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Quantum Fisher Information of Three-Level Atom under the Influence of the Stark Effect and Intrinsic Dechorence

S. Jamal Anwar , **M. Ramzan and M. Khalid Khan** *

Department of Physics, Quaid-i-Azam University, Islamabad 15320, Pakistan
* Correspondence: kk@hotmail.com

Abstract: We study the dynamical evolution of quantum Fisher information (QFI) and von Neumann entropy (VNE) for a three-level atomic system interacting with the single-mode coherent field in the presence of the Stark effect and intrinsic decoherence (ID) with and without atomic motion. The effect of the ID is significant on the VNE and QFI for a three-level atom in the absence of atomic motion. It is observed that in the case of a three-level atomic system in the presence of ID, the decay of QFI and VNE is rapid and significant but no prominent effect of the Stark effect is observed. Hence, for a three-level atom, the decay of quantum entanglement (QE) with respect to time is very fast and rapid in the absence of atomic motion with an increasing value of ID. Moreover, ID is not suitable to maintain the QE for three-level atomic systems in the absence of atomic motion. The Stark effect has no significant effect on the QE. In the case of three-level atoms, ID and the Stark do not affect the periodic nature of QFI and VNE with time evolution in the presence of atomic motion. The periodic response of QFI and VNE is observed under the effect of the Stark effect and ID in the presence of a motion of a three-level atom. The QE sudden death and birth is observed in the presence of atomic motion. Therefore, the ID with the Stark effect is suitable to sustain and maintain the QE in the presence of atomic motion for three-level atomic systems. These results show the strong dependence of QFI and VNE on the Stark effect and ID.

Keywords: quantum entanglement (QE); quantum fisher information (QFI); the Stark effect; von Neumann entropy (VNE)

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1. Introduction

Quantum entanglement (QE) plays a crucial character in quantum information [1,2]. Therefore, more efforts were made to investigate the formation of the QE in various physical atom-field models and to initiate top QE quantifiers as entropy [3,4] and negativity [5,6]. Entropy is very useful in explaining field and atom correlations and the quantum effects in their connection. In particular, entropy is a much better measure of the QE [7–10]. The binomial states were used to show states intermediate between the coherent and the number states. They are considered in the free-electron laser [11] and have essential quantum characteristics such as anti-bunching [12] and squeezing [13,14]. Recently, a great deal of work is performed on QE in the case of non-inertial frames. Entanglement measures of W-state in non-inertial frames are studied by Ariadna J and co-workers [15]. Qiang, WC., Dong, Q., Mercado Sanchez, M.A. et al. studied QE property of the Werner state in accelerated frames [16]. Qian Dong et al. studied tripartite QE measures of generalized GHZ State in uniform acceleration [17].

Fisher information (FI) was first introduced by Fisher as the main quantity in estimation theory [18]. The members of probability distributions can be detected by FI. In this regard, FI can perform optimal tasks in detecting the optimal measurement of the considered quantum systems from the point of view of the theory of quantum estimation. The parameter estimation problem being the more significant is the main support of various science and technology branches. The phase estimation plays the essential role of state generation, decoherence and loss [19–24]. The lower bound of Quantum Fisher information (QFI) can be obtained from the quantum version of the Cramér–Rao (CR) inequality [25]. Interestingly, the mathematical treatments of the lower bound in physical models have been clarified [26,27], in this way the optimal resource for the phase estimation has been discussed [28]. Fisher information is a metric for figuring out how much information an observable variable x has about an unknown parameter of a distribution that represents X in mathematics. For example, it is the variance of the score or the expected value of the observed data [14,29]. As a result, a greater FI suggests that we can more precisely anticipate the parameter [30]. Quantum Fisher information (QFI) is FI's quantum extension, which is stated in terms of quantum Cramer–Rao bound [31–33]. According to the quantum Cramer–Rao inequality [34], the value of QFI provides an upper bound on the parameter precision. Therefore, the larger (QFI) indicates higher precision. QFI has been widely researched from a variety of perspectives. It has been discussed how to increase estimation precision [35,36], measurement effectiveness [37,38], and QFI function self-utilization [39,40]. It is known that the atomic parameter estimation, nonclassical properties of the field, and its atomic version of the QFI are essential tools in some applications of quantum optics and information processing. QFI is a useful quantifier to estimate different information resources, such as the entanglement, quantum speedup [41], and quantum phase transition [42]. It also has practical applications in quantum metrology [43], involves high precision.

