TOPICS IN FIELD THEORY

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Abstract

This thesis consists of different projects in classical and quantum field theory on a curved background which are connected by a common usage of theoretical ideas as well as calculation techniques. The introduction clarifies such connections. In the second chapter we apply the Dirac constraint formalism to the second and first order $1+1$ dimensional gravitational action coupled to a scalar field. The third chapter is devoted to a calculation of the one loop effective action for a spinor field coupled to a constant background chiral vector field. In the fourth chapter we find a new expression for the running coupling through the conformal anomaly in a strong background gauge field. We compare this expression with the value obtained by the standard procedure. This expression should be useful for finding all loop order contributions to the effective action of a gauge theory. The second part of the thesis is devoted to the applied aspects of the AdS/CFT correspondence and related issues. In the fifth chapter we review a three dimensional model of a strongly correlated holographic plasma dual to a $3+1$ dimensional gravity model of exotic black holes. Then we consider the extension of the Correlated Stability Conjecture which attempts to connect mechanical instabilities of black holes with the thermodynamical instabilities of the related holographic plasma by adding to the theory additional conserved chargers connected with the values of the scalar field at the boundary of a black brane. Also, we propose a simple thermodynamic model to test the generalized conjecture. Then, we focus on some transport properties of the model. Namely we check Eling-Oz formula for the bulk-to-shear viscosity of holographic plasmas for different temperature regimes. The last chapter is devoted to a CFT calculation of the entanglement entropy on a sphere for free theories. We try to connect the results with the calculations based on AdS/CFT approach. For this purpose the heat kernel technique is used in this chapter. In our discussion proposals for future work are stated.

Keywords: Constraint formalism, gauge theories, effective action, holographic strongly correlated plasma, heat kernel

Co-Authorship Statement

I would like to acknowledge Gerry McKeon, Alex Buchel, Fernando Brandt, Rob Myers and Misha Smolkin for collaboration. Chapter two, three, five and six are based on the articles 4, 1, 3, 2 respectively cited in CV. Chapter four is based on the paper submitted to Physics Letters B and it is available online in arXiv:1206.6096. Chapter seven is based on the paper being prepared. The contribution of the author can be considered as equal as the contributions of the coauthors. Chapters two and four are based on papers done together with Gerry McKeon. The practical contribution of the author to chapter two was done in calculations for the second order formalism and finding constraints for the first order formalism. Chapter three is based on the paper written with Fernando Brandt and Gerry McKeon. In this chapter the author contributed in calculations of the Schwinger expansion for the axial model. In chapter four the main contribution was in investigating of N^pLL calculations, calculating S parameters using symbolic computer programs and comparing a new expression for the running coupling with a standart expression. Chapter five is based on the paper written with Alex Buchel. In chapter five the main contribution was in testing different Landau-Ginzburg models and calculating the extended Hessian for the Exotic Black Hole model. Chapter 6 was made completely by the author using a formulation of the problem by Alex Buchel. Chapter seven is based on work with Rob Myers and Misha Smolkin. In this chapter the author did calculations of the Renyi entropy for fermions and vectors in four dimensional space which were generalized later for arbitrary dimensions.

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Chapter 1

Introduction

One of the fundamental problems of modern high energy physics is to link quantum field theory with gravity. A semi-classical approach to this problem has been considered since the 1940's, in which one examines quantum fluctuations of matter fields on a fixed gravitational background. The emission of Hawking radiation by black holes is one outcome of this approach [1]. Attempts to quantize pure gravity have met with some success but they are faced with quite serious obstacles associated with renormalization [2]. In an alternative approach using string theory, gravitational excitations are treated in the same way as particle excitations; both of these arise from exciting fundamental strings. The outstanding problem in this approach is finding the low energy limit of the theory and dimensional reduction of the original string theory from ten to four dimensions, as only in ten dimensions can superstring theory be consistently quantized. In an attempt to find a way of addressing these problems, in the chapter two we carefully consider the classical action for the bosonic string. In string theory we treat the string coordinates in N dimensions as N scalar fields f^a $(a = 1...N)$, propagating on a curved two dimensional surface identified with the world sheet of the string. We employ the Dirac constraint formalism in order to determine all gauge symmetries present in the action. We show when using the second order formalism for the gravitational field that not only diffeomorphism symmetry but also Weyl symmetry can be obtained directly from the constraint formalism. We then demonstrate an inequivalence of the first and second order two dimensional gravitational action, when coupled to a scalar field f^a, It is expected that the classical results we have obtained may shed light on the problem present when this model is fully quantized.

A major achievement of String Theory is the AdS/CFT correspondence (as a particular realization of the holographic idea [3],[6]. This correspondence connects gravity in the bulk and Conformal Field Theory on its boundary. Techniques arising from AdS/CFT correspondence allow one to solve problems arising in conformal field theories as well as systems with broken conformal symmetry. It may shed light on the origin of processes occurring in heavy-ion collision at the RHIC experiment, which is presumably related to the processes in the early Universe. Results show that the quark-gluon plasma in the RHIC experiment exhibits strong coupling dynamics [4]. Thus, the AdS/CFT approach is useful for the interpretation of the experiment. Especially, the most exciting result is the observation of the low shear viscosity (η) to entropy (s) ratio, which is predicted from holographic calculations. More specifically, the theoretical result following from using AdS/CFT is that $\frac{\eta}{s} = \frac{1}{4}$ [5] for a wide range of holographic plasmas. In chapter five we review a $2+1$ dimensional model of strongly coupled plasma which is related to $3+1$ gravitational theory with scalar hairs by using AdS/CFT. First, we give the counterargument for the correlated stability conjecture (CSC), establishing a connection between the thermodynamic instabilities in holographic models and the gravitational instabilities. The simple CSC associated with thermodynamic instabilities caused by negative specific heat was proven to be invalid using several holographic systems [9],[10] . We check the generalized CSC. In this proposal translationary invariant horizons of the black holes in the gravitational model we are interested in having scalar hair whose asymptotic parameters can be interpreted as additional charges leading to a generalization of the thermodynamic stability criterion. This example shows that we have to be careful in transferring gravitational properties to the field theory side. Also, we review a new conjecture of Martinez and Emparan connecting ghost excitations in the system with thermodynamical instabilities. The basis for this conjecture is the observation that the black hole ghost excitation wavelength can have arbitrary large magnitude.

In chapter six we check a new formula of Eling and Oz for the bulk-to-shear viscosity ratio for systems involving holographic plasmas [7]. Their formula gives a simple answer for bulk viscosity through the scalar field evaluated at the horizon of the black brain, the temperature of the plasma dual to the black brane, and the speed of sound waves in the plasma. The expression for the bulk-to-shear viscosity employs the values of scalar fields only at the horizon. It is an intriguing result, as the values of scalar fields at the boundary should capture the microscopic scales of the theory. We find the perfect agreement of the numerical calculations of the bulk-to-shear viscosity with Eling-Oz formula for different temperature regimes.

Another connection between gravity and field theory is the conformal anomaly. The breaking of a classical symmetry at the quantum level is due to dependence of the path integral measure used in quantization on gauge transformations of the fields which leave the classical action invariant [11],[12],[13]. We employ the anomaly in conformal symmetry (arising due to the necessity of introducing a renormalization scale parameter into the theory) to find an alternative expression for the running gauge coupling in chapter four. In this new expression for the running coupling, the reciprocal of a power series in the log of the field strength is derived.

In chapter three we consider the one loop effective action in four dimensional Euclidean space for a strong background chiral field coupled to a spinor field. When the mass of the spinor is non-zero, one must expand in powers of the axial field, but can keep all powers of a constant background vector field when computing the effective action. We can reproduce the result for an axial anomaly in a plane wave background field if we use the Schwinger expansion [8] for the quantum action we have derived. Also, we discuss the analogous problem in two dimensions, where the effective action we obtained has a simple form. The four dimensional result contains the parts indicating infrared divergences in the system and can not be obtained by summing Feynman diagrams. (Alternatively, we can say that we should sum an infinite number of diagrams).

In conformal field theory on a curved background the anomaly is given in terms of the curvature tensor [13]. In the context of AdS/CFT in most cases it is easier to perform calculations on the gravity side. In chapter six we will obtain the expression for entanglement entropy for free conformal theories to connect them with holographic calculations. We do field theory calculations on fixed gravitational backgrounds (in AdS space), using heat kernel techniques, which is equivalent to summing all one loop corrections to the theory. (The heat kernel expansion can be considered as a particular case of the Schwinger expansion we applied in chapter three). In general, calculations of the entanglement and Renyi entropy are connected with Riemann surface of on the and conical singularities in the geometry. The calculations in this chapter are important because they give a particular test of the proposal by Casini, Huerta and Myers [14] that the calculation of entanglement entropy for spherical entangling surface in a flat space can be mapped to the calculation of the thermal entropy on a hyperbolic space. The method works perfectly for scalar and fermionic fields (by comparison our results with holographic and direct computations of the entanglement and Renyi entropies) but we have questions that arise for vector fields. Namely, it is not clear if the introduction of the cut-off in the hyperbolic space can lead to boundary effects.

The idea of the renormalization group runs through the last the last four chapters. In chapter four the renormalization scale leads to the conformal anomaly, in chapters five and six we have mixing of relevant and irrelevant operators which deforms a CFT under RG dynamics.

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Chapter 2

Scalar fields on a curved two dimensional background

2.1 Introduction

Scalar fields on a curved background have received considerable attention because of their relationship with bosonic string theory [21]. One normally focuses on the quantum properties of string theory (such as the absence of the conformal anomaly only if the dimension of the target space exceeds four), but it is both interesting and important to have an understanding of the classical canonical structure of this model if one is to truly comprehend the implications of the quantization procedure. In this chapter we undertake the task of applying Dirac's analysis of constrained systems [1–6] to the problem of N scalar fields on a curved two dimensional manifold. We focus in particular on the first class constraints that appear and what they tell us about the gauge invariance present in the theory. A number of novel features arise.

Generally, in any discussion of metric fields on a two dimensional space, the action for the metric is ignored as the Einstein-Hilbert (EH) action $\sqrt{-g}g^{\mu\nu}R_{\mu\nu}(g_{\alpha\beta})$ in two dimensions (2D), when treated as a function of the metric $g_{\mu\nu}$ alone (the second order form), is a pure surface term and has no dynamical degrees of freedom (The metric $g_{\mu\nu}$ has inverse $g^{\mu\nu}$ and determinant g, $R_{\mu\nu}$ is a Riemann tensor). In the second order form of the action, the only dynamical field in the metric $g_{\mu\nu}$ and no term in the action contains more than two derivatives of this field. We note though that this lack of dynamics does not mean that it cannot be quantized; this has been studied in refs. [30, 31] using analysis of Becchi, Rouet, Stora and Tyutin (BRST). There has also been a discussion of the canonical structure of the first order EH action in 2D [7]. The first class constraints that occur have been shown to imply that there is an invariance under the gauge transformation

$$g_{\mu\nu} \to g_{\mu\nu} + \omega_{\mu\nu} \tag{2.1}$$

which is consistent with there being no degrees of freedom present in the action (the gauge function $\omega_{\mu\nu}$ is an arbitrary symmetric tensor of rank two). Normally when a matter field is coupled with a gauge field (e.g. the electron is coupled to a photon), any gauge invariance present in the uncoupled gauge field action is respected by the action in which the coupling is present. In this case however, the coupling of N scalars $f^a, (a = 1, 2...N)$ to the metric $g_{\mu\nu}$ through the Lagrangian

$$\mathcal{L}_f = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\partial_\mu f^a \partial_\nu f^a, \tag{2.2}$$

while being diffeomorphism invariant, does not respect the symmetry of eq. (2.1). In this paper, we first address the problem of disentangling how supplementing the second order EH action in 2D by the action of eq. (2.2) alters the constraint structure of the theory and thereby leads to a new gauge invariance that is distinct from that of eq. (2.1).

The problem of reconciling the gauge invariance present in the action for the free gauge field with that occurring when it is coupled to a matter field becomes even more interesting when the free gauge action is the first order EH action in 2D. By the first order form of the action, we mean that the affine connection and the metric are taken to be independent fields, and no term in the action contains more than one derivative of these fields. We first note that this action, $\sqrt{-g}g^{\mu\nu}R_{\mu\nu}(\Gamma^\lambda_{\alpha\beta})$, is not equivalent to the second order form, unlike what occurs in $D > 2$ dimensions [14, 15]. This is because the affine connection $\Gamma^\lambda_{\mu\nu}$ is no longer given by the Christoffel symbol

$$\left\{ \begin{array}{c} \lambda \\ \mu\nu \end{array} \right\} = \frac{1}{2} g^{\lambda\sigma} \left(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma} \right) \tag{2.3}$$

but rather

$$\Gamma^\lambda_{\mu\nu} = \left\{ \begin{array}{c} \lambda \\ \mu\nu \end{array} \right\} + \delta^\lambda_\mu \xi_\nu + \delta^\lambda_\nu \xi_\mu - g_{\mu\nu}\xi^\lambda \tag{2.4}$$

(where ξ^λ is an arbitrary vector) when solving the equation of motion for $\Gamma^\lambda_{\mu\nu}$. We first consider the implication of having this extra field arising in the model. We then review analysis [8–13] which shows that the canonical structure of the first order EH action in 2D shows that there are no physical degrees of freedom in the model despite it not being topological, and that the first class constraint that arise result in a novel gauge transformation

$$\delta h^{\mu\nu} = -\left(\epsilon^{\mu\rho}h^{\nu\sigma} + \epsilon^{\nu\rho}h^{\mu\sigma}\right)\omega_{\rho\sigma} \tag{2.5}$$

$$\delta G^{\lambda}_{\mu\nu} = -\epsilon^{\lambda\rho}\omega_{\mu\nu,\rho} - \epsilon^{\rho\sigma}\left(G^{\lambda}_{\mu\rho}\omega_{\nu\sigma} + G^{\lambda}_{\nu\rho}\omega_{\mu\sigma}\right) \tag{2.6}$$

where $\epsilon^{01} = -\epsilon^{10} = 1$, $\epsilon^{00} = \epsilon^{11} = 0$, $h^{\mu\nu} = \sqrt{-g}\,g^{\mu\nu}$ and $G^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \frac{1}{2}\left(\delta^{\lambda}_{\mu}\Gamma^{\rho}_{\rho\nu} + \delta^{\lambda}_{\nu}\Gamma^{\rho}_{\rho\mu}\right)$. This is distinct from the manifest diffeomorphism invariance present. We then address the problem of seeing how the first class constraints that lead to eqs. (2.5),(2.6) are modified when the free action for $h^{\mu\nu}, G^{\lambda}_{\mu\nu}$ is supplemented by

$$\mathcal{L}_f = \frac{1}{2}h^{\mu\nu}\partial_\mu f^a\,\partial_\nu f^a. \tag{2.7}$$

A number of interesting features arise in the course of applying the Dirac constraint formalism to these two models in which a scalar field propagates on a curved surface. First of all, when there are N scalar fields, the constraints and their associated gauge conditions combine to leave just $2N-4$ dynamical degrees of freedom in the theory. If $N=1$ there are a negative number of degrees of freedom which is a refection of the fact that in this case the equation of motion for the metric imply that f is a constant and is not dynamical. Secondly, when one considers either the first or second order EH action to be the action for the gauge field coupled to the scalar matter field, the number of first class constraints in each generation is not the same. For $N=1$, there are in the case of the second order EH action, three primary and two secondary first class constraints, while with the first order EH action there are three primary and secondary first class constraints and two tertiary first class constraints. Consequently, when using these constraints to find the gauge invariance that they imply to be present in the initial action, one finds that the techniques of both Castellani (C) (refs. [16, 17]) and of Henneaux, Teitelboim and Zanelli (HTZ) (refs. [18, 19]) do not lead to a unique gauge transformation. Neither diffeomorphism invariance not conformal invariance are implied by these first class constraints; indeed for the first order action the gauge generator derived from the first class constraints implies that the scalar field and affine connections mix under a gauge transformations.

In the next two sections we present a canonical analysis of a scalar field on a curved background, using the second, then the first, order EH action for the metric, including a discussion of the gauge transformations implied by the first class constraints. In an appendix, the way in which the first class constraints can be used to find the generator of the gauge transformation is outlined, using both the approach of C [16, 17] and of HTZ [18, 19].

2.2 Second order EH Action and Scalar Fields

We begin by first reviewing how the second order EH action in 2D can be treated using the Dirac constraint formalism [7], despite it being a topological theory (*i.e.*, a theory with no degrees of freedom as its Lagrangian is a total derivative). We then couple the metric to a scalar field and consider how this affects the gauge invariance of eq. (2.1).

The second order EH action is

$$S_{EH} = \int dx \sqrt{-g}\, g^{\mu\nu} R_{\mu\nu} \tag{2.8}$$

where

$$R_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu,\lambda} - \Gamma^{\lambda}_{\lambda\mu,\nu} + \Gamma^{\lambda}_{\lambda\sigma}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\lambda}_{\sigma\mu}\Gamma^{\sigma}_{\lambda\nu} \tag{2.9}$$

and $\Gamma^{\lambda}_{\mu\nu} = \left\{ \begin{array}{c} \lambda \\ \mu\nu \end{array} \right\}$, so that by eq. (2.3), $\Gamma^{\lambda}_{\mu\nu}$ is expressed in terms of $g_{\mu\nu}$. In any dimension [20]

$$\begin{aligned} &\sqrt{-g}\, g^{\mu\nu} \left(\Gamma^{\lambda}_{\mu\nu,\lambda} - \Gamma^{\lambda}_{\lambda\mu,\nu}\right) \\ &\quad = \left(\sqrt{-g}\, g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu}\right)_{,\lambda} - \left(\sqrt{-g}\, g^{\mu\nu}\Gamma^{\lambda}_{\lambda\mu}\right)_{,\nu} \\ &\quad - 2\sqrt{-g}\, g^{\mu\nu}\left(\Gamma^{\lambda}_{\lambda\sigma}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\lambda}_{\sigma\mu}\Gamma^{\sigma}_{\lambda\nu}\right) \end{aligned} \tag{2.10}$$

and hence if surface terms are discarded, then S_{EH} can be replaced by the non-covariant action

$$S^{(2)}_{\Gamma\Gamma} = -\int dx \sqrt{-g}\, g^{\mu\nu}\left(\Gamma^{\lambda}_{\lambda\sigma}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\lambda}_{\sigma\mu}\Gamma^{\sigma}_{\lambda\nu}\right). \tag{2.11}$$

It is this form of the action that was used by Dirac in the analysis of the canonical structure of the EH action in 4D [22]. (See also refs [32, 33].) We too will use it as the initial action for analyzing the EH action in 2D.

In 2D, eq. (2.11) becomes

$$\begin{aligned} = \frac{1}{2}\int dx(-g)^{-3/2} &\left[g_{11,0}\left(g_{01}g_{00,1} - g_{00}g_{01,1}\right)\right. \\ &+ g_{00,0}\left(g_{11}g_{01,1} - g_{01}g_{11,1}\right) \\ &\left.+ g_{01,0}\left(g_{00}g_{11,1} - g_{11}g_{00,1}\right)\right]. \end{aligned} \tag{2.12}$$

If one were to choose conformal coordinates so that $g_{00} = -g_{11} = \rho(x)$, $g_{01} = 0$ as in [21], then $S_{\Gamma\Gamma}$ vanishes. However, if $g_{01} \neq 0$ then $S_{\Gamma\Gamma}$ is amenable to canonical analysis [7]. However, it becomes apparent that $S_{\Gamma\Gamma}$ itself is a surface term if we adopt the coordinates [22]

$$\delta = \frac{-\sqrt{-g}}{g_{11}} \quad , \rho = \frac{g_{01}}{g_{11}} \quad , g_{11} \tag{2.13}$$

so that

$$S^{(2)}_{\Gamma\Gamma} = \int dx \frac{1}{\delta^2}\left(\delta_{,0}\rho_{,1} - \rho_{,0}\delta_{,1}\right)$$
$$= \int dx \left[\left(\frac{\rho_{,0}}{\delta}\right)_{,1} - \left(\frac{\rho_{,1}}{\delta}\right)_{,0}\right]. \tag{2.14}$$

We will not employ the variables δ and ρ in our canonical analysis; they simply serve to simplify the demonstration that $S^{(2)}_{\Gamma\Gamma}$ is a surface term. They do appear in ref. [25] though.

From eq. (2.12), we see that the canonical momenta conjugate to the components of the metric yield a set of primary constraints

$$\chi^{11} = \pi^{11} - \frac{1}{2(-g)^{3/2}}\left(g_{01}g_{00,1} - g_{00}g_{01,1}\right) \tag{2.15a}$$
$$\chi^{00} = \pi^{00} - \frac{1}{2(-g)^{3/2}}\left(g_{11}g_{01,1} - g_{01}g_{11,1}\right) \tag{2.15b}$$
$$\chi^{01} = \pi^{01} - \frac{1}{2(-g)^{3/2}}\left(g_{00}g_{11,1} - g_{11}g_{01,1}\right) \tag{2.15c}$$

where $(\pi^{11}, \pi^{00}, \pi^{01})$ are the canonical momenta conjugate to (g_{11}, g_{00}, g_{01}) respectively. These are constraints as one cannot solve for $g_{\mu\nu,0}$ in terms of $\pi^{\mu\nu}$. (If one were to simply discard the action of eq. (2.12) because of its topological nature, then we would merely have $\chi^{11} = \pi^{11}$, $\chi^{00} = \pi^{00}$ and $\chi^{01} = \pi^{01}$.) The Poisson Bracket (PB) of any two of these constraints vanishes. Furthermore, the canonical Hamiltonian vanishes. Consequently there are three primary first class constraints and no secondary constraints associated with $S^{(2)}_{\Gamma\Gamma}$ using any of the techniques of refs. [16–19] one finds the generator of gauge transformations to be

$$G = \int dx \left[\omega_{11}\chi^{11} + \omega_{00}\chi^{00} + \omega_{01}\chi^{01}\right] \tag{2.16}$$

which results in

$$\delta g_{\mu\nu} = \omega_{\mu\nu} \tag{2.17}$$

as in eq. (2.1). Eq. (2.17) also would follow from just taking $\chi^{11} = \pi^{11}$, $\chi^{00} = \pi^{00}$ and $\chi^{01} = \pi^{01}$, as is appropriate if were to discard the action all together because of it being topological.

We note that with these first class constraints of eq. (2.15) and the three associated gauge conditions, these are six restrictions on the six canonical variables ($g_{\mu\nu}$ and $\pi^{\mu\nu}$) in phase space, leaving no physical degrees of freedom. Supplementing $S^{(2)}_{\Gamma\Gamma}$ with the action for a massless scalar field f [23]

$$S_f = \frac{1}{2}\int dx \sqrt{-g}\, g^{\mu\nu} f_{,\mu} f_{,\nu} \tag{2.18}$$

we find that the momentum conjugate to f is

$$p = \sqrt{-g}\left(g^{00}f_{,0} + g^{01}f_{,1}\right) = \frac{1}{\sqrt{-g}}\left(-g_{11}f_{,0} + g_{01}f_{,1}\right) \tag{2.19}$$

so that the part of the canonical Hamiltonian arising from S_f in eq. (2.18) is

$$\mathcal{H}_c = \delta S + \rho I\!\!P \tag{2.20}$$

where S and $I\!\!P$ are two new secondary constraints

$$S = \frac{1}{2}\left(p^2 + f_{,1}^2\right) \tag{2.21}$$

$$I\!\!P = pf_{,1}. \tag{2.22}$$

We note that although only the combinations δ and ρ enter both eqs. (2.14) and (2.20), all three components of $h^{\mu\nu}$ appear in the initial action of eqs. (2.11) and (2.18). These three must be all included as fields in the canonical analysis. In ref. [21], a special "conformal gauge" was used to dispense with the "conformal factor" contribution to the action of eq. (2.18), reducing the number of independent components of the metric from three to two. However, choosing a "gauge" at the outset of any canonical analysis is inconsistent with Dirac's procedure [1–5].

Using test functions as in ref. [24] we find the Poisson Brackets (PB)

$$\begin{aligned}\{S(x), S(y)\} &= \left(-I\!\!P(x)\partial_1^y + I\!\!P(y)\partial_1^x\right)\delta(x-y) \\ &= \{I\!\!P(x), I\!\!P(y)\} \qquad (2.23a)\\ \{I\!\!P(x), S(y)\} &= \left(-S(x)\partial_1^y + S(y)\partial_1^x\right)\delta(x-y) \\ &= \{S(x), I\!\!P(y)\} \qquad (2.23b)\end{aligned}$$

and thus no tertiary constraints arise.

With eqs. (2.15),(2.21),(2.22) we see that there are now five first class constraints, which when combined with five associated gauge conditions, leaves us with ten restrictions on the eight variables in phase space $g_{\mu\nu}$, f and their associated momenta. If the single scalar field f in eq. (2.6) were replaced by N scalars f^a $(a = 1, 2...N)$ in an $O(N)$ symmetric fashion, there still would be ten constraints in phase space, but there would now be $2N + 6$ variables, leaving $2N - 4$ net physical degrees of freedom. Only if $N > 2$ are there true physical degrees of freedom.

The general form of the gauge generator for $S_{\Gamma\Gamma}^{(2)} + S_f$, when using the HTZ approach

[18, 19], is

$$G_{HTZ} = \int dx (A_{11}\chi^{11} + A_{00}\chi^{00} + A_{01}\chi^{01} \\ + B_S S + B_{I\!P} I\!P) \tag{2.24}$$

with (A_{11}, A_{00}, A_{01}) being found in terms of B_S and $B_{I\!P}$ by using eq. (A5). (The quantities μ^{a_N} with $N = 2$ in eq. (A5) are identified with the gauge functions $B_S(x,t)$, $B_{I\!P}(x,t)$ in eq. (2.24)) (In ref. [25] no consistent way of deriving the generator of gauge transformations was used; its form is merely postulated.)

Together, eqs. ((2.15), (2.20), (2.23)) lead to eq. (A5) being satisfied to order S and $I\!P$ provided

$$\begin{aligned}(B_{I\!P})_{,0} &+ B_S\left(-\frac{\sqrt{-g}}{g_{11}}\right)_{,1} - (B_S)_{,1}\left(-\frac{\sqrt{-g}}{g_{11}}\right) \\ &+ B_{I\!P}\left(\frac{g_{01}}{g_{11}}\right)_{,1} - (B_p)_{,1}\left(\frac{g_{01}}{g_{11}}\right) \\ &+ \frac{1}{g_{11}}\left(\frac{g_{01}}{g_{11}}A_{11} - A_{01}\right) = 0\end{aligned} \tag{2.25a}$$

and

$$\begin{aligned}(B_S)_{,0} &+ B_S\left(\frac{g_{01}}{g_{11}}\right)_{,1} - (B_S)_{,1}\frac{g_{01}}{g_{11}} \\ &+ B_{I\!P}\left(-\frac{\sqrt{-g}}{g_{11}}\right)_{,1} - (B_{I\!P})_{,1}\left(-\frac{\sqrt{-g}}{g_{11}}\right) \\ &+ \left[-\frac{\sqrt{-g}}{g_{11}^2} - \frac{g_{00}}{2g_{11}\sqrt{-g}}\right]A_{11} \\ &- \frac{1}{2\sqrt{-g}}A_{00} + \frac{g_{01}}{g_{11}\sqrt{-g}}A_{01} = 0.\end{aligned} \tag{2.25b}$$

As there are only two secondary constraints following from three primary constraints, eq. (2.25) does not uniquely fix A_{00}, A_{11} and A_{01} in terms of B_S and B_p.

