,, \tag{2.1.9}$$

where the last inclusion is only an equality for surjective induced maps. By comparison of their dimensions, we then arrive at the claim.

The second claim is a generalisation of the first, which we can prove similarly:

Given any vector $v\in\ker\left(M-\lambda\mathbb{1}\right)^{l}$ with the feature that $\left(M-\lambda\mathbb{1}\right)^{l-1}v\neq0$, meaning that

$$v\in\ker\left(M-\lambda\mathbb{1}\right)^{l}\Big/\ker\left(M-\lambda\mathbb{1}\right)^{l-1}\,, \tag{2.1.10}$$

we know that for $v_p:=\left(M-\lambda\mathbb{1}\right)^{p}v$ and $l>p$ the following holds:

$$\begin{aligned}\left(M-\lambda\mathbb{1}\right)^{l-p}v_p&=0\,,\\ \left(M-\lambda\mathbb{1}\right)^{l-p-1}v_p&\neq0\,.\end{aligned} \tag{2.1.11}$$

This is another way of saying that we have

$$v_p\in\ker\left(M-\lambda\mathbb{1}\right)^{l-p}\Big/\ker\left(M-\lambda\mathbb{1}\right)^{l-p-1}\,. \tag{2.1.12}$$

With this latter ingredient, we can construct the following chain of maps, in a similar vein to before:

$$\left(M-\lambda\right)^{p}:\ker\left(M-\lambda\mathbb{1}\right)^{l}\Big/\ker\left(M-\lambda\mathbb{1}\right)^{l-1}\to\ker\left(M-\lambda\mathbb{1}\right)^{l-p}\Big/\ker\left(M-\lambda\mathbb{1}\right)^{l-p-1}\,. \tag{2.1.13}$$

As we mentioned before we have $l>p$, which using the same isomorphism theorem to before leads us to:[10]

$$\ker\left(M-\lambda\mathbb{1}\right)^{l}\Big/\ker\left(M-\lambda\mathbb{1}\right)^{l-1}\simeq\mathrm{Im}(M-\lambda\mathbb{1})^{p}\subset\ker\left(M-\lambda\mathbb{1}\right)^{l-p}\Big/\ker\left(M-\lambda\mathbb{1}\right)^{l-p-1}\,. \tag{2.1.14}$$

Defining $p\in\mathbb{N}$ in such a way that $l-p=n$, we arrive at the following implication for the dimensions: $dim\left\{ker(M-\lambda\mathbb{1})^{l}\right\}-dim\left\{ker(M-\lambda\mathbb{1})^{l-1}\right\}\leq dim\left\{ker(M-\lambda\mathbb{1})^{n}\right\}-dim\left\{ker(M-\lambda\mathbb{1})^{n-1}\right\}$ □

This theorem is saying that, given a certain geometric multiplicity α_i associated to an eigenvalue λ_i, when we go deeper into the flag of generalised eigenspaces, at every step there are at most α_i new generalised eigenvectors that can appear. Even more so, the second part of the theorem implies that the number of linearly independent new generalised eigenvectors decreases with increasing rank.

[10] Obviously, $\left(\ker\left(M-\lambda\mathbb{1}\right)^{l}\Big/\ker\left(M-\lambda\mathbb{1}\right)^{l-1}\right)\Big/\ker\left(M-\lambda\mathbb{1}\right)^{p}=\ker\left(M-\lambda\mathbb{1}\right)^{l}\Big/\ker\left(M-\lambda\mathbb{1}\right)^{l-1}$ if $p<l$.

The eigenvectors of a Hermitian matrix are orthogonal - in this context, we mean orthogonal in the sense that $v_1^\dagger v_2 = 0$.[11] Hermitian matrices do not possess any proper generalised eigenvectors, since they are a priori diagonalisable. For our study of non-Hermitian matrices, we should not be so lucky: Even for non-Hermitian matrices whose true eigenvectors are orthogonal we have that, while the generalised eigenvectors are linearly independent, they do not need to be orthogonal.[12] We can recover a sense of orthogonality for generalised eigenvectors, but we need to further broaden the notion of eigenvectors first.

Definition 1. *Analogous to the notion of a usual* (right) *eigenvector of M, there exists the notion of* left *eigenvectors, defined to be the right eigenvectors of the Hermitian conjugate of M, meaning a vector that fulfils*

$$M^\dagger \hat{v} = \lambda^* \hat{v} \Leftrightarrow \hat{v}^\dagger M = \lambda \hat{v}^\dagger. \tag{2.1.15}$$

Consequently, a generalised left eigenvector of rank n of M associated to the eigenvalue λ satisfies the relation

$$(M^\dagger - \lambda^*)\hat{v}_{i\alpha}^{(n)} = \hat{v}_{i\alpha}^{(n-1)}. \tag{2.1.16}$$

For Hermitian matrices, i.e. $M = M^\dagger$, we also have that left eigenvectors and right eigenvectors are equivalent.[13] This is obviously not true for non-Hermitian matrices, but we can still find an interesting relation amongst the two different types of eigenvectors for any matrix M:

Theorem 2.1.2. *The left and right generalised eigenvectors of a matrix M can be chosen in such a way that the following* biorthogonality relations *hold:*

$$(\hat{v}_{i,\alpha}^{(n_{i,\alpha}+1-p)})^\dagger \cdot v_{j,\beta}^{(q)} = \delta_{ij}\delta_{\alpha\beta}\delta_{pq} \ , \tag{2.1.17}$$

where $n_{j,\alpha}$ is the size of the Jordan block associated with the α-th geometric multiplicity of λ_j.

Proof. We first prove a small lemma for this:

Lemma 2.1.3. *Given a left eigenvector $\hat{v}_{i,\alpha}$ associated to the eigenvalue λ_i and a right eigenvector $v_{j,\beta}$ associated to the eigenvalue $\lambda_j \neq \lambda_i$. We then have $\hat{v}_{i,\alpha} \perp v_{j,\beta}$.*

Proof.

$$0 = \hat{v}_{i,\alpha}^\dagger M v_{j,\beta} - \hat{v}_{i,\alpha}^\dagger M v_{j,\beta} = (\lambda_i - \lambda_j)\hat{v}_{i,\alpha}^\dagger \cdot v_{j,\beta} \ , \tag{2.1.18}$$

as we are considering $\lambda_i \neq \lambda_j$, the only possibility is that $\hat{v}_{i,\alpha}$ and $v_{j,\beta}$ are orthogonal. ∎

To make a similar statement of generalised eigenvectors, we will make use of the following two relations

$$\begin{aligned}(\lambda_i - \lambda_j)(\hat{v}_{i,\alpha}^{(1)})^\dagger \cdot v_{j,\beta}^{(p)} = (\hat{v}_{i,\alpha}^{(1)})^\dagger M v_{j,\beta}^{(p)} - (\hat{v}_{i,\alpha}^{(1)})^\dagger \lambda_j v_{j,\beta}^{(p)} = (\hat{v}_{i,\alpha}^{(1)})^\dagger (M - \lambda_j \mathbb{1}) v_{j,\beta}^{(p)} = (\hat{v}_{i,\alpha}^{(1)})^\dagger \cdot v_{j,\beta}^{(p-1)} \ , \\ (\hat{v}_{i,\alpha}^{(q-1)})^\dagger \cdot v_{j,\beta}^{(p)} + (\lambda_i - \lambda_j)(\hat{v}_{i,\alpha}^{(q)})^\dagger \cdot v_{j,\beta}^{(p)} = (\hat{v}_{i,\alpha}^{(q)})^\dagger (M - \lambda_j \mathbb{1}) v_{j,\beta}^{(p)} = (\hat{v}_{i,\alpha}^{(q)})^\dagger \cdot v_{j,\beta}^{(p-1)} \ .\end{aligned} \tag{2.1.19}$$

The first equation in (2.1.19) allows us to deduce that a (true) left eigenvector associated to λ_i is orthogonal to generalised right eigenvectors associated to an eigenvalue $\lambda_j \neq \lambda_i$: Given that we know

[11] In terms of notation, we shall sometimes make the $\cdot$ when referring to scalar products explicit when it is beneficial to the readability, and sometimes we shall drop it when the situation allows it. The operations we refer to, however, are identical.

[12] For the case of a matrix with Jordan block sizes not exceeding 2, there is a more explicit way to reconstruct orthogonality.

[13] As left eigenvectors are technically covectors, we cannot claim them to be equal. However, where this abuse of notation generates no confusion, we shall still write equalities for the sake of brevity.

from before that $(\hat{v}_{i,\alpha}^{(1)})^\dagger \cdot v_{j,\beta}^{(1)} = 0$, we can simply inductively (over p) reason that indeed $(\hat{v}_{i,\alpha}^{(1)})^\dagger \cdot v_{j,\beta}^{(p)} = 0$.[14] Similarly, the second equation of (2.1.19) then extends this orthogonality to hold amongst generalised eigenvectors of any rank: For $p = 2$, we can inductively infer orthogonality with any q by making use of the second equation; starting from $q = 2$, one can infer that $(\hat{v}_{i,\alpha}^{(2)})^\dagger \cdot v_{j,\beta}^{(2)} = 0$ (as all the other summands of that equation vanish), and then move on $q \to q+1$ drawing the analogous conclusions. This induction over q is then performed for each increasing value of p. This gives us the claim for $\lambda_j \neq \lambda_i$.

For coinciding $\lambda_i = \lambda_j$, we have the following equality that holds for all $k, l, p, q \in \mathbb{N}$ simultaneously (and, of course, transitively):

$$(\hat{v}_{j,\alpha}^{(p)})^\dagger (M - \lambda_j \mathbb{1})^{k+l} v_{j,\beta}^{(q)} = \begin{cases} (\hat{v}_{j,\alpha}^{(p)})^\dagger \cdot v_{j,\beta}^{(q-k-l)} \\ (\hat{v}_{j,\alpha}^{(p-k-l)})^\dagger \cdot v_{j,\beta}^{(q)} \\ (\hat{v}_{j,\alpha}^{(p-k)})^\dagger \cdot v_{j,\beta}^{(q-l)} \end{cases} . \tag{2.1.20}$$

As this holds for all $k, l, p, q \in \mathbb{N}$, it particularly holds for the case of $k = p-1$ and $l > 0$, which gives us:

$$0 = (\hat{v}_{j,\alpha}^{(p)})^\dagger \cdot v_{j,\beta}^{(q-p-l+1)} = (\hat{v}_{j,\alpha}^{(-l+1)})^\dagger \cdot v_{j,\beta}^{(q)} = (\hat{v}_{j,\alpha}^{(1)})^\dagger \cdot v_{j,\beta}^{(q-l)} , \tag{2.1.21}$$

because $p + l - 1 \geq p$ and thus $(\hat{v}_{j,\alpha}^{(p)})^\dagger (M - \lambda_j \mathbb{1})^{p+l-1} = 0$. Thus, we have that an eigenvector is orthogonal to all the generalised eigenvectors except for the ones of maximal rank n_j, since $n_j \geq q$ as the rank of the generalised eigenvector is bounded by the geometric multiplicity.

If the geometric multiplicity of the eigenvalue is one, our claim is proven: In this case, $\hat{v}_{j,\alpha}^{(1)}$ is orthogonal to all right generalised eigenvectors except the one of maximal rank $v_{j,\beta}^{(n_j)}$. As this set of generalised eigenvectors spans the full n_j-dimensional vector space and $\hat{v}_{j,\alpha}^{(1)}$ is non-vanishing, $(\hat{v}_{j,\alpha}^{(1)})^\dagger v_{j,\beta}^{(n_j)} \neq 0$, $\hat{v}_{j,\alpha}^{(1)}$ can be normalised such that $(\hat{v}_{j,\alpha}^{(1)})^\dagger v_{j,\beta}^{(n_j)} = 1$. Additionally, by considering the above equation for the extremal case of $n_j = q$, $(\hat{v}_{j,\alpha}^{(1+l)})^\dagger v_{j,\beta}^{(n_j-l)} = 1$ can be derived with the other possible combination of parameters giving us zero.

For the above argument to close our proof, we assumed the geometric multiplicity to be one. If it exceeds that, the proof is analogous - but a bit more technical - further details on this are provided in the appendix A.1. This concludes our proof. □

While it might seem so at first, biorthogonality does not imply that $\hat{v}_{i,\alpha}^{(n_{i,\alpha}+1-p)} = v_{i,\alpha}^{(p)}$ due to the generalised eigenvectors generally not being an orthogonal basis, unfortunately. However, there are two important exceptions:

Corollary 2.1.3.1. *The following equalities between left and right generalised eigenvectors of maximal rank hold:* $\hat{v}_{i,\alpha}^{(n_{i,\alpha})} = v_{i,\alpha}^{(1)}$ *and* $\hat{v}_{i,\alpha}^{(1)} = v_{i,\alpha}^{(n_{i,\alpha})}$.

Proof. The set of generalised eigenvectors form a basis of the vector space corresponding to its Jordan block, and since any left generalised eigenvector to the same block lies within the same vector space, we can expand it in terms of right generalised eigenvectors

$$\hat{v}_{i,\alpha}^{(n_{i,\alpha}+1-p)} = \sum_{k,\gamma,l} m_{i,k,\alpha,\gamma,p,l} v_{k,\gamma}^{(l)} . \tag{2.1.22}$$

[14] It is immediate that one is analogously able to prove that we have $(\hat{v}_{i,\alpha}^{(p)})^\dagger \cdot v_{j,\beta}^{(1)} = 0$. Consider the left-right symmetric of the first equation, i.e.

$$(\lambda_j - \lambda_i)(\hat{v}_{i,\alpha}^{(p)})^\dagger \cdot v_{j,\beta}^{(1)} = (\hat{v}_{i,\alpha}^{(p)})^\dagger M v_{j,\beta}^{(1)} - (\hat{v}_{i,\alpha}^{(p)})^\dagger \lambda_i v_{j,\beta}^{(1)} = (\hat{v}_{i,\alpha}^{(p)})^\dagger (M - \lambda_i \mathbb{1}) v_{j,\beta}^{(1)} = (\hat{v}_{i,\alpha}^{(p-1)})^\dagger \cdot v_{j,\beta}^{(1)} .$$

Once more, from the fact that we have $(\hat{v}_{i,\alpha}^{(1)})^\dagger \cdot v_{j,\beta}^{(1)} = 0$, we can inductively infer that $(\hat{v}_{i,\alpha}^{(p)})^\dagger \cdot v_{j,\beta}^{(1)} = 0$ holds.

Then applying biorthogonality, we find that the coefficients m have to satisfy the relation

$$\delta_{ij}\delta_{\alpha\beta}\delta_{pq} = \sum_{k,\gamma,l} m_{i,k,\alpha,\gamma,p,l}(v^{(l)}_{k,\gamma})^\dagger v^{(p)}_{j,\beta} \ . \tag{2.1.23}$$

We cannot set m to a Kronecker delta δ, because the generalised eigenvectors are not a priori orthogonal. For true eigenvectors, the above restriction does not apply: As the linear combination of a generalised eigenvector of rank q and a true eigenvector is again a generalised eigenvector of rank q without changing the vector they are associated with, we are able to construct linear combinations of generalised eigenvectors that are orthogonal to all the generalised eigenvectors of rank 1, i.e. true eigenvectors. With this in mind, we can indeed set m to a Kronecker delta for the case of $p = 1$, i.e. $\hat{v}^{(n_{i,\alpha})}_{i,\alpha} = v^{(1)}_{i,\alpha}$. The other relation is then obtained by expanding a right generalised eigenvector in terms of the basis of left generalised eigenvectors, and then repeating the same arguments. □

2.2 Lie Algebras

In this first section, we shall elaborate on the general theory of *Lie algebras*, a topic arguably familiar to most readers.

Definition 2. *A vector space* $\mathfrak{g}$ *over* $\mathbb{K}$ *together with a bilinear map* $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *is called a* Lie algebra *iff*

- $[\cdot,\cdot]$ *is alternating*[15]
- $[\cdot,\cdot]$ *fulfils the Jacobi identity, i.e.*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \ \forall X,Y,Z \in \mathfrak{g}, \tag{2.2.1}$$

in which case we call $[\cdot,\cdot]$ *the Lie bracket.*

For later convenience, we shall also quickly define two important types of Lie algebras.

Definition 3. *Let* $\mathfrak{g}$ *be a Lie algebra over* $\mathbb{K}$*. We call* $\mathfrak{g}$ solvable *iff the so-called derived series*

$$\mathfrak{g} \geq [\mathfrak{g},\mathfrak{g}] = \mathcal{D}^1\mathfrak{g} \geq [[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]] = \mathcal{D}^2\mathfrak{g} \geq [[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]],[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]]] = \mathcal{D}^3\mathfrak{g} \geq ... \tag{2.2.2}$$

eventually arrives at the zero subalgebra $\mathfrak{o} = \{0\}$*. Similarly, we call a Lie algebra* $\mathfrak{g}$ nilpotent *iff the so-called lower central series*

$$\mathfrak{g} \geq [\mathfrak{g},\mathfrak{g}] = \mathcal{D}_1\mathfrak{g} \geq [\mathfrak{g},[\mathfrak{g},\mathfrak{g}]] = \mathcal{D}_2\mathfrak{g} \geq [\mathfrak{g},[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]] = \mathcal{D}_3\mathfrak{g} \geq [\mathfrak{g},[\mathfrak{g},[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]]] \geq ... \tag{2.2.3}$$

eventually arrives at the zero subalgebra $\mathfrak{o}$*. From this definition, it is also evident that* $\mathcal{D}^1\mathfrak{g} = \mathcal{D}_1\mathfrak{g}$*, and that every nilpotent Lie algebra is solvable due to the inclusion* $\mathcal{D}_k\mathfrak{g} \supset \mathcal{D}^k\mathfrak{g}$*. The reverse implication, however, does generally not hold.*

We can now introduce the adjoint map, i.e. $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ via $\mathrm{ad}X(Y) = [X,Y] \ \forall X,Y \in \mathfrak{g}$, as well as an important definition relating to this:

[15] In this context, the alternativity property means $[X,X] = 0 \ \forall X \in \mathfrak{g}$. For $\mathrm{char}(\mathbb{K}) = 0$, this is equivalent to saying it is antisymmetric.

Definition 4. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$. We call a subalgebra $\mathfrak{h} \leq \mathfrak{g}$*[16] *the (maximal)* Cartan *subalgebra of $\mathfrak{g}$ if it is a nilpotent subalgebra that is self-normalising, in the sense*

$$\underbrace{\{a \in \mathfrak{g} | [a, \mathfrak{h}] \subset \mathfrak{h}\}}_{=:\mathfrak{N}_{\mathfrak{g}}(\mathfrak{h})} = \mathfrak{h}, \tag{2.2.4}$$

where we call $\mathfrak{N}_{\mathfrak{g}}(\mathfrak{h})$ the normalisator of $\mathfrak{h}$ in $\mathfrak{g}$. Additionally (for vanishing characteristic of the field), $ad\mathfrak{h} : \mathfrak{g} \to \mathfrak{g}$ is diagonalisable.

We now make another definition that will be important for us to distinguish certain Lie algebras that, as a class, share useful properties:

Definition 5. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$, and let $\mathfrak{i} \leq \mathfrak{g}$ be a subalgebra. We call $\mathfrak{i}$ an* ideal *of $\mathfrak{g}$ iff we have*

$$[\mathfrak{i}, \mathfrak{g}] \subseteq \mathfrak{i} \ , \tag{2.2.5}$$

and we call $\mathfrak{i}$ proper *iff $\mathfrak{i} \neq \{0\}, \mathfrak{g}$, as both of those sets are always ideals, trivially.*

With this definition at hand, we can state that a complex Lie algebra is *simple* iff it is non-abelian and contains no proper ideals. Similarly, we call a complex Lie algebra *semi-simple* if the bilinear form defined via $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, $\kappa(X, Y) = \mathrm{Tr}(\mathrm{ad}X\ \mathrm{ad}Y)$ is non-degenerate. This specific bilinear form is called the *Killing form*, and it is of central importance in Lie theory. Then, for semi-simple Lie algebras over $\mathbb{C}$, it can be shown that every Cartan subalgebra is abelian, which we shall keep in mind. In this thesis, Cartan elements will be of immense importance, as the energy is represented by an element in the Cartan subalgebra $\mathfrak{h}$. For complex Lie algebras $\mathfrak{g}$, we could have stated the semisimplicity condition equivalently as

- $\mathfrak{g}$ having no proper abelian ideals, or
- $\mathfrak{g}$ having no proper solvable ideals, or
- $\mathfrak{g}$ being the sum of simple complex Lie algebras.

In what follows, we assume $\mathbb{K} = \mathbb{C}$ and $\mathfrak{g}$ to be semi-simple and non-trivial, and we denote its dual algebra as $\mathfrak{h}^*$. For $H \in \mathfrak{h}$ and $\alpha \in \mathfrak{h}^*$, the generalised $\alpha(H)$-eigenspace of $\mathrm{ad}H$ is $\mathfrak{g}_{\alpha(H)}(H) = \{X \in \mathfrak{g} | (\mathrm{ad}H - \alpha(H)\ \mathrm{id})^{\dim(\mathfrak{g})} X = 0\}$. With this, we can introduce the following:

Definition 6. *Given a $\mathbb{K}$-Lie algebra $\mathfrak{g}$ and defining $\mathfrak{g}_\alpha := \bigcap_{H \in \mathfrak{h}} \mathfrak{g}_{\alpha(H)}(H)$ for $\alpha \in \mathfrak{h}^*$, one can show that*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha \ . \tag{2.2.6}$$

For $\mathfrak{g}_\alpha \neq 0$, we call α a root *of $\mathfrak{g}$ and $\mathfrak{g}_\alpha$ the corresponding* root space*. The totality of all roots is denoted by the set $\Phi^{\mathfrak{g}}$.*

Furthermore, by choosing an element $\tilde{H} \in \mathfrak{h}$, [17] *this induces a partial ordering on $\mathfrak{h}^*$ via*

$$\alpha \leq \beta \Leftrightarrow \alpha(\tilde{H}) \leq \beta(\tilde{H}) \ , \tag{2.2.7}$$

[16] A subalgebra is a subvector space $\mathfrak{h} \subseteq \mathfrak{g}$ with $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$. As we did in (3), we denote this by $\mathfrak{h} \leq \mathfrak{g}$.

[17] With the restriction that $\tilde{H} \notin \ker\alpha\ \forall \alpha \in \Phi^{\mathfrak{g}} \subseteq \mathfrak{h}^*$.

which allows us to decompose the roots into positive and negative ones, i.e. $\Phi^{\mathfrak{g}} = \Phi^{\mathfrak{g}}_+ \cup \Phi^{\mathfrak{g}}_-$*, where*

$$\alpha \in \Phi^{\mathfrak{g}}_{\pm} \Leftrightarrow \alpha(\tilde{H}) \begin{cases} > 0 & for + \\ < 0 & for - \end{cases} \quad as\ one\ might\ expect. \tag{2.2.8}$$

Likewise, this allows us to decompose the Lie algebra itself into $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-$*, where* $\mathfrak{g}_{\pm} = \bigoplus_{\alpha \in \Phi^{\mathfrak{g}}_{\pm}} \mathfrak{g}_{\alpha}$*. Finally, we say* $\alpha^s \in \Phi^{\mathfrak{g}}_+$ *is a* simple root *iff it cannot be written as the sum of two elements of* $\Phi^{\mathfrak{g}}_+$ *with positive coefficients. We denote the totality of all simple roots as* $\Phi^{\mathfrak{g}}_s \subseteq \Phi^{\mathfrak{g}}_+ \subset \Phi^{\mathfrak{g}}$*.*

It can also be shown that we have $|\Phi^{\mathfrak{g}}_s| = \dim\mathfrak{h}^* = \dim\mathfrak{h} := r$, the rank of $\mathfrak{g}$. The latter statement of sums in $\mathfrak{h}^*$ connects to the commutator relation $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.

It is easy to see that every root space $\mathfrak{g}_\alpha$ of a semi-simple Lie algebra is 1-dimensional. Thus, for all positive simple roots $\{\alpha^s_1, ..., \alpha^s_r\}$, we find (simple) generators $\{E_1, ..., E_r\}$ (and their negative-rooted pendants $\{F_1, ..., F_r\}$). For a Cartan basis $\text{span}(H_1, ..., H_r) = \mathfrak{h}$, we define the Cartan matrix as $A_{ij} := \alpha^s_j(H_i)$ [18], which leads us to define the following:

Definition 7. *We call a set of generators* $\{E_i, F_i, H_i\}$ *for* $i = 1, .., rank\ \mathfrak{g}$ Chevalley-Serre generators *if they fulfil*

$$\begin{aligned} [H_i, H_j] &= 0\ , \\ [H_i, E_j] &= A_{ij} E_j\ , \\ [E_i, F_j] &= \delta_{ij} H_j\ , \\ [H_i, F_j] &= -A_{ij} F_j\ , \end{aligned} \tag{2.2.9}$$

where span$(H_1, ..., H_r) = \mathfrak{h}$*. So far, we have not dealt with composite (i.e. non-simple) generators, and the a priori non-zero commutators they might produce. For this, we introduce the* Serre relations*, given by*

$$\begin{cases} (adE_i)^{N_{ij}} E_j = 0 \\ (adF_i)^{N_{ij}} F_j = 0 \end{cases} \quad where\ i \neq j\ and\ N_{ij} = \begin{cases} 1 & for\ A_{ii} = A_{ij} = 0 \\ 2 & for\ A_{ii} = 0, A_{ij} \neq 0 \\ 1 - 2\frac{A_{ij}}{A_{ii}} & for\ A_{ii} \neq 0 \end{cases} . \tag{2.2.10}$$

Naively, one might interpret the above as a prescription stating that every complex semi-simple Lie algebra $\mathfrak{g}$ with rank r can be seen as r different $\mathfrak{sl}_2$-triplets whose interaction is given by A_{ij}. While this might be in essence correct, this interpretation might lead one to believe that e.g. $\mathfrak{sl}_3$ is the sum of two copies of $\mathfrak{sl}_2$, which of course dimensionally does not suffice. Nevertheless, $\mathfrak{sl}_2$, being a very straightforward rank $r = 1$ Lie algebra, plays an important role for considering Lie algebras of higher rank.

While this concludes our general discussion of the structure of Lie algebras, we want to add a very brief note as to why it is a rather rewarding task to consider Lie algebras of low dimension, before moving on to the example of $\mathfrak{sl}_3$ and carrying on with our algebraic discussion. The Lie (super)algebras that will be pertinent to our project are all of relatively low dimension. The fact that there are not a large number of distinct low-dimensional - or, more precisely, small rank - Lie algebras can best be seen by looking at the Dynkin classification, which we shall briefly do.

[18] One needs to be careful with this definition of the Cartan matrix, but for our purposes, it will do. There also exist notions of symmetrised and unsymmetrised Cartan matrices, for a more comprehensive discussion, please see [FH04].

Firstly, we state that we can use a well-known invariant bilinear form, the Killing form κ, to construct a scalar product on $\mathfrak{h}^*$, essentially by first restricting $\kappa\big|_{\mathfrak{h}\times\mathfrak{h}}$ to the Cartan subalgebra[19], and then taking its dual inverse, such that we end up with a scalar product $\langle\cdot,\cdot\rangle : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$, with the property that

$$A_{ij} = \alpha_j^s(H_i) = 2\frac{\langle\alpha_i^s, \alpha_j^s\rangle}{\langle\alpha_i^s, \alpha_i^s\rangle} \ . \tag{2.2.11}$$

One can show that this[20] implies $A_{ii} = 2$ and, for $i \neq j$,

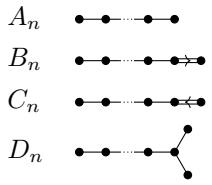
$$(A_{ij}, A_{ji}) \in \{(0,0), (-1,-1), (-2,-1), (-3,-1)\} \ , \tag{2.2.12}$$

which simplifies the situation (see e.g. [Bei15]).

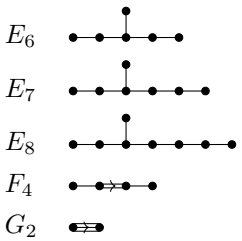
Cartan matrices, as constrained as we stated them to be, can be represented by Dynkin diagrams. Given a specific Cartan matrix A_{ij}, they can be constructed as follows:

- For a rank r Lie algebra $\mathfrak{g}$, first, we must draw r nodes, one for each diagonal element $A_{ii} = 2$. Each node carries a (mental) label from 1 to r.
- Nodes no. i and j are connected by $\max(|A_{ij}|, |A_{ji}|) \in \{0, 1, 2, 3\}$ edges, where $i \neq j$.
- In case of $(A_{ij}, A_{ji}) \in \{(-2,-1), (-3,-1)\}$, the edge carries an arrowhead pointing from node number i to j for $|A_{ji}| > |A_{ij}|$.

As it turns out, there are only so many different Dynkin diagrams - and correspondingly semi-simple complex Lie algebras - that can exist: There are 4 distinct infinite families A_n, B_n, C_n, D_n, and a handful of exceptional cases E_6, E_7, E_8, F_4, G_2. Their diagrammatic representation can be seen in 2.2.



(a) The infinite families



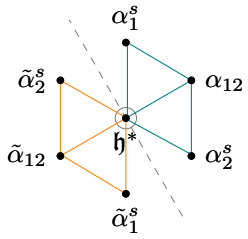
(b) The exceptional cases

Figure 2.1: The different Dynkin diagrams

With regard to the Cartan structure, A_n, B_n, C_n, D_n correspond to $\mathfrak{sl}_{n+1}, \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{so}_{2n}$, respectively. From this, it is also obvious that for the small-rank Lie algebras we have e.g. $A_1 = B_1 = C_1$ and $B_2 = C_2$.

Lastly, we shall just quickly illustrate what we have discussed here by making an example, namely $\mathfrak{sl}_3$. It has rank $r = 2$.

[19] Whose representation matrix is, naturally, related to the Cartan matrix.
[20] We again need to stress that we are only interested in the finite-dimensional case.



(a) The root system of $\mathfrak{sl}_3$.

(b) The Dynkin diagram of $\mathfrak{sl}_3$.

Figure 2.2: The Lie algebraic structure of $\mathfrak{sl}_3$.

In figure 2.2a, we see the positive (teal) and negative (orange) roots of $\mathfrak{sl}_3$, where a superscript s indicates that they are simple roots. For instance, α_1^s, α_2^s are the positive simple roots, while the root $\alpha_{12} = \alpha_1^s + \alpha_2^s$ is composite, which can also be seen by $[E_{\alpha_1^s}, E_{\alpha_2^s}] = E_{\alpha_{12}}$. $\mathfrak{sl}_3$ has a rather trivial Dynkin diagram, as can be seen in figure 2.2b, stemming from the Cartan matrix

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} . \tag{2.2.13}$$

With respect to simple complex Lie algebras, one can state that $\mathfrak{g}$ is simple iff its Dynkin diagram is connected, indicating to us that e.g. $D_2 = \mathfrak{so}(4)$ is not simple.

This concludes our discussion about ordinary (bosonic) Lie algebras. Before moving on to the generalisation of these notions in form of the super-case, we shall briefly introduce the very basic ingredients of *representation theory*.

Definition 8. *Let $\mathfrak{g}$ be a Lie algebra and V a vector space over $\mathbb{K}$. A linear map ρ,*

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V) = End(V) \ , \tag{2.2.14}$$

is called a representation *of $\mathfrak{g}$ on V iff it is a Lie algebra homomorphism, i.e.*

$$\rho([X, Y]) = [\rho(X), \rho(Y)] \ \forall X, Y \in \mathfrak{g} \ , \tag{2.2.15}$$

where the Lie brackets are defined on $\mathfrak{g}$ and $\mathfrak{gl}(V)$ (the latter of which can be understood as the commutator for us). If ρ is a injective map, we call the representation faithful. *In mathematics, the tuple (ρ, V) is referred to as a $\mathfrak{g}$-module.*[21]

Obviously, there exist many possible representations for any given Lie algebra - e.g. the trivial representation $\rho \equiv 0$ always exists for any $\mathfrak{g}$. There exists one particular notion that distinguishes one type of representation from others that we shall highlight.

Definition 9. *Let $\mathfrak{g}$ be a Lie algebra and V a vector space over $\mathbb{K}$, and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a representation. A subspace $W \subset V$ is* invariant *under ρ iff have*

$$\rho(X)w \in W \ \forall w \in W \ \text{and} \ \forall X \in \mathfrak{g} \ . \tag{2.2.16}$$

$\tilde{\rho} = \rho\big|_W$ is then a subrepresentation. *A representation $\rho \not\equiv 0$ on V is called* irreducible *iff V has no proper invariant subspaces. Otherwise, we call it* reducible. *In the language of modules, we call an irreducible representation a* simple module. *Additionally, if (a reducible) ρ on V can be decomposed into a direct sum irreducible subrepresentations, we call it* completely reducible.

[21] In the literature, the nomenclature is sometimes used in a confusing way, by e.g. referring to V as the representation itself.

One important result within the field of representation theory is that for complex semi-simple Lie algebras, every reducible representation is also completely reducible.

Definition 10. *For a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1$, we define the degree of an element $|A| = a$ for $A \in \mathfrak{g}^a$, $a = 0, 1$. We say $\mathfrak{g}$ is a* Lie superalgebra *iff it is equipped with a bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ if it fulfils ($\forall\, X, Y, Z \in \mathfrak{g}$)*

- $|[X, Y]| = |X| + |Y| \mod 2$*, i.e. it respects the grading,*
- $[X, Y] = (-1)^{1+|X||Y|}[Y, X]$*, i.e. it is super-alternating,*
- $(-1)^{|X||Z|}[X, [Y, Z]] + (-1)^{|Y||X|}[Y, [Z, X]] + (-1)^{|Z||Y|}[Z, [X, Y]] = 0$*, the super-Jacobi identity.*

In this case, we call $[\cdot, \cdot]$ a (Lie) superbracket.[22]

The entire discussion about definitions, Cartan classifications, Chevalley bases, roots and Dynkin families generalised straightforwardly (or linearly, more precisely) to the super case, which is why we will not reiterate this in a comprehensive "super-introduction" here. However, we would not do the topic justice if we did not explain some important super-notions.

For this, it is helpful to introduce a prime example: Given a (finite-dimensional) superspace $V = V_0 \oplus V_1$ with homogeneous basis $\mathfrak{B}_{(m|n)} = e_1, ..., e_m, e_{m+1}, ..., e_{m+n}$ (where the first m basis elements span V_0, and the latter n basis elements V_1), we can equip the superalgebra $\mathrm{End}(V)$ with the standard Lie superbracket for associative superalgebras

$$[a, b] = ab - (-1)^{|a||b|}ba, \tag{2.2.17}$$

it has the structure of a $(m|n)$-dimensional (or $m + n$-dimensional) Lie superalgebra and is commonly referred to as $\mathfrak{gl}(m|n)$. In the basis $\mathfrak{B}_{(m|n)}$, we can identify the elements of $\mathfrak{gl}(m|n)$ with blockmatrices of the form

$$\mathfrak{gl}(m|n) \doteq \left\{ \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \right\}, \tag{2.2.18}$$

where even and odd parts - i.e. the parts with respective $0, 1 \mod 2$ grading - of it can be represented by[23]

$$\mathfrak{gl}(m|n)_0 \doteq \left\{ \left[\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right] \right\}, \qquad \mathfrak{gl}(m|n)_1 \doteq \left\{ \left[\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array}\right] \right\}, \tag{2.2.19}$$

where the dimensions of the submatrices A, B, C, D should be self-evident. All of the relevant Lie superalgebras will be subalgebras of $\mathfrak{gl}(m|n)$ or closely related to them, so we can stick with this identification with blockmatrices for the following definitions.

Definition 11. *For a given element $X \in \mathfrak{gl}(m|n)$ realised through*

$$X \doteq \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \tag{2.2.20}$$

we define the supertransposition through

$$st: \ \mathfrak{gl}(m|n) \to \mathfrak{gl}(m|n)$$

[22] There exist many superbracket choices in many instances.

[23] Note here that the odd part does not form a subalgebra, whereas the even part does.

$$X \mapsto X^{st}, \tag{2.2.21}$$

where $X^{st} := \left[\begin{array}{c|c} A^t & -C^t \\ \hline B^t & D^t \end{array}\right]$. *For supermatrices* X, Y, *the supertransposition fulfils the identity* $(XY)^{st} = (-1)^{|X||Y|} Y^{st} X^{st}$, *which naturally reduces to the known property* $(M_1 M_2)^t = M_2^t M_1^t$ *for ordinary matrices* M_i.

For the above X, *there also exists a super-equivalent of the trace:*

$$\begin{aligned} sTr: \ \mathfrak{gl}(m|n) &\to \mathbb{K} \\ X &\mapsto sTr(X), \end{aligned} \tag{2.2.22}$$

where $sTr(X) := Tr(A) - Tr(D)$. *For the Lie superbracket on* $\mathfrak{gl}(m|n)$, *the supertrace fulfils the identity* $sTr([X, Y]) = 0$.

Using the supertrace, we can define an important subalgebra of $\mathfrak{gl}(m|n)$:

$$\mathfrak{sl}(m|n) := \left\{ X \in \mathfrak{gl}(m|n) \;\middle|\; \mathrm{sTr}(X) = 0 \right\}. \tag{2.2.23}$$

While one can prove that $\mathfrak{sl}(m|n)$ is simple for $m + n > 1$ and $m \neq n$, for $m = n$ we have $\mathrm{sTr}(\mathbb{1}) = 0$, thus $\mathfrak{sl}(m|m)$ has a 1-dimensional centre $\mathfrak{Z} = \mathrm{span}(\mathbb{1})$, which makes the algebra non-simple ([FSS96]). It makes therefore sense to look at the projective algebra

$$\mathfrak{psl}(m|m) := \mathfrak{sl}(m|m)/\mathfrak{Z} \tag{2.2.24}$$

instead. $\mathfrak{psl}(m|m)$ is simple for $m > 1$.

As mentioned before, not all Lie superalgebras will be relevant for our project. In fact, we will be focusing our efforts to *basic* Lie superalgebras, meaning the ones whose even parts are reductive and who admit to a so-called invariant (super)form. For a given Lie superalgebra $\mathfrak{g}$, this means that a bilinear form

$$(\cdot, \cdot): \ \mathfrak{g} \times \mathfrak{g} \to \mathbb{K} \tag{2.2.25}$$

fulfils

$$([X, Y], Z) + (-1)^{|X||Y|}(Y, [X, Z]) = 0 \ \ \forall X, Y, Z \in \mathfrak{g}\ , \tag{2.2.26}$$

which also can be reformulated by making use of the properties of the Lie superbracket as

$$([X, Y], Z) = (X, [Y, Z]) \ \ \forall X, Y, Z \in \mathfrak{g}\ . \tag{2.2.27}$$

In most of the literature, to be a basic Lie superalgebra, this invariant form is required to be non-degenerate. In the super-case, the Killing form of $\mathfrak{g}$ is defined as follows:

$$\kappa(X, Y) = \mathrm{sTr}(\mathrm{ad}X \mathrm{ad}Y) \text{ for } X, Y \in \mathfrak{g}\ . \tag{2.2.28}$$

The Killing form is invariant in the above sense, and also invariant under automorphisms of $\mathfrak{g}$. Furthermore, for simple Lie superalgebras $\mathfrak{g}$, the Killing form is graded symmetric satisfying $(X, Y) = (1)^{|X||Y|}(Y, X)\ \forall X, Y$,[24] invariant, and fulfils $\kappa(\mathfrak{g}_0, \mathfrak{g}_1) = 0$. Additionally,

- for $\mathfrak{g}$ of the type $A_{m,n\neq m}, B_{m,n}, C_{n+1}, D_{m,n\neq m-1}$, or F_4, G_3, its Killing form is non-degenerate, and

[24] This is also sometimes referred to as "supersymmetric" or "super-symmetric".

- for $\mathfrak{g}$ of the type $A_{m,m}, D_{m+1,m}, D_{2,1;\alpha}$, its Killing form is identically zero.

In this thesis, most of the above algebras will not play a role and we shall forego an extensive exposition of their definitions and properties, and the reader is referred to consult [Kac77] for a comprehensive discussion.

The notion of (simple) roots $\Delta_{(s)} = \{\alpha_1, ..., \alpha_r\}$, Cartan subalgebras $\mathfrak{h}$, Serre relations and generators E_α, F_α stays mostly the same for the super case, with the difference being now that there are even and odd generators and roots and the Lie bracket gets replaced by the Lie superbracket. Reminiscent of the ordinary case, all basic Lie superalgebras admit a non-degenerate inner product $(.,.) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ with $(E_{\alpha_k}, E_{-\alpha_l}) = (E_{\alpha_k}, E_{-\alpha_k})\delta_{kl}$ and $(H_k, H_l) = (E_{\alpha_k}, E_{-\alpha_k})A_{kl}$, where A is the Cartan matrix analogously. Except for the simple Lie superalgebras of type $A_{n,n}, D_{n+1,n}, D_{2,1;\alpha}$, this inner product always coincides with the Killing form (that is non-degenerate in these cases).

In the Cartan theory of Lie superalgebras, there is one subtlety to be addressed: Generally, there exist an infinite number of choices of generators and roots for any given Lie (super)algebra a priori. For a basic Lie superalgebra, there always exists a simple root system Δ_s which has the fewest odd roots.[25] We call this root system the distinguished root system Δ_s^d of the Lie superalgebra, which gives rise to the distinguished Cartan matrix. Lastly, as in the bosonic case, these Cartan matrices can be symmetrised canonically, but we will skip the procedural details of this and assume the Cartan matrices A^s to be symmetrised wherever necessary.

As a last step, let us discuss briefly the Dynkin classification in the super-case. We consider a basic Lie superalgebra $\mathfrak{g}$ of dimension n and rank $r = \dim(\mathfrak{h})$, with simple root system Δ_s and symmetric Cartan matrix $A^s_{kl} = (\alpha_k, \alpha_l)$. The Dynkin-recipe is then the following:

1. For each even root in Δ_s, we draw a white node. For the odd roots in Δ_s, we have to distinguish between roots of length zero ($A^s_{kk} = 0$) and of positive length ($A^s_{kk} \neq 0$) - for the former, we draw a grey (or marked) node, for the latter we draw a black node. The nodes are (mentally) numbered by the index of their corresponding root.

2. Nodes k and l are then linked by z_{kl} lines, with

   - $z_{kl} = \frac{2|A^{kl}|}{\min(|A^s_{kk}|, |A^s_{ll}|)}$ for $A^s_{kk} \neq 0 \neq A^s_{ll}$,
   - $z_{kl} = \frac{2|A^{kl}|}{\min_{A^s_{ii} \neq 0}(|A^s_{ii}|)}$ for $A^s_{kk} \neq 0, A^s_{ll} = 0$,
   - $z_{kl} = |A^s_{kl}|$ for $A^s_{kk} = 0 = A^s_{ll}$.

3. Lastly, we add an arrow to the line between nodes k and l,

   - pointing from k to l for $A^s_{kk} \neq 0 \neq A^s_{ll}$ and $|A^s_{kk}| > |A^s_{ll}|$, or for $A^s_{kk} = 0 \neq A^s_{ll}$ and $|A^s_{ll}| > 2$,
   - pointing from l to k for $A^s_{kk} = 0 \neq A^s_{ll}$ and $|A^s_{ll}| > 2$.

As we mentioned before, there are many different, inequivalent choices for simple root systems for any basic Lie superalgebra. Had we started this procedure with the distinguished one Δ_s^d, we would have constructed the distinguished Dynkin diagram for $\mathfrak{g}$. In this kind of diagram, the fewest odd roots are

[25] Although this may only seem to be a mathematical curiosity, it actually has important physical implications. One example can be found in [HS19] or [Sei19], where depending on the Dynkin diagram used to deform the string background, it will satisfy supergravity equations or generalised supergravity equations.

pictured. For completeness' sake, we list the Dynkin super-families

$$\left.\begin{matrix} A_{m,n} \\ B_{m,n} \\ D_{m\neq 1,n} \\ C_{n+1} \end{matrix}\right\} \text{ corresponding to } \left\{\begin{matrix} \mathfrak{sl}_{m+1,n+1} \\ \mathfrak{osp}_{2m+1,2n} \\ \mathfrak{osp}_{2m,2n} \\ \mathfrak{osp}_{2,2n} \end{matrix}\right. ,$$

and the exceptional super-cases $D_{2,1;\alpha}$ for $\alpha \notin \{0,-1\}$ (which can be understood as a deformation of $D(2,1)$), F_4 and G_3. Again, for a comprehensive discussion on this, we refer the reader to [Kac77], [FSS96] and [Ser17]. For our purposes, the $A_{m,n}$ family will be the most important one, but we outline the diagrammatic overview of the basic cases in 2.3 (see also [FSS96]).

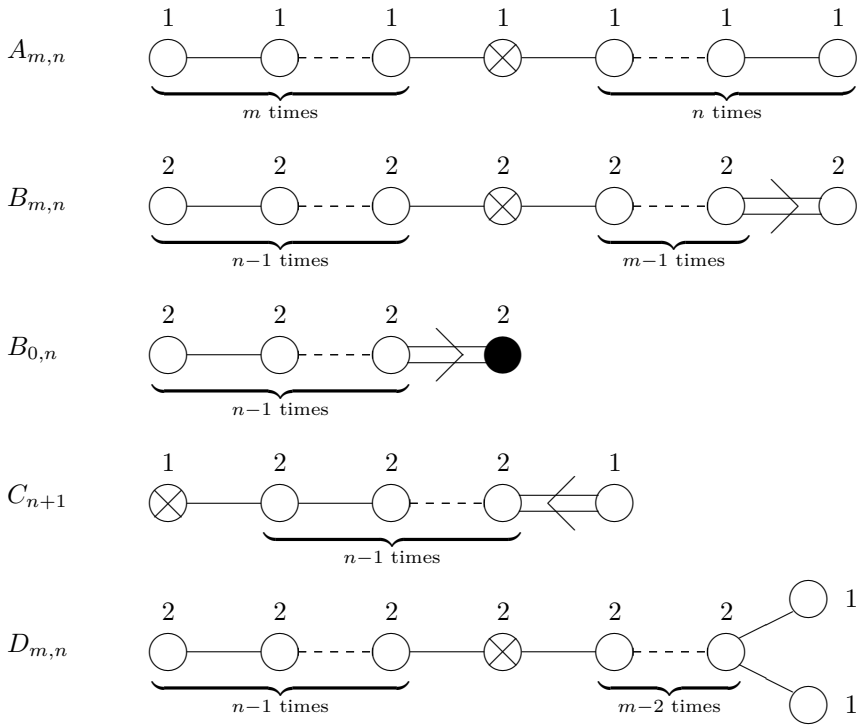


Figure 2.3: Distinguished Dynkin diagrams of the infinite families of the basic Lie superalgebras.

The above shows again the infinite families, and in 2.4 we find the exceptional cases.

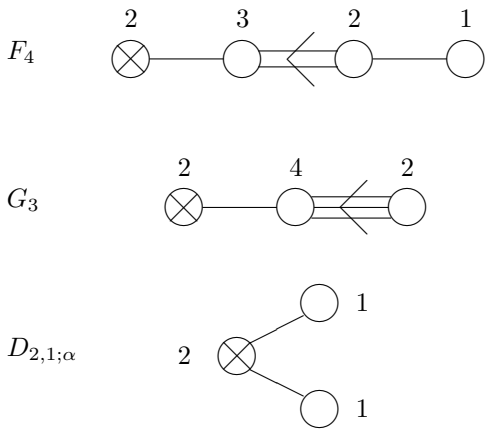


Figure 2.4: Distinguished Dynkin diagrams of the exceptional cases of the basic Lie superalgebras.

From a representation theoretic perspective, for the practical scope of this thesis, there are no particularly intricate differences: The grading of a superalgebra has to be respected by the representation space we choose and the representation we construct - and we will straightforwardly see this in the representations we are dealing with in chapter 4. The only subtlety that frequently appears within the context of superalgebras that we shall point out is the one of *short and long representations*. We first recall one important fact about semi-simple complex Lie algebras, namely that every representation that is not irreducible is completely reducible. For simple complex basic Lie superalgebras, the situation is unfortunately not quite the same, as there exist many representations for them that are reducible but not completely reducible. This fact is what prompts us to introduce the notion of *typical* and *atypical* representations of superalgebras.[26] The situation as it presents itself for semi-simple complex Lie algebras finds its analogue in the typical representation of superalgebras. We refrain from going beyond the scope of this thesis by means of providing a detailed exposition on highest weight representations, and simply state that if (ρ, V_λ) is an irreducible highest-weight representation of a superalgebra $\mathfrak{g}$ with the property that any highest-weight representation of $\mathfrak{g}$ to the same highest weight λ can be written as the direct sum $V_\lambda \oplus W$ (where W is some other representation of $\mathfrak{g}$), then (ρ, V_λ) is a typical representation. If an irreducible representation of $\mathfrak{g}$ is not typical, then we call it atypical. For a more comprehensive discussion on these results, please see [Kac78]. In the language of physicists, typical and atypical modules are referred to as long and short representations, respectively. For a representation to be short, it has to fulfil a so-called *shortening condition*, and shortening conditions play an important role in the discussion of physical representations within string theory (see e.g. [Tor11], [FZ00], [AGH97]), restricting the representation to some extent.[27]

Lastly, representations of the algebraic objects we are about to introduce work straightforwardly as expected, with some of the theorems and lemmata also holding for analogous structures.[28] We shall therefore not reintroduce all of the representation theoretic definitions for them.

[26] Strictly speaking, we are here talking about basic classical simple complex Lie superalgebras, which is a mouthful.

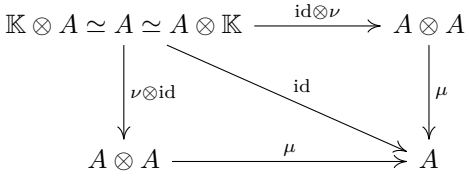
[27] Long and short representations share a meaningful connection with the appearance of anomalous dimensions, see e.g. [DO03].

[28] Schur's lemma, for example, also finds its analogue for algebras.

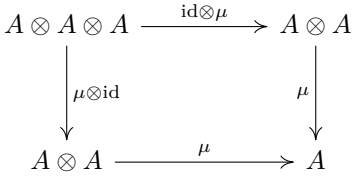
2.3 Algebras, Coalgebras and Bialgebras

We will now take a step back in our (Lie) algebraic discussion, and we shall introduce some more fundamental algebraic notions (for additional examples and explanations, [Kas95] is a good pedagogical introduction). We start with the most fundamental structure, the algebra.

Definition 12. *Let A be a vector space over $\mathbb{K}$, and $\mu : A \otimes A \to A$ and $\upsilon : \mathbb{K} \to A$ be maps. (A, μ, υ) is called an* (associative) algebra *iff the following diagrams commute*



unital



associative

In this case, we call μ the multiplication (or product) and υ the unit.

The diagram regarding unitality translates to the following relation,

$$\mu \circ (\nu \otimes id) = id = \mu \circ (id \otimes \nu) \ . \tag{2.3.1}$$

The diagram regarding associativity implies that we have

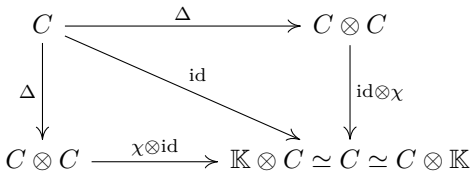
$$\mu \circ (\mu \otimes id) = \mu \circ (id \otimes \mu) \ . \tag{2.3.2}$$

We call an algebra (A, μ, υ) commutative *or* abelian *iff $\mu(x, y)$ is symmetric under the exchange of x and y. Defining the permutation map $\pi_{X,X} : X \otimes X \to X \otimes X$, $a \otimes b \mapsto b \otimes a$, we can also say that (A, μ, υ) is commutative iff $\mu = \mu \circ \pi_{A,A}$.*

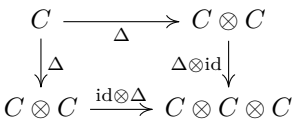
Lastly, we call a map $F : A \to A'$ an algebra morphism *iff $F \circ \mu = \mu' \circ (F \otimes F)$ and $\upsilon' = F \circ \upsilon$.*

We now introduce a concept that is in a sense dual to the one of the algebra - the coalgebra.

Definition 13. *Let C be a vector space over $\mathbb{K}$, and let $\Delta : C \to C \otimes C$ and $\chi : C \to \mathbb{K}$ be maps. We call (C, Δ, χ) a* coalgebra *iff the following diagrams commute*



counital



coassociative

In this case, we call Δ the coproduct (or comultiplication) and χ the counit. We call (C, Δ, χ) cocommutative iff $\pi_{C,C} \circ \Delta = \Delta$. We shall henceforward denote the opposite coproduct as $\pi_{C,C} \circ \Delta = \Delta^{op}$. These two diagrams are, similarly to before, implying the equations[29]

$$(id \otimes \chi) \circ \Delta = id = (\chi \otimes id) \circ \Delta \ , \tag{2.3.3}$$

[29] One could also dualise the unitality diagram by flipping all the arrows and arrive at the way of representing the counitality relations.

and

$$(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta \ , \tag{2.3.4}$$

respectively.[30]

We call a map $F : C \to C'$ a coalgebra homomorphism *iff $\Delta' \circ F = (F \otimes F) \circ \Delta$ and $\chi' \circ F = \chi$ hold.*

For expressions involving the coproduct, Sweedler ([SU69]) introduced a useful notation. For $\Delta(c) \in C \otimes C$, we write (in three variants)

$$\Delta(c) = \sum_k c'_k \otimes c''_k = \sum_{(c)} c' \otimes c'' = c^{(1)} \otimes c^{(2)} \ . \tag{2.3.5}$$

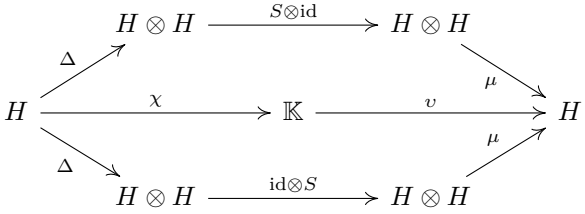
Finally, we mend the notions of algebras and coalgebras together and arrive at something new: the notion of a bialgebra.

Definition 14. *If (B, μ, υ) is an algebra, (B, Δ, χ) is a coalgebra and μ, υ are coalgebra homomorphisms*[31]*, respectively, then we call $(B, \mu, \upsilon, \Delta, \chi)$ a* bialgebra.[32]

2.4 Hopf Algebras

Having gone through a plethora of new algebraic definitions, let us now finally come to the pièce de résistance of this algebraic discussion: the introduction of Hopf algebras. Hopf algebras will be of singular importance for the latter part of this report. For a very brief and pedagogical introduction that is pertinent to our efforts, we refer the reader to [Hec18], and also [PTV13].

Definition 15. *Let $(H, \mu, \upsilon, \Delta, \chi)$ be a bialgebra and let $S : H \to H$ be a map. We call $(H, \mu, \upsilon, \Delta, \chi, S)$ a* Hopf algebra *iff the following diagram commutes*



antipodal

In this case, we call S the antipode. Hopf homomorphisms are just maps that are algebra and coalgebra homomorphisms that are compatible with the antipode(s). The above diagram can be expressed with the

[30] Notice that we will use id, $\mathbb{1}$ and 1 all meaning essentially the same thing - fortunately, there will never be an ambiguity on what it means. However, we shall try to use the $\mathbb{1}$ symbol when talking about evaluated expressions for instance, and id when we want to stress the identity as a map.

[31] This is equivalent to Δ, χ being algebra homomorphisms.

[32] There is a subtlety to be understood in the above statement: Using for example the definition of a coalgebra homomorphism as introduced before, one would try to combine the maps μ and Δ via $\Delta \circ \mu = (\mu \otimes \mu) \circ \Delta'$ - which is ill-defined. This is actually understood as coalgebra homomorphism $\mu : B \otimes B \to B$, where the (B, Δ, χ) is the coalgebra-part of the bialgebra, and $(B \otimes B, \Delta \otimes \Delta, \chi \otimes \chi)$ is the tensored coalgebra that is the codomain of the μ-map mentioned before. The statement that μ is a coalgebra homomorphism then translates to the following condition

$$\Delta \circ \mu = (\mu \otimes \mu) \circ (\Delta \otimes \Delta) \ ,$$

which is a map from $B \otimes B \to B \otimes B$.

equations

$$S \star \mathbb{1} = \mu \circ (S \otimes id) \circ \Delta = \upsilon \circ \chi \ , \tag{2.4.1}$$

$$\mathbb{1} \star S = \mu \circ (id \otimes S) \circ \Delta = \upsilon \circ \chi \ , \tag{2.4.2}$$

where the $\star$-operation is the convolution *defined by the equations above.*

Definition 16. *Let $(H, \mu, \upsilon, \Delta, \chi, S)$ be a Hopf algebra. We call H* quasi-triangular *if there exists an invertible element $\mathcal{R} \in H \otimes H$ such that*

- $\Delta^{op}(h)\mathcal{R} = \mathcal{R}\Delta(h) \ \forall h \in H$,
- $(\Delta \otimes \mathbb{1})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}$,
- $(\mathbb{1} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}$,

where $\mathcal{R}_{ij} \in H \otimes H \otimes H$ has $\mathcal{R}^{(1)}$ as the i-th tensor factor, $\mathcal{R}^{(2)}$ as the j-th tensor factor, and $\mathbb{1}$ as the remaining tensor factor, e.g. $\mathcal{R}_{12} = \mathcal{R} \otimes \mathbb{1}$. In this case, we call $\mathcal{R}$ the universal R-matrix of H. Leaning on this superscript-notation, we may introduce the notation

$$\mathcal{R} = \sum_i \mathcal{R}_i^{(1)} \otimes \mathcal{R}_i^{(2)} \ ,$$

wherever it is convenient. In this context, the superscript (i) refers to the order of tensorial factors (exactly as before), whereas the subscript i refers to the summand of the linear decomposition.

For such an $\mathcal{R}$, the above axioms and properties of a Hopf algebra automatically imply on the one hand

$$\begin{aligned}
[(\pi \circ \Delta) \otimes \mathrm{id}]\,(\mathcal{R}) =& \sum_i [\Delta^{\mathrm{op}} \otimes \mathrm{id}]\,(\mathcal{R}_i^{(1)} \otimes \mathcal{R}_i^{(2)}) \\
=& \sum_i \Delta^{\mathrm{op}}(\mathcal{R}_i^{(1)}) \otimes \mathcal{R}_i^{(2)} \\
=& \sum_i \mathcal{R}\Delta(\mathcal{R}_i^{(1)})\mathcal{R}^{-1} \otimes \mathcal{R}_i^{(2)} \\
=& \mathcal{R}_{12} \left[\sum_i \Delta(\mathcal{R}_i^{(1)}) \otimes \mathcal{R}_i^{(2)} \right] \mathcal{R}_{12}^{-1} \\
=& \mathcal{R}_{12}\,[(\Delta \otimes \mathrm{id})(\mathcal{R})]\,\mathcal{R}_{12}^{-1} \\
=& \mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23}\mathcal{R}_{12}^{-1} \ ,
\end{aligned} \tag{2.4.3}$$

where we have in the third step that

$$\begin{aligned}
\mathcal{R}_{12} \left[\sum_i \Delta(\mathcal{R}_i^{(1)}) \otimes \mathcal{R}_i^{(2)} \right] \mathcal{R}_{12}^{-1} =& \sum_i (\mathcal{R} \otimes \mathrm{id}) \left[\Delta(\mathcal{R}_i^{(1)})\mathcal{R}^{-1} \otimes \mathcal{R}_i^{(2)} \right] \\
=& \sum_i \mathcal{R}\Delta(\mathcal{R}_i^{(1)})\mathcal{R}^{-1} \otimes \mathcal{R}_i^{(2)} \ .
\end{aligned} \tag{2.4.4}$$

However, on the other hand we also have

$$\begin{aligned}
[(\pi \circ \Delta) \otimes \mathrm{id}]\,(\mathcal{R}) =& \pi_{12}(\Delta \otimes \mathrm{id})(\mathcal{R}) \\
=& \pi_{12}(\mathcal{R}_{13}\mathcal{R}_{23}) \\
=& \mathcal{R}_{23}\mathcal{R}_{13} \ .
\end{aligned} \tag{2.4.5}$$

Equating both these expressions, we then get by transitivity that

$$\begin{aligned}\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23}\mathcal{R}_{12}^{-1} =& \mathcal{R}_{23}\mathcal{R}_{13}\\ \Leftrightarrow\\ \mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} =& \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}\ ,\end{aligned} \tag{2.4.6}$$

where we used the invertibility of the $\mathcal{R}$-matrix. We call the last equality the so-called the Yang-Baxter equation (YBE). This equation will be important, even central, in the next chapter when we talk about integrability. Quasi-triangularity also implies the following antipodal relations:

$$\begin{aligned}(\chi\otimes\mathrm{id})(\mathcal{R}) = (\mathrm{id}\otimes\chi)(\mathcal{R}) = \mathbb{1}\ ,\\ \mathcal{R} = (\mathrm{id}\otimes S)(\mathcal{R}^{-1})\ ,\ \mathcal{R}^{-1} = (S\otimes\mathrm{id})(\mathcal{R})\ ,\\ (S\otimes S)(\mathcal{R}) = \mathcal{R}\ .\end{aligned} \tag{2.4.7}$$

The first two equations follow from counitality and from the quasi-triangularity relations that involve $\mathcal{R}$, in the sense that we have:

$$\begin{aligned}\mathcal{R} &= (\mathrm{id}\otimes\mathrm{id})(\mathcal{R})\\ &= ((\mathrm{id}\otimes\chi)\circ\Delta)\otimes\mathrm{id})(\mathcal{R})\\ &= ((\mathrm{id}\otimes\chi)\otimes\mathrm{id})(\mathcal{R}_{13}\mathcal{R}_{23})\\ &= \mathcal{R}\underbrace{((\mathrm{id}\otimes\chi)\otimes\mathrm{id})(\mathcal{R}_{23})}_{\text{has to be }=\mathbb{1}}\ .\end{aligned} \tag{2.4.8}$$

Since the first tensorial factor in $\mathcal{R}_{23}$ is the identity, and the latter two are just $\mathcal{R}$, we have that the following needs to hold,

$$(\chi\otimes\mathrm{id})(\mathcal{R}) = \mathbb{1}\ , \tag{2.4.9}$$

proving our claim. The equation $(\mathbb{1}\otimes\chi)\mathcal{R} = \mathbb{1}$ is proved in a similar way. Furthermore, the relations in the second line can be proved as follows by recalling the antipodal relations proposed before, namely via the following:

$$\begin{aligned}\mathbb{1} &= (\nu\otimes\mathrm{id})(\underbrace{(\chi\otimes\mathrm{id})(\mathcal{R})}_{=\mathbb{1}})\\ &= (\nu\circ\chi\otimes\mathrm{id})(\mathcal{R})\\ &= ((\mu\circ(S\otimes\mathrm{id}\circ\Delta))\otimes\mathrm{id})(\mathcal{R})\\ &= ((\mu\circ(S\otimes\mathrm{id}))\otimes\mathrm{id})(\mathcal{R}_{13}\mathcal{R}_{23})\\ &= \underbrace{((\mu\circ(S\otimes\mathrm{id}))\otimes\mathrm{id})(\mathcal{R}_{13})}_{(S\otimes\mathrm{id})(\mathcal{R})}\mathcal{R}\ ,\end{aligned} \tag{2.4.10}$$

where we again made use of the fact where the identity appears in the triple tensor products.[33] The last equation implies that we have

$$(S\otimes\mathrm{id})(\mathcal{R}) = \mathcal{R}^{-1}\ . \tag{2.4.11}$$

[33] To get from the penultimate to the last line of this equation, we shall make things explicit in index notation for the curious reader. With the expansion of the $\mathcal{R}$-matrix in some appropriate basis $\mathcal{R} = R_{ij}\ X_i\otimes X_j$ (and we shall refrain from displaying the identity with Kronecker-indices for ease of reading), we deduce the following:

$$\begin{aligned}((\mu\circ(S\otimes\mathrm{id}))\otimes\mathrm{id})(\mathcal{R}_{13}\mathcal{R}_{23}) &= ((\mu\circ(S\otimes\mathrm{id}))\otimes\mathrm{id})\big((R_{ij}\ X_i\otimes\mathrm{id}\otimes X_j)(R_{ab}\ \mathrm{id}\otimes X_a\otimes X_b)\big)\\ &= ((\mu\circ(S\otimes\mathrm{id}))\otimes\mathrm{id})\big(R_{ij}R_{ab}\ X_i\otimes X_a\otimes X_jX_b\big)\\ &\overset{(*)}{=} R_{ij}R_{ab}\ S(X_i)X_a\otimes X_jX_b\\ &= (R_{ij}\ S(X_i)\otimes X_j)\,(R_{ab}\ X_a\otimes X_b)\end{aligned}$$

The other equation on line two is proved in an analogous way. Lastly, the third equation is proved using the ones appearing on the second line, namely via:

$$\begin{aligned}\mathcal{R} &= (\mathrm{id}\otimes S)(\mathcal{R}^{-1})\\ &= (\mathrm{id}\otimes S)((S\otimes \mathrm{id})(\mathcal{R}))\\ &= (S\otimes S)(\mathcal{R})\ .\end{aligned} \tag{2.4.12}$$

So far, all things that we discussed here foots on the fact that we have coassociativity. However, there exist analogous notions of bialgebras and Hopf algebras that survive even if we relax this condition. While this might seem like something solely interesting to purely algebraic considerations, as we shall later see, it will play a curious role in this thesis. For this particular topic, we recommend the mathematically inclined reader to have a look at [Hal02], whose definitions we will use and where a comprehensive discussion can be found, but also [Dri98], [CBPV19], [MS92], and a perhaps more relevant approach is presented in [DZ19].

Definition 17. *A* quasi-Hopf algebra H *satisfies all algebraic axioms of a Hopf algebra but for the ones concerning coassociativity. A quasi-Hopf algebra satisfies the relaxed set of the following conditions:*

- $(id\otimes\Delta)\circ\Delta = \Phi\big[\,((\Delta\otimes id)\circ\Delta)\,\big]\Phi^{-1}$,
- $(id\otimes\chi)\circ\Delta = (\chi\otimes id)\circ\Delta = id$,
- $\big[(id\otimes id\otimes\Delta)\Phi\big]\big[(\Delta\otimes id\otimes id)\Phi\big] = \big[(id\otimes\Phi)\big]\big[(id\otimes\Delta\otimes id)\Phi\big]\big[(\Phi\otimes id)\big]$,
- $(id\otimes\chi\otimes id)\Phi = id\otimes id$,

where $\Phi\in H\otimes H\otimes H$ *is an invertible element we call the coassociator. For* $\Phi\equiv 1$*, we recover a coassociative Hopf algebra.*[34]

Similarly to the coassociative case, for quasi-Hopf algebras, there exists also a notion of quasi-triangularity and, subsequently, a quasi-YBE.

Definition 18. *We call a quasi-Hopf algebra* H quasi-triangular *iff there exists an invertible element* $\mathcal{R}\in H\otimes H$ *called again the universal R-matrix of* H*, fulfilling*

- $\Delta^{op}(h)\mathcal{R} = \mathcal{R}\Delta(h)\ \forall h\in H$,
- $(\Delta\otimes id)\mathcal{R} = \Phi^{312}\mathcal{R}_{13}(\Phi^{132})^{-1}\mathcal{R}_{23}\Phi^{123}$,

$$\begin{aligned}&= (S\otimes\mathrm{id})(\mathcal{R})\mathcal{R}\\ &= ((\mu\circ(S\otimes\mathrm{id}))\otimes\mathrm{id})(\mathcal{R}_{13})\mathcal{R}\ ,\end{aligned}$$

In the three equalities following $\overset{(*)}{=}$, we made the multiplication μ implicit factors together, i.e. $\mu\circ(a\otimes b) = ab$, for presentation purposes.

[34] The above equations in bulletpoints 1, 3 and 4 are to be understood in the following way, respectively:

- For $h\in H$, this equation is to be understood as
$$((\mathrm{id}\otimes\Delta)\circ\Delta)(h) = \Phi\big[\,((\Delta\otimes\mathrm{id})\circ\Delta)\,(h)\big]\Phi^{-1}$$
- This equation is to be understood as written not as a map, meaning it is just an element (or multiplicative factor) in the quartic tensorial space of H.
- This equation is to be understood as written not as a map, meaning it is just an element (or multiplicative factor) in the double tensorial space of H.

- $(id \otimes \Delta)\mathcal{R} = (\Phi^{231})^{-1}\mathcal{R}_{13}\Phi^{213}\mathcal{R}_{12}(\Phi^{123})^{-1}$,

where $\Phi^\sigma = \sum_i \Phi_i^{\sigma^{-1}(1)} \otimes \Phi_i^{\sigma^{-1}(2)} \otimes \Phi_i^{\sigma^{-1}(3)}$ *for* $\Phi = \Phi^{123} = \sum_i \Phi_i^{(1)} \otimes \Phi_i^{(2)} \otimes \Phi_i^{(3)}$ *and* $\sigma \in S_3$. *For a quasi-triangular quasi-Hopf algebra, the universal* $\mathcal{R}$*-matrix fulfils the quasi Yang-Baxter Equation*

$$\mathcal{R}_{12}\Phi^{312}\mathcal{R}_{13}(\Phi^{132})^{-1}\mathcal{R}_{23}\Phi^{123} = \Phi^{321}\mathcal{R}_{23}(\Phi^{231})^{-1}\mathcal{R}_{13}\Phi^{213}\mathcal{R}_{12} \ . \tag{2.4.13}$$

We can easily see that for the coassociative case, this reduces to the ordinary Yang-Baxter equation.

Technically, we additionally have to require the existence of a *quasi-antipode* S accompanied by two $\alpha, \beta \in H$, i.e. a triple (S, α, β)) satisfying

$$\begin{aligned} \sum_i S(a_i^{(1)})\alpha a_i^{(2)} &= \chi(a)\alpha \ , \\ \sum_i \alpha a_i^{(1)}\beta S(\alpha a_i^{(2)}) &= \chi(a)\beta \ , \end{aligned} \tag{2.4.14}$$

for all $\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)}$, as well as

$$\begin{aligned} \sum_i \Phi_i^{(1)}\beta S(\Phi_i^{(2)})\alpha\Phi_i^{(3)} &= id \ , \\ \sum_i S(\Phi_i^{-1\,(1)})\alpha\Phi_i^{-1\,(2)}\alpha S(\Phi_i^{-1\,(3)}) &= id \ , \end{aligned} \tag{2.4.15}$$

where we used the notation

$$\Phi = \Phi^{123} = \sum_i \Phi_i^{(1)} \otimes \Phi_i^{(2)} \ , \Phi^{-1} = (\Phi^{-1})^{123} = \sum_i \Phi_i^{-1\,(1)} \otimes \Phi_i^{-1\,(2)} \otimes \Phi_i^{-1\,(3)} \ , \tag{2.4.16}$$

for the coassociator and its inverse. However, if for $\alpha = \beta = id$ the equations in (2.4.14) still hold, then the antipode-coassociator equations in (2.4.15) heavily simplify and can be more compactly written as

$$\begin{aligned} \mu(\mathbb{1} \otimes \mu)\Big[S \otimes \mathbb{1} \otimes S\Big]\Phi^{-1} &= \mathbb{1} \\ \mu(\mu \otimes \mathbb{1})\Big[\mathbb{1} \otimes S \otimes \mathbb{1}\Big]\Phi &= \mathbb{1} \ . \end{aligned} \tag{2.4.17}$$

As it turns out, the only instance where we shall ever make use of quasi-triangular quasi-Hopf algebras, we are allowed to make this simplification, as we shall see in chapter 4.

This concludes our algebraic discussion in this preliminary part of the thesis. We again note that the generalisation is straightforward for the case of graded algebra: Wherever tensor products appear, we have to make use of graded tensor products in their stead, and rather than permutation maps we have to use graded permutation maps, and so on - the exposition stays the same mutatis mutandis.

3 | Integrability

"It is better to solve one problem five different ways, than to solve five problems one way."

– George Polya

Integrability is incredibly rare in the vast landscape of theories and models in physics (see e.g. [MM74]), especially within higher-dimensional theories.[1] They form a special class of physical models that can be solved *exactly*, rather than perturbatively or in the form of some kind of approximation. Though rare, integrability appears in classical mechanics and field theory, condensed matter physics and, more recently, in gauge theory and string theory - most importantly in the AdS/CFT context.

The ambitious goal of this chapter is to provide the reader with an introduction to the vast topic of integrability, brief and fragmentary as it may be. We will do this in two subsections: One dealing with *classical* integrability, one with *quantum* integrability - where we also shall emphasise the aspects most relevant to our later considerations. For the keenly interested reader, there are many references to choose from that cover these topics more extensively, e.g. [Tor16], [Lam15], and an excellent course whose notes can be found in [Bei16]. In our presentation of this we will at some points loosely follow the approaches of these references.

3.1 Classical Integrability

The natural way to start reviewing classical integrability is in the context of Hamiltonian mechanics.

Given a $2n$-dimensional *phase space manifold* M with *(canonical) coordinates* q_i and *momenta* p_i with $i \in \{1, ..., n\}$, and a *Hamiltonian*[2] $H : M \to \mathbb{R}$, a curve $\gamma = (q_i(t), p_i(t)) \subset M$ is a solution to the Hamiltonian system iff it fulfils the equations of motion

$$\frac{\partial H}{\partial p_i} = \dot{q}_i, \qquad \frac{\partial H}{\partial q_i} = -\dot{p}_i, \tag{3.1.1}$$

which in terms of *Poisson-Hamilton brackets* $\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$ can be expressed as

$$\{H, q_i\} = -\frac{d}{dt} q_i, \qquad \{H, p_i\} = -\frac{d}{dt} p_i. \tag{3.1.2}$$

Given initial conditions at some initial time $t = t_0$, one wants to find a general expression for the solutions $(q_i(t), p_i(t))$ at all times[3]. Given all the Hamiltonian systems possible, it is frequently very difficult or infeasible to find a closed expression for a general solution in terms of elementary functions. Therefore, it makes sense to take a step back and see if the Hamiltonian system can be analysed by

[1] See also: Coleman-Mandula Theorem [CM67].

[2] We assume, for this introduction, our H to be time-independent, i.e $\frac{\partial H}{\partial t}$ vanishes.

[3] Depending on the situation, one could also look for this on some interval $t \in (a, b)$ instead.

different means - finding conserved quantities, properties such as periodicity or boundedness of the solution, behaviour near critical points, and so on.

We now arrive at a very important notion in classical integrability in Hamiltonian systems: the existence of integrals of motion. A function $F(q_i, p_i)$ on M is called an *integral of motion* iff

$$\{F, H\} = \frac{d}{dt}F = 0, \tag{3.1.3}$$

where we also assumed F to be time-independent in the same sense that H is assumed to be. It is immediate that H is an integral of motion every Hamiltonian system possesses. Now, if the $2n$-dimensional phase space manifold M features n independent, smooth integrals of motion whose Poisson pairwise brackets vanish, then M is (classically) integrable. A system of this kind can be solved by solving a (finite) number of integrals and algebraic equations, so-called quadratures.[4] These so-called *Liouville-integrable* systems have many interesting mathematical properties, most of which we shall not allude to in this introduction - for further reading, see e.g. [BBT03].