Quantum von Neumann is considered the main entanglement measure for pure states. The correlation between the quantum entanglement measured by the von Neumann entropy and Wehrl entropy has been taken into account [44,45]. On the other hand, the correlation between QE and Fisher information (FI), as we know about a certain parameter in a quantum state, has not been studied widely. However, there are some studies to quantify pure state entanglement by using FI. In this regard, the QE evaluation with atomic classical Fisher information has been investigated [27,46,47]. It has been shown that QE of a two-level atom can be quantified by atomic FI and their marginal distribution. Furthermore, the correlation between the FI and QE during the time evolution for a trapped ion in the laser field. It is found that FI is an important tool to study single qubit dynamics as an indicator of QE under certain conditions [48]. Additionally, the time evolution of the QFI of a system whose dynamics are described by the phase-damped model has been studied [49]. It observed that there is an interesting monotonic relation between the QFI and non-local correlation behavior measured by the negativity depending on choosing the estimator parameter during the time evolution.

The interactions between atoms and the electromagnetic field are the roots of quantum optics and fundamental resources of quantum information and there are plenty of various models that describe these types of interactions. One of the most important types is a system consisting of a three-level atom that interacts with single or multi-mode fields [50–52]. As an example, Obada et al. [53] discussed the behaviour of non-classical correlation for a three-level atom interacting with non-linear two-cavity modes. The Kerr Medium's effect on the QE that is generated between a Cascade-type of a three-level atom and a bimodal cavity field is discussed by Teng et al. [54]. Mortezapour et al. [55] investigated the QE phenomena generated between an atomic system consisting of three levels prepared initially on the configuration with two non-linear cavity modes in the presence of a classical driving field. The phenomena of the sudden death/birth of different dimensions of atomic systems interacting with a classical driven classical field are investigated by Metwally et al. [56]. However, these models have been expanded to hold: the atomic motion [57], damping field via Caldirola and Kanai (CK) Hamiltonian [58], classical homogeneous gravitational field [59] and correlated two-mode field [60]. The dynamics of the QE in these systems are investigated for different considerations as two-cavity modes with parametric down-conversion [57] and time-dependent field [61].

In open quantum systems where the dissipative effects, a type of decoherence destroys the QE and mixture of the system and also it weakens the useful quantum properties of the system and its sub-systems, such as the quantum correlations. There are several methods to study quantum decoherence that are responsible for the transition between quantum and classical systems. Intrinsic decoherence is one of them that is based on the modified Schrodinger equation which reduces to the Milburn equation that describes the intrinsic decoherence [62,63]. In this modification, the quantum coherence is automatically destroyed as the system evolves. Intrinsic decoherence (ID) is a mechanism by which the system loses its coherence because of its intrinsic degrees of freedom. Several models of ID have been proposed [64–66], and one of them is the Milburn decoherence model. Milburn proposed a simple modification to the Schrodinger equation based on the assumption that the change in the state of the system, i.e., $\rho(t) \longrightarrow \rho(t+\tau)$ is uncertain and occurs with a probability of $0 < p(\tau) < 1$ on a sufficiently small time scale [62]. The change of state is always certain in standard quantum mechanics and $p(\tau) = 1$. In the last decade, the ID of quantum coupled discrete systems has been widely studied [67–70], but the majority of those works dealt with diffusion approximation, i.e., first order-correction of the von Neumann equation. A.B.A. Mohamed observed non-local correlation and quantum discord in two atoms in the non-degenerate model [71]. Mohamed, A.B.A., Eleuch, H. described optical tomography dynamics induced by qubit-resonator interaction under intrinsic decoherence [72]. A.B.A. Mohamed et al. studied quantum coherence induced by a flux qubit coupled by a resonator coherent field through a two-photon interaction [73]. A.B.A. Mohamed and H.A. Hessian discussed non-classicality in an open two-mode parametric amplifier cavity containing a qutrit system [74].

In atomic physics, the Stark effect is the split and shift of a spectral line into several components in the presence of an electric field. The amount of splitting is called the Stark shift [75]. It is analogous to the Zeeman effect where a spectral line is split into several components in the presence of a magnetic field. Its effect has been studied on some phenomena in different quantum systems. The interaction between a two-level atom and coherent field has been studied [76], where the effects of Kerr-like medium and Stark shift parameters on Wehrl entropy and field purity are discussed. The effect of Stark shift terms on the interaction between a radiation field and two two-level atoms is studied [77]. The dynamic of quantum Fisher information and entanglement for three- and four-level atomic systems interacting with a coherent field under the effect of Stark shift and Kerr medium is considered in [78]. Additionally, the entanglement for two three-level atomic systems influenced by Stark shift and Kerr-like medium has been studied [79]. The dynamics of quantum entanglement for an N-level atomic system interacting with a coherent field in the presence of Stark effects and Kerr-like medium has been studied [80].