In any case, eq. (2.25) is difficult to deal with, so we will employ the approach of Castellani which involves equations of the form of eq. (A12). In this approach, the form of the primary constraints that are used affects the form of the gauge generator [26]. We find it most convenient to use as primary constraints expressions suggested by the momenta conjugate to ρ, δ and g_{11} under a canonical transformation:

$$\overline{\chi}^{\rho} = 2\chi^{00}g_{01} + \chi^{01}g_{11} \tag{2.26a}$$

$$\overline{\chi}^{\delta} = 2\chi^{00}\sqrt{-g} \tag{2.26b}$$

$$\overline{\chi}^{11} = \chi^{11} + \chi^{00}\left(\frac{g_{00}}{g_{11}}\right) + \chi^{01}\left(\frac{g_{01}}{g_{11}}\right) \tag{2.26c}$$

so that

$$\{\overline{\chi}^{\rho}, \mathcal{H}_c\} = -I\!P, \quad \left\{\overline{\chi}^{\delta}, \mathcal{H}_c\right\} = -S, \quad \left\{\overline{\chi}^{11}, \mathcal{H}_c\right\} = 0. \tag{2.27}$$

In eq. (A12), derived by using the approach of C [16, 17], we take

$$G_1^{\rho} = \overline{\chi}^{\rho} \tag{2.28}$$

so that

$$G_0^{\rho} + \{G_1^{\rho}, H_T\} = \text{p.c.}$$

which leads to (p.c. is for primary constraints)

$$G_0^{\rho}(x) = I\!P(x) + \int dy\,[\alpha_{\rho\rho}(x-y)\overline{\chi}^{\rho}(y) + \alpha_{\rho\delta}(x-y)\overline{\chi}^{\delta}(y) + \alpha_{\rho 11}(x-y)\overline{\chi}^{11}(y)\Big]. \tag{2.29}$$

In turn, we must now have by eq. (A12)

$$\{G_0^{\rho}, H_T\} = \text{p.c.} \tag{2.30}$$

which fixes

$$\int dx \epsilon^{\rho}(x) G_0^{\rho}(x) = \int dx\left[\epsilon^{\rho} I\!P + \overline{\chi}^{\rho}\left(\epsilon^{\rho}_{,1}\left(\frac{g_{01}}{g_{11}}\right) - \epsilon^{\rho}\left(\frac{g_{01}}{g_{11}}\right)_{,1}\right) + \overline{\chi}^{\delta}\left(\epsilon^{\rho}_{,1}\left(\frac{-\sqrt{-g}}{g_{11}}\right) - \epsilon^{\rho}\left(\frac{-\sqrt{-g}}{g_{11}}\right)_{,1}\right)\right]. \tag{2.31}$$

So also, if

$$G_1^{\delta} = \overline{\chi}^{\delta} \tag{2.32}$$

then eq. (A12) leads to

$$\int dx \epsilon^{\delta}(x) G_0^{\delta}(x) = \int dx\left[\epsilon^{\delta} S + \overline{\chi}^{\delta}\left(\epsilon^{\delta}_{,1}\left(\frac{g_{01}}{g_{11}}\right) - \epsilon^{\delta}\left(\frac{g_{01}}{g_{11}}\right)_{,1}\right) + \overline{\chi}^{\rho}\left(\epsilon^{\delta}_{,1}\left(\frac{-\sqrt{-g}}{g_{11}}\right) - \epsilon^{\delta}\left(\frac{-\sqrt{-g}}{g_{11}}\right)_{,1}\right)\right]; \tag{2.33}$$

we finally obtain the full generator

$$
\begin{aligned}
G_C = \int dx \Big\{ & \epsilon^\rho I\!P + \epsilon^\delta S + \epsilon^{11}\overline{\chi}^{11} \\
& + \overline{\chi}^\rho \Bigg(\epsilon^\rho_{,1} \left(\frac{g_{01}}{g_{11}}\right) - \epsilon^\rho \left(\frac{g_{01}}{g_{11}}\right)_{,1} \\
& \quad + \epsilon^\delta_{,1} \left(\frac{-\sqrt{-g}}{g_{11}}\right) - \epsilon^\delta \left(\frac{-\sqrt{-g}}{g_{11}}\right)_{,1} \Bigg) \\
& + \overline{\chi}^\delta \Bigg(\epsilon^\delta_{,1} \left(\frac{g_{01}}{g_{11}}\right) - \epsilon^\delta \left(\frac{g_{01}}{g_{11}}\right)_{,1} \\
& \quad + \epsilon^\rho_{,1} \left(\frac{-\sqrt{-g}}{g_{11}}\right) - \epsilon^\rho \left(\frac{-\sqrt{-g}}{g_{11}}\right)_{,1} \Bigg) \\
& + \dot{\epsilon}^\rho \overline{\chi}^\rho + \dot{\epsilon}^\delta \overline{\chi}^\delta \Big\}
\end{aligned}
\tag{2.34}
$$

by eq. (A10).

A third approach is to find the gauge generator, again using the HTZ approach of eq. (A5), but this time employing the primary constraints of eq. (2.26) so that

$$\overline{G}_{HTZ} = \int dx \left(\overline{A}_\rho \overline{\chi}^p + \overline{A}_\delta \overline{\chi}^\delta + \overline{A}_{11}\overline{\chi}^{11} + \overline{B}_S S + \overline{B}^{I\!P} I\!P \right) \tag{2.35}$$

in place of eq. (2.24). Eq. (A5) results in

$$
\begin{aligned}
\frac{\partial \overline{B}_S}{\partial t} - \overline{A}_\delta + \overline{B}_S \left(\frac{g_{01}}{g_{11}}\right)_{,1} - \overline{B}_{S,1} \left(\frac{g_{01}}{g_{11}}\right) & \\
+ \overline{B}_{I\!P} \left(\frac{-\sqrt{-g}}{g_{11}}\right)_{,1} - \overline{B}_{I\!P,1} \left(\frac{-\sqrt{-g}}{g_{11}}\right) &= 0
\end{aligned}
\tag{2.36a}
$$

and

$$
\begin{aligned}
\frac{\partial \overline{B}_{I\!P}}{\partial t} - \overline{A}_\rho + \overline{B}_{I\!P} \left(\frac{g_{01}}{g_{11}}\right)_{,1} - \overline{B}_{I\!P,1} \left(\frac{g_{01}}{g_{11}}\right) & \\
+ \overline{B}_S \left(\frac{-\sqrt{-g}}{g_{11}}\right)_{,1} - \overline{B}_{S,1} \left(\frac{-\sqrt{-g}}{g_{11}}\right) &= 0.
\end{aligned}
\tag{2.36b}
$$

From eqs. (2.34) and (2.36) we see that $G_C = \overline{G}_{HTZ}$.

With the generator G_{HTZ} of eq. (2.24), we find that

$$
\begin{aligned}
\delta f &= \{f, G_{HTZ}\} \\
&= B_S p + B_{I\!P} f_{,1}
\end{aligned}
\tag{2.37}
$$

which by eq. (2.19) becomes

$$= B_S \sqrt{-g}\, g^{00} f_{,0} + \left(B_S \sqrt{-g}\, g^{01} + B_{I\!P}\right) f_{,1}. \tag{2.38}$$

This is identical to the diffeomorphism transformation

$$\delta f = \eta^0 f_{,0} + \eta^1 f_{,1} \tag{2.39}$$

provided

$$B_S = -\frac{\sqrt{-g}}{g_{11}}\eta^0 \tag{2.40}$$
$$B_{I\!P} = \eta^1 + \frac{g_{01}}{g_{11}}\eta^0. \tag{2.41}$$

Eq. (2.25) cannot be uniquely solved for A_{11}, A_{00} and A_{01} in terms of B_S and $B_{I\!P}$, but a particular solution with B_S and $B_{I\!P}$ given by eqs. (2.40) and (2.41) is

$$A_{11} = 2g_{01}\eta^0_{,1} + 2g_{11}\eta^1_{,1} + \eta^0 g_{11,0} + \eta^1 g_{11,1} \tag{2.42a}$$
$$A_{00} = 2g_{01}\eta^1_{,0} + 2g_{00}\eta^0_{,0} + \eta^1 g_{00,1} + \eta^0 g_{00,0} \tag{2.42b}$$
$$\begin{aligned} A_{01} = {} & g_{00}\eta^0_{,1} + g_{01}\left(\eta^0_{,0} + \eta^1_{,1}\right) + g_{11}\eta^1_{,0} \\ & + \eta^0 g_{01,0} + \eta^1 g_{01,1}. \end{aligned} \tag{2.42c}$$

These expressions are consistent with $\delta g_{\mu\nu} = \{g_{\mu\nu}, G_{HTZ}\}$ giving the diffeomorphism transformation

$$\delta g_{\mu\nu} = g_{\mu\rho}\eta^\rho_{,\nu} + g_{\nu\rho}\eta^\rho_{,\mu} + \eta^\rho g_{\mu\nu,\rho}. \tag{2.43}$$

An additional solution to eq. (2.25) is

$$B_S = B_{I\!P} = 0 \tag{2.44}$$

$$A_{00} = \Lambda g_{00}, \quad A_{11} = \Lambda g_{11}, \quad A_{01} = \Lambda g_{01} \tag{2.45}$$

so that

$$\delta g_{\mu\nu} = \{g_{\mu\nu} G_{HTZ}\} = \Lambda g_{\mu\nu}, \tag{2.46}$$

where the gauge function Λ is a scalar. This is the Weyl conformal (scale) invariance. The transformations generated by G_{HTZ} have also been found in ref. [23], and can also be found using G_C and $\overline{G}_{HTZ}$.

We now consider gauge invariance in two dimensions when a massless scalar field is coupled to the metric and the EH action is first order. Some aspects of this action were considered in ref. [12].

2.3 First Order EH Action and Scalar Fields

In d dimensions, the action of eq. (2.8) can be written

$$S_{hG} = \int d^d x \, h^{\mu\nu} \left(G^\lambda_{\mu\nu,\lambda} + \frac{1}{d-1} G^\lambda_{\lambda\mu} G^\sigma_{\sigma\nu} - G^\lambda_{\sigma\mu} G^\sigma_{\lambda\nu} \right), \tag{2.47}$$

where $h^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu}$ and $G^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \frac{1}{2}\left(\delta^\lambda_\mu \Gamma^\rho_{\rho\nu} + \delta^\lambda_\nu \Gamma^\rho_{\rho\mu}\right)$. We begin by examining the equations of motion that follow from this form of the first order EH action before considering its canonical structure. From eq. (2.47), the equations of motion for $G^\lambda_{\mu\nu}$ is

$$\begin{aligned} h^{\mu\nu}_{,\lambda} - \frac{1}{d-1} \left(\delta^\mu_\lambda h^{\nu\alpha} + \delta^\nu_\lambda h^{\mu\alpha} \right) G^\beta_{\alpha\beta} \\ + G^\mu_{\lambda\alpha} h^{\nu\alpha} + G^\nu_{\lambda\alpha} h^{\mu\alpha} = 0 \end{aligned} \tag{2.48}$$

from which it follows immediately that

$$G^\beta_{\alpha\beta} = -\frac{1}{2}\left(\frac{d-1}{d-2}\right) h_{\rho\sigma} h^{\rho\sigma}_{,\alpha}, \tag{2.49}$$

provided $d \neq 2$. Substitution of eq. (2.49) into eq. (2.48) gives

$$\begin{aligned} h^{\mu\nu}_{,\lambda} + \frac{1}{2(d-2)} \left(\delta^\mu_\lambda h^{\nu\alpha} + \delta^\nu_\lambda h^{\mu\alpha} \right) h_{\rho\sigma} h^{\rho\sigma}_{,\alpha} \\ + G^\mu_{\lambda\alpha} h^{\nu\alpha} + G^\nu_{\lambda\alpha} h^{\mu\alpha} = 0 \end{aligned} \tag{2.50}$$

which when combined with equations for $h^{\nu\lambda}_{,\mu}$ and $h^{\lambda\mu}_{,\nu}$ leads to

$$\begin{aligned} G^\lambda_{\mu\nu} = \frac{1}{2} h^{\lambda\rho} \left(h_{\mu\rho,\nu} + h_{\nu\rho,\mu} - h_{\mu\nu,\rho} \right) \\ - \frac{1}{2(d-2)} h_{\mu\nu} h^{\lambda\rho} h_{\alpha\beta} h^{\alpha\beta}_{,\rho}. \end{aligned} \tag{2.51}$$

For $d \neq 2$, this is equivalent to having $\Gamma^\lambda_{\mu\nu} = \left\{ \begin{array}{c} \lambda \\ \mu\nu \end{array} \right\}$. From eqs. (2.49),(2.51) it is apparent that $d = 2$ dimensions is special. To find $G^\lambda_{\mu\nu}$in terms of $h^{\mu\nu}$ when $d = 2$, we return to eq. (2.48). If $d = 2$, then eq. (2.48) leads to a consistency condition on the equations of motion for $G^\lambda_{\mu\nu}$

$$h_{\mu\nu} h^{\mu\nu}_{,\lambda} = \frac{1}{\Delta} \Delta_{,\lambda} = 0 \quad (\Delta \equiv \det h^{\mu\nu}) \tag{2.52}$$

in place of eq. (2.49). Eq. (2.52) is consistent with

$$\Delta = (\det h^{\mu\nu}) = -(-\det g_{\mu\nu})^{\frac{d}{2}-1} \tag{2.53}$$

when $d=2$.

If now we set

$$G^{\lambda}_{\mu\nu} = \frac{1}{2}h^{\lambda\rho}\left(h_{\mu\rho,\nu} + h_{\nu\rho,\mu} - h_{\mu\nu,\rho}\right) + h_{\mu\nu}X^{\lambda} \tag{2.54}$$

where X^{λ} is an arbitrary vector, then when $d=2$

$$\begin{aligned} &-\left(\delta^{\mu}_{\lambda}h^{\nu\alpha} + \delta^{\nu}_{\lambda}h^{\mu\alpha}\right)G^{\beta}_{\alpha\beta} + G^{\mu}_{\lambda\alpha}h^{\nu\alpha} + G^{\nu}_{\lambda\alpha}h^{\mu\alpha} \\ &\quad = -\frac{1}{2}\left(\delta^{\mu}_{\lambda}h^{\nu\alpha} + \delta^{\nu}_{\lambda}h^{\mu\alpha}\right)h^{\sigma\rho}h_{\sigma\rho,\alpha} - h^{\mu\nu}_{\ ,\lambda} \end{aligned} \tag{2.55}$$

and hence eq. (2.54) satisfies eq. (2.48) provided eq. (2.52) is also satisfied. Arbitrariness is also present in $\Gamma^{\lambda}_{\mu\nu}$ [14, 15] when $d=2$ if the equation of motion for $\Gamma^{\lambda}_{\mu\nu}$ that follows from the first order form of the EH action in terms of $\Gamma^{\lambda}_{\mu\nu}$ and $g_{\mu\nu}$ is solved to give eq. (2.3). Substitution of eq. (2.3) into the first order form of the EH action in terms of $\Gamma^{\lambda}_{\mu\nu}$ and $g_{\mu\nu}$ yields the second order form of the two dimensional EH action with all dependence on the arbitrary vector ξ^{λ} dropping out. In contrast, substitution of eq. (2.54) into eq. (2.47) with $d=2$ leads to

$$\begin{aligned} &\int dx^2 [h^{\mu\nu}\left(G^{\lambda}_{\mu\nu,\lambda} + G^{\lambda}_{\lambda\mu}G^{\sigma}_{\sigma\nu} - G^{\lambda}_{\sigma\mu}G^{\sigma}_{\lambda\nu}\right)] \\ &\quad = \int dx^2 \left[\left(2X^{\lambda} + \frac{1}{2\Delta}h^{\lambda\rho}\Delta_{,\rho} + h^{\lambda\rho}h^{\sigma\tau}h_{\rho\sigma,\tau}\right)_{,\lambda} \right. \\ &\qquad - \frac{1}{\Delta}X^{\lambda}\Delta_{,\lambda} + \frac{1}{4\Delta^2}h^{\mu\nu}\Delta_{,\mu}\Delta_{,\nu} \\ &\qquad \left. + \frac{1}{4}h^{\mu\nu}h^{\alpha\beta}_{,\mu}h_{\alpha\beta,\nu} + \frac{1}{2}h_{\mu\nu}h^{\alpha\mu}_{\ ,\beta}h^{\beta\nu}_{\ ,\alpha}\right]. \end{aligned} \tag{2.56}$$

Upon dropping the total derivatives in eq. (2.56), we see that X^{λ} remains as a Lagrange multiplier that ensures that eq. (2.52) is satisfied. Thus the role of X^{λ} in eq. (2.54) is different from that of ξ^{λ} in eq. (2.3).

We now perform a canonical analysis of S_{hG} when $d=2$. In order to do this we rewrite eq. (2.47) as

$$\begin{aligned} S_{hr} = \int d^2x \Big[&- G^0_{00}h_{,0} - 2G^0_{01}h^1_{,0} - G^0_{11}h^{11}_{,0} \\ &- G^1_{00}(h_{,1} + 2hG^0_{01} + 2h^1G^0_{11}) \\ &- 2G^1_{01}(h^1_{,1} - hG^0_{00} + h^{11}G^0_{11}) \\ &- G^1_{11}(h^{11}_{,1} - 2h^1G^0_{00} - 2h^{11}G^0_{01})\Big]. \end{aligned} \tag{2.57}$$

$$(h = h^{00}, \quad h^1 = h^{01})$$

From eq. (2.57) it follows that the momenta conjugate to (h, h^1, h^{11}) are

$$\pi = -G^0_{00}, \quad \pi_1 = -2G^0_{11}, \quad \pi_{11} = -G^0_{11} \tag{2.58}$$

respectively. The momenta conjugate to the "Lagrange multiplier" fields ($\xi^1 = G^1_{00}, \quad \xi = 2G^1_{01}, \quad \xi_1 = G^1_{11}$) are zero; these primary constraints lead to the secondary constraints

$$\phi_1 = h_{,1} - h\pi_1 - 2h^1\pi_{11} \tag{2.59a}$$

$$\phi = h^1_{,1} + h\pi - h^{11}\pi_{11} \tag{2.59b}$$

$$\phi^1 = h^{11}_{,1} + 2h^1\pi + h^{11}\pi_1. \tag{2.59c}$$

(These fields ξ^1, ξ, ξ_1 are in fact treated as degrees of freedom, and are not merely Lagrange multipliers as is done in refs. [34, 35].) This constraint structure leads to the gauge transformation of eqs. (2.4),(2.5) [7–12]. We see that despite the fact that $G^1_{\mu\nu}$ is a "Lagrange multiplier" field, its transformation under eq. (2.5) is not merely an arbitrary shift, demonstrating why it needs to be treated as a dynamical variable whose associated canonical momentum vanishes. Under this transformation

$$\delta\Delta = 0 \tag{2.60}$$

and, according to eq. (2.54),

$$\begin{aligned}\delta X^\mu &= \delta\left(h^{\mu\nu}G^\lambda_{\lambda\nu} - \frac{1}{2\Delta}h^{\mu\nu}\Delta_{,\nu}\right) \\ &= -h^{\mu\nu}\epsilon^{\lambda\sigma}\omega_{\nu\lambda,\sigma} + \epsilon^{\mu\nu}\omega_{\nu\lambda}h^{\lambda\sigma}G^\rho_{p\sigma} \\ &\quad - h^{\mu\nu}G^\lambda_{\rho\nu}\epsilon^{\rho\sigma}\omega_{\lambda\sigma} - \frac{1}{2\Delta}\left(\epsilon^{\mu\lambda}h^{\sigma\nu} + \epsilon^{\nu\lambda}h^{\sigma\mu}\right)\Delta_{,\nu}.\end{aligned} \tag{2.61}$$

Let us now supplement the action of eq. (2.47) with $d = 2$ by

$$S_f = \frac{1}{2}\int dx^2\, h^{\mu\nu}f_{,\mu}f_{,\nu}. \tag{2.62}$$

The canonical momenta if $h^{\mu\nu}$, $G^\lambda_{\mu\nu}$ and f are all independent fields given by

$$p = \frac{\partial\mathcal{L}}{\partial f_{,0}} = hf_{,0} + h^1 f_{,1} \tag{2.63}$$

$$\Pi^{\mu\nu}_\lambda = \frac{\partial\mathcal{L}}{\partial G^\lambda_{\mu\nu,0}} = 0 \tag{2.64}$$

as well as (π, π_1 and π_{11}).

The canonical Hamiltonian is

$$\mathcal{H}_C = \frac{1}{h}\Sigma + \left(\frac{-h^1}{h}\right)I\!P + \xi^1\phi_1 + \xi\phi + \xi_1\phi^1, \tag{2.65}$$

where

$$\Sigma = \frac{1}{2}(p^2 - \Delta f_{,1}^2) \tag{2.66}$$

and $I\!P$ is given in eq. (22). We now will show that ϕ^1, ϕ, ϕ_1, $I\!P$ and Σ are all first class constraints.

The primary constraints

$$\Pi_1^{\mu\nu} = 0 \tag{2.67}$$

are first class as the momenta associated with $G_{\mu\nu}^{\lambda}$ are immediately seen to vanish; they lead to the secondary first class constraints

$$\phi_1 = \phi = \phi^1 = 0. \tag{2.68}$$

One can show that

$$\left\{\phi_1, \phi^1\right\} = 2\phi \qquad , \left\{\phi, \phi^1\right\} = \phi^1, \qquad \{\phi_1, \phi\} = \phi_1 \tag{2.69}$$

$$\{\phi_1, \Delta\} = \{\phi, \Delta\} = \left\{\phi^1, \Delta\right\} = 0 \tag{2.70}$$

$$\Delta_{,1} = h\phi^1 + h^{11}\phi_1 - 2h^1\phi, \tag{2.71}$$

and, by using test functions as in ref. [24],

$$\{\Sigma(x), \Sigma(y)\} = (\Delta(x)I\!P(x)\partial_1^y - \Delta(y)I\!P(y)\partial_1^x)\delta(x-y) \tag{2.72}$$

This is not identical to the algebra of eq. (2.23a) unless $\Delta = -1$. In addition we have

$$\begin{aligned} &\{\Sigma(x), I\!P(y)\} \\ &\quad = \left[(-\Sigma(x)\partial_1^y + \Sigma(y)\partial_1^x) + \frac{1}{2}f_{,1}^2\Delta_{,1}\right]\delta(x-y) \end{aligned} \tag{2.73a}$$

$$\begin{aligned} &\{I\!P(x), \Sigma(y)\} \\ &\quad = \left[-\Sigma(x)\partial_1^y + \Sigma(y)\partial_1^x - \frac{1}{2}f_{,1}^2\Delta_{,1}\right]\delta(x-y) \end{aligned} \tag{2.73b}$$

Only if $\Delta_{,1} = 0$ does eq. (2.73a) reduce to the algebra of eq. (2.23b) for the tertiary first class constraints Σ and $I\!P$.

As was the case when we considered coupling N scalars to the metric field in section 2, the EH action by itself has no net physical degrees of freedom, while with the N scalar fields there are $2N - 4$ net physical degrees of freedom. (As mentioned above, when $N = 1$ the equation of motion show that f is constant and hence is not dynamical).

If the equation of motion were invoked so that by eq. (2.52) Δ would be constant, then h, h^1 and h^{11} would not be independent, nor by eq. (2.71) would ϕ_1, ϕ and ϕ^1.

However, we will not impose this condition so that all components of $h^{\mu\nu}$ are independent. (One could also ensure that Δ is constant by using a Lagrange multiplier.)

Using the HTZ approach, [18, 19] the generator of a gauge transformation is, by eq. (A2), of the form

$$G = \int dx (a^1\Pi_1 + a\Pi + a_1\Pi^1 + b^1\phi_1 + b\phi + b_1\phi^1 + c_\Sigma\Sigma + c_{I\!P} I\!P) \tag{2.74}$$

where Π_1, and Π and Π^1 are the momenta conjugate to ξ^1, ξ and ξ_1 respectively. By eqs. (2.65),(2.69),(2.70),(2.71),(2.72),(2.73a) it follows that

$$\begin{aligned}&\left\{G, \int dy \mathcal{H}_c\right\} = \int dx \Big\{-a^1\phi_1 - a\phi - a_1\phi^1 \\ &+ (b^1\xi - b\xi^1)\phi_1 + 2(b^1\xi_1 - b_1\xi^1)\phi + (b\xi_1 - b_1\xi)\phi^1 \\ &+ \frac{1}{h^2}(bh + 2b_1h^1)\Sigma \\ &+ \frac{1}{h^2}\left[-hh^1b - h^2b^1 + (hh^{11} - 2h^{1^2})b_1\right] I\!P \\ &+ \left[\Delta\left(c_{\Sigma,1}(\frac{1}{h}) - c_\Sigma(\frac{1}{h})_{,1}\right) + c_{I\!P,1}(\frac{h^1}{h}) - c_{I\!P,1}(\frac{h^1}{h})_{,1}\right] I\!P \\ &+ \left[c_{\Sigma,1}(\frac{h^1}{h}) - c_\Sigma(\frac{h^1}{h})_{,1} - c_{I\!P,1}(\frac{1}{h}) + c_{I\!P}(\frac{1}{h})_{,1}\right]\Sigma \\ &- \frac{1}{2}\Delta_{,1}f_{,1}\left(\frac{h^1}{h}c_\Sigma + \frac{1}{h}c_{I\!P}\right)\Big\}\end{aligned} \tag{2.75}$$

provided we ignore possible dependence of (a^1, a, a_1) and (b^1, b, b_1) on dynamical variables. (In the HTZ approach, $(c_\Sigma, c_{I\!P})$ are chosen to be independent of dynamical variables.)

Eq. (A5) to orders Σ and $I\!P$ respectively gives

$$\begin{aligned}\frac{\partial c_\Sigma}{\partial t} + \left[+c_{\Sigma,1}(\frac{h^1}{h}) - c_\Sigma(\frac{h^1}{h})_{,1} - c_{I\!P,1}(\frac{1}{h}) + c_{I\!P}(\frac{1}{h})_{,1}\right]& \\ + \frac{1}{h^2}(bh + 2b_1h^1) = 0&\end{aligned} \tag{2.76}$$

$$\begin{aligned}\frac{\partial c_{I\!P}}{\partial t} + \left[\Delta\left(c_{\Sigma,1}(\frac{1}{h}) - c_\Sigma(\frac{1}{h})_{,1}\right) + c_{I\!P,1}(\frac{h^1}{h}) - c_{I\!P}(\frac{h^1}{h})_{,1}\right]& \\ + \frac{1}{h^2}[-hh^1b - h^2b^1 + (hh^{11} - 2h^{1^2})b_1] = 0&\end{aligned} \tag{2.77}$$

which relate (b^1, b, b_1) to $(c_\Sigma, c_{I\!P})$. These equations are altered when $(c_\Sigma, c_{I\!P})$ depend on (h, h^1, h^{11}) by terms linear in (ξ^1, ξ, ξ_1).