We can approach this in a more algebraic way, for which we need to introduce some new notions first, the first of which is the so-called Lax pair: We call two $k \times k$-matrices L, M with entries that depend on q_i, p_i *Lax-related*, or more simply the *Lax pair*, iff the resulting Lax equations,

$$\frac{dL}{dt} = [M, L], \tag{3.1.4}$$

are equivalent to the Hamiltonian equations of motion. Provided such L, M exist for a given Hamiltonian system, we can automatically generate a family of conserved quantities $F_m = \mathrm{Tr}L^m$, which is an integral of motion by virtue of the cyclicity of the trace, as we can easily derive:

$$\begin{aligned}
\frac{dF_m}{dt} &= \mathrm{Tr}\left(\frac{d}{dt}L^m\right) \\
&= \sum_{k=0}^{m-1} \mathrm{Tr}\left(L^k \left(\frac{dL}{dt}\right) L^{m-k-1}\right) \\
&= \sum_{k=0}^{m-1} \mathrm{Tr}\left(L^k [M, L] L^{m-k-1}\right) \\
&= \sum_{k=0}^{m-1} \left(\mathrm{Tr}\left(L^k M L^{m-1}\right) - \mathrm{Tr}\left(L^{k+1} M L^{m-k-1}\right)\right) \\
&= 0\ .
\end{aligned} \tag{3.1.5}$$

Finding such a Lax pair is often difficult and by no means straightforward, as it is not unique and there is no canonical way to construct it and k is undetermined a priori.

If we want our Lax pair to describe an integrable system, we have not specified how the fact that integrals of motion F_m should pairwise Poisson-commute manifests in the Lax formalism. For this to be the case, we require the existence of the *classical r-matrix* which fulfils

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2], \tag{3.1.6}$$

where we introduced the tensorial notations $L_1 = L \otimes \mathbb{1}$ and $L_2 = \mathbb{1} \otimes L$ as well as $r_{12} = r$ and $r_{21} = \pi(r_{12})$, where π is the permutation. In this context, we recommend the reader to read the first chapter in [ARU20] which outlines this theory masterfully, and whose explanation we shall draw

[4] Keep in mind that these integrals can be tricky: In the case of a massive particle in a central potential $V(r)$ with Lagrangian $2L = m\dot{r}^2 + mr^2\dot{\phi}^2 - 2V(r)$, we can find an expression for the angular coordinate ϕ. However, depending on the potential $V(r)$, the resulting integrals cannot be expressed using elementary functions.

inspiration from. Note that r lives in the tensor product space with two factors (i.e. $r \in M_{k\times k} \otimes M_{k\times k}$). The Jacobi identity associated with the above equation (3.1.6) lives in the double tensor product space with three factors. This means that we also defined now

$$L_1 = L \otimes \mathbb{1} \otimes \mathbb{1} \ , \tag{3.1.7}$$

$$L_2 = \mathbb{1} \otimes L \otimes \mathbb{1} \ , \tag{3.1.8}$$

$$L_3 = \mathbb{1} \otimes \mathbb{1} \otimes L \ . \tag{3.1.9}$$

The Jacobi identity of (3.1.6) constrains the r-matrix in the following way:

$$0 = [L_1, [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] + \{L_2, r_{13}\} - \{L_3, r_{12}\}] \ + \ \text{cyclical permutations} \ . \tag{3.1.10}$$

In order to further illustrate that the above equation follows from the Jacobi identity, it is easiest to make the underlying calculation more explicit. The Jacobi identity of (3.1.6) is given by:

$$\{L_1, \{L_2, L_3\}\} \ + \ \text{cyclical permutations} \ = 0 \ . \tag{3.1.11}$$

Using (3.1.6), we have

$$\{L_1, [r_{23}, L_2] - [r_{32}, L_3]\} \ + \ \text{cyclical permutations} \ = 0 \ , \tag{3.1.12}$$

or, more explicitly,

$$\{L_1, r_{23}L_2 - L_2 r_{23} - r_{32}L_3 + L_3 r_{32}\} \ + \ \text{cyclical permutations} \ = 0 \ . \tag{3.1.13}$$

Besides the Jacobi identity, the $\{\cdot,\cdot\}$-bracket also fulfils the Leibniz property in both arguments. For the second argument, it looks as follows for arbitrary arguments f, g, h of the underlying space:

$$\{f, gh\} = \{f, g\}h + g\{f, h\} \ . \tag{3.1.14}$$

With this, (3.1.13) becomes the following:

$$\begin{aligned} &\{L_1, r_{23}\}L_2 + r_{23}\{L_1, L_2\} - \{L_1, L_2\}r_{23} - L_2\{L_1, r_{23}\} \\ &\qquad - \{L_1, r_{32}\}L_3 - r_{32}\{L_1, L_3\} + \{L_1, L_3\}r_{32} + L_3\{L_1, r_{32}\} \ + \ \text{cyclical permutations} \ = 0 \ . \end{aligned} \tag{3.1.15}$$

Using now (3.1.6) for all brackets of the form $\{L_i, L_j\}$, we arrive at the following expression:

$$\begin{aligned} &[\{L_1, r_{23}\}, L_2] - [\{L_1, r_{32}\}, L_3] \\ &\quad + r_{23}r_{12}L_1 - r_{23}L_1 r_{12} - r_{23}r_{21}L_2 + r_{23}L_2 r_{21} - r_{12}L_1 r_{23} + L_1 r_{12}r_{23} + r_{21}L_2 r_{23} - L_2 r_{21}r_{23} \\ &\quad - r_{32}r_{13}L_1 + r_{32}L_1 r_{13} + r_{32}r_{31}L_3 - r_{32}L_3 r_{31} + r_{13}L_1 r_{32} - L_1 r_{13}r_{32} - r_{31}L_3 r_{32} + L_3 r_{31}r_{32} \\ &\qquad + \ \text{cyclical permutations} \ = 0 \ . \end{aligned} \tag{3.1.16}$$

Now we need to make two important comments: Firstly, looking at (3.1.10), as the commutator with the L_i is the outermost bracket, we would want the L_i's to appear on the "edge" as a factor of the terms for this. To this end, we observe that L_k and r_{ij} commute provided $i \neq k \neq j$ due to their tensorial structure, which means e.g. $L_1 r_{23} = r_{23} L_1$. Wherever we will move the L_i's to the right, we colour the term in blue, and wherever to the left, we will use the colour orange. Some of the terms above have L_i's sandwiched in the middle, which cannot be moved to the edge of the term, e.g. $r_{23}L_2 r_{21}$. We want to see that these terms cancel out, as they are not appearing in (3.1.10). Secondly, while we have so far only

a third of the terms explicit (as the other two thirds are cyclically related to the former), for the next intermediate steps we will make all terms explicit to allow for the possibility that for some cancellations or combinations related to a term in the "first third" might otherwise not explicitly appear. With these considerations, we have:

$$
\begin{aligned}
0 =&[\{L_1, r_{23}\}, L_2] - [\{L_1, r_{32}\}, L_3] + [\{L_2, r_{31}\}, L_3] \underline{-[\{L_2, r_{13}\}, L_1] + [\{L_3, r_{12}\}, L_1]} - [\{L_3, r_{12}\}, L_2] \\
&\underline{+r_{23}r_{12}L_1 - r_{23}L_1r_{12}} - r_{23}r_{21}L_2 \cancel{+r_{23}L_2r_{21}} \boxed{1} \quad \underline{-r_{12}L_1r_{23} + L_1r_{12}r_{23}} \cancel{+r_{21}L_2r_{23}} \boxed{6} - L_2r_{21}r_{23} \\
&\underline{-r_{32}r_{13}L_1 + r_{32}L_1r_{13}} + r_{32}r_{31}L_3 \cancel{-r_{32}L_3r_{31}} \boxed{2} \quad \underline{+r_{13}L_1r_{32} - L_1r_{13}r_{32}} \cancel{-r_{31}L_3r_{32}} \boxed{3} + L_3r_{31}r_{32} \\
&+ r_{31}r_{23}L_2 - r_{31}L_2r_{23} - r_{31}r_{32}L_3 \cancel{+r_{31}L_3r_{32}} \boxed{3} \quad -r_{23}L_2r_{31} + L_2r_{23}r_{31} \cancel{+r_{32}L_3r_{31}} \boxed{2} - L_3r_{32}r_{31} \\
&- r_{13}r_{21}L_2 + r_{13}L_2r_{21} \underline{+r_{13}r_{12}L_1} \cancel{-r_{13}L_1r_{12}} \boxed{4} \quad +r_{21}L_2r_{13} - L_2r_{21}r_{13} \cancel{-r_{12}L_1r_{13}} \boxed{5} \underline{+L_1r_{12}r_{13}} \\
&+ r_{12}r_{31}L_3 - r_{12}L_3r_{31} \underline{-r_{12}r_{13}L_1} \cancel{+r_{12}L_1r_{13}} \boxed{5} \quad -r_{31}L_3r_{12} + L_3r_{31}r_{12} \cancel{+r_{13}L_1r_{12}} \boxed{4} \underline{-L_1r_{13}r_{12}} \\
&- r_{21}r_{32}L_3 + r_{21}L_3r_{32} + r_{21}r_{23}L_2 \cancel{-r_{21}L_2r_{23}} \boxed{6} \quad +r_{32}L_3r_{21} - L_3r_{32}r_{21} \cancel{-r_{23}L_2r_{21}} \boxed{1} + L_2r_{23}r_{21} \ .
\end{aligned}
\tag{3.1.17}
$$

Above, we can see that indeed all "L_i sandwich terms" cancel, and if we collect illustratively all terms that could be related to an outer L_1 commutator (which were underlined in the equations above), we get:

$$
\begin{aligned}
&-[\{L_2, r_{13}\}, L_1] + [\{L_3, r_{12}\}, L_1] + r_{23}r_{12}L_1 - r_{32}r_{13}L_1 + r_{13}r_{12}L_1 - r_{12}r_{23}L_1 \\
&-r_{12}r_{13}L_1 + r_{13}r_{32}L_1 - L_1r_{23}r_{12} + L_1r_{12}r_{23} + L_1r_{32}r_{13} - L_1r_{13}r_{32} + L_1r_{12}r_{13} - L_1r_{13}r_{12} \ ,
\end{aligned}
\tag{3.1.18}
$$

which is exactly the term explicitly written in (3.1.10). For the remaining two thirds of the terms, the situation is analogous and thus (3.1.10) is obtained this way.

If r does not depend on the dynamics, then we can disregard the Poisson bracket terms in the above equation, as it is just equal to

$$
0 = [L_1, [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}]] \ + \ \text{cyclical permutations} \tag{3.1.19}
$$

in this case. While this is of course not exhausting all possible solutions of r, the above equation is fulfilled if

$$
0 = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] \tag{3.1.20}
$$

holds. This equation is called the *classical Yang-Baxter equation*. In many instances - and in the immediate surrounding of the famous *Belavin-Drinfeld theorems* - one encounters r-matrices that fulfil $r_{12} = -r_{21}$, which simplifies subsequent calculations hugely.

More often than not, we can parametrise L, M such that it depends on one common variable, often denoted as u, a *spectral parameter*. All equations and quantities (such as r) related to L, M then depend on this auxiliary parameter,

$$
\frac{dL(u)}{dt} = [M(u), L(u)], \tag{3.1.21}
$$

and for r which lives in the tensored space (and has thus one parameter per each space)

$$
[r_{12}(u_1, u_2), r_{13}(u_1, u_2)] + [r_{12}(u_1, u_2), r_{23}(u_2, u_3)] + [r_{13}(u_1, u_3), r_{23}(u_2, u_3)] = 0. \tag{3.1.22}
$$

Most of the time, the r-matrix can be written in difference form, meaning that $r(u_1, u_2) = r(u_1' - u_2')$ for some other spectral parameter u'. The reason why this can often be achieved is related to results by Belavin and Drinfeld, which we shall not reiterate in full - the interested reader is recommended to consult [BD98] and [BD83] in this context.

As the Lax matrix L can depend on u, so can the family of conserved quantities F_m it generates. All in all, this means given such a parameter u, our whole problem now has a functional character, and we can take a look at the analytic properties of our u-dependent quantities and make use of tools from functional and complex analysis. In fact, the analytic study of the eigenvalues $\lambda_j(u)$ of $L(u)$, amongst others, has proven to be a fruitful and interesting field of research (c.f. the so-called *spectral curve*).

In much the same way when one goes from classical harmonic oscillators to classical field theory - before any kind of quantisation is even thought of - we now go from coordinates q_i, p_i to fields ϕ and conjugate/derivative quantities related to it, which leaves us with infinitely many degrees of freedom and, thus, an infinite-dimensional phase space manifold M.[5] On the one hand, this changes things considerably, on the other hand not so much: We still have the analogous notions of integrability. There are a few new formulations, such as the zero-curvature formulation, that essentially defines a new local gauge field A, whose defining property is that its associated connection $\mathcal{L}_A = \dot{A}_x(u) - A_t'(u) + [A_x(u), A_t(u)]$ vanishes iff the equations of motion are satisfied, i.e. as the entries of A involve fields and quantities of our theory, the resulting equations from imposing flatness of $\mathcal{L}_A$ are the Hamilton-Lagrange equations exactly. Using this *Lax connection*, one can also define a field-theoretic version of the Lax pair (c.f. monodromy matrix). A stellar and succinct review of this can be found in [BBT03]. However, in this project, this will not be of utmost importance, and we shall distance ourselves from the Lax formalism, and only mention this in the context of the historical exposition of integrability within AdS/CFT. This concludes our introduction of classical integrability.

3.2 Quantum Integrability

In this section, we would like to introduce integrability in a quantum sense, however, this is more difficult than it might seem: To formulate a (universally accepted) definition of quantum integrability for a quantum system is rather difficult. What we can do is look for the usual suspects that we associate with a notion of quantum integrability, and identifying and elaborating on these indicators is what this subchapter is going to address.

From quantum theory, be it quantum mechanics or quantum field theory, we recall the existence of the scattering matrix, or in short, the S-matrix. The S-matrix relates free states of the asymptotic past and free states of the asymptotic future, thus containing information about the interaction.[6] Given the S-matrix of a quantum theory, we define the R-matrix by

$$R = \pi(S), \tag{3.2.1}$$

where π is the permutation, either graded or ungraded, depending on whether there are fermionic elements in our theory.

Mathematically speaking, the R-matrix is an endomorphism between tensorial spaces $R : V \otimes V \to V \otimes V$, that can depend on parameters underlying the fields of V, i.e. $R(u_1, u_2)$ (c.f. spectral parameters of the classical r-matrix).

[5] Which also makes M less useful to think of.

[6] We shall not delve into the long but beautiful motivation for S-matrices in either of these realms, for a reference to this, see e.g. [Sch07] or [Sch14].

Within the framework of integrable theories, the factorisability of the S-matrix plays an important rôle (and the R-matrix, subsequently), which is to say that higher S-matrices can be reduced to the product of 2-particle S-matrices. In 3.1, we are looking at a $3 \to 3$-process in this context.

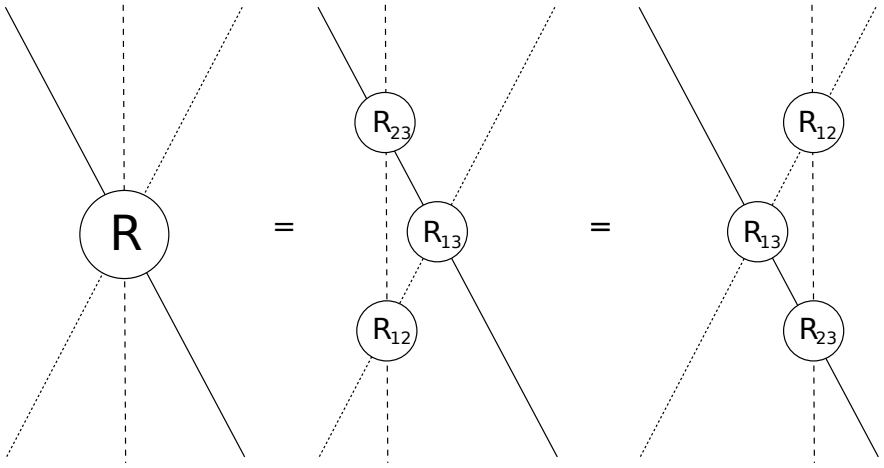


Figure 3.1: Factorisability of a 3-particle process.

We can see this as a graphical representation of the *quantum Yang-Baxter equation*:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{3.2.2}$$

We will revisit this equation later on in this project.

Quantum integrable models are usually characterised by a solution to the quantum YBE (ABA, parametrisation of the Hamiltonian, etc). The nomenclature of this equation is by no means coincidental - let us demonstrate the correspondence of both the classical and quantum analogues by the following consideration: Start with the (quasi-classical) $\hbar$-expansion $R = \mathbb{1} + \hbar r + \hbar^2 \rho + O(\hbar^3)$ and look at the right-hand side and left-hand side of the quantum YBE (see [Kup99]):

$$\begin{aligned} R_{12}R_{13}R_{23} = \mathbb{1} &+ \hbar\,(r_{12} + r_{13} + r_{23}) \\ &+ \hbar^2\,(\rho_{12} + \rho_{13} + \rho_{23}) \\ &+ \hbar^2\,(r_{12}r_{13} + r_{12}r_{23} + r_{13}r_{23}) \\ &+ O(\hbar^3)\ , \end{aligned} \tag{3.2.3}$$

and

$$\begin{aligned} R_{23}R_{13}R_{12} = \mathbb{1} &+ \hbar\,(r_{23} + r_{13} + r_{12}) \\ &+ \hbar^2\,(\rho_{23} + \rho_{13} + \rho_{12}) \\ &+ \hbar^2\,(r_{23}r_{13} + r_{23}r_{12} + r_{13}r_{12}) \\ &+ O(\hbar^3)\ . \end{aligned} \tag{3.2.4}$$

Comparing both sides of the equation, we see that they differ (in order $\hbar^2$) exactly by

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \ , \tag{3.2.5}$$

which is exactly the classical YBE for r.

The property of a theory not to allow particle production in processes also has a deep connection with integrability - one illustrative example of this can be found within $1+1$ dimensional QFT. If one starts with the Lagrangian of a ϕ^4 theory,

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \lambda\frac{\phi^4}{4!} \ , \tag{3.2.6}$$

supplemented by the usual Feynman diagrammatics, one can then start to compute the Feynman amplitudes of scattering processes. We shall do so for one specific setup at tree level. If one starts with 2 particles as incoming legs, the number of outgoing legs has to be even in this theory. As our interaction vertex is of order 4, going beyond the $2 \to 2$ process to the $2 \to 4$ process, one can show that the $2 \to 4$ amplitude is constant and can be cancelled by adding another interaction vertex of the form

$$-\frac{\lambda^2}{m^2}\frac{\phi^6}{6!} \ , \tag{3.2.7}$$

to the Lagrangian. We define

$$\beta^2 = \frac{\lambda}{m^2} \ , \tag{3.2.8}$$

with which the modified Lagrangian then takes the following form:[7]

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \lambda\frac{\phi^4}{4!} - \frac{\lambda^2}{m^2}\frac{\phi^6}{6!} \\ &= \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{\beta^2}\left(\frac{1}{2}\beta^2\phi^2 + \frac{1}{4!}\beta^4 + \frac{1}{6!}\beta^6\phi^6\right) \ .\end{aligned} \tag{3.2.9}$$

Having dealt with the $2 \to 4$ process, one can then go to the $2 \to 6$ process and find again that its amplitude can again by counteracted by the addition of another interaction term, this time of order ϕ^8. This procedure then goes on for all $2 \to 2n$ processes with $n \in \mathbb{N}$, and after an infinite series of added interaction terms, the Lagrangian then takes the form

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{\beta^2}\left(\cosh(\beta\phi) - 1\right) \ , \tag{3.2.10}$$

which after the substitution $\beta \to i\beta$ reduces to the known sine-Gordon model,

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{\beta^2}\left(\cos(\beta\phi) - 1\right) \ . \tag{3.2.11}$$

We know that the sine-Gordon theory is a prototypical example of an integrable model, and we were led to this theory by the sole imposition of there being no other 2 particle processes apart from the $2 \to 2$ process, which forbids particle production. A more detailed exposition of this argument can be found in [Dor96], whose structure we tried to follow. More generally, the feature of no particle production can be seen as a consequence of factorisability (reflected in the R-matrix YBE), which is the central equation in quantum integrability.

Rather than try to provide a general exposition on the notion of quantum integrability, it is more instructive if we illustrate what we have discussed so far by means of an example that will also be imperative for us to be familiar with to understand the latter parts of this thesis.

[7] The fact that we can add terms of this form to our Lagrangian without spoiling renormalisability is connected to the fact that we are formulating our theory in a $1+1$ dimensional space.

3.3 Spin Chains and Bethe Ansätze

Having talked about quantum integrability, it is only appropriate to also talk about some of the models where integrability can be found. Though few and far between, integrable systems can appear in many different settings. One prominent and historical example of quantum mechanical models that are known to frequently feature integrability are *spin chains*. A fantastic introduction to this topic was laid down in [Bei16], and further references are [Fad96] and [LM16]. We will follow these references for the motivation and structure of this chapter, as well as [Sta05] and [BS05].

Originally, the concept of a spin chain arose within the study of magnetic materials, which was assumed to be a system with a certain number (finite or infinite) of sites that possess a particular spin configuration. The totality and interplay of these configurations (described by the Hamiltonian) then gives rise to the energy of the system.

A priori, any Hamiltonian that has terms connecting any two sites is an admissible operator, however, more often than not one focuses on Hamiltonians that only exhibit nearest-neighbour interaction terms. Of course, there exist integrable spin chains with larger interaction range,[8] but the treatment of those spin chains is less standard ([KL22]) and we shall forget about the existence of these spin chains for the scope of this chapter.

3.3.1 The Heisenberg Spin Chain

The $\mathfrak{su}(2)$ Case

A pedagogical and prototypical type of a spin chain is the so-called *Heisenberg spin chain*. In order to define this model, we need to define the space where a potential Hamiltonian can act on. We assume that for one site, the spin configuration state $|\chi\rangle$ can be any (complex) linear combination of $|0\rangle$ and $|1\rangle$ - meaning that $|\chi\rangle \in V \equiv \mathbb{C}^2$. This space of $L \in \mathbb{N}$ sites is then just the L-fold tensor product of V, $V^L = \bigotimes_{k=1}^{L} V_k \equiv \mathbb{C}^{2^L}$. For our case, we will always assume the chain to be periodic, meaning we understand any label $k \equiv k \mod L$. A convenient basis to choose for V^L is given by simple states, meaning states of the form

$$|\chi_1 \cdots \chi_L\rangle\,, \text{ where } \chi_i \in \{0,1\}\ . \tag{3.3.1}$$

V^L is our (Fock) vector space of states, and as with every vector space, we can straightforwardly define linear operators on it. The most important one is the *Hamiltonian* $H : V^L \to V^L$, that can be decomposed into a sum of local Hamiltonians $H_{i,j} : V_i \otimes V_j \to V_i \otimes V_j$,

$$H = \sum_{k=1}^{L} H_{k,k+1}\ , \tag{3.3.2}$$

where again the indices are to be understood mod L. The local Hamiltonians of the Heisenberg spin chain are given by

$$H_{i,j} = \lambda_0(\underbrace{\mathbb{1}_i \otimes \mathbb{1}_j}_{\mathbb{1}_{i,j}}) + \lambda_x(\underbrace{\sigma_i^x \otimes \sigma_j^x}_{\sigma_{i,j}^x}) + \lambda_y(\underbrace{\sigma_i^y \otimes \sigma_j^y}_{\sigma_{i,j}^y}) + \lambda_z(\underbrace{\sigma_i^z \otimes \sigma_j^z}_{\sigma_{i,j}^z})\ , \tag{3.3.3}$$

where the subindices are referring to the space (i.e. the site) on which a certain operator acts, and σ^i are the $\mathfrak{su}(2)$ Pauli matrices. One can show that this Hamiltonian gives rise to a quantum integrable model

[8] Inozemsev's spin chain, for example, has a Hamiltonian kernel for which every site interacts with the other, see e.g. [BFGLR10].

for any value of the constants λ_i, however, we will focus on the particular case where $2\lambda_0 = -2\lambda_x = -2\lambda_y = -2\lambda_z = 1$.[9] With this choice that is referred to as the XXX Heisenberg spin chain, our local Hamiltonians reduce to:

$$H_{i,j} = \mathbb{1}_{i,j} - \mathbb{P}_{i,j} \ , \tag{3.3.4}$$

which is manifestly $\mathfrak{su}(2)$-invariant, with $\mathbb{P}_{i,j}$ being the permutation operator of the indexed spaces and $\mathbb{1}_{i,j}$ acting as the identity on all sites

3.3.2 The Coordinate Bethe Ansatz

Now that we have defined the space and the Hamiltonian, we can finally state and tackle the spectral problem, i.e. the problem of diagonalising the Hamiltonian. Rather than making this a purely computational problem of linear algebra, in the Coordinate Bethe Ansatz (CBA), we proceed by making a reasonable ansatz for our eigenstates, and perhaps adding corrections to it later on. In our approach, we start by defining the vacuum state via:

$$|0\rangle = \underbrace{|0\cdots 0\rangle}_{L \text{ times}} \ , \tag{3.3.5}$$

with vanishing energy as $H_{i,i+1}|0\rangle = (\mathbb{1}_{i,i+1} - \mathbb{P}_{i,i+1})|0\rangle = |0\rangle - |0\rangle = 0$. Since we have that $[\sum_i \sigma_i^z, H] = 0$, we will categorise our eigenstates in terms of how many $|1\rangle$ excitations or spin flips we add to $|0\rangle$, and we shall denote this number with M. [10] The vacuum eigenstate with the energy $E = 0$ solves the case for $\boxed{\text{M=0}}$.

For $\boxed{\text{M=1}}$, we first observe that there exists another operator that commutes with our Hamiltonian H - the shift operator U: The shift operator in the spin chain picture can be defined as the discrete version of the operator generated by the momentum, which means that its eigenstates have plane wave form. For $M = 1$, we can propose the eigenstates of H to have the following form:

$$|p\rangle = \sum_{k=1}^{L} e^{ipk} \underbrace{|0\cdots 0 \overbrace{1}^{\text{site } k} 0\cdots 0\rangle}_{=:|k\rangle} \ , \tag{3.3.6}$$

and when acting on this state with the Hamiltonian, we get

$$H|p\rangle = \sum_{k=1}^{L} e^{ipk}(\underbrace{1 - e^{ip}}_{\text{from } H_{k-1,k}} + \underbrace{1 - e^{-ip}}_{\text{from } H_{k,k+1}})|k\rangle = \underbrace{4\sin^2\left(\frac{p}{2}\right)}_{E_{M=1}(p)}|p\rangle \, , \tag{3.3.7}$$

which indeed proves that $|p\rangle$ is an eigenstate associated to eigenvalue $E(p)$. For closed chains, the fact that L shifts should leave a state invariant further puts a constraint on the permitted values of p, in the sense that this condition quantises the momentum to fulfil $p = \frac{2\pi m}{L}$, $m \in \{0, \cdots, L-1\}$.

For $\boxed{M = 2}$, we start with the following expansion:

$$|\Psi\rangle = \sum_{1 \le k_1 < k_2 \le L} \Psi(k_1, k_2) \underbrace{|0\cdots 0 \overbrace{1}^{\text{site } k_1} 0\cdots 0 \overbrace{1}^{\text{site } k_2} 0\cdots 0\rangle}_{=:|k_1,k_2\rangle} . \tag{3.3.8}$$

[9] Notice here that one of the constants λ_i was free to choose from the beginning (non-trivially), as overall normalisations of the Hamiltonian only scale the spectrum, but do not alter the spectral problem. Additionally, we are also free to shift λ_0 without any such change.

[10] We can do this because the Hamiltonian H commutes with the total spin operator, as mentioned before, thus the eigenstates of H must also be arranged in terms of M.

Now, when acting with our Hamiltonian H on a state like this, we need to distinguish two cases; when $k_2 = k_1 + 1$, and when $k_2 > k_1 + 1$ because of its nearest neighbour structure. Then, $H|\Psi\rangle = E|\Psi\rangle$ takes the form:

$$E\Psi(k_1,k_2) = \begin{cases} 2\Psi(k_1,k_2) - \Psi(k_1-1,k_2) - \Psi(k_1+1,k_2) + 2\Psi(k_1,k_2) - \Psi(k_1,k_2-1) - \Psi(k_1,k_2+1) \\ 2\Psi(k_1,k_2) - \Psi(k_1-1,k_2) - \Psi(k_1,k_2+1) \ , \end{cases} \tag{3.3.9}$$

for $k_2 > k_1 + 1$ and $k_2 = k_1 + 1$, respectively. Bethe proposed an ansatz for this (see [Bet31]),[11]

$$\Psi(k_1,k_2) = e^{ip_1k_1+ip_2k_2} + S(p_2,p_1)e^{ip_2k_1+ip_1k_2} \ . \tag{3.3.10}$$

Without any restriction on the function $S(p_2,p_1)$, this ansatz plugged into the first equation of (3.3.9) gives us the 2-particle energy, which is just $E_{M=2}(p_1,p_2) = E_{M=1}(p_1) + E_{M=1}(p_2)$.[12] The second equation of (3.3.9) then gives us the $\mathfrak{su}(2)$ S-matrix,

$$S(p_2,p_1) = -\frac{e^{ip_1+ip_2} - 2e^{ip_2} + 1}{e^{ip_1+ip_2} - 2e^{ip_1} + 1} \ , \tag{3.3.11}$$

In the case of $M = 1$, the L-fold shifting provided us with a quantisation constraint for our momenta - so far for the $M = 2$ case, our momenta values are unconstrained. We can make a similar argument again, though: We are defining our quantities always mod L, which means that $k_1 \equiv k_1 + L$, and likewise $k_2 \equiv k_2 - L$, which is something our wave function needs to reflect, i.e. $\Psi(k_1,k_2) \equiv \Psi(k_2,k_1+L)$. This then gives rise to the conditions:

$$e^{ip_1L} = S(p_1,p_2) \ , \qquad\qquad e^{ip_2L} = S(p_2,p_1) \ . \tag{3.3.12}$$

We call these conditions the *Bethe equations*, and with those restrictions on our momentum variables, we have solved the spectral problem.

For $\boxed{M \geq 3}$, the story can be generalised quite straightforwardly. We start with an expansion for the eigenstate in line with what we did above:

$$|\Psi\rangle = \sum_{1\leq k_1<\cdots<k_M\leq L} \Psi(k_1,\cdots,k_M)|k_1,\cdots,k_M\rangle \ , \tag{3.3.13}$$

for which we again use Bethe's ansatz of plane waves,

$$\Psi(k_1,\cdots,k_M) = \sum_{\sigma\in S_M} A_\sigma e^{ip_{\sigma(1)}k_1+\cdots+p_{\sigma(M)}k_M} \ . \tag{3.3.14}$$

Using the Schrödinger equation, we get that the energy dispersion relation is again just given by the sum of the M 1-particle energies. It also constrains the coefficients, as for e.g. $M = 3$, the Schrödinger equation also implies[13]

$$S(p_1,p_2) = \frac{A_{213}}{A_{123}} = \frac{A_{321}}{A_{312}} \ ,$$
$$S(p_1,p_3) = \frac{A_{312}}{A_{132}} = \frac{A_{231}}{A_{213}} \ ,$$

[11] This ansatz is also in line with features of integrability: For an $\mathfrak{su}(2)$ integrable scattering system, momenta either exchange or stay the same. The first summand represents the former in a scattering process, while the second summand represents the latter.

[12] The fact that the energy can be written in a decomposed way like this is a sign of integrability.

[13] The subindices of the A follow from the elements of S_3.

$$S(p_2, p_3) = \frac{A_{321}}{A_{231}} = \frac{A_{132}}{A_{123}} \ . \tag{3.3.15}$$

Imposing periodicity then gives rise to the following equations:

$$\begin{aligned} e^{ip_1 L} &= \frac{A_{132}}{A_{321}} = \frac{A_{123}}{A_{132}} \ , \\ e^{ip_2 L} &= \frac{A_{213}}{A_{132}} = \frac{A_{231}}{A_{312}} \ , \\ e^{ip_3 L} &= \frac{A_{321}}{A_{213}} = \frac{A_{312}}{A_{123}} \ . \end{aligned} \tag{3.3.16}$$

Moving to general values of M, the computations are similar and the Bethe equations can be generalised to take the following form[14] given by

$$e^{ip_j L} = \prod_{\substack{m=1 \\ m \neq j}}^{M} S(p_j, p_m) \ , \tag{3.3.17}$$

where $m = 1, \cdots, M$, S is the same 2-particle S-matrix as above. From an eigenvalue perspective, this concludes our discussion for general M. However, we have not made explicit how we would extract the eigenstate coefficients for this case. The end result is the same as for the $M = 3$ case above: It indeed turns out that, for the $\mathfrak{su}(2)$ case, the conditions given by the nearest-neighbour Hamiltonian equation together with the periodicity equation suffice. This means that for every M, enough equations can be generated with this to find all the necessary coefficients this way.

The $\mathfrak{su}(3)$ case

So far, we have discussed the case of two flavour excitations, meaning that each spin site can either be $|0\rangle$ or $|1\rangle$. We now delve into a system of higher rank - meaning that we additionally allow another flavour, $|2\rangle$, to appear on the sites.[15] Provided we choose a Hamiltonian analogous to the one for $\mathfrak{su}(2)$, i.e. being built from a sum over the local Hamiltonians as in (3.3.4), we will still find integrability, factorisability will apply, and, in order to analyse the many-body processes, it suffices to study the 2-body case. As the other cases are already encompassed in our discussion, it is enough for us to analyse the case of one excitation of $|1\rangle$ and $|2\rangle$ each to exhaustively describe the 2-body problem of this higher rank. We denote by M the total number of non-$|0\rangle$ excitations, and by K the number of $|2\rangle$ excitations. For the $\boxed{M = 2, K = 1}$ case, we make the following ansatz:

$$|\Psi\rangle = \begin{bmatrix} |12\rangle \\ |21\rangle \end{bmatrix} = \begin{bmatrix} \sum_{1 \leq k_1 < k_2 \leq L} \Psi_{12}(k_1, k_2) |\cdots 0 \underbrace{1}_{k_1} 0 \cdots 0 \underbrace{2}_{k_2} 0 \cdots \rangle \\ \sum_{1 \leq k_1 < k_2 \leq L} \Psi_{21}(k_1, k_2) |\cdots 0 \underbrace{2}_{k_1} 0 \cdots 0 \underbrace{1}_{k_2} 0 \cdots \rangle \end{bmatrix} . \tag{3.3.18}$$

The crucial difference to the discussion before is that the 2-body S-matrix has to take into account that our excitations are of distinguishable flavour - something that our ansatz of course has to reflect, too.

[14] The fact that the M particle process can be described in terms of the 2-particle variables again foots on integrability and factorisability.

[15] We are well advised to point out one subtlety here: In both the case of higher rank (i.e. going from the fundamental $\mathfrak{su}(2) \to \mathfrak{su}(3)$) and higher spin (i.e. going from $\mathfrak{su}(2)$ spin $\frac{1}{2} \to \mathfrak{su}(2)$ spin 1), we are dealing with representations on spaces with 3 flavour states. However, they are not equivalent: In the $\mathfrak{su}(2)$ spin 1, we still have only one raising operator that can be applied twice on the lowest weight state, whereas in the $\mathfrak{su}(3)$ we have two (both of which can only be applied once on the lowest weight state) due to their ranks being different.

Having now an ansatz for an eigenstate, we get two equations for the cases $k_2 > k_1+1$ and $k_2 = k_1+1$ when acting with the Hamiltonian on it ($H|\Psi\rangle = E|\Psi\rangle$):

$$E\Psi_{12}(k_1,k_2) = \begin{cases} 2\Psi_{12}(k_1,k_2) - \Psi_{12}(k_1-1,k_2) - \Psi_{12}(k_1+1,k_2) \\ \qquad\qquad +2\Psi_{12}(k_1,k_2) - \Psi_{12}(k_1,k_2-1) - \Psi_{12}(k_1,k_2+1) \\ 3\Psi_{12}(k_1,k_2) - \Psi_{12}(k_1-1,k_2) - \Psi_{12}(k_1,k_2+1) - \Psi_{21}(k_1,k_2)\ , \end{cases} \tag{3.3.19}$$

and two more equivalent equations for swapping $\Psi_{21} \leftrightarrow \Psi_{12}$. Once more we make an appropriate ansatz inspired by the one from Bethe for the position space wave functions:

$$\begin{aligned} \Psi_{12}(k_1,k_2) &= A_{12}e^{ip_1k_1+ip_2k_2} + \tilde{A}_{12}e^{ip_2k_1+ip_1k_2}\ , \\ \Psi_{21}(k_1,k_2) &= A_{21}e^{ip_1k_1+ip_2k_2} + \tilde{A}_{21}e^{ip_2k_1+ip_1k_2}\ . \end{aligned} \tag{3.3.20}$$

The A and $\tilde{A}$ terms again symbolise the cases where the momenta just pass through each other and are exchanged, respectively. The first equation ($k_2 > k_1 + 1$) of (3.3.19) again reassures us that the dispersion relation is given by

$$E = \sum_{j=1}^{M} 4\sin^2\left(\frac{p_j}{2}\right)\ , \tag{3.3.21}$$

where we still have $M = 2$ for the moment but the generalisation is immediate. The latter equation in (3.3.19) (with $k_2 = k_1 + 1$) is again addressing the case when particles hit each other and exchange their momenta in a non-diffractive way,[16] but this time, the exchange of momenta also can come with an exchange of the flavour. This again is a restriction on our coefficients $A, \tilde{A}$, and can choose the following way to relate them:

$$\begin{bmatrix} \tilde{A}_{21} \\ \tilde{A}_{12} \end{bmatrix} = \begin{bmatrix} T_{12}^{12}(p_2,p_1) & R_{12}^{21}(p_2,p_1) \\ R_{21}^{12}(p_2,p_1) & T_{21}^{21}(p_2,p_1) \end{bmatrix} \begin{bmatrix} A_{12} \\ A_{21} \end{bmatrix}\ . \tag{3.3.22}$$

T and R are the transmission and reflection amplitudes, respectively, meaning the amplitudes when particle pass through each other or back-scatter. Equation (3.3.19) (i.e. the case $k_2 = k_1 + 1$) can then be solved, yielding

$$R_{12}^{21}(p_1,p_2) = R_{21}^{12}(p_1,p_2) = -\frac{(1-e^{ip_1})(1-e^{ip_2})}{e^{ip_1+ip_2} - 2e^{ip_2} + 1}\ , \tag{3.3.23}$$

$$T_{12}^{12}(p_1,p_2) = T_{21}^{21}(p_1,p_2) = \frac{e^{ip_1}e^{ip_2}}{e^{ip_1+ip_2} - 2e^{ip_2} + 1}\ . \tag{3.3.24}$$

With this result, we can then finally state the 2-body S-matrix in the basis of the 2-particle states $|22\rangle, |12\rangle, |21\rangle, |11\rangle$:

$$S(p_i,p_j) = \left[\begin{array}{c|cc|c} S_{22}^{22}(p_i,p_j) & & & \\ \hline & T_{12}^{12}(p_i,p_j) & R_{12}^{21}(p_i,p_j) & \\ & R_{21}^{12}(p_i,p_j) & T_{21}^{21}(p_i,p_j) & \\ \hline & & & S_{11}^{11}(p_i,p_j) \end{array}\right] =: S_{i,j}\ , \tag{3.3.25}$$

where with the methods from the previous chapter we have

$$S_{11}^{11}(p_i,p_j) = S_{22}^{22}(p_i,p_j) = -\frac{e^{ip_i+ip_j} - 2e^{ip_i} + 1}{e^{ip_i+ip_j} - 2e^{ip_j} + 1}\ , \tag{3.3.26}$$

[16] As in the $\mathfrak{su}(2)$ case, non-diffractivity (and in the same sense integrability) is an assumption when making Bethe ansätze of this form. While the specific momenta are not conserved, the conservation of the total set of the M momenta $\{p_1, \cdots, p_M\}$ as an unordered (and consequently also the total sum of them) is.

as this can just be seen as two lower-rank $\mathfrak{su}(2) \leq \mathfrak{su}(3)$ subsectors that we already solved before. One can check that, when evaluated on a basis of $2^{M=3} = 8$ three particle states[17] that the above S-matrix satisfies the quantum YBE that we know from previous chapters, as well as unitarity, i.e.

$$S_{3,2}S_{3,1}S_{2,1} = S_{2,1}S_{3,1}S_{3,2} \ , \tag{3.3.27}$$

$$S_{1,2}S_{2,1} = \mathbb{1} \ . \tag{3.3.28}$$

These properties of factorisability are ultimately the features of integrability that allow us to reduce the setting of an arbitrary number of particles to the 2-particle case - which is more easily analysed. For the general M particle case, the position space wave function comes with the following Bethe-style ansatz (with $k_i < k_{i+1}$):

$$\Psi_{\cdots 2122\cdots}(k_1, \cdots, k_M) = \sum_{\sigma \in S_M} A^{(\sigma)}_{\cdots 2122\cdots} e^{i \sum_{j=1}^{M} p_{\sigma(j)} k_j} \ . \tag{3.3.29}$$

We again derive some kind of quantisation condition for the momenta that appear in our spectral parametrisations, as they are a priori unconstrained. This is the case because so far we have not made use of any boundary conditions - which is another way of saying that we are working within the framework of an infinite chain that can (and most of the time, will) indeed feature a non-discrete spectrum. If we demand periodic boundary conditions on $|\Psi\rangle$, with the analogous arguments as we made before, we arrive at the Bethe equations for our case, explicitly,

$$e^{ip_j L} |\Psi\rangle = S_{j,j+1} \cdots S_{j,M} S_{j,1} \cdots S_{j,M} \cdots S_{j,j-1} |\Psi\rangle \ , \tag{3.3.30}$$

with the subtle difference that this time around, these equations are equations of *matrices*. This concludes the generalisation for the higher rank case for arbitrary M to the extent that we will need.

For our spectral discussion, we have left out one detail in the $\mathfrak{su}(3)$ case: While the spectrum is determined with the methods we established, we have not explained in a concise manner how we get the eigenstates in an explicit way. This is because this discussion involves further intricacies - especially for the case of twisted spin chains - that will be the cornerstone of our discussion in the chapter revolving around our publication [NGW22].

3.4 Why Integrability?

In the former part of this thesis, we introduced different AdS/CFT correspondences, however, we have not motivated in a concise way why integrability can be particularly useful in this context. For one, integrability has been observed in both the string side (see e.g. [BPR04]) and the CFT side (see e.g. [MZ03]) - given how rare integrability is, the appearance of it on both sides of the correspondence can be seen as an indication that there is connection that relates them. To discuss the appearance of integrability within one instance of an AdS/CFT correspondence in more detail, we shall outline in 3.2 a map of the different parameters of the theories that appear, what regimes they control and how they are connected:[18]

[17] These are the 8 states in the $\mathfrak{su}(2)^{\otimes 3} \subset \mathfrak{su}(3)^{\otimes 3}$. Likewise, for the $\mathfrak{su}(2)$, the S-matrix is given by a phase as $\mathfrak{u}(1)^{\otimes 3} \subset \mathfrak{su}(2)^{\otimes 3}$.

[18] For an even more comprehensive discussion on this, we refer the reader to [BAA+11].

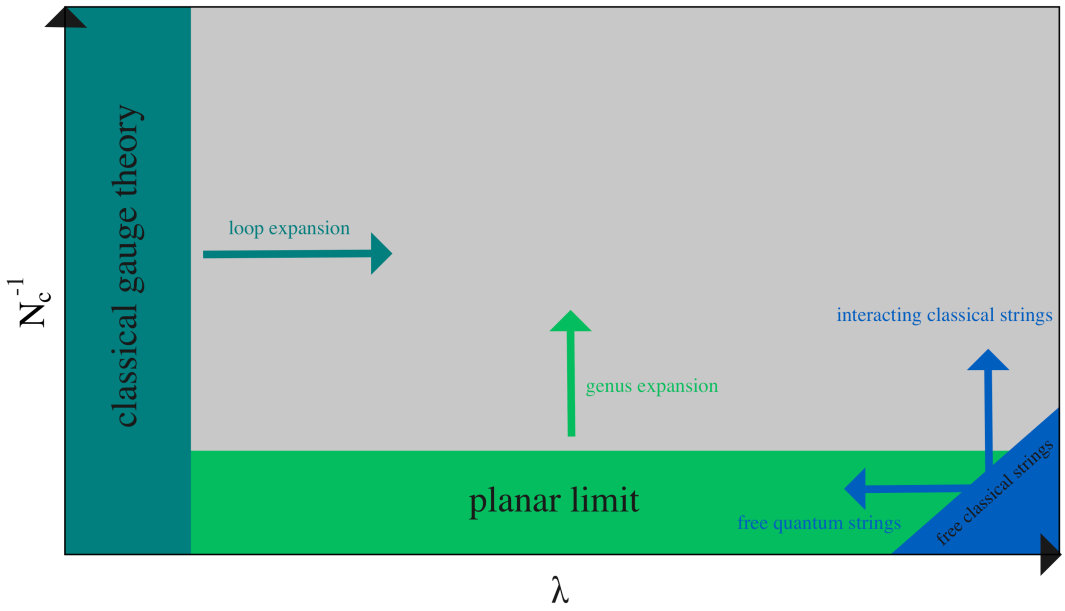


Figure 3.2: Parameter space map $\mathcal{N} = 4$ SYM or strings on $AdS_5 \times S^5$.

Starting on the left-hand side of our map, we find the regime of weak coupling (with small $\lambda = g_{\text{YM}}^2 N_c$), corresponding to *classical gauge theory*. In this region, going to higher loop orders in the Feynman expansion will generate perturbatively accurate results in g_{YM} (but exact in N_c). On the diametrically opposite side at large λ (and small g_{str}), we are in the regime of *free classical strings*, where we can move perturbatively around the $\lambda = \infty, g_{\text{str}} = 0$ point via an expansion in λ (i.e. the σ-model receiving quantum corrections) or an expansion in g_{str} (i.e. the genus/handle expansion of the worldsheet). Lastly, the *planar limit*, where we have $N_c^{-1} \to 0$, we have discussed before that planar diagrams dominate, but expanding away from this limit would mean also taking diagrams of non-planarity degree > 0 into account. The grey area in the map corresponds to the region where we do not have straightforward limits that allow us to probe for information on both models, so there is a large perturbative region where the above approaches are well-separated, unfortunately - in very general regimes, we do not really know how to handle these theories. Integrability is in a sense a global property, and we hope integrability can connect the regimes to which we have perturbative access to the ones we do not. And indeed, the tools of integrability have proven themselves useful in the context of AdS_5, and recently, developments in AdS_3 have sparked new hope for the lower dimensional sibling.[19] Many integrability techniques (such as Bethe ansätze of different kinds, spectral curve approaches, et cetera) have been successfully applied to different parts of the above map, and a comprehensive discussion on all techniques and appearances would go beyond the scope of this introduction. However, it is sensible for us to illustrate that integrable features on both sides of the correspondence exist and motivate some of our approaches to analysing the correspondence that are to follow.

3.4.1 Spin Chains and Conformal Field theories

So far, we have motivated and established the connection between the AdS-part and the CFT-part of the duality and how it arises within the different contexts of $\text{AdS}_5/\text{CFT}_4$ and $\text{AdS}_3/\text{CFT}_2$ settings. Aside from these two obvious entry points, there is another approach to shed light on (one side of) this duality:

[19] For a more comprehensive introduction on this lower dimensional duality, we refer to [BSZ10].

The study of certain spin chains. To justify why we dedicated a considerable part of this preliminary chapter as well as the thesis later on to spin chains and tools to analyse them, we shall now try to explain this seemingly implausible connection that spin chains and conformal field theories share. In this effort, we shall - for illustrative reasons - again stick to the case of AdS_5/CFT_4, where the conformal field theory part is given by $\mathcal{N} = 4$ SYM. References that we refer the reader to and whose pedagogical structure we will draw inspiration from include [Min12], [Min06].

In conformal field theories, the crucial new ingredient on the level of the algebra is D, the generator associated to dilations. Conformal scalings act on coordinates via

$$x \to \lambda x \ , \tag{3.4.1}$$

where λ is a constant. These scale transformations preserve "angles", but obviously do not preserve the metric/distances. In quantum field theories generally and conformal field theories more particularly, we deal with the notion of *local operators* that are evaluated at some specific spacetime point x. It is common to see that local operators are just defined via their charges/quantum numbers that they have with respect to the symmetries of the system. For a local operator $\mathcal{O}(x)$, the generator D acts via conjugation (on the level of the group), meaning

$$\mathcal{O}(x) \to \lambda^{-iD}\mathcal{O}(x)\lambda^{iD} = \lambda^{-\Delta}\mathcal{O}(\lambda x) \ , \tag{3.4.2}$$

where we call the spectral quantity Δ the *conformal dimension*.[20] On the level of the representation of the algebra, we have then

$$[D, \mathcal{O}(x)] = i\left[-\Delta\mathcal{O}(x) + x\frac{\partial}{\partial x}\mathcal{O}(x)\right] \ . \tag{3.4.3}$$

Using the Jacobi identity as well as the conformal commutation relations, one can then prove that, if a given local operator $\mathcal{O}(x)$ has conformal dimension Δ, then $[K_\mu, \mathcal{O}(x)]$, the commutator of a generator of a special conformal transformation with this local operator, has conformal dimension $\Delta - 1$:

$$\begin{aligned}
[D, [K_\mu, \mathcal{O}(0)]] &= [[D, K_\mu], \mathcal{O}(0)] + [K_\mu, [D, \mathcal{O}(0)]] \\
&= i[K_\mu, \mathcal{O}(0)] - i\Delta[K_\mu, \mathcal{O}(0)] \\
&= -i\left(\Delta - 1\right)[K_\mu, \mathcal{O}(0)] \ ,
\end{aligned} \tag{3.4.4}$$

where we made use of the fact that Δ does not depend on spacetime and the fact that at the origin, we have

$$[D, \mathcal{O}(0)] = -i\Delta\mathcal{O}(0) \ . \tag{3.4.5}$$

With this property, we can define *primary operators* $\tilde{\mathcal{O}}(0)$ as the local operators that are annihilated by the special conformal transformations and have a definite conformal dimension Δ, i.e.

$$\begin{aligned}
[K_\mu, \tilde{\mathcal{O}}(0)] &= 0 \ , \\
[D, \tilde{\mathcal{O}}(0)] &= -i\Delta\tilde{\mathcal{O}}(0) \ .
\end{aligned} \tag{3.4.6}$$

While primary operators might seem scarce as they need to fulfil an intricate set of conditions, we can define other operators from primary operators, that inherit some of their properties: A *descendant* operator associated to $\tilde{\mathcal{O}}(x)$ is generated by taking $\tilde{\mathcal{O}}(x)$ and commuting it with the other generators

[20] The terms scaling and dilation, as well as scaling dimension and conformal dimension, are used interchangeably within QFT. At times, the latter notion is just referred to as dimension.

of the conformal algebra.[21] Additionally, we call a primary operator *chiral* iff it is annihilated by some part of the supersymmetry, having as a consequence that the scaling dimension does not receive any (quantum) corrections. Chiral primaries are the prime ingredients to build local operators from - keeping in mind that while the scaling dimensions of chiral operators does not receive corrections, other operators built from several chiral operators might. The primary operator, which forms the highest weight of the representation, together with its descendants forms an irreducible representation of $PSU(2,2|4)$.[22] Apart from being a defining property for local operators and their descendants, the conformal dimension appears in other places too in a crucial way: Within a conformal field theory, the observables associated to local operators that we are interested in are give by the n-point correlation function,

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle \ . \tag{3.4.7}$$

Lest we forget that we are dealing with a QFT that possesses conformal symmetry - this heavily restricts how correlation functions can look like: For the case of two scalar local operators, translational, rotational and conformal invariance imply that the 2-point function has to be of the following form

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \rangle \propto \frac{1}{x_{12}^{\Delta}} \ , \tag{3.4.8}$$

where we define $x_{12} = x_1 - x_2$, as translational invariance implies dependence on the difference of two spacetime points only. The correlation functions of primary operators determine those of their descendants, whose functional form is (in part) characterised by the conformal dimension they carry. It is therefore hugely important to know about the conformal dimension spectrum of our operators if we want to compute the theory.

We now make more explicit what we expect of these local operators $\mathcal{O}(x)$ in our $\mathcal{N} = 4$ SYM setting. Imposing gauge invariance on the physical observables coming from the theory, one quickly finds that the local operators that fit the bill are given by products of traces of fields that transform under the gauge group in a covariant way. For *single trace operators (STO)*, we can prove this quite straightforwardly: Let us start with a scalar field ϕ.[23] In $\mathcal{N} = 4$, the fields in question transform all under the adjoint representation, implying the gauge transformation behaviour

$$\phi \to \phi + [a, \phi] \ , \tag{3.4.9}$$

where a generates the gauge transformation applied (and might be local) and ϕ is a covariant scalar. Its covariant derivative we define via

$$D_\mu \phi = \partial_\mu \phi + [\phi, A_\mu] \ , \tag{3.4.10}$$

where the gauge connection (non-covariantly) transforms as $A_\mu \to A_\mu + \partial_\mu a + [a, A_\mu]$. Expressed with the exponential transformation $E = e^a$, we have the following transformation behaviours:

$$\phi \to E \phi E^{-1} \ , \tag{3.4.11}$$

$$A_\mu \to E A_\mu E^{-1} + (\partial_\mu E) E^{-1} \ . \tag{3.4.12}$$

[21] One can show that the supercharges have scaling dimension $\frac{1}{2}$ and one can build operators with $\frac{1}{2}$ higher dimension.

[22] In general, these representations are infinite dimensional, but by imposing some conditions on the primary operator $\tilde{\mathcal{O}}(x)$, the amount of independent descendants can be restricted - for example by letting $\tilde{\mathcal{O}}(x)$ commute with (some of) the supercharges. This is why, for superconformal theory, the crucial ingredients are the superconformal primaries, see [Ebe21] for a comprehensive discussion on this.

[23] We shall drop the spacetime dependence of the field ϕ, the generator a, and the gauge connection A_μ.

Now, we can see why the covariant derivative is a suitable building block in gauge theories, because the trace of it is gauge invariant:

$$D_\mu \phi \to E D_\mu \phi E^{-1} \ . \tag{3.4.13}$$

Thus, the single trace operator of fields Ξ_i which are either covariant fields or covariant derivatives thereof,

$$\mathcal{O}(x) = \mathrm{Tr}\left[\Xi_1 \cdots \Xi_L\right] \ , \tag{3.4.14}$$

is a gauge invariant quantity.[24] From this point, we could analyse the possible operators that could appear within our theory because, as it turns out, there are only so many, although infinitely many, allowed distinct combinations that the rules of the theory allow us to build. However for the sake of illustrating the connection between spin chains and CFT this is not necessary, thus let us assume them to have the form above for our arguments.[25] As we mentioned before, in our specific model, we classify our STO in terms of the sextuplet of charges given by the rank 6 Cartan subalgebra of the bosonic symmetry group $SU(2,2) \times SU(4) \subset PSU(2,2|4)$,

$$(\Delta, \overbrace{S_1, S_2}^{\text{Lorentz } SO(1,3) \text{ spins that sit in } SU(2,2)} ; \underbrace{J_1, J_2, J_3}_{R\text{-charges that sit in } SU(4)}) \ . \tag{3.4.15}$$

The spins and R-charges can be computed in the usual way, the (bare) dimension Δ_0, meaning $\Delta(g_{YM} \to 0)$, can be guessed by looking at the way they appear in the Lagrangian: For scalar fields ϕ in $3+1$ dimensions, we know that the kinetic term together with the measure of the action integral has to be dimensionless. Thus,

$$0 = [d^4x (\partial\phi)^2] = [d^4x] + 2[\partial\phi] = -4 + 2 + 2[\phi] \Rightarrow [\phi] = 1 = \Delta_0(\phi) \ . \tag{3.4.16}$$

Expanding our (scalar) single trace operators in a basis representative of the $SO(6)$ labels that they carry,

$$\mathcal{O}st = \alpha_{i_1,\cdots,i_L} \mathrm{Tr}\left[\bar{\phi}^{i_1} \cdots \bar{\phi}^{i_L}\right] \ , \tag{3.4.17}$$

with $i_j \in \{1, \cdots, 6\}$, and $\alpha_{i_1,\cdots,i_L}$ being the $SO(6)$ indices, allows us to write $\alpha \in \bigotimes_{k=1}^{L} \mathbb{C}^2_k$. We can therefore interpret this $SO(6)$-tensor system as spin set up with a natural Hilbert space structure.

We are now going to be brief in our exposition of what initially was the great feat of the authors in [MZ03]: To show that there exists an isomorphic identification of the $\mathcal{N} = 4$ SYM spectral dimension problem (for scalars) and the Hamiltonian spectral problem of an $SO(6)$ spin chain with nearest neighbour interaction - in the planar limit. If one starts with a (more general and normalised) form of a scalar operator,

$$\mathcal{O}_\alpha(x) = \frac{(4\pi^2)^{\frac{L}{2}}}{\sqrt{C_\alpha N^L}} \mathrm{Tr}\left[\phi_{i_1} \cdots \phi_{i_L}\right] \ , \tag{3.4.18}$$

where we compactly write $\alpha = i_1 \cdots i_L$, similarly to before. Turning the coupling off, we find a tree-level 2-point correlation function of the form

$$\left\langle \mathcal{O}_\alpha(x_1) \bar{\mathcal{O}}^\beta(x_2) \right\rangle \big|_{\text{tree-level}} = \frac{\delta^{j_1}_{i_1} \cdots \delta^{j_L}_{i_L} + \sigma(\beta)\text{-terms}}{C_\alpha} \frac{1}{(x_{12}^2)^{\Delta_0}} \ , \tag{3.4.19}$$

[24] This is even more straightforward if the induced adjoint representation on the level of the group is considered, since it acts conjugatively.

[25] One can also prove that, in the $N \to \infty$ limit, the dimension of a product of STO equals the sum of the dimension of the factors, so the spectral information of local operators (that are in their most general form products of STO) is fully described by the spectral information of STO, with some caveats.

where $\beta = j_1 \cdots j_L$, $\Delta_0 = L$ for L scalar fields, the constants C_α are a factor from symmetry, and the $\sigma(\beta)$-terms refer to cyclic permutations of the $\delta^{j_1}_{i_1} \cdots \delta^{j_L}_{i_L}$ factor in its $SO(6)$ indices stored in β. If we want to compute the 1-loop contribution to our 2-point correlation function, the large N limit heavily simplifies our problem, because (for the justifiable assumption that $L << N$) the index contractions that correspond to non-planar diagrams get suppressed by a factor of $\frac{1}{N^2}$. We end up summing only the scalar diagrammatic contractions corresponding to a gluon exchange, the self-energy corrections and Φ^4 interaction terms contributing involving adjacent fields, and end up with the following expression (after some compact rewriting and introducing the t'Hooft coupling λ as a convenient expansion parameter in the $N \to \infty$ limit):

$$\langle \mathcal{O}_\alpha(x_1)\bar{\mathcal{O}}^\beta(x_2)\rangle\big|_{\text{1-loop}} = \frac{\lambda}{16\pi^2}\frac{\log\left(\Lambda^2 x_{12}^2\right)}{x_{12}^{2L}}\sum_{k=1}^{L}(2P_{k,k+1} - K_{k,k+1} - 1 + C)\left[\frac{\delta^{j_1}_{i_1}\cdots\delta^{j_L}_{i_L}}{\sqrt{C_\alpha C^\beta}} + \sigma(\beta)\text{-terms}\right] , \tag{3.4.20}$$

where C is a constant, and $P_{k,k+1}$ and $K_{k,k+1}$ are the permutation and trace operators that exchange and contract the $SO(6)$ flavour indices of the k-th and $k+1$-th slot site within the trace, respectively,

$$P_{k,k+1}\delta^{j_1}_{i_1}\cdots\delta^{j_{k-1}}_{i_{k-1}}\delta^{j_k}_{i_k}\delta^{j_{k+1}}_{i_{k+1}}\cdots\delta^{j_L}_{i_L} = \delta^{j_1}_{i_1}\cdots\delta^{j_{k-1}}_{i_{k-1}}\delta^{j_k}_{i_{k+1}}\delta^{j_{k+1}}_{i_k}\cdots\delta^{j_L}_{i_L} \tag{3.4.21}$$

$$K_{k,k+1}\delta^{j_1}_{i_1}\cdots\delta^{j_{k-1}}_{i_{k-1}}\delta^{j_k}_{i_k}\delta^{j_{k+1}}_{i_{k+1}}\cdots\delta^{j_L}_{i_L} = \delta^{j_1}_{i_1}\cdots\delta^{j_{k-1}}_{i_{k-1}}\delta_{i_k i_{k+1}}\delta^{j_k j_{k+1}}\cdots\delta^{j_L}_{i_L} , \tag{3.4.22}$$

meaning that $K_{k,k+1}$ also alters the tensorial degree of the quantity (as contraction always does).[26] Adding the tree-level and one-loop contributions of the 2-point correlation function that gives us summa summarum:

$$\langle \mathcal{O}_\alpha(x_1)\bar{\mathcal{O}}^\beta(x_2)\rangle\big|_{\text{tree-level + 1-loop}} = \frac{1}{x_{12}^{2L}}\left[1 - \frac{\lambda}{16\pi^2}\log\left(\Lambda^2 x_{12}^2\right)\sum_{k=1}^{L}(2P_{k,k+1} - K_{k,k+1} - 1 + C)\right]\delta^{j_1}_{i_1}\cdots\delta^{j_L}_{i_L} + \sigma(\beta)\text{-terms} . \tag{3.4.23}$$

had we started with (3.4.8) and expanded then for relatively small $\gamma << \Delta_0$ (from small coupling g_{YM}, with Δ_0 being the bare dimension and γ being the anomalous dimension with $\Delta = \Delta_0 + \gamma$), we would have ended up with

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle \propto \frac{1}{x_{12}^{\Delta}} \simeq \frac{1}{x_{12}^{\Delta_0}}\left[1 - \gamma\log\left(\Lambda^2 x_{12}^2\right)\right] , \tag{3.4.24}$$

we can see that, in spirit, we are able to replace γ with the operator

$$\Gamma = \frac{\lambda}{16\pi^2}\sum_{k=1}^{L}(-2P_{k,k+1} + K_{k,k+1} + 1 - C) , \tag{3.4.25}$$

and through this transform the problem of finding the anomalous conformal dimensions of our theory from a diagrammatic computation into the spectral problem of the matrix Γ. Even more so, we can therefore map the entire problem onto one of spin chains: Any scalar STO containing L fields[27] can be identified with a state of a Hilbert space $V = V_1 \otimes \cdots \otimes V_L$, where every V_i is the representation space of an $SO(6)$ representation (c.f. (3.4.17)), and V can be identified with the state space of a 1-dimensional spin chain of length L. On this space, Γ acts as an operator

$$\Gamma : V_1 \otimes \cdots \otimes V_L \to V_1 \otimes \cdots \otimes V_L , \tag{3.4.26}$$

[26] One should note that, however, $K_{k,k+1}$ is not a true contraction in the differential geometric sense.

[27] More directly, the identification can be made for the previously defined α-tensor.

and one can show that Γ is Hermitian and that it commutes with the shift operator U that we introduce and discuss later in the chapter about spin chain methods. We can therefore see Γ as the Hamiltonian of the so-defined spin chain, and due to the operator structure of $P_{k,k+1}$ and $K_{k,k+1}$, it only carries nearest neighbour interactions amongst the spin chain sites. This Hamiltonian is integrable (see [Res83], [Res85]), and we may make use of the toolset of spin chain integrability techniques, such as Bethe ansätze.

We have so far only illustrated the correspondence only for an algebraic subsector, but there have been efforts to extend that: At one loop, the above identification has been extended to the full superconformal $PSU(2,2|4)$ symmetry (c.f. [Bei04] [BKS03]), although a similarly extensive description for arbitrary loops is still to be developed. However, all of these efforts have the analogous goal: At any given higher loop order, we would just be confronted again with another spectral problem given by an effective Hamiltonian.

For the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence, CFT-based spin chain developments are, unfortunately, few and far between.[28] The reason here again is connected to the insufficient knowledge in detail about this CFT side: In contrast to its higher dimensional sibling, within the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence we find that it can be supported by both an *RR* and *NSNS* flux. Usual superstring techniques work for the sole appearance of RR fluxes, however, the symmetric orbifold CFT that we mentioned before is built on the NSNS formulation.

3.4.2 String Theory

When motivating the *AdS/CFT* correspondence and its origins, we came across the 't Hooft limit involving the 't Hooft coupling λ. Taking particular limits within analysing a theory allows us to focus on the behaviour of the model in certain regimes. The Lagrangian structure of both the $\mathrm{AdS}_5 \times S^5$ and the $\mathrm{AdS}_3 \times S^3 \times M_4$ (super)string is very involved in the general case, but reduces to a more manageable form expanding around some useful limiting parameters. For the former background, a prominent approach is taking the *BMN* limit, after Berenstein, Maldacena and Nastase (see [BMN02]). In their seminal paper, the authors choose the limit where the angular momentum J and N have the following behaviour:

$$\begin{aligned} J &\to \infty \ , \\ N &\to \infty \ , \\ \frac{N}{J^2} &\to \text{fixed} \ . \end{aligned} \tag{3.4.27}$$

The strings the authors are considering then give rise to a vacuum ground state that, a priori, does not receive quantum corrections (c.f. *BPS* states), and on which the spectrum of excitations can be built using creation operators. The specific string setup they choose, which is related to point-like strings in a pp-wave background, then lets us find the following energy relation:

$$E - J = \sum_n \sqrt{1 + \frac{4\pi g N n^2}{J^2}} N_n \ , \tag{3.4.28}$$

where E is the energy of the corresponding string state, g is the string coupling, n is the mode number associated to the Fourier expansion in modes and N_n is the number of oscillators. In this limit, the AdS radius is given by

$$R_{\mathrm{AdS}} = \sqrt[4]{4\pi g N \alpha'^2} \ . \tag{3.4.29}$$

[28] Nevertheless, there have been developments of spin chains from the string theory perspective (see [BOSS+13a], [BOSS+13b], [BOSSS14b], [BOSSS14a], [LOSSS15], [FS22a], [FS22c], [FS22b] and [BOSS+17]). However, there are also issues with these spin chains due to the presence of massless modes (see [AA16]).

In this regime, the Hamiltonian that lends itself to the description is the light-cone one, which takes the form

$$p^- = \frac{E-J}{2} \ , \tag{3.4.30}$$

and for the ground state BMN vacuum we have that $p^- = 0$, meaning that in that case we have $E = J$. In this particular regime of our theory, the spectral information is directly accessible by applying bosonic and fermionic creation operators to this BMN vacuum, and more importantly, for the unique BMN vacuum we can identify the ground state (with light-cone momentum p^+) $|0, p_+\rangle$ with the scalar chiral primary operator of $\mathcal{N} = 4$ SYM consisting of J scalar complex fields, i.e.

$$\underbrace{|0, p_+\rangle}_{\text{the l.c. string side}} \quad \leftrightarrow \quad \underbrace{\frac{1}{\sqrt{JN^J}} \mathrm{Tr}\left(Z^J\right)}_{=\mathcal{O}_J, \text{ the CFT side}} \ . \tag{3.4.31}$$

The conformal dimension Δ of $\mathcal{O}_J$ is then equivalent to the energy E of $|0, p_+\rangle$. Due to the chirality of the primary operator, the conformal dimension is protected from quantum corrections, and likewise due to the BPS-nature of the string state, the energy will not receive quantum corrections either - further reassuring us in this identification. Going beyond the ground state, while the string state can just be manipulated by creation operators, the chiral operator on the field theory side gets decorated with fields different from Z. In summary, while in the previous section we motivated a close connection between local operators and spin chain states, here the connection is between local operators and string states. In some sense, the BMN picture was a bellwether for the later application of spin chains and their integrability techniques to AdS/CFT correspondences.

So far, we have discussed this relation for the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence, and much of the discussion is analogous for the massive part of the spectrum of $\mathrm{AdS}_3/\mathrm{CFT}_2$. However, the fact that we have massless excitations within the setting of $\mathrm{AdS}_3/\mathrm{CFT}_2$ makes things arguably more intricate, both on the CFT level as well as in the relationship amongst spin chains and $\mathrm{AdS}_3/\mathrm{CFT}_2$ consequentially, as we mentioned before. However, both the $\mathrm{AdS}_5/\mathrm{CFT}_4$ and the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence share a different link to integrability: For each of their background worldsheet σ-models, classical integrability has been proven. For the $\mathrm{AdS}_5 \times S^5$ case (endowed with a particular RR-flux), the Lax connection has been found (see [MSW02], [BPR04], [AAT05], [AFPZ07], [AFZ07]), where in the latter references the authors proved that the resulting S-matrix can be made equivalent to the one of the corresponding CFT of the correspondence. For the $\mathrm{AdS}_3/\mathrm{CFT}_2$ case, the construction of the Lax connection is more intricate due to the presence of the massless modes that are absent in the $\mathrm{AdS}_5/\mathrm{CFT}_4$ case. However, classical integrability of the GS supercoset action featuring a crucial $\mathbb{Z}_4$ symmetry has also been proven for the AdS_3 backgrounds of our interest (see [SW12], [BSZ10], [CZ12], for a comprehensive contextual introduction also [Sfo15]).

4 | The q-Poincaré Algebra

"Un tas de pierres cesse d'être un tas de pierres, dès qu'un seul homme le contemple avec, en lui, l'image d'une cathédrale." ["A rock pile ceases to be a rock pile the moment a single man contemplates it, bearing within him the image of a cathedral."]

– Antoine de Saint-Exupéry

Paul Dirac proposed what is often referred to as the relativistic dispersion relation, or more canonically energy-momentum relation,

$$E_{\text{rel}}^2 = p^2 + m^2 \ , \tag{4.0.1}$$

which is well known due to its abundant occurrence in modern quantum field theory. However, this is not the dispersion relation that appears within the AdS/CFT context. Within the CBA and spin chain considerations we have made in the previous chapter in the context of $\text{AdS}_5/\text{CFT}_4$, had we gone beyond the first loop order of and considered the all-loop case of the $\mathfrak{su}(2) \equiv \mathfrak{so}(3) \leq \mathfrak{so}(6)$ subsector of the spin chain, we would have encountered a dispersion relation of the form

$$E^2 \propto m^2 + 4h^2 \sin^2 \frac{p}{2} \ , \tag{4.0.2}$$

where h is constant and $m = \pm 1$. The $\text{AdS}_3/\text{CFT}_2$ case is arguably more difficult and richer. Most importantly, here we are allowed to assume massless modes in the spectrum, leaving us with a massless dispersion relation of the form[1]

$$E^2 \propto \sin^2 \frac{p}{2} \ . \tag{4.0.3}$$

In our efforts, we will solely focus on the massless sector of the $\text{AdS}_3/\text{CFT}_2$ scattering.

AIt was [GH07], amongst others, who pointed out that one can interpret the connection of E_{rel} with E - essentially a difference between having p and $\sin \frac{p}{2}$ - as a so-called q-deformation: One can indeed see that $[p]_{e^{\frac{i}{2}}} = \frac{e^{\frac{i}{2}p} - e^{-\frac{i}{2}p}}{e^{\frac{i}{2}} - e^{-\frac{i}{2}}} \propto \sin \frac{p}{2}$, where we introduced notation that will be clarified later on. The relativistic dispersion relation is a crucial element in a theory with Poincaré symmetry, embodied in the dispersion relation by the Poincaré algebra. It is then a natural question to ask for a different theory with q-related dispersion relation E: Does a q-analogue of the Poincaré algebra exist as a symmetry of this system? E and p, which are Lorentz quantities, appear in a natural way in our theory, so we only have to see whether we can define a boost operator $\mathfrak{J}$ in our theory to complement them.

We will define these q-analogues, or q-deformations, more concisely later on, but we shall point out that they have been studied within the context of quasi-triangular Hopf algebras long ago (see e.g. [Jim85], [Dri86]). This has also been previously analysed in the Poincaré context (c.f. [LRNT91]).

[1] For more details on the emergence of the magnon dispersion relation within AdS/CFT and the spectral differences of the AdS_5 and AdS_3 cases (massive and massless) within our field of research, one can consult [BDS04], [AF09] or [BAA+11], to name a few references.

In our analysis of our q-analogue of the Poincaré symmetry, we will mostly be concerned with one generator: the boost $\mathfrak{J}$. In [GH07], the authors considered a $1+1$-dimensional Poincaré algebra in the $\mathcal{N}=4$ SYM context of AdS_5/CFT_4 which they identify with the $E_q(1,1)$ algebra, which is a deformation of the pseudoeuclidean algebra. They then introduce the boost generator as the translation generator of an elliptic rapidity variable z, meaning $\mathfrak{J}=\frac{\partial}{\partial z}$ and transforming

$$\mathfrak{J}: z \mapsto z + c \ . \tag{4.0.4}$$

Here, the dispersion relation of the present excitations was identified with the quadratic Casimir of the deformed algebra via $C=\mathfrak{H}^2+h^2(e^{ip/2}-e^{-ip/2})^2$, with $\mathfrak{H}$ playing the role of the energy generator, and h is connected to the tuning of the deformation. Building upon this, supersymmetric considerations of this have been made in [You07], although falling short in lifting the entire algebra to a Hopf structure. The work presented in [BT18] further refined this idea by analysing the coproduct of the boost more closely. One might also add that our particular interest in boost operators is by no means novel: Boost operators have been studied in different contexts, such as the study of σ-models and spin chains (see e.g. [SW83], [BBL08], [KY11]).

Within the context of AdS_3/CFT_2, we again have the main difference of massless modes being allowed, which makes the theory richer (see e.g. [OSST13]), yet sometimes more involved to deal with. In our case, focussing on massless excitations will be of crucial importance and allow for additional freedoms in the choice of the Hopf algebraic embedding of the energy. We adopt the motivations outlined for the AdS_5/CFT_4 case, following [BT18], as well as the problem studied in [ST16], and revisited in [BST18]. They studied q-deformations in the massless sector of the AdS_3/CFT_2 integrable scattering problem, where they also underline the intricacies that the presence of a boost operator in this context brings.