The QFI and VNE quantum systems like three-level atoms with ID in the presence of the Stark effect have not been investigated so far on a large scale. This motivated us to investigate a three-level atomic system with ID in the presence of the Stark effect. It is very important to study the behaviour of QE for three-level atomic systems in the presence of environmental effects such as the Stark effect and ID. Mahmoud Abdel-Aty, S. Furuichi and A-S. F. Obada studied the influences of non-linearity, Stark shifts and detuning on the degree of entanglement [81]. Baghshahi H. R. et al. studied entropy squeezing and atomic inversion in the k-photon Jaynes Cummings model in the presence of the Stark shift and a Kerr medium [82]. The Dynamics and maintenance of bipartite entanglement via the Stark shift effect inside the dissipative reservoirs is studied by S. Golkar1 and M. K. Tavassoly [83], A-S. F. Obada1, S. Abdel-Khalek, E. M. Khalil, and S. I. Ali studied effects of Stark shift and decoherence terms on the dynamics of phase-space entropy of the multiphoton Jaynes Cummings model [84].

The present paper aims to find the QE dynamics of the three-level atomic system with atomic motion in the presence of the Stark effect and ID. We investigate the dynamics of VNE and QFI for moving three-level atomic systems under the influence of the ID and Stark effect. It is seen that the ID and Stark play a dominant role during the time evolution of the quantum system. The dynamics of QFI are heavily influenced by the Stark effect in the presence of ID. Moreover, the Stark effect is more prominent on the VNE in the presence of the ID without atomic motion. Finally, the three-level atomic system is found highly sensitive to these environmental effects.

The paper is organized as follows. In Section 2, we present the model Hamiltonian and interaction dynamics of the coherent field of moving-level atomic systems influenced by the ID and the Stark effect. In Section 3, numerical results and discussions are presented. In Section 4, we present a brief conclusion.

2. Hamiltonian Model

We consider moving three-level atoms entering the cavity at time $t = 0$ in the superposition state in the presence of the Stark effect influenced by the ID. We consider the cascade configuration of moving a three-level atomic system in the presence of the Stark effect with ID.

The total Hamiltonian of the system $\hat{H}_T$ under the RWA for a particular system can be described as [85]

$$\hat{H}_T = \hat{H}_{Atom-Field} + \hat{H}_I. \tag{1}$$

where $\hat{H}_{Atom-Field}$ is representing the Hamiltonian for the non-interacting atom and field, and the interaction part is given by $\hat{H}_I$. We will write $\hat{H}_{Atom-Field}$ as

$$\hat{H}_{Atom-Field} = \sum_j \omega_j \hat{\sigma}_{j,j} + \nu \hat{a}^\dagger \hat{a}, \tag{2}$$

where $\hat{\sigma}_{j,j} = |j\rangle\langle j|$ are described as population operators for the jth level ν is the frequency of oscillation of field mode. The interaction Hamiltonian of a three-level atomic system for the non-resonant case can be written as [85]

$$\hat{H}_I = \Omega(t)\Big[\hat{a}(|2\rangle\langle 3|e^{i\Delta t} + \hat{a}|1\rangle\langle 2|e^{-i\Delta t}) + h.c\Big]. \tag{3}$$

where $\hat{a}$ and $\hat{a}^\dagger$ is the annihilation (creation) operator of the field mode. Two is the middle energy level, one is the upper energy level and two is the lower energy level. Δ is the detuning of the field mode.

In the case of the Stark effect, the interaction Hamiltonian can be written as

$$\hat{H}_I = \Omega(t)\Big[\hat{a}(|2\rangle\langle 3|e^{i\Delta t} + \hat{a}|1\rangle\langle 2|e^{-i\Delta t}) + h.c\Big] + \beta \hat{a}^{\dagger 2} \hat{a}^2, \tag{4}$$

β is the parameter of the Stark effect and the detuning parameter is described as

$$\Delta = \omega - (\omega_2 - \omega_1) = \omega - (\omega_2 - \omega_1). \tag{5}$$