We find that much like eq. (2.38)

$$\delta f = \{f, G\} = (c_\Sigma h)f_{,0} + (c_\Sigma h^1 + c_{I\!P})f_{,1} \tag{2.78}$$

which reduce to eq. (2.39) provided c_Σ and $c_{I\!P}$ acquire dependence on h^1 and h,

$$c_\Sigma = \eta^0/h \tag{2.79}$$

$$c_{I\!P} = \eta^1 - h^1\eta^0/h. \tag{2.80}$$

If c_Σ and $c_{I\!P}$ have this form, then eqs. (2.76) and (2.77) acquire extra contributions on the left side of

$$-\frac{\eta^0\xi}{h} - \frac{2h^1\eta^0\xi_1}{h^2} \tag{2.81}$$

and

$$\frac{h^1\eta^0}{h}\xi + \eta^0\xi^1 + \frac{1}{h^2}(2h^{1^2} - hh^1)\eta^0\xi_1 \tag{2.82}$$

respectively. Upon substituting eqs. (2.78),(2.78) into eqs. (2.76),(2.77) when supplemented by eqs. (2.79),(2.80) we find two equations for b, b_1 and b^1 that are consistent with taking

$$b = \eta^0_{,0} + \eta^1_{,1} + \eta^0\xi \tag{2.83}$$

$$b_1 = \frac{1}{2h^1}\left(\eta^0 h_{,0} + \eta^1 h_{,1} - 2h^1\eta^0_{,1} - 2h\eta^0_{,0}\right) + \eta^0\xi_1 \tag{2.84}$$

$$\begin{aligned} b^1 = {}& \frac{1}{h^1}\left(\eta^1_{,0}h^1 - \eta^0_{,0}h^{11}\right) + \frac{h^{11}}{2hh^1}\left(\eta^1 h_{,1} + \eta^0 h_{,0}\right) \\ & - \frac{1}{h}(\eta^1 h^1_{,1} + \eta^0 h^1_{,0}) + \eta^0\xi^1. \end{aligned} \tag{2.85}$$

With (b, b_1, b^1) given by eqs. (2.81),(2.82),(2.83) we find that

$$\begin{aligned} \delta h = \{h, G\} = {}& -h\eta^0_{,0} + h\eta^1_{,1} + \eta^0 h_{,0} + \eta^1 h_{,1} \\ & - 2h^1\eta^0_{,1} + \eta^0(h\xi + 2h^1\xi_1) \end{aligned} \tag{2.86}$$

$$\begin{aligned} \delta h^1 = \left\{h^1, G\right\} = {}& -h\eta^1_{,0} + \eta^1 h^1_{,1} + \eta^0 h^1_{,0} - h^{11}\eta^0_{,1} \\ & + \eta^0(-h\xi^1 + h^{11}\xi_1) \end{aligned} \tag{2.87}$$

$$\begin{aligned} \delta h^{11} = \left\{h^{11}, G\right\} = {}& -2h^1\eta^1_{,0} + h^{11}\eta^0_{,0} - h^{11}\eta^1_{,1} \\ & + h^{11}_{,0}\eta^0 + h^{11}_{,1}\eta^1 - \frac{1}{h}(\Delta_{,0}\eta^0 + \Delta_{,1}\eta^1) \\ & + \eta^0(-2h^1\xi^1 - h^{11}\xi). \end{aligned} \tag{2.88}$$

From eq. (2.43), under a diffeomorphism transformation

$$\delta h^{\mu\nu} = h^{\mu\lambda}\theta^\nu_{,\lambda} + h^{\nu\lambda}\theta^\mu_{,\lambda} - (h^{\mu\nu}\theta^\lambda)_{,\lambda} \tag{2.89}$$

which is the transformation of eqs. (2.86),(2.88),(2.89) provided

$$\theta^\lambda = -\eta^\lambda \ , \qquad \Delta_{,0} = \Delta_{,1} = 0 \ \text{ and } \ \xi^1 = \xi = \xi_1 = 0.$$

An additional solution to eqs. (2.76),(2.77) is

$$c_\Sigma = c_{I\!P} = 0\ , \qquad b = \frac{-2b_1 h^1}{h}\ , b^1 = \frac{h^{11} b_1}{h} \tag{2.90}$$

so that

$$b^1\phi_1 + b\phi + b_1\phi^1 = \frac{b_1}{h}\Delta_{,1}\ , \tag{2.91}$$

and hence

$$\delta h^{\mu\nu} = \{h^{\mu\nu}, G\} = 0. \tag{2.92}$$

Finding the variation of $G^\lambda_{\mu\nu}$ requires knowing the coefficients (a^1, a, a_1) in eq. (2.74). These are found by considering these terms in eq. (A5) proportional to (ϕ^1, ϕ, ϕ_1). By eq. (2.75), these are respectively given by

$$\frac{\partial b_1}{\partial t} - a_1 + (b\xi_1 - b_1\xi) - \frac{1}{2}f^2_{,1}(h^1 c_\Sigma + c_{I\!P}) = 0 \tag{2.93a}$$

$$\frac{\partial b}{\partial t} - a + 2(b^1\xi_1 - b_1\xi^1) + f^2_{,1}\frac{h^1}{h}(h^1 c_\Sigma + c_{I\!P}) = 0 \tag{2.93b}$$

$$\begin{aligned}\frac{\partial b^1}{\partial t} - a^1 + (b^1\xi - b\xi^1)\\ - \frac{1}{2}f^2_{,1}\frac{h^{11}}{h}(h^1 c_\Sigma + c_{I\!P}) = 0\end{aligned} \tag{2.93c}$$

provided we ignore terms in $\{G, \mathcal{H}_c\}$ that are linear in (ϕ^1, ϕ, ϕ_1) on account of the dependency of (b^1, b, b_1) on (h, h^1, h^{11}) following from eqs. (2.76), (2.77). If one were to supplement eqs. (2.92),(2.93) with terms

$$\begin{aligned}\phi^1\left\{b_{1,}\phi^1\xi_1 + \phi\xi + \phi_1\xi^1\right\} + \phi\left\{b, \phi^1\xi_1 + \phi\xi + \phi_1\xi^1\right\}\\ + \phi_1\left\{b^1, \phi^1\xi_1 + \phi\xi + \phi_1\xi^1\right\}\end{aligned} \tag{2.94}$$

in order to take into account the dependency of (b_1, b, b^1) on (h, h^1, h^{11}), and use eqs. (2.81),(2.82),(2.83) for (b_1, b, b^1), one encounters ill defined PBs of the form $\{h_{,0}, \pi\}$ indicating a breakdown of the HTZ procedure for finding the generator of a gauge transformation that leads to eq. (A5).

However, it is possible to overcome this shortcoming of the HTZ approach for finding the generator of a gauge transformation. If instead of eqs. (A3) to avoid time derivatives in PBs, one were to take the change in a dynamical variable A to be given by

$$\delta A = \nu^{a_i}\{A, \gamma_{a_i}\} \tag{2.95}$$

so that ν^{a_i} is not affected when one computes the PB, then the change in the extended action of eq. (A1) would be

$$\delta S_E = \int dt \Big[- v^{a_i} \Big(\left\{ \gamma_{a_i}, p^j \right\} \dot{q}_j - \left\{ \gamma_{a_i}, q_j \right\} \dot{p}^i \\ - \left\{ \gamma_{a_i}, q_j \right\} \frac{\partial H_c}{\partial q_i} - \left\{ \gamma_{a_i}, p^j \right\} \frac{\partial H_c}{\partial p^j} \\ - U^{a_j} \left\{ \gamma_{a_i}, \gamma_{a_j} \right\} \Big) - \delta U^{a_i} \gamma_{a_i} \Big] \tag{2.96}$$

provided we do an integration by parts, dropping the surface term. Eq. (2.96) further reduces to

$$\delta S_E = \int dt \Big[- v^{a_i} \Big(\frac{\partial \gamma_{a_i}}{\partial q_j} \dot{q}_j + \frac{\partial \gamma_{a_i}}{\partial p^j} \dot{p}^j \\ - \left\{ \gamma_{a_i}, H_c + U^{a_j} \gamma_{a_j} \right\} \Big) - \delta U^{a_i} \gamma_{a_j} \Big] \tag{2.97}$$

as u^{a_j} is not dynamical; a further integration by parts without keeping the surface terms leads to

$$\delta S_E = \int dt \left[+ \gamma_{a_i} \frac{D\nu^{a_i}}{Dt} + \nu^{a_i} \left\{ \gamma_{a_i}, H_c + U^{a_j} \gamma_{a_j} \right\} - \delta U^{a_i} \gamma_{a_i} \right] \tag{2.98}$$

which is almost identical to eq. (A4). However, the coefficients ν^{a_i} are not involved in the evaluation of any PBs.

For the system we have been considering, we can employ eq. (2.98) to find the gauge transformation of a dynamical variable A

$$\delta A = \overline{a}^1 \left\{ A, \Pi_1 \right\} + \overline{a} \left\{ A, \Pi \right\} + \overline{a}_1 \left\{ A, \Pi^1 \right\} \\ + \overline{b}^1 \left\{ A, \phi_1 \right\} + \overline{b} \left\{ A, \phi \right\} + \overline{b}_1 \left\{ A, \phi^1 \right\} \\ + \overline{c}_\Sigma \left\{ A, \Sigma \right\} + \overline{c}_{I\!P} \left\{ A, I\!P \right\}. \tag{2.99}$$

Eq. (2.98), when used in the same way eq. (A4) has been used by HTZ [18, 19] fixes $(\overline{b}^1, \overline{b}, \overline{b}_1)$ in terms of $(\overline{c}_\Sigma, \overline{c}_{I\!P})$ by eqs. (2.76), (2.77) and in turn determines $(\overline{a}^1, \overline{a}, \overline{a}_1)$ by eqs. (2.92), (2.93).

We find that, for example, that eq. (2.95) leads to

$$\delta G^1_{01} = \overline{a} \left\{ \frac{1}{2} \xi, \Pi \right\} \tag{2.100}$$

which, by eq. (93b) becomes

$$= \frac{1}{2} \left[\frac{\partial \overline{b}}{\partial t} + 2(\overline{b}^1 \xi_1 - \overline{b}_1 \xi^1) + f^2_{,1} \frac{h^1}{h} (h^1 \overline{c}_\Sigma + \overline{c}_{I\!P}) \right]. \tag{2.101}$$

Eqs. (2.78),(2.78),(2.81)-(2.83) in turn show that eq. (2.101) reduces to

$$\begin{aligned}\delta G^1_{01} &= \frac{1}{2}\left[\left(\eta^0_{,0} + \eta^1_{,0} + 2\eta^0 G^1_{01}\right)_{,0} + 2(\frac{1}{h^1}(\eta^1_{,0}h^1 - \eta^0_{,0}h^{11})\right. \\ &+ \frac{h^{11}}{2hh^1}(\eta^1 h_{,1} + \eta^0 h_{,0}) - \frac{1}{h}(\eta^1 h^1_{,1} + \eta^0 h^1_{,0}))G^1_{11} \\ &- \left(\frac{1}{h^1}\right)\left(\eta^0 h_{,0} + \eta^1 h_{,1} - 2h\eta^0_{,1} - 2h\eta^0_{,0}\right) G^1_{00} \\ &+ \left. f^2_{,1}\frac{h^1}{h}\eta^1\right] \end{aligned} \tag{2.102a}$$

Similarly, we find that

$$\begin{aligned}\delta G^1_{00} &= \overline{a}_1\left\{\xi^1, \Pi_1\right\} \\ &= \frac{\partial \overline{b}_1}{\partial t} + (\overline{b}\xi_1 - \overline{b}_1\xi) - \frac{1}{2}f^2_{,1}(h^1 c_\Sigma + c_{I\!P}) \end{aligned} \tag{2.102b}$$

and

$$\begin{aligned}\delta G^1_{11} &= \overline{a}^1\left\{\xi_1, \Pi^1\right\} = \frac{\partial \overline{b}^1}{\partial t} + (\overline{b}^1\xi - \overline{b}\xi^1) \\ &- \frac{1}{2}f^2_{,1}\frac{h^{11}}{h}(h^1 c_\Sigma + c_{I\!P}) \end{aligned} \tag{2.102c}$$

Eqs. (2.102a),(2.102b),(2.102c) have a term proportional to $f^2_{,1}$; similarly by eqs. (2.95),(2.66), δG^0_{00} has a term proportional to $-\frac{1}{2}h^{11}c_\Sigma f^2_{,1}$. It is apparent that $\delta G^\lambda_{\mu\nu}$ always has a contribution proportional to $f^2_{,1}$. This mixing of the affine connection and scalar field under a gauge transformation is somewhat unusual. The change in $G^\lambda_{\mu\nu}$ under a diffeomorphism is

$$\begin{aligned}\delta G^\lambda_{\mu\nu} = -G^\lambda_{,\mu\nu} + \frac{1}{2}\left(\delta^\lambda_\mu \theta^\rho_{,\nu\rho} + \delta^\lambda_\nu \theta^\rho_{,\mu\rho}\right) - \theta^\rho G^\lambda_{\mu\nu,\rho} \\ + G^\rho_{\mu\nu}\theta^\lambda_{,\rho} - \left(G^\lambda_{\mu\rho}\theta^\rho_{,\nu} + G^\lambda_{\nu\rho}\theta^\rho_{,\mu}\right) \end{aligned} \tag{2.103}$$

which does not mix $G^\lambda_{\mu\nu}$ and $f_{,1}$.

It is also possible to use the approach of [16,17] to find the gauge generator associated with $S_{hG} + S_f$ when $d = 2$. In eq. (A12), $N = 2$ since there are tertiary constraints. With $G_2 = \Pi^1$ and $\mathcal{H}_c$ given by eqs. (2.65), it follows from

$$G_1 + \{G_2, H_c\} \approx p.c. \tag{2.104}$$

that

$$\begin{aligned}G_1(x) = \phi^1(x) + \int dy[\alpha^1(x-y)\Pi_1(y) \\ + \alpha(x-y)\Pi(y) + \alpha_1(x-y)\Pi^1(y)]; \end{aligned} \tag{2.105}$$

next

$$G_0 + \{G_1, H_c\} \approx p.c. \tag{2.106}$$

leads to

$$\begin{aligned} G_0 = &\int dy \Big[\beta^1(x-y)\Pi_1(y) \\ &+ \beta(x-y)\Pi(y) + \beta_1(x-y)\Pi^1(y) \\ &+\alpha^1(x-y)\phi_1(y) + \alpha(x-y)\phi(y) + \alpha_1(x-y)\phi^1(y)\Big] \\ &+ 2\xi^1(x)\phi(x) + \xi(x)\phi^1(x) \\ &- \frac{2h^1(x)\Sigma(x)}{h^2(x)} + \left(\frac{2h^{1^2}(x) - h(x)h^{11}(x)}{h^2(x)}\right) I\!P(x). \end{aligned} \tag{2.107}$$

The final condition

$$\{G_0, H_c\} \approx p.c. \tag{2.108}$$

is satisfied to orders Σ, $I\!P$, ϕ^1, ϕ and ϕ_1 respectively provided

$$\begin{aligned} &\frac{\alpha}{h} + \frac{2h^1\alpha_1}{h^2} + \frac{4\xi^1}{h} + \frac{6h^1\xi}{h^2} + \left(\frac{8h^{1^2} - 2hh^{11}}{h^3}\right)\xi_1 \\ &- 2\left(\frac{h^{1^2}}{h^2}\right)_{,1}\frac{1}{h} + \left(\frac{hh^{11}}{h}\right)_{,1}\frac{1}{h^2} = 0 \end{aligned} \tag{2.109a}$$

$$\begin{aligned} &-\alpha^1 - \frac{h^1\alpha}{h} + \left(\frac{-2h^{1^2} + hh^{11}}{h^2}\right)\alpha_1 - \frac{4h^1\xi^1}{h} \\ &+ \left(\frac{-6h^{1^2} + 3hh^{11}}{h^2}\right)\xi + \left(\frac{-8h^{1^2} + 6hh^{11}}{h^3}\right)(h^1\xi_1) \\ &- \frac{h^{11}}{h}\left(\frac{h^1}{h}\right)_{,1} + \left(\frac{2h^{1^2} - hh^{11}}{h^2}\right)_{,1}\left(\frac{h^1}{h}\right) = 0 \end{aligned} \tag{2.109b}$$

$$\begin{aligned} &-\beta_1 + \alpha\xi_1 - \alpha_1\xi + 2\xi^1\xi_1 - \xi^2 + \frac{h^{11}}{2h}f^2_{,1} \\ &\quad + \{\alpha_{,1}H_c\} = 0 \end{aligned} \tag{2.110a}$$

$$\begin{aligned} &-\beta + 2(\alpha^1\xi_1 - \alpha_1\xi^1 - \xi\xi^1) - \frac{h^1h^{11}}{h^2}f^2_{,1} \\ &\quad + \{\alpha, H_c\} = 0 \end{aligned} \tag{2.110b}$$

$$\begin{aligned} &-\beta^1 + \alpha^1\xi - \alpha\xi^1 - 2\xi^{1^2} + \frac{h^{11^2}f^3_{,1}}{2h^2} \\ &\quad + \left\{\alpha^1, H_c\right\} = 0. \end{aligned} \tag{2.110c}$$

In exactly, the same way we find that if $G_2 = \Pi$, then

$$G_1 = \phi + \int dy(\alpha^1\Pi_1 + \alpha\Pi + \alpha_1\Pi^1) \tag{2.111}$$

$$\begin{aligned} G_0 = \int dy \left[\beta^1\Pi_1 + \beta\Pi + \beta_1\Pi^1 + \alpha^1\phi_1 + \alpha\phi + \alpha_1\phi^1\right] \\ + \xi^1\phi_1 - \xi_1\phi^1 - \frac{1}{h}(\Sigma - h^1 I\!P) \end{aligned} \tag{2.112}$$

with

$$\frac{\alpha}{h} + \frac{2h^1\alpha_1}{h^2} + \frac{\xi}{h} = 0 \tag{2.113a}$$

$$\begin{aligned} -\frac{\alpha^1}{h} - \frac{h^1\alpha}{h} + \frac{-2h^{1^2} + hh^{11}}{h^2}\alpha_1 \\ - 2\xi^1 - \frac{h^1\xi}{h} = 0 \end{aligned} \tag{2.113b}$$

$$-\beta_1 + \xi_1\alpha - \xi\alpha_1 + \xi\xi_1 + \{\alpha_1, H_c\} = 0 \tag{2.114a}$$

$$-\beta + 2\left(\xi_1\alpha^1 - \xi^1\alpha_1 + 2\xi^1\xi_1\right) + \{\alpha, H_c\} = 0 \tag{2.114b}$$

$$-\beta^1 + \left(\xi\alpha^1 - \xi^1\alpha + \xi\xi^1\right) + \left\{\alpha^1, H_c\right\} = 0. \tag{2.114c}$$

Finally, if $G_2 = \Pi_1$, then we find that

$$G_1 = \phi_1 + \int dy \left[\alpha^1\Pi_1 + \alpha\Pi + \alpha_1\Pi^1\right] \tag{2.115}$$

$$\begin{aligned} G_0 = \int dy \left[\beta^1\Pi_1 + \beta\Pi + \beta_1\Pi^1 + \alpha^1\phi_1 + \alpha\phi + \alpha_1\phi^1\right] \\ - \xi\phi_1 - 2\xi_1\phi + I\!P \end{aligned} \tag{2.116}$$

and so

$$\frac{\alpha}{h} + \frac{2h^1}{h^2}\alpha_1 - \frac{2\xi_1}{h} + \left(\frac{1}{h}\right)_{,1} = 0 \tag{2.117a}$$

$$\begin{aligned} -\alpha^1 + \alpha_1\left(\frac{-2h^{1^2} + hh^{11}}{h^2}\right) - \frac{h^1}{h}\alpha + \xi \\ + \frac{2h^1}{h}\xi_1 - \left(\frac{h^1}{h}\right)_{,1} = 0 \end{aligned} \tag{2.117b}$$

$$-\beta_1 + \xi_1\alpha - \xi\alpha_1 - 2\xi_1^2 - \frac{1}{2}f_{,1}^2 + \{\alpha_1, H_c\} = 0 \tag{2.118a}$$

$$-\beta + 2\xi_1\alpha^1 - 2\xi^1\alpha_1 - 2\xi\xi_1 + \frac{h^1}{h}f_{,1}^2 + \{\alpha, H_c\} = 0 \tag{2.118b}$$

$$-\beta^1 + \xi\alpha^1 - \xi^1\alpha + 2\xi^1\xi_1 - \xi^2 - \frac{h^{11}}{2h}f_{,1}^2 + \left\{\alpha^1, H_c\right\} = 0. \tag{2.118c}$$

In the instance where $G_2 = \Pi^1$, the two conditions of eqs. (2.109a),(2.109b) do not fix α^1, α and α_1 uniquely; however eqs. (2.110) do determine β^1, β and β_1 in terms of α^1, α and α. This lack of uniqueness in the gauge generator is a consequence of there being but two tertiary first class constraints following from the three primary first class constraints. The same pattern is repeated when $G_2 = \Pi$ (eqs. (2.113a),(2.114a) and $G_2 = \Pi$, (eqs. (2.117a),(2.118a). In each case though, β^1, β and β_1 depend on $f_{,1}^2$ in such a way that the transformation $\delta G_{\mu\nu}^{\lambda}$ depends on $f_{,1}^2$ as was the case when the HTZ approach to finding a gauge generator was used.

2.4 Discussion

In this chapter we have closely followed the Dirac constraint formalism [1–6] to analyze the gauge structure of a two dimensional massless scalar field in curved space. Though it has long been recognized that this is related to the bosonic string [21] and that this is a system involving constraints, it does not appear that a full constraint analysis has been performed on this system. It always appears that some fields have been eliminated by choosing to work in a "convenient" gauge before the constraints are identified, or that the generator of gauge transformations is postulated rather than derived from the first class constraints (see for example ref. [25]).

In this analysis we have included the EH action in second order form [7], even though it normally is dropped since it does not contain any dynamical degrees of freedom. This suggests that we also consider the first order EH action whose canonical structure in the absence of matter leads to a gauge invariance generated by the first class constraints that appears distinct from diffeomorphism invariance, and which accounts for the absence of dynamical degrees of freedom [8–13]. (We might also look at other actions for the two dimensional metric field be considered, such as the Weyl scalar invariant action which involves a vector field [27].) One peculiarity in our canonical analysis is that by adding the scalar field f, two degrees of freedom are added in phase space, but this also results in two more first class constraints (either S and $I\!P$ or Σ and $I\!P$ for the second order and first order EH actions respectively) which when combined with the associated gauge conditions, leads to a negative number of degrees of freedom (-2) in phase space. This issue was raised but not satisfactorily resolved in ref. [12]. If there are N scalars f^a and the kinetic term for these scalars were $O(N)$ symmetric, then there are ten restrictions on $2N+6$ fields in phase space, leaving $2N-4$ independent degrees of freedom. There are also $2N-4$ net degrees of freedom when using the first order form of the EH action.

The problem with having an unexpected number of degrees of freedom (especially when $N=1$) is implicit in all discussions of the canonical structure of the bosonic string that we have encountered in the literature (see for example ref. [25]) but no satisfactory resolution of the problem has been provided. In particular, if $N=1$, it would seem that the first class constraints of eqs. (2.21),(2.22), or eqs. (2.22),(2.26) would require imposing a gauge fixing that would over determine f and its conjugate momentum p. For $N=26$ there is a positive number of degrees of freedom (48) even after a gauge is chosen and this problem of over determination of $f^{(a)}$ and $p^{(a)}$, $(a=1...26)$ does not arise. Consequently, the bosonic string does not suffer from this particular inconsistency. In fact though, one should not be surprised that if $N=1$ there are no degrees of freedom associated with the scalar f, as the equation of motion for $h^{\mu\nu}$ that follows from eq. (2.7) is $(\partial_\mu f)(\partial_\nu f)=0$ which implies that f does not propagate. The equation of motion that follows from $g_{\mu\nu}$ in eq. (2.2) is $\partial_\mu\partial_\nu f - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha f\partial_\beta f = 0$ which has the same implications. For $N>1$ fields, $f^{(a)}$ is not necessarily a constant in order to satisfy the equations of motion for the metric.

Our analysis displays some interesting features of the approaches of C and HTZ to finding the gauge generator from the first class constraints. First of all, it is apparent from our discussion of the gauge generator when the EH action is second order that the actual form of the generator is dependent on how the constraints are chosen. When using the method of C, which form of the primary constraints is chosen is important (as was pointed out in ref. [26]) while the form of the gauge generator found using the approach of HTZ is different when different linear combinations of constraints of the highest order are employed.

The diffeomorphism invariance manifestly present in the initial Lagrangian is only recovered when using the second order form of the EH action if the gauge parameters associated with the secondary constraints are field dependent (which is contrary to the HTZ approach). There is also a residual symmetry occurring in this case. This additional symmetry resulting from the gauge generator is the Weyl scale symmetry. Thus both diffeomorphism invariance and Weyl scale invariance are gauge symmetries.

The HTZ formalism, when applied to first order form of the EH action plus the action for a scalar field, yields the diffeomorphism transformation for the scalar field only if the gauge parameters associated with the tertiary constraints are again field dependent. The resulting equations for the gauge parameters associated with primary constraints involves ill defined PBs that can be avoided by slightly modifying the HTZ procedure. When this is done, the resulting gauge transformation is unusual as it mixes the affine connection and the scalar field in an non-polynomial fashion. We have attempted unsuccessfully to find such a gauge invariance directly from the action given in eqs. (2.47),(2.62).

Of course, once the canonical structure of these models is disentangled, their quantization is to be considered. This may have implications for bosonic string theory.

2.5 Appendix. The Gauge Generator

When one is presented with a Lagrangian $L(q_i(t), \dot{q}_i(t))$, passing to the Hamiltonian formalism is straightforward unless the equations defining the canonical momenta $p^i = \partial L(q_i, \dot{q}_i)/\partial \dot{q}_i$ cannot be solved for $\dot{q}_i$ in terms of q_i and p^i. In this case, one must use the Dirac constraint formalism [1–6]. (For a discussion of the history of the constraint formalism, see ref. [28].)

If one encounters first class constraints γ_{a_j} in the j^{th} generation arising from the inability to solve for velocities $\dot{q}_i$ in terms of p_i from the defining equation $p_i = \frac{\partial L}{\partial \dot{q}_i}$[2], then the "extended action" is

$$S_E = \int_{t_i}^{t_f} dt \left[p^i \dot{q}_i - H_c(q_i, p^i) - U^{a_j}(t) \gamma_{a_j}(q_i, p^i) \right]; \tag{A1}$$

where H_c is the canonical Hamiltonian. (Primary constraints are of the first "generation", secondary constraints are of the second "generation" etc.) If a gauge generator G is a linear combination of first class constraints as in the HTZ approach [18, 19]

$$G = \mu^{a_j} \left(q_i(t), p^i(t), U^{\alpha_j}(t), t \right) \gamma_{a_j}(q_i(t), p^i(t)) \tag{A2}$$

[2] We assume all second class constraints have been used to eliminate some of the degrees of freedom and that the Dirac Brackets (DB) for the remaining variables are identical to their Poisson Brackets (PB).

so that the change in a dynamical variable A is given by the PB[3]

$$\delta A = \{A, G\} \tag{A3}$$

then this results in

$$\delta S_E = \int_{t_i}^{t_f} dt \left[\frac{D\mu^{a_j}}{Dt}\gamma_{a_j} + \left\{G, H_c + U^{a_j}\mu_{a_j}\right\} - \delta U^{a_j}\gamma_{a_j}\right] \tag{A4}$$

where δU^{a_j} is the corresponding change in the Lagrange multiplier U^{a_j} and D/Dt is the time derivative induced by the implicit time dependence through $U^{a_j}(t)$ and the explicit time dependence. (The time dependence of γ^{a_j} through $p^i(t)$ and $q^i(t)$ is canceled by the PB $\{\int dtp^i\dot{q}_i, \mu^{a_j}\}$.) Surface terms at $t = t_i, t_f$ are dropped in eq. (A4).