4.1 The Algebra and its Satellites

4.1.1 The Algebra

Recall that our setting is $AdS_3\times S^3$ superstring theory, the algebra in question for the scattering problem can be reduced to just studying $\mathfrak{su}(1|1)^2=\mathfrak{su}(1|1)_L\oplus\mathfrak{su}(1|1)_R$. Throughout this report, when we use the index A we refer to left or right handednesses, i.e. $A=L,R$. Introducing this notation, the non-vanishing relations and generators of our algebra are the following:

$$\begin{aligned} \{\mathfrak{Q}_A,\mathfrak{S}_A\} &= \mathfrak{H}_A \ , \quad [\mathfrak{J}_A,p_A] = i\mathfrak{H}_A \ , \quad [\mathfrak{J}_A,\mathfrak{H}_A] = i\mathfrak{H}_A\Phi_A \ , \\ [\mathfrak{J}_A,\mathfrak{Q}_A] &= \phi_A^Q\,\mathfrak{Q}_A \ , \qquad [\mathfrak{J}_A,\mathfrak{S}_A] = \phi_A^S\,\mathfrak{S}_A \ . \end{aligned} \tag{4.1.1}$$

The grading of the respective elements can be inferred from the nature of the (anti-)commutation relations they obey. A priori, ϕ_A^Q, ϕ_A^S and Φ_A are functions of the *momentum* generators p_A, with no immediate restrictions, and $\mathfrak{J}_A$ are the *boost* generators. However, as we want our superalgebra to satisfy super-Jacobi identities[2], they fulfil

$$i\mathfrak{H}_A\Phi_A = [\mathfrak{J}_A,\mathfrak{H}_A] = [\mathfrak{J}_A,\{\mathfrak{Q}_A,\mathfrak{S}_A\}] = \{[\mathfrak{J}_A,\mathfrak{Q}_A],\mathfrak{S}_A\}+\{[\mathfrak{J}_A,\mathfrak{S}_A],\mathfrak{Q}_A\} = (\phi_A^Q+\phi_A^S)\mathfrak{H}_A \ . \tag{4.1.2}$$

So, we already got acquainted with one central element: the energy Cartan element $\mathfrak{H}_A$ that contains the dispersion relation.[3]

[2] We shall drop the "super" from now on where it is obvious.

[3] Actually, also p_A is central, though not as interesting as $\mathfrak{H}_A$. Whenever we refer to the centrality of operators in this context, we mean that a generator is central with respect to every operator *but* the boost generator.

In addition, we are interested in centrally-extending our $\mathfrak{su}(1|1)^2$ to $\mathfrak{su}(1|1)^2_{c.e.}$ by assuming the anti-commutators involving fermionic generators of different handedness not to vanish:

$$\{\mathfrak{Q}_L, \mathfrak{Q}_R\} \;=\; \mathfrak{P}\;, \qquad \{\mathfrak{S}_L, \mathfrak{S}_R\} \;=\; \mathfrak{K}\;. \tag{4.1.3}$$

Central extensions have also played a role within the context of AdS_5 (see [BdLH16]), and we can interpret their appearance as our approach reflecting the presence of quantum effects.[4]

For the sake of clarity in nomenclature, let us summarise the different algebra "types" that appear:[5]

$$\mathfrak{su}(1|1) \xrightarrow[\text{extension}]{\text{central}} \mathfrak{su}(1|1)_{c.e.} \xrightarrow[\text{boost/momenta}]{\text{adding}} \text{Boost superalgebras } \mathfrak{B}\;.$$

4.1.2 Outer Automorphism Symmetry

$\mathfrak{su}(1|1)^2_{c.e.}$ has another fascinating feature, as was discussed in e.g. [Reg16]: $\mathfrak{su}(1|1)^2_{c.e.}$ possesses a large outer automorphism group, $GL(2)^2$. This means that it features an outer symmetry that, when acting on the generators, leaves the algebraic relations invariant. Automorphisms have been proven to be vital within the field we are studying, see e.g. [BdLH16] and [BHH17]. Within our context, this outer $GL(2)^2$-action is defined as:

$$\begin{pmatrix}\mathfrak{Q}_L\\ \mathfrak{S}_R\end{pmatrix} \longmapsto \lambda \begin{pmatrix}\mathfrak{Q}_L\\ \mathfrak{S}_R\end{pmatrix}\;, \qquad\qquad \begin{pmatrix}\mathfrak{S}_L\\ \mathfrak{Q}_R\end{pmatrix} \longmapsto \rho \begin{pmatrix}\mathfrak{S}_L\\ \mathfrak{Q}_R\end{pmatrix}\;, \tag{4.1.4}$$

where (λ, ρ) is in $GL(2)^2$. In addition, we can define an algebra from the group action in the following way

$$\begin{aligned}
&[t^\lambda_0, \mathfrak{Q}_L] = [t^\lambda_3, \mathfrak{Q}_L] = \mathfrak{Q}_L\;, \quad [t^\lambda_0, \mathfrak{S}_R] = -[t^\lambda_3, \mathfrak{S}_R] = \mathfrak{S}_R\;, \quad [t^\lambda_+, \mathfrak{S}_R] = \mathfrak{Q}_L\;, \quad [t^\lambda_-, \mathfrak{Q}_L] = \mathfrak{S}_R\;,\\
&[t^\rho_0, \mathfrak{Q}_R] = [t^\rho_3, \mathfrak{Q}_R] = \mathfrak{Q}_R\;, \quad [t^\rho_0, \mathfrak{S}_L] = -[t^\rho_3, \mathfrak{S}_L] = \mathfrak{S}_L\;, \quad [t^\rho_+, \mathfrak{Q}_R] = \mathfrak{S}_L\;, \quad [t^\rho_-, \mathfrak{S}_L] = \mathfrak{Q}_R\;.
\end{aligned} \tag{4.1.5}$$

The commutation relations of the outer symmetry generators with p vanish, and the commutation relations with the central elements can be inferred by means of super-Jacobi identities, always being of the schematic form:

$$[t, \{\mathfrak{F}, \mathfrak{F}'\}] = -\{[t, \mathfrak{F}], \mathfrak{F}'\} + \{\mathfrak{F}, [t, \mathfrak{F}']\}\;, \tag{4.1.6}$$

where t is any of the outer symmetry generators, and $\mathfrak{F}, \mathfrak{F}'$ are fermionic generators that in a super-bracket $\{\cdot,\cdot\}$ give rise to a central element. One example of this relation is given by[6]

$$[t^\rho_-, \mathfrak{H}_L] = [t^\rho_-, \{\mathfrak{Q}_L, \mathfrak{S}_L\}] = \mathfrak{P}\;. \tag{4.1.7}$$

The action of the outer symmetry generators on the boost generators follows from the action on its ingredient generators - we will see that the boost generators will always be chosen to consist of central

[4] We make one small remark regarding the underlying algebra we established: It is clearly not *simple*. One easy way to see this is that $(\mathfrak{p})\mathfrak{su}(1|1)_{c.e.}$ possesses a centre, which already constitutes a non-trivial ideal. Disregarding the center, even $(\mathfrak{p})\mathfrak{su}(1|1)$ is not simple. Simple algebras and their quasi-triangularity properties have been studied and classified within this context in more detail, and further analysed (see [KT94], [BD98], [BD83]), but we cannot build upon this in a direct way.

[5] We will sometimes refer to $\mathfrak{su}(1|1)_{c.e.}$ as a "boost algebra" already, as we will - in the setting of the 2-dimensional short representation - not introduce separate symbolical nomenclature for the superalgebra enhanced by the boost.

[6] Here, the second superbracket vanishes, as

$$[t^\rho_-, \mathfrak{H}_L] = [t^\rho_-, \{\mathfrak{Q}_L, \mathfrak{S}_L\}] = -\{\underbrace{[t^\rho_-, \mathfrak{Q}_L]}_{=0}, \mathfrak{S}_L\} + \{\mathfrak{Q}_L, \underbrace{[t^\rho_-, \mathfrak{S}_L]}_{=\mathfrak{Q}_R}\} = \mathfrak{P}\;.$$

.

elements and differential p-operators, whose commutations relations with the outer symmetry generators we have already discussed.

Given our notation, one can infer that the generator basis for one $GL(2)$ given by t_0, t_3, t_+, t_- can be understood as the following set of matrices (see 4.1.4)

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} . \tag{4.1.8}$$

It is worth singling out one special linear combination of symmetry generators, namely what is commonly referred to as the hypercharge operator $\mathfrak{B} = 2i(t_0^\lambda - t_0^\rho)$:

$$[\mathfrak{B}, \mathfrak{Q}_L] = 2i\mathfrak{Q}_L , \qquad [\mathfrak{B}, \mathfrak{S}_L] = -2i\mathfrak{S}_L , \qquad [\mathfrak{B}, \mathfrak{Q}_R] = -2i\mathfrak{Q}_R , \qquad [\mathfrak{B}, \mathfrak{S}_R] = 2i\mathfrak{S}_R . \tag{4.1.9}$$

Interestingly enough, the action of $GL(2)^2$ leaves the $\mathfrak{B}$-charge invariant.

4.2 The 2-Dimensional Short Representation

4.2.1 The Representation

In a first step, we consider the above algebra to admit a certain representation, more precisely a 2-dimensional short one.[7] In a $1+1$-dimensional space with basis $\{|\Phi\rangle, |\Psi\rangle\}$ with charges $\{i, -i\}$ under $\mathfrak{B}$, we define the following representation for the fermionic elements

$$\mathfrak{S}_L \equiv \mathfrak{Q}_R \equiv \sqrt{\frac{E(p)}{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \qquad \mathfrak{Q}_L \equiv \mathfrak{S}_R \equiv \sqrt{\frac{E(p)}{2}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \tag{4.2.1}$$

which is a representation that is well-defined if the two copies are considered independently. With the above representation, we consequently have that $\mathfrak{H} = \mathfrak{H}_L = \mathfrak{H}_R = h \sin \frac{p}{2} = \frac{1}{2}E(p)$, where $E(p)$ is the energy and p the momentum without any reference to handedness.

What we left out so far, however, is how a boost operator of handedness A acts on a boost operator of handedness $B \neq A$. Within this particular representation, this question can be answered immediately and we can consistently set

$$\begin{aligned} [\mathfrak{J}_A, \mathfrak{Q}_B] &= \frac{i}{2\sqrt{\mu}} \frac{e^{i\frac{p}{2}} + e^{-i\frac{p}{2}}}{2} \mathfrak{Q}_B , \qquad [\mathfrak{J}_A, \mathfrak{S}_B] = \frac{i}{2\sqrt{\mu}} \frac{e^{i\frac{p}{2}} + e^{-i\frac{p}{2}}}{2} \mathfrak{S}_B , \\ [\mathfrak{J}_L, \mathfrak{P}] &= [\mathfrak{J}_R, \mathfrak{P}] = [\mathfrak{J}_L, \mathfrak{K}] = [\mathfrak{J}_R, \mathfrak{K}] = -[\mathfrak{J}_A, \mathfrak{H}_B] = \frac{e^{-ip} - e^{ip}}{2\mu} , \end{aligned} \tag{4.2.2}$$

where $A \neq B$, keeping in mind that $\mathfrak{H}_{A,B}$ contains the (massless) dispersion relation E, and we have $\mu = \frac{4}{h^2}$. We will later see that, for the algebraic relations of $\mathfrak{J}$ to be satisfied, it will always feature a ∂_p term, and the commutator of the boost operator with a generator involving (or depending on) p will just be given by the derivative action. Furthermore, it is sensible for us to comment on the appearance of the momentum generator p: A priori, we view p like any other generator from our algebra, but then also see that the momentum appears in exponential functions within our algebraic relations, dispersions relations and, later on, coproduct braiding factors. If featured in this non-trivial way, we always have to think about the momentum as being evaluated, meaning that the corresponding momentum charge (i.e. eigenvalue) takes its place - for unambiguous well-definedness.

[7] In [Reg16], short representations are referred to as atypical modules.

4.2.2 Different Hopf Algebra Structures

As a next step, we want to endow our superalgebra with a Hopf structure and find suitable R-matrices. This mostly boils down to finding consistent coproducts for our algebra. We will do this once for the undeformed algebra we defined in (4.1.1) and (4.2.2), and once for a q-deformed algebra we introduce later. For the former, we will only review briefly the modus operandi and the results we found in [NGTW20], which is where a full discussion can be found. For the latter, whose approach is a bit more intricate but grosso modo quite the same, we shall be more detailed in our explanation.

The Undeformed Case

We start with the very general coproduct ansatz for our fermionic elements

$$\Delta\mathfrak{S}_L = \mathfrak{S}_L \otimes e^{iap/4} + e^{icp/4} \otimes \mathfrak{S}_L \ , \qquad \Delta\mathfrak{Q}_L = \mathfrak{Q}_L \otimes e^{ibp/4} + e^{idp/4} \otimes \mathfrak{Q}_L \ , \tag{4.2.3}$$

where a, b, c, d are constants. Keep in mind that defining coproducts for the fermionic generators immediately defines the coproduct structure for all central elements. We know p to be primitive, i.e. $\Delta p = p \otimes \mathbb{1} + \mathbb{1} \otimes p$, so apart from the fermionic elements, there will only be $\Delta\mathfrak{J}_A$ left to be determined. The constants in (4.2.3) are restricted by $[\Delta\mathfrak{Q}_A, \Delta\mathfrak{S}_A] = \Delta\mathfrak{H}_A$, which we shall make explicit by example

$$\begin{aligned}
\{\Delta\mathfrak{Q}_A, \Delta\mathfrak{S}_A\} =& \big(\mathfrak{Q}_A \otimes e^{ibp/4} + e^{idp/4} \otimes \mathfrak{Q}_A\big)\big(\mathfrak{S}_A \otimes e^{iap/4} + e^{icp/4} \otimes \mathfrak{S}_A\big) \\
&+ \big(\mathfrak{S}_A \otimes e^{iap/4} + e^{icp/4} \otimes \mathfrak{S}_A\big)\big(\mathfrak{Q}_A \otimes e^{ibp/4} + e^{idp/4} \otimes \mathfrak{Q}_A\big) \\
=& \mathfrak{Q}_A\mathfrak{S}_A \otimes e^{(a+b)p/4} + \mathfrak{Q}_A e^{icp/4} \otimes \mathfrak{S}_A e^{ibp/4} - \mathfrak{S}_A e^{idp/4} \otimes e^{iap/4}\mathfrak{Q}_A + e^{i(c+d)p/4} \otimes \mathfrak{Q}_A\mathfrak{S}_A \\
&+ \mathfrak{S}_A\mathfrak{Q}_A \otimes e^{(a+b)p/4} + \mathfrak{S}_A e^{idp/4} \otimes e^{iap/4}\mathfrak{Q}_A - \mathfrak{Q}_A e^{icp/4} \otimes \mathfrak{S}_A e^{ibp/4} + e^{i(c+d)p/4} \otimes \mathfrak{S}_A\mathfrak{Q}_A \\
=& \underbrace{\{\mathfrak{Q}_A, \mathfrak{S}_A\}}_{=\mathfrak{H}_A} \otimes e^{i(a+b)p/4} + e^{i(c+d)p/4} \otimes \underbrace{\{\mathfrak{Q}_A, \mathfrak{S}_A\}}_{=\mathfrak{H}_A} \\
=& \mathfrak{H}_A \otimes e^{i(a+b)p/4} + e^{i(c+d)p/4} \otimes \mathfrak{H}_A \\
=& \Delta\mathfrak{H}_A \ .
\end{aligned} \tag{4.2.4}$$

The condition $\Delta\mathfrak{H}_A = \Delta^{\mathrm{op}}\mathfrak{H}_A$ is necessary to hold for the existence of an R-matrix, since $\mathfrak{H}_A$ is central, since we have

$$\Delta^{\mathrm{op}}(\mathfrak{H}_A)R = R\Delta(\mathfrak{H}_A) = \Delta(\mathfrak{H}_A)R \ , \tag{4.2.5}$$

from the centrality of $\mathfrak{H}_A$, and since R is invertible, we have indeed $\Delta\mathfrak{H}_A = \Delta^{\mathrm{op}}\mathfrak{H}_A$. This means we want

$$\Delta\mathfrak{H} = \mathfrak{H} \otimes e^{i(a+b)p/4} + e^{i(c+d)p/4} \otimes \mathfrak{H} \ , \qquad \Delta^{\mathrm{op}}\mathfrak{H} = e^{i(a+b)p/4} \otimes \mathfrak{H} + \mathfrak{H} \otimes e^{i(c+d)p/4} \tag{4.2.6}$$

to coincide, which happens if[8]

$$a + b = c + d \ . \tag{4.2.7}$$

At first, it might seem that we have four independent (physical) parameters a, b, c, d that span out all possible coproducts. However, this not quite accurate: Redefining our supercharges via the exponential shift

$$\mathfrak{S}_L \to \tilde{\mathfrak{S}}_L = e^{ia_S}\mathfrak{S}_L \ ,$$

[8] Note that, depending on the form of the $\mathfrak{H}$, this is a sufficient condition but not necessarily a requirement for $\Delta\mathfrak{H}_A = \Delta^{\mathrm{op}}\mathfrak{H}_A$ to hold - see the *bosonically braided* family introduced below.

$$\mathfrak{Q}_L \to \tilde{\mathfrak{Q}}_L = e^{ia_Q}\mathfrak{Q}_L \ , \tag{4.2.8}$$

and requiring that $a_S = -a_Q$, as we do not want the eigenvalues of the energy Cartan generator to change, we can see that (4.2.6) is unchanged regardless of whether we choose the redefined supercharges or the original ones. A straightforward way to see this is the following relation, of which we will also make use later on,[9] namely

$$\Delta(e^{i\alpha p}\mathfrak{T}) = e^{i\alpha p} \otimes e^{i\alpha p}\Delta(\mathfrak{T}) \ , \tag{4.2.9}$$

where $\mathfrak{T}$ is any element of the Hopf algebra. Thus, one realises that the only physical directions in parameter space are along $a - c$ and $b - d$ - a fact that we will come back to later on.

At this point it is imperative to emphasise once again that the results dependent on the coproduct of the energy are very much dependent on the form of the energy $\mathfrak{H}$: This analysis heavily relies on the fact that $\mathfrak{H} \propto e^{\frac{i}{2}p} - e^{\frac{-i}{2}p}$, as for generic forms of $\mathfrak{H}$ the coproduct choice we made above would not yield $\Delta\mathfrak{H} = \Delta^{\mathrm{op}}\mathfrak{H}$, which needs to hold for all central elements. We will elaborate on this further when we specify scenarios different than the massless one in the outlook. Summa summarum, this constrains our ansatz to appear in two families:

- The *bosonically unbraided* family, $a+b = c+d = 0$, which contains the trivial braiding $a, b, c, d = 0$ and the braiding $a = -b = -c = d = -1$, which are the ones most commonly used in literature.
- The *bosonically braided* family, $a + b = 2$ and $c + d = -2$, as well as its parity transform obtained by the parity transformation $p \to -p$.

For this undeformed present case, we shall be brief in our explanations, and mention the crucial results succinctly. Later, when dealing with the q-deformed case, we will be more comprehensive in the explanation of our modus operandi, which will be much the same for either case.

For the R-matrix, we start with an ansatz (c.f. 6-vertex models)

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r_{11} & r_{12} & 0 \\ 0 & r_{21} & r_{22} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \ . \tag{4.2.10}$$

Having the coproducts for $\mathfrak{X} = \mathfrak{Q}_A, \mathfrak{S}_A$ at hand, one can make use of the quasi-cocommutativity condition that $\Delta^{\mathrm{op}}\mathfrak{X}_A R = R\Delta\mathfrak{X}_A$. One would best go about this by evaluating this condition on all 2-particle basis states, i.e.

$$\langle\chi_1\chi_2|\Delta^{\mathrm{op}}\mathfrak{X}_A R - R\Delta\mathfrak{X}_A|\chi_3\chi_4\rangle = 0 \ , \chi_i \in \{\Phi, \Psi\}. \tag{4.2.11}$$

These equations give rise to the following R-matrices:

[9] This relation amongst primitive elements and group like coproducts holds more generally. This is the only instance where we will make use of it, however. It can easily be proven the following way:

$$\begin{aligned} e^{i\alpha p} \otimes e^{i\alpha p} =& \left(e^{i\alpha p} \otimes \mathbb{1}\right)\left(\mathbb{1} \otimes e^{i\alpha p}\right) \\ =& \left(\left(\sum_n \frac{(i\alpha p)^n}{n!}\right) \otimes \mathbb{1}\right)\left(\mathbb{1} \otimes \left(\sum_m \frac{(i\alpha p)^m}{m!}\right)\right) \\ =& \left(e^{i\alpha p \otimes \mathbb{1}}\right)\left(e^{\mathbb{1} \otimes i\alpha p}\right) \\ =& e^{i\alpha p \otimes \mathbb{1} + \mathbb{1} \otimes i\alpha p} \\ =& e^{i\alpha\Delta(p)} \ . \end{aligned}$$

bosonically braided family	bosonically unbraided family
$r_{11} = \frac{e^{-\frac{i}{4}p_1(x+y)} \sin\frac{p_2}{2} - e^{-\frac{i}{4}p_2(x+y)} \sin\frac{p_1}{2}}{\sin\frac{p_1+p_2}{2}}$	$r_{11} = \frac{-e^{\frac{i}{4}(b+c)p_2} \sin\frac{p_1}{2} + e^{\frac{i}{4}(b+c)p_1} \sin\frac{p_2}{2}}{\sin\frac{p_1}{2} + \sin\frac{p_2}{2}}$
$r_{12} = \frac{2e^{-\frac{i}{8}(p_1-p_2)(x-y)}\sqrt{\sin\frac{p_1}{2}\sin\frac{p_2}{2}}}{\sin\frac{p_1+p_2}{2}} \cos\left(\frac{(x+y)(p_1-p_2)-2(p_1+p_2)}{8}\right)$	$r_{12} = \frac{2e^{\frac{i}{8}[(b+d)p_1+(a+c)p_2]} \cos\left[\frac{a-c}{8}(p_1-p_2)\right]\sqrt{\sin\frac{p_1}{2}\sin\frac{p_2}{2}}}{\sin\frac{p_1}{2} + \sin\frac{p_2}{2}}$
$r_{21} = \frac{2e^{\frac{i}{8}(p_1-p_2)(x-y)}\sqrt{\sin\frac{p_1}{2}\sin\frac{p_2}{2}}}{\sin\frac{p_1+p_2}{2}} \cos\left(\frac{(x+y)(p_1-p_2)+2(p_1+p_2)}{8}\right)$	$r_{21} = \frac{2e^{\frac{i}{8}[(a+c)p_1+(b+d)p_2]} \cos\left[\frac{a-c}{8}(p_1-p_2)\right]\sqrt{\sin\frac{p_1}{2}\sin\frac{p_2}{2}}}{\sin\frac{p_1}{2} + \sin\frac{p_2}{2}}$
$r_{22} = -\frac{e^{\frac{i}{4}p_1(x+y)} \sin\frac{p_2}{2} - e^{\frac{i}{4}p_2(x+y)} \sin\frac{p_1}{2}}{\sin\frac{p_1+p_2}{2}}$	$r_{22} = \frac{e^{\frac{i}{4}(a+d)p_2} \sin\frac{p_1}{2} - e^{\frac{i}{4}(a+d)p_1} \sin\frac{p_2}{2}}{\sin\frac{p_1}{2} + \sin\frac{p_2}{2}}$

Here, we re-expressed our braiding parameters as $a = 1 + x, c = -1 - y$ with respect to our original parameters for the bosonically unbraided family.[10]

It is sensible to first make a few comments about what we expect the *boost* operator $\mathfrak{J}$ to look like. Here, we can let ourselves be inspired by the approaches we outlined in the motivation for this problem, such as [GH07], and thus a particularly useful representation of the $\mathfrak{J}$ is in differential form:

$$\mathfrak{J}_A = i\mathfrak{H}_A\partial_p \ . \tag{4.2.12}$$

Apart from producing consistent commutation relations, it makes sense for the boost operator to have this form. As conventional Lorentz boosts can just be understood as transformations that lead to translations in the rapidity variable, the differential nature of the representation is to be expected. If we want the boost to feature the momentum differential (rather than some uniformised rapidity as in [GH07] as mentioned before), the appearance of the energy as a prefactor is naturally warranted. Later on, this so-called differential representation will be examined more closely. On the level of the coproduct, we can expect the algebraic form to be

$$\Delta\mathfrak{J}_A = i(\Delta\mathfrak{H}_A)(\Delta\partial_p) + \text{ tail.} \tag{4.2.13}$$

The tail will involve fermionic elements, the most important condition that $\Delta\mathfrak{J}_A$ remains bosonic, meaning that the tail has to contain an even number of fermionic generators. We start with a very general ansatz that fulfils this:

$$\begin{aligned}\Delta\mathfrak{J}_A = {} & A(p_1,p_2)\ (\partial_{p_1} + \partial_{p_2}) + B(p_1,p_2)\ (\partial_{p_1} - \partial_{p_2}) \\ & + C(p_1,p_2)\,\mathfrak{S}\otimes\mathfrak{Q} + D(p_1,p_2)\,\mathfrak{Q}\otimes\mathfrak{S} + F(p_1,p_2)\mathfrak{B}\otimes\mathbb{1} + G(p_1,p_2)\mathbb{1}\otimes\mathfrak{B}\ .\end{aligned} \tag{4.2.14}$$

[10] It might be useful at this point to give an example of how the evaluated coproduct of a fermionic element looks like in the 4×4 matrix representation that we use in the quasi-cocommutativity equation. For $\mathfrak{Q}_R$ (4.2.1), we arrive at

$$\Delta\mathfrak{Q}_R = \begin{pmatrix} 0 & e^{-\frac{i}{4}p_1(1+y)}E(p_2) & e^{\frac{i}{4}p_2(1+x)}E(p_1) & 0 \\ 0 & 0 & 0 & e^{-\frac{i}{4}p_1(1+x)}E(p_1) \\ 0 & 0 & 0 & -e^{-\frac{i}{4}p_1(1+y)}E(p_2) \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\Delta^{\mathrm{op}}\mathfrak{Q}_R = \begin{pmatrix} 0 & e^{\frac{i}{4}p_1(1+x)}E(p_2) & e^{-\frac{i}{4}p_2(1+y)}E(p_1) & 0 \\ 0 & 0 & 0 & e^{-\frac{i}{4}p_1(1+y)}E(p_1) \\ 0 & 0 & 0 & -e^{\frac{i}{4}p_1(1+x)}E(p_2) \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where one needs to pay attention to the fermionic signs arising in the computations.

Having this at hand, once again, we impose the coproduct to be an algebra (homo)morphism.[11] In this case, this is to say that we want $[\Delta\mathfrak{J}_A, \Delta\mathfrak{T}] = \Delta[\mathfrak{J}_A, \mathfrak{T}]$ to hold for any fermionic operator $\mathfrak{T}$. After solving some involved system of equation, we find the following coefficients:

bosonically braided family

$$
\begin{aligned}
A &= \frac{ih}{2}\sin\frac{p_1+p_2}{2}\\
B &= \frac{ih\beta(p_1,p_2)}{2}\sin\frac{p_1}{2}\sin\frac{p_2}{2}\\
C &= \frac{e^{\frac{i}{4}[p_2(x+1)+p_1(y-1)]}}{8}\Big[\Upsilon(p_1,p_2)\\
&\quad - i\cot\frac{p_1}{2} + i\cot\frac{p_2}{2} - i\beta(p_1,p_2)\frac{\cot\frac{p_1}{2}+\cot\frac{p_2}{2}-i(x+y)}{\cot\frac{p_1}{2}+\cot\frac{p_2}{2}}\Big]\\
D &= \\
&\frac{e^{\frac{i}{4}[p_2(1-x)-p_1(y+1)]}}{8}\Big[4-\Upsilon(p_1,p_2)\\
&\quad - i\cot\frac{p_1}{2} + i\cot\frac{p_2}{2} - i\beta(p_1,p_2)\frac{\cot\frac{p_1}{2}+\cot\frac{p_2}{2}+i(x+y)}{\cot\frac{p_1}{2}+\cot\frac{p_2}{2}}\Big]\\
F &= \frac{ih}{8}\Big[e^{-\frac{i}{2}p_1}[\Upsilon(p_1,p_2)-2]\sin\frac{p_2}{2}\\
&\quad - x\sin\frac{p_1+p_2}{2} + \beta(p_1,p_2)\frac{xe^{\frac{i}{2}p_2}\sin\frac{p_1}{2} - ye^{-\frac{i}{2}p_1}\sin\frac{p_2}{2}}{\cot\frac{p_1}{2}+\cot\frac{p_2}{2}}\Big]\\
G &= \frac{ih}{8}\Big[e^{\frac{i}{2}p_2}[\Upsilon(p_1,p_2)-2]\sin\frac{p_1}{2}\\
&\quad + y\sin\frac{p_1+p_2}{2} - \beta(p_1,p_2)\frac{xe^{\frac{i}{2}p_2}\sin\frac{p_1}{2} - ye^{-\frac{i}{2}p_1}\sin\frac{p_2}{2}}{\cot\frac{p_1}{2}+\cot\frac{p_2}{2}}\Big]
\end{aligned}
$$

bosonically unbraided family

$$
\begin{aligned}
A &= \frac{ih}{2}\Big(\sin\frac{p_1}{2}+\sin\frac{p_2}{2}\Big)\\
B &= ih\cot\frac{p_1-p_2}{2}\cos\frac{p_1-p_2}{2}\Big(\cos\frac{p_1}{2}+\cos\frac{p_2}{2}-2\cos\frac{p_1+p_2}{2}\Big)\\
C &= \frac{e^{\frac{i}{4}(ap_2-cp_1)}}{8(\cos\frac{p_1}{2}-\cos\frac{p_2}{2})}\Big(\Big(2i\sec\frac{p_1}{2}\sin\frac{3p_1}{2}+a-c-\Upsilon(p_1,p_2)\Big)\cos\frac{p_2}{2}\\
&\quad + \Big(2i\sec\frac{p_2}{2}\sin\frac{3p_2}{2}+a-c+\Upsilon(p_1,p_2)\Big)\cos\frac{p_1}{2}\Big)\\
D &= \frac{e^{-\frac{i}{4}(ap_2-cp_1)}}{8(\cos\frac{p_1}{2}-\cos\frac{p_2}{2})}\Big(\Big(2i\sec\frac{p_1}{2}\sin\frac{3p_1}{2}+a-c+\Upsilon(p_1,p_2)\Big)\cos\frac{p_2}{2}\\
&\quad + \Big(2i\sec\frac{p_2}{2}\sin\frac{3p_2}{2}+a-c-\Upsilon(p_1,p_2)\Big)\cos\frac{p_1}{2}\Big)\\
F &= \frac{ih}{32(\cos\frac{p_1}{2}-\cos\frac{p_2}{2})}\Big(4\cos\frac{p_1+p_2}{2}\Big(a\sin\frac{p_1}{2}+c\sin\frac{p_2}{2}\Big)\\
&\quad - 2(a+c)\cos\frac{p_1}{2}\sin\frac{p_2}{2} - 2a\sin p_1 + (a-c)\sin p_2\\
&\quad + 2\Big(\cos\frac{p_1}{2}-\cos\frac{p_2}{2}\Big)\Upsilon(p_1,p_2)\sin\frac{p_2}{2}\Big)\\
G &= \frac{ih}{32(\cos\frac{p_1}{2}-\cos\frac{p_2}{2})}\Big(2(a+c)\cos\frac{p_1}{2}\sin\frac{p_2}{2}\\
&\quad - 4\cos\frac{p_1+p_2}{2}\Big(a\sin\frac{p_1}{2}+c\sin\frac{p_2}{2}\Big)\\
&\quad + (a-c)\sin p_1 + 2c\sin p_2\\
&\quad + 2(\cos\frac{p_1}{2}-\cos\frac{p_2}{2})\Upsilon(p_1,p_2)\sin\frac{p_1}{2}\Big)
\end{aligned}
$$

In either of the cases, $\Upsilon(p_1,p_2)$ and $\beta(p_1,p_2)$ are an arbitrary even and odd function, respectively, representing some residual freedom unfixed by the requirement of the algebra homomorphism property of the coproduct: The former ambiguity arises from the fact that the algebra homomorphism property of the boost coproduct for $\mathfrak{Q}$ compared to $\mathfrak{S}$ does not give rise to independent constraints on $\Delta\mathfrak{J}$, while the other comes from the fact that the quasi-cocommutativity relation with the R-matrix allows us to add an *op*-invariant term to it. By making use of our representation and seeing how the respective coefficients appear in the Boost coproduct, we can construct operators from these functions, which we denote by $\hat{\Upsilon}$ and $\hat{\beta}$. We can make the ambiguities explicit, and express the quasi-cocommutativity relation as

$$(\Delta^{\mathrm{op}}\mathfrak{J} + \hat{\beta}^{\mathrm{op}} + \hat{\Upsilon}^{\mathrm{op}})R = R(\Delta\mathfrak{J} + \hat{\beta} + \hat{\Upsilon})\ . \tag{4.2.15}$$

For the bosonically braided family, we have

$$
\begin{aligned}
\hat{\beta} = \beta(p_1,p_2)\ (\partial_{p_1}-\partial_{p_2}) - \beta(p_1,p_2)e^{\frac{i}{4}(p_2-p_1)}\frac{\csc\frac{p_1}{2}\csc\frac{p_2}{2}}{4h}\\
\times\frac{\cot\frac{p_1}{2}+\cot\frac{p_2}{2}-i(x+y)}{\cot\frac{p_1}{2}+\cot\frac{p_2}{2}}\Big[e^{\frac{i}{4}(xp_2+yp_1)}\,\mathfrak{S}\otimes\mathfrak{Q} + e^{-\frac{i}{4}(xp_2+yp_1)}\,\mathfrak{Q}\otimes\mathfrak{S}\Big]\ ,
\end{aligned}
\tag{4.2.16}
$$

as well as

$$
\begin{aligned}
\hat{\Upsilon} = \Upsilon(p_1,p_2)e^{\frac{i}{4}(p_2-p_1)}\ \Big[e^{-\frac{i}{4}(xp_2+yp_1)}\mathfrak{S}\otimes\mathfrak{Q} - e^{-\frac{i}{4}(xp_2+yp_1)}\,\mathfrak{Q}\otimes\mathfrak{S}\Big] +\\
+\Upsilon(p_1,p_2)\frac{ie^{-\frac{i}{2}p_1}}{2}\,\mathcal{B}\otimes\mathfrak{H} + \Upsilon(p_1,p_2)\frac{ie^{\frac{i}{2}p_2}}{2}\,\mathfrak{H}\otimes\mathcal{B}\ .
\end{aligned}
\tag{4.2.17}
$$

The *op*-quantities related to the above are obtained by exchanging $p_1 \leftrightarrow p_2$ and add fermionic signs where necessary.

[11] In the literature, the words homomorphism and morphism are used interchangeably oftentimes. While we introduced functions like the above as (co)algebra morphisms, we will refer to this as the homomorphism property, as is standard in the literature (outside of purely algebraic fields such as category theory).

For a particular choice of $\hat{\Upsilon}$ and $\hat{\beta}$ ($x = y = 0$), we can write a kind of evolution equation for R using $\hat{\beta}$:

$$(\partial_{p_1} - \partial_{p_2})R - [\hat{\beta}, R] = 0 \ , \tag{4.2.18}$$

and likewise, $\hat{\Upsilon}$ also gives rise to an interesting equation,

$$(\partial_{p_1} - \partial_{p_2})R - (\Upsilon - \Upsilon^{\mathrm{op}}) = 0 \ , \tag{4.2.19}$$

if we fix

$$\Upsilon(p_1, p_2) = \frac{\cos\frac{p_1 - p_2}{2}}{4ih \sin\frac{p_1}{2} \sin\frac{p_2}{2} \sin\frac{p_1+p_2}{2}} \ . \tag{4.2.20}$$

For the bosonically unbraided family, we find

$$\begin{aligned}\hat{\Upsilon} = \Upsilon(p_1, p_2) e^{\frac{i}{4}(dp_1 + ap_2)}\, \mathfrak{S} \otimes \mathfrak{Q} - \Upsilon(p_1, p_2) e^{\frac{i}{4}(cp_1 + bp_2)}\, \mathfrak{Q} \otimes \mathfrak{S} + \\ + \Upsilon(p_1, p_2) \frac{i e^{\frac{i}{4}(a+b)p_1}}{2}\, \mathcal{B} \otimes \mathfrak{H} + \Upsilon(p_1, p_2) \frac{i e^{\frac{i}{4}(a+b)p_2}}{2}\, \mathfrak{H} \otimes \mathcal{B} \ .\end{aligned} \tag{4.2.21}$$

We also find an evolution-like equation for R, this time for a particular choice of $\hat{\Upsilon}$,

$$(\partial_{p_1} - \partial_{p_2})R - [\hat{\Upsilon}, R] = 0 \ , \tag{4.2.22}$$

in the setting of

$$\begin{aligned} a = -b = c \mp 2 = d = \pm 1 \ , \\ \Upsilon = \frac{\pm e^{\frac{i}{4}(p_1+p_2)} \left(\sin\frac{p_1}{2} + \sin\frac{p_2}{2}\right)}{4h \sin\frac{p_1}{2} \sin\frac{p_2}{2} \sin\frac{p_1+p_2}{4}} \ . \end{aligned} \tag{4.2.23}$$

Unfortunately, we still lack a good explanation for both the occurrence of these ambiguities in the coefficients, as well as for their curious connection with the R-matrices. We plan to perhaps address this in the future.

Deformed case

We now turn our attention to the q-deformed case, whose motivation we outlined in the very beginning of this chapter. We first define *q-analogues* or *q-deformed quantities* via

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \ . \tag{4.2.24}$$

One can easily see that we have

$$[x]_{q\to 1} = \lim_{q\to 1} \frac{q^x - q^{-x}}{q - q^{-1}} = x \ . \tag{4.2.25}$$

With this, the $\mathfrak{su}(1|1)$ algebra gets deformed to:

$$\{\mathfrak{Q}, \mathfrak{S}\} = [\mathfrak{H}]_q \ , \tag{4.2.26}$$

with the following new q-action for the boost operator defined as follows

$$[\mathfrak{J}, \mathfrak{Q}] = \phi_Q(p)\mathfrak{Q} = \frac{\beta[2\mathfrak{H}]_q \sin p}{4[\mathfrak{H}]_q^2}\mathfrak{Q} \ , \qquad [\mathfrak{J}, \mathfrak{S}] = \phi_S(p)\mathfrak{S} = \frac{\beta[2\mathfrak{H}]_q \sin p}{4[\mathfrak{H}]_q^2}\mathfrak{S} \ ,$$

$$[\mathfrak{J},p] \;=\; \alpha[\mathfrak{H}]_q \;, \qquad\qquad [\mathfrak{J},\mathfrak{H}] \;=\frac{q-q^{-1}}{\log q}\beta\sin p \;, \tag{4.2.27}$$

where both α and β are constants depending on the deformation q and the coupling constant h. Similarly to how the previous algebra appears in the context of strings in $\mathrm{AdS}_3\times S^3$, this algebra appears in the context of a well-known algebra called η-deformation, $\left(\mathrm{AdS}_5\times S^5\right)_\eta$, where the deformation parameters q and η are related as $q=e^{-\frac{2\eta}{g(1+\eta^2)}}$, with g being the string tension. Furthermore, in the following, we are going to identify $\mathfrak{H}=E/2$ per our dispersion relation, exactly as we did in the undeformed case before.

Regarding the representation, we can use the same one as for the undeformed case, with the sole change of $\mathfrak{H}(p)\to[\mathfrak{H}(p)]_q$. Much like we did before, we can again make the distinction between the cases of braided and unbraided energy, which we will now analyse step by step. That is the main reason why we are considering these two algebras together. In either case, rather than analysing general coproduct braiding with parameters a,b,c,d, we focus on one representative case each.

In the unbraided energy case, we have the following Hopf algebraic structure to deal with:

$$\begin{aligned}\Delta\mathfrak{S}=\mathfrak{S}\otimes e^{-\frac{i}{4}p}q^{-\frac{E}{4}}+e^{\frac{i}{4}p}q^{\frac{E}{4}}\otimes\mathfrak{S}\;,\qquad \Delta\mathfrak{Q}=\mathfrak{Q}\otimes e^{\frac{i}{4}p}q^{-\frac{E}{4}}+e^{-\frac{i}{4}p}q^{\frac{E}{4}}\otimes\mathfrak{Q}\;,\\ \Delta p=p\otimes\mathbb{1}+\mathbb{1}\otimes p\;,\qquad \Delta[\mathfrak{H}]_q=[\mathfrak{H}]_q\otimes q^{\frac{E}{2}}+q^{-\frac{E}{2}}\otimes[\mathfrak{H}]_q\;.\end{aligned} \tag{4.2.28}$$

Making again the same ansatz for the R-matrix and imposing quasi-cocommutativity with the fermionic elements, we arrive at the following R-matrix elements:

$$\begin{aligned}r_{11}&=\frac{e^{\frac{ip_1}{2}}\left[\frac{E_2}{2}\right]_q-e^{\frac{ip_2}{2}}\left[\frac{E_1}{2}\right]_q}{\left[\frac{E_1+E_2}{2}\right]_q}\;,\\ r_{12}&=\frac{\left(e^{\frac{i}{2}(p_1-p_2)}q^{-\frac{E_1+E_2}{4}}+e^{-\frac{i}{2}(p_1-p_2)}q^{\frac{E_1+E_2}{4}}\right)\sqrt{\left[\frac{E_1}{2}\right]_q\left[\frac{E_2}{2}\right]_q}}{\left[\frac{E_1+E_2}{2}\right]_q}\;,\\ r_{21}&=\frac{\left(e^{\frac{i}{2}(p_1-p_2)}q^{\frac{E_1+E_2}{4}}+e^{-\frac{i}{2}(p_1-p_2)}q^{-\frac{E_1+E_2}{4}}\right)\sqrt{\left[\frac{E_1}{2}\right]_q\left[\frac{E_2}{2}\right]_q}}{\left[\frac{E_1+E_2}{2}\right]_q}\;,\\ r_{22}&=\frac{e^{\frac{ip_2}{2}}\left[\frac{E_1}{2}\right]_q-e^{\frac{ip_1}{2}}\left[\frac{E_2}{2}\right]_q}{\left[\frac{E_1+E_2}{2}\right]_q}\;.\end{aligned} \tag{4.2.29}$$

One can check that in the limit of no deformation, i.e. $q\to 1$, as the R-matrix above we recover the one we obtained when we considered the undeformed case (for the same choice of braiding parameters). Regarding the coproduct for the boost, we start with the following ansatz (resembling the ansatz we chose before):

$$\begin{aligned}\Delta\mathfrak{J}=&A(p_1,p_2)\;(\partial_{p_1}+\partial_{p_2})+B(p_1,p_2)\;(\partial_{p_1}-\partial_{p_2})\\ &+C(p_1,p_2)e^{i\frac{p_1+p_2}{4}}\,\mathfrak{S}\otimes\mathfrak{Q}+D(p_1,p_2)e^{-i\frac{p_1+p_2}{4}}\,\mathfrak{Q}\otimes\mathfrak{S}+F(p_1,p_2)\mathfrak{B}\otimes\mathbb{1}+G(p_1,p_2)\mathbb{1}\otimes\mathfrak{B}\;,\end{aligned} \tag{4.2.30}$$

where we, with great foresight, already extracted some bosonic, op-invariant factors out of D and C. Again, in order to arrive at an algebraically consistent expression for our coproduct, it is enough to check the algebra homomorphism property with the supercharges as well as the momentum operator - this fixes the action on the central elements automatically. Furthermore, we glossed over the fact that while Δp is unambiguously defined, the story for $\Delta\sin p$ is not quite the same: While one could think about expanding $\sin p$ as a series and then assuming that the algebra homomorphism property for Δ to hold for polynomials of arbitrary order, and thus arrive at $\sin\Delta p$. One can see that $\left[\frac{E}{2}\right]_q\cos(\frac{p}{2})$,

$\sin(\frac{p}{2})\partial_p \left[\frac{E}{2}\right]_q$ or $\partial_p \left[\frac{E}{2}\right]_q^2$ are indistinguishable at the level of the representation, but they have different coproducts. Whatever choice one prefers to make, we shall denote it by $\Delta \sin p$. In any case, we arrive at the following solutions for the coefficients of $\Delta\mathfrak{J}$:

$$A(p_1,p_2) = \frac{\alpha}{2}\Delta[E]_q = \frac{\alpha}{2}[E_1+E_2]_q \tag{4.2.31}$$

$$B(p_1,p_2) = \frac{\alpha}{2}\,\frac{2(\Delta\sin p) - [E_1+E_2]_q\left(\frac{\sin p_1}{[E_1]_q} - \frac{\sin p_2}{[E_2]_q}\right)}{\frac{\sin p_1}{[E_1]_q} - \frac{\sin p_2}{[E_2]_q}} \tag{4.2.32}$$

$$\frac{C(p_1,p_2)-D(p_1,p_2)}{2} = q^{\frac{E_1-E_2}{4}}\,\frac{iB(p_1,p_2)+\Upsilon(p_1,p_2)}{\left[\frac{E_1+E_2}{2}\right]_q}\ , \tag{4.2.33}$$

where a similar ambiguity or freedom is present as we have seen in the undeformed case. This time, we shall make it explicit:

$$\hat{\Upsilon} = \frac{q^{\frac{E_1-E_2}{4}}\Upsilon(p_1,p_2)}{\left[\frac{E_1+E_2}{2}\right]_q}\left(e^{i\frac{p_1+p_2}{4}}\,\mathfrak{S}\otimes\mathfrak{Q} + e^{-i\frac{p_1+p_2}{4}}\,\mathfrak{Q}\otimes\mathfrak{S} + \frac{q^{\frac{E_1-E_2}{4}}}{2i}\mathfrak{B}\otimes[\mathfrak{H}]_q + \frac{q^{-\frac{E_1-E_2}{4}}}{2i}\,[\mathfrak{H}]_q\otimes\mathfrak{B}\right)\ , \tag{4.2.34}$$

where again $\Upsilon(p_1,p_2)$ is some arbitrary, op-invariant function. Sadly, par contre to the undeformed case, we still have not found a relation between the ambiguity of this case and the corresponding R-matrix, be it in the form of an evolution equation or by any other means.

For the case where the energy coproduct has a non-trivial braiding, our approach is much the same. The fermionic coproducts for the braided energy case that we shall consider read:

$$\begin{aligned}\Delta\mathfrak{S} &= \mathfrak{S}\otimes e^{-\frac{i}{4}p}q^{-\frac{E}{4}} + e^{\frac{i}{4}p}q^{\frac{E}{4}}\otimes\mathfrak{S}\ , \qquad \Delta\mathfrak{Q} = \mathfrak{Q}\otimes e^{-\frac{i}{4}p}q^{\frac{E}{4}} + e^{\frac{i}{4}p}q^{-\frac{E}{4}}\otimes\mathfrak{Q}\ ,\\ \Delta p &= p\otimes\mathbb{1}+\mathbb{1}\otimes p\ , \qquad \Delta[\mathfrak{H}]_q = [\mathfrak{H}]_q\otimes e^{-\frac{i}{2}p} + e^{\frac{i}{2}p}\otimes[\mathfrak{H}]_q\ ,\end{aligned} \tag{4.2.35}$$

where, keep in mind, this is again just one representative choice of braiding parameters. Imposing again that the R-matrix be quasi-cocommutative with our algebra elements, we arrive at:

$$\begin{aligned} r_{11} &= \frac{q^{-\frac{E_1}{2}}\sin\frac{p_2}{2} - q^{-\frac{E_2}{2}}\sin\frac{p_1}{2}}{\sin\frac{p_1+p_2}{2}}\ ,\\ r_{12} &= \frac{\left(e^{\frac{i}{4}(p_1+p_2)}q^{-\frac{E_1-E_2}{4}} + e^{-\frac{i}{4}(p_1+p_2)}q^{\frac{E_1-E_2}{4}}\right)\sqrt{\sin\frac{p_1}{2}\sin\frac{p_2}{2}}}{\sin\frac{p_1+p_2}{2}}\ ,\\ r_{21} &= \frac{\left(e^{\frac{i}{4}(p_1+p_2)}q^{\frac{E_1-E_2}{4}} + e^{-\frac{i}{4}(p_1+p_2)}q^{-\frac{E_1-E_2}{4}}\right)\sqrt{\sin\frac{p_1}{2}\sin\frac{p_2}{2}}}{\sin\frac{p_1+p_2}{2}}\ ,\\ r_{22} &= \frac{q^{\frac{E_2}{2}}\sin\frac{p_1}{2} - q^{\frac{E_1}{2}}\sin\frac{p_2}{2}}{\sin\frac{p_1+p_2}{2}}\ . \end{aligned} \tag{4.2.36}$$

Once more, we make an ansatz for what we expect the boost coproduct to look like:

$$\begin{aligned}\Delta\mathfrak{J} &= A(p_1,p_2)\ (\partial_{p_1}+\partial_{p_2}) + B(p_1,p_2)\ (\partial_{p_1}-\partial_{p_2})\\ &+ C(p_1,p_2)q^{-\frac{E_1+E_2}{4}}\,\mathfrak{S}\otimes\mathfrak{Q} + D(p_1,p_2)q^{\frac{E_1+E_2}{4}}\,\mathfrak{Q}\otimes\mathfrak{S} + F(p_1,p_2)\mathfrak{B}\otimes\mathbb{1} + G(p_1,p_2)\mathbb{1}\otimes\mathfrak{B}\ ,\end{aligned} \tag{4.2.37}$$

where we again already extracted some bosonic factors out of some of the coefficients. Using this, we arrive at the following solution:

$$\begin{aligned} A(p_1,p_2) &= \frac{\alpha}{2}[E]_q(p_1+p_2)\ ,\\ B(p_1,p_2) &\propto A(p_1,p_2)\left[\left(\frac{\phi_Q(p_1)}{\alpha[E_1]_q}+\frac{i}{4}\right)e^{\frac{i}{4}(p_1+p_2)}\left[\frac{E_1}{2}\right]_q + \left(\frac{\phi_Q(p_2)}{\alpha[E_2]_q}-\frac{i}{4}\right)e^{-\frac{i}{4}(p_1+p_2)}\left[\frac{E_2}{2}\right]_q\right]\end{aligned}$$

$$
\begin{aligned}
-\phi_Q(p_1+p_2)\left[e^{\frac{i}{4}(p_1+p_2)}\left[\frac{E_1}{2}\right]_q+e^{-\frac{i}{4}(p_1+p_2)}\left[\frac{E_2}{2}\right]_q\right]&=0\ ,\\
C(p_1,p_2)&=-\frac{1}{2i}\left(A(p_1,p_2)\log q\frac{\partial_{p_2}E_2}{4}+\frac{F-G}{2}e^{-\frac{i}{4}(p_1+p_2)}\left[\frac{E_2}{2}\right]_q\right)\ ,\\
D(p_1,p_2)&=\frac{1}{2i}\left(A(p_1,p_2)\log q\frac{\partial_{p_1}E_1}{4}-\frac{F-G}{2}e^{\frac{i}{4}(p_1+p_2)}\left[\frac{E_1}{2}\right]_q\right)\ ,\\
\frac{F(p_1,p_2)+G(p_1,p_2)}{2}&=-e^{-\frac{i}{4}(p_1-p_2)}\frac{iA(p_1,p_2)}{2\left[\frac{E(p_1+p_2)}{2}\right]_q}\left(2-i\cot\frac{p_1}{2}+i\cot\frac{p_2}{2}\right)\ ,\\
\frac{F(p_1,p_2)-G(p_1,p_2)}{2}&=\frac{e^{-\frac{i}{4}(p_1-p_2)}}{\left[\frac{E(p_1+p_2)}{2}\right]_q}\left(A(p_1,p_2)\log q\frac{\partial_{p_1}E_1-\partial_{p_2}E_2}{4}+\Upsilon(p_1,p_2)\right)\ ,
\end{aligned}
\tag{4.2.38}
$$

where we abbreviated $E_i = E(p_i)$. We need to address some particularities of this solution: The $|\Phi\Phi\rangle$ component of the quasi-cocommutativity condition fixes all the functions appearing in our ansatz - however, the resulting coproduct does not fulfil the remaining components of said condition, meaning we find a contradiction. Furthermore, the $\hat{\Upsilon}$ operator does not seem to quasi-cocommute with the R-matrix as it did before.

The braided and deformed case bears yet another important distinction compared to the cases analysed before: The coproduct choices we have made turn out to be non-coassociative. Thus, not all Hopf algebraic requirements are fulfilled. We are going to address this in the subsection to follow.

4.2.3 A Non-Coassociative Curiosity

One can readily check that the coproduct of $q^{\frac{E}{4}}$ is not group-like, namely $\Delta q^{\frac{E}{4}} \neq q^{\frac{E}{4}} \otimes q^{\frac{E}{4}}$.[12] As the coproduct of the supercharges that we proposed involves this factor, this implies that $(\mathbb{1}\otimes\Delta)\Delta\mathfrak{Q} \neq (\Delta\otimes\mathbb{1})\Delta\mathfrak{Q}$ and $(\mathbb{1}\otimes\Delta)\Delta\mathfrak{S} \neq (\Delta\otimes\mathbb{1})\Delta\mathfrak{S}$.[13]

In chapter 2, we introduced the non-coassociative analogues of Hopf algebras - quasi-Hopf algebras. We can find a coassociator for our case that is a hopeful candidate to let us define a quasi-Hopf algebra with the structure we have, and it has the following form:

$$
\Phi = \exp(\mathfrak{B}\otimes\omega_{23}+\omega_{12}\otimes\mathfrak{B}), \qquad \omega_{jk} = \frac{1}{8i}\log q\Big(E(p_j+p_k)-E(p_j)-E(p_k)\Big)\mathbb{1}\otimes\mathbb{1}\ . \tag{4.2.39}
$$

The coassociator is measuring in a very direct sense by how much the coproduct fails to be coassociative. In our case, this is directly visible by the abstract relation:

$$
\omega = \frac{1}{8i}\log q(\Delta(E)-\Delta_{\text{trivial}}(E))\ , \tag{4.2.40}
$$

where $\Delta_{\text{trivial}}(x) = x\otimes\mathbb{1}+\mathbb{1}\otimes x$. To see why this coassociator is indeed correct, the Weyl-type relations:

$$
e^{\alpha\mathfrak{B}}\,\mathfrak{Q} = \mathfrak{Q}\,e^{\alpha(\mathfrak{B}+2i\mathbb{1})}, \qquad e^{\alpha\mathfrak{B}}\,\mathfrak{S} = \mathfrak{S}\,e^{\alpha(\mathfrak{B}-2i\mathbb{1})}\ , \tag{4.2.41}
$$

are helpful. The pentagon relation can be proven straightforwardly, i.e.

$$
\Big[(\mathbb{1}\otimes\mathbb{1}\otimes\Delta)\Phi\Big]\Big[(\Delta\otimes\mathbb{1}\otimes\mathbb{1})\Phi\Big] = (\mathbb{1}\otimes\Phi)\Big[(\mathbb{1}\otimes\Delta\otimes\mathbb{1})\Phi\Big](\Phi\otimes\mathbb{1})\ , \tag{4.2.42}
$$

and one can see that we also have

$$
(\mathbb{1}\otimes\chi\otimes\mathbb{1})\Phi = \mathbb{1}\otimes\mathbb{1}\ , \tag{4.2.43}
$$

[12] See footnote 9 for an elaboration on the origin of group-like coproducts as the exponential of primitive ones, and the inverse implication thereof.

[13] As will be made more explicit later on as well as in the appendix A, this arising non-coassociativity only appears when the deformation is turned on, connected to the facts that p is primitive and the energy corpoduct is braided here. This is most evidently visible in 4.2.39, which trivialises for the undeformed case of $q = 1$ (or for a primitive E, for that matter).

$$(\mathbb{1}\otimes\chi)\Delta = (\chi\otimes\mathbb{1})\Delta = \mathbb{1}\ , \tag{4.2.44}$$

the latter of which we already knew to be the case for the counit χ, as the quasi-coassociative structure does not enter this equation (for more detail, see the appendix A).

Finally, the Hopf algebraic relations for the antipode S,

$$\mu\circ(S\otimes\mathrm{id})\circ\Delta = \upsilon\circ\chi\ , \tag{4.2.45}$$
$$\mu\circ(\mathrm{id}\otimes S)\circ\Delta = \upsilon\circ\chi\ , \tag{4.2.46}$$

evaluated on any generator x of our algebra give

$$\begin{aligned}\mu(S\otimes\mathbb{1})\Delta(x) &= 0\ ,\\ \mu(\mathbb{1}\otimes S)\Delta(x) &= 0\ .\end{aligned} \tag{4.2.47}$$

From this, we can directly conclude that for $x = p$, which is primitive, we have

$$\begin{aligned}0 &= \mu(S\otimes\mathbb{1})\Delta(p)\\ &= \mu(S\otimes\mathbb{1})(p\otimes\mathbb{1}+\mathbb{1}\otimes p)\\ &= \mu(S(p)\otimes\mathbb{1}+\mathbb{1}\otimes p)\\ &= S(p)+p\ ,\end{aligned} \tag{4.2.48}$$

from which we can follow that $S(p) = -p$. With this result, we can also conclude that $S(E) = -E$ since $[\frac{E}{2}]_q \propto \sin\left(\frac{p}{2}\right)$ is an odd function of p and we have $-[\frac{E}{2}]_q = [\frac{-E}{2}]_q$. The antipode-coassociator relations our antipode needs to fulfil, as mentioned before, are then[14][15]

$$\mu(\mathbb{1}\otimes\mu)\Big[S\otimes\mathbb{1}\otimes S\Big]\Phi^{-1} = \mathbb{1} \tag{4.2.49}$$
$$\mu(\mu\otimes\mathbb{1})\Big[\mathbb{1}\otimes S\otimes\mathbb{1}\Big]\Phi = \mathbb{1}\ . \tag{4.2.50}$$

In order to understand this equation for our concrete coassociator, we will need to make the following considerations: Let us, for instance, start with the equation

$$\mu(\mu\otimes\mathbb{1})\Big[\mathbb{1}\otimes S\otimes\mathbb{1}\Big]\Phi = \mathbb{1}\ . \tag{4.2.51}$$

If we evaluate this on a given sate, for example $|\Phi\Phi\Phi\rangle$, we have to first consider what the antipode does to what is to its right-hand side:

$$\begin{aligned}\Big[\mathbb{1}\otimes S\otimes\mathbb{1}\Big]\Phi|\Phi\Phi\Phi\rangle &= \Big[\mathbb{1}\otimes S\otimes\mathbb{1}\Big]\exp(\mathfrak{B}\otimes\omega_{23}+\omega_{12}\otimes\mathfrak{B})|\Phi\Phi\Phi\rangle\\ &= \Big[\mathbb{1}\otimes S\otimes\mathbb{1}\Big]\exp(\mathbb{1}\otimes\omega_{23}+\omega_{12}\otimes\mathbb{1})|\Phi\Phi\Phi\rangle\\ &= \exp\Big(\mathbb{1}\otimes\frac{\log q}{8i}\left(E(-p_2+p_1)+E(p_2)-E(p_1)\right)\\ &\qquad +\frac{\log q}{8i}\left(E(-p_2+p_3)+E(p_2)-E(p_3)\right)\otimes\mathbb{1}\Big)|\Phi\Phi\Phi\rangle\ .\end{aligned} \tag{4.2.52}$$

The term $\mu(\mu\otimes\mathbb{1})$ then, in terms of domains and codomains, takes a 3-particle space and returns a 1-particle space - it therefore "collapses" all of the momentum labels into one, i.e. $p_2 = p_1 = p_3$. We see then that the exponent of the above vanishes. This is the case for any state we evaluate the equation (and

[14] Recall here that our multiplication is associative, and we can choose either one of $\mu(\mathbb{1}\otimes\mu) = \mu(\mu\otimes\mathbb{1})$ in the relations - whichever lends itself more conveniently.

[15] As briefly alluded to before in the preliminary part of this thesis, in the non-coassociative context the antipode appears as a triplet (S,α,β), with generally more complicated relations to fulfil than the ones we prescribe (see e.g. [Dri98]).

its partner equation) on: The antipode will always flip the sign of the momentum/energy corresponding to one of the tensorial spaces appearing in the exponent of $\Phi^{(-1)}$, meaning the exponent will always be proportional to

$$E(-p_i + p_j) + E(p_i) - E(p_j) \ . \tag{4.2.53}$$

Again, once the nested multiplication is performed, all indices of the tensorial spaces have to coincide, and thus all exponents of $\Phi^{(-1)}$ become zero.[16] With this, (4.2.49) is clearly fulfilled, and we can thus conclude we have a true quasi-Hopf algebra on our hands.

The natural question we ask ourselves now is whether our quasi-Hopf algebra is also quasi-triangular. For the R-matrix we introduced above, one can indeed prove that the generalised Yang-Baxter equation holds for the above coassociator. More interestingly, even though we are dealing with a non-coassociative case, this R-matrix also satisfies the regular Yang-Baxter equation of quasi-triangular Hopf algebras. This rather curious result is owed to the structure of the coassociator: Although Φ is not proportional to $\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}$, it is purely diagonal and thus rather simple. The hands-on computation of the generalised Yang-Baxter equation revealed that, evaluated on 3-particle states, each of its components is indeed equal to its regular counterpart times $q^{E/2}$ factors. To make things more explicit, let us start with a given 3-particle state $|\chi_1\chi_2\chi_3\rangle$ and let us denote the generalised Yang-Baxter equation

$$\mathbf{GYBE} := R_{12}\Phi^{312}R_{13}(\Phi^{132})^{-1}R_{23}\Phi^{123} - \Phi^{321}R_{23}(\Phi^{231})^{-1}R_{13}\Phi^{213}R_{12} \ . \tag{4.2.54}$$

Evaluating this equation on all 3-particle states (which is a lengthy calculation that we discuss in the appendix), we indeed find that for our coassociator we can write more compactly that

$$\langle\chi_1'\chi_2'\chi_3'|\exp(-\omega_{12}\otimes\mathfrak{B})\mathbf{GYBE}\exp(-\omega_{12}\otimes\mathfrak{B})|\chi_1\chi_2\chi_3\rangle = \langle\chi_1'\chi_2'\chi_3'|\mathbf{YBE}|\chi_1\chi_2\chi_3\rangle \ , \tag{4.2.55}$$

$\forall\ |\chi_1'\chi_2'\chi_3'\rangle$ and $|\chi_1\chi_2\chi_3\rangle$ 3-particle states, where

$$\mathbf{YBE} := R_{12}R_{13}R_{23} - R_{23}R_{13}R_{12} \ , \tag{4.2.56}$$

denotes the regular Yang-Baxter equation. In particular, this implies that, indeed, for our case, the **GYBE** and **YBE** are algebraically equivalent.

This quasi-Hopf structure, as it turns out, can be reduced to a twist. If a coassociator is induced by a twist $\mathcal{F}$, it can be written as

$$\Phi = \mathcal{F}_{23}(\mathbb{1}\otimes\Delta)(\mathcal{F})(\Delta\otimes\mathbb{1})(\mathcal{F}^{-1})\mathcal{F}_{12}^{-1} \ , \tag{4.2.57}$$

from one can check that $\mathcal{F} = q^{\mathfrak{B}\otimes\alpha-\alpha\otimes\mathfrak{B}}$ generates a coassociator of the form

$$\Phi = q^{\mathfrak{B}\otimes(\Delta\alpha-\mathbb{1}\otimes\alpha-\alpha\otimes\mathbb{1})+(\Delta\alpha-\mathbb{1}\otimes\alpha-\alpha\otimes\mathbb{1})\otimes\mathfrak{B}} \ , \tag{4.2.58}$$

for any $[\alpha,\mathfrak{B}]=0$. We can see that the coassociator from our quasi-Hopf algebra corresponds to setting $\alpha = \frac{E(p)}{8i}$. Undoing this twist implies stripping the coproduct entirely of all $q^{E/4}$ factors, leaving only the braiding factors that depend on the momentum. We can clearly see in this example how twists

[16] The reason why in (4.2.52) chose the specific way of evaluation to show this is the following: A priori, the term

$$\mu(\mu\otimes\mathbb{1})\Big[\mathbb{1}\otimes S\otimes\mathbb{1}\Big]\Phi$$

needs to be evaluated on a 1-particle state - this way, the calculation gets less straightforward. It is easier to first evaluate the term $\Big[\mathbb{1}\otimes S\otimes\mathbb{1}\Big]\Phi$ on a generic 3-particle state (after which the expression depends on three spectral parameters p_1,p_2,p_3), and only then let the $\mu(\mu\otimes\mathbb{1})$ "collapse" it and give zero once only one spectral parameter is present - given the specific form of the exponent in (4.2.52).

dramatically alter the physics, since removing the non-coassociativity lands us in a completely different model with a different R-matrix, $\tilde{R} = \mathcal{F}_{21}^{-1} R \mathcal{F}_{12}$, that, as one expects, still fulfils the ordinary Yang-Baxter equation.

It would be interesting to investigate what precise conditions a coassociator has to fulfil in order for the algebra to be non-coassociative (i.e. the coassociator not being the identity), but the YBE and GYBE still coinciding, as was the case for us, and, ultimately, if this has a physical explanation or reasoning. One could state a conjecture that block-diagonal coassociators of a similar form to ours fulfil exactly that.

4.3 The Generalised Boost Superalgebra

In this section, we are going to generalise the algebra we have worked with thus far. While we have up to this point assumed that the two momenta of the $\mathfrak{su}(1|1)_L \oplus \mathfrak{su}(1|1)_R$ are identical, i.e. $p_L = p_R$. We will now drop this assumption, and allow the two momenta of the $\mathfrak{su}(1|1)$ algebras to be a priori uncorrelated.

In the previous part of this chapter, we circumvented the question of how a boost generator $\mathfrak{J}$ acted on remaining generators of different handedness, by looking at a particular representation. Given now our assumption of uncorrelated p_L, p_R, we will answer this question in the most general way possible.

Inspired by what we know about energy and dispersion relations, we assume the Cartan energy generators $\mathfrak{H}_A$ to be positive even functions of the momentum generators in the sense that $\mathfrak{H}_A = \mathfrak{H}_A(p_A)$. Drawing inspiration from a differentially acting form of the boost, we postulate the action of a boost of one handedness on the momentum generator of opposite handedness to be

$$[\mathfrak{J}_L, p_R] = i\mathfrak{H}_L \mathfrak{d}_{LR} \ , \qquad [\mathfrak{J}_R, p_L] = i\mathfrak{H}_R \mathfrak{d}_{RL} \ , \tag{4.3.1}$$

where the functions $\mathfrak{d}_{AB}$ resemble momentum Jacobians and are assumed to commute with $\mathfrak{H}_A$. Determining these functions will be instrumental in getting insight into the possible algebraic structures that allow for a boost operator of any handedness to act consistently. As we assume the other generators of the algebra to be momentum-dependent as well, we generalise the above relation to

$$\mathfrak{H}_B[\mathfrak{J}_A, X_B] = \mathfrak{H}_A \mathfrak{d}_{AB}[\mathfrak{J}_B, X_B] \ , \tag{4.3.2}$$

where X_B represents a generator of the algebra with well-defined handedness $B \neq A$, and the action on central elements with no defined handedness is deduced by means of either Jacobi identities or Leibniz rules, such as $[\mathfrak{J}_L, \mathfrak{P}] = (\phi_L^Q \mathfrak{H}_R + \phi_R^Q \mathfrak{H}_L \mathfrak{d}_{LR})\mathfrak{P}$.

The equation (4.3.2) is supposed to hold for all generators X_B with definite handedness in our algebra, that also includes boosts $\mathfrak{J}_B$. Obviously, since $[\mathfrak{J}_B, \mathfrak{J}_B]$ vanishes, by (4.3.2) we also expect $[\mathfrak{J}_A, \mathfrak{J}_B]$ to vanish. There is a second argument to be made for $[\mathfrak{J}_A, \mathfrak{J}_B] = 0$, namely the underlying $\mathbb{Z}_2$-symmetry given by the $L \leftrightarrow R$ exchange, which would require that $[\mathfrak{J}_L, \mathfrak{J}_R] = [\mathfrak{J}_R, \mathfrak{J}_L]$ for the (bosonic) $\mathfrak{J}_A$, which together with the antisymmetry of the superbracket for bosonic elements implies $[\mathfrak{J}_L, \mathfrak{J}_R] = 0$. We will revisit this point when we see how that is non-trivially realised in a particular representation.

We now elaborate on the functions $\mathfrak{d}_{AB}$, which control how the left-handed and right-handed side of our algebra interact with each other. The Jacobi identities involving $\mathfrak{J}_A, \mathfrak{J}_B, p_B$ give us restrictions for $\mathfrak{d}_{AB}$ $(A \neq B)$

$$\begin{aligned} 0 &= [\mathfrak{J}_L, [\mathfrak{J}_R, p_R]] - [\mathfrak{J}_R, [\mathfrak{J}_L, p_R]] + [p_R, [\mathfrak{J}_L, \mathfrak{J}_R]] \\ &= [\mathfrak{J}_L, i\mathfrak{H}_R] - [\mathfrak{J}_R, i\mathfrak{H}_L \mathfrak{d}_{LR}] \ , \end{aligned} \tag{4.3.3}$$

where we used the argument above to assume the pure-boost commutators to vanish. This, after evaluating the commutators, gives us:[17]

$$-i\mathfrak{H}_L[\mathfrak{J}_R,\mathfrak{d}_{LR}]+\mathfrak{H}_R\mathfrak{d}_{RL}\Phi_L\mathfrak{d}_{LR}=\mathfrak{H}_L\mathfrak{d}_{LR}\Phi_R\ . \tag{4.3.4}$$

Of course, there are more equations for the $\mathfrak{d}_{AB}$ given by additional Jacobi identities generated, such as:

$$\begin{aligned}
[\mathfrak{J}_L,[\mathfrak{J}_R,\mathfrak{H}_R]]&=[\mathfrak{J}_R,[\mathfrak{J}_L,\mathfrak{H}_R]]\ ,\\
[\mathfrak{J}_L,\mathfrak{H}_R\Phi_R]&=[\mathfrak{J}_R,\mathfrak{H}_L\mathfrak{d}_{LR}\Phi_R]\ ,\\
[\mathfrak{J}_L,\mathfrak{H}_R]\Phi_R+\mathfrak{H}_R[\mathfrak{J}_L,\Phi_R]&=[\mathfrak{J}_R,\mathfrak{H}_L\mathfrak{d}_{LR}]\Phi_R+\mathfrak{H}_L\mathfrak{d}_{LR}[\mathfrak{J}_R,\Phi_R]\ ,\\
\mathfrak{H}_L\mathfrak{d}_{LR}\Phi_R^2&=-i\mathfrak{H}_L[\mathfrak{J}_R,\mathfrak{d}_{LR}]\Phi_R+\mathfrak{H}_R\mathfrak{d}_{RL}\Phi_L\Phi_R\mathfrak{d}_{LR}\ ,
\end{aligned} \tag{4.3.5}$$

and their $L\leftrightarrow R$ exchanges. As an example, the last equation of the above was obtained the following way: Starting with the third equation, i.e.

$$\begin{aligned}
[\mathfrak{J}_L,\mathfrak{H}_R]\Phi_R+\mathfrak{H}_R[\mathfrak{J}_L,\Phi_R]&=[\mathfrak{J}_R,\mathfrak{H}_L\mathfrak{d}_{LR}]\Phi_R+\mathfrak{H}_L\mathfrak{d}_{LR}[\mathfrak{J}_R,\Phi_R]\\
&=[\mathfrak{J}_R,\mathfrak{H}_L\mathfrak{d}_{LR}]\Phi_R+\mathfrak{H}_R[\mathfrak{J}_L,\Phi_R]\ ,
\end{aligned} \tag{4.3.6}$$

and noting the following relation

$$\mathfrak{H}_L[\mathfrak{J}_R,\mathfrak{H}_L]=\mathfrak{H}_R\mathfrak{d}_{RL}[\mathfrak{J}_L,\mathfrak{H}_L]=i\mathfrak{H}_R\mathfrak{d}_{RL}\mathfrak{H}_L\Phi_L\ , \tag{4.3.7}$$

we can show that:

$$[\mathfrak{J}_L,\mathfrak{H}_R]\Phi_R+\cancel{\mathfrak{H}_R[\mathfrak{J}_L,\Phi_R]}^{\text{cancels}}=[\mathfrak{J}_R,\mathfrak{H}_L]\mathfrak{d}_{LR}\Phi_R+\mathfrak{H}_L[\mathfrak{J}_R,\mathfrak{d}_{LR}]\Phi_R+\cancel{\mathfrak{H}_L\mathfrak{d}_{LR}[\mathfrak{J}_R,\Phi_R]}^{\text{cancels}}\ , \tag{4.3.8}$$

leaving us with:

$$\underbrace{[\mathfrak{J}_L,\mathfrak{H}_R]}_{=i\mathfrak{H}_L\mathfrak{d}_{LR}\Phi_R}\Phi_R=\underbrace{[\mathfrak{J}_R,\mathfrak{H}_L]}_{=i\mathfrak{H}_R\mathfrak{d}_{RL}\Phi_L}\mathfrak{d}_{LR}\Phi_R+\mathfrak{H}_L[\mathfrak{J}_R,\mathfrak{d}_{LR}]\Phi_R\ , \tag{4.3.9}$$

giving us the wanted relation. However, all these relations above do not add any further restrictions on our first Jacobi identity, so it suffices to consider (4.3.4).

(4.3.4) gives us the guideline to find the different solutions for $\mathfrak{d}_{AB}$, and we can see five distinct ones right away: A first solution is the trivial case $\mathfrak{d}_{AB}=0$. A second solution is $\mathfrak{d}_{RL}=0$ and $\mathfrak{d}_{LR}=\zeta\mathfrak{H}_R$, and a third solution is obtained from the second one by swapping handedness, where ζ is assumed to be a constant with respect to the momentum. The remaining two solutions are given by $\mathfrak{d}_{AB}=1$ and $\mathfrak{d}_{AB}=-1$.