The coupling constant for the atom and field is g, $\Omega(t)$ represents the shape function of the cavity-field mode [86] and the motion of the atom is along the z-axis. For particular interest

$$\begin{aligned} \Omega(t) &= g\ \sin(p\pi vt/L) \text{ in the presence of atomic motion }, p \neq 0 \\ \Omega(t) &= g \text{ in the absence of atomic motion } p = 0 \end{aligned} \tag{6}$$

where the velocity of motion of the atom is v and p denotes half of the number of wavelengths of the mode in the cavity and L describes the length of the cavity along the z-direction. Now take the velocity of an atom as $v = gL\,/\pi$ which gives us

$$\Omega_1(t) = \int_0^t \Omega(\tau)d\tau = \frac{1}{p}(1 - \cos(p\pi vt/L) \text{ for } p \neq 0 \tag{7}$$

$$= gt \text{ for } p = 0. \tag{8}$$

To find the phase shift parameter as precisely as possible, we consider the optimal input state as

$$|\Psi(0)\rangle_{\text{Opt}} = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) \otimes |\alpha\rangle \tag{9}$$

where $|1\rangle$ and $|2\rangle$ describe the states of the atom and α is the coherent state of the input field given as

$$|\alpha\rangle = \sum_{n=0}^{\infty} \alpha^n \sqrt{e^{-|\alpha|^2}/n!}|n\rangle. \tag{10}$$

We consider a single-atom phase gate that introduces the phase shift as

$$\hat{U}_\phi = |1\rangle\langle 1| + e^{i\phi}|2\rangle\langle|2|, \tag{11}$$

$|\Psi(0)\rangle$ is obtained from the operation of the single-atom phase gate on $|\Psi(0)\rangle_{\text{Opt}}$

$$\hat{U}_\phi|\Psi(0)\rangle_{\text{Opt}} = |\Psi(0)\rangle \tag{12}$$

$$= \frac{1}{\sqrt{2}}(|1\rangle + e^{i\phi}|2\rangle) \otimes |\alpha\rangle \tag{13}$$

After the operation of the phase gate, the system will interact with a field. For the time-dependent case, the wave function can be characterized by the transformation matrix $\hat{U}(t)$ as given by

$$|\Psi(t)\rangle = \hat{U}(t)|\Psi(0)\rangle, \tag{14}$$

One can write the explicit expression of the density matrix as

$$\hat{\rho}(t) = \sum_{m,n}^{N} |\Psi_n(t)\rangle\langle\Psi_n(t)|\hat{\rho}(0)|\Psi_m(t)\rangle\langle\Psi_m(t)| \tag{15}$$

and in terms of ID, the density matrix can be as [87]

$$\hat{\rho}(t) = \sum \exp\left(\frac{\gamma t}{2}(E_m - E_n)^2 - i(E_m - E_n)t\right) \times \langle\Psi_m|\hat{\rho}(0)|\Psi_n\rangle|\Psi_m\rangle\langle\Psi_n|, \tag{16}$$

where $E_{m,n}$ and $\Psi_{m,n}$ are the eigenvalues and the corresponding eigenvectors of H_I, respectively, and γ is the parameter of ID.

In this fashion, we can define the QFI of a bipartite density operator $\rho_{AB.}$ in terms of ϕ as [88]

$$I_{QF}(t) = I(\phi, t) = \text{Tr}[\rho_{AB}(\phi, t)\{L^2(\phi, t)\}], \tag{17}$$

where $L(\phi, t)$ is the quantum score [89] (the symmetric logarithmic derivative) which can be found as

$$\frac{\partial\rho(\phi, t)}{\partial\phi} = \frac{1}{2}[L(\phi, t)\rho_{AB}(\phi, t) + \rho_{AB}(\phi, t)L(\phi, t)]. \tag{18}$$

Similarly, the VNE can be defined as

$$S_A = -\text{Tr}(\rho_A \ln \rho_A) = -\sum_i r_i \ln r_i, \tag{19}$$

where r_i are the eigenvalues of the atomic density matrix $\rho_A = \text{Tr}_B(\rho_{AB})$.

Now, the influence of different environmental parameters γ, ϕ and p on the evolution of the QFI and VNE is presented in the next section.

3. Numerical Results and Discussion

In this section, we present the results for the dynamical evolution of QFI and VNE for moving three-level atomic system interacting with a coherent field with the Stark effect in the presence of the ID. We study the time evolution of QFI and VNE of the three-level atomic system in the presence of the Stark effect and ID with and without atomic motion.