One can move from the extended action of eq. (A1) to the "total action" S_T by setting $U^{a_j} = \delta U^{a_j} = 0$ for $j \geq 2$. This total action has the same invariance as the classical action $\int dtL$ [29]. Consequently one can find invariance of the classical action by determining the functions $\mu^{a_j}(j = 1, 2 \ldots N)$ in eq. (A2) by solving

$$\frac{D\mu^{a_j}}{Dt}\gamma_{a_j} + \{G, H_c + U^{a_1}\gamma_{a_1}\} - \delta U^{a_1}\gamma_{a_1} = 0 \tag{A5}$$

systematically; as eq. A(5) ensures that S_T remains invariant; μ^{a_N} is taken to be an arbitrary function of time, $\mu^{a_{N-1}}$ is fixed in terms of μ^{a_N}; $\mu^{a_{N-2}}$ is fixed in terms of $\mu^{a_{N-1}}$ etc.

An approach to finding the gauge invariance in a system with a denumerable number of degrees of freedom in which the Lagrangian is at most linear in time derivatives appears in refs. [34, 35]. However, this discussion does not consider the possibility of tertiary constraints (which occur in the first order form of the EH action in $D > 2$ dimensions [36, 37]) nor is it extendable to deal with such constraints. Also, it does not exploit the fact that it is the total action, not the extended action, that has the same invariances as the classical action in order to find dependence of the gauge transformation on the time derivative of the gauge functions.

In the approach of C [16, 17], the generator G is found by considering the Hamiltonian equations of motion. If both (q_i, p^i) and $(q_1 + \alpha_i, p^i + \beta^i)$ are solutions, then

$$\alpha_i = \{q_i, G\} = \frac{\partial G}{\partial p^i}, \qquad \beta^i = \left\{p^i, G\right\} = -\frac{\partial G}{\partial q_i}. \tag{A6}$$

We now have the general equation $\frac{dA}{dt} \approx \{A, H_T\} + \frac{\partial A}{\partial t}$ which means that eq. (A6) leads to

$$\dot{\alpha}_i \approx \left\{\frac{\partial G}{\partial p^i}, H_T\right\} + \frac{\partial^2 G}{\partial t \partial p^i}, \quad \dot{\beta}^i \approx -\left\{\frac{\partial G}{\partial q_i}, H_T\right\} - \frac{\partial^2 G}{\partial t \partial q_i}. \tag{A7}$$

[3] The PB is defined to be $\{A(q_i, p_i), B(q_i, p_i)\} = \sum_i (A_{,q_i}B_{,p_i} - A_{,p_i}B_{,q_i})$

(The weak inequality $A \approx B$ holds when the primary constraints vanish.) In addition, the equations of motion themselves lead to

$$\dot{q}_i + \dot{\alpha}_i \approx \frac{\partial}{\partial p^i} H_T(q_i + \alpha_i, p^i + \beta^i),$$

$$\dot{p}^i + \dot{\beta}^i \approx -\frac{\partial}{\partial q_i} H_T(q_i + \alpha_i, p^i + \beta^i) \tag{A8}$$

which to lowest order becomes

$$\dot{\alpha}_i \approx \frac{\partial}{\partial p^i}\left(\frac{\partial H_T}{\partial q_i}\alpha_i + \frac{\partial H_T}{\partial p^i}\beta^i\right),$$

$$\dot{\beta}^i \approx -\frac{\partial}{\partial q_i}\left(\frac{\partial H_T}{\partial q_i}\alpha_i + \frac{\partial H_T}{\partial p^i}\beta^i\right). \tag{A9}$$

We now can equate the expressions for $\dot{\alpha}_i$ and $\dot{\beta}^i$ in eqs. (A7) and (A9) and then use eq. (A6) to eliminate α_i and β^i. If the gauge generator is expanded

$$G = \epsilon(t)G_0 + \dot{\epsilon}(t)G_1 + \ldots \epsilon^{(N)}(t)G_N \tag{A10}$$

when there are $N+1$ generations of constraints, then we find that

$$\epsilon\{G_0, H_T\} + \dot{\epsilon}[G_0 + \{G_1, H_T\}] + \ddot{\epsilon}[G_1 + \{G_2, H_T\}]$$

$$+ \ldots \epsilon^{(N)}[G_{N-1} + \{G_N, H_T\}] + \epsilon^{(N+1)}[G_N] \approx 0. \tag{A11}$$

Eq. (2.11) can be satisfied iteratively by taking

$$G_N \approx \text{ (primary constraints)} \tag{A12}$$

$$G_{N-1} + \{G_N, H_T\} \approx \text{ (primary constraints)}$$

$$\{G_0, H_T\} \approx \text{ (primary constraints)}.$$

Only primary constraints appear in eq. (A12) as in eq. (A7) the weak inequality need only hold on the constraint surface on which the primary constraints vanish.

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Chapter 3

Electrodynamics in a Background Chiral Field

3.1 Introduction

Parity violating interactions with a spinor field yield several interesting consequences, among them an anomalous divergence in the axial current [1–3] and the absence of bound states in a "Coulomb" axial potential [4, 5]. In this chapter we consider the one loop effective action for a spinor field in the presence of a constant background chiral vector field. The analogous situation in which the interaction is parity conserving is well known [1, 6–8].

3.2 Effective Action

If a spinor ψ is in the presence of a background vector field V^μ and a background axial field A^μ in four dimensional Euclidean space we have the Lagrangian

$$\mathcal{L} = \psi^\dagger \left[\left(\not{p} - \not{W}_+ P_+ - \not{W}_- P_- \right) - m \right] \psi \tag{3.1}$$

where $p = -i\partial$ and $W_\pm = V \pm A$ are chiral fields. (The notation used is in the appendix.) The effective action is then given by the one loop expression followed from the path integral representation of the effective action

$$\Gamma_4 = \ln \det \left(\not{p} - \not{W}_+ P_+ - \not{W}_- P_- - m \right). \tag{3.2}$$

We now rewrite Eq. (3.2) as

$$\Gamma_4 = \left[\ln\det\left(\not{p} - \not{W}_+ P_+ - \not{W}_- P_-\right) + \ln\det\left(1 - \frac{m}{\not{p} - \not{W}_+ P_+ - \not{W}_- P_-}\right)\right] \tag{3.3}$$

and then expand the second term in Eq. (3.3) so that

$$\ln\det\left(1 - \frac{m}{\not{p} - \not{W}_+ P_+ - \not{W}_- P_-}\right) = -\mathrm{tr}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{m}{\not{p} - \not{W}_+ P_+ - \not{W}_- P_-}\right)^n . \tag{3.4}$$

We now rewrite

$$\begin{aligned}\frac{1}{\not{p} - \not{W}_+ P_+ - \not{W}_- P_-} &= \frac{1}{\not{p}}\,\frac{1}{1 - \frac{1}{\not{p}}\left(\not{W}_+ P_+ + \not{W}_- P_-\right)} \\ &= \frac{1}{\not{p}}\sum_{n=0}^{\infty}\left[\frac{1}{\not{p}}\left(\not{W}_+ P_+ + \not{W}_- P_-\right)\right]^n\end{aligned}$$

which by the properties of the projection operators $P_\pm$ becomes

$$\begin{aligned}&= \frac{1}{\not{p}}\sum_{n=0}^{\infty}\left[\left(\frac{1}{\not{p}}\not{W}_+\right)^n P_+ + \left(\frac{1}{\not{p}}\not{W}_-\right)^n P_-\right] \\ &= \frac{1}{\not{p} - \not{W}_+}P_+ + \frac{1}{\not{p} - \not{W}_-}P_- .\end{aligned} \tag{3.5}$$

Similarly, we have for the first term in Eq. (3.3)

$$\begin{aligned}\ln\det\left(\not{p} - \not{W}_+ P_+ - \not{W}_- P_-\right) &= \mathrm{tr}\left[\ln\,\not{p} - \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{\not{p}}\not{W}_+ P_+ + \frac{1}{\not{p}}\not{W}_- P_-\right)^n\right] \\ &= \mathrm{tr}\left[\left(\ln(\not{p} - \not{W}_+)\right)P_+ + \left(\ln(\not{p} - \not{W}_-)\right)P_-\right].\end{aligned} \tag{3.6}$$

Together, Eqs. (3.3), (3.4), (3.5) and (3.6) show that

$$\begin{aligned}\Gamma_4 = \mathrm{tr}\Bigg[&(\ln\,\not{\Pi}_+)P_+ + (\ln\,\not{\Pi}_-)P_- - \frac{m}{1}\left(\frac{1}{\not{\Pi}_+}P_+ + \frac{1}{\not{\Pi}_-}P_-\right) \\ &- \frac{m^2}{2}\left(\frac{1}{\not{\Pi}_-}\frac{1}{\not{\Pi}_+}P_+ + \frac{1}{\not{\Pi}_+}\frac{1}{\not{\Pi}_-}P_-\right) \\ &- \frac{m^3}{3}\left(\frac{1}{\not{\Pi}_+}\frac{1}{\not{\Pi}_-}\frac{1}{\not{\Pi}_+}P_+ + \frac{1}{\not{\Pi}_-}\frac{1}{\not{\Pi}_+}\frac{1}{\not{\Pi}_-}P_-\right) \\ &- \dots\Bigg]\end{aligned} \tag{3.7}$$

where $\Pi_\pm \equiv p - W_\pm$.

If we now use the identity

$$\mathrm{tr}X = \frac{1}{2}\mathrm{tr}\left[X + \gamma^5 X\gamma^5\right] \tag{3.8}$$

then we see that terms in Eq. (3.7) with odd powers of m vanish. This reduces Eq. (3.7) to

$$\Gamma_4 = \frac{1}{2}\mathrm{tr}\left\{\left[\ln\left(\not{\Pi}_+^2\left(1-\frac{m^2}{\not{\Pi}_-\not{\Pi}_+}\right)\right)\right]P_+ + \left[\ln\left(\not{\Pi}_-^2\left(1-\frac{m^2}{\not{\Pi}_+\not{\Pi}_-}\right)\right)\right]P_-\right\}. \qquad (3.9)$$

Under "charge conjugation" we find that

$$\begin{aligned} & C^{-1}\left(\not{p}-\not{W}_+P_+-\not{W}_-P_- - m\right)C \\ & \quad = \left[\not{p}+\not{W}_+P_-+\not{W}_-P_+ - m\right]^T \end{aligned} \qquad (3.10)$$

and so Eq. (3.2) is symmetric under the replacement $W_\pm \to -W_\mp$. (In ref. [9] the fact that $p^{\mu T} = -p^\mu$ was ignored.)

3.3 Explicit Evaluation of the Effective Action

Evaluation of Γ in Eq. (3.9) in closed form when $m^2 \neq 0$ involves having to determine $\mathrm{tr}\ln(\not{\Pi}_\pm\not{\Pi}_\mp - m^2)$. If $\not{W}_\pm \neq \not{W}_\mp$ this is prohibitively difficult, even if $W_\pm = \pm A$. In this case we must consider

$$\mathrm{tr}\ln\left[(\not{p}\pm\not{A})(\not{p}\mp\not{A}) - m^2\right] = \mathrm{tr}\ln\left[\left(p^\mu \mp i\sigma^{\mu\nu}A^\nu\right)^2 + 2A^2 \pm iA^\lambda_{,\lambda} - m^2\right] \qquad (3.11)$$

which, though it is well suited for a perturbative expansion in powers of A^μ [10, 11], does not lend itself to being evaluated even when A^μ corresponds to there being a constant field strength.

However, if $m^2 = 0$, or if Eq. (3.9) were expanded to some finite order in powers of m^2, then one is faced with evaluation of only $\frac{1}{2}(\Lambda_+ + \Lambda_-)$ where $\Lambda_\pm = \mathrm{tr}\left[\ln\not{\Pi}_\pm^2\right]P_\pm$. In refs. [1, 6, 7], it is shown that since $(\not{p}-\not{V})^2 = (p^\mu - V^\mu)^2 - \frac{1}{2}\sigma^{\mu\nu}F^{\mu\nu}$ ($F = \partial\wedge V$) the gamma matrix trace occurring in $\Lambda_\pm$ involves

$$\begin{aligned} \mathrm{tr}\, e^{\frac{1}{2}F^{\mu\nu}\sigma^{\mu\nu}t}P_\pm = \mathrm{tr}\Bigg\{ & \cosh K_- P_+ + \cosh K_+ P_- \\ & + \frac{t}{2}\sigma^{\mu\nu}F^{\mu\nu}\left(\frac{\sinh K_-}{K_-}P_+ + \frac{\sinh K_+}{K_+}P_-\right)\Bigg\}P_\pm \\ = {} & 4\cosh K_\mp \end{aligned} \qquad (3.12)$$

where $K_\pm^2 = \frac{t^2}{2}\left[F^{\mu\nu}F^{\mu\nu} \pm F^{\mu\nu}F^{*\mu\nu}\right]$. We thus see that the presence of the chiral projection operator $P_\pm$ in Eq. (3.9) serves to eliminate the contribution of $\cosh K_\pm$ as well as $\sinh K_+$ and $\sinh K_-$, leaving only $4\cosh K_\mp$.

The background field strength $W_\pm$ in the gauge $x \cdot W_\pm = 0$ can be expanded in powers of the field strength $F_\pm$ [12–14],

$$W^\mu_\pm = \sum_{n=0}^{\infty} \frac{-1}{n!(n+2)} x^\nu x^{\lambda_1} \dots x^{\lambda_n} \; F_\pm^{\mu\nu,\lambda_1\dots\lambda_n}(0). \tag{3.13}$$

The first term in Eq. (3.13) corresponds to a constant background field as discussed in refs. [1, 6, 7]; higher contributions are dealt with in refs. [8, 15–17]. Other special background field configurations have been considered [1, 8, 18–20].

If $m^2 = 0$ and $W_\pm = \pm A$, then we have a purely axial coupling and

$$\Gamma_A^{(0)} = \frac{1}{2}\mathrm{tr}\left[\left(\ln(\not{p} - \not{A})^2\right) P_+ + \left(\ln(\not{p} + \not{A})^2\right) P_-\right]. \tag{3.14}$$

If A^μ is in the gauge $x \cdot A = 0$ so that it is expressed in the form of Eq. (3.13) then gauge invariance is manifestly preserved since A^μ is expressed in terms of the field strength. If we then expand $\Gamma_A^{(0)}$ with this background field using the Schwinger expansion as in ref. [1, 21], then the three point function $\langle AAA \rangle$ vanishes. However, again computing $\langle AAA \rangle$ but with plane wave background axial fields, the three point function is consistent with the axial anomaly [1–3].

If $m^2 \neq 0$ when $W_\pm = \pm A$ then Eq. (3.9) reduces to

$$\begin{aligned}\Gamma_A = \frac{1}{2}\mathrm{tr}\Bigg\{ &\left[\ln\left((\not{p} + \not{A})(\not{p} - \not{A}) - m^2\right)\right] P_+ + \left[\ln\left((\not{p} - \not{A})(\not{p} + \not{A}) - m^2\right)\right] P_- \\ &+ \frac{1}{2}\left[\ln(\not{p} - \not{A})^2 - \ln(\not{p} + \not{A})^2\right]\gamma_5\Bigg\}.\end{aligned} \tag{3.15}$$

There doesn't appear to be a way of evaluating this in closed form when even $A^\mu = -\frac{1}{2}F^{\mu\nu}x^\nu$ if $m^2 \neq 0$, though with this background field $\langle AAA \rangle = 0$. With a plane wave background field the axial anomaly can however be recovered [21] when $\langle AAA \rangle$ is computed by applying the Schwinger expansion [1] to Eq. (3.15).

Although it doesn't appear to be feasible to compute Γ_4 when there is a constant strength $\partial^\mu A^\nu - \partial^\nu A^\mu$ in Eq. (3.1), we can consider the case in which Γ_4 is restricted to being linear in the external axial field and the vector field is taken to be constant. In this case we begin by using Eq. (3.8) to write

$$\Gamma_4 = \frac{1}{2}\ln\det\left[(\not{p} - \not{V} - \not{A}\gamma^5)^2 - m^2\right]. \tag{3.16}$$

Dropping those terms in Eq. (3.12) that cannot contribute to the contribution to Γ_4 that are linear in A_μ, we see that upon letting $m^2 \to -m^2$,

$$\Gamma_4 \approx \frac{1}{2}\ln\det\Big[(p-V)^2 + m^2 - \frac{1}{2}F^{\mu\nu}\sigma^{\mu\nu} + iA^{\mu,\mu}\gamma^5 + i\sigma^{\mu\nu}\left(2A^\mu p^\nu + \frac{i}{2}G^{\mu\nu} - 2A^\mu V^\nu\right)\gamma^5\Big] \tag{3.17}$$

where $F^{\mu\nu} = \partial^\mu V^\nu - \partial^\nu V^\mu$ and $G^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. If we now employ operator regularization [1] to expand Γ_4 in Eq. (3.17) to the term linear in A_μ, we need the equations [21]

$$\begin{aligned}\frac{1}{2}\ln\det(H_0+H_1) &= -\frac{1}{2}\frac{d}{ds}\Big|_0 \mathrm{tr}\frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\, e^{-(H_0+H_1)t} \\ &= -\frac{1}{2}\frac{d}{ds}\Big|_0 \frac{1}{\Gamma(s)}\int_0^\infty dt\ t^{s-1}\ \mathrm{tr}\Big[e^{-H_0 t} + \frac{(-t)}{1}\ e^{-H_0 t}H_1 \\ &\quad + \frac{(-t)^2}{2}\int_0^1 du\ e^{-(1-u)H_0 t}H_1 e^{-uH_0 t}H_1 + \dots\Big].\end{aligned} \tag{3.18}$$

Upon using Eq. (3.18), Eq. (3.17) reduces to

$$\Gamma_4 \approx \frac{1}{2}\frac{d}{ds}\Big|_0 \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^s\ \mathrm{tr}\ e^{-[(p-V)^2+m^2-\frac{1}{2}F^{\mu\nu}\sigma^{\mu\nu}]t}\Big[iA^\mu_{,\mu} + i\sigma^{\lambda\sigma}\left(2A^\lambda p^\sigma + \frac{i}{2}\ G^{\lambda\sigma} - 2A^\lambda V^\sigma\right)\Big]\gamma^5\ . \tag{3.19}$$

If $F^{\mu\nu}$ is constant, then by Eqs. (3.37) and (3.38) this becomes

$$\begin{aligned} &= \frac{1}{2}\ \frac{d}{ds}\Big|_0\ \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^s\ \mathrm{tr}\ e^{[(p-V)^2+m^2]t}\Big[(\cosh K_-)P_+ + (\cosh K_+)P_- \\ &\quad + \left(\frac{\sinh K_-}{K_-}P_+ + \frac{\sinh K_+}{K_+}P_-\right) w^{\mu\nu}\sigma^{\mu\nu}\Big] \\ &\quad \Big[iA^\lambda_{,\lambda} + i\sigma^{\lambda\sigma}\left(2A^\lambda p^\sigma + \frac{i}{2}G^{\lambda\sigma} - 2A^\lambda V^\sigma\right)\Big]\gamma_5 \end{aligned} \tag{3.20}$$

where $w^{\mu\nu} = \frac{1}{2}F^{\mu\nu}t$ and $K_\pm^2 = 2(w^{\alpha\beta}w^{\alpha\beta} \pm w^{*\alpha\beta}w^{\alpha\beta})$.

Evaluating the γ-matrix traces in Eq. (3.20) leads to

$$\begin{aligned} &= \frac{d}{ds}\Big|_0\ \frac{i}{\Gamma(s)}\int_0^\infty dt\ t^s\ \mathrm{tr}\ e^{-[(p-V)^2+m^2]t}\Big\{(\cosh K_- - \cosh K_+)A^\lambda_{,\lambda} \\ &\quad + 2\left[\left(\frac{\sinh K_-}{K_-} - \frac{\sinh K_+}{K_+}\right)w^{\lambda\sigma} - 2\left(\frac{\sinh K_-}{K_-} + \frac{\sinh K_+}{K_+}\right)w^{*\lambda\sigma}\right] \\ &\quad \left[2A^\lambda p^\sigma + \frac{i}{2}G^{\lambda\sigma} - 2A^\lambda V^\sigma\right]\Big\}. \end{aligned} \tag{3.21}$$

When $V^\mu = -\frac{1}{2}F^{\mu\nu}x^\nu$, then the result of Schwinger [1]

$$\langle x|e^{-(p-V)^2 t}|y\rangle = \frac{i}{(4\pi t)^2}\exp\left(i\int_y^x dz\cdot V(z)\right)e^{-L(t)} \qquad (3.22)$$
$$\exp\left(-\frac{1}{4}(x-y)\cdot F\cdot\cot(Ft)\cdot(x-y)\right)$$

can be used to compute the functional trace in Eq. (3.21). (Here we have $L(t) = \frac{1}{2}\mathrm{tr}\ln((Ft)^{-1}\sin(Ft))$ and one should notice the Wick rotation applied to the original formula in [1]) In particular, it follows from Eq. (3.22) that

$$\begin{aligned}\mathrm{tr}\ e^{-(p-V)^2 t}A^\lambda p^\sigma &= \mathrm{tr}\int dz\langle x|e^{-(p-V)^2 t}|z\rangle i\partial_y^\sigma\langle z|A^\sigma|y\rangle \\ &= \int dx\int dy\ \delta(x-y)i\partial_y^\sigma\Bigg[\frac{i}{(4\pi t)^2}\exp\left(i\int_y^x dzV(z)\right)e^{-L(t)} \\ &\qquad \exp\left(-\frac{1}{4}(x-y)\cdot F\cdot\cot(Ft)\cdot(x-y)\right)A^\lambda(y)\Bigg] \\ &= \frac{i}{(4\pi t)^2}e^{-L(t)}\int dx\left[V^\sigma(x)A^\lambda(x) + i\partial_x^\sigma A^\lambda(x)\right]. \qquad (3.23)\end{aligned}$$

Substitution Eqs. (3.22) and (3.23) into Eq. (3.21) leads to

$$\begin{aligned}\Gamma_4 \approx \frac{-1}{(4\pi)^2}\frac{d}{ds}\bigg|_0\frac{1}{\Gamma(s)}\int_0^\infty dt\ t^{s-2}e^{-L(t)-m^2 t}\int dx\Bigg\{(\cosh K_- - \cosh K_+)A^{\mu,\mu}(x) \\ -\frac{i}{2}G^{\lambda\sigma}(x)t\left[\left(\frac{\sinh K_-}{K_-} - \frac{\sinh K_+}{K_+}\right)F^{\lambda\sigma} - \left(\frac{\sinh K_-}{K_-} + \frac{\sinh K_+}{K_+}\right)F^{*\lambda\sigma}\right]\Bigg\}. \qquad (3.24)\end{aligned}$$

Expanding Eq. (3.24) to lowest order in $F^{\lambda\sigma}$ results in

$$\begin{aligned}\Gamma_4 &\approx \frac{1}{(4\pi)^2}\frac{d}{ds}\bigg|_0\frac{1}{\Gamma(s)}\int_0^\infty dt\ t^{s-2}e^{-m^2 t}\int dx\Bigg[\frac{1}{2}t^2 F^{\lambda\sigma}F^{*\lambda\sigma}A^{\mu,\mu}(x) \\ &\qquad - it\,G^{\lambda\sigma}(x)F^{*\lambda\sigma}\Bigg] \\ &= \frac{1}{(4\pi)^2}\int dx\left[\frac{1}{m^2}F^{\lambda\sigma}F^{*\lambda\sigma}A^{\mu,\mu} + i(\ln m^2)G^{\lambda\sigma}F^{*\lambda\sigma}\right]. \qquad (3.25)\end{aligned}$$

Neither term in Eq. (3.25) would arise from the calculation of one-loop Feynman diagrams with plane wave external fields. For $F_{\mu\nu}$ being a constant field, the first term in Eq. (3.25) is a total derivative. When either F or G (or both) are non-constant the second term is also a total derivative.

3.4 The Two-dimensional Limit

Two dimensional models have long been a convenient testing ground for ideas in quantum field theory. The two dimensional limit of massive electrodynamics has been considered in refs. [22, 23]. In this case we have analytic final expressions for the effective action. If there is an axial coupling between the spinor and an external axial field, this leads to the one-loop effective action

$$\Gamma_2 = \ln\det(\not{p} - \not{A}\sigma^3 - m) \qquad (p \equiv -i\partial). \tag{3.26}$$

However, as $\gamma^\mu\sigma^3 = \epsilon^{\mu\nu}\gamma_\nu$, this becomes

$$\Gamma_2 = \ln\det(\not{p} - A_\mu\epsilon^{\mu\nu}\gamma_\nu - m). \tag{3.27}$$

Consequently, if the background field A_μ corresponds to a constant field strength $A_\mu = -\frac{1}{2}F_{\mu\nu}x^\nu = -\frac{f}{2}\epsilon_{\mu\nu}x^\nu$, then Eq. (3.27) reduces to

$$\Gamma_2 = \ln\det\left(\not{p} - \frac{f}{2}\not{x} - m\right) \tag{3.28}$$

which is what would be obtained if there were a parity conserving coupling with an external vector field $V_\mu = \frac{1}{4}f\partial_\mu(x^2)$ which corresponds to a pure gauge field. This effective action should thus be independent of f, which we will show explicitly by using Schwinger's technique [1].

If now

$$\Pi_\mu = p_\mu - \frac{f}{2}x_\mu \tag{3.29}$$

then Eq. (3.28) becomes

$$\Gamma_2 = \ln\det{}^{1/2}(\not{\Pi} + m)(\not{\Pi} - m) = \frac{1}{2}\ln\det(\Pi^2 - m^2) \tag{3.30}$$

upon using the two dimensional analogue of Eq. (3.8) and

$$[\Pi_\mu, \Pi_\nu] = 0. \tag{3.31}$$

Regulating Γ_2 using the ζ-function [24, 25] we have

$$\Gamma_2 = -\frac{1}{2}\frac{d}{ds}\bigg|_0 \frac{1}{\Gamma(s)}\mathrm{tr}\int_0^\infty d\,it(it)^{s-1}e^{i(m^2-\Pi^2)t}. \tag{3.32}$$

To evaluate the functional trace in Eq. (3.32), we use the Hamiltonian approach of ref. [1], defining

$$\langle x(t)|y(0)\rangle = \langle x|e^{-iHt}|y\rangle \tag{3.33}$$

with

$$H = -\Pi^2\,. \tag{3.34}$$

The equations

$$i\frac{\partial \Pi^{\mu}(t)}{\partial t} = [\Pi^{\mu}(t), H] \quad (3.35a)$$
$$i\frac{\partial x^{\mu}}{\partial t} = [x^{\mu}(t), H] \quad (3.35b)$$

can be integrated to give

$$\Pi^{\mu}(t) = \Pi^{\mu}(0) \quad (3.36a)$$
$$x^{\mu}(t) = -2\Pi_{\mu}(0). \quad (3.36b)$$

Since Eq. (3.36) is identical to the equations that arise if $f = 0$, we see that the effective action in two dimensions for a spinor in the presence of a constant background axial field is just that of a free field.