Let us now check more thoroughly to see if there is any other possibility. For this, we will start by showing that any possible solution for $\mathfrak{d}_{AB}$ has to fall into one of two categories: either $\mathfrak{d}_{LR}\mathfrak{d}_{RL}=1$, or at least one of the functions $\mathfrak{d}_{LR}=0$ is vanishing. To see this, we need to analyse the the equation

$$\mathfrak{d}_{LR}\mathfrak{d}_{RL}(1-\mathfrak{d}_{LR}\mathfrak{d}_{RL})(\mathfrak{H}_L\Phi_R-\mathfrak{d}_{RL}\Phi_L\mathfrak{H}_R)=0\ , \tag{4.3.10}$$

as well as its handedness-swap. To derive them, one starts by considering the Leibniz rule, i.e.

$$[\mathfrak{J}_R,\mathfrak{d}_{LR}\mathfrak{d}_{RL}]=[\mathfrak{J}_R,\mathfrak{d}_{LR}]\mathfrak{d}_{RL}+\mathfrak{d}_{LR}[\mathfrak{J}_R,\mathfrak{d}_{RL}]\ . \tag{4.3.11}$$

[17] For this, as well as for the equation before, we are using the following relations:

$$[\mathfrak{J}_L,\mathfrak{H}_L]=i\mathfrak{H}_L\Phi_L\ ,\ [\mathfrak{J}_R,\mathfrak{H}_R]=i\mathfrak{H}_R\Phi_R\ ,\ [\mathfrak{J}_R,\mathfrak{H}_L]=i\mathfrak{H}_R\mathfrak{d}_{RL}\Phi_L\ ,\ [\mathfrak{J}_L,\mathfrak{H}_R]=i\mathfrak{H}_L\mathfrak{d}_{LR}\Phi_R\ .$$

For the next step, it is in this context prudent to remember that the action of the boost operator on $\mathfrak{d}_{RL}$ and its handedness-swap was consistently found using Jacobi identities, but is not initially covered by (4.3.2), and we need to impose this to be the case for mathematical coherence. Similarly, we also need to impose the equations

$$\begin{aligned}\mathfrak{H}_L[\mathfrak{J}_R,\mathfrak{d}_{RL}] &= \mathfrak{d}_{RL}\mathfrak{H}_R[\mathfrak{J}_L,\mathfrak{d}_{RL}] \ ,\\ \mathfrak{H}_L[\mathfrak{J}_R,\mathfrak{d}_{LR}\mathfrak{d}_{RL}] &= \mathfrak{d}_{RL}\mathfrak{H}_R[\mathfrak{J}_L,\mathfrak{d}_{LR}\mathfrak{d}_{RL}] \ ,\end{aligned} \tag{4.3.12}$$

which is just an extension of the already imposed (4.3.2) to the $\mathfrak{d}$-algebra ingredients above. These equations can then, with a little bit of algebra, be combined: Multiplying (4.3.11) with $-i\mathfrak{H}_L$, we get

$$-i\mathfrak{H}_L[\mathfrak{J}_R,\mathfrak{d}_{LR}\mathfrak{d}_{RL}] = \underbrace{-i\mathfrak{H}_L[\mathfrak{J}_R,\mathfrak{d}_{LR}]\mathfrak{d}_{RL}}_{①}\underbrace{-i\mathfrak{H}_L\mathfrak{d}_{LR}[\mathfrak{J}_R,\mathfrak{d}_{RL}]}_{②} \ . \tag{4.3.13}$$

The term ① can be easily evaluated by (4.3.4) directly, while for the term ② we need a handedness-swap of (4.3.4), i.e.

$$-i\mathfrak{H}_R[\mathfrak{J}_L,\mathfrak{d}_{RL}] + \mathfrak{H}_L\mathfrak{d}_{LR}\Phi_R\mathfrak{d}_{RL} = \mathfrak{H}_R\mathfrak{d}_{RL}\Phi_L \ , \tag{4.3.14}$$

together with the first equation in (4.3.12) in order to get the wanted handedness of the boost generator and indices of the $\mathfrak{d}$, giving us in total

$$-i\mathfrak{H}_L[\mathfrak{J}_R,\mathfrak{d}_{LR}\mathfrak{d}_{RL}] = \mathfrak{d}_{LR}\mathfrak{d}_{RL}\mathfrak{H}_L\Phi_R(1-\mathfrak{d}_{LR}\mathfrak{d}_{RL}) \ , \tag{4.3.15}$$

and similarly for its handedness-swap. By the second equation of (4.3.12), we know that $\mathfrak{H}_L[\mathfrak{J}_R,\mathfrak{d}_{LR}\mathfrak{d}_{RL}]-\mathfrak{d}_{RL}\mathfrak{H}_R[\mathfrak{J}_L,\mathfrak{d}_{LR}\mathfrak{d}_{RL}]=0$, which when evaluated leads us to the equation

$$\underbrace{\mathfrak{d}_{LR}\mathfrak{d}_{RL}(1-\mathfrak{d}_{LR}\mathfrak{d}_{RL})}_{\boxed{1}}\underbrace{(\mathfrak{H}_L\Phi_R-\mathfrak{d}_{RL}\Phi_L\mathfrak{H}_R)}_{\boxed{2}} = 0 \ , \tag{4.3.16}$$

and the analogous equation for the handedness-swap. As we are operating in an integral domain, we can look at the factors $\boxed{1}$, $\boxed{2}$ separately. The above consistency conditions can be simultaneously fulfilled for $\mathfrak{d}_{LR}=0$, $\mathfrak{d}_{RL}=0$ or $(1-\mathfrak{d}_{LR}\mathfrak{d}_{RL})=0$ if we require the vanishing of $\boxed{1}$. One can show that no further solutions for $\mathfrak{d}_{AB}$ come from requiring simultaneously

$$\begin{aligned}\mathfrak{H}_L\Phi_R\mathfrak{d}_{RL}-\mathfrak{H}_R\Phi_L &= 0 \ ,\\ \mathfrak{H}_R\Phi_L\mathfrak{d}_{LR}-\mathfrak{H}_L\Phi_R &= 0 \ ,\end{aligned} \tag{4.3.17}$$

which would originate from assuming $\boxed{1}\neq 0$: By taking e.g. the first equation in (4.3.17) and left-multiplying them by $\mathfrak{d}_{LR}$, we get:

$$\mathfrak{H}_L\Phi_R\mathfrak{d}_{RL}\mathfrak{d}_{LR} = \mathfrak{H}_R\Phi_L\mathfrak{d}_{LR} \overset{(*)}{=} \mathfrak{H}_L\Phi_R \Rightarrow \mathfrak{d}_{RL}\mathfrak{d}_{LR} = 1 \ , \tag{4.3.18}$$

where we used the second equation of (4.3.17) after the equality $\overset{(*)}{=}$ and always assume that $\mathfrak{d}$ commute with the central elements (and consequently, Φ). By the deduction that $\mathfrak{d}_{RL}\mathfrak{d}_{LR}=1$, we proved our initial claim that no further solutions arise.

So far we have found five solutions, and the defining equations are relatively transparent if we assume one of the $\mathfrak{d}_{AB}$ to vanish. Let us see if there exist any other and less obvious cases that fulfil the condition $\mathfrak{d}_{LR}\mathfrak{d}_{RL}=1$. Then, the constraining equation reduces to

$$\mathfrak{H}_L[\mathfrak{J}_R,\mathfrak{d}_{LR}] = i\mathfrak{H}_L\mathfrak{d}_{LR}\Phi_R - i\mathfrak{H}_R\Phi_L \ . \tag{4.3.19}$$

We can now define an auxiliary $\mathfrak{d}$ variable via $\mathfrak{d}_{LR} = \hat{\mathfrak{d}}_{LR}\mathfrak{H}_R$. Together with the above equation, we then also have

$$-i\mathfrak{H}_L[\mathfrak{J}_R, \mathfrak{d}_{LR}] = -i\mathfrak{H}_L\mathfrak{H}_R[\mathfrak{J}_R, \hat{\mathfrak{d}}_{LR}] + \mathfrak{H}_L\mathfrak{H}_R\hat{\mathfrak{d}}_{LR}\Phi_R = \mathfrak{H}_L\mathfrak{H}_R\hat{\mathfrak{d}}_{LR}\Phi_R - \mathfrak{H}_R\Phi_L \ . \tag{4.3.20}$$

Therefore, $i\mathfrak{H}_L[\mathfrak{J}_R, \hat{\mathfrak{d}}_{LR}] = \Phi_L$ notably simplifying the constraint. Using Jacobi identities or Leibniz rules, one can then also prove that $\mathfrak{d}_{LR}\mathfrak{H}_L$ is central with respect to all elements of the algebra including the boosts,

$$\begin{aligned}\mathfrak{H}_R[\mathfrak{J}_L, \hat{\mathfrak{d}}_{LR}\mathfrak{H}_L] &= \mathfrak{H}_R[\mathfrak{J}_L, \hat{\mathfrak{d}}_{LR}]\mathfrak{H}_L + \mathfrak{H}_R\hat{\mathfrak{d}}_{LR}[\mathfrak{J}_L, \mathfrak{H}_L] = \hat{\mathfrak{d}}_{LR}\mathfrak{H}_L[\mathfrak{J}_R, \hat{\mathfrak{d}}_{LR}]\mathfrak{H}_L + i\mathfrak{H}_R\hat{\mathfrak{d}}_{LR}\mathfrak{H}_L\Phi_L \\ &= -i\hat{\mathfrak{d}}_{LR}\mathfrak{H}_R\Phi_L\mathfrak{H}_L + i\mathfrak{H}_R\hat{\mathfrak{d}}_{LR}\mathfrak{H}_L\Phi_L = 0 \ .\end{aligned} \tag{4.3.21}$$

From this centrality property, we deduce that $\hat{\mathfrak{d}}_{LR}\mathfrak{H}_L$ needs to be proportional to the identity, with a proportionality constant we shall call $\zeta \in \mathbb{C}$. This then gives us the relation,

$$\mathfrak{H}_L\mathfrak{d}_{LR} = \zeta\mathfrak{H}_R \ , \tag{4.3.22}$$

which constitutes another solution. This is leaving us with six possible solutions, which we classify as:

$$\textit{separable} \text{ algebras} = \begin{cases} \mathfrak{B}_1^s : \ \mathfrak{d}_{AB} = 0 \\ \mathfrak{B}_2^s : \ \mathfrak{d}_{LR} = 0 \ \textit{and} \ \mathfrak{d}_{RL} = \zeta\mathfrak{H}_L \\ \mathfrak{B}_{\bar{2}}^s : \ \mathfrak{d}_{RL} = 0 \ \textit{and} \ \mathfrak{d}_{LR} = \zeta\mathfrak{H}_R \end{cases}$$

$$\textit{differential} \text{ algebras} = \begin{cases} \mathfrak{B}_1^d : \ \mathfrak{d}_{AB} = +1 \\ \mathfrak{B}_{\bar{1}}^d : \ \mathfrak{d}_{AB} = -1 \\ \mathfrak{B}_2^d : \ \mathfrak{H}_L\mathfrak{d}_{LR} = \zeta\mathfrak{H}_R \end{cases} \ .$$

This is a nomenclature that lends itself to our situation nicely, *separable* and *differential*, but we shall go into more detail on that in due time.[18]

Before delving into all of these algebras, and in spite of the fact that we want to lay our focus on a representation-independent analysis, we shall describe in greater detail what is arguably the most important representation of our algebra - the differential representation:

$$\mathfrak{J}_A = i\mathfrak{H}_A\frac{d}{dp_A} \ . \tag{4.3.23}$$

The differential is to be understood like a material derivative e.g. from fluid mechanics, in the sense that

$$\frac{d}{dp_R} = \frac{\partial}{\partial p_R} + \frac{dp_L}{dp_R}\frac{\partial}{\partial p_L} \ . \tag{4.3.24}$$

$\mathfrak{d}_{AB}$, $\mathfrak{P}$ and $\mathfrak{K}$ depend explicitly on both momenta, for which this representation is convenient. Thinking of the boost as some kind of decorated derivative, we can understand

$$\Phi_A = \mathfrak{H}'_A = \frac{d\mathfrak{H}_A}{dp_A} \ , \tag{4.3.25}$$

and $\mathfrak{d}_{LR}$ and $\mathfrak{d}_{RL}$ as nothing but momentum derivatives of generators. In addition, $\mathfrak{d}_{AB}$ assume the rôle of Jacobians. In this sense, we can also display the material derivative nature of the boost as

$$\frac{d}{dp_R} = \frac{\partial}{\partial p_R} + \mathfrak{d}_{RL}\frac{\partial}{\partial p_L} \ . \tag{4.3.26}$$

[18] While the $\mathfrak{B}_2^d$ can consistently be formulated for any value of $\zeta \in \mathbb{C}$, for it to be a true differential algebra, we need to impose $\zeta^2 = 1$.

In this representation, we can also write our postulated starting equation in (4.3.4) and its $L \leftrightarrow R$ symmetric as

$$\mathfrak{H}_L\mathfrak{H}_R\frac{d\mathfrak{d}_{LR}}{dp_R} = (\mathfrak{H}_L\Phi_R - \mathfrak{H}_R\Phi_L\mathfrak{d}_{RL})\,\mathfrak{d}_{LR}\ , \qquad \mathfrak{H}_L\mathfrak{H}_R\frac{d\mathfrak{d}_{RL}}{dp_L} = (\mathfrak{H}_R\Phi_L - \mathfrak{H}_L\Phi_R\mathfrak{d}_{LR})\,\mathfrak{d}_{RL}\ . \tag{4.3.27}$$

The six solutions naturally acquire an induced representation in this way. $\mathfrak{B}_1^s$ is the case where p_R and p_L are independent of each other. $\mathfrak{B}_2^s$ and $\mathfrak{B}_{\bar{2}}^s$ imply instead $p_R = \pm p_L$ and $\mathfrak{H}_R(p_R = \pm p_L) = \lambda\mathfrak{H}_L$, for λ constant.

Furthermore, we can infer the algebraic action of the outer symmetry generators on the boost operator within this differential framework, since it reduces to computing the action of the outer automorphisms on $\mathfrak{H}_A$, as e.g.

$$[t_-^\rho, \mathfrak{J}_L] = [t_-^\rho, i\mathfrak{H}_L\frac{d}{dp_L}] = i[t_-^\rho, \mathfrak{H}_L]\frac{d}{dp_L}\ , \tag{4.3.28}$$

and where we have $[t_-^\rho, \mathfrak{H}_L] = \mathfrak{P}$.

$[\mathfrak{J}_L, \mathfrak{J}_R]$ should vanish in this representation too, as we generally stated, and we quickly want to demonstrate that this is indeed the case. The differential representation for the boost features a material derivative, and as $\mathfrak{d}_{AB}$ depends on the momenta p_A, p_B, $[\mathfrak{J}_L, \mathfrak{J}_R]$ also features such derivatives, as for example (with (4.3.25) in mind)

$$[\mathfrak{J}_L, \mathfrak{J}_R]\Big|_{\partial_L \text{ coeff.}} = -\mathfrak{H}_L\mathfrak{H}_R(\partial_{p_L}\mathfrak{d}_{RL}) - \mathfrak{H}_L\mathfrak{d}_{LR}(\partial_{p_R}\mathfrak{H}_R)\mathfrak{d}_{RL} - \mathfrak{H}_L\mathfrak{d}_{LR}\mathfrak{H}_R(\partial_{p_R}\mathfrak{d}_{RL}) + \mathfrak{H}_R\mathfrak{d}_{RL}(\partial_{p_L}\mathfrak{H}_L), \tag{4.3.29}$$

which is exactly the right-hand equation (4.3.27) after expressing the material derivative in terms of partial derivatives - and thus vanishes. Analogously, the vanishing of the ∂_{p_R} coefficient can be proven by a $L \leftrightarrow R$ handedness exchange.

Above, we were able to see that a decisive difference of the material derivative with respect to the ordinary (holonomic) partial derivatives is that two material derivatives might not commute due to the possible momentum-dependence of $\mathfrak{d}_{AB}$. However, this in turn is an essential ingredient for the vanishing of $[\mathfrak{J}_L, \mathfrak{J}_R] = 0$.

The Different Algebras

We have established the six different solutions to our algebraic problem, and divided them into two different categories: If (at least) one of the $\mathfrak{d}_{AB}$ vanishes, we are dealing with a *separable* algebra, and *differential* if it fulfils the relation $\mathfrak{d}_{AB}\mathfrak{d}_{BA} = 1$. We shall now examine more closely the cases we have established.

Separable Algebras

We shall start our closer inspection with the $\mathfrak{B}_1^s$ case. The algebraic relations are as follows:

$$\begin{aligned}
[\mathfrak{J}_A, p_A] &= i\mathfrak{H}_A\ , & [\mathfrak{J}_A, \mathfrak{H}_A] &= i\mathfrak{H}_A\Phi_A\ ,\\
[\mathfrak{J}_L, p_L] &= \mathfrak{H}_L\ , & [\mathfrak{J}_R, p_R] &= \mathfrak{H}_R\ ,\\
[\mathfrak{J}_A, \mathfrak{Q}_A] &= \phi_A^Q\,\mathfrak{Q}_A\ , & [\mathfrak{J}_A, \mathfrak{S}_A] &= \phi_A^S\,\mathfrak{S}_A\ ,\\
[\mathfrak{J}_L, X_R] &= 0\ , & [\mathfrak{J}_R, X_L] &= 0\ ,\\
[\mathfrak{J}_A, \mathfrak{P}] &= \phi_A^Q\mathfrak{P}\ , & [\mathfrak{J}_A, \mathfrak{K}] &= \phi_A^S\mathfrak{K}\ ,
\end{aligned} \tag{4.3.30}$$

where X_A is any generator with well-defined handedness A. In the language of the differential representation, we can see this case as p_L and p_R being wholly uncorrelated, making all Jacobians (and cross commutators, by consequence) vanish.

In the first chapter, we spoke at length about the importance of dispersion relations. And as we will see, the separation of this algebra is best explained when we assume relativistic $\mathfrak{H}_A^2 = p_A^2 + m^2$. Focusing on the bosonic part of our algebra spanned by boost generators, momentum generators and energy generators, we see that this subalgebra reduces to a known finite-dimensional Lie algebra, as $p_A = \mathfrak{H}_A \Phi_A$. We then want to compare the case to the standard solvable Lie algebras of dimension 6, and use the notation and classification of [Tur90] with basis choice

$$\begin{aligned} x_{1,2} &= -i\mathfrak{J}_{L,R} \ , \\ n_{1,2} &= \mathfrak{H}_{R,L} - p_{R,L} \ , \\ n_{3,4} &= \mathfrak{H}_{R,L} + p_{R,L} \ , \end{aligned} \tag{4.3.31}$$

where the n_i span the nilradical (which in this case forms an abelian subalgebra) and the x_i form its algebraic complement (that itself does not form a subalgebra). Our algebra corresponds to $N_{6,1}^{\alpha\beta\gamma\delta}$ with $\alpha = \delta = 0$ and $\beta = \gamma = -1$ in the notation of [Tur90]. $N_{6,1}^{\alpha\beta\gamma\delta}$, however, is indecomposable for $\gamma^2 + \delta^2 \neq 0$ and $\alpha\beta \neq 0$, which is not our case and implies that our algebra corresponds to a direct sum of the 3-dimensional left-handed and right-handed sides. For $\mathfrak{H}_A \Phi_A \propto [p_A]_q$, this is the case as well, which corresponds to the magnonic dispersion relation by selecting a particular value of q, i.e. $\mathfrak{H}_A^2 = h_A^2 \sin^2 \frac{p_A}{2} + m^2$ with h_A a constant, as we've already elaborated on in the motivation.

As mixed-handed commutators vanish in $\mathfrak{B}_1^s$, the functions ϕ_A^Q and ϕ_A^S (as defined in (4.1.1)) in this setting can be constrained more straightforwardly. The algebraic relations (4.3.30) together with 4.1.2 constrain the expressions for the central elements:

$$i\mathfrak{K}\mathfrak{P}\Phi_A = \mathfrak{K}\left(\phi_A^Q \mathfrak{P}\right) + \mathfrak{P}\left(\phi_A^S \mathfrak{K}\right) = [\mathfrak{J}_A, \mathfrak{P}] + [\mathfrak{J}_A, \mathfrak{K}] \ , \tag{4.3.32}$$

since Φ_A depends on p_A only, while $\mathfrak{P}$ and $\mathfrak{K}$ can depend on both momenta. However, the Jacobi identity involving two boosts of different handednesses and one supercharge gives the restriction

$$[\mathfrak{J}_A, \phi_B^Q] = [\mathfrak{J}_A, \phi_B^S] = 0 \ , \tag{4.3.33}$$

for $A \neq B$. Assuming we can write the boost to be proportional to a derivative of the momentum, this can be interpreted as the functions ϕ_A^S and ϕ_A^Q only depending on the momentum p_A. This forces the central elements to be separable in the momenta, i.e. $\mathfrak{P} = \mathfrak{P}_R \mathfrak{P}_L$ and $\mathfrak{K} = \mathfrak{K}_R \mathfrak{K}_L$. In the differential representation of the boost, this is immediate:

$$i\mathfrak{H}_A \frac{d}{dp_A}\mathfrak{P} = [\mathfrak{J}_A, \mathfrak{P}] = \phi_A^Q \mathfrak{P} \implies i\frac{d}{dp_A}\log\mathfrak{P} = \underbrace{\mathfrak{H}_A^{-1}\phi_A^Q}_{\textit{only} \text{ depends on } p_A} \ . \tag{4.3.34}$$

The same argument can be made without referring to this assumption, as (4.3.33) still holds since the adjoint action of $\mathfrak{J}_A$ only vanishes on p_A-independent elements because of $\mathfrak{d}_{AB} = 0$. Thus, the same argument follows as $\mathfrak{J}_A$ vanishes on $\mathfrak{P}^{-1}[\mathfrak{J}_B, \mathfrak{P}]$ for $A \neq B$.

Sadly, the central elements in AdS_3 physics are $\propto \sin(\frac{p_L + p_R}{4})$, thus the $\mathfrak{B}_1^s$ algebra is not suitable for any serious application in this regard.

For the $\mathfrak{B}_2^s$ algebra, we can see that this case can be transformed back to the $\mathfrak{B}_1^s$ algebra by redefining one of the boosts in terms of the $\mathfrak{B}_1^s$-boosts $\mathfrak{J}_A^0$

$$\hat{\mathfrak{J}}_R = \mathfrak{J}_R^0 - \zeta \mathfrak{H}_R \mathfrak{J}_L^0 \ , \tag{4.3.35}$$

so all of the reasoning applies here with minimal changes. Similarly can be done for the $\mathfrak{B}_2^s$ with the redefinition

$$\hat{\mathfrak{J}}_L = \mathfrak{J}_L^0 - \zeta \mathfrak{H}_L \mathfrak{J}_R^0 \ . \tag{4.3.36}$$

In all of the $\mathfrak{B}^s$ algebras, it is fairly easy to see that $[\mathfrak{J}_A, \mathfrak{J}_B] = 0$, as mixed commutators vanish for the "maternal" $\mathfrak{B}_1^s$ algebra.

The $\mathfrak{B}_{1,\bar{1}}^d$ algebras

From the differential perspective, $\mathfrak{d}_{AB} = \pm 1$ implies that we have either $p_L = p_R$ or $p_L = -p_R$. This means that either the two copies built on each $\mathfrak{su}(1|1)$ are exactly the same or one is the parity-transformed of the other. This solution only exists provided

$$\mathfrak{H}_R\left[p_R(p_L)\right] = \mathfrak{d}_{LR} \frac{h_R}{h_L} \mathfrak{H}_L(p_L) \ , \tag{4.3.37}$$

with h_A constants such that $\text{sign}(\mathfrak{d}_{AB} \frac{h_B}{h_A}) = +1$, as can be easily understood in the framework of the differential representation. After performing the identifications (4.3.37), the algebra can be simplified to

$$\begin{aligned}
\{\mathfrak{Q}_A, \mathfrak{S}_A\} &= \mathfrak{H}_A \ , \quad [\mathfrak{J}_A, p_A] = i\mathfrak{H}_A \ , \quad [\mathfrak{J}_A, \mathfrak{H}_A] = i\mathfrak{H}_A \Phi_A \ , \\
[\mathfrak{J}_A, \mathfrak{Q}_A] &= \phi_A^Q \mathfrak{Q}_A \ , \qquad [\mathfrak{J}_A, \mathfrak{S}_A] = \phi_A^S \mathfrak{S}_A \ , \\
\{\mathfrak{Q}_L, \mathfrak{Q}_R\} &= \mathfrak{P} \ , \qquad \{\mathfrak{S}_L, \mathfrak{S}_R\} = \mathfrak{K} \ , \\
h_A[\mathfrak{J}_B, \mathfrak{P}] &= \left(h_L \phi_R^Q + h_R \phi_L^Q\right) \mathfrak{P} \ , \qquad h_A[\mathfrak{J}_B, \mathfrak{K}] = \left(h_L \phi_R^S + h_R \phi_L^S\right) \mathfrak{K} \ , \\
[\mathfrak{J}_A, X_B] &= \frac{h_A}{h_B} [\mathfrak{J}_B, X_B] \ ,
\end{aligned} \tag{4.3.38}$$

with $A, B = L, R$, $A \neq B$, and X being any generator with well-defined handedness. Here, we explicitly have made use of $h_A \mathfrak{H}_B = \mathfrak{d}_{AB} h_B \mathfrak{H}_A$. The above relations also imply $[\mathfrak{J}_L, \mathfrak{J}_R] = 0$, as $\mathfrak{J}_L$ and $\mathfrak{J}_R$ also coincide for $\mathfrak{d}_{AB} = \pm 1$.

Once more, concentrating on the 6-dimensional algebra formed by the bosonic generators $\mathfrak{J}_A$, $\mathfrak{H}_A$ and p_A with relativistic dispersion relation $\mathfrak{H}_A^2 = p_A^2 + m^2$, it is now reduced to a 5-dimensional algebra by identifying $\mathfrak{H}_R = \mathfrak{H}_L = \mathfrak{H}$. We find that this algebra has a 2-dimensional centre spanned by the generators $p_L \mp p_R$ and $\mathfrak{J}_L - \mathfrak{J}_R$. In the algebra of $\mathfrak{B}_1^d$, the remaining 3-dimensional algebra obtained after we mod-out the centre is the following:

$$\left[\frac{\mathfrak{J}_L + \mathfrak{J}_R}{2}, \frac{p_L + p_R}{2} \pm \mathfrak{H}\right] = \pm \left(\frac{p_L + p_R}{2} \pm \mathfrak{H}\right) \ , \tag{4.3.39}$$

which correspond to one of the four irreducible 3-dimensional solvable algebras presented in [dG05]. This construction can be extended to different dispersion relations, thanks to the restriction $\mathfrak{H}_R = \mathfrak{H}_L$, which keeps $p_L \mp p_R$ and $\mathfrak{J}_L - \mathfrak{J}_R$ as central elements.

Differential Representation

Let us briefly examine the differential representation for the $\mathfrak{B}_1^d$ algebra. We are going to choose the following particular dependence on the momenta for the central generators, motivated by the AdS_3 setting:

$$\mathfrak{H}_L = h_L \left|\sin \frac{p_L}{2}\right| \mathbb{1} \ , \qquad \mathfrak{H}_R = \mathfrak{d}_{LR}\, h_R \left|\sin \frac{p_R}{2}\right| \mathbb{1} \ , \qquad \mathfrak{K} \propto \mathfrak{P} \propto \left|\sin\left(\frac{p_L + \mathfrak{d}_{LR} p_R}{4}\right)\right| \mathbb{1} \ , \tag{4.3.40}$$

where $\mathbb{1}$ is the identity matrix in 1|1-dimensional space, and h_L and h_R are positive constants.[19] Moreover, we set the two boost to be equal up to a multiplicative factor $h_R\mathfrak{J}_L \equiv h_L\mathfrak{J}_R$, which is consistent with the constraints imposed by the Jacobi identities specialised to the algebra in question. This, together with the constraint $p_L = \pm p_R$, also implies $\mathfrak{d}_{LR}h_R\mathfrak{H}_L = h_L\mathfrak{H}_R$ for this representational setting.

We shall now assume that $p_L \equiv p_R \equiv p > 0$, in order to simplify the expressions that have absolute values - this is without loss of generality. The supercharges can be rescaled by setting $\phi^Q = \phi^S$, and thus:

$$[\mathfrak{J}_A, \mathfrak{Q}_B] = \frac{i\Phi_A}{2}\,\mathfrak{Q}_B \ , \tag{4.3.41}$$

$$[\mathfrak{J}_A, \mathfrak{S}_B] = \frac{i\Phi_A}{2}\,\mathfrak{S}_B \ , \tag{4.3.42}$$

for arbitrary A, B. We now want to redefine our supercharges in such a way that their momentum dependence can be viewed separately to their purely algebraic features, by:

$$\mathfrak{Q}_L = \sqrt{\alpha h_L \sin\frac{p}{2}}\hat{Q}_L \ , \qquad \mathfrak{Q}_R = \sqrt{\beta h_R \sin\frac{p}{2}}\hat{Q}_R \ ,$$
$$\mathfrak{S}_L = \sqrt{\frac{h_L}{\alpha}\sin\frac{p}{2}}\hat{S}_L \ , \qquad \mathfrak{S}_R = \sqrt{\frac{h_R}{\beta}\sin\frac{p}{2}}\hat{S}_R \ , \tag{4.3.43}$$

where α and β are real numbers and $\{\hat{Q}_L, \hat{S}_L\} = \{\hat{Q}_R, \hat{S}_R\} = \mathbb{1}$ as a consequence of the algebraic relation for $\mathfrak{H}_A$. For the other two central elements $\mathfrak{P}$ and $\mathfrak{K}$, we can proceed with the same motivation and see

$$\mathfrak{P} = \left|h_L h_R \alpha\beta\right| \sin\frac{p}{2}\hat{P} \ , \qquad \mathfrak{K} = \left|h_L h_R \frac{1}{\alpha\beta}\right| \sin\frac{p}{2}\hat{K} \ . \tag{4.3.44}$$

Using the fermionic anticommutation relations and the fact that $\hat{P}, \hat{K} \propto \mathbb{1}$ from centrality, we can find constants γ and η such that we can write

$$\{\hat{Q}_L, \hat{Q}_R\} = \hat{P} = \gamma\mathbb{1} \ , \qquad \{\hat{S}_L, \hat{S}_R\} = \hat{K} = \eta\mathbb{1} \ . \tag{4.3.45}$$

We can set one of the parameters, e.g. γ, to be equal to 1 by redefining $\hat{Q}_A \to \frac{1}{\sqrt{\gamma}}\hat{Q}_A$ and $\hat{S}_A \to \sqrt{\gamma}\hat{S}_A$, however, there is no way to simultaneously be able to set $\eta = 1$ and $\gamma = 1$. Recalling the anticommutation relations of our algebra, one can see that η is connected to the shortening condition: More concretely, $\eta = 1$ is a precondition for the representation to be short. Letting η thus be unconstrained, the hatted quantities give rise to the following relations for any values of x and y:

$$\{(1 + x\eta)\hat{Q}_L - (1 + x)\hat{S}_R, x\hat{S}_L + \hat{Q}_R\} = 0 \ , \tag{4.3.46}$$

$$\{y\hat{Q}_L + \hat{S}_R, (y + 1)\hat{S}_L - (y + \eta)\hat{Q}_R\} = 0 \ . \tag{4.3.47}$$

Therefore, the relevant relations in this representation are:[20]

$$\left[\mathfrak{J}_L + \mathfrak{J}_R, \frac{p_L + p_R}{2}\right] = \mathfrak{H}_L + \mathfrak{H}_R \ , \qquad [\mathfrak{J}_L + \mathfrak{J}_R, \mathfrak{H}_L + \mathfrak{H}_R] = i(\mathfrak{H}_L + \mathfrak{H}_R)(\Phi_L + \Phi_R) \ ,$$
$$\{\hat{Q}_L, \hat{S}_L\} = \{\hat{Q}_R, \hat{S}_R\} = \{\hat{Q}_L, \hat{Q}_R\} = \eta^{-1}\{\hat{S}_L, \hat{S}_R\} = \mathbb{1} \ , \tag{4.3.48}$$

which constitutes a complete separation of the $\mathfrak{su}(1|1)$ algebra from the modified Poincaré algebra, as we initially expected. η here is chosen such that the anti-commutation relations can be normalized in this way, making α and β obsolete.

[19] $h_L + h_R$ is usually identified as a coupling constant appearing in the dispersion relation of the fundamental excitations in the context of the AdS_3/CFT_2 duality.

[20] We could just as well state the algebra without reference to handedness for this particular case.

The $\mathfrak{B}_2^d$ algebra

Using our paramount equation (4.3.2) for the $\mathfrak{B}_2^d$ algebra, we arrive at:

$$[\mathfrak{J}_A, X_B] = \Big(\zeta + (1-\zeta)\mathfrak{d}_{AB}\Big)[\mathfrak{J}_B, X_B] \ , \tag{4.3.49}$$

for any generator X and any A, B. No linear combination of p_L and p_R is central, thus our momentum centre is empty. However, we can construct the boost operators $\mathfrak{J}_L$ and $\mathfrak{J}_R$ of this algebra from $\mathfrak{J}_L^0$ and $\mathfrak{J}_R^0$ from the $\mathfrak{B}_1^s$ algebra by making the following identifications:

$$\begin{aligned} \mathfrak{J}_L &= \mathfrak{J}_L^0 + \zeta\mathfrak{J}_R^0 \ , \\ \mathfrak{J}_R &= \mathfrak{J}_R^0 + \zeta\mathfrak{J}_L^0 \ . \end{aligned} \tag{4.3.50}$$

The above also implies $[\mathfrak{J}_L, \mathfrak{J}_R] = 0$, as we can just re-express $[\mathfrak{J}_L, \mathfrak{J}_R]$ in terms of $[\mathfrak{J}_L^0, \mathfrak{J}_R^0] = 0$.

We shall omit the discussion of the differential representation for this algebra, as it follows from the considerations that we have already made in this regard.

4.4 The Coproduct Map for the Boost Operator for the $\mathfrak{B}_{1,\bar{1}}^d$ Algebras

In the previous chapter within the framework of the $1+1$-dimensional short representation, we have already acquainted ourselves with the intricacies and non-trivial structure that arise when trying to construct and understand the coproduct of the boost. In this chapter, we shall do the same from a representation-independent point of view. As we will see, we have to go beyond the elements of our algebra in our construction, and we have to make use of elements of the $GL(2)^2$ outer automorphism generators of $\mathfrak{su}(1|1)^2_{c.e.}$. This is in analogy to what has been observed in [Hec18], but in the study of the universal R-matrix of $\mathfrak{psu}(2|2)_{c.e.}$ of the AdS_5 scattering problem. There, it also was found that it is essential to utilise the generators associated to the outer automorphisms, which are dual in the sense of the non-degenerate Killing form to the central elements.

4.4.1 The Braided Energy Coproduct Case

In the case of our energy coproduct being braided, we start off with the ansatz

$$\Delta\mathfrak{S}_L = \mathfrak{S}_L \otimes e^{ip_L/4} + e^{-ip_L/4} \otimes \mathfrak{S}_L \ , \qquad \Delta\mathfrak{Q}_L = \mathfrak{Q}_L \otimes e^{ip_L/4} + e^{-ip_L/4} \otimes \mathfrak{Q}_L \ , \tag{4.4.1}$$

with similar coproducts for the right supercharges, with p_L substituted by $\pm p_R$. From this, our central elements are forced to take the form

$$\begin{aligned} \mathfrak{H}_L \propto \Big|e^{ip_L/2} - e^{-ip_L/2}\Big|\mathbb{1} \qquad \mathfrak{H}_R \propto \Big|e^{ip_R/2} - e^{-ip_R/2}\Big|\mathbb{1} \ , \\ \mathfrak{P} \propto \mathfrak{K} \propto \big(e^{i(p_L \pm p_R)/4} - e^{-i(p_L \pm p_R)/4}\big)\mathbb{1} \ , \end{aligned} \tag{4.4.2}$$

where the $\pm$ corresponds to the sign of the respective R-handed coproduct choice. These constraints are due to the fact that central generators of the algebra need to be of cocommutative form.

In each of our algebras, we face different challenges regarding the coproduct prescription. In $\mathfrak{B}_1^s$, $\mathfrak{P}$ and $\mathfrak{K}$ need to be separable to be consistent, while the coproduct imposes them to be a trigonometric function of the sum or difference of the two momenta. These two restrictions together make the situation a bit intricate, but a simple way to simultaneously accommodate for them is to set them to zero and consider

the algebra $\mathfrak{d}_{AB} = 0$ as separate $\mathfrak{su}(1|1)$ algebras. Furthermore, we run into trouble for $p_L \pm p_R = 0$, but this is just a consequence of $\Delta\mathfrak{P}$ and $\Delta\mathfrak{K}$ becoming trivial in such cases, therefore eliminating the restriction that allowed to fix them, as we will see later.

Looking where we came from, in the setting of the representation in the previous chapter with $\mathfrak{B}_1^d$ and $\mathfrak{S}_R \equiv \mathfrak{Q}_L, \mathfrak{S}_L \equiv \mathfrak{Q}_R$, we had

$$\Delta\mathfrak{J} = \Delta_0\mathfrak{J} + \frac{e^{-\frac{i}{4}p} \otimes e^{\frac{i}{4}p}}{4} \left[\mathfrak{S} \otimes \mathfrak{Q} + \mathfrak{Q} \otimes \mathfrak{S}\right] \ , \tag{4.4.3}$$

where the derivative part is given by $\Delta_0\mathfrak{J} = \mathfrak{J}_A \otimes \cos\frac{p}{2} + \cos\frac{p}{2} \otimes \mathfrak{J}_A$.

This map did its job within the borders of the short representation, but for a generic representation, the following, unfortunately, does not vanish any more

$$\left[\left(\mathfrak{S}_L \otimes \mathfrak{Q}_L + \mathfrak{Q}_L \otimes \mathfrak{S}_L\right), \Delta\mathfrak{Q}_R\right] = e^{-\frac{i}{4}p_R}\mathfrak{S}_L \otimes \mathfrak{P} + \mathfrak{P} \otimes e^{\frac{i}{4}p_R}\mathfrak{S}_L \ . \tag{4.4.4}$$

However, when computing the commutator of a coproduct of the boost with $\Delta\mathfrak{Q}_R$, we would want the terms involving $\mathfrak{S}_L$ to disappear completely. Unfortunately, no matter how creative we try to be with the tools and ingredients provided to us from within the algebra $\mathfrak{su}(1|1)^2$, we cannot find a remedy for this. Thus, we need to go beyond that: We need to make use of the outer automorphisms to get rid of the unwanted contributions:

$$\mathfrak{tail}_L = \mathfrak{S}_L \otimes \mathfrak{Q}_L + \mathfrak{Q}_L \otimes \mathfrak{S}_L - \alpha_R \left(\mathfrak{P} \otimes t_+^\rho + \mathfrak{K} \otimes t_+^\lambda\right) - \beta_R \left(t_+^\rho \otimes \mathfrak{P} + t_+^\lambda \otimes \mathfrak{K}\right) \ , \tag{4.4.5}$$

where $\alpha_A = -e^{\frac{i}{4}p_A} \otimes e^{\frac{i}{4}p_A}$ and $\beta_A = e^{-\frac{i}{4}p_A} \otimes e^{-\frac{i}{4}p_A}$. Now, the situation seems to be rectified: The commutators with all the labelled R fermionic generators vanish and the commutator with L fermionic generators, which were already zero, are not modified. For the right-handed copy of the algebra, we can similarly write:

$$\mathfrak{tail}_R = \mathfrak{S}_R \otimes \mathfrak{Q}_R + \mathfrak{Q}_R \otimes \mathfrak{S}_R - \alpha_L \left(\mathfrak{P} \otimes t_-^\lambda + \mathfrak{K} \otimes t_-^\rho\right) - \beta_L \left(t_-^A \otimes \mathfrak{P} + t_-^\rho \otimes \mathfrak{K}\right) \ . \tag{4.4.6}$$

This expression commutes with the two labelled L fermionic generators, and the two ingredients we just introduced will form a centrepiece in our tactic to tackle to problem of the $\mathfrak{d}_{AB} = \pm 1$ construction.

The $\mathfrak{B}_1^s$ Algebra

The $\mathfrak{B}_1^s$ algebra will be our first focus point, as the restriction $\mathfrak{P} = \mathfrak{K} = 0$ completely separates the two algebras. This makes our job easier: We can just consider equation 4.4.3 and write the appropriate subindices. Mostly for purposes that will be clear later, we will upgrade the fermionic tail with the expressions we wrote above even though the new terms give no contribution for the case at hand:

$$\begin{aligned}
\Delta\mathfrak{J}_L(\mathfrak{d}_{AB} = 0) &= \Delta_0\mathfrak{J}_L(\mathfrak{d}_{AB} = 0) + \frac{e^{-ip/4} \otimes e^{ip/4}}{4}\mathfrak{tail}_L \ , \\
\Delta\mathfrak{J}_R(\mathfrak{d}_{AB} = 0) &= \Delta_0\mathfrak{J}_R(\mathfrak{d}_{AB} = 0) + \frac{e^{-ip/4} \otimes e^{ip/4}}{4}\mathfrak{tail}_R \ .
\end{aligned} \tag{4.4.7}$$

The $\mathfrak{B}_1^d$ Algebra

Let us now focus on the $\mathfrak{B}_1^d$ algebra. By the identifications that the algebra allows us to make, we can set $\mathfrak{J}_L = \mathfrak{J}_R = \mathfrak{J}$, then, the action of $\mathfrak{J}$ is perfectly analogous to what the action of $\mathfrak{J}_L + \mathfrak{J}_R$ is in the case where $\mathfrak{d}_{LR} = \mathfrak{d}_{RL} = 0$, which we just discussed. At the level of the algebra, this relation is unproblematic. At the level of the Hopf algebra, the situation changes: The $\mathfrak{d}_{LR} = \mathfrak{d}_{RL} = 0$ case

exhibits problems between coproducts and the constraint of the central elements, so we are forced to set the latter to zero. However, if we want to draw some parallels between how the boost operators act for the different values of $\mathfrak{d}_{AB}$ rather than redevelop all ingredients and tools for each algebraic case from scratch, the upgraded tails we used in equation (4.4.7) are necessary. One needs to be aware that by simply adding two $\mathfrak{d}_{LR} = \mathfrak{d}_{RL} = 0$ boost coproducts (or boost coproducts of any kind, for that matter), the derivative term would appear twice, hence it is necessary to subtract the so-obtained result by one $\Delta_0\mathfrak{J}$. This happens because at the end we have to identify $p_L = p_R$, which makes it appear twofold. Therefore, the combination of generators that makes the coproduct map a homomorphism and that reduces to the correct expression for the short representation is

$$\Delta\mathfrak{J}(\mathfrak{d}_{AB} = 1) = \Delta\mathfrak{J}_L(\mathfrak{d}_{AB} = 0) + \Delta\mathfrak{J}_R(\mathfrak{d}_{AB} = 0) - \Delta_0\mathfrak{J} \ , \tag{4.4.8}$$

$$\begin{aligned}\Delta\mathfrak{J} = \Delta_0\mathfrak{J} + \mathfrak{tail}_L + \mathfrak{tail}_R = \Delta_0\mathfrak{J} + \frac{e^{-\frac{i}{4}p}\otimes e^{\frac{i}{4}p}}{4}\{&\mathfrak{S}_L\otimes\mathfrak{Q}_L + \mathfrak{Q}_L\otimes\mathfrak{S}_L + \mathfrak{S}_R\otimes\mathfrak{Q}_R + \mathfrak{Q}_R\otimes\mathfrak{S}_R \\ &-\alpha\left[\mathfrak{P}\otimes(t^\rho_+ + t^\lambda_-) + \mathfrak{K}\otimes(t^\lambda_+ + t^\rho_-)\right] - \beta\left[(t^\rho_+ + t^\lambda_-)\otimes\mathfrak{P} + (t^\lambda_+ + t^\rho_-)\otimes\mathfrak{K}\right]\} \ .\end{aligned} \tag{4.4.9}$$

To illustrate that this does indeed render the results of the $1+1$-dimensional short representation once this representation is chosen, we shall make things more explicit. This representation is potentially problematic as not all the outer automorphism generators are well-defined there, however, the identification $\mathfrak{P} \equiv \mathfrak{H}_A \equiv \mathfrak{K}$ gives us

$$\Delta\mathfrak{J} = \Delta_0\mathfrak{J} + \frac{e^{-\frac{i}{4}p}\otimes e^{\frac{i}{4}p}}{4}\left[2\mathfrak{S}\otimes\mathfrak{Q} + 2\mathfrak{Q}\otimes\mathfrak{S} - \alpha\mathfrak{H}\otimes\mathfrak{T} - \beta\mathfrak{T}\otimes\mathfrak{H}\right] \ , \tag{4.4.10}$$

with $\mathfrak{T} = t^\lambda_+ + t^\lambda_- + t^\rho_+ + t^\rho_-$, which is well-defined in this representation. Indeed, we can easily see that $[\mathfrak{T}, \mathfrak{Q}] = \mathfrak{Q}$ and $[\mathfrak{T}, \mathfrak{S}] = \mathfrak{S}$, which implies that

$$[2\mathfrak{S}\otimes\mathfrak{Q} + 2\mathfrak{Q}\otimes\mathfrak{S} - \alpha\mathfrak{H}\otimes\mathfrak{T} - \beta\mathfrak{T}\otimes\mathfrak{H}, \Delta X] = [\mathfrak{S}\otimes\mathfrak{Q} + \mathfrak{Q}\otimes\mathfrak{S}, \Delta X] \ , \tag{4.4.11}$$

for either $X = \mathfrak{Q}$ or $\mathfrak{S}$. Thus, our coproduct map for the boost for the $\mathfrak{d}_{LR} = \mathfrak{d}_{RL} = 1$ reduces to the appropriate one in this representation, which supports that our result is indeed correct.

The $\mathfrak{B}^d_1$ Algebra

In the $\mathfrak{B}^d_1$ algebra, (4.4.2) appears to fix two of the central elements to be trivial in this case, by virtue of how the momenta are correlated. However, this is not true as the coproduct structure allows us more freedom as the central elements $\mathfrak{P}$ and $\mathfrak{K}$, compared to the $\mathfrak{B}^s_1$ case, now fulfil

$$\begin{aligned}\Delta\mathfrak{P} &= \mathfrak{P}\otimes 1 + 1\otimes\mathfrak{P} \ , \\ \Delta\mathfrak{K} &= \mathfrak{K}\otimes 1 + 1\otimes\mathfrak{K} \ .\end{aligned} \tag{4.4.12}$$

Then, we can fix the four central elements to be

$$\mathfrak{H}_L \propto \mathfrak{H}_R \propto \mathfrak{P} \propto \mathfrak{K} \propto (e^{ip/2} - e^{-ip/2})\mathbb{1} \ , \tag{4.4.13}$$

and a similar approach to the $\mathfrak{B}^s_1$ algebra can be used.

4.4.2 The Unbraided Energy Coproduct Case

The bosonically unbraided coproduct is defined by the following choice of coproducts for the fermionic generators [21]

$$\Delta\mathfrak{Q}_A = \mathfrak{Q}_A\otimes e^{i\frac{p}{4}} + e^{-i\frac{p}{4}}\otimes\mathfrak{Q}_A \ ,$$

[21] Of course, there are many equivalent coproduct choices to choose from, we choose the most natural one.

$$\Delta\mathfrak{S}_A = \mathfrak{S}_A \otimes e^{-i\frac{p}{4}} + e^{i\frac{p}{4}} \otimes \mathfrak{S}_A \ . \tag{4.4.14}$$

In contrast with the bosonically braided case, here the coproduct of the boost also involves the hypercharge operator, as we have already seen in the short representation (thus the universal expression must account for that, also). In order to construct an analogous expression to $\mathfrak{tail}_A$ for this case, we also need to substitute any $\mathfrak{B}$ operator present in the tail of $\Delta\mathfrak{J}_A$ by another operator that acts as the usual hypercharge on A-handed operators, whereas they would have to vanish on operators with the opposite handedness. We call these operators $\mathfrak{B}_A$ and they can be written in terms of the outer automorphisms generators as

$$\begin{aligned} -i\mathfrak{B}_R &= t^\lambda_0 - t^\rho_0 - t^\lambda_3 - t^\rho_3 \ , \\ -i\mathfrak{B}_L &= t^\lambda_0 - t^\rho_0 + t^\lambda_3 + t^\rho_3 \ , \end{aligned} \tag{4.4.15}$$

which allows us to write

$$\begin{aligned} \mathfrak{tail}_L =& G\left[\mathfrak{B}_L \otimes \mathbb{1} - \mathbb{1} \otimes \mathfrak{B}_L\right] + F_+\left[\mathfrak{S}_L \otimes \mathfrak{Q}_L - \beta_R t^\rho_+ \otimes \mathfrak{P} + \beta_R \mathfrak{K} \otimes t^\lambda_+\right] \\ &+ F_-\left[\mathfrak{Q}_L \otimes \mathfrak{S}_L - \alpha_R \mathfrak{P} \otimes t^\rho_+ + \alpha_R t^\lambda_+ \otimes \mathfrak{K}\right] \ , \end{aligned} \tag{4.4.16}$$

$$\begin{aligned} \mathfrak{tail}_R =& G\left[\mathfrak{B}_R \otimes \mathbb{1} - \mathbb{1} \otimes \mathfrak{B}_R\right] + F_+\left[\mathfrak{Q}_R \otimes \mathfrak{S}_R + \alpha_L t^\rho_- \otimes \mathfrak{K} - \alpha_L \mathfrak{P} \otimes t^\lambda_-\right] \\ &+ F_-\left[\mathfrak{S}_R \otimes \mathfrak{Q}_R + \beta_L \mathfrak{K} \otimes t^\rho_- - \beta_L t^\lambda_- \otimes \mathfrak{P}\right] \ , \end{aligned} \tag{4.4.17}$$

where $G, F_\pm$ are suitable functions of the momenta and we once more have $\alpha_A = -e^{\frac{i}{4}p_A} \otimes e^{\frac{i}{4}p_A}$ and $\beta_A = e^{-\frac{i}{4}p_A} \otimes e^{-\frac{i}{4}p_A}$.

Having constructed the $\mathfrak{tail}_A$-equivalent for this fermionic coproduct, we can follow the prescription we used above, giving us

$$\Delta\mathfrak{J}_L(\mathfrak{d}_{AB}=0) = \Delta_0\mathfrak{J}_L(\mathfrak{d}_{AB}=0) + \mathfrak{tail}_L \ , \qquad \Delta\mathfrak{J}_R(\mathfrak{d}_{AB}=0) = \Delta_0\mathfrak{J}_R(\mathfrak{d}_{AB}=0) + \mathfrak{tail}_R \ , \tag{4.4.18}$$

$$\begin{aligned} \Delta\mathfrak{J}(\mathfrak{d}_{AB}=1) =& \Delta\mathfrak{J}_L(\mathfrak{d}_{AB}=0) + \Delta\mathfrak{J}_R(\mathfrak{d}_{AB}=0) - \Delta_0\mathfrak{J} \\ =& \Delta_0\mathfrak{J} + G\left(\mathfrak{B} \otimes \mathbb{1} - \mathbb{1} \otimes \mathfrak{B}\right) \\ &+ F_+\left(\mathfrak{S}_L \otimes \mathfrak{Q}_L + \mathfrak{Q}_R \otimes \mathfrak{S}_R - \beta\left(t^\rho_+ \otimes \mathfrak{P} - \mathfrak{K} \otimes t^\lambda_+\right) - \alpha\left(\mathfrak{P} \otimes t^\lambda_- - t^\rho_- \otimes \mathfrak{K}\right)\right) \\ &+ F_-\left(\mathfrak{Q}_L \otimes \mathfrak{S}_L + \mathfrak{S}_R \otimes \mathfrak{Q}_R - \alpha\left(\mathfrak{P} \otimes t^\rho_+ - t^\lambda_+ \otimes \mathfrak{K}\right) - \beta\left(t^\lambda_- \otimes \mathfrak{P} - \mathfrak{K} \otimes t^\rho_-\right)\right) \ , \end{aligned} \tag{4.4.19}$$

keeping in mind that we have imposed the identifications $p_L = p_R \equiv p$ as well as $\mathfrak{B}_R + \mathfrak{B}_L = \mathfrak{B}$.

With this, we have analysed all the pertinent cases and can conclude this line of investigation.

5 | Deformed AdS_3 in the Landau-Lifshitz Limit

"Es gibt zwei Arten, sein Leben zu leben: entweder so, als wäre nichts ein Wunder, oder so, als wäre alles ein Wunder." ["There are only two ways to live your life. One is as though nothing is a miracle. The other is as though everything is a miracle."]

– Albert Einstein

5.1 The Setting

As was alluded to in the introductory part of this thesis, the powerful tools of integrability have been astoundingly effective and successful in the setting of $AdS_5 \times S^5$, and more recently also within the $AdS_3 \times S^3 \times M_4$ context (we want to cite [BAA+11] for an extensive review on the former). Given this amazing track record, it makes sense to, when looking for specific backgrounds to study, look out for integrability as a feature. One approach to this is to study novel backgrounds with the somewhat starry-eyed hope that they can be proven to be integrable - in the sense that the generated physical theory is integrable - thus having the powerful integrability toolset at one's disposal. However, in the totality of all possible backgrounds to study, integrability is likely to be a rare occurrence, unfortunately (see e.g. [MM74]). A more prudent course of action is to pick a starting point, i.e. an integrable background, that we already analysed and see if we can change it into something new that still exhibits the initial features that we want it to preserve. Following the previous chapters of this thesis, this is a roundabout way of saying that we want to *deform* a background in an integrable way. A methodological way of tackling this is needed, and has been done (see [Kli02], and then [DMV13], [DMV14], [KMY14b], [KMY14a]) - leading to so-called *Yang-Baxter deformed* backgrounds. In contrast to the AdS_5 setting, the $AdS_3 \times S^3 \times T^4$ space admits two independent Yang-Baxter deformations, meaning that two a priori independent deformation parameters appear in the model.[1] Additionally, the $AdS_3 \times S^3 \times T^4$ background is supported by both an RR and an NS-NS flux, which yield an additional deformation parameter (and, like the YB deformations, is integrable [CZ12]). Since both of these ways of deforming an $AdS_3 \times S^3 \times T^4$ background do not depend on one another, we may hope they can also be applied in tandem, leading to an integrable background that is deformed in a three-fold way. This is indeed the case (see e.g. [DHK+19]), and we shall refer to backgrounds deformed in this particular way as *3-deformed*.[2] It is this setting that we will delve into with our analysis that is to follow based on our publication [GW21].

[1] This is due to the (super)coset structure of the embedded superbackground for $AdS_3 \times S^3 \times T^4$, see the introductory chapter for details.

[2] In this exposition of these backgrounds, we will refrain from addressing the question of whether they are suitable supergravity backgrounds. This discussion involves a lot of subtleties, such as the particular and inequivalent fermionic

5.2 The three-deformed $\mathbb{R} \times S^3$ action in the Landau-Lifshitz Limit

We already stated the (bosonic) Polyakov action before, this time however, we add the so-called *Kalb-Ramond* term to it,

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d\tau \int d\sigma \left(\sqrt{-\det \gamma} \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu} + \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu} \right), \tag{5.2.1}$$

where we re-expressed the string tension using a rescaled parameter λ, γ is the worldsheet metric, g the target space metric and B the Kalb-Ramond field. We shall always stick to choosing $\alpha, \beta, \cdots$ as worldsheet coordinate labels, and $\mu, \nu, \cdots$ as target space coordinate labels. Abbreviating worldsheet coordinate derivatives, we shall refer to τ-derivatives of a quantity x as $\dot{x}$, and σ-derivatives as x'. Since B is a two-form, we need to introduce the completely alternating tensor symbol ϵ, where we choose the convention

$$\epsilon^{\tau\sigma} = -\epsilon^{\sigma\tau} = 1. \tag{5.2.2}$$

The metric and B-field we are interested in, g and B, written as infinitesimal form elements, appear in the following way:

$$\begin{aligned} ds^2 = &\frac{1}{F_A}\Big[\frac{1 - q^2\rho^2(1+\rho^2)}{1+\rho^2} \, d\rho^2 - 2q\chi_- \rho(1+\rho^2) \, d\rho \, dt + 2q\chi_+ \rho^3 \, d\rho \, d\psi \\ &- \big(1 + \chi_-^2(1+\rho^2)\big)(1+\rho^2) \, dt^2 + 2\chi_+\chi_- \rho^2(1+\rho^2) \, dt \, d\psi + \rho^2(1 - \rho^2\chi_+^2) \, d\psi^2 \Big], \\ &+ \frac{1}{F_S}\Big[\frac{1 + q^2 r^2(1-r^2)}{1-r^2} \, dr^2 - 2q\chi_- r(1-r^2) \, dr \, d\phi_1 - 2q\chi_+ r^3 \, dr \, d\phi_2 \\ &+ \big(1 + \chi_-^2(1-r^2)\big)(1-r^2) \, d\phi_1^2 + 2\chi_+\chi_- r^2(1-r^2) \, d\phi_1 \, d\phi_2 + r^2(1 + \chi_+^2 r^2) \, d\phi_2^2 \Big] , \end{aligned} \tag{5.2.3}$$

$$\begin{aligned} dB = &-\frac{a\,q}{F_A}(1+\rho^2)\Big[2 - \rho^2 q^2 - (2+\rho^2)\chi_-^2 - \rho^2\chi_+^2\Big] \, dt \wedge d\psi \\ &- \frac{a\,q}{F_S}(1-r^2)\Big[2 + r^2 q^2 + (r^2-2)\chi_-^2 + r^2\chi_+^2\Big] \, d\phi_1 \wedge d\phi_2 , \end{aligned} \tag{5.2.4}$$

where

$$F_A = 1 - \chi_+^2\rho^2 + \chi_-^2(1+\rho^2) - q^2\rho^2(1+\rho^2) , \tag{5.2.5}$$

$$F_S = 1 + \chi_+^2 r^2 + \chi_-^2(1-r^2) + q^2 r^2(1-r^2) , \tag{5.2.6}$$

$$a = \frac{1}{\sqrt{\big(q^2 + \chi_+^2 + \chi_-^2\big)^2 + 4\big(q^2 - \chi_+^2\chi_-^2\big)}} . \tag{5.2.7}$$

Here, $\{t, \rho, \psi\}$ are the coordinates of the AdS_3 part, whereas $\{r, \phi_1, \phi_2\}$ are the ones of the S^3. $\chi_\pm$ are the deformation parameters originating from the two-fold YB deformation, and q the one from the flux-based deformation.

In any study of a classical Lagrangian, looking for conserved charges from the beginning might simplify the analysis to follow. In our case, four of the coordinates, $Q = \{t, \psi, \phi_1, \phi_2\}$, are cyclical, and translations in these variables leave the action invariant, thus quantities of the form $\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ are constant for any $q \in Q$.[3] The conserved charges associated to shifts in angles are associated with angular momenta (S, J_1, J_2), whereas shifts in time as usual correspond to conservation of the energy E.

Dynkin diagram one chooses at the foundation of the YB deformation (for both the two-fold and one-fold case, see [Sei19] and [HS19], respectively) and properties such as unimodularity.

[3] This assumption is correct only if the equations of motion are of Euler-Lagrange form.

Computing these conserved charges (as well as defining calligraphic analogues that are normalised by $\sqrt{\lambda}$), we arrive at the following expressions: [4]

$$\begin{aligned}
E &= \sqrt{\lambda}\mathcal{E} \\
&= \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi}\frac{d\sigma}{F_A}\Big\{q\chi_-\rho(1+\rho^2)\dot{\rho} - \chi_-\chi_+\rho^2(1+\rho^2)\dot{\psi} + [1+\chi_-^2(1+\rho^2)](1+\rho^2)\dot{t} \\
&\qquad - aq(1+\rho^2)[2-\rho^2q^2-(2+\rho^2)\chi_-^2-\rho^2\chi_+^2]\psi'\Big\}, \qquad (5.2.8)\\
S &= \sqrt{\lambda}\mathcal{S} \\
&= \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi}\frac{d\sigma}{F_A}\Big\{-q\chi_+\rho^3\dot{\rho} - \chi_-\chi_+\rho^2(1+\rho^2)\dot{t} - \rho^2(1-\rho^2\chi_+^2)\dot{\psi} \\
&\qquad - aq(1+\rho^2)[2-\rho^2q^2-(2+\rho^2)\chi_-^2-\rho^2\chi_+^2]t'\Big\}, \qquad (5.2.9)\\
J_1 &= \sqrt{\lambda}\mathcal{J}_1 \\
&= \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi}\frac{d\sigma}{F_S}\Big\{-[1+\chi_-^2(1-r^2)](1-r^2)\dot{\phi_1} - \chi_+\chi_-r^2(1-r^2)\dot{\phi_2} + q\chi_-r(1-r^2)\dot{r} \\
&\qquad - aq(1-r^2)[2+r^2q^2+(r^2-2)\chi_-^2+r^2\chi_+^2]\phi_2'\Big\}, \qquad (5.2.10)\\
J_2 &= \sqrt{\lambda}\mathcal{J}_2 \\
&= \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi}\frac{d\sigma}{F_S}\Big\{-r^2(1+\chi_+^2r^2)\dot{\phi_2} - \chi_+\chi_-r^2(1-r^2)\dot{\phi_1} + q\chi_+r^3\dot{r} \\
&\qquad + aq(1-r^2)[2+r^2q^2+(r^2-2)\chi_-^2+r^2\chi_+^2]\phi_1'\Big\}. \qquad (5.2.11)
\end{aligned}$$

We have already mentioned that the T^4 coset factor will not carry any dynamics in our case, there is another simplification we are going to make: We want to truncate the AdS_3 factor by setting $\rho \to 0$, heavily simplifying our metric.[5] Thus, the only coordinate from the AdS_3 part surviving is the time, contributing with a $-dt^2$ factor in the metric. This reduces the dynamical part of our product manifold to $\mathbb{R}\times S^3 \subset \mathrm{AdS}_3\times S^3\times T^4$.

In this study, we want to analyse the S-matrix that arises from the 3-deformed model. This action, rich and intricate as it may be, is too complicated still to analyse in the detail we would like to (see [BMSS20]). Therefore, we will choose an approach that makes physical quantities more computationally accessible - by constructing an effective field theory associated to it. For the case of $\mathrm{AdS}_5\times S^5$, although less rich in deformations, this has already been done in a particular way (see e.g. [ST04] or [KRT04] for related work) that we shall try to imitate called the *Landau-Lifshitz limit*, or LL-limit in short. In this limit, one assumes the total angular momentum, i.e. $J = J_1 + J_2$, and string tension to be large. The construction starts by defining the "fast" and "slow" coordinates of the theory, for instance given by $\phi_1 \mp \phi_2$ for us, respectively, and in the case of $\mathrm{AdS}_5\times S^5$, the leading contribution essentially describes a (classical) ferromagnet.

Before performing any expansion, there is an important topic to discuss. On the worldsheet side, the metric γ is so far undetermined and to be constrained by the action as the other variables are. However, due to our reparametrisation symmetry, we are allowed to pick any particular gauge. While - a priori - the gauge choice should be irrelevant for any physical quantities we compute, it turns out that some gauge choices are more apt than others for our expansion. The simplest and most natural choice of gauge

[4] Here, there was a typo in our original paper [GW21] in the definition of E.

[5] For this to be a consistent truncation, $\rho = 0$ needs to be an admissible solution to the (untruncated) equations of motions of our model. Luckily, in our case this holds true, as one can check by evaluating the equations of motions for ρ.

for γ is the *conformal gauge*, meaning

$$\gamma^{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.2.12}$$

which heavily simplifies the metric part of the Polyakov action to

$$S_g = \frac{\sqrt{\lambda}}{4\pi} \int d\tau \int d\sigma \left(-\dot{X} \cdot \dot{X} + X' \cdot X' \right), \tag{5.2.13}$$

where the $\cdot$-product signifies the implicit contraction with the (curved) target space metric, as it did in the introduction. With the residual conformal diffeomorphism symmetry, we then would also choose $t = \kappa\tau$, exhausting our full reparametrisation freedom, which also fixed $\mathcal{E} = \kappa$ and $\mathcal{S} = 0$. The natural expansion parameter within the conformal gauge is κ, however, we are interested in taking the limit of large J (or $\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2$, equivalently). To do so, we would need to make use of less trivial so-called *non-diagonal uniform gauge* (see [KRT04] for the appearance of this in the $\mathrm{AdS}_5 \times S^5$ case), making computational efforts more difficult and the procedure we described before inadmissible. However, we are bound to make a curious finding: At leading order, the expansions in inverse powers of κ and $\mathcal{J}$ agree (see (2.28) in [KRT04], where $\tilde{\lambda} \propto \kappa^{-2}$):

$$\lambda J^{-2} = \kappa^{-2} + \mathcal{O}(\kappa^{-4}). \tag{5.2.14}$$

This implies that it is pragmatic for us to simply expand the action in conformal gauge in inverse powers of κ and limit our considerations to the leading order terms only. We will follow this approach for the remainder of this chapter.

We have to identify some of the coordinate combinations to be fast and slow to then re-express our Lagrangian accordingly. To this end, we introduce slow coordinates α, β and rewrite[6]

$$\begin{aligned} \phi_1 &= \kappa\tau + \alpha + \beta \,, \\ \phi_2 &= \kappa\tau + \alpha - \beta \,. \end{aligned} \tag{5.2.15}$$

In these coordinates, our Lagrangian takes the following form:

$$\begin{aligned} \frac{4\pi}{\sqrt{\lambda}} F_S \mathcal{L} = {} & \frac{\left[q^2 r^2 \left(r^2 - 1\right) - 1\right] \left(\dot{r}^2 - r'^2\right)}{r^2 - 1} + \left(1 - r^2\right) \left[1 + \chi_-^2 \left(1 - r^2\right)\right] \left[\left(\kappa + \dot{\alpha} + \dot{\beta}\right)^2 - \left(\alpha' + \beta'\right)^2\right] \\ & - 2\chi_- \chi_+ \left(r^2 - 1\right) r^2 \left[(\kappa + \dot{\alpha})^2 - \dot{\beta}^2 - \alpha'^2 + \beta'^2\right] + r^2 \left(1 + \chi_+^2 r^2\right) \left[\left(\kappa + \dot{\alpha} - \dot{\beta}\right)^2 - \left(\alpha' - \beta'\right)^2\right] \\ & - 2aq \left(1 - r^2\right) \left[r^2 \left(\chi_+^2 - \chi_-^2 + q^2\right) + 2\chi_-^2 + 2\right] \left[\beta' \left(\kappa + \dot{\alpha}\right) - \dot{\beta}\alpha'\right] - \kappa^2 F_S \,, \end{aligned} \tag{5.2.16}$$

where we can see (only) the lattermost term to originate from the time contribution from the AdS_3 part.

5.2.1 Leading Order in κ

Rescaling $\tau \to \kappa\tau$ and $\partial_\tau \to \frac{\partial_\tau}{\kappa}$ and expanding (5.2.16) in (inverse) powers of κ, we arrive at the following:

$$\mathcal{L} = \sum_k \frac{\mathcal{L}^{(k)}}{\kappa^n} \,,$$

[6] In the cited literature, it is common to first express the fast coordinates as $u \pm \beta$, and then later fix $u = t - \alpha$, with u exhibiting the fast behaviour. Given the static choice we made above, we shall express our coordinates in their final form right away.

$$\mathcal{L}^{(j<2)} = 0 \ ,$$
$$\mathcal{L}^{(2)} = r^2(1-r^2)\frac{(\chi_+ - \chi_-)^2 + q^2}{F_S} \ , \tag{5.2.17}$$

with odd orders in κ vanishing. We immediately are confronted with an issue: There are no time derivatives appearing at leading order, i.e. in $\mathcal{L}^{(2)}$, and only constant radius solutions would emerge from this alone.

To arrive at the above expression, we have assumed that the deformation parameters $\chi_\pm$ and q are of order κ^0. A priori, this assumption was only justified by convenience, but we now need to make an adjustment to it: In the concrete case above, we want to "downgrade" the leading order contribution to appear in $\mathcal{L}^{(0)}$ only, where we encounter dynamics (τ-derivatives) for the first time. Performing the rescaling

$$\chi_\pm \to \frac{\chi_\pm}{\kappa} \ ,$$
$$q \to \frac{q}{\kappa} \ , \tag{5.2.18}$$

would indeed produce the desired result. However, this assumption would give rise to a different problem: Recall that the B-field term only appears with a factor of aq in front, then the above redefinition would yield

$$aq \simeq \frac{1}{2}, \tag{5.2.19}$$

where the above equality is true in the limit large κ. Instead, the redefinition

$$\chi_\pm \to \frac{\chi_\pm}{\kappa} \ ,$$
$$q \to \frac{q}{\kappa^3} \ , \tag{5.2.20}$$

however, does the trick: The flux term appears now with a tunable parameter at leading order. If we further disregard total derivative terms with respect to time of α in the Lagrangian (as they leave the equations of motion invariant), we arrive at the following expression for the leading order Lagrangian:

$$\mathcal{L}^{(0)} = -2(1-2r^2)\dot{\beta} + \frac{r'^2}{1-r^2} + \alpha'^2 + \beta'^2 + 2(1-2r^2)\alpha'\beta' + 4\tilde{q}(1-r^2)\beta' - (\chi_+ - \chi_-)^2 r^2(1-r^2) + \mathcal{O}\left(\frac{1}{\kappa^2}\right) \ , \tag{5.2.21}$$

where $\tilde{q} = \frac{q}{\sqrt{(\chi_+^2 - \chi_-^2)^2}}$. For the non-linear sigma model that we are studying, there exists an additional set of constraints called the *Virasoro constraints*. In our gauge choice, they can be written as[7]

$$\dot{X} \cdot X' = 0, \tag{5.2.22}$$
$$\frac{1}{2}(\dot{X}^2 + X'^2) = 0. \tag{5.2.23}$$

For our metric and fields, we arrive at:

$$[\alpha' + (1-2r^2)\beta']\kappa = \mathcal{O}\left(\frac{1}{\kappa}\right), \tag{5.2.24}$$
$$\frac{r'^2}{1-r^2} - (\chi_+ - \chi_-)^2 r^2(1-r^2) + 2[\dot{\alpha} + (1-2r^2)\dot{\beta}] + (1-2r^2)[\alpha'^2 + \beta'^2] + 2\alpha'\beta' = \mathcal{O}\left(\frac{1}{\kappa^2}\right) \ . \tag{5.2.25}$$

[7]Here, the prefactor $\frac{1}{2}$ is convention, as that is how it appears for the stress-energy tensor (which is the variation of the action with respect to the metric) to vanish in the bosonic Polyakov framework.

The former of the two equations implies $\alpha' = -(1-2r^2)\beta' + \mathcal{O}\left(\frac{1}{\kappa^2}\right)$, from which (5.2.21) yields upon substitution:

$$\mathcal{L}^{(0)} = -2(1-2r^2)\dot{\beta} + 4r^2(1-r^2)\beta'^2 + 4\tilde{q}(1-r^2)\beta' + \frac{r'^2}{1-r^2} - (\chi_+ - \chi_-)^2 r^2(1-r^2) + \mathcal{O}\left(\frac{1}{\kappa^2}\right) . \tag{5.2.26}$$

We have performed several limits and rescalings to our fields and Lagrangian, it might therefore be prudent to have a sanity check available as to whether what we ended up with is still sensible. Indeed, looking at the result of [KY14] in (2.28), which describes the leading-order action of a fast-moving string in $\mathbb{R} \times S^3$,

$$\mathcal{L} = -\frac{1}{4}\kappa^2 \varkappa^2 \sin^2 2\theta + 2\kappa\left[\dot{\tilde{\varphi}}_1 + \cos 2\theta \dot{\varphi}_2\right] - \theta'^2 - \sin^2 2\theta \varphi_2'^2 , \tag{5.2.27}$$

we can identify the above with (5.2.26) with the identification of $\sin\theta = r$, helped by the fact that then $\sin^2 2\theta = 4r^2(1-r^2)$, $\cos 2\theta = (1-2r^2)$ and $\theta'^2 = \frac{r'^2}{1-r^2}$, and recalling that we need to κ-rescale τ-derivatives and the deformation parameter $\kappa^2\varkappa^2 = \chi_+$, and that $2\dot{\tilde{\varphi}}_1$ is a total time derivative.[8]

We now arrived at a Lagrangian that features two *real* fields (as we were able to completely eliminate α), and ultimately, we would like to compute the S-matrix associated to it. In that light, it is more useful to merge β and r into one single *complex* field via the definition:

$$\begin{aligned} \phi &= \sqrt{1-r^2} e^{2i\beta} , \\ \phi^\dagger &= \sqrt{1-r^2} e^{-2i\beta} . \end{aligned} \tag{5.2.28}$$

Re-expressing (5.2.26) in terms of this new field, we arrive at

$$\begin{aligned} \mathcal{L}^{(0)} = &\frac{i(1-2|\phi|^2)\phi^\dagger\dot{\phi}}{2|\phi|^2} - \frac{i(1-2|\phi|^2)\phi\dot{\phi}^\dagger}{2|\phi|^2} + i\tilde{q}\phi^\dagger\phi' - i\tilde{q}\phi\phi^{\dagger\prime} - (\chi_+ - \chi_-)^2|\phi|^2(1-|\phi|^2) \\ &- \frac{1}{2}(1-|\phi|^2)\phi'\phi^{\dagger\prime} - \frac{(\phi^\dagger)^2(2-|\phi|^2)\phi'^2 + 2\phi'\phi^{\dagger\prime} + \phi^2(2-|\phi|^2)(\phi^{\dagger\prime})^2}{4(1-|\phi|^2)} . \end{aligned} \tag{5.2.29}$$

We started with a real Lagrangian containing real fields only and rewrote them in terms of one complex field, the resulting Lagrangian, however, should still be a real expression in its totality. The is appearing might seem disconcerting, however, one can find that our Lagrangian is of Hermitian form. In the beginning of this chapter, we tried to motivate why expanding in inverse powers of κ is the most natural approach for the worldsheet gauge we chose. With our next goal being the computation of S-matrix elements, it makes sense to choose a formulation for large J and λ, and we want to decompactify the coordinate σ. With large J, one method to do this is by redefining $\sigma \to \frac{\sigma}{J}$ (and keeping the scaling difference of σ and τ, also imposing $\tau \to \frac{\tau}{J^2}$). As a consequence, the deformation parameters receive a rescaling as well:

$$\tilde{q} \to J\tilde{q} \tag{5.2.30}$$

$$\chi_\pm \to J\chi_\pm \tag{5.2.31}$$

Now we will follow the approach from section 3 of [MTT05]: The expansion around $\phi = 0$ corresponds to evaluating the corrections around the BMN solution. As the overall J factor in front of the action in the Landau-Lifshitz limit has the role of the Planck constant, we can choose to expand around $\phi = 0$ by rescaling

$$\phi \to \frac{\phi}{\sqrt{J}} . \tag{5.2.32}$$

[8] Since the authors in [KY14] use a different coordinate parametrisation (e.g. defining their S^3 angles with a minus sign), there are some relative sign differences in two of the terms in their Lagrangian compared to ours.

The power of J that accompanies a given term is then a simple indicator of the order in which any products of ϕ and $\phi^\dagger$ appear, and if we keep contributions up to quartic order in the fields, meaning quadratic order in $\frac{1}{J}$, and adding a total time derivative $-2\dot{\beta}$ to make a form of the kinetic term easier to deal with, we arrive at

$$\begin{aligned} J^{-1}\mathcal{L}^{(0)} =& \left(\phi^\dagger(-i\partial_t + i\tilde{q}\partial_\sigma)\phi - \phi(-i\partial_t + i\tilde{q}\partial_\sigma)\phi^\dagger - (\chi_+ - \chi_-)^2\phi^\dagger\phi - \phi^{\dagger\prime}\phi'\right) \\ & - \frac{1}{2J}\left[(\phi^\dagger)^2\phi'^2 + \phi^2(\phi^{\dagger\prime})^2 - 2(\chi_- - \chi_+)^2\phi^2(\phi^\dagger)^2\right] + \mathcal{O}\left(\frac{1}{J^3}\right) . \end{aligned} \tag{5.2.33}$$

We will only use quartic interaction term for our S-matrix calculations regardless of the order in κ, and we shall later see that the diagrams generated by this interaction term are indeed sufficient to compute the contribution to the 2-body S-matrix that we are going to look for. Thus, while higher interaction terms in $\frac{1}{J}$ exist, we do not need them for our purposes.