In Figures 1–3, we plot QFI and VNE vs time for two- and three-level atomic system interacting with a coherent field for $|\alpha|^2 = 6$ and the phase shift parameter $\phi = 0$ (left panel) and $\pi/4$ (right panel) for $\beta = 0.3, 1$ and 3 and ID parameter $\gamma = 0.00001, 0.0001$ and 0.001 without atomic motion, i.e., $p = 0$. The effect of the Stark shift is not seen as significant on the VNE and QFI of the three-level atom in the absence of atomic motion. It is observed that ID influenced the QFI and VNE and on the increasing value of ID parameter γ, the decay is observed in both QFI and VNE with the evolution of time. At the value $\gamma = 0.001$, the QFI and VNE decrease more rapidly with the evolution of time. It means QE is decreasing rapidly with the increasing value of the ID parameter. However, at $\beta = 0.3, 1$ and 3, the Stark shift does not influence the QE significantly, but the ID influences the QE strongly.

The periodic behaviour of QFI and VNE is suppressed under the effect of the Stark effect with ID in the absence of atomic motion. It is observed the ID affects the QFI and VNE at larger values but the increasing value of the Stark effect does not affect QFI and VNE significantly. It means the Stark effect does not affect the QE of the three-level atomic system.

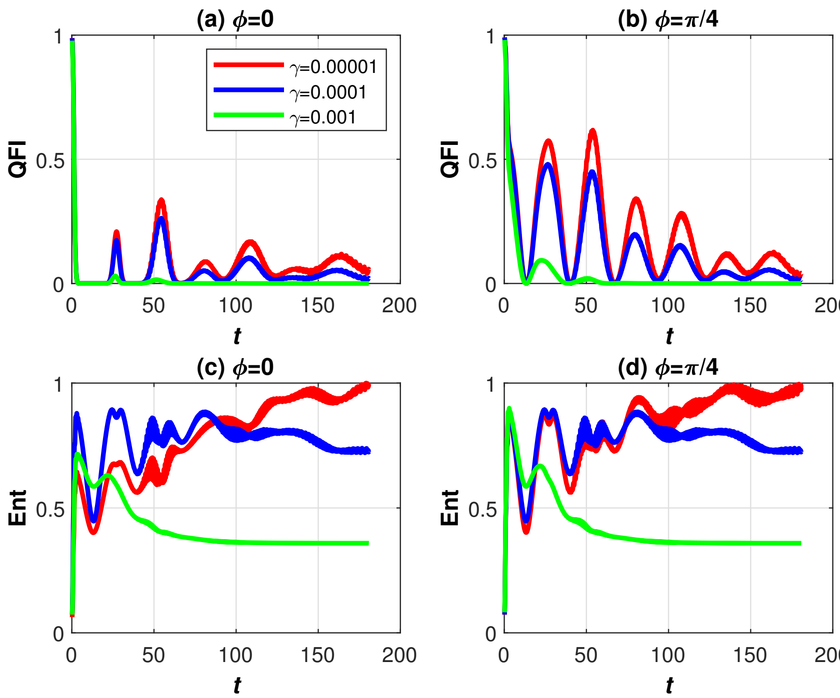

Figure 1. (Colour online) The QFI (upper panel) and VNE (lower panel) as a function of time for three-level atom interacting with a coherent field for $\alpha = 6$ in the presence of the Stark effect ($\beta = 0.3$), $\gamma = 0.00001, 0.0001$ and 0.001 and the phase shift estimator parameter $\phi = 0$ (left panel) and $\pi/4$ (right panel). The atomic motion parameter p is neglected.