3.5 Conclusions

We thus see that the one-loop effective action for a spinor in the presence of a constant background chiral field is closely related to that of considered in refs. [1, 6–8] provided $m^2 = 0$. The case in which $m^2 \neq 0$ in four dimensions has not as yet been given in closed form. Higher order calculations in both the number of loops and also in the number of background axial fields, or those involving non-constant background fields are currently being considered, as is that all-orders approach in the presence of a weak background field [26, 27].

We note the use of projection operators in conjunction with background gauge fields in ref. [28].

3.6 Appendix. Conventions for Dirac Matrices and Projectors.

In four dimensional Euclidean space we have the conventions

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2\delta^{\mu\nu}, \qquad [\gamma^{\mu}, \gamma^{\nu}] = 2i\sigma^{\mu\nu}$$

$$\left[\sigma^{\mu\nu}, \sigma^{\lambda\sigma}\right] = 2i\left(\delta^{\mu\lambda}\sigma^{\nu\sigma} - \delta^{\mu\sigma}\sigma^{\nu\lambda} + \delta^{\nu\sigma}\sigma^{\mu\lambda} - \delta^{\nu\lambda}\sigma^{\mu\sigma}\right)$$

$$\left\{\sigma^{\mu\nu}, \sigma^{\lambda\sigma}\right\} = 2\left(\delta^{\mu\lambda}\delta^{\nu\sigma} - \delta^{\mu\sigma}\delta^{\nu\lambda}\right) - 2\epsilon^{\mu\nu\lambda\sigma}\gamma^5$$

$$\gamma^\alpha\gamma^\beta\gamma^\lambda = \delta^{\alpha\beta}\gamma^\lambda - \delta^{\alpha\lambda}\gamma^\beta + \delta^{\beta\lambda}\gamma^\alpha - \epsilon^{\alpha\beta\lambda\rho}\gamma^\rho\gamma^5$$

$$\epsilon^{1234} = 1, \qquad \gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4, \qquad \mathrm{tr}\,\gamma^5 = 0$$

$$\sigma^{\mu\nu}\gamma^5 = \epsilon^{\mu\nu\lambda\sigma}\sigma^{\lambda\sigma}.$$

$$P_\pm = \frac{1 \pm \gamma^5}{2} \quad , (P_\pm)^2 = P_\pm \quad , \quad P_\pm P_\mp = 0$$

$$P_\pm\gamma^\mu = \gamma^\mu P_\mp \quad , \quad P_\pm\gamma^5 = \gamma^5 P_\pm.$$

These show that if

$$e^{\lambda w^{\mu\nu}\sigma^{\mu\nu}} = (A_+(\lambda)P_+ + A_-(\lambda)P_-) + (B_+(\lambda)P_+ + B_-(\lambda)P_-)w^{\mu\nu}\sigma^{\mu\nu} \tag{3.37}$$

then the differential equation

$$\frac{d}{d\lambda}e^{\lambda w^{\mu\nu}\sigma^{\mu\nu}} = w^{\mu\nu}\sigma^{\mu\nu}e^{\lambda\sigma^{\mu\nu}w^{\mu\nu}}$$

leads to

$$\dot{A}_\pm = K_\mp^2 B_\pm \ , \qquad \dot{B}_\pm = A_\pm \qquad (A_\pm(0) = 1,\ B_\pm(0) = 0)$$

where $K_\pm^2 = 2(w^{\mu\nu}w^{\mu\nu} \pm w^{\mu\nu}w^{*\mu\nu})$ and $w^{*\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}w^{\lambda\sigma}$. These have the solution when $\lambda = 1$

$$A_\pm = \cosh K_\mp \qquad B_\pm = \frac{\sinh K_\mp}{K_\mp}. \tag{3.38}$$

The "charge conjugation" matrix C satisfies $C^{-1}\gamma^\mu C = -\gamma^{\mu T}, C^{-1}\gamma^5 C = \gamma^{5T}$.

In two dimensional Minkowski space, we take

$$g^{00} = 1 = -g^{11} \quad \text{and} \quad \gamma^0 = \sigma^1\ , \gamma^1 = i\sigma^2 \quad \text{so that}$$
$$\text{if} \quad \epsilon^{01} = 1 = \epsilon_{10},\ \text{then} \quad \gamma^\mu\gamma^\nu = g^{\mu\nu} - \epsilon^{\mu\nu}\sigma^3 \ \text{ and } \ \gamma^\mu\sigma^3 = \epsilon^{\mu\nu}\gamma_\nu$$

(where σ^i is a Pauli spin matrix).

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Chapter 4

On Determining the Running Coupling from the Effective Action

4.1 Introduction

It has been long known that the introduction of a renormalization scale μ leads to a conformal anomaly. More explicitly, the trace of the energy-momentum tensor is no longer zero but rather is proportional to the renormalization group β-function [1]. From this result, one can show that the effective action for a gauge theory can be written in terms of the running gauge coupling when considered as a function of a strong background field [2]. At the same time, the effective action satisfies the renormalization group equation, which leads to explicit summation of all its leading-log (LL), next-to-leading-log (NLL) etc. contributions [3]. In this chapter we exploit these two different expressions for the effective action to obtain a novel expression for the running gauge coupling. This appears in eq. (4.9) below and the bulk of this chapter deals with the sum appearing in the denominator on the right side of this equation. We relate this new expansion to one previously derived by systematically solving the usual differential equation for the running coupling.

4.2 The Running Coupling and the Effective Action

If the effective Lagrangian L is treated as a function of μ (the renormalization scale), $F_{\mu\nu}$ (the constant background field strength) and λ (the gauge coupling), then we have the renormalization group equation:

$$\mu\frac{dL}{d\mu}=\left(\mu\frac{\partial}{\partial\mu}+\beta(\lambda)\frac{\partial}{\partial\lambda}+\gamma(\lambda)F_{\mu\nu}\frac{\partial}{\partial F_{\mu\nu}}\right)L(\lambda,F_{\mu,\nu},\mu)=0. \tag{4.1}$$

Since $\lambda F_{\mu\nu}$ is not renormalized [4] it follows that $\beta(\lambda)=-\lambda\gamma(\lambda)$ and equation (4.1) becomes

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(\lambda)\left(\frac{\partial}{\partial\lambda}-\frac{2}{\lambda}\Phi\frac{\partial}{\partial\Phi}\right)\right]L=0, \tag{4.2}$$

where $\Phi=F_{\mu\nu}F^{\mu\nu}$.

For strong background fields (*i.e.*, $\lambda\Phi\gg\mu^2$)

$$L=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}T_{n,m}\lambda^{2n}t^m\Phi \tag{4.3}$$

where $t=\frac{1}{4}\ln\left(\frac{\lambda^2\Phi}{\mu^4}\right)$ [5]. If $S_n(\lambda^2t)=\sum_{m=0}^{\infty}T_{n+m,m}(\lambda^2t)^m$ ($n=0$ is LL, $n=1$ is NLL etc.), then eq. (4.2) leads to the nested equations ($n=0,1,2\ldots$)

$$-\frac{d}{d\xi}S_n(\xi)+2\sum_{\rho=0}^{n}b_{2\rho+3}\left[\xi\frac{d}{d\xi}+(n-\rho-1)\right]S_{n-\rho}=0 \tag{4.4}$$

where $\beta(\lambda)=\sum_{\rho=0}^{\infty}b_{2\rho+3}\lambda^{2\rho+3}$ and $\xi=\lambda^2t$. The boundary condition for these equations is $S_n(\xi=0)=T_{n,0}$. Solutions for $n=0,1,2$ are respectively given by

$$S_0=-T_{0,0}w \tag{4.5a}$$

$$S_1=\frac{T_{0,0}b_5}{b_3}\ln|w|+T_{1,0} \tag{4.5b}$$

$$S_2=-\frac{T_{2,0}}{w}+\frac{b_7}{b_3}T_{0,0}\left(\frac{1+w}{w}\right)-\left(\frac{b_5}{b_3}\right)^2T_{0,0}\left(\frac{\ln|w|+(1+w)}{w}\right) \tag{4.5c}$$

where $w=-1+2b_3\xi$. (Eq. (4.5) corrects errors in ref. [3].) For the solutions of eq. (4.4) for $S_n(n=3\ldots6)$ see the appendix.

An alternate expression for the effective action that follows from the conformal anomaly is [2]

$$L=-\frac{1}{4}\frac{\lambda_0^2}{\bar{\lambda}^2(t)}\Phi \tag{4.6}$$

where the running coupling $\bar{\lambda}(t)$ satisfies

$$\frac{d\bar{\lambda}(t)}{dt}=\beta(\bar{\lambda}(t))\qquad(\bar{\lambda}(t=0)=\lambda_0) \tag{4.7}$$

Eq. (4.6) satisfies (4.1) provided $\mu = \mu_0$ is fixed. In ref. [3] it is shown that eqs. (4.3) and (4.6) are consistent provided

$$T_{n,0} = -\frac{1}{4}\delta_{n,0}. \tag{4.8}$$

Furthermore, these two equations show that

$$\bar{\lambda}^2(t) = \frac{-\lambda_0^2}{4}\left[\sum_{n=0}^{\infty} S_n(\lambda_0^2 t)\lambda_0^{2n}\right]^{-1}. \tag{4.9}$$

More explicitly, from eqs. ((4.5),(4.8),(4.9)) it follows that

$$\begin{aligned}\bar{\lambda}^2(t) = \lambda_0^2\Bigg[&(1-2b_3\lambda_0^2) + \lambda_0^2\left(\frac{b_5}{b_3}\ln|{-1}+2b_3\lambda_0^2 t|\right) \\ &+ \lambda_0^4\left(\frac{b_7}{b_3}\frac{2b_3\lambda_0^2 t}{-1+2b_3\lambda_0^2 t} - \left(\frac{b_5}{b_3}\right)^2\frac{\ln|{-1}+2b_3\lambda_0^2 t| + 2b_3\lambda_0^2 t}{-1+2b_3\lambda_0^2 t}\right) + \dots\Bigg]^{-1}\end{aligned} \tag{4.10}$$

This rather unusual expression for $\bar{\lambda}^2(t)$ can be composed with what can be obtained directly from eq. (4.7). For a lowest order solution, from

$$\frac{d\bar{\lambda}^2(t)}{dt} = b_3\bar{\lambda}^3(t) \tag{4.11a}$$

we easily find that

$$\bar{\lambda}^2(t) = \frac{\lambda_0^2}{1-2b_3\lambda_0^2 t} \tag{4.11b}$$

while if we go the next order

$$\frac{d\bar{\lambda}(t)}{dt} = b_3\bar{\lambda}^3(t) + b_5\bar{\lambda}^5(t) \tag{4.12a}$$

it follows that

$$\frac{d\bar{\lambda}^2}{\bar{\lambda}^2\left[b_3 + b_5(\bar{\lambda}^2)^2\right]} = 2dt$$

which, when integrated, yields $\bar{\lambda}(t)$ in terms of a Lambert $W-$ function [6]. Eq. (4.11b) is identical to the lowest order contribution to eq. (4.10), while eq. (4.7) yields no closed form expression when b_3, b_5 are non-zero.

However, eq. (4.10) can be related to what is obtained from a perturbative solution to eq (4.7) which is found in the following systematic way. We begin by letting $x = \bar{\lambda}^2$ and $2b_{2\rho+3} = \beta_\rho(\rho = 0, 1, 2\dots)$ so that eq. (4.7) becomes [7]

$$\frac{dx}{dt} = x^2(\beta_0 + \beta_1 x + \beta_2 x^2 + \dots) \tag{4.13}$$

If we now rescale $t \to t/\epsilon, x \to \epsilon x$, then make the expansion $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots$ $(x_n(t=0) = x\delta_{n,0})$ we find that at successive orders in ϵ,

$$\frac{dx_0}{dt} = \beta_0 x_0^2 \tag{4.14a}$$

$$\frac{dx_1}{dt} = \beta_0 x_0^2 + 2\beta_1 x_0 x_1 \tag{4.14b}$$

$$\frac{dx_2}{dt} = \beta_0(x_1^2 + 2x_0x_2) + 3\beta_1 x_1 x_0^2 + \beta_4 x_0^4 \tag{4.14c}$$

Solving these equations in turn leads to

$$x_0 = \frac{x}{1 - \beta_0 x t} \tag{4.15a}$$

$$x_1 = -x^2 \frac{\beta_1}{\beta_0} \frac{\ln|1 - \beta_0 x t|}{(1 - \beta_0 x t)^2} \tag{4.15b}$$

etc.

The solutions for $x_n (n = 2 \ldots 5)$ are given in the appendix.

An alternate approach is to systematically solving eq. (4.7) is to write (in analogy with eq. (4.3) [8])

$$x(\mu_0) = x(\mu) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \tau_{n,m} x^n(\mu) \ln^m \left(\mu^2/\mu_0^2\right) \tag{4.16a}$$

$$\equiv \sum_{n=0}^{\infty} \sigma_n(\zeta) x^{n+1}(\mu) \qquad (\sigma_n(0) = \delta_{n0}) \tag{4.16b}$$

where $\zeta = x(\mu) \ln \left(\mu^2/\mu_0^2\right)$. If now $\beta(x) = x^2 \sum_{n=0}^{\infty} \beta_n x^n$ and

$$\mu^2 \frac{d}{d\mu^2} x(\mu_0) = 0 \tag{4.17a}$$

$$\mu^2 \frac{d}{d\mu^2} x(\mu) = \beta\left(x(\mu)\right) \tag{4.17b}$$

then we see that

$$(1 + \beta_0 \zeta)\sigma_0' = -\beta_0 \sigma_0 \tag{4.18a}$$

$$(1 + \beta_0 \zeta)\sigma_1' + 2\beta_0 \sigma_1 = (-\beta_1 \sigma_0 - \beta_1 \zeta \sigma_0') \tag{4.18b}$$

$$(1 + \beta_0 \zeta)\sigma_2' + 3\beta_0 \sigma_2 = (-\beta_2 \sigma_0 - \beta_2 \zeta \sigma_0') + (-2\beta_1 \sigma_1 - \beta_1 \zeta \sigma_1') \tag{4.18c}$$

These equations have the solutions

$$\sigma_0 = (1 + \beta_0 \zeta)^{-1} \tag{4.19a}$$

$$\sigma_1 = -\left(\frac{\beta_1}{\beta_0}\right)^2 \frac{\ln|1 + \beta_0 \zeta|}{(1 + \beta_0 \zeta)^2} \tag{4.19b}$$

$$\sigma_2 = \left(\left(\frac{\beta_1}{\beta_0}\right)^2 - \frac{\beta_2}{\beta_0}\right)\left(\frac{1}{(1+\beta_0\zeta)^2} - \frac{1}{(1+\beta_0\zeta)^3}\right) \tag{4.19c}$$
$$- \left(\frac{\beta_1}{\beta_0}\right)^2 \frac{1}{(1+\beta_0\zeta)^3}\left(\ln|1+\beta_0\zeta| - \ln^2|1+\beta_0\zeta|\right)$$

etc.

These solutions to eq. (4.18) are seen to be equivalent to those of eq. (4.14).

With the solution to eq. (4.7) given by eq. (4.15) (or alternatively eq. (4.19)), we find that this is equivalent to the expression for the running coupling given by eq. (4.9) where the running coupling appearing in eq. (4.9) is expanded in powers of λ_0^2. This holds true to the order that we have computed (λ_0^{12}) and we anticipate that it would be true to all orders in λ_0^2. Eq. (4.9) is unusual in that the dependence of $\bar{\lambda}^2(t)$ on t is exclusively in the denominator.

The sums $\sum_{n=0}^{\infty} S_n(\lambda^2 t)\lambda^{2n}\Phi$ and $\sum_{n=0}^{\infty}\sigma_n(\zeta)x^{n+1}$ in eqs. (4.3) and (4.16a),(4.16b) represent leading-log (LL) contributions (for $n=0$), next-to-leading-log (NLL) contributions (for $n=1$) and, in general, N^pLL contribution (for $n=p$) for L and $\bar{\lambda}^2$ respectively. It proves possible to use the renormalization group equation to perform parts of these sums, as was done in ref. [9] when considering the effective potential.

We illustrate this by first considering $\sigma_n(\zeta)$. From eqs. (4.16b) and (4.17a),(4.17b) we find that

$$\left[(1+\beta_0\zeta)\frac{d}{d\zeta} + (n+1)\beta_0\right]\sigma_n + \sum_{\rho=1}^{n}\beta_\rho\left[\zeta\frac{d}{d\zeta} + (n+1-\rho)\right]\sigma_{n-\rho} = 0 \tag{4.20}$$

(This generalizes eq. (4.18a),(4.18b),(4.18c).) The general form of $\sigma_n(\zeta)$ that follows from eq. (4.20) is

$$\sigma_n = \sum_{i=0}^{n}\sum_{j=0}^{i}\sigma_{i,j}^n \frac{L^j}{U^{i+1}} \tag{4.21}$$

where $U = 1+\beta_0\zeta$ and $L = \ln U$. Substitution of eq. (4.21) into eq. (4.20) leads to the recursion relation

$$\beta_0\left[(j+1)\sigma_{i,j+1}^n + (n+1-i)\sigma_{i,j}^n\right] + \sum_{\rho=1}^{n}\beta_\rho\Big[-(j+1)\sigma_{i-1,j+1}^{n-\rho} + (i-1)\sigma_{i-1,j}^{n-\rho} \tag{4.22}$$
$$+(j+1)\sigma_{i,j+1}^{n-\rho} - i\sigma_{i,j}^{n-\rho} + (n+1-\rho)\sigma_{i,j}^{n-\rho}\Big] = 0.$$

If in eq. (4.22) we set $i=n+1$, then

$$\sigma_{n+1,j+1}^n = \rho_1\left[\frac{n}{j+1}\sigma_{n,j}^{n-1} - \sigma_{n,j+1}^{n-1}\right] \tag{4.23}$$

where $\rho_n = -\beta_n/\beta_0$. If in eq. (4.23), we set $j = n-1$, then

$$\sigma_{n+1,n} = \rho_1 \sigma^{n-1}_{n,n-1} = (\rho_1)^n \sigma^0_{10} = (\rho_1)^n \tag{4.24}$$

as by eq. (4.19a), $\sigma^0_{10} = 1$. Restricting σ^n_{ij} in eq. (4.21) to $\sigma^n_{n,n+1}$, we find from eq. (4.16b) that

$$\begin{aligned} x(\mu_0) &= \sum_{n=0}^{\infty} \rho_1^n \frac{L^n}{U^{n+1}} x^n(\mu) \\ &= \frac{x(\mu)}{U - \rho_1 L x(\mu)} \end{aligned} \tag{4.25}$$

or, more explicitly (reversing the roles of μ and μ_0)

$$x(\mu) = \frac{x(\mu_0)}{1 - \beta_0 \ln\left(\frac{\mu^2}{\mu_0^2}\right) + \frac{\beta_1}{\beta_0} \ln\left(1 - \beta_0 \ln\left(\frac{\mu^2}{\mu_0^2}\right)\right) x(\mu_0)} \tag{4.26}$$

which is consistent with eq. (4.10).

If $j = n-2$ in eq. (4.23), an explicit expression for $\sigma^n_{n+1,n-1}$ can be found following the approach of ref. [5]; this further modifies the expression for $x(\mu)$ in eq. (4.26).

In a similar fashion, one can use eq. (4.4) to see that

$$S_n(\xi) = \sum_{i=0}^{n} \sum_{j=0}^{i} S^n_{ij} \frac{L^j}{w^{i-1}}; \tag{4.27}$$

in analogy with eq. (4.22) we find that

$$\begin{aligned} (j+1)S^n_{i,j+1} + (n-i)S^n_{ij} + \sum_{\rho=1}^{n-1} \chi_{2\rho+3} \Big[&(j+1)S^{n-\rho}_{i-1,j+1} - (i-2)S^{n-\rho}_{i-1,j} \\ &+ (j+1)S^{n-\rho}_{i,j+1} + (n-\rho-i)S^{n-\rho}_{ij} \Big] = 0, \end{aligned} \tag{4.28}$$

where $\chi_{2\rho+3} = b_{2\rho+3}/b_3$ $(\rho = 1, 2 \ldots)$. For $i = n$ and $j = n-1$, eq. (4.28) reduces to

$$S^n_{n,n} - \chi_5 \frac{(n-2)}{n} S^{n-1}_{n-1,n-1} = 0. \tag{4.29}$$

As $S^0_{0,0} = \frac{1}{4}$ (by eqs. (4.5a),(4.8)), we see by eq. (4.29) that $S^1_{1,1} = -\chi_5/4$, $S^n_{n,n} = 0$ $(n \geq 2)$. If we only consider the contributions to S_n coming from $S^n_{n,n}$, it follows from eq. (4.9) that

$$\bar{\lambda}^2(t) = -\frac{\lambda_0^2}{4} \left[\frac{1}{4} w - \frac{\chi_5}{4} (\ln w) \lambda_0^2 \right]^{-1} \tag{4.30}$$

which is identical to eq. (4.26) upon using the relation between x and λ, as well as β_i and χ_i.

Further results that follow from eq. (4.28) are

$$S^2_{2,0} = -\frac{1}{4}(\chi_7 - \chi_5^2), \tag{4.31a}$$

$$S^3_{3,1} = -\frac{\chi_5\chi_7}{4} \tag{4.31b}$$

$$S^n_{n,n-2} = -\frac{\chi_5^{n-2}\chi_7}{4} - \frac{\chi_5^n}{4}\left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-2}\right) \quad (n \geq 4) \tag{4.31c}$$

$$S^n_{n-1,n-1} = 0 \quad (n \geq 1) \tag{4.32}$$

$$S^2_{1,0} = \frac{\chi_5^2}{4} - \frac{\chi_7}{4}, \tag{4.33a}$$

$$S^3_{2,1} = 0 \tag{4.33b}$$

$$S^n_{n-1,n-3} = \frac{1}{4}\left(\chi_7\chi_5^{n-2} - \chi_5^n\right) \quad (n \geq 3). \tag{4.34}$$

These contributions to L in eq. (4.3) can now be easily summed. (For the contribution of eq. (4.31c) see the appendix.) The final result for L/Φ coming from eqs. (4.31),(4.32),(4.33),(4.34) is the following

$$\begin{aligned} L/\Phi = \frac{1}{4}\Bigg[& w - \chi_5 \ln w\lambda^2 + (\chi_5^2 - \chi_7)\left(\frac{1+w}{w}\right)\lambda^4 \\ & -\frac{\lambda^4}{w}\frac{1}{1-\lambda^2\ln w/w}\left(\lambda^2 \ln w/w - \ln\left(1 - \lambda^2 \ln w/w\right)\right) \\ & +\frac{\lambda^6}{w}\left(\chi_7\chi_5 - \chi_5^3\right)\left(1 - \lambda^2\chi_5 \ln w/w\right)^{-1}\Bigg] \end{aligned} \tag{4.35}$$

where the last two terms are due to the contributions from all N^pLL.

4.3 Discussion

By exploiting the conformal anomaly, the effective action for a constant external gauge field can be expressed in terms of the running coupling. We have used this result to find an alternative expression for the running coupling that is perturbatively equivalent to the usual solutions to eq. (4.7).

We have also shown how portions of all N^pLL contributions to the running coupling can be summed.

4.4 Appendix

Let us obtain the expression for the effective Lagrangian in eq. (4.6) from confromal anomaly using the approach of ref. [10]. The trace anomaly for the energy-momentum tensor is the following

$$\left\langle \Theta^{\mu}_{\mu} \right\rangle = \frac{\beta(\bar{\lambda}(t))}{2\bar{\lambda}(t)} \frac{\lambda_0^2}{\bar{\lambda}(t)^2}. \tag{4.36}$$

Also, using the effective Lagrangian for a constant background field strength one can find

$$\langle \Theta^{\mu\nu} \rangle = -\eta^{\mu\nu} L + 2\frac{\partial L}{\partial \eta_{\mu\nu}} \tag{4.37}$$

Thus we can see that eq. (4.6) is valid.