After having introduced yet another expansion parameter, it might be prudent to recapitulate all the expansion parameters, where they come from, what scale they control and what their purpose is: We began by wanting to analyse our model within the effective theory framework of the Landau-Lifshitz limit. To this end, we demand the angular momentum J and the string tension λ to be large in such a way that the "dimensionless" angular momentum, $\mathcal{J}$, is also large. Our gauge choice then made us choose (large) κ instead of the angular momentum as the initial expansion parameter of our Lagrangian - but fortunately the two expansions coincide at leading order. After calculating within the large κ framework and performing some rescalings, J appears in front of the Lagrangian and serves as pseudo $\hbar$-constant for the expansion of our fields around the BMN solution, and then indicates the interaction order of our effective Lagrangian featuring ϕ (i.e. whether the contribution of a certain $\frac{1}{J}$ is quadratic, quartic, et cetera). Simultaneously, the other parameter that sets the LL scale, λ, is then used as the expansion parameter of our vertices, kinematical factors and dispersion relations - our ingredients for the Feynman diagrammatics - and finally the S-matrix elements. The different rescalings and particular expansion parameters we were focusing on also allowed us to compare our results with other authors that used those expansion parameters.

5.2.2 Next-to-leading Order in κ

So far, our considerations ultimately regarded only the leading order, we now want to expand our analysis beyond this. With the first step being again the unaltered Lagrangian,

$$\begin{aligned} \frac{4\pi}{\sqrt{\lambda}}F_S\mathcal{L} =& \frac{\left[q^2r^2\left(r^2-1\right)-1\right]\left(\dot{r}^2 - r'^2\right)}{r^2-1} + \left(1-r^2\right)\left[1+\chi_-^2\left(1-r^2\right)\right]\left[\left(\kappa+\dot{\alpha}+\dot{\beta}\right)^2 - \left(\alpha'+\beta'\right)^2\right] \\ & - 2\chi_-\chi_+\left(r^2-1\right)r^2\left[(\kappa+\dot{\alpha})^2 - \dot{\beta}^2 - \alpha'^2 + \beta'^2\right] + r^2\left(1+\chi_+^2r^2\right)\left[\left(\kappa+\dot{\alpha}-\dot{\beta}\right)^2 - \left(\alpha'-\beta'\right)^2\right] \\ & - 2aq\left(1-r^2\right)\left[r^2\left(\chi_+^2-\chi_-^2+q^2\right)+2\chi_-^2+2\right]\left[\beta'\left(\kappa+\dot{\alpha}\right) - \dot{\beta}\alpha'\right] - \kappa^2F_S , \end{aligned} \tag{5.2.34}$$

with the same rescalings in τ and the deformation parameters, we arrive the NLO contribution in κ orders of

$$\begin{aligned} \mathcal{L}^{(2)} =& (\chi_- - \chi_+)^2r^2(1-r^2)(\chi_+^2r^2 + \chi_-^2(1-r^2)) + \frac{\dot{r}^2}{1-r^2} + \frac{\chi_+^2r^2 - \chi_-^2(1-r^2)}{1-r^2}r'^2 + \dot{\alpha}^2 \\ & + (\chi_- - \chi_+)^2r^2(1-r^2)\alpha'^2 + \dot{\beta}^2 + 2(\chi_- - \chi_+)^2r^2(1-r^2)\dot{\beta} \\ & + 2\tilde{q}(1-r^2)(4\tilde{q}^2 + (\chi_+^2 - \chi_-^2)r^2)\beta' + (\chi_-\chi_+)^2r^2(1-r^2)\beta'^2 \\ & - 2(1-r^2)\alpha'((\chi_-^2 - \chi_+^2)r^2\beta' - 2\tilde{q}\dot{\beta}) \end{aligned}$$

$$+\dot{\alpha}((1-2r^2)\dot{\beta}-(1-r^2)((\chi_- -\chi_+)^2r^2+2\tilde{q}\beta')). \tag{5.2.35}$$

The same modus operandi applies: Now we want to get rid of the coordinate α in our Lagrangian by making use of the Virasoro constraints - this time expanding them up to second order in $\frac{1}{\kappa}$. The crossed Virasoro constraint (i.e. the one involving terms that feature both t and σ derivatives) serves to eliminate spatial derivatives of α, whereas the non-crossed Virasoro constraint serves to eliminate time derivatives (in both the Lagrangian and the aforementioned constraint for α'). It is prudent to recall one thing in this instance: $\mathcal{L}^{(0)}$ does not feature any $\dot{\alpha}$ terms (except for a total derivative $-2\dot{\alpha}$ that we can disregard). This implies that for the non-crossed Virasoro constraint to second order,

$$\frac{r'^2}{1-r^2}-(\chi_+-\chi_-)^2r^2(1-r^2)+2[\dot{\alpha}+(1-2r^2)\dot{\beta}]+(1-2r^2)[\alpha'^2+\beta'^2]+2\alpha'\beta'+\frac{C}{\kappa^2}=\mathcal{O}\left(\frac{1}{\kappa^4}\right), \tag{5.2.36}$$

any term C would neither contribute in $\mathcal{L}^{(0)}$ nor $\mathcal{L}^{(2)}$ upon resubstitution for $\dot{\alpha}$, and we can therefore stick to the leading order expression for this constraint. The same holds for the other one: For the Virasoro constraint with cross-derivatives we go one order beyond, yielding

$$\begin{aligned}0+\mathcal{O}\left(\frac{1}{\kappa^4}\right)=&\left[\left(1-2r^2\right)\beta'+\alpha'\right]+\frac{1}{\kappa^2}\left\{\alpha'\left[(\chi_-^2-4\chi_-\chi_++\chi_+^2)r^2(r^2-1)+\dot{\alpha}+(1-2r^2)\dot{\beta}\right]\right.\\&\left.+\left[-(\chi_-^2-\chi_+^2)r^2(r^2-1)+(1-2r^2)\dot{\alpha}+\dot{\beta}\right]\beta'-\frac{r'\dot{r}}{r^2-1}\right\}.\end{aligned} \tag{5.2.37}$$

In the constraint above, $\dot{\alpha}$ appears in the second order $\frac{1}{\kappa}$ term only. Thus, we only need to substitute the leading order expression for $\dot{\alpha}$ from the other Virasoro constraint in order to consistently make (5.2.37) free from $\dot{\alpha}$ appearances, and then solving the resulting equation for α' as we did before. However, this time as the leading order expression for $\dot{\alpha}$ involves α', (5.2.37) would be an equation cubic in α' upon this substitution - rather tedious to solve. Lest we forget: We are only looking for an expression for α' that is correct *up to the order we are considering*. With this in mind, we can eliminate α' from our constraint for $\dot{\alpha}$ *first*, only using the leading order expression in $\frac{1}{\kappa}$ for α' since we substitute it into the constraint for $\dot{\alpha}$ that we also only need up to leading order. With this subtle trick, we first end up with an α-free expression for $\dot{\alpha}$,

$$\dot{\alpha}=\frac{\left(r^2-1\right)\left(\left(4r^2-2\right)\dot{\beta}-r^2\left(r^2-1\right)\left((\chi_- -\chi_+)^2+\left(4-8r^2\right)\beta'^2\right)\right)+r'^2}{2\left(r^2-1\right)}, \tag{5.2.38}$$

that we can insert into the constraint for α' without generating a cubic equation for it. Imposing then α' to be of the form $\alpha'=A_0+\frac{A_2}{\kappa^2}$ (which again, is correct because we are looking for a result up to this order) and solving for the coefficients A_i, we arrive at an expression for α' that we can use in our Lagrangian to take care of the α'-appearances,

$$\begin{aligned}\alpha'=&-(1-2r^2)\beta'\\&-\frac{1}{(1-r^2)\kappa^2}\left(\dot{r}r'-2r^2(1-r^2)^2((\chi_+^2-4\chi_+\chi_--\chi_-^2)-\chi_-(\chi_- -2\chi_+)-2\dot{\beta})\beta'\right),\end{aligned} \tag{5.2.39}$$

while the isolated total derivative $\dot{\alpha}$-appearances can be disregarded and for the residual (non-leading order) appearances, we can use (5.2.38), and arrive at a Lagrangian without any α at all:

$$\mathcal{L}^{(2)}=\frac{\dot{r}^2}{1-r^2}-4r^2(1-r^2)\dot{\beta}^2+\cdots, \tag{5.2.40}$$

where, for brevity, we only display the terms involving a time derivative here. Our Lagrangian now features two fields only, r and β, and is in a sense much simpler than what we started with. There is a

different issue that we have not addressed so far: After the above substitutions and expansions, our $\mathcal{L}^{(2)}$ features squares of time derivatives of our fields, i.e. $\dot{r}^2$ and $\dot{\beta}^2$. Non-standard contributions like these make further computations more difficult. Additionally, if there was a way to construct a Lagrangian that is (classically) equivalent to ours without these squared derivatives, we could make an argument that these squared derivative terms are in a sense unphysical and an artefact of our particular choice of fields and expansion. Indeed, there exists a classical field redefinition such that these unwanted terms are no longer present - we will follow the methodology summarised in the appendix of [KRT04]. Since we want to change our Lagrangian at order κ^{-2}, the way we redefine our fields will reflect this, i.e.

$$\begin{aligned} \beta &\to \beta + \frac{\tilde{\beta}}{\kappa^2} + \dots \ , \\ r &\to r + \frac{\tilde{r}}{\kappa^2} + \dots \ . \end{aligned} \tag{5.2.41}$$

These redefinitions affect the Lagrangian expansion via

$$\mathcal{L}(\beta + \frac{\tilde{\beta}}{\kappa^2}, r + \frac{\tilde{r}}{\kappa^2}) = \mathcal{L}^{(0)}(\beta + \frac{\tilde{\beta}}{\kappa^2}, r + \frac{\tilde{r}}{\kappa^2}) + \frac{\mathcal{L}^{(2)}(\beta + \frac{\tilde{\beta}}{\kappa^2}, r + \frac{\tilde{r}}{\kappa^2})}{\kappa^2} + \cdots \tag{5.2.42}$$

$$\downarrow$$

$$\mathcal{L}(\beta + \frac{\tilde{\beta}}{\kappa^2}, r + \frac{\tilde{r}}{\kappa^2}) = \mathcal{L}^{(0)}(\beta, r) + \frac{\mathcal{L}^{(2)}(\beta, r) + \tilde{\beta}\delta_\beta \mathcal{L}^{(0)}(\beta, r) + \tilde{r}\delta_r \mathcal{L}^{(0)}(\beta, r)}{\kappa^2} + \dots \ , \tag{5.2.43}$$

where δ_F is the variational derivative with respect to F, and we shall drop the dependence of the Lagrangians on arguments for simplicity's sake from now on. While this is a legitimate way to redefine our fields, we have not explained why we have hope for this to be our remedy. The crucial observation is that our Lagrangian, i.e. the first two leading order, depend at most linearly on the time derivatives of the fields, meaning it is of the form

$$\mathcal{L}^{(0)} = A(r, \beta)\dot{\beta} + B(r, \beta)\dot{r} + V(r, \beta) \ , \tag{5.2.44}$$

which implies the variational derivatives to be at most linearly dependent on time derivatives as well

$$\delta_\beta \mathcal{L}^{(0)} = \partial_\beta A(r, \beta)\dot{\beta} + \partial_\beta B(r, \beta)\dot{r} + \partial_\beta V - \frac{\partial}{\partial \tau} A(r, \beta) \ , \tag{5.2.45}$$

$$\delta_r \mathcal{L}^{(0)} = \partial_r A(r, \beta)\dot{\beta} + \partial_r B(r, \beta)\dot{r} + \partial_r V - \frac{\partial}{\partial \tau} B(r, \beta) \ , \tag{5.2.46}$$

or, more concretely,

$$\frac{1}{8}\delta_\beta \mathcal{L}^{(0)} = \left[\left(2 - 4r^2\right)\beta' - \tilde{q}\right] r r' + \left(1 - r^2\right) r^2 \beta'' + r\dot{r} \ , \tag{5.2.47}$$

$$\delta_r \mathcal{L}^{(0)} = r\left\{-2\left[(\chi_- - \chi_+)^2 - 4\tilde{q}\beta' + 4\beta'^2\right] + \frac{2\left(r'\right)^2}{\left(r^2 - 1\right)^2} - 8\dot{\beta}\right\} + 4r^3\left((\chi_- - \chi_+)^2 + 4\beta'^2\right) - \frac{2r''}{r^2 - 1} \ . \tag{5.2.48}$$

Thus, it is possible to find two functions $\tilde{\beta}$ and $\tilde{r}$ such that

$$\mathcal{L}^{(2)} + \tilde{\beta}\delta_\beta \mathcal{L}^{(0)} + \tilde{r}\delta_r \mathcal{L}^{(0)} \tag{5.2.49}$$

contains no square time derivatives. We directly find as solutions for our wanted field redefinitions

$$\tilde{\beta} = \frac{r'\left[\tilde{q} + \left(4r^2 - 2\right)\beta'\right] + r\left(r^2 - 1\right)\beta''}{8r\left(r^2 - 1\right)} + \frac{\dot{r}}{8r(r^2 - 1)} \ ,$$

$$8\tilde{r} = r'' - r(1 - r^2)\left[((1 + 2r^2)(\chi_+ + \chi_-)^2 - 4\chi_-(\chi_- + 2\chi_+ r^2) - 4\tilde{q}\beta' + 4(1 - 2r^2)\beta''\right]$$

$$+\frac{rr'^2}{(1-r^2)}+4r(1-r^2)\dot{\beta}\ . \tag{5.2.50}$$

With the prescription we established in (5.2.41) and the so-generated new fields, we find for our next-to-leading order Lagrangian the expression

$$\begin{aligned}\mathcal{L}^{(2)}=&-\frac{r'^4}{4(1-r^2)^3}+\frac{r'^2\left[(1-r^2)(\chi_+^2r^2(2-r^2)-2\chi_+\chi_-r^2(1-r^2)+\chi_-^2(2-r^4))+rr''\right]}{2(1-r^2)^2}\\&+\frac{4\tilde{q}^2r'^2+(r'')^2}{4(1-r^2)}+\frac{1}{4}(\chi_--\chi_+)^2r^2(1-r^2)[(\chi_--\chi_+)^2(1+r^2-r^4)-4\chi_-\chi_+]\\&+\frac{1}{2}(\chi_-^2-\chi_+^2)rr''+\text{ terms involving }\beta\ ,\end{aligned} \tag{5.2.51}$$

which is now free from squared time derivatives. With this Lagrangian, we then make use of the same way of re-expressing β and r simultaneously with a single complex field, i.e.

$$\begin{aligned}\phi&=\sqrt{1-r^2}e^{2i\beta}\ ,\\ \phi^\dagger&=\sqrt{1-r^2}e^{-2i\beta}\ .\end{aligned} \tag{5.2.52}$$

and, as before, rescale as we did in (5.2.30) for t and the deformation parameters, and for large J, we then arrive at the following expression for the next-to-leading order Lagrangian written in terms of the field ϕ:

$$\mathcal{L}=\frac{\sqrt{\lambda}\kappa}{4\pi}\left[\mathcal{L}^{(0)}+\lambda^{-1}\mathcal{L}^{(2)}+\lambda^{-2}\mathcal{L}^{(4)}+\dots\right]\ , \tag{5.2.53}$$

$$\begin{aligned}J^{-1}\mathcal{L}^{(2)}=&\frac{1}{4}\Big[(\phi^{\dagger\prime\prime}\phi''+(\chi_-^2-\chi_+^2)^2|\phi|^2-(\chi_-^2-\chi_+^2)(\phi^{\dagger\prime\prime}\phi+\phi^\dagger\phi'')-2i\tilde{q}(\phi^{\dagger\prime\prime}\phi'+\phi^{\dagger\prime}\phi'')\\&+4(\chi_+^2+\tilde{q}^2)\phi^{\dagger\prime}\phi'+8i\tilde{q}^3(\phi^{\dagger\prime}\phi-\phi^\dagger\phi')\Big]+\frac{1}{8J}\Big[2(\phi^{\dagger\prime})^2\phi'^2-2\phi^\dagger\phi^{\dagger\prime\prime}\phi'^2-2(\phi^{\dagger\prime})^2\phi\phi''\\&+4\phi^{\dagger\prime}\phi'\left(\phi^\dagger\phi''+\phi^{\dagger\prime\prime}\phi\right)+(\phi^\dagger)^2\phi''^2+(\phi^{\dagger\prime\prime})^2\phi^2+(\chi_-^2-\chi_-\chi_++\chi_+^2)[(\phi^\dagger)^2\phi'^2+(\phi^{\dagger\prime})^2\phi^2]\\&-8\chi_-\chi_+(\chi_--\chi_+)^2|\phi|^4-4i\tilde{q}(\phi^{\dagger\prime}\phi-\phi^\dagger\phi')[(\chi_--\chi_+)^2|\phi|^2+2\phi^{\dagger\prime}\phi']+4\tilde{q}^2[(\phi^\dagger)^2\phi'^2\\&+(\phi^{\dagger\prime})^2\phi^2]\Big]+\mathcal{O}\left(\frac{1}{J^2}\right)\ ,\end{aligned} \tag{5.2.54}$$

where the $\frac{1}{\kappa^2}$ expansion became $\frac{1}{\lambda}$ after the rescaling of the fields.

5.3 Canonical Quantisation of the Field Theory and S-matrix

5.3.1 Canonical Quantisation and Dispersion Relation

Following [BMSS20], we want to compute the S-matrix related to the Lagrangian analysis that we conducted, and later cross-check their results with ours. To this end, we shall fall back on one of the standard tools in QFT that we mentioned briefly in the introductory chapter: Canonical Quantisation. Our Lagrangian, after all the rescalings we performed and limits we assumed, is akin to the one for a complex scalar field,[9] and we shall borrow a lot of methodology from the quantisation of such a field for our purposes. The standard mode expansion of a complex scalar field (in two dimensions) is given by:

$$\phi(t,\sigma)=\int\frac{dp}{\sqrt{2\pi\omega(p)}}\left(a_pe^{-i\omega(p)t+ip\sigma}+b_p^\dagger e^{i\omega(p)t-ip\sigma}\right)\ , \tag{5.3.1}$$

[9] Or, to be precise, the positive-energy part thereof and after adding a boundary term to it, to get the canonical representation.

$$\phi^\dagger(t,\sigma) = \int \frac{dp}{\sqrt{2\pi\omega(p)}} \left(b_p e^{-i\omega(p)t+ip\sigma} + a_p^\dagger e^{i\omega(p)t-ip\sigma} \right) , \tag{5.3.2}$$

where $a_p^{(\dagger)}$ and $b_p^{(\dagger)}$ are the creation and annihilation mode operators. While we remain inspired by this, our case allows for a handy simplification of this ansatz. Let us consider our Lagrangian again,

$$\begin{aligned} J^{-1}(\mathcal{L}^{(0)} + \lambda^{-1}\mathcal{L}^{(2)}) =& \phi^\dagger(-i\partial_t + i\tilde{q}\partial_\sigma)\phi - \phi(-i\partial_t + i\tilde{q}\partial_\sigma)\phi^\dagger - (\chi_+ - \chi_-)^2\phi^\dagger\phi - \phi^{\dagger\prime}\phi' \\ & - \frac{1}{2J}\left[(\phi^\dagger)^2\phi'^2 + \phi^2(\phi^{\dagger\prime})^2 - 2(\chi_- - \chi_+)^2\phi^2(\phi^\dagger)^2\right] \\ & + \frac{1}{4\lambda}\Big((\phi^{\dagger\prime\prime}\phi'' + (\chi_-^2 - \chi_+^2)^2|\phi|^2 - (\chi_-^2 - \chi_+^2)(\phi^{\dagger\prime\prime}\phi + \phi^\dagger\phi'') - 2i\tilde{q}(\phi^{\dagger\prime\prime}\phi' + \phi^{\dagger\prime}\phi'') \\ & + 4(\chi_+^2 + \tilde{q}^2)\phi^{\dagger\prime}\phi' + 8i\tilde{q}^3(\phi^{\dagger\prime}\phi - \phi^\dagger\phi')\Big) + \frac{1}{8J\lambda}\Big(2(\phi^{\dagger\prime})^2\phi'^2 - 2\phi^\dagger\phi^{\dagger\prime\prime}\phi'^2 - 2(\phi^{\dagger\prime})^2\phi\phi'' \\ & + 4\phi^{\dagger\prime}\phi'\left(\phi^\dagger\phi'' + \phi^{\dagger\prime\prime}\phi\right) + (\phi^\dagger)^2\phi''^2 + (\phi^{\dagger\prime\prime})^2\phi^2 + (\chi_-^2 - \chi_-\chi_+ + \chi_+^2)[(\phi^\dagger)^2\phi'^2 + (\phi^{\dagger\prime})^2\phi^2] \\ & - 8\chi_-\chi_+(\chi_- - \chi_+)^2|\phi|^4 - 4i\tilde{q}(\phi^{\dagger\prime}\phi - \phi^\dagger\phi')[(\chi_- - \chi_+)^2|\phi|^2 + 2\phi^{\dagger\prime}\phi'] + 4\tilde{q}^2[(\phi^\dagger)^2\phi'^2 \\ & + (\phi^{\dagger\prime})^2\phi^2]\Big) + \mathcal{O}\left(\frac{1}{J^2}\right) , \end{aligned} \tag{5.3.3}$$

where the blue part is the part of our Lagrangian that is quadratic in fields, and after normalising it in a canonical way by dividing it by a factor of -2 (to get a canonically normalised kinetic term), we shall from now onwards refer to it as our quadratic (free) Lagrangian $\mathcal{L}_F$. There is, however, a decisive difference between the relativistic Lagrangian of the standard complex field and $\mathcal{L}_F$: In the former, the kinetic term features a second order time derivative, i.e. within the contraction $\cdots - \phi\partial^2\phi^\dagger \cdots$, whereas our Lagrangian only features a single first order time derivative. This allows us to work with mode expansions that only feature positive frequencies for ϕ:

$$\phi(t,\sigma) = \int \underbrace{\frac{dp}{2\pi} a_p e^{-i\omega(p)t+ip\sigma}}_{=:\hat{\phi}} , \qquad \phi^\dagger(t,\sigma) = \int \underbrace{\frac{dp}{2\pi} a_p^\dagger e^{i\omega(p)t-ip\sigma}}_{=:\hat{\phi}^\dagger} , \tag{5.3.4}$$

with $[a_p, a_k^\dagger] = (2\pi)\delta(p-k)$ canonically satisfied. We can then define a ground state $|0\rangle$ such that $\phi(t,\sigma)|0\rangle = 0$ holds.

Defining our (now quantised) fields in this way, we can find the dispersion frequency relation $\omega(p)$ using the equations of motion for one of our fields, e.g. [10]

$$\frac{\partial\mathcal{L}_F}{\partial\phi^\dagger} - \partial_t\frac{\partial\mathcal{L}_F}{\partial\dot{\phi}^\dagger} - \partial_\sigma\frac{\partial\mathcal{L}_F}{\partial\phi'^\dagger} = 0, \tag{5.3.5}$$

with the observation that

$$\begin{aligned} \frac{\partial^n\hat{\phi}}{\partial t^n} &= (-i\omega(p))^n\hat{\phi} , \\ \frac{\partial^n\hat{\phi}}{\partial\sigma^n} &= (ip)^n\hat{\phi} , \\ \frac{\partial^n\hat{\phi}^\dagger}{\partial t^n} &= (i\omega(p))^n\hat{\phi} , \\ \frac{\partial^n\hat{\phi}^\dagger}{\partial\sigma^n} &= (-ip)^n\hat{\phi} , \end{aligned} \tag{5.3.6}$$

we finally get, acting on a state, the following equation for the dispersion relation at leading order:

$$\left[-\omega(p) - \tilde{q}p - (\chi_+ - \chi_-)^2\right] - \left[\omega(p)\right] - \left[\tilde{q}p + p^2\right] = 0$$

[10] The contraction $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)$ is to be understood as a sum, and not as a metric contraction - this has already happened on the level of the Lagrangian.

$$\Leftrightarrow$$

$$2\omega(p) = p(p-2\tilde{q}) + (\chi_- - \chi_+)^2 \ . \tag{5.3.7}$$

For the $\mathcal{O}(\lambda^{-1})$ order, the calculation is analogous, and summa summarum we get the following expression for the dispersion relation

$$\omega(p) = \frac{(\chi_- - \chi_+)^2 + p(p-2\tilde{q})}{2} - \frac{(\chi_-^2 - \chi_+^2)^2 + p^2(p-2\tilde{q})^2 + 2(\chi_-^2 + \chi_+^2)p^2 - 16\tilde{q}^3 p}{8\lambda} + \mathcal{O}\left(\frac{1}{\lambda^2}\right) \ . \tag{5.3.8}$$

In [BMSS20], the authors computed the (exact) dispersion relation within the framework of light-cone gauge quantisation,

$$\Omega_-(p) = \sqrt{p^2 - 2aqp(2 + q^2 + \chi_-^2 + \chi_+^2) + q^2 + (1+\chi_+^2)(1+\chi_-^2)} - \chi_+\chi_- \ . \tag{5.3.9}$$

though in order to draw a comparison of this with our result, we need to make the same assumptions that are made in [BMSS20], namely small momentum and deformation, achieved by continuing to assume λ is large and setting in their result

$$\begin{aligned} q &\to \frac{q}{\sqrt{\lambda^3}} \ , \\ \chi_\pm &\to \frac{\chi_\pm}{\sqrt{\lambda}} \ , \\ p &\to \frac{p}{\sqrt{\lambda}} \ , \end{aligned} \tag{5.3.10}$$

giving us[11]

$$\Omega_-\left(\frac{p}{\lambda}\right) - 1 = \frac{\omega(p)}{\lambda} + \mathcal{O}(\lambda^{-3}) \ , \tag{5.3.11}$$

where the offset by 1 comes from the subtraction of the fast coordinate. Thus, we can see that our results coincide in the first two orders in $\frac{1}{\lambda}$. We therefore establish consistency between their approach and ours.

Having now established the free (quadratic) part of our Lagrangian, $\mathcal{L}_F$ that we work with, canonically quantised the complex field ϕ appearing in it after identifying its equations of motion and dispersion relation, it is time to deal with a quantity that lies at the heart of QFT: The propagator. The propagator, or the 2-point correlation function for the (free) field ϕ, represents the amplitude for a given particle to travel from one point in spacetime to another - which is another way of saying the amplitude that a particle is annihilated in one point and a particle created at another point. Without going into too much detail about Green's functions and similar considerations, knowing that our quantised fields (in their mode expansions) are constructed precisely to describe these events, it makes sense for our propagator to be of the form[12]

$$D(t,\sigma) = \langle 0|T\left\{\phi(t,\sigma)\phi^\dagger(0,0)\right\}|0\rangle \ , \tag{5.3.12}$$

where T stands for the time-ordering. There are different ways to account for this time ordering, and for the order of succession in which these events happen, leading to different propagators describing different amplitudes of scenarios. Canonically, there are three different propagators: The *advanced*, *retarded* and

[11] In [BMSS20], the flux deformation parameter q is accompanied by an additional sign factor that we need to take into account when making this comparison.

[12] Here, we use the fact that our propagator has to be (Lorentz-)translation invariant and therefore only needs to depend on the difference of two spacetime points, which allows us to place one on the origin.

Feynman propagators.[13] The advanced propagator $D_A(t,\sigma)$ described the scenario where the particle is created at σ before being annihilated at 0, while the retarded propagator $D_R(t,\sigma)$ represents annihilation before creation. For the standard complex scalar field with squared time derivatives appearing in its Lagrangian, the (Feynman) propagator features two poles consequently - switching to momentum space via a Fourier transform and looking at its equation of motions shows that imminently.[14] The equations of motion for $\mathcal{L}_F$ only feature a single time derivative, leading to a simple pole in the propagator.[15] $D_R(t,\sigma)$ only features one pole, and the time ordering of this choice can be represented with the insertion of a Heaviside function:

$$\begin{aligned} D_R(t,\sigma) &= \Theta(t)\langle 0|\phi(t,\sigma)\phi^\dagger(0,0)|0\rangle \\ &= \Theta(t)\langle 0|\left(\int \frac{dp}{2\pi} a_p e^{-i\omega(p)t+ip\sigma}\right)\left(\int \frac{dp'}{2\pi} a^\dagger_{p'}\right)|0\rangle \\ &= \Theta(t)\langle 0|\left(\int \frac{dpdp'}{(2\pi)^2}(a^\dagger_{p'}a_p + 2\pi\delta(p-p'))e^{-i\omega(p)t+ip\sigma}\right)|0\rangle \ . \end{aligned} \tag{5.3.13}$$

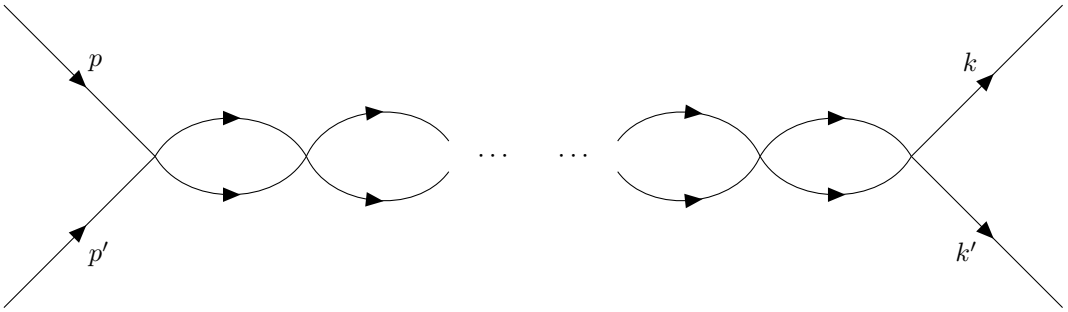
The term involving the annihilation and creation operator vanishes, and we can use the fact that we represent a Heaviside function in the complex plane as

$$2\pi\Theta(t-t') = i\int d\Omega' \frac{e^{-i\Omega'(t-t')}}{\Omega'+i\epsilon} \ , \tag{5.3.14}$$

and then arrive at after redefining $\Omega = \omega(p) + \Omega'$:

$$\begin{aligned} D_R(t,\sigma) &= \int \frac{dpd\Omega}{(2\pi)^2} e^{-i\Omega t+ip\sigma}\frac{i}{\Omega-\omega(p)+i\epsilon} \\ &= \int \frac{dpd\Omega}{(2\pi)^2} e^{-i\Omega t+ip\sigma}\tilde{D}_R(\Omega,p) \ , \end{aligned} \tag{5.3.15}$$

where $\tilde{D}_R(\Omega,p)$ is the retarded propagator in Fourier space. Thus far, the features of our quantised fields, their mode expansions in the interaction picture, and their propagators have been similar to the analysis in section 3 of [KZ06], and their observations are applicable to our case, too. Firstly, the 2-body S-matrix only consists of a sum of bubble diagrams:



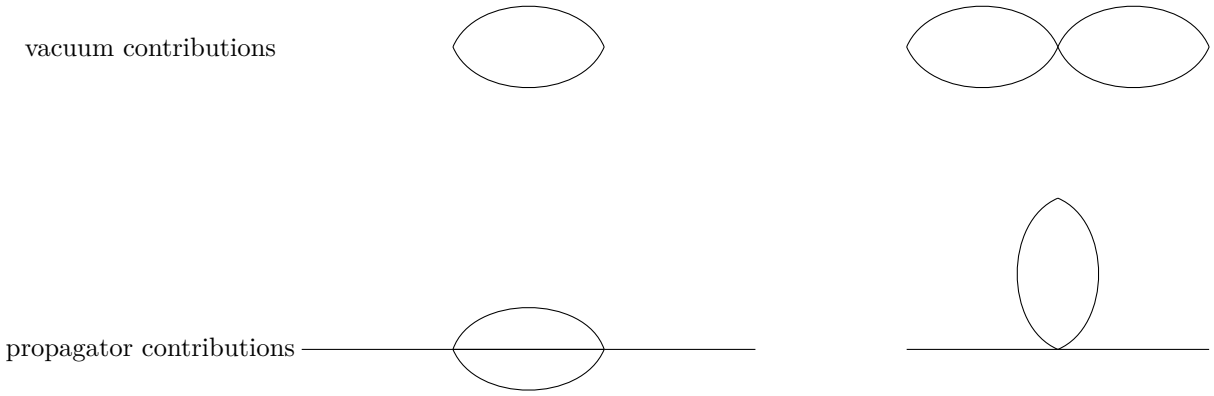
The reason that only bubble diagrams contribute is fairly easy to see: If we were to truncate any diagram at any point, due to $U(1)$ charge conservation of the Lagrangian (i.e. the invariance under $\phi \to e^{ia}\phi$), the number of forward-time propagators minus the number of backward-time propagators

[13] The sum of the former two equal the sum of Feynman and anti-Feynman propagators.

[14] Here, what we left out for brevity would be brilliant to illustrate this point: In the language of Green's functions, in momentum space the equation (of motion) for the propagator function can be directly "inverted" using distributions, and double derivatives that lead to squared appearances of the momentum variable leading in turn to double poles, loosely speaking.

[15] Similar to the argument and situation in [KZ06], we argue that the particles we created cannot travel back in time, which makes the choice of propagator that we are about to make sensible.

has to be a constant for any truncation point. In addition, we established before that, due to our field annihilating the ground state, our particles do not travel backwards in time, which means we only have forward-time propagators in our diagrams. Since we also can truncate a diagram at the very beginning, i.e. when only external legs are present, this means that the number of these propagators is always equal to the number of incoming or outgoing external legs of the diagram. As we are considering the 2-body S-matrix, only quartic vertices will be present and only Lagrangian interacting terms involving (no more than) four fields shall be relevant. This leaves us with the only possibility of bubble diagrams. Furthermore, this has a similar consequence on the non-renormalisability of the ground state and the propagator/Green's function: For there to be any correction to the energy, for the ground state, there would need to exist non-trivial vacuum diagram contributions. Likewise for the propagator, there would need to be higher order 1-particle propagator contributions arising from our theory for there to be any corrections (c.f. *mass-renormalisation* of the propagator). However, those diagrams precisely, e.g.



are not allowed in our theory. The most straightforward way to see this again is by considering the total number of propagating lines at any truncation point: For the vacuum contributions, we start and end with 0 external legs, but have non-zero external legs if we were to truncate it at any point in the middle. Similarly, while propagator diagrams always have one initial and final external leg, higher order contributions will feature point-loops that cause the number of the external legs to be higher if we were to truncate the diagram at any point involving a loop.

In our Lagrangian, the quartic contribution (orange) is

$$\begin{aligned}
J^{-1}(\mathcal{L}^{(0)}+\lambda^{-1}\mathcal{L}^{(2)}) =&\phi^\dagger(-i\partial_t+i\tilde{q}\partial_\sigma)\phi-\phi(-i\partial_t+i\tilde{q}\partial_\sigma)\phi^\dagger-(\chi_+-\chi_-)^2\phi^\dagger\phi-\phi^{\dagger\prime}\phi^\prime\\
&-\frac{1}{2J}\left[(\phi^\dagger)^2\phi^{\prime 2}+\phi^2(\phi^{\dagger\prime})^2-2(\chi_--\chi_+)^2\phi^2(\phi^\dagger)^2\right]\\
&+\frac{1}{4\lambda}\Big((\phi^{\dagger\prime\prime}\phi^{\prime\prime}+(\chi_-^2-\chi_+^2)^2|\phi|^2-(\chi_-^2-\chi_+^2)(\phi^{\dagger\prime\prime}\phi+\phi^\dagger\phi^{\prime\prime})-2i\tilde{q}(\phi^{\dagger\prime\prime}\phi^\prime+\phi^{\dagger\prime}\phi^{\prime\prime})\\
&+4(\chi_+^2+\tilde{q}^2)\phi^{\dagger\prime}\phi^\prime+8i\tilde{q}^3(\phi^{\dagger\prime}\phi-\phi^\dagger\phi^\prime)\Big)+\frac{1}{8J\lambda}\Big(2(\phi^{\dagger\prime})^2\phi^{\prime 2}-2\phi^\dagger\phi^{\dagger\prime\prime}\phi^{\prime 2}-2(\phi^{\dagger\prime})^2\phi\phi^{\prime\prime}\\
&+4\phi^{\dagger\prime}\phi^\prime\left(\phi^\dagger\phi^{\prime\prime}+\phi^{\dagger\prime\prime}\phi\right)+(\phi^\dagger)^2\phi^{\prime\prime 2}+(\phi^{\dagger\prime\prime})^2\phi^2+(\chi_-^2-\chi_-\chi_++\chi_+^2)[(\phi^\dagger)^2\phi^{\prime 2}+(\phi^{\dagger\prime})^2\phi^2]\\
&-8\chi_-\chi_+(\chi_--\chi_+)^2|\phi|^4-4i\tilde{q}(\phi^{\dagger\prime}\phi-\phi^\dagger\phi^\prime)[(\chi_--\chi_+)^2|\phi|^2+2\phi^{\dagger\prime}\phi^\prime]+4\tilde{q}^2[(\phi^\dagger)^2\phi^{\prime 2}\\
&+(\phi^{\dagger\prime})^2\phi^2]\Big)+\mathcal{O}\left(\frac{1}{J^2}\right)\ , \qquad (5.3.16)
\end{aligned}$$

which we shall refer to as V_4, which itself is expanded in λ^{-1},

$$V_4=\sum_{i=0}^{\infty}V_4^{(i)}\lambda^{-i}\ ,\ \text{and more revelantly for us,}$$

$$V_4 = V_4^{(0)} + \frac{V_4^{(1)}}{\lambda} + \mathcal{O}(\lambda^{-2}) \ . \tag{5.3.17}$$

As we alluded to before, the order of the appearing fields can more easily be seen by the order of $\frac{1}{J}$ that is accompanying a given term, as is evident above.

5.3.2 The 2-body S-matrix

The creation and annihilation operators $a_p^\dagger$, a_p in the (quantised) mode expansion create and annihilate a state with momentum p and energy $\omega(p)$, respectively. The 2-particle in-states and out-states that we need in order to compute 2-body S-matrix elements later can be defined the following way,

$$|p,p'\rangle = a_p^\dagger a_{p'}^\dagger|0\rangle \ , \qquad \langle k,k'| = \langle 0|a_{k'}a_k \ . \tag{5.3.18}$$

With our quartic vertex contribution that incorporates the relevant interacting terms, we define the S-matrix via

$$S(p,p',k,k') = \langle k,k'|Te^{-i\int dtd\sigma V_4}|p,p'\rangle \ . \tag{5.3.19}$$

In our expansion approach for the S-matrix, its leading-order in λ consists of all the bubble diagrams with $V_4^{(0)}$ vertices (i.e. such that the total order of any particular contributing term stays $\mathcal{O}(\lambda^0)$)). For the $\mathcal{O}(\lambda^{-1})$) contribution, we will have to take all diagrams into account that feature one single $V_4^{(1)}$ vertex and all others being $V_4^{(0)}$ vertices (i.e. such that the total order of any particular contributing term stays $\big[\mathcal{O}(\lambda^{-1}))\big]\big[\mathcal{O}(\lambda^0))\big] = \big[\mathcal{O}(\lambda^{-1}))\big]$). For each order in λ, this combinatorial approach in building the terms works analogously.

In our action, we integrate over the worldsheet coordinates t,σ, but our action does not feature them in an *explicit* way - which makes the action as a whole invariant under shifts in these coordinates. For the quantities conjugate to them, this leads to conservation laws at *every vertex*. The S-matrix is therefore expected to include the following delta functions:

$$\delta(\omega(p)+\omega(p')-\omega(k)-\omega(k'))\delta(p+p'-k-k') \tag{5.3.20}$$

Using some re-expression tricks, we can reformulate this product of δ-functions into an easier expression involving a kinematical factor K: For any $\delta(f(x))$-function whose conditional argument $f \in C^2(\mathbb{R})$ only has simple roots $\{x_1,\cdots x_n\}$ [16], then

$$\delta(f(x)) = \sum_{j=1}^{n} \frac{\delta(x_j - x)}{|f'(x_j)|}. \tag{5.3.21}$$

For $\delta(\omega(p)+\omega(p')-\omega(k)-\omega(k'))$, i.e. $f(p) := \omega(p)+\omega(p')-\omega(k)-\omega(k')$, the expressions get a bit complicated. For the sake of the argument, let us assume an easy homogeneously quadratic dispersion relation $\omega(p) = p^2$, and then argue why for our dispersion relation, the same result holds. Firstly, $\delta(p+p'-k-k')$ implies that we can re-express $p' = k+k'-p$ in $f(p)$, leading us to:

$$\frac{1}{2}f(p) = p^2 - kp - k'p + kk', \quad \frac{1}{2}f'(p) = 2p - k - k', \tag{5.3.22}$$

whose two simple poles are at $p=k$ and $p=k'$. Thus, we have so far that (again with $p' = k+k'-p$)

$$\delta(\omega(p)+\omega(p')-\omega(k)-\omega(k')) = \frac{\delta(p-k)}{2|k-k'|} + \frac{\delta(p-k')}{2|k'-k|}. \tag{5.3.23}$$

[16] We also assume the derivative of the argument function not to vanish at the root points, for well-definedness.

We reinstate the appearance of p' now, and remember that we have another δ-function as a factor, and can establish that

$$\begin{aligned}\delta(p-k)\delta(p+p'-k-k') &= \delta(p-k)\delta(p'-k')\\ \delta(p-k')\delta(p+p'-k-k') &= \delta(p-k')\delta(p'-k),\end{aligned} \tag{5.3.24}$$

which finally gives us for (5.3.20)

$$\begin{aligned}\delta(\omega(p)+\omega(p')-\omega(k)-\omega(k'))\delta(p+p'-k-k') &= \frac{\delta(p-k)\delta(p'-k')+\delta(p-k')\delta(p'-k)}{2|k-k'|}\\ &= \underbrace{\frac{1}{\left|\frac{d\omega(p)}{dp}-\frac{d\omega(p')}{dp'}\right|}}_{K(p,p')}(\delta(p-k)\delta(p'-k')+\delta(p-k')\delta(p'-k))\ .\end{aligned} \tag{5.3.25}$$

One can check that for the expression of our $\omega(p)$ in (5.3.8), we arrive at the same final expression. This comes from the fact that the δ-function for total momentum conservation is the same (and a linear constraint for p' in terms of p) and $f(p)$ gives rise to an analogous pole structure, leading to (5.3.24) and eventually the same kinematical factor in terms of $\omega(p)$.

Our focus is still expressions *only up to a certain order in terms of our expansion parameter*, and if we substitute our expression for $\omega(p)$ into the expression for our kinematical factor, we get

$$K(p,p') = \underbrace{\frac{1}{p-p'}}_{K^{(0)}} + \frac{1}{\lambda}\underbrace{\frac{p^2+pp'+p'^2+\chi_-^2+\chi_+^2-3\tilde{q}(p+p')+2\tilde{q}^2}{2(p-p')}}_{K^{(1)}} + \mathcal{O}\left(\frac{1}{\lambda^2}\right)\ . \tag{5.3.26}$$

Leading Order in λ

In a first step, we single out the leading order quartic vertex in λ that we need in order to generate the tree-level contribution[17] and diagrammatically illustrate it in 5.1,

$$-\langle k,k'|V_4^{(0)}|p,p'\rangle = -V_4^{(0)}(p,p',k,k') = 2(\chi_- - \chi_+)^2 + pp' + kk'\ . \tag{5.3.27}$$

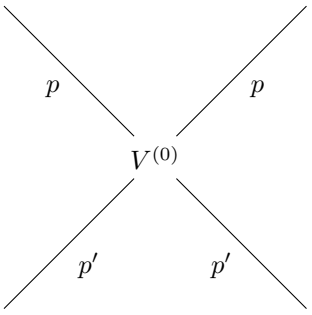


Figure 5.1: The diagram of (5.3.27), already imposing $\{k,k'\} = \{p,p'\}$, as we will later discuss amply. We will not make the diagrams that differ only by a permutation of momenta pictorially explicit.

In the expression of $K(p,p')$, we also only need the leading order term, and arrive at the expression of the tree-level matrix element (having in mind the momentum implications from the δ-functions):

$$S^{(0)}_{\text{tree}} = -\frac{i}{2}\left(V_4^{(0)}(p,p',p,p') + V_4^{(0)}(p,p',p',p)\right)K^{(0)}(p,p')$$

[17] The symmetry factor for standard diagrams is $(2!)^2$, which is needed for the computation of the correct contributions. For simplicity's sake, we will suppress the $1/J$ factors, as they can be reinstated at the end of the computations by substituting $i \to i/J$ (if the complex units are consistently carried).

$$= -iV_4^{(0)}(p,p',p,p')K(p,p') = 2i\frac{pp' + (\chi_- - \chi_+)^2}{p-p'} \ . \tag{5.3.28}$$

From [BMSS20], we can take the S-matrix element S_{YY} to compare our expression against. To this end, we need to perform the following rescaling again:[18]

$$\begin{aligned} p_i &\to \frac{p_i}{\lambda} \ , \\ \chi_\pm &\to \frac{\chi_\pm}{\lambda} \ , \\ q &\to \frac{q}{\lambda^3} \ . \end{aligned} \tag{5.3.29}$$

Then, the leading order in large λ gives us

$$S_{YY}^{(0)} = i\left[\frac{4(\chi_- - \chi_+)^2 + (p+p')^2}{2(p-p')} + (\alpha - \frac{1}{2})(p-p')\right] \ , \tag{5.3.30}$$

and when setting the light-cone gauge parameter to $\alpha = 0$, our S-matrix elements coincide.

Staying at the leading order in λ for the vertex, we go one order beyond in the order of the diagram we are considering, as illustrated in 5.2 below.

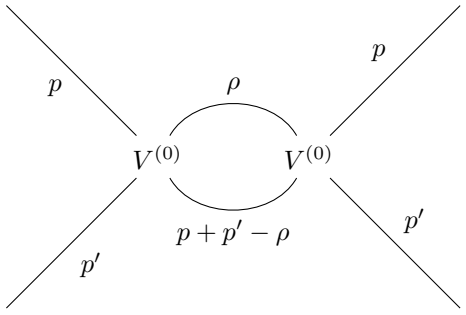


Figure 5.2: The diagram corresponding to the above.

We therefore need to perform the following loop integral

$$\begin{aligned} I^{(1)} &= \int \frac{d\Omega d\rho}{(2\pi)^2} \tilde{D}_R(\Omega,\rho)\tilde{D}_R(\omega(p)+\omega(p')-\Omega-\omega(\rho), p+p'-\rho)\left[V_4^{(0)}(p,p',\rho,p+p'-\rho)\right]^2 \\ &= \int \frac{d\Omega d\rho}{(2\pi)^2} \frac{[2(\chi_- - \chi_+)^2 + pp' + \rho(p+p'-\rho)]^2}{[\Omega - \omega(\rho) + i\epsilon][\omega(p)+\omega(p') - \Omega - \omega(p+p'-\rho) + i\epsilon]} \ , \end{aligned} \tag{5.3.31}$$

which has two simple poles, at $\Omega_1 = \omega(\rho) - i\epsilon$ and at $\Omega_2 = \omega(p) + \omega(p') - \omega(p+p'-\rho) + i\epsilon$, as can be seen in 5.3.

[18] Here, λ in [BMSS20] has a different meaning. Additionally, the appendix in [BMSS20] has typos whenever Ω_i is addressed, which should be kept in mind when performing the calculation.

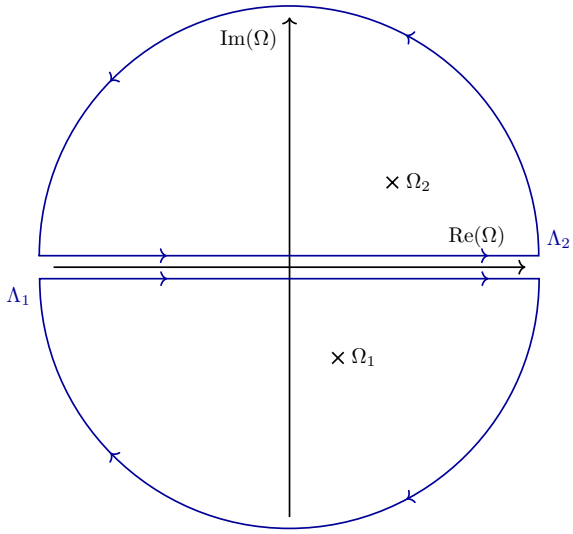


Figure 5.3: The poles of $I^{(1)}$.

We can compute $I^{(1)}$ using the residue theorem, as the integration over the arcs tends to zero for arc radii $r_i \to \infty$. It does not matter which integration contour we choose, so we shall go with Λ_1:

$$\begin{aligned} I^{(1)} &= 2\pi i \lim_{\Omega \to \Omega_1} (\Omega - \Omega_1) \int \frac{d\rho}{(2\pi)^2} \frac{[2(\chi_- - \chi_+)^2 + pp' + \rho(p + p' - \rho)]^2}{[\Omega - \omega(\rho) + i\epsilon][\omega(p) + \omega(p') - \Omega - \omega(p + p' - \rho) + i\epsilon]} \\ &= \int \frac{d\rho}{2\pi} \frac{i[2(\chi_- - \chi_+)^2 + pp' + \rho(p + p' - \rho)]^2}{\omega(p) + \omega(p') - \omega(\rho) - \omega(p + p' - \rho) + 2i\epsilon} \ . \end{aligned} \tag{5.3.32}$$

Inserting the leading order in λ of $\omega(p)$ into the equation, we can see that the resulting integral is divergent by mere power counting:

$$\begin{aligned} I^{(1)} &\approx \int \frac{d\rho}{2\pi} \frac{-[2(\chi_- - \chi_+)^2 + pp' + \rho(p + p' - \rho)]^2}{p(p - 2\tilde{q}) + p'(p' - 2\tilde{q}) - \rho(\rho - 2\tilde{q}) - (p + p' - \rho)(p + p' - \rho - 2\tilde{q}) + \mathcal{O}(\lambda^{-1}) + 2i\epsilon} \\ &= \int \frac{d\rho}{2\pi} \frac{-[2(\chi_- - \chi_+)^2 + pp' + \rho(p + p' - \rho)]^2}{p^2 + p'^2 - \rho^2 - (p + p' - \rho)^2 + \mathcal{O}(\lambda^{-1}) + 2i\epsilon} \ . \end{aligned} \tag{5.3.33}$$

Again, we approach this integral using contour integration: Here, we implicitly assume $p > p'$. The poles of ρ are at p and p', and which pole receives a positive and negative imaginary part from the $i\epsilon$ prescription we chose depends on how we interpret $|p - p'|$, meaning which of the two momenta is larger. In our case of $p > p'$, the pole at p has a positive imaginary ϵ correction.

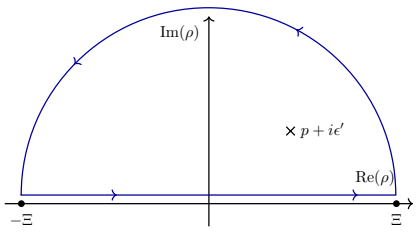


Figure 5.4: The pole of ρ at p, the ϵ' here represents some momentum-rescaled quantity that leads to an imaginary part in the upper half plane allowing us to choose the upper half-circle as a consistent arc.

As illustrated in 5.4, we close the arc parametrised as a half-circle in the upper half plane, and expand

the (divergent) arc part in terms of a sharp cut-off parameter Ξ we introduce:

$$I^{(1)} \approx \underbrace{\operatorname*{Res}_{\rho\to p} \frac{1}{2\pi} \frac{i[2(\chi_- - \chi_+)^2 + pp' + \rho(p+p'-\rho)]^2}{p^2+p'^2-\rho^2-(p+p'-\rho)^2+\mathcal{O}(\lambda^{-1})+2i\epsilon}}_{\text{residue part } (R)}$$
$$- \underbrace{\int_0^\pi \Xi e^{i\theta} \frac{d\theta}{2\pi} \frac{-[2(\chi_- - \chi_+)^2 + pp' + \Xi e^{i\theta}(p+p'-\Xi e^{i\theta})]^2}{p^2+p'^2-\Xi^2 e^{2i\theta}-(p+p'-\Xi e^{i\theta})^2+\mathcal{O}(\lambda^{-1})+2i\epsilon}}_{\text{arc part } (A)} . \tag{5.3.34}$$

For the residue part, as we are dealing with a simple pole, we can simply evaluate it as

$$\begin{aligned} R &= 2\pi i \lim_{\substack{\rho\to p \\ \epsilon\to 0^+}} \frac{i[2(\chi_- - \chi_+)^2 + pp' + \rho(p+p'-\rho)]^2}{p^2+p'^2-\rho^2-(p+p'-\rho)^2+\mathcal{O}(\lambda^{-1})+2i\epsilon} \\ &= 2(p-p')\left(\frac{pp' + (\chi_- - \chi_+)^2}{p-p'}\right)^2 , \end{aligned} \tag{5.3.35}$$

while the arc part gives us once expanded in powers of Ξ:

$$\begin{aligned} A &= \int_0^\pi \Xi e^{i\theta} \frac{d\theta}{2\pi} \frac{-[2(\chi_- - \chi_+)^2 + pp' + \Xi e^{i\theta}(p+p'-\Xi e^{i\theta})]^2}{p^2+p'^2-\Xi^2 e^{2i\theta}-(p+p'-\Xi e^{i\theta})^2+\mathcal{O}(\lambda^{-1})+2i\epsilon} \\ &\approx \int_0^\pi \Xi e^{i\theta} \frac{d\theta}{2\pi} \left[\frac{\Xi^2 e^{2i\theta}}{2} + \frac{(p+p')}{2}\Xi e^{i\theta} + 2(\chi_- - \chi_+)^2 + pp' - \frac{(p+p')^2}{4} + \frac{p^2+p'^2}{4} + 0\cdot\Xi^{-1} + \mathcal{O}(\Xi^{-2})\right] . \end{aligned} \tag{5.3.36}$$

As a regularisation scheme, we choose to impose that positive exponential power integrands of ρ to vanish, i.e. $\int d\rho \rho^\alpha = 0$ for $\alpha \geq 0$. For Ξ, this lets us disregard all Ξ^n with $n > 0$ in the integral. Thus, we have that $A = 0$, and finally for the total $I^{(1)}$

$$I^{(1)} = (p-p')\left(\frac{2(\chi_- - \chi_+)^2 + 2pp'}{p-p'}\right)^2 . \tag{5.3.37}$$

From the two vertices in the diagram we get an imaginary unit factor each of $(-i)$, and from the overall δ-functions we get the kinematical factor $K(p,p')$ for the S-matrix element. Lastly, not forgetting that the diagram illustrating the process has a symmetry factor of 2 that we need to divide by, we arrive at:

$$S^{(0)}_{\text{1-loop}} = \frac{(-i)^2}{2} K(p,p') I^{(1)} = -2\left(\frac{(\chi_- - \chi_+)^2 + pp'}{p-p'}\right)^2 . \tag{5.3.38}$$

We can immediately generalise this computation to the n-bubble case, as each loop bubble contributes with precisely this factor, i.e. [19]

$$S^{(0)}_{\text{(n-1)-loop}} = 2i^n \left(\frac{(\chi_- - \chi_+)^2 + pp'}{p-p'}\right)^n . \tag{5.3.39}$$

Summa summarum, we can sum all the relevant contributions to the S-matrix

$$\begin{aligned} S^{(0)}(p,p') &= 1 + S^{(0)}_{\text{tree}} + \sum_{n=1}^\infty S^{(0)}_{\text{n-loop}} = 1 + 2\sum_{n=0}^\infty i^n \left(\frac{(\chi_- - \chi_+)^2 + pp'}{p-p'}\right)^n \\ &= \frac{p-p' + i[(\chi_- - \chi_+)^2 + pp']}{p-p' - i[(\chi_- - \chi_+)^2 + pp']} = \frac{\frac{1}{p} - \frac{1}{p'} - i\frac{(\chi_- - \chi_+)^2 + pp'}{pp'}}{\frac{1}{p} - \frac{1}{p'} + i\frac{(\chi_- - \chi_+)^2 + pp'}{pp'}} . \end{aligned} \tag{5.3.40}$$

Curiously, this S-matrix does not depend on the deformation parameter q coming from the flux, which might be due to the particular rescalings we performed on all of our deformation parameters. Furthermore, for $\chi_\pm \in \mathbb{R}$, this is a unitary S-matrix. Lastly, in the limit of zero deformation $\chi_\pm \to 0$, we recover the Heisenberg S-matrix, which is consistent with what we expect.

[19] Here, there was a typo in our original paper, as the exponent n of the imaginary unit and the S-matrix factor describes the case of $n-1$ bubbles, rather than n. For the summation of $\sum_{n=0}^\infty$, this is inconsequential, though.

The S-matrix we obtained appears in a very compact form - perhaps even suspiciously so: A priori, one may think that the set of assumptions that led to our S-matrix was incomplete, and that we oversimplified the problem at hand. After all, we are restricting our considerations to a purely bosonic model and consider no coordinate contributions coming from the T^4 factor of our manifold which could potentially lead to other fields contributing to our S-matrix processes. The authors in [BMSS20], for example, consider four possible field excitations rather than just one. However, here we can make use of arguments akin to the ones outlined in [RTT06], where the authors discuss (within the undeformed $\mathrm{AdS}_5 \times S^5$ context) LL-type models with larger sectors containing the $SU(2)$ one. Firstly, each type of excitation is associated to a Dynkin node in its diagram, and thus one can assign an abelian conserved charge to them. Therefore, each appearing term in the Lagrangian has to be uncharged with respect to these charges, and conserve the quantum numbers associated to them. Two important implications follow from this: The first one is the absence of cubic vertices, and the second one is that, because the excitations appearing in this Lagrangian are magnons around the ferromagnetic ground state, we can always choose all present propagators to be of retarded nature. We also have as a consequence that there exist no interaction terms that could correspond to the annihilation of one type of excitation pair and pair-creation of another type of excitation. With the restriction on the vertices and propagators that can appear in our diagrams, we can therefore assert that loop contributions that (for example as intermediaries) contain different excitations than the excitations that appear on their incoming and outgoing legs are indeed absent. Therefore, the S-matrix that we computed is not incomplete.

Next-To-Leading Order in λ

All of our tree level and next-to-leading order level considerations have so far been limited to the case of the leading order λ setting. In order to get the first-order correction in λ to the S-matrix we just computed, we need to be aware that our S-matrix is built out of different ingredients: The vertex contribution, and the kinematical contribution from the rewriting of the δ-functions that appear - they both feature contributions of order $\mathcal{O}(\lambda^{-1})$ as we can see in (5.3.16) and (5.3.26). The first-order correction to the tree-level S-matrix is given by

$$\lambda S^{(1)}_{\text{tree}} = -i\left[K^{(0)}V_4^{(1)}(p,p',p,p') + K^{(1)}V_4^{(0)}(p,p',p,p')\right]. \tag{5.3.41}$$

This expression features no summands involving $K^{(1)}$ and $V_4^{(1)}$ at the same time, as this would result in the term being a correction to the S-matrix of order $\mathcal{O}(\lambda^{-2})$, thus it makes sense for the leading and subleading corrections of the vertex and the kinematical factor to appear in this chiastic way.

While we have a handy expression for $K^{(1)}$, we need to find one for $V_4^{(1)}$:

$$\begin{aligned}4\lambda\langle k,k'|V_4^{(1)}|p,p'\rangle =& 4\lambda V_4^{(1)}(p,p',k,k') = 8\chi_+\chi_-(\chi_- - \chi_+)^2 + pp'k(p+p'+k+4\tilde{q})\\ &+ pp'k'(p+p'+k'+4\tilde{q}) + pkk'(p+k+k'+4\tilde{q}) + p'kk'(p'+k+k'+4\tilde{q})\\ &+ 2\tilde{q}(\chi_- - \chi_+)^2(p+p'+k+k')\\ &- (pp'+kk')\left[pp'+kk'-4\chi_+\chi_- - 4(\chi_- - \chi_+)^2 - 4\tilde{q}^2\right].\end{aligned} \tag{5.3.42}$$

Proceeding methodologically exactly as we did for the leading order λ calculation, with (5.3.41) we then arrive at

$$\begin{aligned}\lambda S^{(1)}_{\text{tree}} =& -i\left[K^{(0)}V_4^{(1)}(p,p',p,p') + K^{(1)}V_4^{(0)}(p,p',p,p')\right]\\ =& -i\frac{2(\chi_+\chi_- + pp')(\chi_- - \chi_+)^2 + pp'(p^2+p'^2) + \tilde{q}(p+p')[2pp' + (\chi_- - \chi_+)^2] + 2pp'\left[\chi_+\chi_- + \tilde{q}^2\right]}{p-p'}\end{aligned}$$

$$+\,i\frac{[pp'+(\chi_--\chi_+)^2][p^2+pp'+p'^2+\chi_-^2+\chi_+^2-3\tilde{q}(p+p')+2\tilde{q}^2]}{(p-p')^2}\ . \tag{5.3.43}$$

While this was straightforward enough, for the case of next-to-leading order λ contributions on the 1-loop level, things are a bit more tricky, as there are three distinct origins of contributing terms, and one is quick to lose oversight.

The first and arguably easiest contribution comes from taking the $\mathcal{O}(\lambda^0)$-order term $I^{(1)}$ that stems from taking $\mathcal{O}(\lambda^0)$-order vertices,[20] but taking the $\mathcal{O}(\lambda^{-1})$-order correction to the kinematical factor, $K^{(1)}$, when constructing the S-matrix:

$$\begin{aligned} S^{(1)}_{\text{1-loop},1} &= \left(\frac{(-i)^2}{2}K(p,p')I^{(0)}\right)\Big|_{\mathcal{O}(\lambda^{-1})} \\ &= \frac{(-i)^2}{2}K^{(1)}(p,p')I^{(1)} \\ &= \frac{(-i)^2}{2}(p-p')K^{(1)}(p,p')\left(\frac{2(\chi_--\chi_+)^2 2pp'}{p-p'}\right)\ . \end{aligned} \tag{5.3.44}$$

Considering just $K^{(0)}$ as contributing kinematical factor, there are still two other contributions at next-to-leading order we need to consider: One from the next-to-leading order contribution to the vertex (leading to $S^{(1)}_{\text{1-loop},2}$) and one from the next-to-leading order contribution to the dispersion relation $\omega(p)$ that appears in the propagators (leading to $S^{(1)}_{\text{1-loop},3}$)). Diagrammatically, this is illustrated in 5.5.

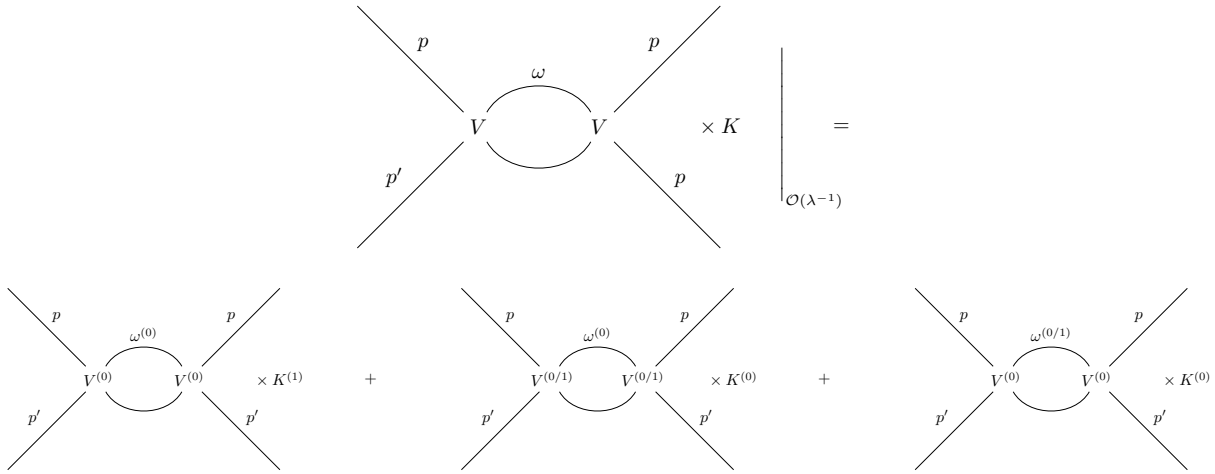


Figure 5.5: Schema of the contributions that lead to $S^{(1)}_{\text{1-loop},1}$, $S^{(1)}_{\text{1-loop},2}$, and $S^{(1)}_{\text{1-loop},3}$, respectively. The index (0/1) refers to the fact that both $\mathcal{O}(\lambda^0)$ and $\mathcal{O}(\lambda^{-1})$ ingredients are present.

The latter two contributions emerge from the following 1-loop integral that we need to analyse

$$I^{(2)} = \int \frac{d\Omega d\rho}{(2\pi)^2}\frac{\left[V_4^{(0)}(p,p',\rho,p+p'-\rho)+\frac{V_4^{(1)}(p,p',\rho,p+p'-\rho)}{\lambda}\right]^2}{[\Omega-\omega(\rho)+i\epsilon][\omega(p)+\omega(p')-\Omega-\omega(p+p'-\rho)+i\epsilon]}. \tag{5.3.45}$$

The $\int d\Omega$ integration can be performed immediately as we did before (with pole in the upper and lower half plane that we can equivalently choose for our Cauchy arc), leading us to:

$$I^{(2)} = \int \frac{d\rho}{2\pi}\frac{i\left[V_4^{(0)}(p,p',\rho,p+p'-\rho)+\frac{V_4^{(1)}(p,p',\rho,p+p'-\rho)}{\lambda}\right]^2}{\omega(p)+\omega(p')-\omega(\rho)-\omega(p+p'-\rho)+2i\epsilon}\ . \tag{5.3.46}$$

[20] And $\mathcal{O}(\lambda^0)$-order dispersion relations in the propagator, consequently.

Using the expansion in λ of the energy, i.e.

$$\omega(p) = \omega^{(0)}(p) + \omega^{(1)}(p)\lambda^{-1} + \mathcal{O}\left(\lambda^{-2}\right), \tag{5.3.47}$$

we arrive at the following expression:

$$\begin{aligned} I^{(2)} =& \int \frac{d\rho}{2\pi} \frac{i\left[V_4^{(0)}(p,p',\rho,p+p'-\rho)\right]^2}{\omega(p)+\omega(p')-\omega(\rho)-\omega(p+p'-\rho)+2i\epsilon} \\ &+ \frac{1}{\lambda}\int \frac{d\rho}{2\pi} \frac{2iV_4^{(0)}(p,p',\rho,p+p'-\rho)V_4^{(1)}(p,p',\rho,p+p'-\rho)}{\omega^{(0)}(p)+\omega^{(0)}(p')-\omega^{(0)}(\rho)-\omega^{(0)}(p+p'-\rho)+2i\epsilon} + \mathcal{O}\left(\frac{1}{\lambda^2}\right) \\ =& \bar{I}^{(2)} + \tilde{I}^{(2)} + \mathcal{O}\left(\frac{1}{\lambda^2}\right). \end{aligned} \tag{5.3.48}$$

The latter term is manifestly of order $\mathcal{O}\left(\frac{1}{\lambda}\right)$, and can be solved similarly to the integral before, just now dealing with a different numerator (again with pole at $\rho = p + i\epsilon'$):

$$\begin{aligned} \lambda\tilde{I}^{(2)} =& \int \frac{d\rho}{2\pi} \frac{2iV_4^{(0)}(p,p',\rho,p+p'-\rho)V_4^{(1)}(p,p',\rho,p+p'-\rho)}{\omega^{(0)}(p)+\omega^{(0)}(p')-\omega^{(0)}(\rho)-\omega^{(0)}(p+p'-\rho)+2i\epsilon} = \frac{2(\chi_- - \chi_+)^2 + 2pp'}{p-p'} \\ &\times \left(2(\chi_+\chi_- + pp')(\chi_- - \chi_+)^2 + pp'(p^2+p'^2) + \tilde{q}(p+p')[2pp' + (\chi_- - \chi_+)^2] + 2pp'\left[\chi_+\chi_- + \tilde{q}^2\right]\right). \end{aligned} \tag{5.3.49}$$

Re-expressing this compactly, this gives rise to the S-matrix contribution:

$$\begin{aligned} S^{(1)}_{\text{1-loop},2} &= \frac{(-i)^2}{2} K^{(0)} \tilde{I}^{(2)} \\ &= \frac{(-i)^2}{2} {K^{(0)}}^2 V_4^{(0)}(p,p',p,p') V_4^{(1)}(p,p',p,p'). \end{aligned} \tag{5.3.50}$$

The integral $\bar{I}^{(2)}$, however, is not yet manifestly of the order we want it to be. We can again use the λ-expansion for $\omega(p)$, and the fact that for large λ, we have

$$\frac{1}{a+\frac{b}{\lambda}} = \frac{1}{a} + \frac{b}{a^2}\lambda^{-1} + \mathcal{O}(\lambda^{-2}). \tag{5.3.51}$$

Applied to our case, we then get:

$$\begin{aligned} \bar{I}^{(2)} =& \underbrace{\int \frac{d\rho}{2\pi} \frac{i\left[V_4^{(0)}(p,p',\rho,p+p'-\rho)\right]^2}{\omega^{(0)}(p)+\omega^{(0)}(p')-\omega^{(0)}(\rho)-\omega^{(0)}(p+p'-\rho)+2i\epsilon}}_{=I^{(1)}, \text{as we calculated before}} \\ &+ \frac{1}{\lambda}\int \frac{d\rho}{2\pi} \frac{i\left[V_4^{(0)}(p,p',\rho,p+p'-\rho)\right]^2\left[\omega^{(1)}(p)+\omega^{(1)}(p')-\omega^{(1)}(\rho)-\omega^{(1)}(p+p'-\rho)\right]}{\left[\omega^{(0)}(p)+\omega^{(0)}(p')-\omega^{(0)}(\rho)-\omega^{(0)}(p+p'-\rho)+2i\epsilon\right]^2} + \mathcal{O}\left(\frac{1}{\lambda^2}\right). \end{aligned} \tag{5.3.52}$$

The leading order λ contribution (that is not of our interest when computing the first order λ correction to the S-matrix) appears in a natural way in this setting. The second term is computable using Cauchy's trick, as well - here we are dealing with second order poles though, and things are a bit involved, but still within computational means. Nevertheless, we shall choose a different approach to evaluate $\bar{I}^{(2)}$: Let us consider the zero-order vertex integral without expanding the dispersion relation $\omega(p)$ in terms of λ, i.e.

$$\int \frac{d\rho}{2\pi} \frac{i\left[V_4^{(0)}(p,p',\rho,p+p'-\rho)\right]^2}{\omega(p)+\omega(p')-\omega(\rho)-\omega(p+p'-\rho)+2i\epsilon}. \tag{5.3.53}$$

This is an integral with simple pole at $\rho = p$ and $\rho = p'$ (ignoring the imaginary ϵ shifts), and for an integration contour closing around the former, we get

$$\int \frac{d\rho}{2\pi} \frac{i\left[V_4^{(0)}(p,p',\rho,p+p'-\rho)\right]^2}{\omega(p)+\omega(p')-\omega(\rho)-\omega(p+p'-\rho)+2i\epsilon} = 2\pi i \lim_{\rho\to p}(\rho-p)\frac{i\left[V_4^{(0)}(p,p',\rho,p+p'-\rho)\right]^2}{\omega(p)+\omega(p')-\omega(\rho)-\omega(p+p'-\rho)+2i\epsilon} \,. \tag{5.3.54}$$

In order to see how the denominator behaves when computing the residue, the following two Taylor expansions (both around the point $\rho = p$) are useful:

$$\begin{aligned} \omega(\rho) &\simeq \omega(p) + (\rho-p)\omega'(p) + \mathcal{O}\left((\rho-p)^2\right) \,, \\ \omega(p+p'-\rho) &\simeq \omega(p') - (\rho-p)\omega'(p') + \mathcal{O}\left((\rho-p)^2\right) \,. \end{aligned} \tag{5.3.55}$$

This means that we have the following:

$$\begin{aligned} \int \frac{d\rho}{2\pi} \frac{i\left[V_4^{(0)}(p,p',\rho,p+p'-\rho)\right]^2}{\omega(p)+\omega(p')-\omega(\rho)-\omega(p+p'-\rho)+2i\epsilon} &= \frac{-\left[V_4^{(0)}(p,p',p,p')\right]^2}{\lim_{\rho\to p}\frac{(\rho-p)\omega'(p)-(\rho-p)\omega'(p')+\mathcal{O}((\rho-p)^2)}{(\rho-p)}+2i\epsilon} \\ &= \frac{\left[V_4^{(0)}(p,p',p,p')\right]^2}{\frac{\partial\omega(q)}{\partial q}\Big|_{q\to p} - \frac{\partial\omega(q)}{\partial q}\Big|_{q\to p'}} \,. \end{aligned} \tag{5.3.56}$$

We are in luck, as this just corresponds to $\left[V_4^{(0)}(p,p',p,p')\right]^2 K(p,p')$, of which we just need to isolate the λ^{-1} term, meaning $K^{(1)}(p,p')\left[V_4^{(0)}(p,p',p,p')\right]^2$. Thus, we get

$$\begin{aligned} S^{(1)}_{\text{1-loop,2}} &= \frac{(-i)^2}{2} K^{(0)} \bar{I}^{(2)} \\ &= \frac{(-i)^2}{2} K^{(0)} K^{(1)}(p,p') \left[V_4^{(0)}(p,p',p,p')\right]^2 . \end{aligned} \tag{5.3.57}$$

Having now all of our S-matrix contributions at hand, we can state our final result for the next-to-leading order S-matrix element:

$$\begin{aligned} \lambda S^{(1)}_{\text{1-loop}} &= \lambda S^{(1)}_{\text{1-loop, 1}} + \lambda S^{(1)}_{\text{1-loop, 2}} + \lambda S^{(1)}_{\text{1-loop, 3}} \\ &= \frac{(-i)^2}{2}\left[\left(K^{(0)}\right)^2 V_4^{(0)}(p,p',p,p') V_4^{(1)}(p,p',p,p') + 2K^{(0)}K^{(1)}\left(V_4^{(0)}(p,p',p,p')\right)^2\right] \,. \end{aligned} \tag{5.3.58}$$

Now, we could in principle repeat the same calculations that we performed for the tree level λ case and arrive at the n-bubble contribution - which, in turn, can be divided into three contributions, depending on whether the next-to-leading order contribution from the kinematical factor, from the vertices or from the dispersion relations are taken into account. Likewise, we could compute further terms of higher order in the Lagrangian and use them to compute higher corrections in λ of the S-matrix. As this is not very illuminating, we shall make here a caesura.