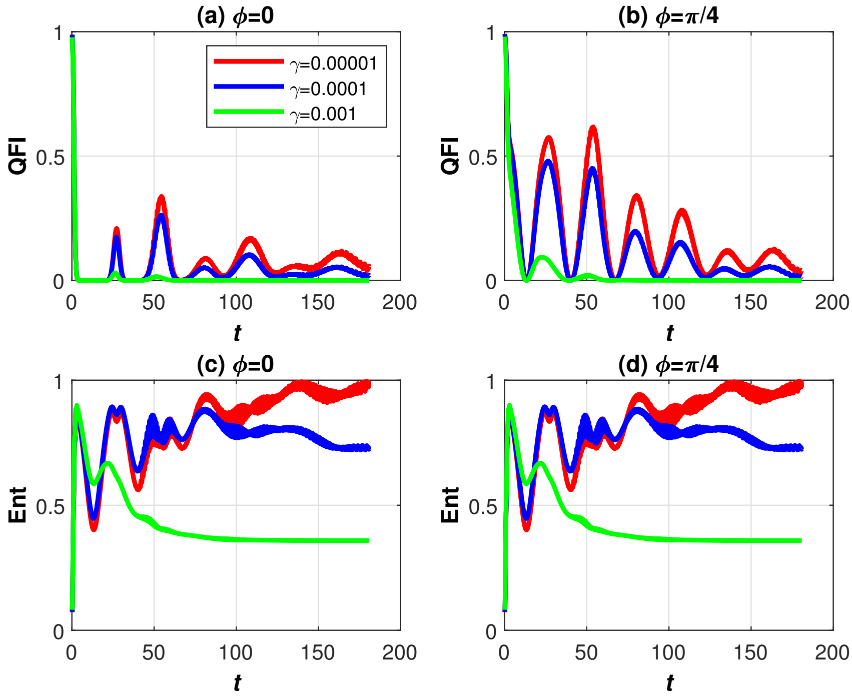

Figure 2. (Colour online) The QFI (upper panel) and VNE (lower panel) as a function of time for three-level atom interacting with a coherent field for $\alpha = 6$ in the presence of the Stark effect ($\beta = 1$), $\gamma = 0.00001, 0.0001$ and 0.001 and the phase shift estimator parameter $\phi = 0$ (left panel) and $\pi/4$ (right panel). The atomic motion parameter p is neglected.

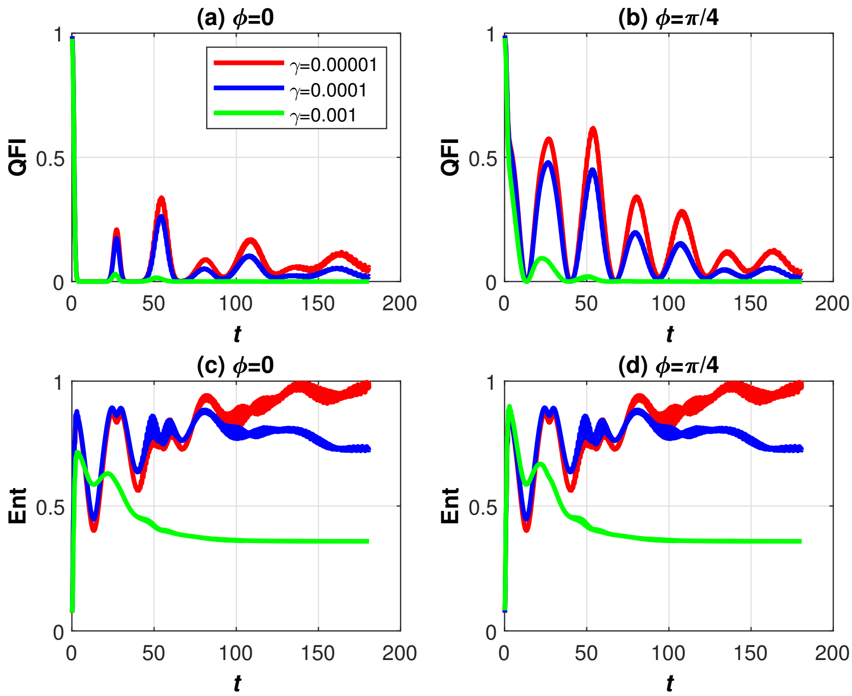

Figure 3. (Colour online) The QFI (upper panel) and VNE (lower panel) as a function of time for four-level atom interacting with a coherent field for $\alpha = 6$ in the presence of the Stark effect ($\beta = 3$), $\gamma = 0.00001, 0.0001$ and 0.001 and the phase shift estimator parameter $\phi = 0$ (left panel) and $\pi/4$ (right panel). The atomic motion parameter p is neglected.

In Figures 4–6, we plot QFI and VNE for three-level atomic systems interacting with the coherent field in the presence of the ID for $\beta = 0.3, 1$ and 3 and $p = 1$. It is observed that in the case of atomic motion ($p = 1$), both QFI and VNE show periodic behaviour with respect to time for three-level atoms. The Stark effect and ID have no significant effect on the periodicity of the VNE and QFI for three-level atom. For three-level atom the periodic behaviour is the same for ID parameter $\gamma = 0.00001, 0.0001$ and 0.001. So it means that in the case of three level-atom, ID and the Stark effect do not affect the periodic nature of QFI and VNE with time evolution. Hence it is observed that sudden death and birth of QE is not affected by ID and the Stark effect for three-level atom in the presence of the atomic motion ($p = 1$). Therefore, it is concluded that the ID with the Stark effect is suitable to sustain and maintain the QE in the presence of atomic motion for the N-level atomic system.