The solutions for $x_n (n = 2 \ldots 5)$ are as follows:

$$x_2 = \frac{1}{\beta_0^2 w^3}\left[x^3\left(\beta_1^2\left(w - \ln^2 w + \ln(w) + 1\right) - \beta_0\beta_2(w+1)\right)\right] \tag{4.38a}$$

$$x_3 = -\frac{1}{2\beta_0^3 w^4}x^4\left[\beta_0^2\beta_3\left(w^2-1\right) + \beta_1^3\left((w+1)^2 + 2\ln^3 w - 5\ln^2 w - 4(w+1)\ln w\right)\right.$$
$$\tag{4.38b}$$
$$\left. - 2\beta_0\beta_2\beta_1(w(w+1) - (2w+3)\ln(w))\right]$$

$$x_4 = \frac{1}{6\beta_0^4 w^5}x^5\left[-2\beta_0^2\left(\beta_0\beta_4\left(w^3+1\right) - \beta_2^2(w-5)(w+1)^2\right)\right. \tag{4.38c}$$
$$-6\beta_0\beta_2\beta_1^2\left(-\left(2w^2+5w+3\right)\ln w + (w-3)(w+1)^2 + 3(w+2)\ln^2 w\right)$$
$$+\beta_1^4\left(-6\left(w^2+5w+4\right)\ln w + (w+1)^2(2w-7) - 6\ln^4 w + 26\ln^3 w + 9(2w+1)\ln^2 w\right)$$
$$\left.+\beta_0^2\beta_3\beta_1\left(4w^3+3w^2-6\left(w^2-2\right)\ln w + 1\right)\right]$$

$$x_5 = -\frac{1}{12\beta_0^5 w^6}x^6\left[\beta_1^5\left(6\left(3w^2+26w+23\right)\ln^2 w + (w+1)^3(3w-17) + 12\ln^5 w - 77\ln^4 w\right.\right.$$
$$\tag{4.38d}$$
$$\left.+(22-48w)\ln^3 w - 2(w+1)^2(4w-11)\ln w\right) + 3\beta_0^3\left(\beta_0\beta_5\left(w^4-1\right) - 2\beta_2\beta_3\left(-w^2+w+2\right)^2\right)$$
$$+\beta_0^2\beta_3\beta_1^2\left(\left(9w^2-22w+23\right)(w+1)^2 + 6\left(3w^2-10\right)\ln^2 w - 2\left(8w^3+15w^2-7\right)\ln w\right)$$
$$-6\beta_0\beta_2\beta_1^3\left((w+1)^2\left(2w^2-8w-3\right) + \left(6w^2+26w+27\right)\ln^2 w + \left(-4w^3+2w^2+30w+24\right)\ln w\right.$$
$$\left.-4(2w+5)\ln^3 w\right) + \beta_0^2\beta_1\left(2\beta_0\beta_4\left(-3w^4-2w^3+2\left(2w^3+5\right)\ln w + 1\right)\right.$$
$$\left.\left.+\beta_2^2(w+1)\left(9w^3-29w^2+\left(-8w^2+44w+100\right)\ln w - 37w + 1\right)\right)\right]$$

The solutions for $S_n (n = 3 \ldots 6)$ are as follows:

$$S_3 = -\frac{1}{8w^2}\left[\chi_9\left(w^2-1\right) - 2\chi_7\chi_5\left(w^2+w-\ln(w)\right) + \chi_5^3\left((w+1)^2 - \ln^2 w\right)\right] \quad (4.39a)$$

$$S_4 = \frac{1}{24w^3}\Big[-2\chi_{11}\left(w^3+1\right) + \chi_9\chi_5\left(4w^3+3w^2+6\ln(w)+1\right) + 2\chi_7^2(w-2)(w+1)^2 \quad (4.39b)$$
$$-6\chi_7\chi_5^2\left((w-1)(w+1)^2 + \ln^2 w - (w+1)\ln(w)\right)$$
$$+\chi_5^4\left((w+1)^2(2w-1) + 2\ln^3 w - 3\ln^2 w - 6(w+1)\ln(w)\right)\Big]$$

$$S_5 = -\frac{1}{48w^4}\Big[\chi_{13}\Big(-6\chi_7\chi_5^3\Big((w+1)^2\left(2w^2-2w-1\right) + \left(-2w^2+2w+4\right)\ln(w) - 2\ln^3 w \quad (4.39c)$$
$$+(2w+5)\ln^2 w\Big) - 3\left(2\chi_7\chi_9\left(w^4-w^2+2w+2\right) - \chi_{13}\left(w^4-1\right)\right)$$
$$+\chi_9\chi_5^2\left(9w^4+8w^3-6\left(w^2-1\right)\ln(w)+12w-18\ln^2 w+11\right)$$
$$+\chi_5\left(2\chi_{11}\left(-3w^4-2w^3+6\ln(w)+1\right) + \chi_7^2(w+1)\left(9w^3-5w^2-13w+24\ln(w)+1\right)\right)$$
$$+\chi_5^5\left((w+1)^3(3w-5) - 3\ln^4 w + 10\ln^3 w + 12(w+1)\ln^2(w) - 6(w+1)^2\ln(w)\right)\Big)\Big]$$

$$S_6 = \frac{1}{240w^5}\Big[-10\chi_7\chi_5^4\Big((w+1)^3\left(6w^2-12w+7\right) + \left(6w^2-3w-9\right)\ln^2 w + 6\ln^4 w \quad (4.39d)$$
$$-2(3w+13)\ln^3 w - 6(w-4)(w+1)^2\ln(w)\Big) + \chi_5^6\Big(3(w+1)^3\left(4w^2-7w-1\right)$$
$$+30\left(w^2+5w+4\right)\ln^2 w + 12\ln^5 w - 65\ln^4 w - 30(2w+1)\ln^3 w$$
$$-10(w+1)^2(2w-7)\ln(w)\Big) + \chi_9\chi_5^3\Big(30\left(w^2-5\right)\ln^2 w + 3(w+1)^2\left(16w^3-17w^2+8w+1\right)$$
$$-10\left(4w^3+3w^2+18w+19\right)\ln(w) + 120\ln^3 w\Big)$$
$$-2\Big(6\chi_{15}\left(w^5+1\right) + 2\chi_7^3(w+1)^3\left(3w^2-9w+13\right) + 2\chi_{11}\chi_7\left(-6w^5+5w^3+15w+14\right)$$
$$-3\chi_9^2\left(2w^5+5w^2-3\right)\Big) + \chi_5^2\Bigg(\chi_7^2\Big(3\left(24w^3-33w^2+2w+39\right)(w+1)^2$$
$$-20\left(w^3-9w^2-15w-5\right)\ln(w) - 60(3w+4)\ln^2 w\Big) + 2\chi_{11}\Big(10\left(w^3+1\right)\ln(w)$$
$$+3\left(-6w^5-5w^4+10w+9\right) - 60\ln^2(w)\Big)\Bigg) + \chi_5\Big(3\chi_{13}\left(8w^5+5w^4+20\ln(w)+3\right)$$
$$-2\chi_7\chi_9(w+1)\left(36w^4-21w^3-14w^2+29w+30(w-4)\ln(w)-14\right)\Big)\Big]$$

We also employ, in evaluating the contributions of eq. (4.31c) to L, the result

$$\begin{aligned}\sum_{n=4}^{\infty} x^n \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n-2}\right) &= \frac{1}{2}(x^4 + x^5 + x^6 + \ldots) + \frac{1}{3}(x^5 + x^6 + \ldots) \\ &= \frac{1}{2}\frac{x^4}{1-x} + \frac{1}{3}\frac{x^5}{1-x} + \ldots \\ &= \frac{x^2}{1-x}(-x - \ln(1-x)). \end{aligned} \tag{4.40a}$$

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Chapter 5

Can the correlated stability conjecture be saved?

5.1 Generalized correlated stability conjecture

A standard claim in classical thermodynamics[1] is that a system is thermodynamically stable if the Hessian $\mathbb{H}^{\mathcal{E}}_{s,Q_A}$ of the energy density $\mathcal{E} = \mathcal{E}(s, Q_A)$ with respect to the entropy density s and charges $Q_A \equiv \{Q_1, \cdots Q_n\}$, *i.e.*,

$$\mathbb{H}^{\mathcal{E}}_{s,Q_A} \equiv \begin{pmatrix} \frac{\partial^2 \mathcal{E}}{\partial s^2} & \frac{\partial^2 \mathcal{E}}{\partial s \partial Q_B} \\ \frac{\partial^2 \mathcal{E}}{\partial Q_A \partial s} & \frac{\partial^2 \mathcal{E}}{\partial Q_A \partial Q_B} \end{pmatrix}, \tag{5.1}$$

does not have negative eigenvalues. In the simplest case $n = 0$, *i.e.*, no conserved charges, the thermodynamic stability implies that

$$0 \; < \; \frac{\partial^2 \mathcal{E}}{\partial s^2} = \frac{T}{c_v}, \tag{5.2}$$

that is the specific heat c_v is positive. In the context of gauge theory/string theory correspondence [7] black holes with translationary invariant horizons in asymptotically anti-de-Sitter space-time are dual (equivalent) to equilibrium thermal states of certain strongly coupled systems. Thus, the above thermodynamic stability criterion should be directly applicable to black branes as well. The correlated stability conjecture (CSC) asserts that it is only when the Hessian (5.1) for a given black brane geometry is positive, the spectrum of on-shell excitations in this background geometry is free from tachyons [1, 26].

[1] Assuming that the temperature is positive.

In the simplest case, *i.e.,* the absence of the chemical potentials, one can trivially identify the classical instabilities of the thermodynamically unstable system [3]. Indeed, since the speed of sound waves squared in this case is $c_s^2 = \frac{s}{c_v}$, the thermodynamic instability of the system ($c_v < 0$) immediately implies that the hydrodynamic (sound) modes are classically unstable.[2] There is no simple argument implying that thermodynamic stability of the system is enough to secure its classical stability; moreover, the instability link with the sound waves does not work in strongly coupled R-charged $\mathcal{N} = 4$ supersymmetric Yang-Mills plasma [6] — here, the speed of sound is always fixed to a conformal value $c_s^2 = \frac{1}{3}$ even though there is an equilibrium branch with $c_v < 0$.

In [5, 6] it has demonstrated that, at least for a canonical interpretation of the black brane thermodynamics, the CSC is violated in the case of black branes with scalar hair that undergo a continuous phase transition. The dual gauge theory picture makes such violation almost self-evident. Indeed, in the vicinity of a continuous phase transition the condensate does not noticeably modify the thermodynamics, and thus should not affect the thermodynamic stability of the system. On the other hand, the phase of the system with the higher free energy is expected to be classically unstable. The condensation of the tachyon should bring the system to the equilibrium phase with the lowest free energy.

The important qualifier for the above counter-examples is the *canonical interpretation* of the corresponding black brane thermodynamics. Specifically, the black branes considered have scalar hair and in the proper boundary (field theoretic) thermodynamic interpretation one has to keep non-normalizable coefficients of the scalars fixed. The reason for this is that these non-normalizable coefficients are dual to mass-scales in the boundary field theory. In thermodynamic stability analysis one naturally would like to keep microscopic mass scales in the field theory fixed. If one abandons the gauge/gravity analogy and considers black branes as thermal systems in higher dimensional general relativity, the motivation for keeping the asymptotic scalar hair parameters fixed is removed. It is an interesting question as to whether these parameters might be treated as generalized charges in the context of thermodynamic stability of translationary invariant horizons in such a way that the CSC is validated[3]. We argue here that CSC generalizations of these type are false.

In the next section we present a simple statistical model in which the generalized thermodynamic and the dynamical (in)stabilities are not correlated. In section 3 we

[2] So, the *generalized* correlated stability conjecture demands identification of the conserved charges in the system leading to the appearance of Q_A in eq. (5.1).

[3] We would like to thank Barak Kol for raising this possibility.

show that the exotic hairy black branes discussed in [6, 9, 10] while classically unstable, are thermodynamically stable in the generalized manner outlined above. In both cases the classical instabilities we identify are long-wavelength, provided, in the statistical model in section 2, $\Lambda \ll T$, and for exotic hairy black branes one stays close to the phase transition.

5.2 Counter-example to generalized CSC in statistical physics

Consider a Landau-Ginsburg model with the following free energy density functional

$$\mathcal{F} = -T^4 + \Lambda^4 + \frac{1}{2}\left(\vec{\nabla}\phi(\vec{x})\right)^2 - \frac{1}{2}\Lambda^2\phi(\vec{x})^2 + \frac{1}{4}\phi(\vec{x})^4\,, \tag{5.3}$$

where Λ is a mass-scale, and $\phi(\vec{x})$ is a dynamical scalar field. For any temperature T, there are three equilibrium states of the system: one unstable $_u$ and two degenerate stable ones $_s$ ($_u$ and $_s$ are subscripts),

$$\begin{aligned}
\langle\phi(\vec{x})\rangle\Big|_{unstable} &= 0\,, \qquad \Rightarrow \qquad \mathcal{F}_u = -T^4 + \Lambda^4\,, \\
\langle\phi(\vec{x})\rangle^{\pm}\Big|_{stable} &= \pm\Lambda\,, \qquad \Rightarrow \qquad \mathcal{F}_s^{\pm} = -T^4 + \frac{3}{4}\Lambda^4\,.
\end{aligned} \tag{5.4}$$

In what follows we focus on the unstable equilibrium. Here, the energy density $\mathcal{E}_u$ is given by

$$\mathcal{E}_u = \frac{3}{2^{8/3}}\ s^{4/3} + \Lambda^4\,, \tag{5.5}$$

where s is the entropy density. It is straightforward to see that whether or not we treat the scale Λ as a generalized charge Q_A in the context of the thermodynamic stability (see (5.1)), this classically unstable equilibrium is thermodynamically stable. In other words, both Hessians $\mathbb{H}_s^{\mathcal{E}_u}$ and $\mathbb{H}_{s,\Lambda}^{\mathcal{E}_u}$ are positive.

Notice that since the energy $\mathcal{E}_s$ of the stable equilibrium is

$$\mathcal{E}_s = \frac{3}{2^{8/3}}\ s^{4/3} + \frac{3}{4}\Lambda^4\,, \tag{5.6}$$

any other definition of the generalized charge $Q_A = f(\Lambda)$ would imply that the two equilibria $_u$ and $_s$ are simultaneously either thermodynamically stable or not[4]. It might be possible to define a generalized charge Q_A which depends both on s and Λ, *i.e.,*

[4] Clearly, we need to restrict definition of Q_A so that $_s$ equilibrium is thermodynamically stable.

$Q_A = f(s,\Lambda)$, so that $_s$ state is thermodynamically stable while the $_u$ state is thermodynamically unstable — it is not clear to us how to make such a definition universally for all statistical systems.

The model (5.3) is probably the simplest example which clearly demonstrates that the thermodynamic and the dynamical (in)stabilities of the system do not generically correlate. Since black holes with translationary invariant horizons in asymptotically anti-de-Sitter space-time are dual (albeit sometimes in a purely phenomenological way) to some strongly coupled field theory, one expects that it should be possible to construct a counter-example of generalized CSC as well. In the next section we show that the generalized CSC[5] is violated for the exotic hairy black branes introduced in [9].

5.3 Counter-example to generalized CSC in gravity

The exotic black hole model we examine is defined by the following effective (3+1)-dimensional gravitational action [8, 12]:

$$S_4 = \frac{1}{2\kappa^2}\int dx^4\,\sqrt{-\gamma}\left[R+6-\frac{1}{2}\left(\nabla\phi\right)^2+\phi^2-\frac{1}{2}\left(\nabla\chi\right)^2-2\chi^2-g\phi^2\chi^2\right], \tag{5.7}$$

where g is a coupling constant.[6] This action is motivated by the $4+1$ dimensional action constructed in [4]

$$S=\int d^5x\,\sqrt{g}\left[R-\frac{1}{2}(\partial_\mu\phi)^2-\frac{1}{2}(\partial\chi)^2-V(\phi,\chi)\right] \tag{5.8}$$

with $V(\phi,\chi) = -\frac{6}{L^2}+\frac{1}{2}m_\phi^2\phi^2+\frac{1}{2}m_\chi^2\chi^2+\frac{g}{4}\phi^2\chi^2$ and $g<0$, describing second order transitions of the $3+1$ dual plasma. Note that ϕ induces a relevant deformation of the dual CFT by an operator $\mathcal{O}_r$ and χ is associated with an irrelevant operator $\mathcal{O}_i$ in the dual gauge theory. The last term in (5.7) involves mixing of $\mathcal{O}_i$ with $\mathcal{O}_r$ under RG dynamics. The central charge of the UV fixed point is defined as [8]

$$c=\frac{192}{\kappa^2}\,, \tag{5.9}$$

This central charge should be understood as a measure of the degrees of freedom in the CFT, which is defined thermodynamically or via two-point correlation functions [13]. We demand the solution to be AdS_4 asymptotically with translationary invariant horizon. For this purpose only the normalizable mode of $\mathcal{O}_i$ is nonzero near the boundary.

[5] The violation of the canonical CSC in this system is shown in [6].

[6] In numerical analysis we set $g=-100$.

The background geometry is defined as

$$ds_4^2 = -c_1(r)^2\, dt^2 + c_2(r)^2\, \left[dx_1^2 + dx_2^2\right] + c_3(r)^2\, dr^2\,, \qquad \phi = \phi(r)\,, \qquad \chi = \chi(r)\,, \tag{5.10}$$

where $r \to \infty$ corresponds to the AdS boundary. Then one can introduce a new radial coordinate x as follows

$$1 - x \equiv \frac{c_1(r)}{c_2(r)}\,, \tag{5.11}$$

so that $x \to 0$ corresponds to the AdS boundary, and $y \equiv 1 - x \to 0$ corresponds to a horizon asymptotic. Afterwards, we introduce $a(x)$ as

$$c_2(x) = \frac{a(x)}{(2x - x^2)^{1/3}}\,, \tag{5.12}$$

The equations of motion (EOMs) obtained from 5.7, with the background ansatz (5.10), define the following expansion of the model parameters

$$\begin{aligned} a =&\alpha\left(1 - \frac{1}{40}\, p_1^2\, x^{2/3} - \frac{1}{18}\, p_1 p_2\, x + \mathcal{O}(x^{4/3})\right)\,, \\ \phi =&p_1\, x^{1/3} + p_2\, x^{2/3} + \frac{3}{20} p_1^3 x + \mathcal{O}(x^{4/3})\,, \\ \chi =&\chi_4\left(x^{4/3} + \left(\frac{1}{7}g - \frac{3}{70}\right) p_1^2\, x^2 + \mathcal{O}(x^{7/3})\right)\,, \end{aligned} \tag{5.13}$$

near the boundary $x \to 0_+$, and

$$a = \alpha\left(a_0^h + a_1^h\, y^2 + \mathcal{O}(y^4)\right)\,, \qquad \phi = p_0^h + \mathcal{O}(y^2)\,, \qquad \chi = c_0^h + \mathcal{O}(y^2)\,, \tag{5.14}$$

near the horizon. Up to the overall scaling factor α the thermodynamics of the black branes can be uniquely specified with 3 UV coefficients $\{p_1, p_2, \chi_4\}$ and 4 IR coefficients $\{a_0^h, a_1^h, p_0^h, c_0^h\}$.

We use the integral of motion [12]

$$(a_0^h)^2 \sqrt{\frac{(6a_0^h)^3(6 + (p_0^h)^2 - 2(c_0^h)^2 - g(p_0^h)^2(c_0^h)^2)}{3a_1^h + a_0^h}} = 6, \tag{5.15}$$

which arises after integration of the EOMs, to find the temperature T and the entropy density s of the black brane solution (5.10):

$$T = \frac{3\alpha}{4\pi (a_0^h)^2}\,, \tag{5.16}$$

$$\hat{s} \equiv \frac{384}{c}\, s = 4\pi\alpha^2\, (a_0^h)^2\,, \tag{5.17}$$

In the dual picture p_1 can be interpreted as the coupling of the operator $\mathcal{O}_r$, p_2 as the expectation value of $\mathcal{O}_r$ and χ_4 as $< \mathcal{O}_i >$ (see [14] for the argumentation). Without the loss of generality, we choose the model with $\dim[\mathcal{O}_r] = 2$. Which means the combination $p_1\alpha$ should be fixed. The free energy density $\mathcal{F}$ and the energy density $\mathcal{E}$ are given by

$$\begin{aligned} \hat{\mathcal{F}} \equiv \frac{384}{c}\, \mathcal{F} =& \alpha^3 \left(2 - \frac{1}{6}p_1 p_2 - \frac{(a_0^h)^3}{2}\sqrt{\frac{6a_0^h(6-2(c_0^h)^2+(p_0^h)^2-g(p_0^h)^2(c_0^h)^2)}{3a_1^h+a_0^h}}\right), \\ \hat{\mathcal{E}} \equiv \frac{384}{c}\, \mathcal{E} =& \alpha^3 \left(2 - \frac{1}{6}p_1 p_2\right). \end{aligned} \tag{5.18}$$

Lastly, we identify Λ,

$$\Lambda \equiv p_1\ \alpha\,, \tag{5.19}$$

with the mass scale of the dual (boundary) field theory. Notice that the scalar field χ can not have a non-zero non-normalizable coefficient as the latter would destroy the asymptotic AdS_4 geometry — near the boundary, the non-normalizable mode of χ behaves as[7] $\chi \sim x^{-1/3}$.

For a given set of $\{\alpha, p_1\}$ there is a discrete set of the remaining parameters

$$\{p_2,\ \chi_4,\ a_0^h,\ a_1^h,\ p_0^h,\ c_0^h\}$$

characterizing black brane solutions.One solution with $c_0^h = 0$ describes the black brane without the condensate of the χ field. All the other solutions have $c_0^h \neq 0$ and describe the "exotic black branes" [8]. This model is interesting because the transition occurs at the high temperatures (rather than the low temperatures). That is the irrelevant operator $\mathcal{O}_i$ obtains nonzero vacuum expectation value for $T > T_c$, spontaneously breaking a discrete $\mathbb{Z}_2$ symmetry of the model. Also, it was shown in [11] that all the exotic black branes contain a tachyonic quasinormal mode. Thus, they are dynamically unstable but thermodynamically stable, thereby, violating the correlated stability conjecture [1, 26]. In the remainder of this section we show that exotic black branes are not only thermodynamically stable in a canonical way [9], they are thermodynamically stable in a generalized way as well, with Λ being treated as a generalized charge.

Given a dataset $\{p_1,\ p_2,\ \chi_4,\ a_0^h,\ a_1^h,\ p_0^h,\ c_0^h\}$ for each of the discrete branches of the black brane solutions we can construct parametric dependence of $\frac{\hat{\mathcal{E}}}{\hat{s}^{3/2}}$ versus $\frac{\Lambda}{\hat{s}^{1/2}}$, *i.e.*, the function $(x, \mathcal{G}(x))$ such that

$$\hat{\mathcal{E}} = \hat{s}^{3/2}\ \mathcal{G}\left(\frac{\Lambda}{\hat{s}^{1/2}}\right). \tag{5.20}$$

[7]Further details of the hairy black brane solutions can be found in [9].

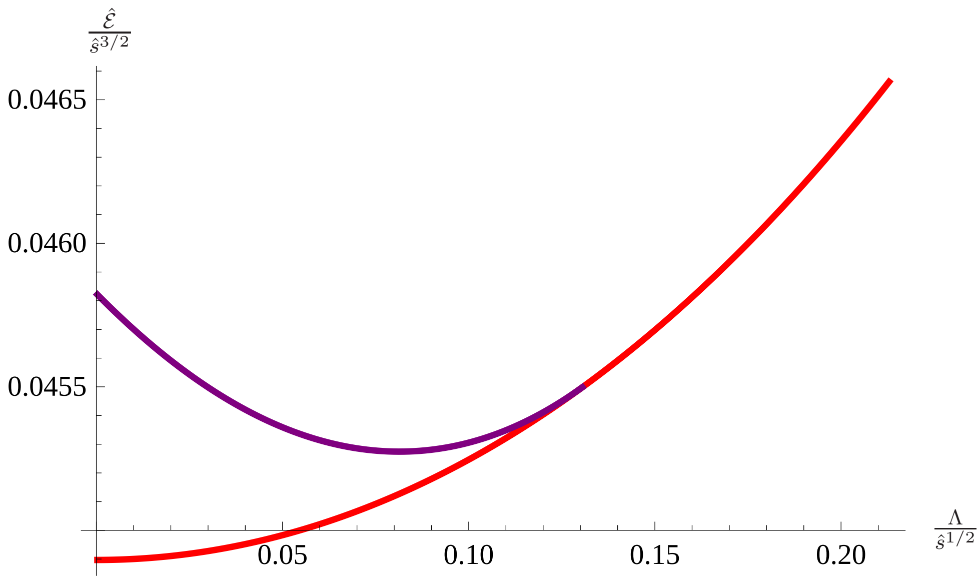

Figure 5.1: (Colour online) The energy density of the black branes with the scalar condensate (purple points) and without the scalar condensate (red points).

Given (5.16)-(5.19) we have

$$\frac{\Lambda}{\hat{s}^{1/2}} = \frac{p_1}{2\pi^{1/2}a_0^h}\,, \tag{5.21}$$

$$\frac{\hat{\mathcal{E}}}{\hat{s}^{3/2}} = \frac{12 - p_1p_2}{48\pi^{3/2}(a_0^h)^3}\,. \tag{5.22}$$

Figure 5.1 presents the function $(x, \mathcal{G}(x))$ for the black branes without the condensate of the χ scalar (the red points), and with the condensate of the χ scalar (purple points).

The following fits to $\mathcal{G}(x)^{red}$ and $\mathcal{G}(x)^{purple}$ are indistinguishable with a naked eye from the data points in Figure 5.1:

$$\begin{aligned} \mathcal{G}(x)^{red} =& 0.0448955 + 0.000128216\; x + 0.0316168\; x^2 + 0.0212735\; x^3\,, \\ \mathcal{G}(x)^{purple} =& 0.0458244 - 0.0130892\; x + 0.0721953\; x^2 + 0.06829850585\; x^3\,. \end{aligned} \tag{5.23}$$

We are now ready to analyze the canonical and the generalized thermodynamic stability criterion for the hairy black branes.

- In the canonical case we require that the Hessian

$$\mathbb{H}_{\hat{s}}^{\hat{\mathcal{E}}} \tag{5.24}$$

be positive, which translates into

$$0 \;<\; \hat{s}^{1/2}\,\frac{\partial^2\hat{\mathcal{E}}}{\partial\hat{s}^2} = \left\{\frac{3}{4}\,\mathcal{G}(x) - \frac{3}{4}\,x\,\mathcal{G}'(x) + \frac{1}{4}x^2\,\mathcal{G}''(x)\right\}\bigg|_{x=\frac{\Lambda}{\hat{s}^{1/2}}}\,. \tag{5.25}$$

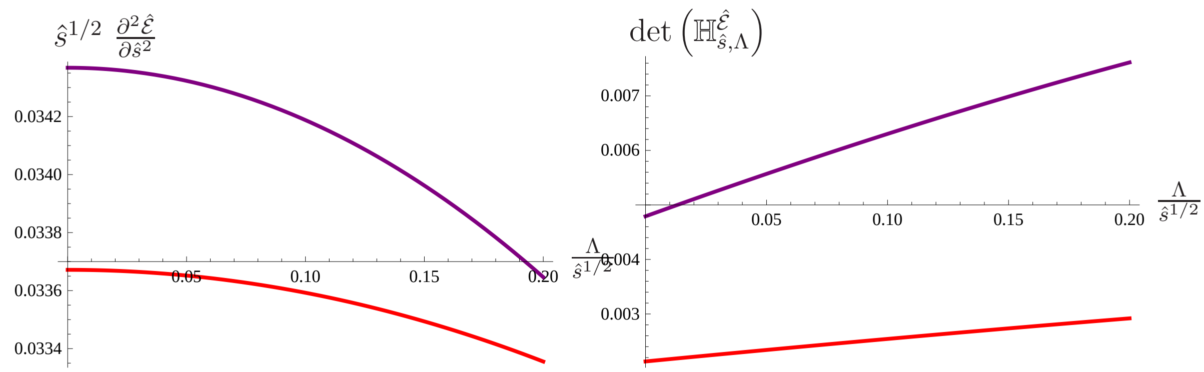

Figure 5.2: (Colour online) Canonical $\hat{s}^{1/2}\ \frac{\partial^2 \hat{\mathcal{E}}}{\partial \hat{s}^2} > 0$ (left plot) and the generalized $\det\left(\mathbb{H}^{\hat{\mathcal{E}}}_{\hat{s},\Lambda}\right) > 0$ (right plot) thermodynamic stability criteria for the dynamically stable (red curves) and the dynamically unstable (purple curves) hairy black branes.

- In the generalized case, the scale Λ is treated as one of the charges Q_A; thus, the thermodynamic stability criterion becomes the positivity of the Hessian

$$\mathbb{H}^{\hat{\mathcal{E}}}_{\hat{s},\Lambda}\,, \tag{5.26}$$

which in addition to (5.25) requires that

$$0\ <\ \det\left(\mathbb{H}^{\hat{\mathcal{E}}}_{\hat{s},\Lambda}\right) = \left\{\frac{3}{4}\ \mathcal{G}''(x)\ \mathcal{G}(x) + \frac{1}{4}x\ \mathcal{G}''(x)\ \mathcal{G}'(x) - \left(\mathcal{G}'(x)\right)^2\right\}\Bigg|_{x=\frac{\Lambda}{\hat{s}^{1/2}}}\,. \tag{5.27}$$

The results of the stability analysis (5.25) and (5.27) are presented in Figure 5.2. Much like in the simple statistical model of section 2 both the canonical and the generalized thermodynamic stability criteria characterize the hairy black branes (with or without the scalar condensate χ) as being stable. As established in [6], the hairy black branes with the non-zero condensate of χ are dynamically unstable. Thus, we conclude that generalizing the thermodynamic stability criterion to include (in an appropriate manner) the asymptotic coefficients of scalar fields sourcing the black branes in an asymptotically anti-de-Sitter space-time can not validate the "Correlated Stability Conjecture".