6 | Jordan Blocks and the Eclectic Spin Chain

"Sólo el misterio nos hace vivir, sólo el misterio." ["Only mystery allows us to live, only mystery."]

– Federico García Lorca

6.1 The Eclectic Spin Chain

In the preliminary chapters of this thesis, we have elaborated on the fact that spin chain Hamiltonians appear in the effective expansion of the dilatation operator acting on STO. We have discussed how this arises in undeformed theories, but as this thesis is revolving about deformed settings, we are also interested whether this method can be extended to theories exhibiting some kind of deformation. This has been done within the context of a special case of a deformed $\mathrm{AdS}_5 \times S^5$ theory called the fishnet theory (see [GK16]), a non-hermitian QFT, which means that the spin chain associated to it has a non-hermitian Hamiltonian. The γ_i-deformation is, roughly speaking, obtained by deforming the product of the fields in the Lagrangian within $\mathcal{N} = 4$ SYM theory in an integrable way - losing conformal invariance outside of the limit of infinite twisting in the process. As we have motivated in the preliminary part of this thesis, these theories can then also be studied by analysing an effective spin chain associated to it, as has been done in e.g. [FSW14] or [ISZ19], and then revisited in [AS21], whose $\mathfrak{su}(3)$-notation we shall follow.[1] On this conformal field theory side (in the limit of the strong twist), we will care about three distinct (bosonic) scalar fields ϕ^i, which the leading order dilatation operator sees as three distinct spin chain states $|i\rangle$, with $i \in \{1, 2, 3\}$, being of the form

$$\mathcal{D} = \mathcal{D}_0 + g^2 \hat{\mathbf{H}}_{\xi_1,\xi_2,\xi_3} + \mathcal{O}(g^4) = \mathcal{D}_0 + g^2 \left[\sum_{l=1}^{L} \hat{\mathbb{P}}^{l,l+1} \right] + \mathcal{O}(g^4) \ , \tag{6.1.1}$$

where we assume the spin chain to be closed - as this will let us connect the spin chain picture to STO later on - and of length L. The first term of the expansion of the dilatation operator $\mathcal{D}$ is the bare scaling dimension of the operator - which is just $\mathcal{D}_0 = L$ for scalar fields like ours. $\hat{\mathbb{P}}^{j,k}$ is an operator that is - or resembles half of - a permutation and whose action is defined as

$$\hat{\mathbb{P}} \, |21\rangle = \xi_3 \, |12\rangle \ , \qquad \hat{\mathbb{P}} \, |32\rangle = \xi_1 \, |23\rangle \ , \qquad \hat{\mathbb{P}} \, |13\rangle = \xi_2 \, |31\rangle \ , \tag{6.1.2}$$

[1] In contrast to the notation we used in the introductory part of this thesis for $\mathfrak{su}(3)$ spin chains, instead of flavours $|0\rangle, |1\rangle, |2\rangle$, in our notation the flavours present on the spin chain sites will be $|1\rangle, |2\rangle, |3\rangle$. This is a mere reflection of the different notations used in this context, and we shall choose ours in such a way that it is consistent with [AS21].

while all other matrix elements vanish. The twisting parameters are allowed to take any complex value $\xi \in \mathbb{C}$. We will define the single trace operator consisting of L ϕ^1 fields as the ground or vacuum state, meaning on the spin chain side that we define

$$|\underbrace{1, \cdots, 1}_{L \text{ times}}\rangle \tag{6.1.3}$$

as the ground state onto which we put excitations. Following (6.1.2), we can identify ϕ^2 with the right-moving excitations and ϕ^3 with the left-moving excitations. This nomenclature comes from the notion of left- and right-movers in CFT, and the fact that our Hamiltonian lets us make this identification straightforwardly. We will denote the number of ϕ^2 states present in a given state as $M - K$, whereas the number of ϕ^3 states is represented by K, such that the total number of excitations, i.e. non-ϕ^1 states is given by M, consistent with the notation we introduced before. For example, the state

$$|1, 2, 1, 1, 2, 3, 3, 1, 1, 1\rangle \tag{6.1.4}$$

is then an $L = 10, M = 4, K = 2$ state. We will always assume $K \leq M - K \leq L - M$ without loss of generality. If we have $K = 0$, then our spin chain only features one type of excitation - this case resembles the much simpler $\mathfrak{su}(2)$ setting that we are already familiar with (barring the new permutation), and the Hamiltonian can be diagonalised straight away. If $K > 0$, however, the situation is quite the opposite: The Hamiltonian is nilpotent in this case, and no nilpotent operator (that is not identically zero) is diagonalisable. The simplest way to see this is to understand in more direct terms what it is that our Hamiltonian does to a given state:

$$|\cdots, 1, \underset{|2\rangle\rightarrow}{2}, 1, \cdots, 1, 2, 3, \cdots, 3, 1, \cdots, \underset{\leftarrow|3\rangle}{1, 3}, \cdots\rangle \ . \tag{6.1.5}$$

Our Hamiltonian moves $|2\rangle$ states to the right, and $|3\rangle$ states to the left. Since $\hat{\mathbb{H}}|23\rangle = 0$, they are not able to pass through each other and have to accumulate at some point, Thus, a "23 wall" (of which there may be several) forms after successive application of the Hamiltonian, meaning[2]

$$|\cdots, 1, \underset{|2\rangle\rightarrow}{1, 1}, \cdots, 1, 1, \underset{23 \text{ wall}}{2, \cdots, 2, 3}, \cdots, 3, 1, \cdots, \underset{\leftarrow|3\rangle}{1, 1}, \cdots\rangle \ . \tag{6.1.6}$$

After having applied the Hamiltonian a sufficient number of times (for which we will provide an upper bound later on), all non-$|1\rangle$ states will have accumulated around the 23 walls and form a locked state, and these locked states vanish when applying the Hamiltonian once more on them). Periodic spin chains featuring this nilpotency property are called eclectic, as coined in [ISZ19].

After a first situational overview, we can more concisely recapitulate our intentions now: With our strongly twisted $\hat{\mathbf{H}}$, we have a linear operator on a finite dimensional vector space whose basis we can write down - we have all the necessary ingredients at hand to start our JNF triangularisation: Find eigenvalues, eigenvectors, hauptvectors, and then generate the necessary JNF basis similarity transformation matrix out of the latter two. We shall not go down this road, instead, we will diagonalise the Hamiltonian associated with the γ_i-deformation, which is diagonalisable in general, and after having solved the spectral problem for this Hamiltonian (which is much simpler due to its semi-simplicity), we will consider the limit of large twists.

Let us now talk about the spin chain associated to the γ_i deformation. In the introduction, we studied the $\mathfrak{su}(3)$ XXX Heisenberg spin chain, whose analysis will lend itself nicely to the case we are

[2] To see this, we also need to keep in mind that our chain is periodic.

about to explore: The effective spin chain that originates from considering γ_i deformation is given by an integrable deformed $\mathfrak{su}(3)$ XXX chain, and in this context we shall refer to the process of deforming it as *twisting*. The maximal number of independent twisting parameters q_i an $\mathfrak{su}(3)$ spin chain can exhibit is 3 - this is because we have three different flavour states $|i\rangle$, and any permutation operator in the Hamiltonian ideally still has to satisfy $\tilde{\mathbb{P}} \circ \tilde{\mathbb{P}} = \mathbb{1}$ (which links any matrix elements to its transposed pendant). In terms of such a maximally twisted permutation operator $\tilde{\mathbb{P}}$, we can write the Hamiltonian we are interested in as

$$\tilde{\mathbf{H}}_{(q_1,q_2,q_3)} = \sum_{l=1}^{L} \tilde{\mathbb{P}}^{l,l+1} \ , \tag{6.1.7}$$

where our twisted permutation operator $\tilde{\mathbb{P}}^{i,j}$ acts non-trivially on sites i and j as follows

$$\begin{aligned}
&\tilde{\mathbb{P}}\,|11\rangle = |11\rangle \ , && \tilde{\mathbb{P}}\,|22\rangle = |22\rangle \ , && \tilde{\mathbb{P}}\,|33\rangle = |33\rangle \ , \\
&\tilde{\mathbb{P}}\,|12\rangle = \frac{1}{q_3}\,|21\rangle \ , && \tilde{\mathbb{P}}\,|23\rangle = \frac{1}{q_1}\,|32\rangle \ , && \tilde{\mathbb{P}}\,|31\rangle = \frac{1}{q_2}\,|13\rangle \ , \\
&\tilde{\mathbb{P}}\,|21\rangle = q_3\,|12\rangle \ , && \tilde{\mathbb{P}}\,|32\rangle = q_1\,|23\rangle \ , && \tilde{\mathbb{P}}\,|13\rangle = q_2\,|31\rangle \ .
\end{aligned} \tag{6.1.8}$$

The so-constructed Hamiltonian is only Hermitian for $q_i^* = q_i^{-1}$, i.e. when the twisting parameters are complex phases $|q_i| = 1$. Fortunately, it is semi-simple for any finite value of the twisting parameters, though, as was remarked in footnote 2 in [AS21].

In the (normalised) limit of strong twisting, we recover the eclectic spin chain Hamiltonian we alluded to before, i.e.

$$\hat{\mathbf{H}}_{\xi_1,\xi_2,\xi_3} = \lim_{\epsilon\to 0} \epsilon \tilde{\mathbf{H}}_{(\frac{\xi_1}{\epsilon},\frac{\xi_2}{\epsilon},\frac{\xi_3}{\epsilon})} \ , \tag{6.1.9}$$

however, the eigenbasis for the Hilbert space stemming from $\tilde{\mathbf{H}}$ ceased to be a basis once the limit of strong twisting is applied: The Bethe vectors (i.e. eigenstates of the energy) of $\tilde{\mathbf{H}}$ for any finite value of $q_i = \frac{\xi_i}{\epsilon}$ coalesce into states of the following form in the limit of vanishing ϵ:

$$|\psi(k)\rangle = \sum_{l=1}^{L} e^{2\pi k l i/L} U^l |1,\ldots,1,2,\ldots,2,3,\ldots,3\rangle \ , \tag{6.1.10}$$

where L is the length of the spin chain, and U is the shift operator whose action is defined via $U|n_1,\ldots,n_{L-1},n_L\rangle = |n_2,\ldots,n_L,n_1\rangle$. Excitations are not able to move from their relative positions in states with this configuration, and we call these states *locked*. They are, of course, eigenstates of the Hamiltonian $\hat{\mathbf{H}}$ for any ξ_1,ξ_2,ξ_3, but since they do not span out our Hilbert space but only a subspace thereof, so we do need more advanced methods to be able to make up for this loss in dimensions, which is going to be the goal of the sections to follow.

6.2 Exceptional Points and Jordan Blocks

6.2.1 A Twisted Warm Up: The $L = 3, M = 2, K = 1$ Case

As we explained before, the strongly twisted limit of $\tilde{\mathbf{H}}$ is non-diagonalisable for $K > 0$, and since $L - M \geq M - K \geq K$, the simplest non-diagonalisable case is $K = 1, M = 2, L = 3$. It is our intention now to compute the eigenvectors of $\tilde{\mathbf{H}}_{q_1,q_2,q_3}$, and then study the spectrum's behaviour as the twisting parameters approach exceptional points, meaning points where the Hamiltonian ceases to be diagonalisable.

Without any previous knowledge about the situation, the most general ansatz we can make for a state of an $L = 3$ spin chain is given by[3]

$$\psi = \sum_{\sigma \in \mathcal{S}_3} \psi(\sigma) |\sigma(1)\sigma(2)\sigma(3)\rangle \ . \tag{6.2.1}$$

The twisted Hamiltonian transforms even elements of $\mathcal{S}_3$ into odd elements of $\mathcal{S}_3$ - this is because it consists of transposition permutations only, and the signature of an element in $\mathcal{S}_n$ is multiplicative. In the basis

$$\{|123\rangle, |231\rangle, |312\rangle, |213\rangle, |132\rangle, |321\rangle\} \ , \tag{6.2.2}$$

it takes the following form

$$\tilde{\mathbf{H}}_{(q_1,q_2,q_3)} = \left(\begin{array}{ccc:ccc} 0 & 0 & 0 & q_3 & q_1 & q_2 \\ 0 & 0 & 0 & q_2 & q_3 & q_1 \\ 0 & 0 & 0 & q_1 & q_2 & q_3 \\ \hdashline \frac{1}{q_3} & \frac{1}{q_2} & \frac{1}{q_1} & 0 & 0 & 0 \\ \frac{1}{q_1} & \frac{1}{q_3} & \frac{1}{q_2} & 0 & 0 & 0 \\ \frac{1}{q_2} & \frac{1}{q_1} & \frac{1}{q_3} & 0 & 0 & 0 \end{array}\right) \ . \tag{6.2.3}$$

As we mentioned before, for any choice of q_i's, this matrix is diagonalisable. The characteristic polynomial (which has to be even for matrices of this form) is,

$$\begin{aligned} \mathrm{char}_{\tilde{\mathbf{H}}_{(q_1,q_2,q_3)}}(x) = x^6 - 9x^4 - \frac{3q_1^5q_2^2q_3^2 + 3q_1^4q_2q_3\left(q_2^3+q_3^3\right) - 18q_1^3q_2^3q_3^3 + 3q_1^2q_2^2q_3^2\left(q_2^3+q_3^3\right) + 3q_1q_2^4q_3^4}{q_1^3q_2^3q_3^3}x^2 \\ - \frac{q_1^6\left(q_2^3+q_3^3\right) - 3q_1^5q_2^2q_3^2 - 3q_1^4q_2q_3\left(q_2^3+q_3^3\right) + q_1^3q_2^6 + 12q_1^3q_2^3q_3^3 + q_1^3q_3^6 - 3q_1^2q_2^2q_3^2\left(q_2^3+q_3^3\right) - 3q_1q_2^4q_3^4 + q_2^3q_3^3\left(q_2^3+q_3^3\right)}{q_1^3q_2^3q_3^3} \ . \end{aligned} \tag{6.2.4}$$

The evenness of the characteristic polynomial for this case forces the eigenvalues to come in pairs,[4]

$$\lambda_1^{\pm} = \pm\sqrt{\sum_{i,j}\frac{q_i}{q_j}} \ , \tag{6.2.5}$$

$$\lambda_2^{\pm} = \pm\frac{1}{\sqrt{2}}\sqrt{9 + \frac{(q_1-q_2)(q_2-q_3)(q_3-q_1)}{q_1q_2q_3} - \sum_{i,j}\frac{q_i}{q_j}} \ , \tag{6.2.6}$$

$$\lambda_3^{\pm} = \pm\frac{1}{\sqrt{2}}\sqrt{9 - \frac{(q_1-q_2)(q_2-q_3)(q_3-q_1)}{q_1q_2q_3} - \sum_{i,j}\frac{q_i}{q_j}} \ . \tag{6.2.7}$$

We can check that, indeed, the Hamiltonian becomes nilpotent in the limit of strong twists, meaning that all its eigenvalues will vanish: $E_i^{\pm} = \lim_{\epsilon\to 0}\epsilon\lambda_i^{\pm} = 0$. The analysis to follow is the same for all three pairs of eigenvectors (and their eigenvalues), but focusing on the eigenvectors associated to the first pair of eigenvalues,

$$v_1^{\pm} = \sqrt{\frac{(q_1+q_2+q_3)^2}{2(q_1+q_2+q_3)^2 + 3(\lambda_1^{\pm})^2}}\left(\frac{q_1+q_2+q_3}{\lambda_1^{\pm}}, \frac{q_1+q_2+q_3}{\lambda_1^{\pm}}, \frac{q_1+q_2+q_3}{\lambda_1^{\pm}}, 1, 1, 1\right) \ , \tag{6.2.8}$$

they become collinear (and thus linearly dependent) in the large twist limit

$$\lim_{\epsilon\to 0} v_1^{\pm} = \frac{\pm 1}{\sqrt{3}}(1,1,1,0,0,0) \ . \tag{6.2.9}$$

[3] Here, once more we are dealing with the change in notation - our choice here merely reflects the most natural choice in each situation.

[4] The sums $\sum_{i,j}$ run over the all possible combinations, also the ones where i and j coincide.

Here, we can explicitly see how we lose vectors of our base when we approach the non-diagonalisable limit, as in the strong twist limit, the Hamiltonian has the following JNF form,

$$\hat{\mathbf{H}} \simeq \left(\begin{array}{cc:cc:cc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) , \tag{6.2.10}$$

meaning that we have three 1-dimensional eigenspaces (represented by one block of size 2 each), and therefore cannot span out the 6-dimensional space. This illustrates that, pairwise, eigenvectors will coalesce and we will only have their distinct true eigenvectors after having taken the limit. We "lose" half of our Hilbert space - as was what the authors observed in [AS21].

We shall try to gain deeper insight as to whether we can make up for these lost dimensions by finding information about the eigensystem of the Hamiltonian *before* the limit is taken. The authors in [GN16] had a similar motivation when analysing a spin chain in a slightly different context, and although the ideas in that work did not treat any Jordan blocks bigger than size 2, their initial intention was instrumental in the formulation of our methodology.

The starting point and guiding philosophy for our approach is that, while the twist parameters still are of finite value, we want to consider *linear combinations* of eigenvectors that coalesce to the same vector and analyse their behaviour once we take our limit. To show in a first instance that there is some promise in this idea, for the case above, if we consider the following linear combinations of eigenvectors[5]

$$v_{\text{example}} = \frac{v_1^+ + v_1^-}{|v_1^+ + v_1^-|} = \frac{1}{\sqrt{3}}\,(0,0,0,1,1,1) \ . \tag{6.2.11}$$

One can check that

$$(\hat{\mathbf{H}} - 0\mathbb{1})^2 v_{\text{example}} = 0 \text{ while } (\hat{\mathbf{H}} - 0\mathbb{1})^1 v_{\text{example}} \neq 0, \tag{6.2.12}$$

making v_{example} a generalised eigenvector of rank 2 of $\lim_{\epsilon\to 0} v_1^+ = \lim_{\epsilon\to 0} v_1^-$. For the other two pairs of eigenvectors, the same procedure reveals all the missing generalised eigenvectors in the limit of strong twisting, and our Hilbert space basis is completed. We, although coincidentally for the moment it might seem, in this way recover information about the generalised eigentheory problem of the strongly twisted Hamiltonian *prior to twisting*, which is precisely what we wanted. We systematise this procedure in a formal way in the next subsection.

We will in the next subsections go on to motivate that the geometric multiplicity of the eigenvalues of the defective matrix in the limit is indicative of how involved our analysis is going to be. As one might expect, the study will turn out to be much more involved as soon as the geometric multiplicity of any eigenvalue exceeds one, as in this case the eigenvalue has several Jordan blocks in the JNF - so when we refer to "the" Jordan block associated to λ, there is ambiguity. Our introductory example might at first glance seem already to be more involved than any preliminary minimal case we could consider, as

[5] In the normalisation that we chose, the linear combination does not depend on the twist parameters anymore - however, we are still in the setting of finite twist.

the three Jordan blocks featured in the JNF are all associated to the eigenvalue 0 - we have geometric multiplicity three! The reason why the eigenvectors associated to the different blocks do not "interfere" with each other is that they are associated with different total twisted momenta, where the momentum operator is the generator of the shift operator U we introduced before. We have $[\hat{\mathbf{H}}, U] = 0$, thus the eigenvectors that correspond to different momenta cannot mix, even if they are associated to the same (vanishing) energy. This is most easily made apparent if we make the representation of the U operator in the basis we used for (6.2.3) explicit,

$$U \doteq \left(\begin{array}{ccc:ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right) . \tag{6.2.13}$$

In the block diagonal form, we can easily see that the eigenvectors in (6.2.8) are also eigenvectors of U, and that each of the three pairs share indeed the same charge under U, explaining the pairwise grouping and non-mixing.

6.2.2 Distinguishable Jordan Blocks

In a first, we limit our considerations to the cases where all the eigenvalues of a matrix have geometric multiplicity one - meaning that at most one Jordan block is generated by each distinct eigenvalue, we call this the *distinguishable case*.

We first introduce the necessary ingredients for the methodology we want to develop: Let $M \in M_N(\mathbb{C})$ be a complex $N \times N$ matrix. Furthermore, let $M = M(\epsilon)$ also continuously depend on a complex parameter ϵ. It is our assumption that $M(\epsilon)$ is semi-simple *almost everywhere*, meaning $\forall \epsilon \in \mathbb{C}$ except for finitely many. We will parametrise the ϵ-arc in such a way, that $M(\epsilon)$ ceases to be diagonalisable at $\epsilon = 0$, forming an exceptional point, where it becomes a defective matrix whose JNF exhibits a single Jordan block of size N. The eigenvectors and eigenvalues of $M(\epsilon)$, v_i and λ_i, naturally depend on ϵ, too: As the entries of $M(\epsilon)$ depend on ϵ, so will the characteristic polynomial and the eigenvector equation, subsequently. We will, however, suppress making this dependence explicit wherever it is natural to do so with the aim of uncluttering our expressions.

Let $\sigma(M(\epsilon)) = \{\lambda_1, \cdots, \lambda_r\}$ be the spectrum of $M(\epsilon)$. Since we assume that in the limit of $\epsilon \to 0$, the matrix is similar to $M(\epsilon) \xrightarrow{\sim} J$, where J is a Jordan block of size N associated to an eigenvalue, let us call it λ. As we have $\sigma(\lim_{\epsilon\to 0} M(\epsilon)) = \sigma(J) = \{\lambda\}$, we have $\lim_{\epsilon\to 0}\{\lambda_1, \cdots, \lambda_r\} = \{\lambda\}$, implying that in the limit all the eigenvalues coincide and we get

$$\lim_{\epsilon\to 0} \lambda_i = \lim_{\epsilon\to 0} \lambda_j = \lambda \tag{6.2.14}$$

for any two eigenvalues λ_i, λ_j of $M(\epsilon)$. This has an important implication on the eigenvectors in the limit: As the relevant functions in our quantities depend continuously on ϵ and products of finite well-defined limits equal the limit of the product, we have that

$$0 = \lim_{\epsilon\to 0} [(M(\epsilon) - \lambda_i \mathbb{1})v_i] = \left[\lim_{\epsilon\to 0}(M(\epsilon) - \lambda_i \mathbb{1})\right] \left[\lim_{\epsilon\to 0} v_i\right] = \left[M(0) - \mathbb{1} \lim_{\epsilon\to 0} \lambda_i\right] \left[\lim_{\epsilon\to 0} v_i\right] . \tag{6.2.15}$$

We argued before that all the eigenvalues coincide in the limit, and therefore $[M(0) - \mathbb{1} \lim_{\epsilon\to 0} \lambda_i]$ is independent of the index i. Furthermore, $\lim_{\epsilon\to 0} v_i$ solves the eigenvector equation for $M(0)$, and we know that $M(0)$ is similar to a Jordan block and thus only has one true eigenvector,[6] implying that all the eigenvectors v_i of $M(\epsilon)$ coincide in the limit as well (up to scalar factors, so they are collinear). What the authors in [AS21] call the "collapse of Bethe vectors" reflects this fact and is only to be expected. As the geometric multiplicity in this setting is one - we are in the distinguishable case - we cannot draw the same conclusion for the indistinguishable case, and things will be more involved later on.

We have now seen how eigenvectors can appear in the limit this way, it is only natural to ask ourselves now whether generalised eigenvectors can appear as limits, too. Let us first look at the following linear combination

$$[M(\epsilon) - \lambda_1 \mathbb{1}][M(\epsilon) - \lambda_2 \mathbb{1}](\alpha_1\, v_1 + \alpha_2\, v_2) = 0\ , \tag{6.2.16}$$

where $\alpha_i \in \mathbb{C}$ might depend on ϵ. In the $\epsilon \to 0$ limit, this equation becomes

$$\begin{aligned}\Big[M(0) - \mathbb{1} \lim_{\epsilon\to 0} \lambda_1\Big] \Big[M(0) - \mathbb{1} \lim_{\epsilon\to 0} \lambda_2\Big] \Big[\lim_{\epsilon\to 0}(\alpha_1\, v_1 + \alpha_2\, v_2)\Big] = \\ = \Big[M(0) - \mathbb{1} \lim_{\epsilon\to 0} \lambda_i\Big]^2 \Big[\lim_{\epsilon\to 0}(\alpha_1\, v_1 + \alpha_2\, v_2)\Big] = 0\ ,\end{aligned} \tag{6.2.17}$$

with λ_i being eigenvalues of $M(\epsilon)$. The last equality in (6.2.17) is the defining equation of generalised eigenvectors of rank 2, and it implies that, somehow, this linear combination in the limit encodes some information about the generalised eigensystem of the matrix we are considering. We have shown above that the linear combination $(\alpha_1\, v_1 + \alpha_2\, v_2)$ satisfies one of the two conditions that could make it a candidate for a rank 2 generalised eigenvector, however, for general values α_i it is just a true eigenvector only, since

$$\lim_{\epsilon\to 0}[M(\epsilon) - \lambda_1 \mathbb{1}](\alpha_1\, v_1 + \alpha_2\, v_2) = \lim_{\epsilon\to 0} \alpha_2 \underbrace{(\lambda_2 - \lambda_1)}_{=0 \text{ in the limit}} v_2 = 0\ . \tag{6.2.18}$$

Therefore, the limit of $(\alpha_1\, v_1 + \alpha_2\, v_2)$ is only part of $\ker\{[M(0) - \mathbb{1} \lim_{\epsilon\to 0} \lambda_i]^2\}$ because it is also part of $\ker\{[M(0) - \mathbb{1} \lim_{\epsilon\to 0} \lambda_i]\}$, but we do not want the latter to be the case. This can be avoided by considering a coefficient α_2 that diverges as $(\lambda_2 - \lambda_1)^{-1}$. The same must hold for α_1. It is particularly useful to assume that indeed both $\alpha_i^{-1} \propto |v_1 - v_2|$, all while assuming that $\lim_{\epsilon\to 0}(\lambda_2 - \lambda_1)(\alpha_1\, v_1 + \alpha_2\, v_2) = 0$ for well-definedness. Conversely, if we find two (finite) α_1 and α_2 such that they do not vanish in the $\epsilon \to 0$ limit and $\lim_{\epsilon\to 0}(\alpha_1\, v_1 + \alpha_2\, v_2) = 0$, then $\lim_{\epsilon\to 0} \frac{\alpha_1\, v_1 + \alpha_2\, v_2}{\lambda_1 - \lambda_2}$ gives us the generalised eigenvector of rank 2, since in this case we have

$$\lim_{\epsilon\to 0}[M(\epsilon) - \lambda_1 \mathbb{1}] \frac{(\alpha_1\, v_1 + \alpha_2\, v_2)}{\lambda_1 - \lambda_2} = \lim_{\epsilon\to 0} \alpha_2\, v_2 \neq 0\ , \tag{6.2.19}$$

and likewise for λ_2 as a starting point.

We need to remind ourselves that we are working with complex vector spaces, meaning that collinear unital vectors can differ by a phase factor (as opposed to just a sign), which means that for coalescing eigenvectors we can have $\lim_{\epsilon\to 0} v_i = e^{i\phi_{ij}} \lim_{\epsilon\to 0} v_j$. We introduce a factor β to take this phase difference

[6] Recall that the number of Jordan blocks in a JNF corresponds to the dimension of the eigenspace associated to an eigenvalue.

into account in the following linear combinations[7]

$$w_{ij}^{\pm} = \frac{v_i \pm \beta_{ji} v_j}{|v_i \pm \beta_{ji} v_j|} \ , \qquad \text{with} \qquad \beta_{ji} = v_j^\dagger v_i \ , \tag{6.2.20}$$

with $|v|^2 = v^\dagger v$ being the usual vector norm. Then, $\lim_{\epsilon\to 0} w_{ij}^{+}$ is equal - up to a potential phase factor - to $\lim_{\epsilon\to 0} v_i$. The expression $\lim_{\epsilon\to 0} w_{ij}^{-}$ is in the limit indeterminate (of type "$\frac{0}{0}$"). With our previous considerations, the latter can then be seen to be the generalised eigenvector of rank 2 when applying L'Hôpital's rule. In proposition 4.3 in [GN16], a similar result was motivated, for which we have now found first-hand explanation. The choice of v_i, v_j is inconsequential, and $\lim_{\epsilon\to 0} w_{ij}^{-}$ will always equal the generalised eigenvector of rank 2 (see 2.1.1). This is a consequence of the fact that we are in the case of geometric multiplicity 1, so we know that we (at most) get one linearly independent eigenvector of rank 2.

Having discussed how we get generalised eigenvectors of rank 2, we now extend this procedure to generalised eigenvectors of rank n: For this to be achievable, we need linear combinations of $M(\epsilon)$ of order n. The "candidate" vector combination from which we want to construct a generalised eigenvector of rank n satisfies

$$\left[M(0) - \mathbb{1} \lim_{\epsilon\to 0} \lambda_i\right]^n \left[\lim_{\epsilon\to 0} \left(\sum_{i=1}^{n} \alpha_i \, v_i\right)\right] = 0 \ , \tag{6.2.21}$$

for any constants α_i provided that v_i are, as before, eigenvectors. Here again, as this is one of the two defining equations for generalised eigenvectors of rank n, so we want to manipulate the combination $\lim_{\epsilon\to 0} (\sum_{i=1}^{n} \alpha_i \, v_i)$ in such a way that we can extract the generalised eigenvectors of rank n from this. The previous method can recursively be extended by substituting the (true) eigenvectors v_i by the appropriate ones[8][9]

$$w_{ij}^{(n)} = \frac{w_{ji}^{(n-1)} - \beta_{kj}^{(n-1)} w_{ki}^{(n-1)}}{|w_{ji}^{(n-1)} - \beta_{kj}^{(n-1)} w_{ki}^{(n-1)}|} \ , \qquad \text{with} \qquad \beta_{kj}^{(n-1)} = (w_{ki}^{(n-1)})^\dagger w_{ji}^{(n-1)} \quad \text{and} \quad w_{ij}^{(0)} = v_i \ . \tag{6.2.22}$$

There are some caveats associated to this: With the above, we are actually solving the condition

$$\left[M(0) - \mathbb{1} \lim_{\epsilon\to 0} \lambda_i\right]^{n+1} w^{(n)} = 0 \ , \tag{6.2.23}$$

which does not provide us with the full picture, as we do impose $[M(0) - \mathbb{1} \lim_{\epsilon\to 0} \lambda_i]^n \, w^{(n)} \neq 0$ (recalling the definition of a generalised eigenvector of rank $n+1$), but we do not impose

$$\left[M(0) - \mathbb{1} \lim_{\epsilon\to 0} \lambda_i\right] w^{(n)} = w^{(n-1)} \ . \tag{6.2.24}$$

By solving for a weaker condition, we do not get "pure" generalised eigenvectors of rank $n+1$, but rather a linear combination of generalised eigenvectors up to rank $n+1$ (as all of them also satisfy the relation (6.2.23)), with the guarantee that there is a non-zero contribution of the generalised eigenvector

[7] Throughout this project, we will encounter normalised limits of this form which, technically speaking, are indefinite - in 1 dimension, the limit

$$\lim_{x\to 0} \frac{x}{|x|}$$

is not defined. This is because the limit value depends on the "direction" with which x is approaching 0. For some directional choices of $x \to 0$, the above limit exists. Likewise, for limits of this form in our context, we just need to think of the limits as directional limits where necessary, as the resulting vector - which might get a different phase depending on the direction we choose - will have the desired properties.

[8] In the introduction (see (2.1.5)), we defined $v^{(n)}$ as the generalised eigenvector of rank n, and here $w_{ij}^{(n)}$ will give rise to the generalised eigenvector of rank $n+1$.

[9] Here, k is not being summed over, but we choose a compact form of notation that lets us drop the k-dependence where it is inconsequential.

of rank $n+1$ in this linear combination. Finally, it goes without saying that we cannot recursively use this process ad infinitum: As a starting point, we used n distinct eigenvectors to arrive at a generalised eigenvector of rank n. As a Jordan block of size N (which $M(\epsilon)$ in the $\epsilon \to 0$ limit is similar to) can only give rise to N independent generalised eigenvectors, we can only repeat this recursion until $w_{ij}^{(N-1)}$. To go beyond $N-1$ as a superindex, we just do not have the necessary ingredients.

Let us state and prove the following proposition that addresses the completeness of our recipe:

Proposition 6.2.1. *Let $n \leq N-1$, and be the setting as above. Then the following hold:*

- *Every $\lim_{\epsilon\to 0} w_{ij}^{(n)}$ is neither zero nor does it diverge.*
- *$\lim_{\epsilon\to 0} w_{ij}^{(n)}$ is orthogonal to any $\lim_{\epsilon\to 0} w_{ij}^{(m)}$ with $m \neq n$.*

Proof.

- All $w^{(n)}$ are normalised for any ϵ, and as the norm is the composition of continuous functions, we know that the limit of the norm is equal to the norm of the limit. Therefore, $\lim_{\epsilon\to 0} w_{ij}^{(n)} \notin \{0, \pm\infty\}$ as it is normalised to be 1 for any value of ϵ.

- $\boxed{n=2, m=1}$ We start with generalised eigenvectors of rank 1 and 2. The claim is then that $(\lim_{\epsilon\to 0} v_j)^\dagger (\lim_{\epsilon\to 0} w_{ij}^{(1)})$ vanishes. Rather than computing this limit, it is more straightforward to equivalently compute $\lim_{\epsilon\to 0}\lim_{\epsilon'\to\epsilon}(v_j(\epsilon))^\dagger (w_{ij}^{(1)}(\epsilon'))$. The $\epsilon' \to \epsilon$ limit can be continuously taken right away, and using (6.2.20), we arrive at

$$(\lim_{\epsilon\to 0} v_j)^\dagger (\lim_{\epsilon\to 0} w_{ij}^{(1)}) = \lim_{\epsilon\to 0}\lim_{\epsilon'\to\epsilon}(v_j(\epsilon))^\dagger (w_{ij}^{(1)}(\epsilon')) = \lim_{\epsilon\to 0}(v_j(\epsilon))^\dagger (w_{ij}^{(1)}(\epsilon)) = \lim_{\epsilon\to 0}\frac{(v_j^\dagger\cdot v_i) - \beta_{ji}(v_j^\dagger\cdot v_j)}{|v_i - \beta_{ji} v_j|}\,. \tag{6.2.25}$$

If we now use the definition of β_{ji} and use the fact that the eigenvectors are normalised, the numerator vanishes for any ϵ, including in the (continuous) point $\epsilon = 0$:

$$\frac{(v_j^\dagger\cdot v_i) - v_j^\dagger\cdot v_j \overbrace{(v_j^\dagger\cdot v_j)}^{=1}}{|v_i - v_j^\dagger\cdot v_j v_j|} = \frac{0}{|v_i - v_j^\dagger\cdot v_j v_j|}\,. \tag{6.2.26}$$

Analogously for $\boxed{n, m = n-1}$, we can more generally prove the orthogonality between the vectors $w_{ij}^{(n)}(\epsilon)$ and $w_{jk}^{(n-1)}(\epsilon)$ in the limit

$$\begin{aligned}(w_{jk}^{(n-1)}(\epsilon))^\dagger (w_{ij}^{(n)}(\epsilon)) &= \frac{(w_{jk}^{(n-1)})^\dagger\cdot w_{ik}^{(n-1)} - \beta_{ji}^{(n-1)}(w_{jk}^{(n-1)})^\dagger\cdot w_{jk}^{(n-1)}}{|w_{ik}^{(n-1)} - \beta_{ji}^{(n-1)} w_{jk}^{(n-1)}|} \\ &= \frac{(w_{jk}^{(n-1)})^\dagger\cdot w_{ik}^{(n-1)} - (w_{jk}^{(n-1)})^\dagger\cdot w_{ik}^{(n-1)} \overbrace{(w_{jk}^{(n-1)})^\dagger\cdot w_{jk}^{(n-1)}}^{=1}}{|w_{ik}^{(n-1)} - (w_{jk}^{(n-1)})^\dagger\cdot w_{ik}^{(n-1)} w_{jk}^{(n-1)}|} \\ &= \frac{0}{|w_{ik}^{(n-1)} - (w_{jk}^{(n-1)})^\dagger\cdot w_{ik}^{(n-1)} w_{jk}^{(n-1)}|}\,,\end{aligned} \tag{6.2.27}$$

which vanishes in the same way.

The remaining scalar products $\boxed{n, m}$ are proven inductively: Using the definition of $w_{ij}^{(m+1)}(\epsilon)$, it is easy to similarly show that $w_{ij}^{(n)}(\epsilon)$ and $w_{ij}^{(m+1)}(\epsilon)$ are orthogonal provided $w_{ij}^{(n)}(\epsilon)$ and $w_{ij}^{(m)}(\epsilon)$ are orthogonal (assuming that the superindices never coincide). With our starting point that we have proven the claim for indices pairs $(n, n-1)$, the just established fact that $(n, m) \Rightarrow (n, m+1)$ and that $(n, m) \Leftrightarrow (m, n)$. Thus, we have the claim for the following index pairs:

$$
\begin{array}{ccccccc}
(1,2) & \longrightarrow & (1,3) & \longrightarrow & \cdots & \longrightarrow & (1,N-1) \\
(2,3) & \longrightarrow & (2,4) & \longrightarrow & \cdots & \longrightarrow & (2,N-1) \\
\vdots & & & & & & \vdots \\
(N-6,N-5) & \longrightarrow & (N-6,N-4) & \longrightarrow & \cdots & \longrightarrow & (N-6,N-1) \\
(N-5,N-4) & \longrightarrow & (N-5,N-3) & \longrightarrow & (N-5,N-2) & \longrightarrow & (N-5,N-1) \\
(N-4,N-3) & \longrightarrow & (N-4,N-2) & & & \longrightarrow & (N-4,N-1) \\
(N-3,N-2) & & & & & \longrightarrow & (N-3,N-1)
\end{array}
$$

and, finally, $(N-2,N-1)$.

This concludes our induction and proves our claim.

□

From this proposition, we can infer an important isomorphism corollary:

Corollary 6.2.1.1. *The* $\text{span}\{\lim_{\epsilon\to 0} v_i, \lim_{\epsilon\to 0} w_{ij}^{(1)}, \ldots, \lim_{\epsilon\to 0} w_{ij}^{(N-1)}\}$ *is a vector space of dimension* N *and isomorphic to the vector space spanned by the set of generalised eigenvectors of* $M(0)$ *in the sense that the identification of each individual* $w^{(n)}$ *with its corresponding* $ker(M-\lambda\mathbb{1})^{(n+1)}/ker(M-\lambda\mathbb{1})^{(n)}$ *can be made.*

Proof. For the resulting $\lim_{\epsilon\to 0} w_{ij}^{(n-1)}$, it is irrelevant what initial choice of eigenvectors we make in their construction, and we shall take one representative for each $w_{ij}^{(n)}$. The previous proposition implies that the N vectors that generate this space are all non-zero and orthogonal, proving the former part of the claim.

$\lim_{\epsilon\to 0} w_{ij}^{(n-1)}$ is a linear combination of generalised eigenvectors of up to rank n. We are still in the case of geometric multiplicity 1, and $\ker\{(M(0)-\lambda\mathbb{1})^n\}/\ker\{(M(0)-\lambda\mathbb{1})^{n-1}\}$ is one-dimensional. Hence, $\ker\{(M(0)-\lambda\mathbb{1})^n\}$ and $\text{span}\{\lim_{\epsilon\to 0} v_i, \lim_{\epsilon\to 0} w_{ij}^{(1)}, \ldots, \lim_{\epsilon\to 0} w_{ij}^{(n-1)}\}$ are isomorphic as vector spaces in the sense that we established that each of these sub-vector spaces are as was stated in the claim. □

Indeed, we have now shown that the vectors $\{\lim_{\epsilon\to 0} v_i, \lim_{\epsilon\to 0} w_{ij}^{(1)}, \ldots, \lim_{\epsilon\to 0} w_{ij}^{(N-1)}\}$ that we generate using our methodology correspond to the full set of generalised eigenvectors of $M(0)$, and we can read off the rank of each generalised eigenvector by the superindex (plus one) we assigned it. However, in addition to the restriction that the geometric multiplicity of the eigenvalue λ is not allowed to be larger than 1, we implicitly also assumed that $M(0)$ is not allowed to feature any other eigenvalues - this of course is not true in general. However, it suffices to analyse each eigenvalue separately, and our methodology is correct as long as all the present eigenvalues have geometric multiplicity one.

From the point of view of constructing an algorithmic method to find the generalised eigensystem of matrix that is non-diagonalisable for some values of the parameter that it depends on, we provided a concise recipe of steps to follow - but some of the steps featured expressions that, while correct, are not particularly "user-friendly" in computations.[10] We shall make two simplifying remarks to make things more computationally accessible:

- In (6.2.20), we always make sure to normalise our vectors - by dividing by the norm. However, this is not wholly necessary: The reason for the normalisation was mainly to provide us with the correct power in ϵ of a given vector. Therefore, instead of dividing by the whole norm in denominator, we can just divide by the ϵ-power that arises from it. We shall make use of this fact while applying our methodology within the context of spin chains later on.

- While we always assume the limits to exist and be well-defined where we take them, there is another subtlety we have not addressed so far: In order to get the generalised eigenvector of rank $n+1$, we need to compute the difference between eigenvectors of a diagonalisable matrix. If however, in the process, we already take the limit at some point when dealing with the $w^{(n-1)}$ vectors and end up continuing our computation with the generalised eigenvectors of the non-diagonalisable matrix in the limit, we potentially end up with the generalised eigenvector of the wrong rank. This is due to the fact that

$$\lim_{\epsilon\to 0}\frac{w_{ji}^{(n-1)}-\beta_{kj}^{(n-1)}w_{ki}^{(n-1)}}{|w_{ji}^{(n-1)}-\beta_{kj}^{(n-1)}w_{ki}^{(n-1)}|}\neq\lim_{\epsilon\to 0}\frac{w_{ji}^{(n-1)}-\lim_{\epsilon\to 0}[\beta_{kj}^{(n-1)}w_{ki}^{(n-1)}]}{|w_{ji}^{(n-1)}-\beta_{kj}^{(n-1)}w_{ki}^{(n-1)}|}\,, \tag{6.2.28}$$

  for general $w^{(n-1)}$. It is therefore imperative to always observe the correct order of "limits", as to avoid spoiling the result, which is a key difference between our method and [GN16]. We illustrate this in the appendix.

6.2.3 Indistinguishable Jordan Blocks

We now want to extend our methodology for the case of a matrix that (at an exceptional point) is defective possessing eigenvalues with geometric multiplicity larger than 1 - so far our recipe and statements depended heavily on this fact. As we shall see in an instant, there are some intricacies associated to the non-distinguishable case, and we will illustrate this both by making examples within this subsection (related to the spin chain scenario we are interested in) and more algebraic ones in the appendix.

We are starting this off by considering the simplest setting where the strong-limit Hamiltonian $\hat{\mathbf{H}}_{\xi_1,\xi_2,\xi_3}$ develops more than one Jordan block (of non-minimal size) and that exhibits the subtlety that we want to illustrate - with the additional assumption that the total twisted momentum of the states vanishes. This is given by the $L=5, M=3, K=1$ spin chain. A priori, this spin chain has

$$\binom{5}{2,2,1}=30 \tag{6.2.29}$$

[10] To mention one such instance: We will later apply our methodology to the context of Bethe ansätze, where the (diagonalisable) eigensystem is already presented in an involved form. With these starting ingredients, it might be difficult to compute scalar products and norms of linear combinations were we to blindly follow our presented expressions thus far.

states that we need to include in our basis. Thanks to the shift operator commuting with the Hamiltonian, we can focus only on the vanishing momentum sector,[11], as the other sectors have the same structure (as discussed in [AS21]), reducing the number of states to $\frac{30}{5} = 6$ - which is manageable as a matrix representation. This case has already been studied in [AS21], and the authors claimed that the Hamiltonian was similar to a JNF with one Jordan block of size 5 and one of size 1, i.e.

$$\tilde{\mathbf{H}}^{L=5,M=3,K=1} \simeq \left(\begin{array}{ccccc:c} & & J_1 & & & 0 \\ & & & & & 0 \\ & & & & & 0 \\ & & & & & 0 \\ & & & & & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & J_2 \end{array}\right) . \tag{6.2.30}$$

Before applying the limit of strong twist, the finite twist (spin chain) Hamiltonian takes the following form for the same restriction of zero total momentum:

$$\tilde{\mathbf{H}}^{L=5,M=3,K=1}_{(q_1,q_2,q_3)} = \begin{pmatrix} 2 & q_3 & q_1 & q_2 & 0 & 0 \\ \frac{1}{q_3} & 0 & q_3 & q_3 & q_1+q_2 & 0 \\ \frac{1}{q_1} & \frac{1}{q_3} & 1 & 0 & q_3 & q_1 \\ \frac{1}{q_2} & \frac{1}{q_3} & 0 & 1 & q_3 & q_2 \\ 0 & \frac{1}{q_1}+\frac{1}{q_2} & \frac{1}{q_3} & \frac{1}{q_3} & 0 & q_3 \\ 0 & 0 & \frac{1}{q_1} & \frac{1}{q_2} & \frac{1}{q_3} & 2 \end{pmatrix} . \tag{6.2.31}$$

In an initial step, we shall still stick to the methodology for the case of geometric multiplicity not exceeding 1 in order to precisely see where our established toolset breaks down, and only then try to find a remedy for it. We first shall use (6.2.31) to make its eigensystem more accessible, as this matrix is otherwise very tedious to diagonalise. Here it is important to remember that $\tilde{\mathbf{H}}^{L=5,M=3,K=1}$ has an auxiliary role in so far as we are only interested in *what* it approaches in the limit, but not *how*. We are therefore free to choose a matrix $A(q)$ that parametrises the approach to the limit in a different, perhaps more convenient way. Therefore, we are free to modify (6.2.31) in whatever way pleases us provided that in some continuous limit (that for simplicity's sake we assume to still be at the same point) it still coincides with the defective limit of $\tilde{\mathbf{H}}^{L=5,M=3,K=1}$. We do this to arrive at a spectral problem that we can actually solve, since for generic characteristic polynomials of degree 6, the solution is not always attainable in terms of elementary functions. The following two modifications fulfil this requirement and are choices for $A(q)$:

$$\tilde{H}^{(1)} = \begin{pmatrix} 2 & q_3 & q_1 & q_2 & 0 & 0 \\ 0 & 0 & q_3 & q_3 & q_1+q_2 & 0 \\ 0 & 0 & 1 & 0 & q_3 & q_1 \\ 0 & 0 & 0 & 1 & q_3 & q_2 \\ 0 & 0 & 0 & 0 & 0 & q_3 \\ 0 & 0 & 0 & 0 & \frac{1}{q_3} & 2 \end{pmatrix} , \qquad \tilde{H}^{(2)} = \begin{pmatrix} 2 & q_3 & q_1 & q_2 & 0 & 0 \\ 0 & 0 & q_3 & q_3 & q_1+q_2 & 0 \\ 0 & 0 & 1 & 0 & q_3 & q_1 \\ 0 & \frac{1}{q_3} & 0 & 1 & q_3 & q_2 \\ 0 & 0 & 0 & 0 & 0 & q_3 \\ 0 & 0 & 0 & 0 & \frac{1}{q_3} & 2 \end{pmatrix} . \tag{6.2.32}$$

Substituting $q_i \to \frac{\xi_i}{\epsilon}$, the six distinct eigenvectors of $\tilde{H}^{(1)}$ are

$$v_1 = (1,0,0,0,0,0) \ ,$$

[11] The zero-momentum restriction corresponds to us restricting our analysis to one block appearing in the Hamiltonian, as the analysis for the different momenta values is analogous. In (6.2.10) for example, the zero momentum restriction corresponds to us only looking at the very first block in the top left corner.

$$
\begin{aligned}
v_2 &= (\xi_3, -2\epsilon, 0, 0, 0, 0) \ , \\
v_3 &= \left(\xi_3^2 + \xi_2\epsilon, -2\xi_3\epsilon, 0, -\epsilon^2, 0, 0\right) \ , \\
v_4 &= \left(\xi_3^2 + \xi_1\epsilon, -2\xi_3\epsilon, 0, -\epsilon^2, 0, 0\right) \ , \\
v_5 &= \Bigg(\xi_3^4 - \frac{1+2\sqrt{2}}{2+\sqrt{2}}(\xi_1+\xi_2)\xi_3^2\epsilon + \frac{2-\sqrt{2}}{2+\sqrt{2}}(\xi_1^2+\xi_2^2)\frac{\epsilon^2}{2}, -\frac{4+3\sqrt{2}}{2+\sqrt{2}}\xi_3^3\epsilon + \frac{8+5\sqrt{2}}{2+\sqrt{2}}(\xi_1+\xi_2)\xi_3\frac{\epsilon^2}{2}, \\
&\quad \frac{\xi_3^2\epsilon^2}{2} + \frac{1-\sqrt{2}}{2}\xi_1\epsilon^3, \frac{\xi_3^2\epsilon^2}{2} + \frac{1-\sqrt{2}}{2}\xi_2\epsilon^3, -\frac{1+\sqrt{2}}{2+\sqrt{2}}\xi_3\epsilon^3, \frac{\epsilon^4}{2+\sqrt{2}}\Bigg) \ , \\
v_6 &= \Bigg(\xi_3^4 - \frac{1-2\sqrt{2}}{2-\sqrt{2}}(\xi_1+\xi_2)\xi_3^2\epsilon + \frac{2+\sqrt{2}}{2-\sqrt{2}}(\xi_1^2+\xi_2^2)\frac{\epsilon^2}{2}, -\frac{4-3\sqrt{2}}{2-\sqrt{2}}\xi_3^3\epsilon + \frac{8-5\sqrt{2}}{2-\sqrt{2}}(\xi_1+\xi_2)\xi_3\frac{\epsilon^2}{2}, \\
&\quad \frac{\xi_3^2\epsilon^2}{2} + \frac{1+\sqrt{2}}{2}\xi_1\epsilon^3, \frac{\xi_3^2\epsilon^2}{2} + \frac{1+\sqrt{2}}{2}\xi_2\epsilon^3, -\frac{1-\sqrt{2}}{2-\sqrt{2}}\xi_3\epsilon^3, \frac{\epsilon^4}{2-\sqrt{2}}\Bigg) \ ,
\end{aligned} \tag{6.2.33}
$$

all approaching $\hat{u}_1$ in the $\epsilon \to 0$ limit, where we introduced the notation

$$
\hat{u}_i = (0, \cdots, 0, \underbrace{1}_{\text{index } i}, 0, \cdots, 0) \ . \tag{6.2.34}
$$

That means in the limit, we are dealing with one true eigenvector. We go on in our recipe and find the following five vectors[12]

$$
w_{i,1}^{(1)} = \frac{v_i - (v_i \cdot v_1)v_1}{\epsilon} \ , \tag{6.2.35}
$$

and we find that they all are proportional to $\hat{u}_2 = (0,1,0,0,0,0)$ in the limit of vanishing ϵ. Going one step further and constructing the four vectors

$$
w_{i,2}^{(2)} = \frac{w_{i,1}^{(1)} - (w_{2,1}^{(1)} \cdot w_{i,1}^{(1)})w_{2,1}^{(1)}}{\epsilon} \ , \tag{6.2.36}
$$

we find that, in the $\epsilon \to 0$ limit, they span out a two-dimensional space given by $\text{span}\{\hat{u}_3, \hat{u}_4\}$, which initially might seem to be contradicting 2.1.1 given that we only found one true eigenvector in the limit. However, one can find that the two vectors $\hat{u}_3, \hat{u}_4$ differ by a true eigenvector, $(0, \xi_2 - \xi_1, \xi_3, -\xi_3, 0, 0)$, of the strongly twisted Hamiltonian. One step further in the recipe, we end up finding

$$
\begin{aligned}
\lim_{\epsilon\to 0} w^{(3)} &= (0,0,0,0,1,0) = \hat{u}_5 \ , \\
\lim_{\epsilon\to 0} w^{(4)} &= (0,0,0,0,0,1) = \hat{u}_6 \ .
\end{aligned} \tag{6.2.37}
$$

Summa summarum, we can display the genealogy of our limits in the following way:

$$
\begin{array}{ccccccccc}
\text{Span}\{\lim v_i\} & & \text{Span}\{\lim w_{ij}^{(1)}\} & & \text{Span}\{\lim w_{ij}^{(2)}\} & & \text{Span}\{\lim w_{ij}^{(3)}\} & & \text{Span}\{\lim w_{ij}^{(4)}\} \\
\hat{u}_1 & \longrightarrow & \hat{u}_2 & \begin{array}{c}\nearrow \\ \searrow\end{array} & \begin{array}{c}\hat{u}_3 \\ \hat{u}_4\end{array} & \begin{array}{c}\searrow \\ \nearrow\end{array} & \hat{u}_5 & \longrightarrow & \hat{u}_6
\end{array}
$$

In this horizontal "family tree" of vectors, we find the true eigenvector to be on the very left. As we move up generation by generation to the right, we get the vectors that our algorithmic framework generates at each subsequent step. Looking at the above diagram, it might be tempting to conclude that the JNF

[12] Notice that here we made use of one of the simplifying prescriptions.

associated to our problem exhibits one Jordan block of size 5 and one Jordan block of size 1. We need to show this explicitly, however, by finding two Jordan chains of generalised eigenvectors satisfying (2.1.5) - one of length 5 and one of length 1. Indeed, this is what we find after a healthy dose of algebra:

$$\begin{aligned}
&\hat{\mathbf{H}}^5 \frac{2\xi_3^3\hat{u}_6 - 3\xi_3^2(\xi_1+\xi_2)\hat{u}_5 + 2\xi_3(\xi_1^2+\xi_2^2+3\xi_1\xi_2)(\hat{u}_3+\hat{u}_4) - 2(\xi_1^3+\xi_2^3+4\xi_1^2\xi_2+4\xi_1\xi_2^2)\hat{u}_2}{4\xi_3^7} = \\
&\hat{\mathbf{H}}^4 \frac{2\xi_3^2\hat{u}_5 - \xi_3(3\xi_1+\xi_2)\hat{u}_4 - \xi_3(\xi_1+3\xi_2)\hat{u}_3 + (\xi_1^2+\xi_2^2+6\xi_1\xi_2)\hat{u}_2}{4\xi_3^5} = \hat{\mathbf{H}}^3 \frac{\xi_3(\hat{u}_3+\hat{u}_4) - (\xi_1+\xi_2)\hat{u}_2}{2\xi_3^3} = \\
&\hat{\mathbf{H}}^2 \frac{\hat{u}_2}{\xi_3} = \hat{\mathbf{H}}\hat{u}_1 = 0 \ , \\
&\hat{\mathbf{H}}[\xi_3(\hat{u}_3-\hat{u}_4) - (\xi_1-\xi_2)\hat{u}_2] = 0 \ ,
\end{aligned} \tag{6.2.38}$$

where we implied $\hat{\mathbf{H}} = \lim_{\epsilon\to 0} \tilde{\mathbf{H}}_{(\frac{\xi_1}{\epsilon}, \frac{\xi_2}{\epsilon}, \frac{\xi_3}{\epsilon})}$. This is consistent with what the authors found in [AS21], but we shall improve on this method later on.

Let us now analyse the case of $\tilde{H}^{(2)}$, whose six eigenvectors are

$$\begin{aligned}
v_1 &= (1,0,0,0,0,0) \ , \\
v_2 &= \left(\frac{(1+\sqrt{5})\xi_3^2 - 2\xi_2\epsilon}{3+\sqrt{5}}, -(1+\sqrt{5})\xi_3\frac{\epsilon}{2}, 0, \epsilon^2, 0, 0 \right) \ , \\
v_3 &= \left(\frac{(1-\sqrt{5})\xi_3^2 - 2\xi_2\epsilon}{3-\sqrt{5}}, -(1-\sqrt{5})\xi_3\frac{\epsilon}{2}, 0, \epsilon^2, 0, 0 \right) \ , \\
v_4 &= (\xi_1-\xi_2, 0, -\epsilon, \epsilon, 0, 0) \ , \\
v_5 &= \Big(2(2-\sqrt{2})\xi_3^4 - [(1-3\sqrt{2})\xi_1 + (3-5\sqrt{2})\xi_2]\xi_3^2\epsilon + [(1+\sqrt{2})\xi_1^2 + (3-\sqrt{2})\xi_1\xi_2 + 4\xi_2^2]\epsilon^2, \\
&\quad - \frac{4(1-\sqrt{2})\xi_3^3\epsilon - 2(3-\sqrt{2})(\xi_1+\xi_2)\xi_3\epsilon^2}{2+\sqrt{2}}, -(1-\sqrt{2})\xi_3^2\epsilon^2 + \xi_1\epsilon^3, \\
&\quad (1-\sqrt{2})[(1-2\sqrt{2})\xi_3^2\epsilon^2 - 4\xi_2\epsilon^3 - (3-\sqrt{2})\xi_1\epsilon^3, (2-\sqrt{2})\xi_3\epsilon^3, \sqrt{2}\epsilon^4] \Big) \ , \\
v_6 &= \Big(-2(2+\sqrt{2})\xi_3^4 + [(1+3\sqrt{2})\xi_1 + (3+5\sqrt{2})\xi_2]\xi_3^2\epsilon - [(1-\sqrt{2})\xi_1^2 + (3+\sqrt{2})\xi_1\xi_2 + 4\xi_2^2]\epsilon^2, \\
&\quad \frac{4(1+\sqrt{2})\xi_3^3\epsilon - 2(3+\sqrt{2})(\xi_1+\xi_2)\xi_3\epsilon^2}{2+\sqrt{2}}, (1+\sqrt{2})\xi_3^2\epsilon^2 - \xi_1\epsilon^3, \\
&\quad -(1+\sqrt{2})[(1+2\sqrt{2})\xi_3^2\epsilon^2 - 4\xi_2\epsilon^3 - (3+\sqrt{2})\xi_1\epsilon^3, -(2+\sqrt{2})\xi_3\epsilon^3, \sqrt{2}\epsilon^4] \Big) \ .
\end{aligned} \tag{6.2.39}$$

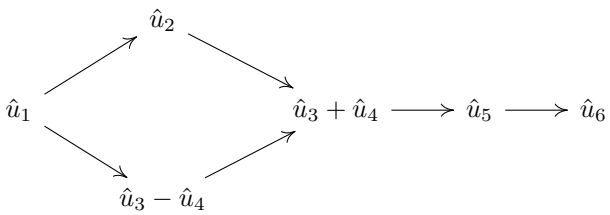
As before, the six eigenvectors all collapse into $\hat{u}_1$ in the $\epsilon \to 0$ limit. However, if we go one step further, the first difference with the $\tilde{H}^{(1)}$ analysis already arises, and we get that the vectors

$$w_{i,1}^{(1)} = \frac{v_i - (v_i \cdot v_1)v_1}{\epsilon} \ , \tag{6.2.40}$$

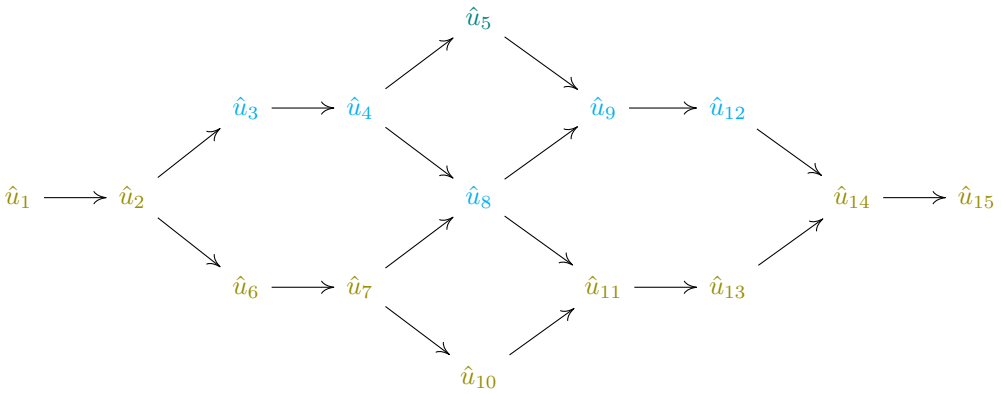
lie within a two-dimensional space spanned by the two vectors $\hat{u}_2$ and $\hat{u}_3 - \hat{u}_4$, which both fulfil the properties of generalised eigenvectors of rank 2. And indeed again, it is true that $\hat{u}_2$ and $\hat{u}_3 - \hat{u}_4$ differ by the true eigenvector $(0, \xi_2 - \xi_1, \xi_3, -\xi_3, 0, 0)$. Continuing the prescriptions in our recipe, we find:

$$\lim_{\epsilon\to 0} w^{(2)} = \hat{u}_3 + \hat{u}_4 \ , \qquad \lim_{\epsilon\to 0} w^{(3)} = \hat{u}_5 \ , \qquad \lim_{\epsilon\to 0} w^{(4)} = \hat{u}_6 \ . \tag{6.2.41}$$

Our "family tree" of generalised eigenvectors then looks like



In the end, the structure of $\tilde{H}^{(2)}$ that we computed is similar to the one of $\tilde{H}^{(1)}$, with the sole difference that instead of a "fake generalised eigenvector of rank 3", we found a "fake generalised eigenvector of rank 2".[13] The results of our analysis are not a particularity of the $L = 5, M = 3, K = 1$ case: Performing the same analysis for the $L = 7, M = 3, K = 1$ case, the structure we would arrive at implies the case to feature one Jordan block of size 9, one Jordan block of size 5, and one Jordan block of size 1, which we colour-code in the genealogical diagram to illustrate our point:[14]



This is consistent with the result the authors found in [AS21], in their Table 1, and the computation in [ACS22].

The Untangling of Mixed Jordan Chains

We have now, as a representative example, analysed some cases where degenerate Jordan blocks appear. We have seen that, although the core assumption of onefold geometric multiplicity that we used in the formal construction of our tools was not fulfilled, our recipe still manages to shed light on the Jordan structure of our cases. We shall now, in more detail, study which steps of our recipe remain untouched by the relaxation to allow higher geometric multiplicities, and which steps we need to alter.

Our starting point will be again a matrix $M(\epsilon)$ that is diagonalisable a.e., but that at $\epsilon = 0$ it becomes defective. This time, we assume geometric multiplicity to exceed one, meaning that the eigenvalue appears in more than one Jordan block of the JNF of the defective matrix. Revisiting (6.2.15),

$$\left[M(0) - \mathbb{1} \lim_{\epsilon\to 0} \lambda_i\right] \left[\lim_{\epsilon\to 0} v_i\right] = 0 \ , \tag{6.2.42}$$

we remind ourselves that this gives us a sufficient (though not necessary in the strict sense) condition implying that, in the limit, any eigenvector of $M(\epsilon)$ becomes an eigenvector of $M(0)$. This still holds for higher geometric multiplicities, naturally, but, we no longer have the promise that whatever vectors we get in this limit suffice to span out the eigenspace $\ker\{(M(0) - \lambda\mathbb{1})\}$, as this eigenspace is now higher-dimensional and we might only get a limiting space by the vectors $[\lim_{\epsilon\to 0} v_i]$ that is one dimensional (for example, all limit eigenvectors might be collinear). One can check that for the $L = 5, M = 3, K = 1$ case, the "naive" limit space of $\text{span}\{\lim_{\epsilon\to 0} v_i\} \subsetneq \ker\{(M(0) - \lambda\mathbb{1})\}$ does not give the full eigenpicture, and true eigenvectors appear as generalised eigenvectors of higher rank. We shall later on refer to this phenomenon of misplaced eigenvectors as *chain mixing*.

[13] By "fake" we are referring to the fact that it these vectors might, mendaciously, give us the impression that there are multiple generalised eigenvectors of a certain rank, but they can be connected by ones of lower rank.

[14] Recall that this is just illustrating how we count, not whether what we get are true or fake eigenvectors of a certain rank.

At its core, this is due to the fact that there exist linear combinations of eigenvectors of $M(\epsilon)$ whose coefficients diverge at the exceptional point $\epsilon = 0$ and fulfil

$$\underbrace{[M(\epsilon) - \mathbb{1}\lambda_1(\epsilon)] \sum_{i=1}^{n-1} \alpha_i v_i(\epsilon) = \mathcal{O}(\epsilon)}_{\text{eigenvector equation fulfilled only for } \epsilon = 0} \qquad \text{while} \qquad \underbrace{\lim_{\epsilon \to 0} \sum_{i=1}^{n-1} \alpha_i v_i(\epsilon) = \mathcal{O}(\epsilon^0)}_{\text{constant } \epsilon \text{ behaviour in the limit}} \quad . \tag{6.2.43}$$

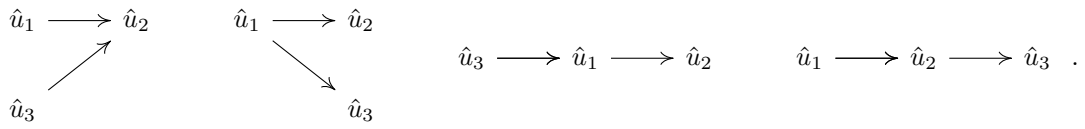
What the above equation describes is that there exist (true) eigenvectors of $M(0)$ that cannot be extended to true eigenvectors of $M(\epsilon)$. A priori, there is no way for us to know which of the true eigenvectors of $M(0)$ cannot be accessed in the limit by true eigenvectors of $M(\epsilon)$ (as this depends on the ϵ-parametrisation of $M(\epsilon)$). We will exemplify this by finding four different parametrisations of matrices that all coincide in the $\epsilon \to 0$ limit to the same matrix,

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad . \tag{6.2.44}$$

Our parametrisations are given by

$$\begin{pmatrix} 0 & 1 & \epsilon^2 \\ \epsilon^2 & \epsilon^4 & \epsilon^2 \\ 0 & 0 & \epsilon^6 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & \epsilon^2 \\ \epsilon^2 & \epsilon^4 & \epsilon \\ 0 & 0 & \epsilon^6 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & \epsilon^2 \\ \epsilon^5 & 0 & 0 \\ \epsilon & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & \epsilon^2 \\ 0 & \epsilon^4 & \epsilon^2 \\ 0 & 0 & \epsilon^6 \end{pmatrix} \quad . \tag{6.2.45}$$

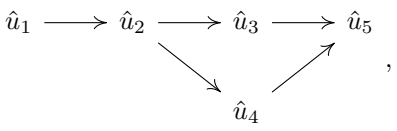
If we were to apply our recipe to these cases, we get, in order, the following genealogical diagrams:



We get four different diagrams! Therefore, having a limit matrix with a JNF possessing geometric multiplicity higher than one makes the analysis less transparent, and it is impossible to separate one Jordan chain of $M(0)$ from the other, as the above diagrams show (hence the term "chain mixing"). The situations only gets direr as Jordan blocks grow bigger; e.g.

$$\begin{pmatrix} \epsilon & 1 & \epsilon & 0 & 0 \\ 0 & \epsilon^4 & 1 & \epsilon & 0 \\ 0 & 0 & \epsilon^7 & \epsilon^9 & 0 \\ 0 & 0 & 0 & \epsilon^6 & 1 \\ 0 & 0 & 0 & 0 & \epsilon^8 \end{pmatrix} \quad , \tag{6.2.46}$$

which has the following genealogical diagram (notice the "double jump" containing the $\hat{u}_4$ vector on the bottom),



,

which implies that we cannot determine whether $\lim_{\epsilon \to 0} w^{(3)} = \hat{u}_5$ is a true eigenvector, a generalised eigenvector of rank 4 or one of rank 2.[15]

[15] We shall make the rectified analysis of this example explicit in the appendix.

Before we lose all hope, let us stress two important observations: The first one is the fact that our methodology is still complete - in the sense that, in the end, the complete generalised eigenspectrum of $M(0)$ will be generated. It is fairly straightforward to see this: Similarly to the case of distinguishable Jordan blocks, this method gives us non-zero, non-divergent and most importantly orthogonal (and, thus, linearly independent) vectors, and we just plainly repeat this process as many times as $M(\epsilon)$ has distinct eigenvectors. Additionally, the assertion that eigenvectors of $M(\epsilon)$ become eigenvectors of $M(0)$ in the $\epsilon \to 0$ limit still holds.

The second key observation is that for any generalised eigenvector of rank n found at a given step, its associated eigenvector of one rank lower has to appear one step prior. This can be understood the following way: Let us assume that we have a "misplaced" eigenvector in the n-th step of our method, then we may find a generalised eigenvector of rank m at the $n+m-1$-th step. By the earlier statement, then we have a linear combination of vectors satisfying:

$$[M(\epsilon) - \mathbb{1}\lambda_n(\epsilon)] \sum_{i=1}^{n} \alpha_i v_i = \mathcal{O}(\epsilon) \ . \tag{6.2.47}$$

If we now define coefficients

$$\beta_i = \gamma \frac{\alpha_i}{\prod_{j=1}^{m}(\lambda_i - \lambda_{n+j-1})} \tag{6.2.48}$$

for $1 \leq i < n$, where λ_j is the eigenvalue associated with v_j and γ is a constant, then the following linear combination of $n+m-1$ vectors fulfils

$$\prod_{j=1}^{m} [M(\epsilon) - \mathbb{1}\lambda_{n+j-1}(\epsilon)] \sum_{i=1}^{n+m-1} \beta_i v_i = \mathcal{O}(\epsilon) \ . \tag{6.2.49}$$

This implies that linear combination of this form contain information about generalised eigenvectors of higher rank, which we might be able to extract by applying the tools we already established.

Let us now motivate how we plan to separate the mixed Jordan chains: If we still were in the case of distinguishable Jordan blocks, then the number of different vectors we obtained in the n-th step would equal $\dim(\ker\{(M(0) - \lambda_i\mathbb{1})^n\}/\ker\{(M(0) - \lambda_i\mathbb{1})^{n-1}\})$. With theorem 2.1.1, we could then assume to find the same number or fewer vectors in the n-th step as in the first step. Therefore, if the dimension of a vector space we obtain in a given step is greater than the dimension of the vector space we obtained one step prior, that is a clear indication that we have found an eigenvector "in disguise".

Our chain mixing problem is then solved by the following consideration: As we are able to find at least some of the true eigenvectors of $M(0)$, it would suffice to untangle the mixed chains if we were able to reconstruct a Jordan chain from its generalised eigenvector of rank 1.

This can be achieved by first using corollary 2.1.3.1 in order to identify a right generalised eigenvector of rank 1 of $M(0)$ with a left generalised eigenvector of maximal rank (i.e. a right generalised eigenvector of highest rank of $M^\dagger(0)$), and then using the definition (2.1.5) we can find a left generalised eigenvector of rank 1. Using then again 2.1.3.1, the vector we found this way can be identified with the generalised eigenvector of maximal rank of the same Jordan chain as the generalised eigenvector of rank 1, that was our initial ingredient. From this point, we simply need to go down in our Jordan chain like in (2.1.5), and we are done. Recapitulating, if v is a generalised eigenvector of rank 1 of $M(0)$, then the generalised eigenvectors of the Jordan chain it is associated with can be generated via

$$M(0)^l \left[M(0)^\dagger\right]^m v \text{ with } 0 \leq l \leq m \ , \tag{6.2.50}$$

and $m \in \mathbb{N}_0$ such that $\left[M(0)^\dagger\right]^m v \neq 0$ while also $\left[M(0)^\dagger\right]^{m+1} v = 0$. The issue of chain mixing was also found in [AS21], where the authors called it *unexpected shortening*, however the authors did not provide

a method to deal with it. We have this addressed this point, and put the final cornerstone in our recipe that now gives us unambiguous results for any Jordan setting.

6.3 Coordinate Bethe Ansatz for Twisted Spin Chains

In the previous sections, we established a comprehensive toolset to construct the generalised eigensystem of a defective matrix by perturbing it in a diagonalisable way, and then using our prescriptions when taking the limit. We now want to apply our machinery to the case of the eclectic spin chain with an unrestricted number of excitations of flavour 2, but only one excitation of flavour 3, which is the $M > 1, K = 1$ case. We want to assume that $M > K$, since the $M = K = 1$ case just featured two flavours on the spin chain (namely flavour 3 and 1), and is therefore diagonalisable and thus there is no need for our machinery to be involved. As was the setting for matrices, we first needed to diagonalise $M(\epsilon)$, and likewise we first construct the eigensystem of the twisted Hamiltonian (6.1.7), still assuming general (non-exceptional) values of the twist parameters. We can then make use of our recipe and analyse the defective $K = 1$ case in full.

Our first step involves then the construction of the eigensystem of (6.1.7) for general length L, featuring excitations of two different flavours (meaning that, with the ground state flavour 1, three distinct flavours are appearing). To this end, we shall use a slightly modified (i.e. twisted) version of the Nested Coordinate Bethe Ansatz (NCBA) that we introduced for the $\mathfrak{su}(3)$ spin chain in 3.3.2. For the original method, we again refer the reader to [BS05], but also to section II.O in [SJS06], although the latter, while more complete, is arguably harder to consult due to its datedness.[16]

Our first step in the CBA was to make an ansatz for the position space wave functions. In this, it was particularly helpful that our Hamiltonian in the untwisted case commuted with the shift operator - hinting at an invariance that we also want the eigensystem to reflect. This is no longer true: the twisted Hamiltonian in (6.1.7) no longer commutes with the (usual) permutation operator, however, it still commutes with the shift operator U that we have defined before. We can still make a plane-wave ansatz (c.f. [Blo29]) in spite of the twisting parameters appearing, although we have to include them in some non-trivial way in our ansatz:

$$\begin{aligned}|\psi_{23}(p_1,p_2)\rangle = &\sum_{1\le n_1<n_2\le L}\left[A_{23}e^{i(p_1n_1+p_2n_2)}\frac{q_3^{n_1}}{q_2^{n_2}}+\tilde{A}_{23}e^{i(p_1n_2+p_2n_1)}\frac{q_3^{n_1}}{q_2^{n_2}}\right]S^{2,+}_{n_1}S^{3,+}_{n_2}|0\rangle\\ &+\sum_{1\le n_1<n_2\le L}\left[A_{32}e^{i(p_1n_1+p_2n_2)}\frac{q_3^{n_2}}{q_2^{n_1}}+\tilde{A}_{32}e^{i(p_1n_2+p_2n_1)}\frac{q_3^{n_2}}{q_2^{n_1}}\right]S^{3,+}_{n_1}S^{2,+}_{n_2}|0\rangle\ ,\end{aligned}\tag{6.3.1}$$

where we again define the pseudo-vacuum[17] as the L-fold tensor product of states of type 1, meaning $|0\rangle = \otimes_{n=1}^{L}|1\rangle$. As a notational difference compared to the introductory chapter, we introduced "creation operators" for the excitations of flavours 2 and 3, that work as

$$S_n^{j,+}|0\rangle = |1,\cdots,1,\underbrace{j}_{\text{at site } n},1,\cdots,1\rangle\ .\tag{6.3.2}$$

To make our notation a bit more economical, we suppress the explicit momentum dependence, which we shall understand to be one on p_1 and p_2 if not specified another way. As in the introduction, we

[16] In [dL07], a version of the Nested Coordinate Bethe Ansatz using ZF operators is presented. However, we will not make use of it.