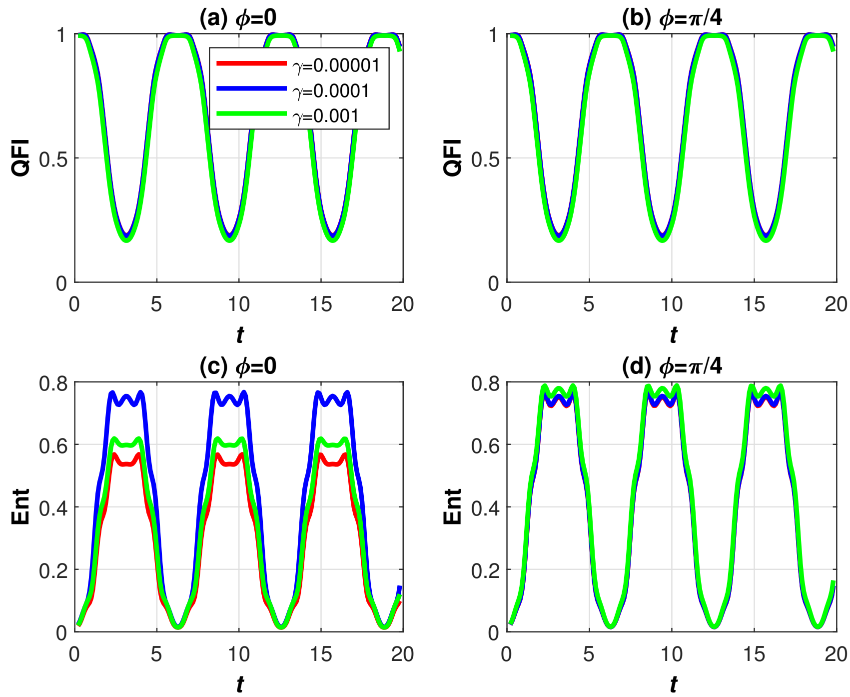

Figure 4. (Colour online) The QFI (upper panel) and VNE (lower panel) as a function of time for two-level atom interacting with a coherent field for $\alpha = 6$ in the presence of the Stark effect ($\beta = 0.3$), $\gamma = 0.00001, 0.0001$ and 0.001 and the phase shift estimator parameter $\phi = 0$ (left panel) and $\pi/4$ (right panel). The atomic motion parameter $p = 1$.

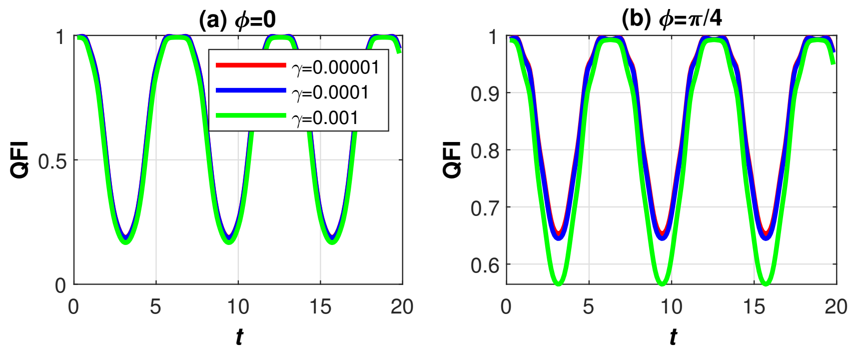

Figure 5. *Cont*.