Much like in the statistical model in section 2, it is clear that, once one is sufficiently close to the transition (so that the tachyon condensate contribution to the thermodynamics is negligible), any redefinition of the generalized charge $Q_A = f(\Lambda)$ would change the thermodynamic stability of both classically stable and unstable phases in identical manner. Thus, insisting that the classically stable phase is thermodynamically stable (in a generalized way) as well would imply that the classically unstable phase is also thermodynamically stable. From this perspective our counter-example of the CSC conjecture is robust with respect to a definition of the generalized charge[8] Q_A. We see that the connection between thermodynamical and mechanical instabilities is not so obvious. However, it is tempting to find a new criteria to identify mechanical instabilities in holographic systems using thermodynamics.

5.4 New conjecture

In the paper [15] Emparan and Martinez provide a new Stability Conjecture - Correlated Hydrodynamic Stability, based on consideration of propagation not the tachyonic but ghost modes. The conjecture states "translationally invariant horizons have massless ghost excitations if and only if they are locally thermodynamically unstable. The ghost is a long-wavelength, low imaginary frequency, hydrodynamic instability of the horizon." The idea is that unstable modes are of two types with respect to the dispersion relation.

$$\omega^2 = c^2 q^2 + m^2 \tag{5.28}$$

$m^2 < 0$ corresponds to a tachyonic mode. This type of excitations was used in the previous analysis. Also we have a zero mode with $\omega = 0$. The second type is the ghost mode with $c^2 < 0$, and in particular the massless ghost has $\mathrm{Im}\omega = -\sqrt{-c^2}q$. The argument for the conjecture is that translationary invariant horizon can have fluctuations of an arbitrary wavelength.

Authors of [15] mention that a tachyionic instability does not necessarily lead to the hydrodynamic ghost instability at very long wavelength. Which is applicable for our exotic black model, where there is a homogeneous tachyonic mode with $\mathrm{Im}\omega(q = 0) = -\sqrt{-m^2}$ in the symmetry broken phase.

[8] As emphasized in section 2 we assume that Q_A depends only on the microscopic scale(s) of the theory.

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Chapter 6

Eling-Oz Formula for Exotic Hairy Black Holes

6.1 Introduction and Summary

A new formula for bulk-to-shear viscosity of strongly coupled gauge theory plasma was proposed by Eling and Oz (EO) [1]. The wide class of the gauge theories are dual (in the context of AdS/CFT [6]) to the following $(d+1)$ gravitational action

$$S = \frac{1}{16\pi}\int \sqrt{-g} d^{d+1}x \left(R - \frac{1}{2}\sum_i (\partial\phi_i)^2 - V(\phi_i)\right) + S_{gauge}. \tag{6.1}$$

They used the null focusing (Raychaudhuri) equation describing the evolution of the horizon entropy, which is equivalent to the viscous fluid entropy balance law. In the absence of chemical potentials for the conserved charges, the formula for the bulk viscosity of the plasma dual to (6.1) takes the following form

$$\frac{\zeta}{\eta} = \sum_i c_s^4 T^2 \left(\frac{d\phi_i^H}{dT}\right)^2 , \tag{6.2}$$

where ϕ_i^H are the scalar field values evaluated at the horizon of the black brain, T is the temperature of plasma dual to the black brane, c_s is the speed of sound waves in plasma. In the same paper the EO formula was verified for a large number of gauge theories dual to string theory at high temperature limit [7–10] and some phenomenological models of gauge/gravity correspondence [11, 12].

The expression for the bulk-to-shear viscosity employs the values of scalar fields only at the horizon. It is an intriguing result as the bulk viscosity in general depends on the energy scale; the boundary data is essential to capture microscopic scales of the theory. That is in contrast with the universality of the shear viscosity calculations [13–15]. In [3] the validity of (6.2) was extended for cascading gauge plasma [16, 17] and $\mathcal{N} = 2^*$ gauge theory plasma [18–20] for all the temperatures.

The correctness of the EO formula for the phenomenological models of gauge/gravity correspondence was also verified in [21]. Particularly, bulk viscosity obtained from the Gubser, Pufu and Rocha (GPR) formula for the GPR model [22] and the Improved Holographic QCD model [23] coincides with the EO formula. The essential feature of the GPR formula (extracted from the holographic Kubo formula) is that it is suitable only for the models with one gravitational scalar field acting as the new radial coordinate. The exotic black hole model is the example of the phenomenological model with several gravitational scalar fields.

We checked the validity of the Eling-Oz formula analytically for the exotic black holes in the high-temperature (conformal) limit. The formula is correct for the intermediate temperatures, the vicinity of the phase transition and for the temperatures up to $\frac{m}{T} < 2.75$, where m is the mass associated with breaking of the conformal symmetry. [1] The correctness of the formula for exotic black holes extends the number of models for which the EO formula is valid for all energy scales. It would be interesting to explain this kind of universality of (6.2) (see also footnote [4] in the text).

While preparing this manuscript, I learned that the authors of [1] proved such a universality of the transport coefficients of the holographic plasmas [24]. They showed that the transport coefficients depend on the boundary conditions, but they are independent of the RG running from UV to IR. Then our work can be considered as a test of the general result of Eling and Oz in [24].

6.2 Bulk viscosity for the exotic hairy black holes

We will use the the exotic black hole model from the previous chapter to find the expression fro the bulk-to-shear viscosity ratio. Let us remind the following effective (3+1)-dimensional gravitational action [2, 12]:

$$S_4 = \frac{1}{2\kappa^2}\int dx^4\,\sqrt{-\gamma}\left[R + 6 - \frac{1}{2}\left(\nabla\phi\right)^2 + \phi^2 - \frac{1}{2}\left(\nabla\chi\right)^2 - 2\chi^2 - g\phi^2\chi^2\right], \tag{6.3}$$

[1] Original calculations of the bulk viscosity were done in [4, 5].

where g is a coupling constant, with the background geometry defined as

$$ds_4^2 = -c_1(r)^2\, dt^2 + c_2(r)^2\, \left[dx_1^2 + dx_2^2\right] + c_3(r)^2\, dr^2\,, \qquad \phi = \phi(r)\,, \qquad \chi = \chi(r)\,, \quad (6.4)$$

where $r \to \infty$ corresponds to the AdS boundary. The thermodynamical parameters are given by the following expressions

$$T = \frac{3\alpha}{4\pi (a_0^h)^2}\,, \qquad (6.5)$$

$$\hat{s} \equiv \frac{384}{c}\, s = 4\pi\alpha^2\, (a_0^h)^2\,. \qquad (6.6)$$

In contrast with the previous chapter, for a given set of $\{\alpha, p_1\}$ there is a discrete set of just three remaining parameters used in the previous chapter.

$$\{a_0^h,\ p_0^h, c_0^h\}$$

allowing us to characterize the thermodynamics of black branes suitable for Eling-Oz formula.

For the exotic black hole model formula (6.2) takes the following simple form

$$\left.\frac{\zeta}{\eta}\right|_{EO} = c_s^4 \tau^2 \left(\left(\frac{dp_0^h}{d\tau}\right)^2 + \left(\frac{dc_0^h}{d\tau}\right)^2 \right)\,, \qquad (6.7)$$

where c_s is a speed of sound defined by

$$c_s^2 = \frac{d(\ln T)}{d(\ln s)}, \qquad (6.8)$$

and τ is an inversed dimensionless temperature, e.g. $\tau = \frac{T_c}{T}$ or $\tau = \frac{\alpha p_1}{T}$. Further, we will check the validity of the formula for the different temperature regimes.

6.3 Explicit analytical check of (6.2) in the conformal limit of a symmetric phase

In this section we briefly repeat the main results for the thermodynamics of the model in the high temperature limit as presented in [12]. These results can be readily applied to supply evidence of (6.7) in the conformal limit. We demonstrate this below. For the sake of simplicity, we consider the symmetric phase only ($\chi = 0$). If we introduce the small deformation parameter δ such that

$$\delta = \frac{m}{T} << 1, \qquad (6.9)$$

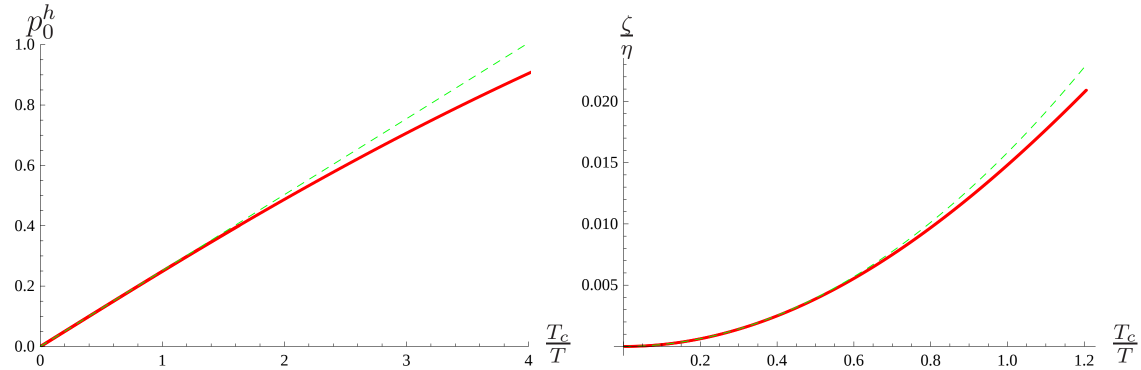

Figure 6.1: (Colour online) Comparison of a scalar field at the horizon and the bulk viscosity computed using quasinormal modes with the high-temperature limit [2]. The dashed green lines represent the conformal limit.

where m is the mass associated with the deformation of CFT. Then one can expand the scalar field ϕ in the EOMs to leading order in δ as [2]

$$\phi(y) = \delta\tilde{\phi}(y). \tag{6.10}$$

Afterwards, we solve the EOMs demanding regularity of the scalar field at the horizon and the first law of thermodynamics to fix the integration constants. [3]

The expansion of the scalar field ϕ near horizon (5.14) assumes that

$$p_0^h = \delta. \tag{6.11}$$

Eventually, it leads to the following expression for the speed of sound in the conformal limit

$$c_s^2 = \frac{1}{2} - \frac{\sqrt{3}}{8\pi}\delta^2 + \mathcal{O}(\delta^4). \tag{6.12}$$

The results for the bulk viscosity in the conformal limit were discussed in [4]. For a symmetric phase at the high temperatures we have

$$\left.\frac{\zeta}{\eta}\right|_{ordered} = \frac{2\pi}{\sqrt{3}}\left(\frac{1}{2} - c_s^2\right) + \mathcal{O}\left(\left(\frac{1}{2} - c_s^2\right)^2\right). \tag{6.13}$$

[2]It is a natural small parameter to expand the solutions of EOMs defining the values of the scalar fields.

[3]In general, p_2 — the expectation value of $\mathcal{O}_r$ is connected to the horizon data through the boundary conditions, regularity of the solution of the EOMs at the horizon and boundary. Particularly, in the conformal limit $p_2 \propto p_0^h$. This can be the qualitative argument that the EO formula captures the UV data from the boundary.

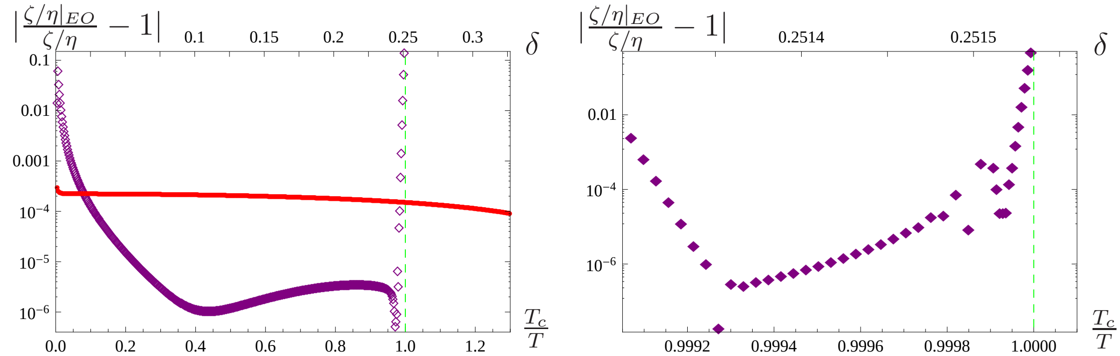

Figure 6.2: (Colour online) Comparison of the EO formula for exotic black holes with the computations using quasinormal modes [4] far from criticality (the left plot) and in the vicinity of the critical point (the right plot). Red circles are for the "ordered" (symmetric) phase, purple diamonds are for the "disordered" (broken) phase. The dashed vertical green line represents the critical point of the theory $T = T_c$.

A a result, the bulk viscosity should be proportional to the square of the deformation parameter, *i.e.*,

$$\frac{\zeta}{\eta} = \frac{1}{4}\delta^2. \tag{6.14}$$

If we substitute (6.11) and (6.12) into (6.7) we recover exactly the same relation ($\tau = \delta$ in this case). This justifies the validity of (6.7) in the conformal limit.

One can see that in the conformal limit the value of the scalar field at the horizon is proportional to $\frac{m}{T}$. In Fig. 6.1 we use a linear approximation for the scalar field at the horizon to establish the connection between δ and $\frac{T_c}{T}$. In addition, on the same figure we compare the bulk viscosity evaluated from the sound waves attenuation coefficient with analytical result (6.14) in the conformal limit.

6.4 Comparison of (6.2) away from criticality for symmetric and unstable phases

Now it is possible to check the formula for an intermediate temperature regime. The original calculations of the thermodynamics and the bulk viscosity were done in [2, 4]. We use a cubic spline approximation for the values of the scalar fields at the horizon to get their derivatives with respect to the inversed temperature. Then we compare the validity of the Eling-Oz formula with respect to two arguments δ and $\frac{T_c}{T}$.

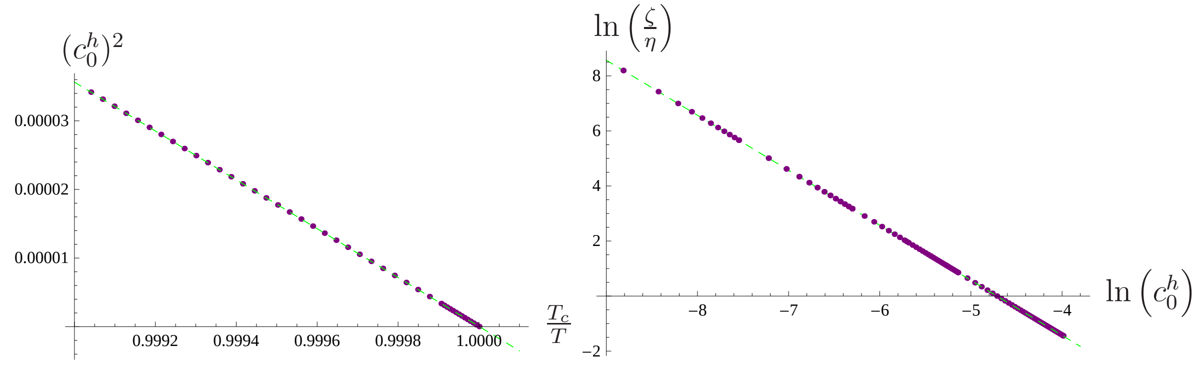

Figure 6.3: (Colour online) The square of a scalar field at the horizon c_0^h as function of the reduced temperature $\frac{T_c}{T}$. The dashed green line is a linear fit to $(c_0^h)^2$ The right plot shows the bulk viscosity dependence on c_0^h. The dashed green line represents a linear fit to the log-log data.

Fig. 6.2 illustrates the absolute value of relative error between the bulk viscosity obtained from quasinormal mode method and the Eling-Oz formula. We can a good agreement between numerical calculations of the bulk-to-shear viscosity and Eling-Oz formula. The error is slightly big for large temperatures due to small values of the scalar fields (which increases numerical errors). But we have the analytic results from the previous section for this region of temperatures. The agreement is also worse in the vicinity of the critical point (due to large values of $\frac{dp_0^h}{d\tau}$). In the next subsection we will improve the agreement in the critical regime.

6.5 Comparison of (6.2) at criticality

The critical behavior of the exotic black branes was discussed in [4]. The peculiar thing in the model is that the bulk-to-shear viscosity diverges in the broken phase at criticality. One can use more detailed data to check the formula close to criticality as it is done in the Fig. 6.2. Alternatively, we can proceed the semi-numerical analysis.

In [4] the authors constructed an exotic model of the second order transition in $d = 3$ at finite temperature and zero chemical potentials. The corresponding conformal field theory in $2+1$ dimensions is deformed by a relevant operator $\mathcal{O}_r$. The expectation value of $\mathcal{O}_i$ acts as the order parameter of the phase transition and it scales as $|t|^{1/2}$ in the vicinity of the critical point, where $t = \frac{T-T_c}{T_c}$. In Fig. 6.3 we plot $\ln\left(\frac{\zeta}{\eta}\right)$ versus $\ln\left(c_0^h\right)$

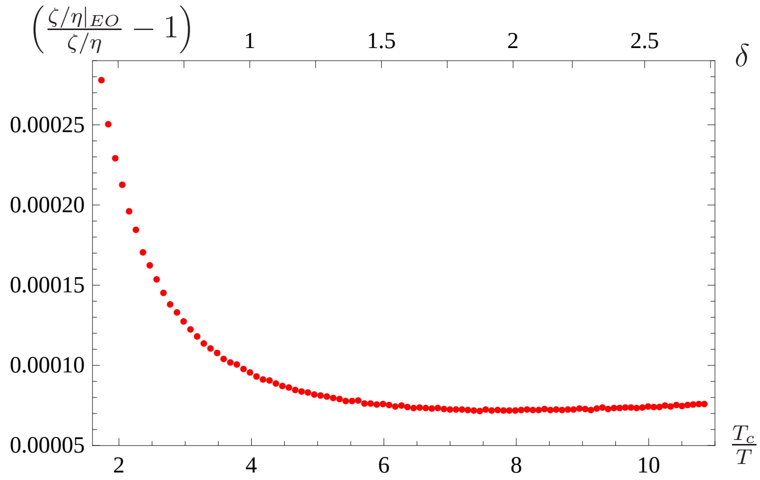

Figure 6.4: Comparison of the EO formula for symmetric phase with the computations using quasinormal modes for the low temperatures.

with a dashed green line which fits the data

$$y = -1.999(8)x - 9.427(8). \tag{6.15}$$

It is clear from the Fig. 6.3 that parameter c_0 has the same critical exponent as $< \mathcal{O}_i >$

$$c_0 \propto |t|^{1/2}. \tag{6.16}$$

For the speed of sound and bulk viscosity it was shown that

$$c_s \propto |t|^0, \qquad \left.\frac{\zeta}{\eta}\right|_{disordered} \propto |t|^{-1}. \tag{6.17}$$

Whereas, formula (6.2) suggests that at criticality

$$\left.\frac{\zeta}{\eta}\right|_{EO} = c_s^4 T_c^2 \left(\frac{c_0}{T - T_c}\right)^2 \propto |t|^{-1}, \tag{6.18}$$

which confirms the correctness of the EO formula at criticality. Also, we expect that the formula will be valid for other symmetry-broken phases.

6.6 Validity of (6.2) for the low temperatures

Let us proceed to the check of the Eling-Oz formula in the case when the stable phase is driven into the low temperature regime. First note that data for the thermodynamic parameters is more discrete, while the number of points for viscosity is significantly reduced (100 points altogether). Furthermore, Fig. 6.4 presents a plot of the relative error with respect to δ and $\frac{T_c}{T}$. One can readily see that formula (2.11) is valid up to $\delta < 2.75$. Therefore, combining the results of [3], we expect that the formula is valid for the whole range of the temperatures.

6.7 Appendix. Hydrodynamics Preliminaries

We will give a short review of hydrodynamics based on the ref. [4]. The local stress-energy tensor of relativistic fluid is given by

$$\begin{aligned} T^{\mu\nu} =& \mathcal{E} u^\mu u^\nu + P(\mathcal{E})\Delta^{\mu\nu} - \eta(\mathcal{E})\sigma^{\mu\nu} - \zeta(\mathcal{E})\Delta^{\mu\nu}\left(\nabla\cdot u\right)\,, \\ \Delta^{\mu\nu} =& g^{\mu\nu} + u^\mu u^\nu\,, \qquad \sigma^{\mu\nu} = \Delta^{\mu\alpha}\Delta^{\nu\beta}(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha) - \frac{2}{d-1}\Delta^{\mu\nu}\Delta^{\alpha\beta}\nabla_\alpha u_\beta\,, \end{aligned} \tag{6.19}$$

where $\mathcal{E}$ and $P(\mathcal{E})$ are the local energy density and pressure, u^μ is the local d-velocity of the plasma, and $\eta(\mathcal{E})$ and $\zeta(\mathcal{E})$ are the shear and the bulk viscosities correspondingly. The propagating sound waves in the plasma obey the following dispersion relation

$$\mathfrak{w} = \pm c_s\ \mathfrak{q} - i\ \Gamma\ \mathfrak{q}^2 + \mathcal{O}\left(\mathfrak{q}^3\right)\,, \tag{6.20}$$

where c_s is the speed of sound and Γ is the sound wave attenuation,

$$c_s^2 = \left(\frac{\partial P}{\partial \mathcal{E}}\right)_T = \frac{s}{c_v}\,, \qquad \Gamma = 2\pi\ \frac{\eta}{s}\left(\frac{d-2}{d-1} + \frac{\zeta}{2\eta}\right)\,, \tag{6.21}$$

and $\mathfrak{w} = \omega/(2\pi T)$ and $\mathfrak{q} = |\vec{q}|/(2\pi T)$.

In the AdS/CFT approach the dispersion relation of sound waves in the plasma is identified with quasinormal modes of the dual gravity theory in the limit where $q \to 0$.

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Chapter 7

Entanglement Entropy for Free Theories

7.1 Introduction

Entanglement and Renyi entropies are the important quantities came from the quantum information theory to solid state physics and field theory. They are useful in characterizing critical behavior of the theory and topological phases. To find an entanglement entropy we have to take a time slice of an entire space and divide it by two parts V and its complement $\bar{V}$. Entanglement entropy between V and $\bar{V}$ is a von Neumann entropy of the density matrix ρ with integrated out degrees of freedom in the complement $\bar{V}$. It is $S_{EE} = -tr(\rho \log \rho)$. AdS/CFT approach tremendously simplify the calculation of entanglement entropy, mainly due to the Ryu-Takayanagi conjecture [2]. Let us consider bulk surfaces m which are "homologous" [3] to the region V in the boundary (in particular $\partial m = \partial V$). Then conjecture says that we have to extremize the area of m to calculate the entanglement entropy:

$$S(V) = \underset{m \sim V}{\text{ext}} \left[\frac{A(m)}{4G_{\text{N}}} \right] . \tag{7.1}$$

However, there is no direct proof of the conjecture and CFT calculations are important in order to compare them with the holographic results.

7.2 Entanglement entropy for a sphere in the case of free fields.

In this section, we provide a derivation of the entanglement entropy for a sphere of radius R embedded in a d–dimensional space-time in the case of free, massless spin $\frac{1}{2}$ and spin 1 fields. As argued in [1], when a spherical entangling surface is concerned, one can use conformal transformation to map the vacuum state of an arbitrary CFT onto a thermal state on the hyperbolic geometry $\mathcal{M} = S_1 \times H^{d-1}$, where S_1 is associated with periodic Euclidean time [1]. As further shown in [1], such a conformal transformation eventually maps the original computation of EE to the computation of thermodynamic entropy of this thermal state. The main issue in the proof was the preserving of the modular Hamiltonian H defining the entanglement entropy under the conformal transformation defined as $\rho = e^{-H}$, where ρ is the reduced density matrix [2]. The radius of the entangling sphere sets both the inverse temperature of the thermal state $\beta = 2\pi R$, as well as the curvature scale of the hyperbolic plane H^{d-1}

$$ds^2 = d\tau^2 + R^2(du^2 + \sinh^2 u\, d\Omega^2_{d-2}) \ . \tag{7.2}$$

Hence, the entanglement entropy is given by

$$S_{EE} = (1 - \beta\ \partial_\beta) \log Z(\beta)\Big|_{\beta=2\pi R} \ , \tag{7.3}$$

where $Z(\beta)$ is the partition function of the system evaluated on the hyperbolic space (7.2).

Moreover, having the partition function $Z(\beta)$ at hand one can evaluate the Renyi entropies of the system [4]

$$S_n = \frac{\log Z(n\beta) - n \log Z(\beta)}{1-n}\Big|_{\beta=2\pi R} \ . \tag{7.4}$$

The EE (7.3) can be recovered from the Renyi entropies (7.4) by taking the limit $n \to 1$.

7.3 Free massless fermions.

We start from reviewing our spinor notation. The spinors are associated with the orthonormal frames, e^μ_a, of (7.2)

$$e^\mu_a e^\nu_b g_{\mu\nu} = \delta_{ab} \ . \tag{7.5}$$

[1] It is also possible to map the initial state to d-dimensional de Sitter space.