[17] Here, we speak of a *pseudo*-vacuum rather than an ordinary one since the signs and constants appearing in our Hamiltonian determine which configuration corresponds to the ground state, meaning there is a priori no unique notion of a ground state (e.g. ferromagnetic vs. anti-ferromagnetic).

analyse as a next step the Schrödinger equation $\tilde{\mathbf{H}}_{(q_1,q_2,q_3)}|\psi_{23}\rangle = E|\psi_{23}\rangle$ - again, we will make the distinction between the case where the well separated case of the 2 and 3 excitations gives rise to the energy equations, yielding

$$E = L - 4 + 2\cos(p_1) + 2\cos(p_2) \ , \tag{6.3.3}$$

which is the same as for the untwisted case (the twist parameters do not make an appearance), and the case where the 2 and 3 excitations are next to each other, giving rise to

$$\begin{pmatrix} \tilde{A}_{32} \\ \tilde{A}_{23} \end{pmatrix} = \begin{pmatrix} \frac{1}{q_1q_2q_3}\frac{e^{ip_2}-e^{ip_1}}{1-2e^{ip_1}+e^{i(p_1+p_2)}} & \frac{-(1-e^{ip_1})(1-e^{ip_2})}{1-2e^{ip_1}+e^{i(p_1+p_2)}} \\ \frac{-(1-e^{ip_1})(1-e^{ip_2})}{1-2e^{ip_1}+e^{i(p_1+p_2)}} & q_1q_2q_3\frac{e^{ip_2}-e^{ip_1}}{1-2e^{ip_1}+e^{i(p_1+p_2)}} \end{pmatrix} \begin{pmatrix} A_{23} \\ A_{32} \end{pmatrix} \ . \tag{6.3.4}$$

So far, we have only dealt with the mixed-flavour case, where the two non-1 excitations of the spin state are of different flavour (which is why the wave function carries the index ψ_{23}). In order to get the full $2 \to 2$ S-matrix, we also need to consider the single-flavour cases, for which we will make the following ansätze:

$$|\psi_{22}\rangle = \sum_{1\le n_1<n_2\le L} \left[A_{22}e^{i(p_1n_1+p_2n_2)}q_3^{n_1+n_2} + \tilde{A}_{22}e^{i(p_1n_2+p_2n_1)}q_3^{n_1+n_2}\right] S_{n_1}^{2,+}S_{n_2}^{2,+}|0\rangle \ , \tag{6.3.5}$$

$$|\psi_{33}\rangle = \sum_{1\le n_1<n_2\le L} \left[A_{33}\frac{e^{i(p_1n_1+p_2n_2)}}{q_2^{n_1+n_2}} + \tilde{A}_{33}\frac{e^{i(p_1n_2+p_2n_1)}}{q_2^{n_1+n_2}}\right] S_{n_1}^{3,+}S_{n_2}^{3,+}|0\rangle \ , \tag{6.3.6}$$

whose dispersion relation and S-matrix element are the same as the (untwisted) $\mathfrak{su}(2)$ spin chain. Thus, the full 2-particle S-matrix for the twisted case is of the following form:

$$S(p_2,p_1) = \begin{pmatrix} -\frac{1-2e^{ip_2}+e^{i(p_1+p_2)}}{1-2e^{ip_1}+e^{i(p_1+p_2)}} & 0 & 0 & 0 \\ 0 & \frac{1}{q_1q_2q_3}\frac{e^{ip_2}-e^{ip_1}}{1-2e^{ip_1}+e^{i(p_1+p_2)}} & \frac{-(1-e^{ip_1})(1-e^{ip_2})}{1-2e^{ip_1}+e^{i(p_1+p_2)}} & 0 \\ 0 & \frac{-(1-e^{ip_1})(1-e^{ip_2})}{1-2e^{ip_1}+e^{i(p_1+p_2}} & q_1q_2q_3\frac{e^{ip_2}-e^{ip_1}}{1-2e^{ip_1}+e^{i(p_1+p_2)}} & 0 \\ 0 & 0 & 0 & -\frac{1-2e^{ip_2}+e^{i(p_1+p_2)}}{1-2e^{ip_1}+e^{i(p_1+p_2)}} \end{pmatrix} . \tag{6.3.7}$$

So far, our momenta are not quantised. As we are dealing with a closed spin chain, we expect our spectrum to end up being discretised. Similar to the untwisted case in our introduction, we do this by imposing a periodicity condition on our wave function (keeping in mind that we are now dealing with a twisted theory). Expressing the wave function as a sum over the sites, $|\psi_{ij}\rangle = \sum_{n_1<n_2}|\psi_{ij}(n_1,n_2)\rangle$, imposing periodicity on this wave function is tantamount to having $|\psi_{ij}(n_1,n_2)\rangle = |\psi_{ij}(n_2,n_1+L)\rangle$, as we already mentioned in the introduction in a different context. With our ansätze for the different wave function components, the so-generated matrix Bethe equations are

$$e^{ip_1L}\begin{pmatrix} q_3^LA_{22} \\ q_3^LA_{23} \\ q_2^{-L}A_{32} \\ q_2^{-L}A_{33} \end{pmatrix} = \begin{pmatrix} \tilde{A}_{22} \\ \tilde{A}_{23} \\ \tilde{A}_{32} \\ \tilde{A}_{33} \end{pmatrix} = S(p_1,p_2)\begin{pmatrix} A_{22} \\ A_{23} \\ A_{32} \\ A_{33} \end{pmatrix} \ , \qquad e^{ip_2L}\begin{pmatrix} q_3^L\tilde{A}_{22} \\ q_2^{-L}\tilde{A}_{23} \\ q_3^L\tilde{A}_{32} \\ q_2^{-L}\tilde{A}_{33} \end{pmatrix} = \begin{pmatrix} A_{22} \\ A_{23} \\ A_{32} \\ A_{33} \end{pmatrix} = S(p_2,p_1)\begin{pmatrix} \tilde{A}_{22} \\ \tilde{A}_{23} \\ \tilde{A}_{32} \\ \tilde{A}_{33} \end{pmatrix} . \tag{6.3.8}$$

We know that our theory factorises, because we still have an integrable theory with an R-matrix, and thus like before we can immediately extend these equations to larger M (and K):

$$e^{ip_kL}q_3^{(3-f_k)L}q_2^{(2-f_k)L} = S(p_k,p_{k+1})\dots S(p_k,p_M)S(p_k,p_1)\dots S(p_k,p_{k-1}) \ . \tag{6.3.9}$$

In (6.3.9), f_k indicates the flavour of the k-th excitation, which consequently gives us a dressing factor q_2^{-L} if $f_k = 3$ and a one of q_3^L if $f_k = 2$, consistently. These additional factors are forcing us to adapt

the regular NCBA (for the $\mathfrak{su}(3)$ spin chain) to accommodate for the twisted nature of our model. For general values of M, the twisted generalisation follows closely our outline in the introduction, and we can postulate the following ansatz:

$$|\psi\rangle = \sum_{k=1}^{M} \sum_{\sigma\in S_M} \psi_k(\sigma) e^{i\sum_j p_{\sigma(j)} n_j} \frac{q_3^{\sum_j n_j}}{(q_2 q_3)^{n_k}} S^{2+}_{n_1} S^{2+}_{n_2} \cdots S^{3+}_{n_k} \cdots S^{2+}_{n_M} |0\rangle \ , \tag{6.3.10}$$

where we kept in mind that we are and will only ever be interested in the case $K = 1$. In (6.3.10), we are adopting the notation of [SJS06] (or are at least inspired by it), and we are re-expressing the coefficients from before via

$$\begin{aligned} \psi_2(id) &= A_{23} \ , \\ \psi_1(id) &= A_{32} \ , \\ \psi_1(\tau) &= \tilde{A}_{32} \ , \\ \psi_2(\tau) &= \tilde{A}_{23} \ , \end{aligned} \tag{6.3.11}$$

where $\tau \in S_2$ is the transposition $\tau = (12) = (21)$ in cycle notation. Acting with the matrix Bethe equations on this, we can extract a recursive recipe out of it that was outlined in [SJS06] and we shall extend to twisted models.

In order to make our computations more compact in their expressions, we shall work with a normalised S-matrix of the following form:

$$\lambda_k q_3^{(3-f_k)L} q_2^{(2-f_k)L} |\psi\rangle = s(p_k, p_{k+1}) \dots s(p_k, p_M) s(p_k, p_1) \dots s(p_k, p_{k-1}) |\psi\rangle \ , \tag{6.3.12}$$

$$s(p_i, p_j) = -\frac{1 - 2e^{ip_j} + e^{i(p_i+p_j)}}{1 - 2e^{ip_i} + e^{i(p_i+p_j)}} S(p_i, p_j) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{q_1 q_2 q_3} \frac{e^{ip_j} - e^{ip_i}}{1 - 2e^{ip_i} + e^{i(p_j+p_i)}} & \frac{(1-e^{ip_j})(1-e^{ip_i})}{1-2e^{ip_i}+e^{i(p_j+p_i)}} & 0 \\ 0 & \frac{(1-e^{ip_j})(1-e^{ip_i})}{1-2e^{ip_i}+e^{i(p_j+p_i)}} & q_1 q_2 q_3 \frac{e^{ip_j} - e^{ip_i}}{1 - 2e^{ip_i} + e^{i(p_j+p_i)}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{6.3.13}$$

In the literature, it is more common to see these quantities written in terms of rapidities rather than in their explicit momenta formulation. We will choose the rapidity variable $e^{ip_i} = \frac{u_i}{u_i+1}$.[18] This makes our expressions much more compact, and the undressed S-matrix is then

$$s(u_i, u_j) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{q_1 q_2 q_3} \frac{u_j - u_i}{u_j - u_i + 1} & \frac{1}{u_j - u_i + 1} & 0 \\ 0 & \frac{1}{u_j - u_i + 1} & q_1 q_2 q_3 \frac{u_j - u_i}{u_j - u_i + 1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{6.3.14}$$

Let us now analyse the product and the action in (6.3.12) and try to untangle it. We shall only focus on coefficients of the form $\psi_k(id)$, and consequently drop the (id) argument for compactness. We then start with the action in (6.3.12) from the inside

$$s(u_k, u_{k+1}) \dots s(u_k, u_M) s(u_k, u_1) \dots \underbrace{s(u_k, u_{k-1}) |\psi\rangle}_{\text{only relevant action on } \psi_{k-1}} \ , \tag{6.3.15}$$

[18] This definition of the rapidity variable is the inverse of the one used in [AS21], coming from the opposite momenta definition that they choose in comparison. With these rapidity variables, we can directly compare our Bethe equations against the ones of [AS21].

which lets us quickly write down the equation for ψ_{k-1},[19]

$$q_3^L \lambda_k \psi_{k-1} = \frac{1}{u_{k-1}-u_k+1}\psi_k + q_1q_2q_3\frac{u_{k-1}-u_k}{u_{k-1}-u_k+1}\psi_{k-1} \ . \tag{6.3.16}$$

Further untangling this product, we can likewise extract equations for the other coefficients ψ_l,

$$q_3^L \lambda_k \psi_{k-l-1} = \frac{1}{u_{k-l-1}-u_k+1}\psi_k^{(l-1)} + q_1q_2q_3\frac{u_{k-l-1}-u_k}{u_{k-l-1}-u_k+1}\psi_{k-l-1} \ , \tag{6.3.17}$$

$$\psi_k^{(l)} = \frac{1}{q_1q_2q_3}\frac{u_{k-l-1}-u_k}{u_{k-l-1}-u_k+1}\psi_k^{(l-1)} + \frac{1}{u_{k-l-1}-u_k+1}\psi_{k-l-1} \ , \tag{6.3.18}$$

where we defined

$$\psi_k^{(1)} = \frac{1}{q_1q_2q_3}\frac{u_{k-1}-u_k}{u_{k-1}-u_k+1}\psi_k + \frac{1}{u_{k-1}-u_k+1}\psi_{k-1} \ , \tag{6.3.19}$$

and, naturally, $k-l-1$ and the other superscripts are integers implicitly understood as mod M. We should also stress that (6.3.17) holds for all l with the sole exception of $l = M-1$ (corresponding to $k-l-1 \bmod M = k$), where one has to substitute q_3^L by q_2^{-L}. This is due to the fact that $f_k = 3$ and $f_l = 2$ if $l \neq k$ for the part of the wavefunction associated with the coefficient ψ_k.

So far, we have only talked about the coefficients appearing in the wave function with argument (id) that we suppressed, but more often than not does $\psi(id)$ carry this argument. Fortunately, there are not one but two ways to get from the (id) to the (σ): Either by using the periodicity condition, or by applying the S-matrix to $\psi(id)$ in an appropriate way (as we implicitly for the $M = 2$ case in (6.3.8), comparing both orders of the momenta).

We can combine (6.3.17) and (6.3.18) by finding equations for $\psi_k^{(l-1)}$ and $\psi_k^{(l)}$ using (6.3.17), and then resubstituting it into (6.3.18), finding the following equation

$$\frac{\psi_{k-l-1}}{\psi_{k-l}} = \frac{1}{q_1q_2q_3}\frac{q_1q_2q_3 + (u_k-u_{k-l})(q_1q_2q_3 - q_3^L\lambda_k)}{q_3^L\lambda_k + (q_1q_2q_3 - q_3^L\lambda_k)(u_k-u_{k-l-1})} \ . \tag{6.3.20}$$

Akin to the considerations the authors made in [SJS06], we can analyse the dependencies of the above equation: the right-hand side depends on indices k and $k-l$, whereas the left-hand side only depends on $k-l$. We therefore argue that, if we define

$$\bar{x} = \frac{q_1q_2q_3}{q_1q_2q_3 - \lambda_k q_3^L} + u_k \ , \tag{6.3.21}$$

then (6.3.20) takes the form[20]

$$\frac{\psi_{k-l-1}}{\psi_{k-l}} = \frac{1}{q_1q_2q_3}\frac{u_{k-l}-\bar{x}}{u_{k-l-1}-\bar{x}+1} \ . \tag{6.3.22}$$

Therefore, $\bar{x}$ must be a constant, or else (6.3.22) would not depend only on the difference $k-l$. We can then solve for the eigenvalue λ_k, we arrive at

$$q_3^L\lambda_k = q_1q_2q_3\frac{\bar{x}-u_k-1}{\bar{x}-u_k} \ . \tag{6.3.23}$$

In [SJS06], the authors would take (6.3.22) to find an expression for a general ψ_l in terms of a ψ_1 (that can be normalised, as the ψ_i form a representative of an "eigenvector"), and then substituting this

[19] We are only allowed to do that because no other s-matrix factors will act on ψ_{k-1}, and we can equate it to the l.h.s. of (6.3.12).

[20] Had we stuck to momenta variables, then (6.3.20) takes the form

$$\frac{\psi_{k-j-1}}{\psi_{k-j}} = \frac{(e^{ip_{k-j-1}}-1)[q_1q_2q_3(1-2e^{ip_{k-j}}+e^{i(p_{k-j}+p_k)}) + q_3^L\lambda_k(e^{ip_k}-e^{ip_{k-j}})]}{q_1q_2q_3(e^{ip_{k-j}}-1)[q_3^L\lambda_k(1-2e^{ip_k}+e^{i(p_{k-j-1}+p_k)}) - q_1q_2q_3(e^{ip_k}-e^{ip_{k-j-1}})]} \ ,$$

which is arguably harder to manipulate in a transparent way.

expression into (6.3.17). The presence of the twist factors does not allow us to do the same; we need to find a way to relate a general ψ_l to ψ_k, from which we have

$$\psi_{k-l} = \left[\frac{1}{(q_1q_2q_3)^l}\prod_{j=0}^{l-1}\frac{u_{k-j}-\bar{x}}{u_{k-j-1}-\bar{x}+1}\right]\psi_k = q_1q_2q_3\,\frac{u_k-\bar{x}+1}{u_{k-l}-\bar{x}}\frac{\psi_k}{\prod_{j=0}^{l}(q_3^L\lambda_{k-j})}\ . \tag{6.3.24}$$

Furthermore, the $\psi_k^{(l)}$ are of the form

$$\psi_k^{(l-1)} = q_1q_2q_3\frac{u_{k-l-1}-\bar{x}+1}{u_k-\bar{x}}\psi_{k-l-1} = \frac{\psi_k}{\prod_{j=1}^{l-1}(q_3^L\lambda_{k-j})}\ . \tag{6.3.25}$$

As a concluding step, we periodically identify the equations where we need to, by imposing the identification of the excitation $M+1$ with the excitation 1. This implies that we need to equate the function $\psi_k^{(M-1)}$ with $q_2^{-L}\lambda_k\psi_k$. This constrains the eigenvalues λ_k, which in turn translates into a constraint for $\bar{x}$

$$\frac{\lambda_k}{q_2^L}\prod_{\substack{j=1\\j\neq k}}^{M}q_3^L\lambda_j = 1 \Longrightarrow \frac{(q_2q_3)^L}{(q_1q_2q_3)^M}\prod_{j=1}^{M}\frac{\bar{x}-u_j}{\bar{x}-u_j-1} = 1\ . \tag{6.3.26}$$

This constraint for the constant $\bar{x}$ is the *auxiliary Bethe equation* of our model, which we have not come across in our introduction. We have now established all three main equations of interest: The Bethe equations, the auxiliary Bethe equations, and the recursive eigenstate equations, and the example that is to follow will put them into context.

If the matrix Bethe equations for the normalised S-matrix are solved, solving then the matrix Bethe equations for the original S-matrix is straightforward: They reduce to the algebraic equations

$$e^{ip_kL} = \lambda_k\prod_{j\neq k}\left(-\frac{1-2e^{ip_k}+e^{i(p_k+p_j)}}{1-2e^{ip_j}+e^{i(p_k+p_j)}}\right)\ , \tag{6.3.27}$$

with λ_k being the computed eigenvalues. Switching to rapidity variables and substituting them into the expression, we arrive at Bethe equations of the following form:

$$\frac{q_3^L}{q_1q_2q_3}\frac{\bar{x}-u_k}{\bar{x}-u_k-1}\prod_{j\neq k}\frac{u_k-u_j+1}{u_k-u_j-1} = \left(\frac{u_k+1}{u_k}\right)^L\ . \tag{6.3.28}$$

Our expressions for the Bethe equations and the auxiliary Bethe equation coincide with the ones appearing in [AS21] (we need to identify our $\bar{x}$ with their $v+1$), which is a great reaffirmation for our results.

6.3.1 Generalised Bethe System for $M = 2$

We have now analysed the diagonalisable case of our (finitely) twisted Hamiltonian, and with this we have all the ingredients at hand that are necessary to find the generalised eigensystem of (6.1.7) for the defective large twist limit q_i, which we shall reparametrise again as $q_i \to \frac{\xi_i}{\epsilon}$ with $\epsilon \to 0$, meaning that the three twist parameters approach infinity at the same rate. In this section, we shall focus on the analysis of the $M = 2$ case, and then analyse the $M = 3$ and $M > 3$ cases in the appendix, as the expressions will prove to be more involved and we need to go on a tangent on algebraic polynomials for the general M case.

We analysed the $M = 2$ case in full, and we now want to study how the A and $\tilde{A}$ coefficients (as first presented in (6.3.1)) and the momenta p_i behave in the limit of large twists. For this, we also need to

study the equations for which the former are solutions in this limit, and will borrow the ϵ-expansions of our quantities from the computation in [AS21],

$$\begin{aligned} u_1 &\approx \epsilon^{\alpha} u_- \\ u_2 &\approx -1 + \epsilon^{\alpha} u_+ \\ \bar{x} &\approx u_2 + \epsilon^{\gamma} \hat{v} \ , \end{aligned} \tag{6.3.29}$$

with the spin chain-specific constants defined as $\alpha = \frac{L-3}{L-1}$ and $\gamma = 2L-6$. In the $M=2$ case, the Bethe equations and the auxiliary Bethe equations take the following form, respectively:[21]

$$\frac{q_3^L}{q_1 q_2 q_3} \frac{\bar{x}-u_1}{\bar{x}-u_1-1} \frac{u_1-u_2+1}{u_1-u_2-1} = \left(\frac{u_1+1}{u_1}\right)^L \ , \tag{6.3.30}$$

$$\frac{(q_2 q_3)^L}{(q_1 q_2 q_3)^2} \frac{\bar{x}-u_1}{\bar{x}-u_1-1} \frac{\bar{x}-u_2}{\bar{x}-u_2-1} = 1 \ . \tag{6.3.31}$$

With the substitutions in (6.3.29), we then arrive at

$$u_-^L = \frac{\xi}{\xi_3^L}(u_- - u_+) \ , \quad (-u_+)^L = \frac{\xi}{\xi_2^L}(u_- - u_+) \ , \quad \hat{v} = -\frac{2\xi_1^L}{\xi^{L-2}} \ , \tag{6.3.32}$$

where we defined $\xi = \xi_1 \xi_2 \xi_3$, and we will also define $Q = q_1 q_2 q_3$ wherever it appears from now on.

Furthermore, the momenta variables have the following ϵ-behaviour:

$$\begin{aligned} e^{ip_1} &= \frac{u_- \epsilon^{\alpha}}{u_- \epsilon^{\alpha} + 1} \approx u_- \epsilon^{\alpha} \ , \\ e^{ip_2} &= \frac{-1 + \epsilon^{\alpha} u_+}{\epsilon^{\alpha} u_+} \approx -u_+^{-1} \epsilon^{-\alpha} \ . \end{aligned} \tag{6.3.33}$$

For the both plane wave factors, we have then the following behaviours, depending solely on the site indices n_1, n_2:

$$\begin{aligned} e^{i(p_1 n_1 + p_2 n_2)} &\sim \epsilon^{\alpha(n_1 - n_2)} \ , \\ e^{i(p_1 n_2 + p_2 n_1)} &\sim \epsilon^{\alpha(n_2 - n_1)} \ . \end{aligned} \tag{6.3.34}$$

Having now analysed how our momentum variables and rapidity variables scale, we now turn our attention to the eigenstate coefficients A and $\tilde{A}$. We already mentioned before that, as these coefficients are nothing else but "vector entries" of an eigenstate whose direction is of our interest, we can choose one of the coefficients freely. To make computations simpler, we choose $A_{23} = 1$, and in the language of the coefficients appearing in the equation of our interest, (6.3.22), we are then interested in the following ratio:

$$\frac{\psi_1}{\psi_2} = \frac{A_{32}}{A_{23}} = A_{32} \ , \tag{6.3.35}$$

which we can then express as

$$A_{32} = \frac{1}{Q} \frac{u_2 - \bar{x}}{u_1 - \bar{x} + 1} \ . \tag{6.3.36}$$

These considerations give us the un-tilded coefficients. The relation in (6.3.4) then gives us the tilded coefficients via

$$\begin{pmatrix} \tilde{A}_{23} \\ \tilde{A}_{32} \end{pmatrix} = \begin{pmatrix} \frac{1}{q_1 q_2 q_3} \frac{u_2 - u_1}{u_2 - u_1 + 1} & \frac{1}{u_2 - u_1 + 1} \\ \frac{1}{u_2 - u_1 + 1} & -q_1 q_2 q_3 \frac{u_2 - u_1}{u_2 - u_1 + 1} \end{pmatrix} \begin{pmatrix} A_{23} \\ A_{32} \end{pmatrix} \ . \tag{6.3.37}$$

[21] These equations are for $k=1$, we arrive at another analogous set of equations for u_2.

We remind ourselves that we are primarily interested in the ϵ-behaviour of our coefficients, which allows us to only care about the (far simpler) relevant parts of the coefficients - meaning for the computation that is to follow, we can limit ourselves to the leading order contributions for small ϵ:

$$\begin{aligned}
\begin{pmatrix} A_{23} \\ A_{32} \end{pmatrix} &= \begin{pmatrix} 1 \\ \frac{1}{q_1 q_2 q_3} \frac{-\hat{v}\epsilon^{\gamma}}{2+(u_- - u_+)\epsilon^{\alpha} - \hat{v}\epsilon^{\gamma}} \end{pmatrix} \approx \begin{pmatrix} 1 \\ -\frac{\hat{v}}{2\xi}\epsilon^{\gamma+3} \end{pmatrix} , \\
\begin{pmatrix} \tilde{A}_{32} \\ \tilde{A}_{23} \end{pmatrix} &= \begin{pmatrix} \frac{\epsilon^{-\alpha}+u_- - u_+}{q_1 q_2 q_3 u_- - q_1 q_2 q_3 u_+} + \frac{-\hat{v}\epsilon^{\gamma-\alpha}}{q_1 q_2 q_3 (u_- - u_+)(2-u_-\epsilon^{\alpha}+u_+\epsilon^{\alpha}-\hat{v}\epsilon^{\gamma})} \\ \frac{\epsilon^{-\alpha}}{u_- - u_+} - \frac{-\hat{v}\epsilon^{\gamma-\alpha}(-1+u_-\epsilon^{\alpha}-u_+\epsilon^{\alpha})}{(u_- - u_+)(2-u_-\epsilon^{\alpha}+u_+\epsilon^{\alpha}-\hat{v}\epsilon^{\gamma})} \end{pmatrix} \approx \begin{pmatrix} \frac{1}{\xi(u_- - u_+)}\epsilon^{3-\alpha} \\ \frac{1}{u_- - u_+}\epsilon^{-\alpha} \end{pmatrix} .
\end{aligned} \tag{6.3.38}$$

We now want to put all of these considerations together and find out, what the ϵ of our mixed-flavour wave function is. We recall that the only ϵ-relevant terms in its expression are the summands, i.e.

$$\begin{aligned}
|\psi_{23}(p_1,p_2)\rangle = \cdots \Big[\overbrace{A_{23} e^{i(p_1 n_1 + p_2 n_2)} \frac{q_3^{n_1}}{q_2^{n_2}}}^{\mathbf{A_{23}}} + \overbrace{\tilde{A}_{23} e^{i(p_1 n_2 + p_2 n_1)} \frac{q_3^{n_1}}{q_2^{n_2}}}^{\tilde{\mathbf{A}}_{\mathbf{23}}} \Big] \cdots \\
+ \cdots \Big[\underbrace{A_{32} e^{i(p_1 n_1 + p_2 n_2)} \frac{q_3^{n_2}}{q_2^{n_1}}}_{\mathbf{A_{32}}} + \underbrace{\tilde{A}_{32} e^{i(p_1 n_2 + p_2 n_1)} \frac{q_3^{n_2}}{q_2^{n_1}}}_{\tilde{\mathbf{A}}_{\mathbf{32}}} \Big] \cdots ,
\end{aligned} \tag{6.3.39}$$

whose powers we can now infer:

$$\begin{aligned}
\mathbf{A_{23}} &\sim \epsilon^{(1-\alpha)(n_2-n_1)} \\
\tilde{\mathbf{A}}_{\mathbf{23}} &\sim \epsilon^{(1+\alpha)(n_2-n_1)-\alpha} \\
\mathbf{A_{32}} &\sim \epsilon^{\gamma+3-(1+\alpha)(n_2-n_1)} \\
\tilde{\mathbf{A}}_{\mathbf{32}} &\sim \epsilon^{3-\alpha-(1-\alpha)(n_2-n_1)} .
\end{aligned} \tag{6.3.40}$$

We can make our lives easier by reminding ourselves of the ranges the indices can take values in: The difference of site indices is bound by $n_2 - n_1 \in \{1, \cdots, L-1\}$ (as $n_2 \geq n_1$), thus with (6.3.39)

- We have that $(1-\alpha)(n_2-n_1) < (1+\alpha)(n_2-n_1) - \alpha$, since $n_2 - n_1 \geq 1$, and thus will $\mathbf{A_{23}}$ always dominate over $\tilde{\mathbf{A}}_{\mathbf{23}}$ in the limit. We can therefore neglect the summands associated to $\tilde{\mathbf{A}}_{\mathbf{23}}$ going forward.
- Likewise, $3 - \alpha - (1-\alpha)(n_2 - n_1) < \gamma + 3 - (1+\alpha)(n_2 - n_1)$ holds, and implies that $\tilde{\mathbf{A}}_{\mathbf{32}}$ is dominating over $\mathbf{A_{32}}$, allowing us to neglect summands associated to the latter.

Summa summarum, we only have two types of summands in the sum for our wave function that are asymptotically relevant,

$$\begin{aligned}
|\psi_{23}\rangle &\approx \sum_{n_1<n_2} \left[A_{23} e^{i(p_1 n_1 + p_2 n_2)} \frac{q_3^{n_1}}{q_2^{n_2}} S^{2,+}_{n_1} S^{3,+}_{n_2} + \tilde{A}_{32} e^{i(p_1 n_2 + p_2 n_1)} \frac{q_3^{n_2}}{q_2^{n_1}} S^{3,+}_{n_1} S^{2,+}_{n_2} \right] |0\rangle \\
&\approx \sum_{n_1<n_2} \left[\frac{(\xi_3 u_-)^{n_1}}{(-\xi_2 u_+)^{n_2}} \epsilon^{(1-\alpha)(n_2-n_1)} S^{2,+}_{n_1} S^{3,+}_{n_2} + \frac{1}{\xi(u_- - u_+)} \frac{(\xi_3 u_-)^{n_2}}{(-\xi_2 u_+)^{n_1}} \epsilon^{3-\alpha-(1-\alpha)(n_2-n_1)} S^{3,+}_{n_1} S^{2,+}_{n_2} \right] |0\rangle .
\end{aligned} \tag{6.3.41}$$

We defined $\alpha = \frac{L-3}{L-1}$, which means it fulfils

$$\begin{aligned}
3 - \alpha - (1-\alpha)(L-1) &= 3 - \alpha - \left(1 - \frac{L-3}{L-1}\right)(L-1) \\
&= 3 - \alpha - L + 1 + L - 3 = 1 - \alpha ,
\end{aligned} \tag{6.3.42}$$

which allows us to write the dominating terms as

$$\frac{|\psi_{23}\rangle}{\epsilon^{1-\alpha}} \approx \left[\sum_{n_1=1}^{L-1} \frac{-1}{\xi_2 u_+}\left(-\frac{\xi_3 u_-}{\xi_2 u_+}\right)^{n_1} S^{2,+}_{n_1} S^{3,+}_{n_1+1}\right]|0\rangle + \frac{1}{\xi(u_- - u_+)}\frac{(\xi_3 u_-)^L}{(-\xi_2 u_+)^1} S^{3,+}_1 S^{2,+}_L |0\rangle \ . \tag{6.3.43}$$

We can combine the latter term with the other summands using the Bethe equation $(-\xi_2 u_+)^L = \xi(u_- - u_+)$, giving rise to the following expression:

$$\frac{|\psi_{23}\rangle}{\epsilon^{1-\alpha}} \approx \frac{-1}{\xi_2 u_+}\sum_{n_1=1}^{L}\left(-\frac{\xi_3 u_-}{\xi_2 u_+}\right)^{n_1} S^{2,+}_{n_1} S^{3,+}_{n_1+1}|0\rangle = |\psi^{(1)}(u_-/u_+)\rangle \ , \tag{6.3.44}$$

where the indices of the position of the operators $S^{i,+}_n$ should, as always, be understood mod L. This result is consistent with [AS21]: $|\psi^{(1)}(u_-/u_+)\rangle$ is indeed nothing else but the locked state the authors also found.

In the previous section, we argued that the defective limit of eigenvectors gives rise to eigenvectors of the defective matrix - we can therefore claim that $|\psi^{(1)}(u_-/u_+)\rangle$ are true eigenvectors of (6.1.7) for its defective limit, where it approaches the strongly twisted Hamiltonian $\hat{\mathbf{H}}_{\xi_1,\xi_2,\xi_3}$. One curiosity that we find is that the eigenstates only depend on the rapidity combination $\frac{u_-}{u_+}$, and not on either $u_\pm$ separately.[22] $\frac{u_-}{u_+}$ is equivalent to the total momentum $p_1 + p_2$ in momentum variables. The first step in our recipe is therefore complete, we have the generalised eigenvectors of rank 1 (i.e. the true eigenvectors) to proceed with our algorithm.

Computing now the vectors $w^{(1)}_{ij}$ as defined in (6.2.20), we need two eigenvectors of the non-defective case that coalesce to the same exceptional point, and then find a linear combination thereof that vanishes in the defective limit. Then, $w^{(1)}_{ij}$ is given by the normalised linear combination that stays finite in the defective limit (where we only care about the correct normalisation in terms of ϵ power, as commented on before). Candidates for two such vectors are now $|\psi_{23}(p_1,p_2)\rangle$ and $|\psi_{23}(p'_1,p'_2)\rangle$, which coalesce to the same eigenvector $|\psi^{(1)}(u_-/u_+)\rangle$ provided their respective total momentum coincide, i.e. if $p_1+p_2 = p'_1+p'_2$. In explicit momentum variables, we can also immediately see that

$$e^{ip_1}|\psi_{23}(p_1,p_2)\rangle - e^{ip'_1}|\psi_{23}(p'_1,p'_2)\rangle = 0 \ . \tag{6.3.45}$$

In the large twist limit, meaning when we have $\epsilon \to 0$, we find that the terms with $n_2 = n_1 + 1$ of $e^{ip_1}|\psi_{23}(p_1,p_2)\rangle$ cancel with the ones from $e^{ip'_1}|\psi_{23}(p'_1,p'_2)\rangle$, meaning that the leading order leftovers of (6.3.45) are given by

$$\begin{aligned}
&e^{ip_1}|\psi_{23}(p_1,p_2)\rangle - e^{ip'_1}|\psi_{23}(p'_1,p'_2)\rangle \approx \\
&\qquad \epsilon^{2-\alpha}\sum_{n_1=1}^{L-2}\left[\frac{u_-}{(-\xi_2 u_+)^2}\left(-\frac{\xi_3 u_-}{\xi_2 u_+}\right)^{n_1} - \frac{u'_-}{(-\xi_2 u'_+)^2}\left(-\frac{\xi_3 u'_-}{\xi_2 u'_+}\right)^{n_1}\right] S^{2,+}_{n_1} S^{3,+}_{n_1+2}|0\rangle \\
&\qquad + \epsilon^{3-(1-\alpha)(L-2)}\left[\frac{u_-}{\xi(u_- - u_+)}\frac{(\xi_3 u_-)^L}{(-\xi_2 u_+)^2} - \frac{u'_-}{\xi(u'_- - u'_+)}\frac{(\xi_3 u'_-)^L}{(-\xi_2 u'_+)^2}\right] S^{3,+}_2 S^{2,+}_L|0\rangle \\
&\qquad + \epsilon^{3-(1-\alpha)(L-2)}\left[\frac{u_-}{\xi(u_- - u_+)}\frac{(\xi_3 u_-)^{L-1}}{(-\xi_2 u_+)^1} - \frac{u'_-}{\xi(u'_- - u'_+)}\frac{(\xi_3 u'_-)^{L-1}}{(-\xi_2 u'_+)^1}\right] S^{3,+}_1 S^{2,+}_{L-1}|0\rangle \ .
\end{aligned} \tag{6.3.46}$$

All of these contributions are of the same ϵ-order, as

$$3-(1-\alpha)(L-2) = 2 + \frac{L-1}{L-1} - \frac{2}{L-1}(L-2)$$

[22] This is to be understood in the sense that any eigenstate can be reduced to only depend on the ratio (u_-/u_+) - of course, we can multiply any given state by an arbitrary combination of either rapidity, this is however only an artefact of the particular normalisation we choose. $|\psi^{(1)}(u_-/u_+)\rangle$ also possesses such a factor, $\frac{1}{\xi_2 u_+}$, but we can normalise that away when analysing rapidity dependencies.

$$= 2 + \frac{L-1-2L+4}{L-1} = 2 + \frac{-L+3}{L-1} = 2-\alpha \ . \tag{6.3.47}$$

Assuming coinciding total momentum, i.e. $\frac{u_-}{u_+} = \frac{u'_-}{u'_+}$, and choosing $u_\pm$ in such a way that the Bethe equations in (6.3.32) hold, we can then recast the before leading order expression into the more compact

$$\frac{e^{ip_1}|\psi_{23}(p_1,p_2)\rangle - e^{ip'_1}|\psi_{23}(p'_1,p'_2)\rangle}{\epsilon^{2-\alpha}} \approx \frac{u_-}{u_+}\frac{u'_+-u_+}{\xi_2^2 u_+ u'_+}\sum_{n_1=1}^{L}\left(-\frac{\xi_3 u_-}{\xi_2 u_+}\right)^{n_1} S^{2,+}_{n_1}S^{3,+}_{n_1+2}|0\rangle = |\psi^{(2)}(u_-/u_+)\rangle \ , \tag{6.3.48}$$

where the leading order exponent is more transparently presented. With the analysis from the previous sections, we can therefore state that $|\psi^{(2)}(u_-/u_+)\rangle$ are the generalised eigenstates of $\hat{\mathbf{H}}_{\xi_1,\xi_2,\xi_3}$, which again only depend on the total momentum combination of the rapidities.

Going one step further, we build the vectors $w^{(2)}_{ij}$, for which we find the following expression

$$\frac{e^{i(p_1+p'_1)}}{e^{ip'_1}-e^{ip_1}}\left(e^{ip_1}|\psi_{23}(p_1,p_2)\rangle - e^{ip'_1}|\psi_{23}(p'_1,p'_2)\rangle\right) - \frac{e^{i(p_1+p''_1)}}{e^{ip''_1}-e^{ip_1}}\left(e^{ip_1}|\psi_{23}(p_1,p_2)\rangle - e^{ip''_1}|\psi_{23}(p''_1,p''_2)\rangle\right) \ , \tag{6.3.49}$$

which we want to take the $\epsilon \to 0$ limit of. We find that, for small ϵ, it behaves as

$$\text{const. } \epsilon^{3-\alpha}\sum_{n_1=1}^{L}\left(-\frac{\xi_3 u_-}{\xi_2 u_+}\right)^{n_1} S^{2,+}_{n_1}S^{3,+}_{n_1+3}|0\rangle = \epsilon^{3-\alpha}|\psi^{(3)}(u_-/u_+)\rangle \ . \tag{6.3.50}$$

Our recipe dictates that we need to take the finite part of the above vector (which we achieve by normalising it), so in this case again, this can be easily read off as the ϵ-dependence is fully separated. Thus, $|\psi^{(3)}(u_-/u_+)\rangle$ is a generalised eigenvector of rank 3 of $\hat{\mathbf{H}}_{\xi_1,\xi_2,\xi_3}$.[23]

We can continue this process for the $M=2, K=1$ case in accordance with our recipe, and we find that the generalised eigenvector of rank k of $\hat{\mathbf{H}}_{\xi_1,\xi_2,\xi_3}$ are

$$|\psi^{(k)}(u_-/u_+)\rangle = \text{const. } \sum_{n_1=1}^{L}\left(-\frac{\xi_3 u_-}{\xi_2 u_+}\right)^{n_1} S^{2,+}_{n_1}S^{3,+}_{n_1+k}|0\rangle \ . \tag{6.3.51}$$

For $k \geq L$, the site indices of the above expression are not sensible, which is why we can assume $k \in \{1, \cdots L-1\}$. Moreover, for any given value of the total momentum we find exactly one such vector at any step of the process.[24] This is an indication that, in the sector of $M=2$ and $K=1$, we find Jordan chains of size $L-1$ for every permitted value of the total momentum. This reaffirms the numerical results from [AS21] and the combinatorial results from [ACS22], which we are in complete agreement with.

We have not mentioned anything regarding the potential danger of chain mixing so far, which might spoil our results if inspected more closely. Fortunately, chain mixing will not be present in our case, and we will show this with the arguments we outlined in the section about indistinguishable Jordan blocks; consider the Hermitian conjugate of $\hat{\mathbf{H}}_{\xi_1,\xi_2,\xi_3}$,

$$\hat{\mathbf{H}}^\dagger_{\xi_1,\xi_2,\xi_3} = \sum_{l=1}^{L}(\hat{\mathbb{P}}^{l,l+1})^\dagger \ , \tag{6.3.52}$$

$$\hat{\mathbb{P}}^\dagger|12\rangle = \xi_3^*|21\rangle \ , \qquad \hat{\mathbb{P}}^\dagger|23\rangle = \xi_1^*|32\rangle \ , \qquad \hat{\mathbb{P}}^\dagger|31\rangle = \xi_2^*|13\rangle \ . \tag{6.3.53}$$

[23] This is always to be understood up to a possible additional contribution proportional to $|\psi^{(2)}(u_-/u_-)\rangle$.

[24] It is important to remember, once again, that the $u_\pm$ parameters need to satisfy the Bethe equations, meaning that these ξ and u combinations are not free (and free to produce pathologies): (6.3.29) tells us that $-\xi_3 u_-/\xi_2 u_+ = \sqrt[L]{1}$, which is precisely the factor appearing as total momentum.

In 2.1.3.1, we proved that the generalised eigenvector of rank 1, $|\psi^{(1)}(u_-/u_+)\rangle$, must behave like a generalised eigenvector of highest rank of $\hat{\mathbf{H}}^\dagger_{\xi_1,\xi_2,\xi_3}$. Moreover, it is easy to see that[25]

$$\begin{aligned}
&\hat{\mathbf{H}}^\dagger_{\xi_1,\xi_2,\xi_3} S_n^{2,+} S_{n+1}^{3,+}|0\rangle = \xi_3^* S_{n-1}^{2,+} S_{n+1}^{3,+}|0\rangle + \xi_1^* S_n^{3,+} S_{n+1}^{2,+}|0\rangle + \xi_2^* S_n^{2,+} S_{n+3}^{3,+}|0\rangle \ , \\
&\hat{\mathbf{H}}^\dagger_{\xi_1,\xi_2,\xi_3}|\psi^{(1)}(u_-/u_+)\rangle = \rho(\xi_2^*+\xi_3^*)|\psi^{(2)}(u_-/u_+)\rangle + \rho'\xi_1^*|\psi^{(L-1)}(u_-/u_+)\rangle \ , \\
&\hat{\mathbf{H}}^\dagger_{\xi_1,\xi_2,\xi_3}\hat{\mathbf{H}}^\dagger_{\xi_1,\xi_2,\xi_3}|\psi^{(1)}(u_-/u_+)\rangle \propto (\xi_2^*+\xi_3^*)^2|\psi^{(3)}(u_-/u_+)\rangle \ , \\
&(\hat{\mathbf{H}}^\dagger_{\xi_1,\xi_2,\xi_3})^k|\psi^{(1)}(u_-/u_+)\rangle \propto (\xi_2^*+\xi_3^*)^k|\psi^{(k+1)}(u_-/u_+)\rangle \ , \\
&(\hat{\mathbf{H}}^\dagger_{\xi_1,\xi_2,\xi_3})^{L-1}|\psi^{(1)}(u_-/u_+)\rangle \propto \hat{\mathbf{H}}^\dagger_{\xi_1,\xi_2,\xi_3}(\xi_2^*+\xi_3^*)^{L-2}|\psi^{(L-1)}(u_-/u_+)\rangle = 0 \ ,
\end{aligned} \tag{6.3.54}$$

with ρ, ρ' being two constants irrelevant to our discussion. Curiously, the above generalised eigenvector of maximal rank is in appearance similar to a locked state, however, the 2's and 3's are swapped. We refer to states of these kind as *anti-locked states* (for obvious nomenclatural reasons), and the authors in [AS21] proved them to be generalised eigenvectors of maximal rank. With this result, we have the generalised eigenvectors of highest rank of the Jordan chain at our hands, and we can apply the reasoning from the section before: By applying (2.1.5) in order to generate the remaining generalised eigenvectors of the Jordan chain;

$$\hat{\mathbf{H}}^k_{\xi_1,\xi_2,\xi_3}|\psi^{(L-1)}(u_-/u_+)\rangle = \rho''(\xi_2+\xi_3)^k|\psi^{(L-k-1)}(u_-/u_+)\rangle + \rho'''\xi_1\delta_{k,1}|\psi^{(1)}(u_-/u_+)\rangle \ , \tag{6.3.55}$$

where ρ'', ρ''' are yet another pair of constants irrelevant to our considerations. Provided $\xi_2 \neq -\xi_3$, we are therefore able to act $L-2$ times with $\hat{\mathbf{H}}_{\xi_1,\xi_2,\xi_3}$ on $|\psi^{(L-1)}(u_-/u_+)\rangle$ and can show that we have Jordan cells of size $L-1$. However, for $\xi_1 \neq 0, \xi_2 = -\xi_3$, our reasoning is does not hold and we have in fact a Jordan chain of size 2 and $L-3$ eigenvectors. Lastly, for $\xi_1 = 0, \xi_2 = -\xi_3$, we do not have any non-trivial Jordan chains and have $L-1$ eigenvectors.

In [ACS22], the authors discussed the issue of chain mixing for $\xi_1 = \xi_2 = 0$ and proved numerically chain mixing not to be present up to $L = 30, M = 6$, although no formal proof was given.

We have now analysed the $M = 2$ case in a comprehensive way and successfully applied the algorithmic machinery that we developed earlier. The remaining cases ($M \geq 3$) are similar in analysis, but arguably feature more involved expressions, and requires us to embark on a satellite discussion about Diophantine equations, removing ourselves a bit from our topic. For these reasons, these cases are discussed - in full - in the appendix.

[25] Nota Bene: Not all the left generalised eigenvectors are orthogonal to the left generalised eigenvector of rank 1, $|\psi^{(L-1)}(u_-/u_+)\rangle$. 2.1.3.1 only requires orthogonality with respect to the eigenvector we are initially starting with, not necessarily with the one associated with the generalised eigenvector of maximal rank we are computing.

7 | Conclusions and Outlook

"The time has come," the walrus said, "to talk of many things: Of shoes and ships - and sealing wax - of cabbages and kings."

– Lewis Carroll, *Alice's Adventures in Wonderland*

7.1 The q-Poincaré Algebra

The work presented in chapter 4 builds upon the considerations of [GH07] and [You07], who were concerned with quantum deformed kinematical (super)algebras. In these two articles, the authors connect the quadratic Casimir operator with the $\mathcal{N} = 4$ SYM dispersion relation, and they search for an operator $\mathfrak{J}$ that acts as a translation of the rapidity variable, i.e. the variable that parametrises and homogenises the dispersion relation. The authors in [ST16] and [BST18] then build on and further extend these works. They find a realisation of a q-Poincaré algebra within the massless AdS_3 scattering problem and study the coproduct of the boost operator $\mathfrak{J}$. Our work is a direct continuation of the latter two works: In the setting of string theories in $AdS_3 \times S^3$ and their q-deformed counterpart (that can be identified with $(AdS_3 \times S^3)_\eta$), we examined the algebras of the associated massless scattering problem - $\mathfrak{su}(1|1)^2$ and $U_q[\mathfrak{su}(1|1)]^2$. In both cases, at first we restrict our algebraic analysis on the choice of a $1+1$-dimensional representation, but we later attempt to go beyond that.

In contrast to massive modes, the massless modes admit two different sets of coproducts, which we denote as bosonically braided and bosonically unbraided. This terminology refers to the braiding of the Cartan (energy) generator coproduct $\Delta\mathfrak{H}$, which are fundamentally inequivalent. Imposing quasi-cocommutativity with the R-matrix, which we assume is of 6-vertex type, we arrive at suitable R matrices for each case. In the q-deformed case, the entries of the R-matrices depend on the deformation parameter as well but reduce consistently to the undeformed case for $q \to 1$. With this information, we construct the coproduct for the boost operators $\mathfrak{J}_A$, which we obtain by imposing the algebra homomorphism property of the coproduct as well as the R-matrix cocommutativity relations. For each of the expressions of $\Delta\mathfrak{J}_A$, we are left with some ambiguities or freedoms, which we decompose in even and odd parts as functions of the momenta, denoted by $\hat{\Upsilon}$ and $\hat{\beta}$, respectively. Both $\hat{\Upsilon}$ and $\hat{\beta}$ fulfil interesting relations with their respective R-matrices, whose origin and significance is not yet fully clear to us.

Of all the cases studied, one deserves special attention. We find that, due to the form of ΔE, the q-deformed bosonically-braided case is not a Hopf algebra, but an example of a quasi-Hopf algebra, as it is not coassociative. This vanishes for $q \to 1$ and is not present for trivial braiding of $\mathfrak{H}_A$, which is why we do not see it in any of the undeformed cases. We are not able to formulate precise physical implications of this non-coassociative structure, and we plan to address this in the future.

Building upon this, in the second part of chapter 4 we try to generalise the results we have obtained

in the $1+1$-dimensional short representation to universal, representation-independent statements.

The first main question we investigated was how the left copy communicates with the right copy of our algebra, especially with respect to the boost generators of different handedness. Starting with the most general algebraic relations and postulates, we arrived at a classification of 6 different types of algebras - mostly by studying the resulting Jacobi identities and general properties of the algebra. These six algebras can be classified into two groups: separable algebras, which can be split into two disjoint subalgebras, and the differential algebras, which fulfil $\mathfrak{d}_{LR}\mathfrak{d}_{RL} = 1$ - this, in a sense, is a version of the inverse function theorem if we understand the boost as a differential operator. Amongst the algebras that we found, there was one special case, namely the one defined by the relation $\mathfrak{H}_L\mathfrak{d}_{LR} = \zeta\mathfrak{H}_R$, that belongs to both algebra types: On the one hand, it is differential because it fulfils the defining equation of differential algebras, $\mathfrak{d}_{LR}\mathfrak{d}_{RL} = 1$. On the other hand, it can be expressed via a generator redefinition based on a separable algebra. This relationship is represented in 7.1 below.

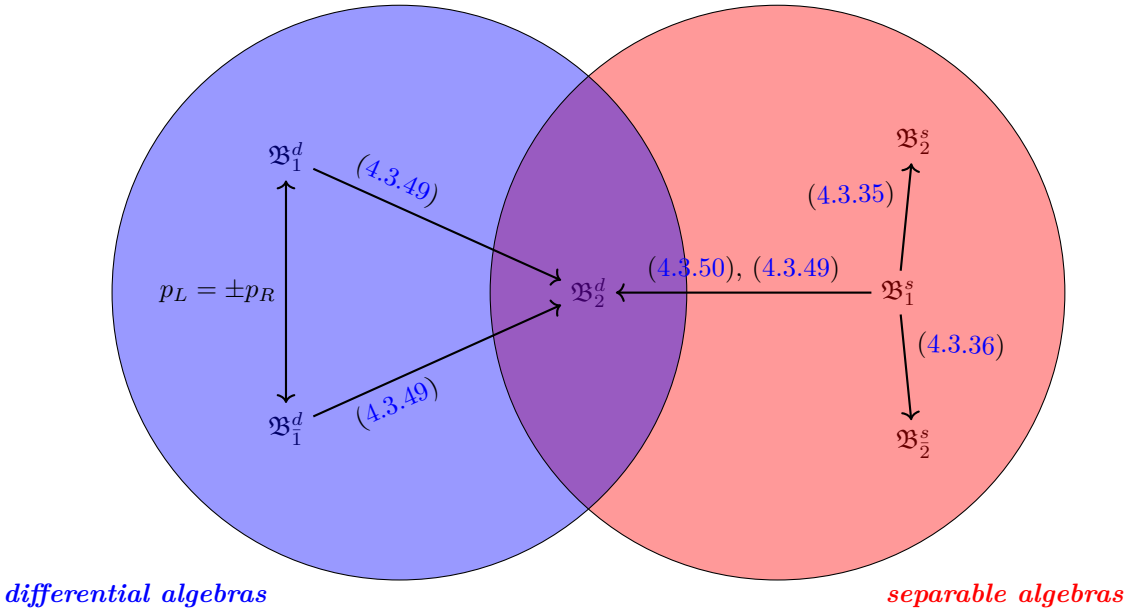


Figure 7.1: **Left**: The differentiable algebras and their relations. **Right**: The genealogy of separable algebras.

Having established all these algebraic structures, we want to establish the corresponding Hopf algebraic structures. In contrast to our analysis within the context of the 2-dimensional short representation, we here need to make use of generators of the outer symmetry group of our algebra order to formulate a universal coproduct for $\mathfrak{J}$. Despite that, the process is the same: we have to impose the coproduct to be an algebra homomorphism to obtain the wanted expressions.

A natural extension of our work relating to our boost algebraic efforts is to consider the massive case. It would be very interesting to analyse the massive case for the 2-dimensional short representation, whose massless sibling we discussed in section 4.2. The main obstruction to be considered in this case is that the dispersion relation changes with respect to the massless case. This has an effect on the kinds of coproducts we can choose: For example, for the following choice for energy coproduct

$$\Delta\mathfrak{H}_A = \mathfrak{H}_A \otimes e^{ip/2} + e^{-ip/2} \otimes \mathfrak{H}_A \ , \tag{7.1.1}$$

we no longer have $\Delta\mathfrak{H}_A = \Delta^{\text{op}}\mathfrak{H}_A$, which we need to hold for central elements. The construction of the massive representation would be an interesting extension of our results we presented here (see [Hoa15] for related work).

We have so far made no representation-independent analyses with respect to the study of quasi-triangularity and R-matrices. This is not a coincidence, rather unfortunately it is more difficult to do so: We expect our centrally extended, q-deformed Poincaré algebra to have a universal R-matrix.[1] There also exists a consistent and systematic way to go about constructing this object - the Drinfeld Double/Khoroshkin-Tolstoy construction [KT94] (with a pedagogical exposition in [Hec18]). Focusing on the $\mathfrak{d}_{LR} = +1$ case, we would cast our (enveloping) algebra into a Serre form, meaning such that the generators or Chevalley-Serre that fulfil the Serre relations. Constructing then a dual algebra $\mathcal{H}^{*}$[2] to our original algebra $\mathcal{H}$, the quantum double $\mathcal{D}(\mathcal{H})$ is defined as the union of these two algebras. This new algebra has a well-defined bilinear, non-degenerate map $\langle\cdot,\cdot\rangle : \mathcal{H}\otimes\mathcal{H}^* \to \mathbb{K}$, which can be used to construct the universal R-matrix.

However, it turns out that this is not quite as easy to achieve in our setting as one could hope: Our algebra is defined through intricate relations, and the presence of outer symmetry generators in coproducts of algebra elements make the construction non-trivial. This hinders us from straightforwardly applying the Drinfeld/Khoroshkin-Tolstoy recipe.

Additionally, it would be also be interesting to find a representation-independent, universal realisation of the coproduct of boost generators for the q-deformed case. However, we recall again that in the tail of the non-q-deformed boost coproduct, the generators of the outer automorphism symmetry appear. Thus, for any further analyses of the universal picture, it would be crucial to understand what form of the outer symmetry of our algebra that can be lifted to the q-deformed case (see [Reg16]).

Lastly, as we mentioned in different points throughout the chapter pertaining to the boost superalgebra constructions, we have not attended to the problem of finding an antipodal map, and so far the algebraic structure we found - though intrinsically consistent - is not a true Hopf algebra per se, but rather a bialgebra. Finding a suitable antipode would lift our structure to the Hopf level, and perhaps this antipodal map would then hold further valuable information on other aspects of this problem, such as the ambiguities $\hat{\Upsilon}$ and $\hat{\beta}$. However, in order to find such a map, we would also have to have a clearer idea about how the bialgebraic elements, most prominently the coproduct and counit, behave with respect to the outer automorphism generators, which is by no means at all trivial.

7.2 Deformed AdS_3 in the Landau-Lifshitz Limit

In chapter 5, our starting point was the 3-deformed $\text{AdS}_3 \times S^3 \times T^4$ background, as studied in [BMSS20]. In our approach, we constructed the effective field theory associated to the Landau-Lifshitz limit of the subspace $\mathbb{R} \times S^3 \subset \text{AdS}_3 \times S^3 \times T^4$, i.e. slow modes around a constant radius classical solution. In particular, we consider the effective Lagrangian at leading and next-to-leading order in κ, the energy constant associated to time-translation invariance. With these classical considerations, we proved that the so-constructed action gives rise to a near-BMN dispersion relations in agreement with the results the authors computed in [BMSS20].

Within the LL regime, we then computed the λ-leading order all-loop for the 2-to-2-particle S-matrix, and got a deformation of the Heisenberg $\mathfrak{su}(2)$ S-matrix (with sole dependence on $\chi_\pm$ out of the three deformation parameters). The tree level contribution to this S-matrix is congruent with [BMSS20] at

[1] In fact, the central-extension here is crucial for this statement to be true.

[2] Technically, one often considers the dual (c)op algebra, i.e. $\mathcal{H}^{*(\text{c})\text{op}}$.

leading order in λ. One order beyond in the string tension λ, the S-matrix we computed only shows dependence on some particular combinations of the deformation parameters $\tilde{q}, \chi_{\pm}$.

One natural continuation of our study involving the Landau-Lifshitz limit of our model is to also investigate whether it shows features of integrability. So far, by only considering 2-to-2-particle scattering, we have not been able to make any statements regarding factorisability of the higher particle S-matrix - a key indicator of quantum integrability. In [MPRW08], the authors compute the 3-to-3-particle S-matrix elements for the Landau-Lifshitz model. Similarly to our analysis, they proceed by means of canonical quantisation of a conveniently defined complex field, a process that requires very involved Feynman diagrammatics and kinematical considerations. They find particular QFT behaviours that hint at the integrability of the theory, such as the absence of particle creation and annihilation (i.e. the set, though not the order, of incoming and outgoing momenta is identical), and a non-trivial mechanism behind the factorisation of the 3-particle S-matrix. Similar considerations can be made for our model, involving the analogous set of computations. If successful, one could then consider a generalisation of this factorisation approach to the N-particle scattering amplitude.

Although it might seem straightforward, going one order beyond in the initial leading order corrections that we limited ourselves to is far from immediate. As we explained, this happens because we cannot continue using conformal gauge, meaning that there would be no way around using a gauge like the non-diagonal uniform one presented in [KRT04]. Going beyond the orders we considered might uncover interesting subtleties of the S-matrix computation that did no appear in our case (such as a dressing phase appearance), although this approach is arguably more involved in terms of the expressions that it produces.

Lastly, we could also perform the analogous computations and considerations for the inverted mode expansion of ϕ, meaning that we would essentially switch the creators and annihilators within the mode expansion (recalling that either choice is legitimate). We could then compare the so-obtained results against the S_{YY} matrix element in [BMSS20], rather than the $S_{\bar{Y}\bar{Y}}$ one as we did for this project. This would then further reaffirm us in our chosen approach.

7.3 Jordan Blocks and the Hypereclectic Spin Chain

In chapter 6, we focused our efforts on two subjects: (perturbative) linear algebra and spin chain applications.

In the former, we devised a recipe to find the generalised eigensystem of non-diagonalisable matrices by mean of perturbations: We started with a one-parameter family of matrices $\{M(\epsilon)|\epsilon \in \mathbb{R}\}$ which only at a chosen point $\epsilon = 0$ gives rise to a defective matrix, but remains diagonalisable in its neighbourhood (illustrated in Figure 7.2). Then, we show that it is necessary to consider appropriate linear combinations of eigenvectors of $M(\epsilon)$ to construct generalised eigenvectors of $M(0)$. In particular, we show that we need a linear combination of at least n different eigenvectors of $M(\epsilon)$ to recover the generalised eigenvector of rank n of $M(0)$.

In our analysis, we found that the cases of the Jordan Normal Form of $M(0)$ having only one or several Jordan blocks associated to the same eigenvalue behave differently, depending on whether we are in the distinguishable or indistinguishable Jordan setting. Starting with the former case, we are able to construct the complete generalised eigensystem of the defective matrix, with our recipe also indicating us what rank and eigenvalue is associated to each generalised eigenvector we construct. For the latter case, however, we saw our recipe break down and in need of some adjustments: While we still generate

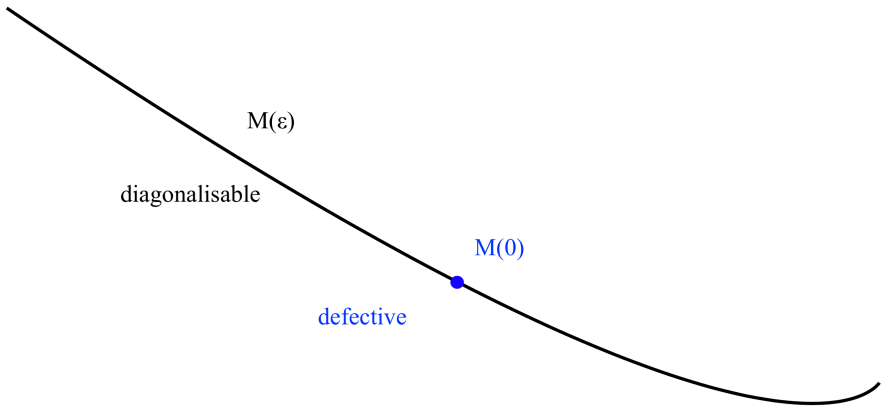


Figure 7.2: The one parameter family of matrices, $M(\epsilon)$, and the defective/exceptional point at $\epsilon = 0$.

the complete set of generalised eigenvectors of the defective matrix this way, the ranks that the "naive" recipe indicates us are wrong. This would make the recipe less useful, as the Jordan structure that we read off the generated genealogical diagram might be wrong. To this end, we supplemented our recipe with an additional step for the indistinguishable case to untangle the mixing of Jordan chains in our genealogical diagrams, an effect that we called "chain mixing" for obvious reasons. With this additional step for the indistinguishable case, we then fully established a complete and exhaustive method to find the generalised eigensystem of a defective matrix regardless of its Jordan structure or form.

With a fully working methodology at hand, we then proceeded to apply this to the case of the eclectic spin chain (see [ISZ19],[ISZ19],[AS21]). In this setting of this spin chain, coalescence of eigenstates of the Hamiltonian appears, as the strongly twisted Hamiltonian ceases to be diagonalisable. Applying our algorithm to this setup with $K = 1$, we find the generalised Bethe system of the spin chain, confirming and extending some of the results of [AS21], where the authors were not able to find the complete generalised eigensystem and were not able to provide an explanation for the unexpected shortening that they found, remedied by the untanglement of the mixed Jordan chains.

Throughout this thesis, we only considered the case of one excitation of type 3, i.e. the case of $K = 1$. In [NG22], the author then extends our analysis to the setting where $K > 1$. In this paper, our results and methodology are extended, and the characterisation of the Jordan chains associated to our specific spin chain setting is completed. It was shown that the information on the Jordan structure for generic values of L, M and K can be condensed into a product of K q-deformed binomial coefficients, in analogy to the single one we found in this thesis for $K = 1$.

Furthermore, there are plenty of potential settings where we can apply our algorithmic methodology to, and some of them also appear within the context of integrable systems: In the papers [dLPP$^+$21], [dLPP$^+$20], [DLPRR20] and [DLPR19], the authors classified all integrable systems with nearest-neighbour interactions exhibiting $\mathfrak{su}(2)(\oplus\mathfrak{su}(2))$ symmetry in a systematic way, and some of the R-matrices therein are defective. This way, the transfer matrix associated to any such R-matrix - whose eigensystem solves the spectral problem of the Hamiltonian (and the commuting charges) - is then a potential place for us to use our algorithmic machinery.

A | Appendix

"Tâchez de garder toujours un morceau de ciel au-dessus de votre vie." ["Always try to keep a patch of sky above your life."]

– Marcel Proust, *Du côté de chez Swann*

A.1 Further Details on the proof of Lemma 2.1.3 for non-singular geometric multiplicity

In this section, we want to outline some further details on the proof of Lemma 2.1.3 for when the geometric multiplicity exceed one. In short, we need to make use of Gram-Schmidt-like orthogonalisation arguments that makes use of the fact that any linear combination of generalised eigenvectors of rank p and true eigenvectors associated to the same eigenvalue is a generalised eigenvector of rank p. While we have already illustrated what orthogonality/duality relations that can be found between eigenvectors of different eigenvalues, we still need to clarify and expand on the relations that can be found *within* same-eigenvalue eigenvectors. We shall make the logic more explicit by starting with the geometric multiplicity 2-case, and for clarity's sake "abbreviate" our notation compared to the notation used in the main text. For the case of geometric multiplicity 2, we take two left eigenvectors to the same eigenvalue λ,

$$v_1, v_2 \ , \tag{A.1.1}$$

and, correspondingly, two right eigenvectors

$$u_1, u_2 \ . \tag{A.1.2}$$

We can remind ourselves that this must be true by recalling that, over $\mathbb{C}$, every matrix is similar (i.e. has the same JNF) as its transpose, because they can be related through a similarity transformation. Therefore, since transposing is relating the notions of left and right eigenvectors, we can conclude that the number of linearly independent ones must be the same. We want to relate a left eigenvector v_1 and a corresponding "top" generalised eigenvector u_1 such that they fulfil

$$v_1^\dagger \cdot u_1 = 1 \ . \tag{A.1.3}$$

Using Gram-Schmidt logic, we can then define a vector in the span of left eigenvectors, i.e. $\tilde{v}_2 = \gamma v_1 + \delta v_2$ with $\gamma, \delta \in \mathbb{C}$, that fulfils

$$\tilde{v}_2^\dagger \cdot u_1 = 0 \ . \tag{A.1.4}$$

Likewise, on the right eigenvector side, we can define a vector in the span of right eigenvectors, i.e. $\tilde{u}_2 = \alpha u_1 + \beta u_2$ with $\alpha, \beta \in \mathbb{C}$, such that

$$\tilde{u}_2^\dagger \cdot v_1 = 0 \ , \tag{A.1.5}$$

that additionally fulfils

$$\tilde{u}_2^\dagger \cdot v_2 = 1 \ , \tag{A.1.6}$$

after normalisation, if necessary. The vectors $\{v_1, \tilde{v}_2\}$ and $\{u_1, \tilde{u}_2\}$ have now the desired properties for the rest of our initially outlined proof to work, and we can conclude the case of geometric multiplicity 2. For even higher geometric multiplicities, this orthogonalisation/dualisation process works analogously: For the case of geometric multiplicity 3, we start with left and right eigenvectors $\{v_1, v_2, v_3\}$ and $\{u_1, u_2, u_3\}$, respectively, start with the initial pairing of v_1 and u_1, proceed identically the same, and then use the same logic for the additionally introduced v_3 and u_3 vectors. This concludes this elaboration.

A.2 The q-Poincaré Algebra

A.2.1 The Coassociator Relations

In what follows, we want to provide more details on the calculation regarding the coassociator in (4.2.39). The relatively simple structure of our coassociator makes it manageable to prove the defining relations in a compact way:

- $\boxed{(\mathrm{id} \otimes \chi \otimes \mathrm{id})\Phi = \mathrm{id} \otimes \mathrm{id}}$ - this computation almost trivialises if we recall that $\chi \equiv 0$ on the whole superalgebra. Relation (4.2.40) (with (4.2.6)) allows us to assume that

$$\frac{8i}{\log q}\omega = \Delta(E) - \Delta_{\mathrm{trivial}}(E) = E \otimes C_1 + C_2 \otimes E \ , \tag{A.2.1}$$

for appropriate C_i. We can see that also $\chi(C_i) = 0$ must hold, since we schematically have

$$\begin{aligned}(\chi \otimes \mathbb{1})(\Delta - \Delta_{\mathrm{trivial}})E &= \cancelto{0}{\chi(E)} \otimes (e^{\alpha p} - 1) + \underbrace{\chi(e^{\beta p} - 1)}_{=1-1=0} \otimes E \ , \\ (\mathbb{1} \otimes \chi)(\Delta - \Delta_{\mathrm{trivial}})E &= E \otimes \underbrace{\chi(e^{\alpha p} - 1)}_{=1-1=0} + (e^{\beta p} - 1) \otimes \cancelto{0}{\chi(E)} \ ,\end{aligned} \tag{A.2.2}$$

where α and β are constants representing the appearing factors, and in the latter arguments we also used that $\chi(p) = 0$. Expanding the coassociator via

$$\Phi = \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} + \mathfrak{B} \otimes \omega + \omega \otimes \mathfrak{B} + \text{higher order exponent terms with } \mathfrak{B}, \omega \ , \tag{A.2.3}$$

we can immediately see that for the subleading terms, χ will always make them vanish as it will hit either an exponential p or an E term. Thus, we end up with

$$\begin{aligned}(\mathrm{id} \otimes \chi \otimes \mathrm{id})\Phi &= \mathbb{1} \otimes \mathbb{1} + 0 + 0 + ... \\ &= \mathbb{1} \otimes \mathbb{1} \ .\end{aligned} \tag{A.2.4}$$

- $\boxed{(\mathrm{id} \otimes \Delta) \circ \Delta = \Phi\left((\Delta \otimes \mathrm{id}) \circ \Delta\right)\Phi^{-1}}$ - here, we only need to check this relation where both sides are evaluated on the supercharges $\mathfrak{Z}$, meaning

$$(\mathrm{id} \otimes \Delta)\Delta(\mathfrak{Z}) \overset{!}{=} \Phi\left((\Delta \otimes \mathrm{id})\Delta(\mathfrak{Z})\right)\Phi^{-1} \ , \tag{A.2.5}$$

as the other coproducts are fixed from theirs. The calculation to prove this coassociator relation in analogous for any of them, so we shall exemplify this using $\mathfrak{Q}$. The left-hand side of the relation then gives us:

$$(\mathrm{id} \otimes \Delta)\Delta(\mathfrak{Q}) = (\mathrm{id} \otimes \Delta)\left(\mathfrak{Q} \otimes e^{-\frac{i}{4}p}q^{\frac{E}{4}} + e^{\frac{i}{4}p}q^{-\frac{E}{4}} \otimes \mathfrak{Q}\right)$$

$$
\begin{aligned}
=&\mathfrak{Q}\otimes\Delta\left(e^{-\frac{i}{4}p}q^{\frac{E}{4}}\right)+e^{\frac{i}{4}p}q^{-\frac{E}{4}}\otimes\mathfrak{Q}\otimes e^{-\frac{i}{4}p}q^{\frac{E}{4}}+e^{\frac{i}{4}p}q^{-\frac{E}{4}}\otimes e^{\frac{i}{4}p}q^{-\frac{E}{4}}\otimes\mathfrak{Q}\\
=&\underbrace{\mathfrak{Q}\otimes\left(e^{-\frac{i}{4}p}\otimes e^{-\frac{i}{4}p}\right)q^{\frac{\Delta(E)}{4}}}_{\mathrm{Co}_1}+\underbrace{e^{\frac{i}{4}p}q^{-\frac{E}{4}}\otimes\mathfrak{Q}\otimes e^{-\frac{i}{4}p}q^{\frac{E}{4}}}_{\mathrm{Co}_2}+\underbrace{e^{\frac{i}{4}p}q^{-\frac{E}{4}}\otimes e^{\frac{i}{4}p}q^{-\frac{E}{4}}\otimes\mathfrak{Q}}_{\mathrm{Co}_3}\ .
\end{aligned}
\tag{A.2.6}
$$

Similarly, the right-hand side of this coassociator relation is given by:

$$
\begin{aligned}
\Phi\left((\Delta\otimes\mathrm{id})\Delta(\mathfrak{Q})\right)\Phi^{-1}=&\underbrace{\Phi(\mathfrak{Q}\otimes e^{-\frac{i}{4}p}q^{\frac{E}{4}}\otimes e^{-\frac{i}{4}p}q^{\frac{E}{4}})\Phi^{-1}}_{\mathrm{Co}_1'}+\underbrace{\Phi(e^{\frac{i}{4}p}q^{-\frac{E}{4}}\otimes\mathfrak{Q}\otimes e^{-\frac{i}{4}p}q^{\frac{E}{4}})\Phi^{-1}}_{\mathrm{Co}_2'}\\
&+\underbrace{\Phi\left(\left(e^{\frac{i}{4}p}\otimes e^{\frac{i}{4}p}\right)q^{-\frac{\Delta(E)}{4}}\otimes\mathfrak{Q}\right)\Phi^{-1}}_{\mathrm{Co}_3'}\ .
\end{aligned}
\tag{A.2.7}
$$

Due to the position of the fermionic charge on the second tensorial slot, the coassociator conjugation does not affect Co_2', and we can immediately see that the terms $\mathrm{Co}_2'=\mathrm{Co}_2$ agree. Regarding Co_1', we find that

$$
\begin{aligned}
\Phi(\mathfrak{Q}\otimes e^{-\frac{i}{4}p}q^{\frac{E}{4}}\otimes e^{-\frac{i}{4}p}q^{\frac{E}{4}})\Phi^{-1}=&e^{\mathfrak{B}\otimes\omega_{23}+\omega_{12}\otimes\mathfrak{B}}(\mathfrak{Q}\otimes e^{-\frac{i}{4}p}q^{\frac{E}{4}}\otimes e^{-\frac{i}{4}p}q^{\frac{E}{4}})e^{-\mathfrak{B}\otimes\omega_{23}-\omega_{12}\otimes\mathfrak{B}}\\
=&e^{\mathfrak{B}\otimes\omega_{23}}(\mathfrak{Q}\otimes e^{-\frac{i}{4}p}q^{\frac{E}{4}}\otimes e^{-\frac{i}{4}p}q^{\frac{E}{4}})e^{-\mathfrak{B}\otimes\omega_{23}}\ ,
\end{aligned}
\tag{A.2.8}
$$

where the other part of the coassociator is irrelevant for this term due to the position of the fermionic generator. We then make use of the Weyl-type relations in (4.2.41), the relation in (4.2.40) and the fact that

$$
e^{\mathbb{1}\otimes M}=\mathbb{1}\otimes e^{M}\ ,
\tag{A.2.9}
$$

giving us

$$
\begin{aligned}
e^{\mathfrak{B}\otimes\omega_{23}}(\mathfrak{Q}\otimes e^{-\frac{i}{4}p}q^{\frac{E}{4}}\otimes e^{-\frac{i}{4}p}q^{\frac{E}{4}})e^{-\mathfrak{B}\otimes\omega_{23}}=&(\mathfrak{Q}\otimes\underbrace{e^{-\frac{i}{4}p}q^{\frac{E}{4}}\otimes e^{-\frac{i}{4}p}q^{\frac{E}{4}}}_{=\left(e^{-\frac{i}{4}p}\otimes e^{-\frac{i}{4}p}\right)q^{\frac{\Delta_{\mathrm{trivial}}(E)}{4}}})\underbrace{q^{\frac{\Delta(E)-\Delta_{\mathrm{trivial}}(E)}{4}}}_{e^{2i\omega_{23}}}\\
=&(\mathfrak{Q}\otimes\left(e^{-\frac{i}{4}p}\otimes e^{-\frac{i}{4}p}\right)q^{\frac{\Delta_{\mathrm{trivial}}(E)}{4}}q^{\frac{\Delta(E)-\Delta_{\mathrm{trivial}}(E)}{4}})\\
=&\mathfrak{Q}\otimes\left(e^{-\frac{i}{4}p}\otimes e^{-\frac{i}{4}p}\right)q^{\frac{\Delta(E)}{4}}\ ,
\end{aligned}
\tag{A.2.10}
$$

which is nothing else than Co_1. In a similar way, $\mathrm{Co}_3=\mathrm{Co}_3'$ is shown.

- Lastly, the pentagon relation $\boxed{(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)\Phi(\Delta\otimes\mathrm{id}\otimes\mathrm{id})\Phi=(\mathrm{id}\otimes\Phi)(\mathrm{id}\otimes\Delta\otimes\mathrm{id})(\Phi\otimes\mathrm{id})}$ can be proved quite directly using similar manipulations. In particular, by expanding both terms and making use of

$$
\begin{aligned}
(\Delta\otimes\mathbb{1})(\omega_{ij})&=\frac{1}{8i}\log q\left(E(p_i+p_j+p_k)-E(p_i+p_j)-E(p_k)\right)\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\ ,\\
(\mathbb{1}\otimes\Delta)(\omega_{ij})&=\frac{1}{8i}\log q\left(E(p_i+p_j+p_k)-E(p_i)-E(p_j+p_k)\right)\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\ .
\end{aligned}
\tag{A.2.11}
$$

Here, we can also immediately see again that the structure of the coassociator trivialises for additive energy, as it does for $q\to 1$. Having this at hand, the pentagon relation follows immediately.

With the relations we used above, it is straightforward to prove the coassociator equations related to quasi-triangularity. However, to see that our quasi-Hopf algebraic setup satisfies the generalised YBE, we

need to make things more explicit. As we mentioned before in the main part of this thesis, our strategy here is to show that, for each of the states we evaluate it on, the GYBE reduces to the ordinary YBE equations - modulo a prefactor in each case. To this end, we want to exemplify the coassociator YBE equations by explicitly evaluating both sides of the GYBE in (2.4.13) applied on some representative set of 3-particle states. To this end, we first introduce our notation for this:

- In order to abbreviate our expressions, we introduce the following notation,

$$\mathcal{E}_{ij} = \mathcal{E}_{ji} = \frac{E_{i+j} - E_i - E_j}{4} \ , \tag{A.2.12}$$

  which then lets the coassociator appear in a more compact way, e.g.

$$\begin{aligned} \Phi^{123} &= q^{\frac{1}{4}\left(\frac{\mathfrak{B}}{2i}\otimes E_{2+3} - E_2 - E_3\right)} q^{\frac{1}{4}\left(E_{1+2} - E_1 - E_2)\otimes\frac{\mathfrak{B}}{2i}\right)} \\ &= q^{\left(\frac{\mathfrak{B}}{2i}\otimes\mathcal{E}_{23}\right)} q^{\left(\mathcal{E}_{12}\otimes\frac{\mathfrak{B}}{2i}\right)} \ . \end{aligned} \tag{A.2.13}$$

- We further abbreviate $\bar{h}_i = \sqrt{h \sin\frac{p_i}{2}}$.

- We will write the R-matrix components as:

$$R = \alpha\, \mathbb{1}\otimes\mathbb{1} + \beta\, \mathfrak{B}\otimes\mathbb{1} + \tilde{\beta}\, \mathbb{1}\otimes\mathfrak{B} + \gamma\, \mathfrak{B}\otimes\mathfrak{B} + s\, \mathfrak{Q}\otimes\mathfrak{S} + \tilde{s}\, \mathfrak{S}\otimes\mathfrak{Q} \ , \tag{A.2.14}$$

  and we shall refer to the specific momenta space dependence in the coefficient functions by use of superindices, e.g. α^{12}.

In total, we have $2^3 = 8$ 3-particle states that we need to test the GYBE on. The two cases of three identical states, $\boxed{|\Phi\Phi\Phi\rangle}$ and $\boxed{|\Psi\Psi\Psi\rangle}$, are immediate when evaluated.

For the remaining two sets of three states each, namely the set of states containing one fermionic or two fermionic states, respectively, we shall fully discuss one representative example. In a second step, we prove the equivalence concretely by eventually showing that the aforementioned prefactor appears on both sides of the GYBE in the same way. To this end, we shall always compute the expressions in all detail, in order to illustrate how computations of this kind work for our representation (recall that the usual YBE computation can also be read off by just setting $\Phi \equiv 1$). Notice also that once the coassociator and coefficients of the R-matrix have been evaluated, they reduce to being scalar functions, and their order is inconsequential; we shall therefore make the evaluation evident with indices and superindices, and change the order of the expressions then in such a way to provide the clearest oversight. In what is to follow, one should remember how the $\mathfrak{B}$ operator acts on the states, which one can infer by considering the algebraic relations and the representation, yielding:

$$\mathfrak{B}|\Phi\rangle = i \ , \qquad \mathfrak{B}|\Psi\rangle = -i \ . \tag{A.2.15}$$

$\boxed{|\Phi\Phi\Psi\rangle}$

For the representative case of one fermion, the left-hand side of the equation yields:

$$\begin{aligned} R_{12}\Phi^{312}R_{13}\Phi^{132^{-1}}R_{23}\Phi^{123}|\Phi\Phi\Psi\rangle = &\left(\alpha^{12} + i\left(\beta^{12} + \tilde{\beta}^{12}\right) - \gamma^{12}\right)\Phi^{312}_{|\Phi\Phi\Psi\rangle}\left(\alpha^{13} + i\left(\beta^{13} - \tilde{\beta}^{13}\right) + \gamma^{13}\right)\Phi^{132}_{|\Phi\Phi\Psi\rangle}{}^{-1} \\ &\times\left(\alpha^{23} + i\left(\beta^{23} - \tilde{\beta}^{23}\right) + \gamma^{23}\right)\Phi^{123}_{|\Phi\Phi\Psi\rangle}|\Phi\Phi\Psi\rangle \\ &+ \Phi^{312}_{|\Psi\Phi\Phi\rangle}\left(\tilde{s}^{13}\bar{h}_1\bar{h}_2\right)\Phi^{132}_{|\Phi\Phi\Psi\rangle}{}^{-1}\left(\alpha^{23} + i\left(\beta^{23} - \tilde{\beta}^{23}\right) - \gamma^{23}\right) \\ &\times\Phi^{123}_{|\Phi\Phi\Psi\rangle}\left(\left(\alpha^{12} - i\left(\beta^{12} - \tilde{\beta}^{12}\right) - \gamma^{12}\right)|\Psi\Phi\Phi\rangle - s^{12}\bar{h}_1\bar{h}_2|\Phi\Psi\Phi\rangle\right) \\ &+ \Phi^{312}_{|\Phi\Psi\Phi\rangle}\left(\alpha^{13} + i\left(\beta^{13} + \tilde{\beta}^{13}\right) + \gamma^{13}\right)\Phi^{132}_{|\Phi\Psi\Phi\rangle}{}^{-1}\tilde{s}^{23}\bar{h}_2\bar{h}_3\Phi^{123}_{|\Phi\Phi\Psi\rangle} \end{aligned}$$

$$\times \left(\left(\alpha^{12} + i \left(\beta^{12} - \tilde{\beta}^{12} \right) - \gamma^{12} \right) |\Phi\Psi\Phi\rangle + \tilde{s}^{12} \bar{h}_1 \bar{h}_2 |\Psi\Phi\Phi\rangle \right) \ . \tag{A.2.16}$$

With (A.2.14) in mind, it is immediate what our evaluation strategy is: We must start with the term that is "closest" to the state and evaluate it. In the above, this was the Φ^{123} which after evaluation carries a label, i.e. $\Phi^{123}_{|\Phi\Phi\Psi\rangle}$. We then work our way to the next term to act on the state, which is R_{23}. Whenever an R-matrix hits our state, our state might "split" into a linear combination of states, which needs to be taken into account for all subsequent terms to hit the states, and a kind of "evaluation tree" forms, gaining a new branching whenever such a split happens. It is therefore hugely important that we keep track of labels and indices, which was the reason why this particular notation was chosen. Similarly, the right-hand side gives us:

$$\begin{aligned}
\Phi^{321} R_{23} \Phi^{231\,-1} R_{13} \Phi^{213} R_{12} |\Phi\Phi\Psi\rangle =& \Phi^{213}_{|\Phi\Phi\Psi\rangle} \left(\alpha^{12} + i \left(\beta^{12} + \tilde{\beta}^{12} \right) + \gamma^{12} \right) {\Phi^{231}_{|\Phi\Phi\Psi\rangle}}^{-1} \left(\alpha^{13} + i \left(\beta^{13} - \tilde{\beta}^{13} \right) - \gamma^{13} \right) \\
& \times \left(\Phi^{321}_{|\Phi\Phi\Psi\rangle} \left(\alpha^{23} + i \left(\beta^{23} - \tilde{\beta}^{23} \right) - \gamma^{23} \right) |\Phi\Phi\Psi\rangle + \Phi^{321}_{|\Phi\Psi\Phi\rangle} \tilde{s}^{23} \bar{h}_2 \bar{h}_3 |\Phi\Psi\Phi\rangle \right) \\
& + \Phi^{213}_{|\Phi\Phi\Psi\rangle} \left(\alpha^{12} + i \left(\beta^{12} + \tilde{\beta}^{12} \right) + \gamma^{12} \right) {\Phi^{231}_{|\Psi\Phi\Phi\rangle}}^{-1} \tilde{s}^{13} \bar{h}_1 \bar{h}_3 \Phi^{321}_{|\Psi\Phi\Phi\rangle} \\
& \times \left(\alpha^{23} + i \left(\beta^{23} + \tilde{\beta}^{23} \right) + \gamma^{23} \right) |\Psi\Phi\Phi\rangle \ .
\end{aligned} \tag{A.2.17}$$

We now only need to compare whether, the coassociator contributions going along with each type of final 3-particle state are the same on both sides of the equations:

- $\Phi\Phi\Psi$ type:

$$\Phi^{312}_{|\Phi\Phi\Psi\rangle} {\Phi^{132}_{|\Phi\Phi\Psi\rangle}}^{-1} \Phi^{123}_{|\Phi\Phi\Psi\rangle} \overset{!}{=} \Phi^{321}_{|\Phi\Phi\Psi\rangle} {\Phi^{231}_{|\Phi\Phi\Psi\rangle}}^{-1} \Phi^{213}_{|\Phi\Phi\Psi\rangle} \ , \tag{A.2.18}$$

  which is fulfilled, most easily checked by comparing the exponents:

$$(-\cancel{\mathcal{E}_{12}} + \cancel{\mathcal{E}_{31}}) - (\cancel{\mathcal{E}_{32}} + \cancel{\mathcal{E}_{13}}) + (\cancel{\mathcal{E}_{23}} - \cancel{\mathcal{E}_{12}}) \overset{!}{=} (\cancel{\mathcal{E}_{13}} - \cancel{\mathcal{E}_{21}}) + (-\cancel{\mathcal{E}_{21}} + \cancel{\mathcal{E}_{32}}) - (\cancel{\mathcal{E}_{31}} + \cancel{\mathcal{E}_{23}}) \ . \tag{A.2.19}$$

- $\Phi\Psi\Phi$ type:

$$\Phi^{312}_{|\Psi\Phi\Phi\rangle} {\Phi^{132}_{|\Phi\Phi\Psi\rangle}}^{-1} \Phi^{123}_{|\Phi\Phi\Psi\rangle} \# + \Phi^{312}_{|\Phi\Psi\Phi\rangle} {\Phi^{132}_{|\Phi\Psi\Phi\rangle}}^{-1} \Phi^{123}_{|\Phi\Phi\Psi\rangle} \#' \overset{!}{=} \Phi^{321}_{|\Phi\Psi\Phi\rangle} {\Phi^{231}_{|\Phi\Phi\Psi\rangle}}^{-1} \Phi^{213}_{|\Phi\Phi\Psi\rangle} \ , \tag{A.2.20}$$

  where, in this instance, we introduced # symbols to indicate that there are different coassociator factors for the different terms appearing on the same side of the equation. This translates to:

$$\begin{aligned}
&[(\cancel{\mathcal{E}_{12}} + \cancel{\mathcal{E}_{31}}) - (\cancel{\mathcal{E}_{32}} + \cancel{\mathcal{E}_{13}}) + (\cancel{\mathcal{E}_{23}} - \cancel{\mathcal{E}_{12}})] \# + [(\cancel{\mathcal{E}_{12}} - \cancel{\mathcal{E}_{31}}) - (\cancel{\mathcal{E}_{32}} - \cancel{\mathcal{E}_{13}}) + (\cancel{\mathcal{E}_{23}} - \cancel{\mathcal{E}_{12}})] \#' \\
&\qquad \overset{!}{=} (\cancel{\mathcal{E}_{13}} - \cancel{\mathcal{E}_{21}}) - (\cancel{\mathcal{E}_{31}} + \cancel{\mathcal{E}_{23}}) + (\cancel{\mathcal{E}_{21}} + \cancel{\mathcal{E}_{32}}) \ .
\end{aligned} \tag{A.2.21}$$

- $\Psi\Phi\Phi$ type:

$$\Phi^{312}_{|\Psi\Phi\Phi\rangle} {\Phi^{132}_{|\Phi\Phi\Psi\rangle}}^{-1} \Phi^{123}_{|\Phi\Phi\Psi\rangle} \# + \Phi^{312}_{|\Phi\Psi\Phi\rangle} {\Phi^{132}_{|\Phi\Psi\Phi\rangle}}^{-1} \Phi^{123}_{|\Phi\Phi\Psi\rangle} \#' \overset{!}{=} \Phi^{312}_{|\Psi\Phi\Phi\rangle} {\Phi^{132}_{|\Phi\Phi\Psi\rangle}}^{-1} \Phi^{123}_{|\Phi\Phi\Psi\rangle} \ , \tag{A.2.22}$$

  translates to

$$\begin{aligned}
&[(\cancel{\mathcal{E}_{12}} + \cancel{\mathcal{E}_{31}}) - (\cancel{\mathcal{E}_{32}} + \cancel{\mathcal{E}_{13}}) + (\cancel{\mathcal{E}_{23}} - \cancel{\mathcal{E}_{12}})] \# + [(\cancel{\mathcal{E}_{12}} - \cancel{\mathcal{E}_{31}}) - (\cancel{\mathcal{E}_{32}} - \cancel{\mathcal{E}_{13}}) + (\cancel{\mathcal{E}_{23}} - \cancel{\mathcal{E}_{12}})] \#' \\
&\qquad \overset{!}{=} (\cancel{\mathcal{E}_{12}} + \cancel{\mathcal{E}_{31}}) - (\cancel{\mathcal{E}_{32}} + \cancel{\mathcal{E}_{13}}) + (\cancel{\mathcal{E}_{23}} - \cancel{\mathcal{E}_{12}}) \ .
\end{aligned} \tag{A.2.23}$$

Thus, we have proven that the YBE and the GYBE are equivalent on the state $|\Phi\Phi\Psi\rangle$, too. For the next state containing two fermions, we shall again show that the appearing factors from the coassociator can be reduced away, but we shall skip the making the first step of the calculation explicit this time.

$\boxed{|\Phi\Psi\Psi\rangle}$

For the representative case of two fermions, the GYBE evaluated on it,

$$R_{12}\Phi^{312}R_{13}\Phi^{132^{-1}}R_{23}\Phi^{123}|\Phi\Psi\Psi\rangle = \Phi^{321}R_{23}\Phi^{231^{-1}}R_{13}\Phi^{213}R_{12}|\Phi\Psi\Psi\rangle \ , \tag{A.2.24}$$

will produce more fermionic signs and is slightly more involved to tackle, but in essence the same. The contributions and factors can be categorised in the following way:

- $\Phi\Psi\Psi$ type:

$$\Phi^{312}_{|\Phi\Psi\Psi\rangle}\Phi^{132}_{|\Phi\Psi\Psi\rangle}{}^{-1}\Phi^{123}_{|\Phi\Psi\Psi\rangle} \overset{!}{=} \Phi^{321}_{|\Phi\Psi\Psi\rangle}\Phi^{231}_{|\Phi\Psi\Psi\rangle}{}^{-1}\Phi^{213}_{|\Phi\Psi\Psi\rangle} \ , \tag{A.2.25}$$

  translates to

$$(\cancel{\mathcal{E}_{23}} - \cancel{\mathcal{E}_{12}}) - (\cancel{\mathcal{E}_{32}} - \cancel{\mathcal{E}_{13}}) + (-\cancel{\mathcal{E}_{12}} - \cancel{\mathcal{E}_{31}}) \overset{!}{=} (-\cancel{\mathcal{E}_{13}} - \cancel{\mathcal{E}_{21}}) - (-\cancel{\mathcal{E}_{31}} + \cancel{\mathcal{E}_{23}}) + (-\cancel{\mathcal{E}_{21}} + \cancel{\mathcal{E}_{32}}) \ . \tag{A.2.26}$$

- $\Psi\Phi\Psi$ type:

$$\Phi^{321}_{|\Psi\Phi\Psi\rangle}\Phi^{231}_{|\Psi\Psi\Phi\rangle}{}^{-1}\Phi^{213}_{|\Phi\Psi\Psi\rangle}\# + \Phi^{321}_{|\Psi\Phi\Psi\rangle}\Phi^{231}_{|\Psi\Phi\Psi\rangle}{}^{-1}\Phi^{213}_{|\Psi\Phi\Psi\rangle}\#' \overset{!}{=} \Phi^{312}_{|\Phi\Psi\Psi\rangle}\Phi^{132}_{|\Phi\Psi\Psi\rangle}{}^{-1}\Phi^{123}_{|\Phi\Psi\Psi\rangle} \ , \tag{A.2.27}$$

  which translates to

$$\begin{aligned}(-\cancel{\mathcal{E}_{13}} - \cancel{\mathcal{E}_{21}}) - (-\cancel{\mathcal{E}_{31}} + \cancel{\mathcal{E}_{23}}) + (-\cancel{\mathcal{E}_{21}} - \cancel{\mathcal{E}_{32}})\# + (\cancel{\mathcal{E}_{13}} - \cancel{\mathcal{E}_{21}}) - (\cancel{\mathcal{E}_{31}} + \cancel{\mathcal{E}_{23}}) + (-\cancel{\mathcal{E}_{21}} - \cancel{\mathcal{E}_{32}})\#' \\ \overset{!}{=} (\cancel{\mathcal{E}_{23}} - \cancel{\mathcal{E}_{12}}) - (\cancel{\mathcal{E}_{23}} - \cancel{\mathcal{E}_{13}}) + (-\cancel{\mathcal{E}_{12}} - \cancel{\mathcal{E}_{31}}) \ .\end{aligned} \tag{A.2.28}$$

- $\Psi\Psi\Phi$ type:

$$\Phi^{321}_{|\Psi\Psi\Phi\rangle}\Phi^{231}_{|\Psi\Psi\Phi\rangle}{}^{-1}\Phi^{213}_{|\Phi\Psi\Psi\rangle}\# + \Phi^{321}_{|\Psi\Psi\Phi\rangle}\Phi^{231}_{|\Psi\Phi\Psi\rangle}{}^{-1}\Phi^{213}_{|\Psi\Phi\Psi\rangle}\#' \overset{!}{=} \Phi^{312}_{|\Psi\Psi\Phi\rangle}\Phi^{132}_{|\Phi\Psi\Psi\rangle}{}^{-1}\Phi^{123}_{|\Phi\Psi\Psi\rangle} \ , \tag{A.2.29}$$

  which translates to

$$\begin{aligned}(-\cancel{\mathcal{E}_{13}} - \cancel{\mathcal{E}_{21}}) - (-\cancel{\mathcal{E}_{31}} + \cancel{\mathcal{E}_{23}}) + (\cancel{\mathcal{E}_{21}} - \cancel{\mathcal{E}_{32}})\# + (\cancel{\mathcal{E}_{13}} - \cancel{\mathcal{E}_{21}}) - (\cancel{\mathcal{E}_{31}} + \cancel{\mathcal{E}_{23}}) + (\cancel{\mathcal{E}_{21}} - \cancel{\mathcal{E}_{32}})\#' \\ \overset{!}{=} (\cancel{\mathcal{E}_{23}} - \cancel{\mathcal{E}_{12}}) - (\cancel{\mathcal{E}_{23}} - \cancel{\mathcal{E}_{13}}) + (\cancel{\mathcal{E}_{12}} - \cancel{\mathcal{E}_{31}}) \ .\end{aligned} \tag{A.2.30}$$

With this, all the states are checked. Notice that, due to statistical conservation, the sectors with $0, 1, 2$ or 3 fermions are only able to be mapped onto (linear combinations of) states of the same sector.

A.3 Spin Chains and Jordan Blocks

In this appendix that concerns chapter 6, we want to both make examples of our algorithmic framework in practice, and point out certain caveats to be aware of during the computation. As with our exposition, we divide this endeavour into the distinguishable and the indistinguishable case.

A.3.1 The Distinguishable Case And A Caveat Regarding Limits

Let us consider the following complex 3×3 matrix:

$$M(\epsilon) = \begin{pmatrix} 0 & 1 & \epsilon^2 \\ \epsilon^2 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} . \tag{A.3.1}$$

$M(\epsilon)$ has the following eigenvalues and eigenvectors

$$\begin{aligned} \lambda_1 &= 0 , & v_1 &= (1, \epsilon^4, -\epsilon^2) , \\ \lambda_2 &= -\epsilon , & v_2 &= (1, -\epsilon, 0) , \\ \lambda_3 &= \epsilon , & v_3 &= (1, \epsilon, 0) . \end{aligned} \tag{A.3.2}$$

We can see that $\lim_{\epsilon \to 0} v_i = (1, 0, 0)$ for the three vectors - so indeed, all eigenvectors collapse to the same eigenvector in the defective limit. We now want to make the subtlety in limit-taking that we alluded to in (6.2.28) explicit while computing the generalised eigenvectors, by also doing it the "wrong way" on the right-hand side below (i.e. by subtracting limits of eigenvectors instead of computing the limit only at the end). Going one rank up to computing $w^{(1)}$, we find

$$\begin{aligned} \frac{v_1 - \beta_{21} v_2}{|v_1 - \beta_{21} v_2|} &\approx (\epsilon^4, 1, -\epsilon) , & \frac{v_1 - \lim_{\epsilon \to 0}(\beta_{21} v_2)}{|v_1 - \beta_{21} v_2|} &\approx (0, -\epsilon^2, 1) , \\ \frac{v_2 - \beta_{32} v_3}{|v_2 - \beta_{32} v_2|} &\approx (-\epsilon, 1, 0) , & \frac{v_2 - \lim_{\epsilon \to 0}(\beta_{32} v_3)}{|v_2 - \beta_{32} v_3|} &\approx (0, 1, 0) . \end{aligned} \tag{A.3.3}$$

Thus, were we to proceed the wrong way, we will incorrectly generate two generalised eigenvectors of rank 2 rather than one - this directly contradicts 2.1.1 as our geometric multiplicity is just one, providing an upper bound for the number of linearly independent generalised eigenvectors that can appear with each step. So, correctly we find that $\lim_{\epsilon \to 0} w_{12}^{(1)} = (0, 1, 0)$ is the generalised eigenvector of rank 2.

Going one step beyond, we then find

$$\frac{w_{12}^{(1)} - \beta_{32} w_{23}^{(1)}}{w_{12}^{(1)} - \beta_{32} w_{23}^{(1)}} \approx (1, 0, 1) , \tag{A.3.4}$$

which is then implying that $\lim_{\epsilon \to 0} w_{12}^{(2)} = (1, 0, 1)$.[1] In [GN16], the authors use the "wrong" method of constructing generalised eigenvectors, however, in their case it is inconsequential due to their Jordan Blocks being of minimal size 2.

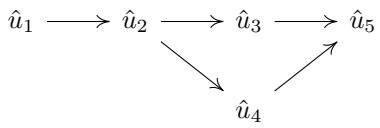
A.3.2 The Indistinguishable Case

In this chapter, we want to fully and correctly examine the example briefly mentioned in the corresponding section 6.2.3 concerning chain mixing, we recall the matrix we mentioned where this issue arises in (6.2.46):

$$M(\epsilon) = \begin{pmatrix} \epsilon & 1 & \epsilon & 0 & 0 \\ 0 & \epsilon^4 & 1 & \epsilon & 0 \\ 0 & 0 & \epsilon^7 & \epsilon^9 & 0 \\ 0 & 0 & 0 & \epsilon^6 & 1 \\ 0 & 0 & 0 & 0 & \epsilon^8 \end{pmatrix} , \tag{A.3.5}$$

giving rise to the following genealogical diagram:

[1] Here, it is again apparent that this is not a "pure" generalised eigenvector of rank 2: It is the sum of a true eigenvector and the "pure" generalised eigenvector of rank 2, $(1, 0, 1) = (1, 0, 0) + (0, 0, 1)$.



We can straightforwardly see that in the defective limit, $M(0) \equiv M$ is in JNF form, and has Jordan blocks of sizes 3 and 2 - not reflected in the diagram above. We need to make use of our remedy in (6.2.50), and proceeding systematically, we find two things at first, namely that we have two admissible m values and associated generalised eigenvectors of rank 1:

$$\begin{aligned} M^\dagger M^\dagger M^\dagger \hat{u}_1 = 0 \neq M^\dagger M^\dagger \hat{u}_1 \ , \\ M^\dagger M^\dagger \hat{u}_4 = 0 \neq M^\dagger \hat{u}_4 \ , \end{aligned} \tag{A.3.6}$$

while also satisfying the eigenvector condition (recall that M is nilpotent and has therefore only 0 as eigenvalues)

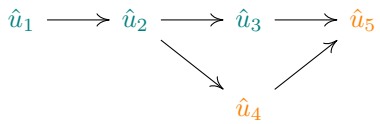
$$M\hat{u}_1 = M\hat{u}_4 = 0 \ . \tag{A.3.7}$$

This implies we have admissible $m \in \{1, 2\}$ for $v \in \{u_4, u_1\}$ in (6.2.50). We can thus generate $|\{1, 2\}| = 2$ untangled chains, and find:

$$m = \text{2-chain} \begin{cases} M^\dagger M^\dagger \hat{u}_1 & = \hat{u}_3 \\ M M^\dagger M^\dagger \hat{u}_1 & = \hat{u}_2 \\ M M M^\dagger M^\dagger \hat{u}_1 & = \hat{u}_1 \end{cases} \tag{A.3.8}$$

$$m = \text{1-chain} \begin{cases} M^\dagger \hat{u}_4 & = \hat{u}_5 \\ M M^\dagger \hat{u}_4 & = \hat{u}_4 \end{cases} \tag{A.3.9}$$

This implies that indeed we have a Jordan chain of length 3 and one of length 2, and we can untangle the genealogical diagram above in the following way:



Using this untangling recipe, any chain mixing can be resolved and the correct Jordan structure computed. With this example, one can also see that the above recipe is consistent with the case where there is no chain mixing, as is to be expected. Of course, the cases of chain mixing are more likely to appear for bigger Jordan blocks, and we have dealt so far with a "minimal" example to illustrate things, but the recipe we introduced works just the same for more intricate Jordan structures, too.

A.3.3 Generalised Eigensystem for $M = 3$

In this section, we will analyse the generalised Bethe system for the $M = 3$ case, whose analysis will closely follow the one for the $M = 2$ case but does come with some new caveats that we shall emphasise before moving on to the case of arbitrary M, on which we shall only make some comments. In either case, our starting case will again be studying both sets of Bethe equations ((6.3.26) and (6.3.28)) in the defective limit, parametrised by $\epsilon \to 0$. Proceeding again as in [AS21], we expand the physical rapidities u_1, u_2, u_3 and the auxiliary rapidity $\bar{x}$ around the defective limit,

$$u_1 \to \epsilon^\alpha u_- + \epsilon^{2\alpha} \tilde{u}_- \ ,$$

$$
\begin{aligned}
u_2 &\to \epsilon^{\alpha} u'_- + \epsilon^{2\alpha} \tilde{u}'_- \ , \\
u_3 &\to -1 + \epsilon^{\beta} u_+ \ , \\
\bar{x} &\to u_3 + \epsilon^{\gamma} \hat{v} \ ,
\end{aligned} \tag{A.3.10}
$$

where we redefine the constants to account for $M \neq 2$,

$$
\begin{aligned}
\alpha &= \frac{L-M-1}{L-M+1} \ , \\
\beta &= \frac{L-3M-3}{L-M+1} \ , \\
\gamma &= 2L-3M \ ,
\end{aligned} \tag{A.3.11}
$$

and the Bethe equations take the following form

$$
\begin{aligned}
u_-^L &= (-1)^{M-1} \frac{\xi}{\xi_3^L} u_+ \ , \quad \frac{u_-^2 - \tilde{u}_-}{u_-^{L+1}} = 2 \frac{\xi_3^L}{\xi} \frac{\tilde{u}_- - \tilde{u}'_-}{u_+} \ , \\
(u'_-)^L &= (-1)^{M-1} \frac{\xi}{\xi_3^L} u_+ \ , \quad \frac{u'^2_- - \tilde{u}'_-}{(u'_-)^{L+1}} = 2 \frac{\xi_3^L}{\xi} \frac{\tilde{u}'_- - \tilde{u}_-}{u_+} \ , \\
-(-u_+)^{L-M+1} &= \frac{\xi_3^L}{2^{M-1}\xi} \hat{v} \ , \quad \hat{v} = -\frac{2^{M-1}\xi_1^L}{\xi^{L-M}} \ .
\end{aligned} \tag{A.3.12}
$$

Expressed in momentum variables rather than rapidities, we have the following relations (using again that $e^{ip_k} = \frac{u_k}{u_k+1}$):

$$
\begin{aligned}
e^{ip_1} &\approx u_- \epsilon^{\alpha} + (\tilde{u}_- - u_-^2)\epsilon^{2\alpha} \ , \\
e^{ip_2} &\approx u'_- \epsilon^{\alpha} + (\tilde{u}'_- - u'^2_-)\epsilon^{2\alpha} \ , \\
e^{ip_3} &\approx -u_+^{-1} \epsilon^{-\beta} \ .
\end{aligned} \tag{A.3.13}
$$

This means that in the plane wave ansatz, where we have in general summands of the form

$$
|\psi\rangle = \cdots \underbrace{e^{i \sum_j p_{\sigma(j)} n_j} \frac{q_3^{\sum_j n_j}}{(q_2 q_3)^{n_k}}}_{\mathbf{F}(\mathbf{M},\sigma)} \cdots \ , \tag{A.3.14}
$$

we have the following behaviour:

$$
\mathbf{F}(\mathbf{3},\mathbf{Id}) = e^{i(p_1 n_1 + p_2 n_2 + p_3 n_3)} \frac{q_3^{n_1} q_3^{n_2}}{q_2^{n_3}} \sim \epsilon^{(\alpha-1)(n_1+n_2)-(\beta-1)n_3} = \epsilon^{(\alpha-1)[(n_1-n_3)+(n_2-n_3)]} \ , \tag{A.3.15}
$$

where we make use of the relation $(M-K)(\alpha-1) = K(\beta-1)$ amongst our constants in the exponents. The Hamiltonian that describes our model is still invariant under the shift operator U, meaning that we would expect only a dependence on the relative positions of the excitations, and indeed the sole dependence on $(n_1 - n_3)$ and $(n_2 - n_3)$ indicates precisely that.

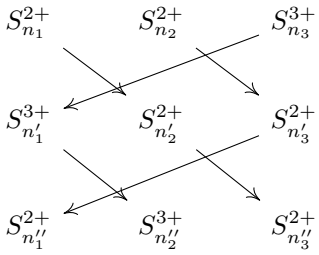
Having now discussed the ϵ-behaviour of the rapidities and the momenta, we now analyse the behaviour of the coefficients $\psi_k(\sigma)$, that were previously in the $M = 2$ case represented by the $A, \tilde{A}$ coefficients. We are again free to normalise one of the coefficients (which corresponds to an over normalisation of the eigenstate), and we shall choose $\psi_3(id) = 1$. For the two cases $k - l = 3, 2$ in (6.3.22), we then have

$$
\begin{aligned}
\frac{\psi_2(id)}{\psi_3(id)} &= \frac{1}{Q} \frac{u_3 - \bar{x}}{u_2 - \bar{x} + 1} \ , \\
\frac{\psi_1(id)}{\psi_2(id)} &= \frac{1}{Q} \frac{u_2 - \bar{x}}{u_1 - \bar{x} + 1} \ .
\end{aligned} \tag{A.3.16}
$$

As for the case of $M = 2$, in this first step we have dealt with the coefficients that carry (id) as an argument. The coefficients carrying transposition $\tau \in S_3$ as arguments, i.e. $\psi_i((jk))$, with i, j, k different integers, are straightforwardly computed, as the transposition occurs amongst labels of excitations of flavour 2, and for these cases we simply have:

$$\psi_i((jk)) = \underbrace{-\frac{1 - 2e^{ip_j} + e^{i(p_j+p_k)}}{1 - 2e^{ip_k} + e^{i(p_j+p_k)}}}_{S_{22}(p_j,p_k)} \psi_i(id) \ . \tag{A.3.17}$$

This is enough to fix $3! = 6$ coefficients of the wave functions, as we have to choose pairwise different integers i, j, k for expressions $\psi_i((jk))$. For all the other cases, we can use the periodicity condition to relate terms of the following form in a particular way:



Examples of such a connection are

$$\begin{aligned}
\psi_1((321)) &= \psi_3(id)\frac{e^{ip_3 L}}{q_2^L} \ , \\
\psi_2((123)) &= \psi_3(id)e^{i(p_2+p_3)L}\frac{q_3^L}{q_2^L} \ .
\end{aligned} \tag{A.3.18}$$

Needing to fix 18 coefficients in total, and having had computed 6 before, for each one of them we can derive 2 more, meaning we exhaust all the freedoms the coefficients might possess and fix them fully. Having now all the coefficients in our quiver, we can start to link them together, and we see that the coefficients associated with $\psi_3(id)$ are the dominant terms. For instance, we can take a look at the part of the wave function associated with the creation operators $S^{2+}_{n_1} S^{2+}_{n_2} S^{3+}_{n_3}$:

$$\begin{aligned}
\psi_3(id)e^{i(p_1 n_1+p_2 n_2+p_3 n_3)}\frac{q_3^{n_1+n_2}}{q_2^{n_3}} &\sim \psi_3((12))e^{i(p_2 n_1+p_1 n_2+p_3 n_3)}\frac{q_3^{n_1+n_2}}{q_2^{n_3}} \sim \epsilon^{(\alpha-1)[(n_1-n_3)+(n_2-n_3)]} \ , \\
\psi_3((123))e^{i(p_2 n_1+p_3 n_2+p_1 n_3)}\frac{q_3^{n_1+n_2}}{q_2^{n_3}} &\sim \psi_1(id)\frac{q_2^L}{e^{ip_1 L}}\epsilon^{\alpha(n_1+n_3)-\beta n_2}\epsilon^{n_1+n_2-n_3} \\
&\sim \epsilon^{\gamma-3}\epsilon^{-(1+\alpha)L}\epsilon^{(\alpha-1)[(n_1-n_3)+(n_2-n_3)]+(\alpha+\beta)(n_3-n_2)} \ , \\
\psi_3((13))e^{i(p_3 n_1+p_2 n_2+p_1 n_3)}\frac{q_3^{n_1+n_2}}{q_2^{n_3}} &\sim \psi_1(id)S_{22}(p_3,p_2)\frac{q_2^L}{e^{ip_1 L}}\epsilon^{\alpha(n_2+n_3)-\beta n_1}\epsilon^{n_1+n_2-n_3} \\
&\sim \epsilon^{\gamma-3}\epsilon^{-\beta}\epsilon^{-(1+\alpha)L}\epsilon^{(\alpha-1)[(n_1-n_3)+(n_2-n_3)]+(\alpha+\beta)(n_3-n_1)} \ , \\
\psi_3((321))e^{i(p_3 n_1+p_1 n_2+p_2 n_3)}\frac{q_3^{n_1+n_2}}{q_2^{n_3}} &\sim \psi_2(id)\frac{q_2^L}{e^{i(p_1+p_2)L}q_3^L}\epsilon^{\alpha(n_2+n_3)-\beta n_1}\epsilon^{n_1+n_2-n_3} \\
&\sim \epsilon^{3}\epsilon^{-2\alpha L}\epsilon^{(\alpha-1)[(n_1-n_3)+(n_2-n_3)]+(\alpha+\beta)(n_3-n_1)} \ , \\
\psi_3((23))e^{i(p_1 n_1+p_3 n_2+p_2 n_3)}\frac{q_3^{n_1+n_2}}{q_2^{n_3}} &\sim \psi_2(id)\frac{q_2^L}{e^{i(p_1+p_2)L}q_3^L}S_{22}(p_1,p_3)\epsilon^{\alpha(n_1+n_3)-\beta n_2}\epsilon^{n_1+n_2-n_3} \\
&\sim \epsilon^{3}\epsilon^{-2\alpha L}\epsilon^{\beta}\epsilon^{\alpha(n_1+n_3)-\beta n_2}\epsilon^{n_1+n_2-n_3} \ .
\end{aligned} \tag{A.3.19}$$

There is one more caveat that we should be aware of: In the defective $\epsilon \to 0$ limit, the momenta p_1 and p_2 coincide, implying that the leading contribution of $\psi_3(id)e^{i(p_1 n_1+p_2 n_2+p_3 n_3)}\frac{q_3^{n_1+n_2}}{q_2^{n_3}}$ exactly cancels the

leading contribution of $\psi_3((12))e^{i(p_2n_1+p_1n_2+p_3n_3)}\frac{q_3^{n_1+n_2}}{q_2^{n_3}}$, as $S_{22}(p,p)=-1$ if the two excitations have identical flavours and coinciding momenta. However, the next-to-leading contribution to the rapidities u_1 and u_2 makes them different, appearing an additional ϵ^α factor compared to our initial computation from before

$$\psi_3(id)e^{i(p_1n_1+p_2n_2+p_3n_3)}\frac{q_3^{n_1+n_2}}{q_2^{n_3}}+\psi_3((12))e^{i(p_2n_1+p_1n_2+p_3n_3)}\frac{q_3^{n_1+n_2}}{q_2^{n_3}}\sim\epsilon^\alpha\epsilon^{(\alpha-1)[(n_1-n_3)+(n_2-n_3)]}\ . \tag{A.3.20}$$

Nonetheless, the situation is much alike to the one we found in the setting of $M=2$, in the sense that we have two dominating contributions, the ones associated with $\psi_3(id)$ and $\psi_3((12))$, and we can restrict our considerations from now on to contributions associated with the coefficient $\psi_3(id)$, as we similarly did in the $M=2$ case for the dominating terms.

Having analysed all the ϵ-behaviours of the coefficients and the rapidities (and momenta), the mise-en-place for our recipe is now complete. In the defective limit, the wave function behaves as

$$\begin{aligned}|\psi\rangle\approx&\sum_{\substack{n_1<n_2<n_3\\1\le n_i\le L}}^{L}\Big[\left(\psi_3(id)e^{i(p_1n_1+p_2n_2)}+\psi_3((12))e^{i(p_2n_1+p_1n_2)}\right)e^{ip_3n_3}\frac{q_3^{n_1+n_2}}{q_2^{n_3}}S^{2,+}_{n_1}S^{2,+}_{n_2}S^{3,+}_{n_3}\\&+\left(\psi_1((321))e^{i(p_1n_2+p_2n_3)}+\psi_1((13))e^{i(p_2n_2+p_1n_3)}\right)e^{ip_3n_1}\frac{q_3^{n_2+n_3}}{q_2^{n_1}}S^{3,+}_{n_1}S^{2,+}_{n_2}S^{2,+}_{n_3}\\&+\left(\psi_2((23))e^{i(p_1n_1+p_2n_3)}+\psi_2((123))e^{i(p_2n_1+p_1n_3)}\right)e^{ip_3n_2}\frac{q_3^{n_1+n_3}}{q_2^{n_2}}S^{2,+}_{n_1}S^{3,+}_{n_2}S^{2,+}_{n_3}\Big]|0\rangle\\\approx&\sum_{\substack{n_1<n_2<n_3\\1\le n_i\le L}}^{L}\epsilon^{\alpha+(\alpha-1)[(n_1-n_3)+(n_2-n_3)]}\Big[(n_1-n_2)\frac{\tilde u_--\tilde u'_-}{u_-}\frac{(\xi_3u_-)^{n_1+n_2}}{(-\xi_2u_+)^{n_3}}S^{2,+}_{n_1}S^{2,+}_{n_2}S^{3,+}_{n_3}\\&+(n_2-n_3)\frac{\tilde u_--\tilde u'_-}{u_-}\cdot\frac{(\xi_3u_-)^{n_2+n_3}}{(-\xi_2u_+)^{n_1+L}}S^{3,+}_{n_1}S^{2,+}_{n_2}S^{2,+}_{n_3}+(n_1-n_3-L)\frac{\tilde u_--\tilde u'_-}{u_-}\frac{(\xi_3u_-)^{n_1+n_3+L}}{(-\xi_2u_+)^{n_2+L}}S^{2,+}_{n_1}S^{3,+}_{n_2}S^{2,+}_{n_3}\Big]|0\rangle\ .\end{aligned} \tag{A.3.21}$$

The permitted combinations of the three different n_i and the parts of the sum that dominate are

$$\begin{aligned}\frac{|\psi\rangle}{\epsilon^{3-2\alpha}}\approx&\left[\sum_{n_1=1}^{L-2}\frac{\xi_3(\tilde u'_--\tilde u_-)}{\xi_2^2u_+^2}\left(-\frac{\xi_3^2u_-^2}{\xi_2u_+}\right)^{n_1}S^{2,+}_{n_1}S^{2,+}_{n_1+1}S^{3,+}_{n_1+2}\right]|0\rangle+\frac{\tilde u'_--\tilde u_-}{u_-}\frac{(\xi_3u_-)^{2L-1}}{(-\xi_2u_+)^{L+1}}\\&\cdot S^{3,+}_1S^{2,+}_{L-1}S^{2,+}_L|0\rangle+\frac{\tilde u'_--\tilde u_-}{u_-}\frac{(\xi_3u_-)^{2L+1}}{(-\xi_2u_+)^{L+2}}S^{2,+}_1S^{3,+}_2S^{2,+}_L|0\rangle\ .\end{aligned} \tag{A.3.22}$$

We can recast all of these contributions into a single sum in the following way:

$$|\psi\rangle\approx\epsilon^{3-2\alpha}\frac{\xi_3(\tilde u'_--\tilde u_-)}{\xi_2^2u_+^2}\sum_{n_1=1}^{L}\left(-\frac{\xi_3^2u_-^2}{\xi_2u_+}\right)^{n_1}S^{2,+}_{n_1}S^{2,+}_{n_1+1}S^{3,+}_{n_1+2}|0\rangle\ , \tag{A.3.23}$$

where again we understand the subindices of the creation operators $S^{i,+}_n$ are to be floored modulo L. Curiously, we again have that, symbolised by the ratio $\frac{u_-^2}{u_+}$, this expression only depends on the total momentum $P_3=p_1+p_2+p_3$. Additionally, once more as in the $M=2$ case, we can see that the (finite part of the) eigenvector corresponds exactly to what in [AS21] is referred to as locked state.

We now go one step further in our recipe, and, analogously to the $M=2$ case, try to furnish a suitable linear combination of eigenvectors, that we just now have computed. If two eigenstates $|\psi(p_1,p_2,p_3)\rangle$ and $|\psi(p'_1,p'_2,p'_3)\rangle$ have the same total momentum, $P_3=P'_3$, then they coalesce to the same (true) eigenstate of the defective strongly twisted Hamiltonian. Similarly to the $M=2$, the linear combination that we want to analyse the limit behaviour of,

$$e^{i(2p_1+p_2)}|\psi(p_1,p_2,p_3)\rangle-e^{i(2p'_1+p'_2)}|\psi(p'_1,p'_2,p'_3)\rangle\ , \tag{A.3.24}$$

then lets us deduce that

$$\frac{e^{i(2p_1+p_2)}|\psi(p_1,p_2,p_3)\rangle - e^{i(2p_1'+p_2')}|\psi(p_1',p_2',p_3')\rangle}{\epsilon^4} \approx \text{const.} \sum_{n_1=1}^{L} \left(-\frac{\xi_3^2 u_-^2}{\xi_2 u_+}\right)^{n_1} S_{n_1}^{2,+} S_{n_1+2}^{2,+} S_{n_1+3}^{3,+}|0\rangle \ . \tag{A.3.25}$$

The finite part of the above state corresponds to a generalised eigenstate of rank 2. Things get more interesting when we turn to constructing the generalised eigenvectors of rank 3. Proceeding along the same lines as in the $M = 2$ case, we end up finding that the suitable linear combination of eigenstates (of the non-defective Hamiltonian) can, in the limit, be written as the linear combination of two excitations that solely depend on P_3,

$$a_1 \left[\sum_{n_1=1}^{L} \left(-\frac{\xi_3^2 u_-^2}{\xi_2 u_+}\right)^{n_1} S_{n_1}^{2,+} S_{n_1+3}^{2,+} S_{n_1+4}^{3,+}|0\rangle\right] + a_2 \left[\sum_{n_1=1}^{L} \left(-\frac{\xi_3^2 u_-^2}{\xi_2 u_+}\right)^{n_1} S_{n_1}^{2,+} S_{n_1+1}^{2,+} S_{n_1+3}^{3,+}|0\rangle\right] \ , \tag{A.3.26}$$

where a_i are coefficients that are functions of the individual momenta, too. This is indicative that there is another true eigenvector hidden somewhere in disguise.

The situation gets more involved the deeper we get into our recipe, and we need to construct a more streamlined approach to find the number of independent vectors that is found at each step. When analysing $|\psi\rangle$ more closely, we find that, similar to before, the relevant ϵ-order contributions of the wave function are controlled by the separation of the excitations, most directly seen by analysing the (twisted) $M = 3$ plane wave factor (as already stated before),

$$\mathbf{F}(\mathbf{3},\mathbf{Id}) = e^{i(p_1 n_1 + p_2 n_2 + p_3 n_3)} \frac{q_3^{n_1} q_3^{n_2}}{q_2^{n_3}} \sim \epsilon^{(\alpha-1)(n_1+n_2)-(\beta-1)n_3} = \epsilon^{(\alpha-1)[(n_1-n_3)+(n_2-n_3)]} \ . \tag{A.3.27}$$

The question regarding the number of independent vectors found at each step can therefore be reformulated as the question of how many solutions does the equation $n = (n_1 - n_3) + (n_2 - n_3)$ possess, for a given suitable n. There is the additional restriction of $1 \leq n_1 < n_2 < n_3 \leq L$, as this is how we build our wave functions (see (6.3.10)), which means that the separations of two states can at most be $L - 1$, but has to be larger than 0. Thus, the set of equations whose number of solutions we are interested in looks as follows:

$$\begin{aligned} (n_3 - n_1) + (n_3 - n_2) &= n \ , \\ 1 \leq (n_3 - n_2) < (n_3 - n_1) &\leq L - 1 \ . \end{aligned} \tag{A.3.28}$$

We are standing on the shoulders of giants, as this is nothing else but a (restricted form of a) *Diophantine equation*, and we shall dedicate a large portion of the remainder of this chapter to the study of such equations.[2] To write this equation in a more compact form, we perform the following change of variables

$$(n_3 - n_2) = x_2 + 1 \ , \qquad (n_3 - n_1) = x_1 + x_2 + 2 \ , \tag{A.3.29}$$

with which our equations that the following form

$$x_1 + 2x_2 = n - 3 = \Delta \ , \qquad x_1 \geq 0 \ , \qquad x_2 \geq 0 \ , \qquad x_1 + x_2 + 2 \leq L - 1 \ . \tag{A.3.30}$$

Furthermore, we can simplify these equations even more by introducing the slack variable (*Schlupfvariable*) $x_0 \geq 0$ that allows us to re-express the latter inequality as an equality, and finally lets us write the

[2] The field of algebraic number theory has a rich history a plethora of interesting approaches to study Diophantine equations. It would go beyond the scope of this thesis to, in addition to algebraic notions already introduced and discussed, fruitlessly attempt to formulate an adequate overview of this fascinating topic. For historical and introductory references on Diophantine equations and related algebraic topics, we refer the reader to [Mor69], [Sma98], [YT04], [VE06], in no particular order.

Diophantine equations in a more standard form,

$$\begin{cases} x_0 + x_1 + x_2 = L - 3 \\ x_1 + 2x_2 = \Delta \\ x_j \geq 0 \end{cases} . \tag{A.3.31}$$

Δ	(x_0, x_1, x_2)	Number of sol.
0	$(3,0,0)$	1
1	$(2,1,0)$	1
2	$(1,2,0)$, $(2,0,1)$	2
3	$(0,3,0)$, $(1,1,1)$	2
4	$(0,2,1)$, $(1,0,2)$	2
5	$(0,1,2)$	1
6	$(0,0,3)$	1

Δ	(x_0, x_1, x_2)	Number of sol.
0	$(4,0,0)$	1
1	$(3,1,0)$	1
2	$(2,2,0)$, $(3,0,1)$	2
3	$(1,3,0)$, $(2,1,1)$	2
4	$(0,4,0)$, $(1,2,1)$, $(2,0,2)$	3
5	$(0,3,1)$, $(1,1,2)$	2
6	$(0,2,2)$, $(1,0,3)$	2
7	$(0,1,3)$	1
8	$(0,0,4)$	1

Table A.1: Solutions to the system of Diophantine equations (A.3.31) for the cases of $L = 6$ and $L = 7$. In both cases, we consider $M = 3$ and $K = 1$.

The above table in A.1 gives us information on the cases of $L = 6$ and $L = 7$. We count the number of different solutions for Δ, as Δ is connected to the order of the ϵ-exponent n, and it is the ϵ-orders that are material to our Jordan construction. For the former case, we find that the system of equations has one (or more) solutions for 7 different values of Δ and two (or more, but in this case no more) solutions for three different values of Δ. This lets us claim that in the $L = 6$ case, the strongly twisted Hamiltonian has a JNF features one Jordan block of size 7, and one of size 3. This is consistent with [AS21]: With the condition of zero total momentum, we have that

$$\frac{\binom{6}{2,1,3}}{6} = 10 = 7 + 3 \ . \tag{A.3.32}$$

Likewise for the $L = 7$, we find that the system of equations admits one (or more) solutions for 9 different Δ, two (or more) solutions for 5 different values of Δ, and three solutions for 1 value of Δ. Similarly, we therefore expect the Hamiltonian to be similar to a JNF matrix with three blocks; one of size 9, one of size 5 and one of size 1. We also have that

$$\frac{\binom{7}{2,1,4}}{7} = 15 = 9 + 5 + 1 \ . \tag{A.3.33}$$

These results are in agreement with the results in [AS21].

We have now exemplified that our method indeed works by discussing the cases of $L = 6$ and $L = 7$, and we now want to analyse the case of general length L. To this end, the realisation that if (x_0, x_1, x_2) is a solution for length L then $(x_0 + 1, x_1, x_2)$ is a solution for length $L + 1$ is particularly helpful. We can start by finding the number of solutions for when we have $x_0 = 0$, and then proceeding recursively, meaning finding solutions for

$$\begin{cases} x_1 + x_2 = L - 3 \\ x_1 + 2x_2 = \Delta \\ x_j \geq 0 \end{cases} . \tag{A.3.34}$$

This system of equation has precisely one solution for any given value of Δ given by

$$x_2 = \Delta + 3 - L \tag{A.3.35}$$

$$x_1 = 2L - 6 - \Delta \ . \tag{A.3.36}$$

Lest we forget about our restriction that $x_i \geq 0$, implying that $\Delta \in \{L-3, \cdots, 2L-6\}$. One can then prove by induction that there are a total number of $\lfloor \frac{\Delta}{2} \rfloor + 1$ solutions for the case where $0 \leq \Delta \leq L-3$, and a total number of $\lfloor L-3-\frac{\Delta}{2} \rfloor + 1$ solution for $L-3 \leq \Delta \leq 2L-6$. $\lfloor x \rfloor$ is the usual floor function, i.e. the function that gives us the largest integer that is smaller (or equal) to x. We shall show that this result can be written more compactly in terms of a special function later on. Once this result is established, the Jordan structure of the spin chain with general length L can be easily computed. We can already see that the system of equations (A.3.31) has one or more solutions for a total of $2L-6+1 = 2L-5$ different values of Δ, two or more solutions for a total of $2L-5-4 = 2L-9$ different values of Δ, three or more solutions for a total of $2L-13$ different values of Δ, et cetera. Concluding, the Jordan structure for the $M=3, K=1$ case is such that the Jordan blocks are of size $2L-5-4n$, with $n \in \mathbb{N}_0$, which is in agreement with [AS21] and [ACS22], reassuring us in our efforts.

A.3.4 Generalised Eigensystem for General M and an Interlude on Gauss Polynomials

For the case of arbitrary M, our approach will be much the same: The relevant terms in the wave function scale as

$$e^{i\sum_i p_i n_i} \frac{\prod_i q_3^{n_i}}{q_2^{n_M} q_2^{n_M}} \sim \epsilon^{(\alpha-1)(\sum_i n_i)-(\alpha+\beta-2)n_M} = \epsilon^{(\alpha-1)\sum_i (n_i - n_M)} \ , \tag{A.3.37}$$

where $K(\beta-1) = (M-K)(\alpha-1)$ lets us combine the exponents. We again want to compute the number of different generalised eigenvectors appearing at each step of our method, which is again equivalent to the solution counting of the (linear) Diophantine equations,

$$\sum_i (n_M - n_i) = n \ , \tag{A.3.38}$$

with the restriction that the excitation separations are bound by $(n_M - n_i) \in \{1, \cdots, L-1\}$ for a spin chain of length L. Akin to the case before, we want to bring the Diophantine equations (and the restrictions) into a more standard form using the variables x_i, defined by

$$n_{i+1} - n_i = x_i + 1 \Leftrightarrow (n_M - n_i) = (M-i) + \sum_{j=i}^{M-1} x_j \ , \tag{A.3.39}$$

and we introduce the slack variable $x_0 \geq 0$ again to transform in the inequality $(n_M - n_1) \leq L-1$ into the equality $(x_0 + n_M - n_1) = L-1$. With these redefinitions, we arrive at the system of linear Diophantine equations

$$\begin{cases} \sum_{j=0}^{M-1} x_j = L - M \\ \sum_{j=0}^{M-1} (j x_j) = \Delta \\ x_j \geq 0 \end{cases} \ . \tag{A.3.40}$$

The following table A.2 shows the solutions for the case $L=8, M=4, K=1$; we can again see that the system of equations possesses one or more solutions for 13 different values of Δ, two or more solutions for 9 different values of Δ, three or more solutions for 7 different values of Δ, four or more solutions for 5 different values of Δ, and five solutions for one particular value of Δ. This implies that we expect the

Jordan chains for this case to have lengths 13, 9, 7, 5 and 1, which matches the results from Table 2 in [AS21].

Δ	(x_0, x_1, x_2, x_3)	Number of sol.
0	$(4,0,0,0)$	1
1	$(3,1,0,0)$	1
2	$(2,2,0,0)$, $(3,0,1,0)$	2
3	$(1,3,0,0)$, $(2,1,1,0)$, $(3,0,0,1)$	3
4	$(0,4,0,0)$, $(1,2,1,0)$, $(2,0,2,0)$, $(2,1,0,1)$	4
5	$(0,3,1,0)$, $(1,2,0,1)$, $(1,1,2,0)$, $(2,0,1,1)$	4
6	$(0,2,2,0)$, $(0,3,0,1)$, $(1,0,3,0)$, $(1,1,1,1)$, $(2,0,0,2)$	5
7	$(0,2,1,1)$, $(0,1,3,0)$, $(1,0,2,1)$, $(1,1,0,2)$	4
8	$(0,1,2,1)$, $(0,0,4,0)$, $(1,0,1,2)$, $(0,2,0,2)$	4
9	$(0,1,1,2)$, $(0,0,3,1)$, $(1,0,0,3)$	3
10	$(0,1,0,3)$, $(0,0,2,2)$	2
11	$(0,0,1,3)$	1
12	$(0,0,0,4)$	1

Table A.2: Solutions to the system of linear Diophantine equations in (A.3.40) for $L = 8$, $M = 4$ and $K = 1$.

Moving on from this case with specified length, let us consider general values of L. Similarly to the $M = 3$ case, we have that if $(x_0, x_1, \ldots, x_{M-1})$ is a solution for length L, then $(x_0 + 1, x_1, \ldots, x_{M-1})$ is a solution for length $L + 1$. Therefore, we start by analysing the $x_0 = 0$ case of (A.3.40) first again and proceed then iteratively:

$$\begin{cases} \sum_{j=1}^{M-1} x_j = L - M \\ \sum_{j=1}^{M-1} [(j-1)x_j] = \Delta + M - L \\ x_j \geq 0 \end{cases}, \tag{A.3.41}$$

where we subtracted the first equation from the second equation in (A.3.40) to arrive at this form. We can see that (A.3.41) corresponds to the system of Diophantine equations for the case $L - 1, M - 1, K = 1$, implying that we can arrive at the number of solutions to (A.3.40) with Δ, L and M by taking the number of solutions for Δ, $L - 1$ and M and adding the number of solutions for $\Delta + M - L$, $L - 1$ and $M - 1$.

Table A.3 below shows the sum rule we just derived for the case of $L = 9, M = 4, K = 1$, where we have Jordan chains of lengths 6, 8, 10, 12 and 16, while the case of $L = 10$, $M = 4, K = 1$ exhibits Jordan chains of lengths 3, 7, 7, 9, 11, 13, 15 and 19, in agreement with [AS21]:

(L,M) \ Δ	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$(8,4)$	1	1	2	3	4	4	5	4	4	3	2	1	1	0	0	0	0	0	0
$(8,3)\cdot q^{8-3}$	0	0	0	0	0	1	1	2	2	3	3	3	2	2	1	1	0	0	0
$(9,4)$	1	1	2	3	4	5	6	6	6	6	5	4	3	2	1	1	0	0	0
$(9,3)\cdot q^{9-3}$	0	0	0	0	0	0	1	1	2	2	3	3	4	3	3	2	2	1	1
$(10,4)$	1	1	2	3	4	5	7	7	8	8	8	7	7	5	4	3	2	1	1

Table A.3: Explicit computation of the number of solutions for the cases $L=9$, $M=4$ and $K=1$ and $L=10$, $M=4$ and $K=1$, starting from the number of solutions for the case with $L=8$, $M=4$ and $K=1$.

Taking one step back, if we consider general values of L, M, we can write the recursive relation that connects solutions for the cases L with $L-1$ by first defining the generating function $F(L,M,x)$ as

$$F(L,M,x)=\sum_{\Delta=0}^{\infty} x^{\Delta}\#\{\text{Solutions of eq. (A.3.40) for given values of } L,M,\Delta\}\ , \tag{A.3.42}$$

that we can use in establishing

$$\begin{aligned} F(L,M,x) &= F(L-1,M,x)+x^{L-M}F(L-1,M-1,x)\\ F(L,2,x) &= \sum_{j=0}^{L-2} x^j\ . \end{aligned} \tag{A.3.43}$$

In (A.3.43), the second equation is the result from the $M=2$ case (where we found a single Jordan chain of length $L-1$), which serves as a base condition for our recursive relation to foot on. This recurrence relation has been studied before in the context of combinatorial analysis: The generating function is nothing else but a type of *Gaussian polynomial* (or *q-deformed binomial coefficient*), i.e.

$$F(L,M,x)=\binom{L-1}{M-1}_x\ , \tag{A.3.44}$$

where we define

$$\binom{m}{r}_q=\frac{(1-q^m)(1-q^{m-1})\dots(1-q^{m-r+1})}{(1-q)(1-q^2)\dots(1-q^r)}\ . \tag{A.3.45}$$

The nomenclature comes from the fact that in the $q\to 1$ limit, we recover the usual binomial coefficient, i.e.

$$\lim_{q\to 1}\binom{m}{r}_q=\binom{m}{r}\ . \tag{A.3.46}$$

We can see that the values in table A.3 are identical to the coefficients of the polynomials $\binom{8-1}{4-1}_x$, $x^{9-4}\binom{8-1}{3-1}_x$, $\binom{9-1}{4-1}_x$, $x^{10-4}\binom{9-1}{3-1}_x$ and $\binom{10-1}{4-1}_x$, respectively. In [ACS22], the authors found this generating function without using integrability, strengthening our confidence in these results as different approaches led to the same findings.

We conclude this section by making some remarks about Gaussian polynomials and prove some of their properties.[3] We start with the fact that it is immediate from (A.3.45) that they fulfil the properties

[3] For a larger discussion on this, we refer the reader to the references [vLW01] and [Sz 13]. We shall stick to their definitions, and also use their schematic sketches of proofs for two of the properties they show in a more detailed way. Further helpful references include [Cig81] and [Kro22].

- Reflection Property:

$$\binom{m}{r}_q = \binom{m}{m-r}_q . \tag{A.3.47}$$

- 1st Pascal identity:

$$\binom{m}{r}_q = q^r \binom{m-1}{r}_q + \binom{m-1}{r-1}_q . \tag{A.3.48}$$

- 2nd Pascal identity:

$$\binom{m}{r}_q = \binom{m-1}{r}_q + q^{m-r} \binom{m-1}{r-1}_q . \tag{A.3.49}$$

In order to get more acquainted with these polynomials, we shall prove these three properties at once:

Proof. In the same order as above, we have:

- For the Reflection property, we can see that

$$\begin{aligned}
\binom{m}{r}_q =& \frac{(1-q^m)(1-q^{m-1})\dots(1-q^{m-r+1})}{(1-q)(1-q^2)\dots(1-q^r)} \\
=& \frac{(1-q^m)(1-q^{m-1})\dots(1-q^{m-r+1})}{(1-q)(1-q^2)\dots(1-q^r)} \times \frac{(1-q^{r+1})\dots(1-q^{m-r})}{(1-q^{r+1})\dots(1-q^{m-r})} \\
=& \frac{(1-q^m)(1-q^{m-1})\dots(1-q^{r+1})}{(1-q)(1-q^2)\dots(1-q^{m-r})} \\
=& \binom{m}{m-r}_q .
\end{aligned} \tag{A.3.50}$$

- The 1st Pascal identity is proven by noticing that we have

$$\begin{aligned}
\binom{m}{r}_q =& \frac{(1-q^m)(1-q^{m-1})\dots(1-q^{m-r+1})}{(1-q)(1-q^2)\dots(1-q^r)} \\
=& (1-q^m) \times \underbrace{\frac{(1-q^{m-1})\dots(1-q^{m-r+1})(1-q^{m-r})}{(1-q)(1-q^2)\dots(1-q^r)}}_{=\binom{m-1}{r}_q, \text{ since } m-r=(m-1)-r+1} \times \frac{1}{1-q^{m-r}} \\
=& \frac{1-q^m}{1-q^{m-r}} \binom{m-1}{r}_q .
\end{aligned} \tag{A.3.51}$$

  Similarly, we also have that

$$\binom{m}{r}_q = \frac{1-q^m}{1-q^r} \binom{m-1}{r-1}_q , \tag{A.3.52}$$

  as well as

$$\binom{m-1}{r-1}_q = \frac{1-q^r}{1-q^{m-r}} \binom{m-1}{r}_q , \tag{A.3.53}$$

  With these identities, we can show that

$$\binom{m}{r}_q = \underbrace{\frac{1-q^m}{1-q^{m-r}}}_{=q^r+\frac{1-q^r}{1-q^{m-r}}} \binom{m-1}{r}_q$$

$$
\begin{aligned}
&= q^r \binom{m-1}{r}_q + \underbrace{\frac{1-q^r}{1-q^{m-r}} \binom{m-1}{r}_q}_{= \binom{m-1}{r-1}_q, \text{ using (A.3.53)}} \\
&= \binom{m}{r}_q = q^r \binom{m-1}{r}_q + \binom{m-1}{r-1}_q \, . \qquad \text{(A.3.54)}
\end{aligned}
$$

- Lastly, we can also prove the 2nd Pascal identity in a similar way, starting with (A.3.52) and proceeding then like

$$
\begin{aligned}
\binom{m}{r}_q &= \underbrace{\frac{1-q^m}{1-q^r}}_{= \frac{1-q^{m-r}}{1-q^r} + q^{m-r}} \binom{m-1}{r-1}_q \\
&= \underbrace{\frac{1-q^{m-r}}{1-q^r} \binom{m-1}{r-1}_q}_{= \binom{m-1}{r}_q, \text{ using the inverted relation of (A.3.53)}} + q^{m-r} \binom{m-1}{r-1}_q \\
&= \binom{m-1}{r}_q + q^{m-r} \binom{m-1}{r-1}_q \, , \qquad \text{(A.3.55)}
\end{aligned}
$$

  concluding our proof.

□

We can also prove that we have $(1-q)(1-q^2)\dots(1-q^r) | (1-q^m)(1-q^{m-1})\dots(1-q^{m-r+1})$, in the polynomial ring $\mathbb{R}[q]$, by applying binomial rules, meaning that they are indeed polynomials (not just meromorphic or rational functions). We shall prove this by induction:

Lemma A.3.1. *$\binom{m}{r}_q$ is a polynomial in q of degree $\deg\left(\binom{m}{r}_q\right) = (m-r)r$ for $r \in \{0, \cdots, m\}$.*

Proof. Since we have that, for $m = 0$,

$$
\binom{0}{0}_q = 1 \, , \qquad \text{(A.3.56)}
$$

which is a polynomial in q, and we can therefore proceed to the induction step over m. With the 2nd Pascal identity for $m+1$,

$$
\binom{m+1}{r}_q = \binom{m}{r}_q + q^{m-r+1} \binom{m}{r-1}_q \, , \qquad \text{(A.3.57)}
$$

we infer that $\binom{m+1}{r}_q$ is a polynomial in q since, by inductive assumption, we assume $\binom{m}{r}_q$ and $q^{m-r+1}\binom{m}{r-1}_q$ to be polynomials in q. By then counting the degree in q of the numerator polynomial and subtracting from it the degree of the denominator polynomial, we get

$$
\underbrace{\frac{1}{2}(1+2m-r)r}_{\text{numerator}} - \underbrace{\frac{1}{2}r(1+r)}_{\text{denominator}} = (m-r)r \, , \qquad \text{(A.3.58)}
$$

proving our claim. We will now make use of these properties to get information regarding our Jordan chains. ■

Additionally, it is for example easy to see that we have

$$
\binom{m}{1}_q = \sum_{j=0}^{m-1} q^j \, , \qquad \text{(A.3.59)}
$$

since we have that $(1+q+q^2+\cdots+q^{m-1})(1-q)=1-q^m$. This Gaussian polynomial's coefficients contain the information about the Jordan cells of the case of length $L=m+1$ and two excitations with different flavours (corresponding to the $M=2, K=1$ that we studied already). The polynomial goes up to order q^{m-1}, and all the coefficients are identical 1, implying that we have a single Jordan chain of length $L-1$ for each value of the total momentum. The iterative application of the 1st Pascal identity on $\binom{m}{1}_q$ then implies

$$\text{Coeff}\left(\binom{m}{r}_q ; q^0\right)=1\ , \tag{A.3.60}$$

where $\text{Coeff}(P(q);q^n)$ represents the coefficient associated to q^n in the polynomial $P(q)$. Moreover, one can prove that the highest non-vanishing coefficient is

$$\text{Coeff}\left(\binom{m}{r}_q ; q^{(m-r)r}\right)=1\ , \tag{A.3.61}$$

implying that the longest Jordan chain for the case of length L, M excitations and $K=1$ has size $(L-M)(M-1)+1$. This result perfectly reproduces the values $L-1$ and $2L-5$ which we found for $M=2$ and $M=3$, respectively.

In the body of the thesis, we proved that our Hamiltonian was nilpotent, but were not able to exactly state that the degree of nilpotency is. We can now produce the upper bound using this result: As we know that the largest Jordan block associated to the Hamiltonian for the spin chain of length L with general M and $K=1$ has size $(L-M)(M-1)+1$, this means that we need to exponentiate the Hamiltonian $(L-M)(M-1)+1$ times for it to vanish. More generally, if we do not know about M, the largest possible nilpotency order our Hamiltonian associated to a spin chain with length L can possess is $\lceil\frac{L+1}{2}\rceil$.

Furthermore, we can argue in the same way as in the $M=2$ case that chain mixing do not spoil the above result. However, this is more easily shown by setting first $\xi_1=\xi_2=0$, as $\hat{\mathbf{H}}^\dagger_{0,0,\xi_3}$ can only move the 2's in the state to the left until they find a 3. By definition, we have a total of $L-M$ 1's and can apply $\hat{\mathbf{H}}^\dagger_{0,0,\xi_3}$ a total of $(M-1)(L-M)$ times, which implies that the Jordan chain has indeed size $(L-M)(M-1)+1$. This proves the conjecture the authors in [AS21] made regarding chain mixing for the longest Jordan chain.

However, we have still not proven that chain mixing is not affecting the chains apart from the longest one, and many of our claims rest on the assumption that chain mixing never enters. However, all the cases we have checked explicitly as well as the before argument leave us hopeful about this conjecture.

Lastly, with a bit of algebra, the following properties can be shown:

- We firstly have that

$$\text{Coeff}\left(\binom{m}{r}_q ; q^1\right)=1\ , \tag{A.3.62}$$

$$\text{Coeff}\left(\binom{m}{r}_q ; q^{(m-r)r-1}\right)=1\ , \tag{A.3.63}$$

  for general r and $m>r$, of which we shall sketch the proof of the former. The latter follows from the palindromic property, that we will prove later on.

  Proof. Our inductive proof starts by (re-)stating that, for all m, we have that $\binom{m}{1}_q=1+q+\cdots$, meaning that already fulfils our claim. By proceeding by induction over r, we now show that if

$\binom{m}{r-1}_q$ fulfils $\text{Coeff}\left(\binom{m}{r-1}_q; q^1\right) = 1$ (with $r \geq 2$), then $\binom{m}{r}_q$ fulfils $\text{Coeff}\left(\binom{m}{r}_q; q^1\right) = 1$ as well. The 1st Pascal identity gives us that

$$\underbrace{\binom{m-1}{r-1}_q}_{=1+q+\cdots} = \binom{m}{r}_q - \underbrace{q^r \binom{m-1}{r}_q}_{\text{Polynomial with lowest degree } q^{r\geq 2}} , \tag{A.3.64}$$

which implies that, since $\binom{m}{r}_q$ is a polynomial, it has to be one with the lowest orders in the exponents starting as $\binom{m}{r}_q = 1 + q + \cdots$, as the equation would otherwise not hold. □

- In a similar way, one can show

$$\text{Coeff}\left(\binom{m}{r}_q ; q^2\right) = 2 , \tag{A.3.65}$$

$$\text{Coeff}\left(\binom{m}{r}_q ; q^{(m-r)r-2}\right) = 2 , \tag{A.3.66}$$

for general $r \geq 2$ and $m > \max(r, 3)$.

- The above two properties imply that we have a second Jordan cell only if $M \geq 3$, and it has size $(L-M)(M-1)-3$ while keeping the total momentum fixed, which is in agreement with the value $2L-9$ we found for the $M = 3$ case.

- One last curiosity that one can prove is

$$\text{Coeff}\left(\binom{m}{r}_q ; q^s\right) = \text{Coeff}\left(\binom{m}{r}_q ; q^{(m-r)r-s}\right) , \tag{A.3.67}$$

meaning that the Gaussian polynomials are palindromic.

Proof. We first recall that $\binom{m}{r}_q$ is indeed a polynomial in q, meaning we have

$$\binom{m}{r}_q = \sum_{s=0}^{(m-r)r} C_{(m,r,s)} q^s , \tag{A.3.68}$$

for some coefficients $C_{(m,r,s)}$. In general, we have that a polynomial $p(q) \in \mathbb{R}[q]$ is palindromic iff the two mirrored polynomials,

$$p(q) = a_0 + a_1 q + \cdots + a_n q^n \tag{A.3.69}$$

and

$$q^n p\left(\frac{1}{q}\right) = a_0 q^n + a_1 q^{n-1} + \cdots + a_n , \tag{A.3.70}$$

coincide. With this convenient statement, we note that one can see that

$$\begin{aligned}
\binom{m}{r}_q &= \frac{1}{q^{(m-r)r}} \frac{(1-q^m)(1-q^{m-1})\dots(1-q^{m-r+1})}{(1-q)(1-q^2)\dots(1-q^r)} \\
&= \frac{q^m(q^{-m}-1)q^{m-1}(q^{-(m-1)}-1)\dots q^{m-r+1}(q^{-(m-r+1)})}{q(q^{-1}-1)q^2(q^{-2}-1)\dots q^r(q^{-r}-1)} \\
&= \frac{q^{\frac{1}{2}(1+2m-r)r}}{q^{\frac{1}{2}r(1+r)}} \underbrace{\frac{(q^{-m}-1)(q^{-(m-1)}-1)\dots(q^{-(m-r+1)})}{(q^{-1}-1)(q^{-2}-1)\dots(q^{-r}-1)}}_{=\binom{m}{r}_{\frac{1}{q}}}
\end{aligned}$$

$$= q^{(m-r)r} \binom{m}{r}_{\frac{1}{q}} , \tag{A.3.71}$$

which in turn implies the equation

$$\binom{m}{r}_{\frac{1}{q}} = \frac{1}{q^{(m-r)r}} \binom{m}{r}_{q} , \tag{A.3.72}$$

proving our claim, as

$$C_{(m,r,s)} = C_{(m,r,(m-r)r-s)} . \tag{A.3.73}$$

□

This implies that the size of all Jordan cells can differ by even numbers only (including 0).

Data Management

“不怕慢，只怕站。” [“Be not afraid of growing slowly; Be afraid only of standing still.’]

– Chinese proverb

No data beyond those presented and cited in this report are needed to validate this study.

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"Physics isn't the most important thing. Love is."

– Richard Feynman

Curriculum Vitae

Name Leander Wyss

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Education

2017-2019 Master of Science in Physics ETH, ETH Zürich, Switzerland
Focus on Theoretical/Mathematical Physics, graduated with 5.54/6 GPA (First Class Equivalent).

2017-2019 Bachelor of Science in Physics, University of Basel, Switzerland
Focus on Theoretical/Mathematical Physics, graduated with 5.7/6 GPA (First Class Equivalent).

Academic Work Experience

2017-2019 Teaching Assistant at the Department of Mathematics at ETH Zürich, Switzerland
Assisting in the Lectures "Linear Algebra 1 and 2".

2017 Teaching Assistant at the Department of Mathematics at the University of Basel, Switzerland
Assisting in the Lecture "Analysis 2".

Conferences (selection)

2022 Invited Speaker at the "Cordas Club" Seminar Series Hosted by the IGFAE in Santiago de Compostela, Spain
Talk Given on "Jordan Vectors and the Eclectic Spin Chain".

2021 Invited Speaker at the "Integrability, Dualities and Deformations" Conference in Santiago de Compostela, Spain
Talk Given on "Boost superalgebras in undeformed and deformed AdS_3/CFT_2".

2021 Invited Speaker at the "South East Mathematical Physics Seminars" Hosted by the University of Kent, UK
Talk Given on "Boost superalgebras in AdS_3/CFT_2".