[2] The reduced density matrix ρ_A for a system with disjoint degrees of freedom A and B is given by $\rho_A = tr_B(\rho)$

The Clifford algebra in the orthonormal frame is generated by d matrices γ^a, satisfying the anticommutation relations

$$\{\gamma^a, \gamma^b\} = 2\delta^{ab} \; . \tag{7.6}$$

The dimension of these matrices is $2^{[\frac{d}{2}]}$, and the associated $d(d-1)/2$ generators of $SO(d)$ are

$$\sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b] \; . \tag{7.7}$$

They satisfy the standard $SO(d)$ commutation rules

$$[\sigma^{ab}, \sigma^{cd}] = \delta^{bc}\sigma^{ad} - \delta^{ac}\sigma^{bd} - \delta^{bd}\sigma^{ac} + \delta^{ad}\sigma^{bc} \; , \tag{7.8}$$

and the commutator of σ^{ab} with γ^c is

$$[\sigma^{ab}, \gamma^c] = \delta^{bc}\gamma^a - \delta^{ac}\gamma^b \; . \tag{7.9}$$

The covariant derivative of a spinor may be written in terms of e^μ_a as follows

$$\nabla_a = e^\mu_a \nabla_\mu \; , \quad \nabla_\mu = \partial_\mu + \frac{1}{2}\sigma^{bc}\omega_{\mu bc} \; , \quad \omega_{\mu bc} = e^\nu_b(\partial_\mu e_{c\nu} - \Gamma^\alpha_{\nu\mu} e_{c\alpha}) \; . \tag{7.10}$$

It satisfies the following anticommutation relations [5]

$$[\nabla_a, \nabla_b]\psi = -\frac{1}{2}R_{abcd}\sigma^{cd}\psi \; . \tag{7.11}$$

In the case of free massless fermions on $\mathcal{M}$, we have

$$Z(\beta) = \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\, e^{-\int_{\mathcal{M}} \bar{\psi}\not\nabla\psi} = \det(\not\nabla) \; , \tag{7.12}$$

where $\not\nabla = \gamma^a\nabla_a$. Since the γ^a matrices are covariantly constant, one can use (7.6) and (7.11) to verify the following identity

$$\not\nabla^2 = (\gamma^a\nabla_a)^2 = \delta^{ab}\nabla_a\nabla_b - \frac{R}{4} \; . \tag{7.13}$$

Hence, the free energy of the free massless fermions can be written as follows

$$-\beta F = \log Z(\beta) = \frac{1}{2}\log\det(\not\nabla\cdot\not\nabla^\dagger) = \frac{1}{2}\,\mathrm{Tr}\log(-\not\nabla^2) = -\frac{1}{2}\int_0^\infty \frac{dt}{t}K(t) \; , \tag{7.14}$$

where $K(t)$ is the trace of the spinor heat kernel associated with operator $(-\not\nabla^2)$

$$K(t) = \mathrm{tr}\int_{\mathcal{M}} K(t,x,x), \tag{7.15}$$

here and thereafter "tr" denotes the trace over spinor indexes. Note that in the case of (7.2) $\nabla_0 = \partial_\tau$ and therefore from (7.13) $\nabla\!\!\!\!/^2 = \partial_\tau^2 + \nabla\!\!\!\!/^2\Big|_{H^{d-1}}$. As a result, one can separate the Euclidean time from the coordinates on H^{d-1} and get

$$K(t) = \operatorname{tr} \int_{S_1} K_{S_1}(t,\tau,\tau) \int_{H^{d-1}} K_{H^{d-1}}(t,x,x) = \operatorname{tr} K_{S_1}(t) K_{H^{d-1}}(t) \ . \tag{7.16}$$

$K_{S_1}(t)$ can be readily evaluated. It is given by an infinite sum of heat kernels of a particle on an infinite line shifted by an integer number of inverse temperatures, $n\beta$, with respect to each other and weighted by $(-1)^n$ to maintain the antiperiodic boundary conditions for the fermions on a circle, namely

$$K_{S_1}(t) = \frac{2\beta}{\sqrt{4\pi t}} \sum_{n=1}^{\infty} (-1)^n e^{-\frac{n^2\beta^2}{4t}} \ \mathbb{I}_{[\frac{d}{2}]}, \tag{7.17}$$

where $\mathbb{I}_{[\frac{d}{2}]}$ is the unit matrix in $2^{[\frac{d}{2}]}$ dimensions. The $n = 0$ term has been dropped from the above expression, as it corresponds to $\beta \to \infty$ (zero temperature) limit and contributes an infinite constant to the free energy.

Furthermore, since the hyperbolic space is homogeneous, the volume factorizes in the heat kernel on H^{d-1}, and we obtain

$$K_{H^{d-1}}(t) = K_{H^{d-1}}(t,x,x)\mathrm{Vol}(H^{d-1}). \tag{7.18}$$

However, the volume of the hyperbolic space diverges, therefore one needs to introduce an infrared cutoff u_{max}, which is closely related to a short distance cut-off δ in the CFT [1, 4]

$$\cosh u_{max} = \frac{R}{\delta} \ . \tag{7.19}$$

As a result, we get

$$\mathrm{Vol}(H^{d-1}) = R^{d-1}\Omega_{d-2} \int_0^{u_{max}} \sinh^{d-2} u \, du = R^{d-1}\Omega_{d-2} \int_{\delta/R}^{1} \frac{(1-y^2)^{\frac{d-3}{2}}}{y^{d-1}} dy \ , \tag{7.20}$$

where

$$\Omega_{d-2} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})} \tag{7.21}$$

is the volume of the unit $(d-2)$-dimensional sphere. Expanding (7.20) in powers of δ/R, yields

$$\mathrm{Vol}(H^{d-1}) = p_1(R/\delta)^{d-2} + p_3(R/\delta)^{d-4} + \ldots + \begin{cases} p_{d-2}(R/\delta) + p_{d-1} + \mathcal{O}(\delta/R), & d : \text{odd} \\ p_{d-3}(R/\delta)^2 + q\log(R/\delta) + \mathcal{O}(1), & d : \text{even} \end{cases} \tag{7.22}$$

with $p_1 = R^{d-1}\Omega_{d-2}(d-2)^{-1}$ and etc., $p_{d-1} = R^{d-1}\pi^{(d-2)/2}\Gamma\left(\frac{2-d}{2}\right)$ and finally $q = 2(-\pi)^{\frac{d-2}{2}}R^{d-1}/\Gamma\left(\frac{d}{2}\right)$.

The heat kernel for even $d = 2m+2$ with $m = 0, 1, 2, ...$, i.e. for odd dimensional hyperbolic space, is given by [6]

$$K_{H^{2m+1}}(t,x,y) = U(x,y)\cosh\frac{\rho}{2}\left(\frac{-1}{2\pi R^2}\frac{\partial}{\partial\cosh\rho}\right)^m\left(\cosh\frac{\rho}{2}\right)^{-1}\frac{e^{-\frac{\rho^2R^2}{4t}}}{(4\pi t)^{1/2}}, \tag{7.23}$$

where ρ is the geodesic distance between x and y in units of R and $U(x,y)$ is the parallel spinor propagator from x to y.

On the other hand, for odd $d = 2m+3$ with $m = 0, 1, 2, ...$, we have [6]

$$K_{H^{2m+2}}(t,x,y) = U(x,y)\cosh\frac{\rho}{2}\left(\frac{-1}{2\pi R^2}\frac{\partial}{\partial\cosh\rho}\right)^m\left(\cosh\frac{\rho}{2}\right)^{-1}f_2(\rho,t), \tag{7.24}$$

where

$$f_2(\rho,t) = \frac{R\sqrt{2}\left(\cosh\frac{\rho}{2}\right)^{-1}}{(4\pi t)^{3/2}}\int_\rho^\infty\frac{\tilde{\rho}\cosh\frac{\tilde{\rho}}{2}e^{-\frac{R^2\tilde{\rho}^2}{4t}}}{\sqrt{\cosh\tilde{\rho}-\cosh\rho}}d\tilde{\rho}\ . \tag{7.25}$$

The structure of $U(x,y)$ is not important for our needs, as we are interested in the limit of coincident points (7.18) in which case $U(x,y)$ transmutes into an identity matrix on a $2^{[d/2]}$–dimensional spinor space. We should note here that according to [6], the dimension of the spinor space associated with (7.23) is twice as small and thus a modification of (7.23) might be expected. However, the same reasoning presented in [6] which leads to (7.23) can be equally well applied to the case considered here without introducing any changes. Henceforth, we consider the case of even d. It follows from (7.23) that for $d = 2m+2$, $K_{H^{d-1}}(x,x,t)$ takes the following general form

$$K_{H^{d-1}}(t,x,x) = \frac{P_m(t/R^2)}{(4\pi t)^{m+1/2}}\mathbb{I}_{m+1}, \tag{7.26}$$

where $P_m(x)$ is a polynomial of degree m with rational coefficients

$$P_m(x) = \sum_{j=0}^m a_j^{(m)}x^j. \tag{7.27}$$

In particular $a_0^{(m)} = 1$, and

$$\begin{aligned} P_0(x) &= 1\ , \\ P_1(x) &= 1+\frac{1}{2}x\ , \\ P_2(x) &= 1+\frac{5}{3}x+\frac{3}{4}x^2\ , \\ P_3(x) &= 1+\frac{7}{2}x+\frac{259}{60}x^2+\frac{15}{8}x^3\ . \end{aligned} \tag{7.28}$$

Substituting (7.16), (7.17) and (7.27) into (7.14), yields

$$\log Z(\beta) = \mathrm{Vol}(H^{d-1})\ \beta^{1-d}\, \frac{2^{d/2}}{\pi^{d/2}} \sum_{j=0}^{(d-2)/2} a_j^{(m)} \left(\frac{\beta}{2R}\right)^{2j} (1-2^{2j+1-d})\Gamma\left(\frac{d}{2}-j\right)\zeta(d-2j). \tag{7.29}$$

Furthermore, using the latter result with (7.4) and (7.22) one can evaluate the universal coefficients in the Renyi entropies

$$S_n = q_d^{(n)} \log(R/\delta) + \text{non-universal terms}\ . \tag{7.30}$$

Thus, for instance

$$\begin{aligned} q_2^{(n)} &= \frac{n+1}{6n}\ , \\ q_4^{(n)} &= -\frac{(n+1)(37n^2+7)}{720n^3}\ , \\ q_6^{(n)} &= \frac{(1+n)(31+276n^2+1221n^4)}{60480n^5}\ . \end{aligned} \tag{7.31}$$

When $n = 1$ these results, using the holography as shown in [1], are proportional to the A-type anomaly, see e.g. [7]. [3]

7.4 Abelian gauge field.

Consider the action for the abelian gauge field A_μ on the hyperbolic space (7.2)

$$S(A_\mu) = \frac{1}{4}\int_{\mathcal{M}} F_{\mu\nu}F^{\mu\nu}\ , \tag{7.32}$$

where as usual[4] $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$. The partition function is given by

$$Z_A(\beta) = \int \mathcal{D}A_\mu e^{-S(A_\mu)}\ . \tag{7.33}$$

This path integral is not well-defined since it overcounts field configurations related by the gauge symmetry. Rather than applying the usual Faddeev-Popov technique to factor out an infinite gauge volume from the path integral we proceed by direct computation instead. However, we also carried out the same computation using the Faddeev-Popov approach in Appendix (7.5).

[3] The conformal anomaly in even dimensions is given by $\langle T^\mu{}_\mu \rangle = \sum B_n\, I_n - 2\,(-)^{d/2} A\, E_d$ where E_d is the Euler density in d dimensions and I_n are the independent Weyl invariants of weight $-d$.

[4] Of course, in the definition of $F_{\mu\nu}$ one can use ordinary derivatives instead.

Let us start from noting that on the product space (7.2) the following identities hold

$$\nabla_\tau = \partial_\tau \ , \quad \nabla_i = \nabla_i\Big|_{H^{d-1}} \ , \quad [\partial_\tau, \nabla_i] = 0 \ , \quad \nabla^j \nabla_j A_\tau = \nabla_S^2 A_\tau \ , \tag{7.34}$$

where ∇_S^2 represents scalar covariant Laplacian on the hyperbolic space H^{d-1}.

Taking (7.33) at face value and integrating over A_τ variable, yields

$$Z_A(\beta) = \prod_\tau \det(-\nabla_S^2)^{-1/2} \int \mathcal{D}\vec{A}\, e^{-S_{eff}(\vec{A})} \ , \tag{7.35}$$

with

$$S_{eff}(\vec{A}) = \int_{\mathcal{M}} \left[\frac{1}{4} F_{ij}F^{ij} + \frac{1}{2}\dot{A}_i \dot{A}^i + \frac{1}{2}\nabla_i \dot{A}^i \nabla_S^{-2} \nabla_j \dot{A}^j \right] \ . \tag{7.36}$$

where dot denotes the derivative with respect to τ.

Now let us decompose $\vec{A}$ into transverse and longitudinal components

$$\begin{aligned} A_i^L &= \nabla_i \nabla_S^{-2} \nabla_j A^j \ , \\ A_i^T &= A_i - A_i^L = (g_{ij} - \nabla_i \nabla_S^{-2} \nabla_j) A^j \ . \end{aligned} \tag{7.37}$$

By definition, the transverse component satisfies Lorentz gauge condition, whereas the gauge field measure breaks into longitudinal and transverse parts

$$\nabla^i A_i^T = 0 \ , \quad \mathcal{D}\vec{A} = \mathcal{D}A^L \mathcal{D}A^T \ . \tag{7.38}$$

Moreover, since $F_{ij} = F_{ij}^T$, the partition function becomes

$$Z_A(\beta) = \prod_\tau \det(-\nabla_S^2)^{-1/2} \int \mathcal{D}A^L \mathcal{D}A^T \, e^{-S_{eff}(\vec{A}^T)} \ , \tag{7.39}$$

with

$$S_{eff}(\vec{A}^T) = \int_{\mathcal{M}} \left[\frac{1}{4} F_{ij}^T F^{T ij} + \frac{1}{2} \dot{A}_i^T \dot{A}^{T i} \right] \ . \tag{7.40}$$

As a result, the longitudinal part of the gauge measure factorizes. This factorization is associated with an infinite gauge volume which naturally arises in the Faddeev-Popov approach.

To make this statement even more evident, let us express the longitudinal component in the following form $A_i^L = \nabla_i \phi$, where $\phi = \nabla_S^{-2} \nabla_j A^j$, then it is clear that ϕ describes gauge modes of the theory and A_i^L is a pure gauge vector field which is responsible for an infinite overcounting in the path integral.

In particular, the following relation holds for the longitudinal part of the measure

$$\mathcal{D}A^L = J\, \mathcal{D}\phi \ , \tag{7.41}$$

where the Jacobian factor J needs to be determined. Introducing the standard inner product into the space of vector fields on H^{d-1} [8]

$$\langle A^{(1)}, A^{(2)} \rangle = \int_{H^{d-1}} A_i^{(1)} A^{(2)i} \ , \tag{7.42}$$

yields

$$\langle A^i, \nabla_i \phi \rangle = \langle -\nabla_i A^i, \phi \rangle \quad \Rightarrow \quad \nabla_i^\dagger = -\nabla_i \ . \tag{7.43}$$

Therefore

$$J = \prod_\tau \det(\nabla^\dagger \nabla)^{\frac{1}{2}} = \prod_\tau \det(-\nabla_S^2)^{\frac{1}{2}} \ . \tag{7.44}$$

Integrating by parts and dropping the surface terms, yields

$$Z_A(\beta) = \int \mathcal{D}\phi \, \mathcal{D}A^T \, e^{-\frac{1}{2}\int_{\mathcal{M}} A_i^T (-g^{ij}\Box + R^{ij}) A_j^T} \ , \tag{7.45}$$

where $\Box = \partial_\tau^2 + g^{ij}\nabla_i\nabla_j$ is the covariant D'Alembert operator acting on vectors, and we have used the following general anticommutation relations

$$[\nabla_\mu, \nabla_\nu] A^\alpha = R^\alpha{}_{\beta\mu\nu} A^\beta \ . \tag{7.46}$$

Discarding the infinite gauge volume from the partition function, gives

$$Z_A(\beta) = \det(D_{ij})_\perp^{-1/2} \ , \tag{7.47}$$

where $D_{ij} = -g_{ij}\Box + R_{ij}$, and the determinant is restricted to the vector fields on H^{d-1} satisfying the gauge condition (7.38).

To evaluate the above functional determinant we resort to the heat kernel approach in which case

$$-\beta F = \log Z_A(\beta) = -\frac{1}{2} \log\det(D_{ij})_\perp = -\frac{1}{2} \operatorname{Tr}\log(D_{ij})_\perp = \frac{1}{2}\int_0^\infty \frac{dt}{t} K(t) \ , \tag{7.48}$$

where $K(t)$ is the trace of the heat kernel associated with $(D)_\perp$.

$$K(t) = \int_{S_1} K_{S_1 ij}(t, \tau, \tau) \int_{H^{d-1}} K^{ji}_{H^{d-1}}(t, x, x) = K_{S_1 ij}(t) K^{ji}_{H^{d-1}}(t) \ . \tag{7.49}$$

The $K_{S_1 ij}(t)$ factor can be evaluated similarly to (7.17). The only difference is that this time there is no need in $(-1)^n$ weights since gauge field obeys periodic boundary conditions. Hence,

$$K_{S_1 ij}(t) = g_{ij} \frac{2\beta}{\sqrt{4\pi t}} \sum_{n=1}^\infty e^{-\frac{n^2\beta^2}{4t}} \ . \tag{7.50}$$

The $n = 0$ term has been dropped from the above expression, as it corresponds to $\beta \to \infty$ (zero temperature) limit and contributes an infinite constant to the free energy.

To compute $K^{ij}_{H^{d-1}}(t)$, we use the results of [9] for even $d \geq 4$ to construct

$$g_{ij}K^{ij}_{H^{d-1}}(t) = \mathrm{Vol}(H^{d-1})\frac{2^{d-3}(d-2)}{\pi\,\Omega_{d-2}}\int_0^\infty d\lambda\,\mu(\lambda)e^{-\left[\lambda^2+\left(\frac{d-4}{2}\right)^2\right]t}\;, \tag{7.51}$$

where the spectral function

$$\mu(\lambda) = \frac{\pi}{\left[2^{d-3}\Gamma\left(\frac{d-1}{2}\right)\right]^2}\sum_{k=0}^{\frac{d-2}{2}} a_k^{(d)}\lambda^{2k}\;, \tag{7.52}$$

with the coefficients $a_k^{(d)}$ defined by

$$\left[\lambda^2+\left(\frac{d-2}{2}\right)^2\right]\prod_{j=0}^{\frac{d-6}{2}}(\lambda^2+j^2) = \sum_{k=0}^{\frac{d-2}{2}} a_k^{(d)}\lambda^{2k}\;. \tag{7.53}$$

The product in the above expression should be omitted when $d = 4$.

Combining altogether, yields

$$g_{ij}K^{ij}_{H^{d-1}}(t) = \mathrm{Vol}(H^{d-1})\frac{(d-2)R^{1-d}}{(4\pi)^{\frac{d-1}{2}}\Gamma(\frac{d-1}{2})}\sum_{k=0}^{\frac{d-2}{2}} a_k^{(d)}\Gamma(k+1/2)\left(\frac{R^2}{t}\right)^{k+1/2}e^{-\frac{(d-4)^2}{4R^2}t}\;, \tag{7.54}$$

and finally

$$\begin{aligned} -\beta F \;&=\; \log Z_A(\beta) = \mathrm{Vol}(H^{d-1})\frac{(d-2)\tilde\beta R^{1-d}}{(4\pi)^{\frac{d-1}{2}}\Gamma(\frac{d-1}{2})\sqrt{\pi}}\times \\ &\sum_{n=1}^{\infty}\sum_{k=0}^{\frac{d-2}{2}} a_k^{(d)}\Gamma(k+1/2)\left(\frac{d-4}{n\tilde\beta}\right)^{k+1}K_{1+k}[(d-4)n\tilde\beta/2] \end{aligned} \tag{7.55}$$

where $\tilde\beta = \beta/R$.

Let us consider $d = 4$. In this case the theory is conformal and there is a significant simplification in the above expression

$$-\beta F = \log Z_A(\beta) = \mathrm{Vol}(H^3)\frac{2\pi^2+15\tilde\beta^2}{90R^3\tilde\beta^3}\;. \tag{7.56}$$

Substituting this result into (7.4) and using (7.20), yields

$$S_n = \mathrm{Vol}(H^3)\frac{(1+n)(1+31n^2)}{360\,n^3R^3\pi} = -\frac{(1+n)(1+31n^2)}{180\,n^3}\log R/\delta + \ldots \tag{7.57}$$

When $n = 1$ these result gives $S_1 = \frac{16}{45}$ while the A-type anomaly for vector fields is equal to $-\frac{31}{45}$. This ambiguity needs further consideration, but preliminary we can say that it is connected with the existence of contributions coming from the boundary of the hyperbolic space occurred after cut-off procedure.

7.5 Appendix. Faddeev-Popov procedure.

In this appendix we confirm the result (7.47) for the partition function using the traditional Faddeev-Popov method.

Integrating (7.32) by parts and dropping the surface terms, yields

$$S(A_\mu) = \frac{1}{2}\int_{\mathcal{M}} \Big[- A^\nu \Box A_\nu + A_\mu \nabla^\mu \nabla^\nu A_\nu + A_\mu A_\nu R^{\mu\nu} \Big] \,, \tag{7.58}$$

where (7.46) has been used. To define the partition function one should include the gauge fixing term, we choose

$$S_{GF}(A_\mu) = \frac{1}{2}\int_{\mathcal{M}} \big(\nabla_\mu A^\mu\big)^2 \,. \tag{7.59}$$

As a result, we obtain

$$Z_A(\beta) = \det(-\Box_S) \int \mathcal{D}A_\mu e^{-\frac{1}{2}\int_{\mathcal{M}} \left[-A^\nu \Box A_\nu + A_\mu A_\nu R^{\mu\nu}\right]} \tag{7.60}$$

where the $\det(-\Box_S)$ prefactor, with $\Box_S$ being the covariant scalar D'Alembert operator on $\mathcal{M}$, corresponds to the standard Faddeev-Popov determinant in the Lorentz gauge (7.59).

We now separate temporal index τ from the spatial indices on the hyperbolic space H^{d-1}

$$Z_A(\beta) = \det(-\Box_S) \int \mathcal{D}A_\tau \mathcal{D}\vec{A}\, e^{-\frac{1}{2}\int_{\mathcal{M}} \left[-A^\tau \Box_S A_\tau - A^i \Box A_i + A_i A_j R^{ij}\right]} \,, \tag{7.61}$$

where (7.34) has been used. The gaussian integral over A_τ can be readily performed, and we get

$$Z_A(\beta) = \det(-\Box_S)^{\frac{1}{2}} \int \mathcal{D}\vec{A}\, e^{-\frac{1}{2}\int_{\mathcal{M}} \left[-A^i \Box A_i + A_i A_j R^{ij}\right]} \,, \tag{7.62}$$

Furthermore, let us decompose $\vec{A}$ into transverse and longitudinal components (7.37), then using (7.34) and (7.46), we obtain

$$Z_A(\beta) = \det(-\Box_S)^{\frac{1}{2}} \int \mathcal{D}A^L \mathcal{D}A^T\, e^{-\frac{1}{2}\int_{\mathcal{M}} \left[A_i^T D^{ij} A_j^T + A_i^L D^{ij} A_j^L\right]} \,. \tag{7.63}$$

The integral over A^T yields a factor of $\det(D_{ij})_\perp^{-1/2}$. On the other hand, the integral over A^L can be carried out if we recall (7.41), (7.44) and use the following identity to simplify the longitudinal part of the action

$$(-g^{ij}\nabla^2 + R^{ij})\nabla_j \phi = -\nabla^i \nabla_S^2 \phi \,. \tag{7.64}$$

Hence,

$$Z_A(\beta) = \det(-\Box_S)^{\frac{1}{2}} \det(D_{ij})_\perp^{-1/2} \prod_\tau \det(-\nabla_S^2)^{\frac{1}{2}} \int \mathcal{D}\phi\, e^{-\frac{1}{2}\int_{\mathcal{M}} \left[\phi \nabla_S^2 \Box_S \phi\right]} \,. \tag{7.65}$$

integrating over ϕ reproduces (7.47).

7.6 Appendix. Heat kernel for four dimensional vectors.

Alternatively, we can evaluate the heat kernel for gauge fields in four dimensions without splitting the components of the vector into transversal and longitudinal parts. Let us start with the expression (7.62) for the heat kernel of the massless vectors in H^3.

$$Z_A(\beta) = \det(-\Box_S)^{\frac{1}{2}} \int D\vec{A}\, e^{-\frac{1}{2}\int_{\mathcal{M}}\left[-A^i \Box A_i + A_i A_j R^{ij}\right]} , \tag{7.66}$$

Which leads to the following expression

$$\log Z_A = \frac{1}{2}\det(-\Box_S) - \frac{1}{2}\det(D_{ij}) = \frac{1}{2}\int_0^\infty \frac{dt}{t}(K^{full}(t) - K^{sc}(t)) , \tag{7.67}$$

where $K^{full}(t) = K_{S_1 ij} K_{H^3}^{ij\ full}$, and the heat kernel $K_{H^3}^{ij\ full}$ includes both transversal and longitudinal components of the vector.

$$K^{sc}(t) = \int_{S_1} K^{sc}_{S_1}(t,\tau,\tau) \int_{H^3} K^{sc}_{H^3}(t,x,x) = K^{sc}_{S_1}(t) K^{sc}_{H^{d-1}}(t) . \tag{7.68}$$

$K^{sc}(t)$ is the trace of the heart kernel on a three-dimensional hyperbolic space for scalars. To compute it we use the expression for the heat kernel in H^3 [15]

$$K^{sc}_{H^3}(t,x,y) = \frac{1}{(4\pi t)^{\frac{3}{2}}} \frac{\rho}{\sinh\rho} e^{-(t/R^2) - \frac{\rho^2 R^2}{t}}, \tag{7.69}$$

where ρ is the geodesic distance between x and y in H^3 space, expressed in units of R. This expression gives

$$K^{sc}_{H^3}(t,x,x) = \frac{e^{-(t/R^2)}}{(4\pi t)^{\frac{3}{2}}} . \tag{7.70}$$

The heat kernel for the three dimensional vectors was calculated in [16].

$$g_{ij} K^{ij}_{H^3}(t) = \mathrm{Vol}(H^3) \frac{e^{-(t/R^2)} + 2 + 4(t/R^2)}{(4\pi t)^{\frac{3}{2}}}. \tag{7.71}$$

Thus, for the partition function we have

$$\log Z = \frac{1}{2}\mathrm{Vol}(H^3) 2\beta \sum_{n=1}^{\infty} \int_0^\infty e^{-\frac{n^2\beta^2}{4t}} \frac{2 + 4(t/R^2)}{(4\pi)^2 t^3} , \tag{7.72}$$

then,

$$\log Z = \mathrm{Vol}(H^3)\beta \sum_{n=1}^{\infty} \frac{1}{(4\pi)^2} \left(2\left(\frac{n^2\tilde{\beta}^2 R^2}{4}\right)^{-2} \Gamma(2) + \frac{4}{R^2}\left(\frac{n^2\tilde{\beta}^2 R^2}{4}\right)^{-1} \Gamma(1) \right) , \tag{7.73}$$

after summation

$$\log Z = \tilde{\beta} R \mathrm{Vol}(H^3) \frac{1}{(4\pi)^2} \left(\frac{16}{45}\left(\frac{\pi}{\tilde{\beta}R}\right)^4 + \frac{8}{3R^2}\left(\frac{\pi}{\tilde{\beta}R}\right)^2 \right) , \tag{7.74}$$

which leads to the same result in (7.56)

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Chapter 8

Discussion

In the second chapter we showed inequivalence of first and second order formalism for a gravity in $1 + 1$ dimensions when coupled to a scalar field on a curved background. This inequivalence is related to the Weyl and diffeomorphism symmetry present in the action. Future work will involve an adding fermion degrees of freedom to the action and to see how the two formalisms are related. This is important for understanding "real" supersymetric string theory. In the next section we found a relatively simple expressions for the effective action in four and two dimensions for a constant $U(1)$ background axial field coupled to a spinor. If the spinor s massive the axial field is treated perturbatively while a background constant vector field can be trated exactly. In the fourth chapter we were able to find all loop corrections to the effective action for a gauge field. This is acomplished by using the renormalization group equation. By combining this result with the conformal anomaly in this theory, we have obtained a novel expression for the running gauge coupling. In the AdS/CFT part of the thesis one question that arises is finding an analogue to the Eling-Oz formula for the shear-to-bulk viscosity ratio involving expansion parameters of scalar field on the boundary. Eling-Oz proved that the transport coefficients depend on the boundary conditions but do not depend on the RG running from UV to IR. Therefore, it should a connection between horizon and boundary data of the black hole, which may give a new formula for the bulk-to-shear viscosity ratio. In our calculations of the entanglement entropy for free fields we plan to see if the boundary term gives any contribution to the expression for the Renyi and entanglement entropies. This is perhaps connected to the appropriate choice of the type of the boundary conditions.

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