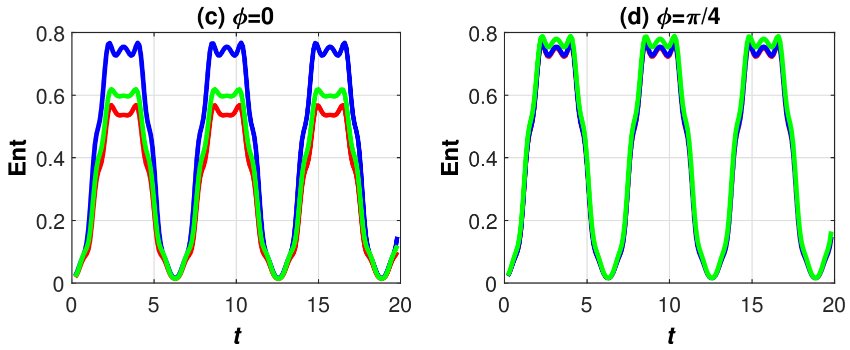

Figure 5. (Colour online) The QFI (upper panel) and VNE (lower panel) as a function of time for three-level atom interacting with a coherent field for $\alpha = 6$ in the presence of the Stark effect ($\beta = 1$), $\gamma = 0.00001, 0.0001$ and 0.001 and the phase shift estimator parameter $\phi = 0$ (left panel) and $\pi/4$ (right panel). The atomic motion parameter $p = 1$.

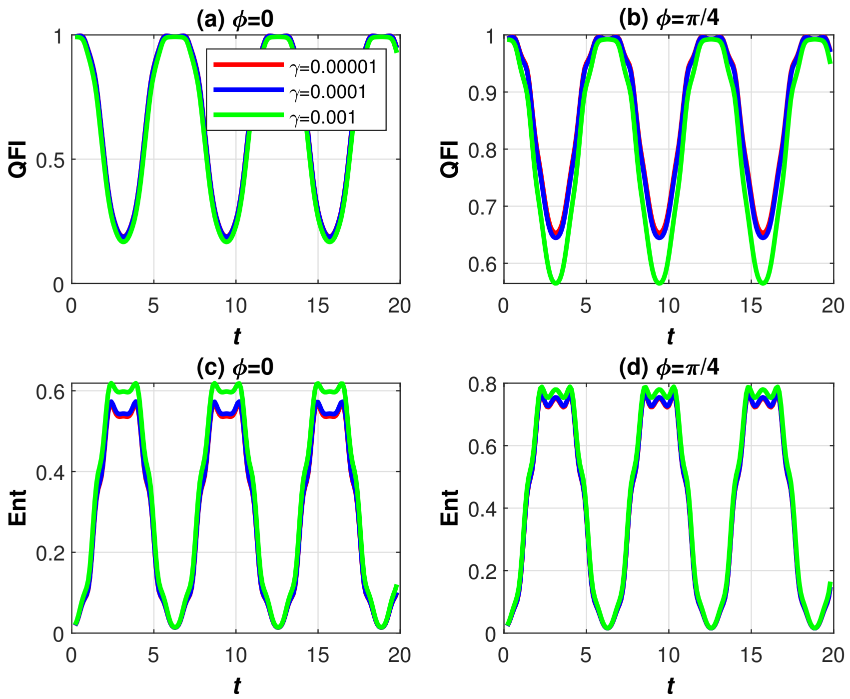

Figure 6. (Colour online) The QFI (upper panel) and VNE (lower panel) as a function of time for three-level atom interacting with a coherent field for $\alpha = 6$ in the presence of the Stark effect ($\beta = 3$), $\gamma = 0.00001, 0.0001$ and 0.001 and the phase shift estimator parameter $\phi = 0$ (left panel) and $\pi/4$ (right panel). The atomic motion parameter $p = 1$.

4. Conclusions

We have investigated the dynamics of VNE and QFI for the three-level atomic system affected by the Stark effect and ID. The time evolution of QFI and VNE for three-level atomic systems influenced by ID and the Stark was examined with and without atomic motion. It is observed that the ID has a prominent role during the time evolution of the quantum system but the Stark effect does not influence the three-level atomic system significantly. The effect of ID is seen prominent on the VNE as compared to the QFI in the absence of the atomic motion but the Stark effect shows no prominent effect. The influence of the Stark effect is not found significant on the VNE and QFI of the three-level atom in the absence of atomic motion. It is observed that in the case of larger values of the ID parameter γ especially at $\gamma = 0.001$, the VNE decreases more rapidly as compared to the QFI. It means QE is influenced by a larger value ID parameter and it reduces rapidly. So it is concluded that for the three-level atomic system, the QE does not maintain itself at increasing values of ID but increasing values of the Stark parameter β does not affect the QE. Hence, the Stark effect does not influence the QE of the three-level atom, so the Stark effect is useful to maintain QE but ID is not suitable to sustain QE. For a three-level atom, ID and the Stark effect do not affect the periodic nature of QFI and VNE with time evolution in the presence of atomic motion. It is observed that sudden death and birth of QE is not affected by ID and the Stark effect for the three-level atom in the presence of the atomic motion ($p = 1$). The periodic response of QFI and VNE is not suppressed under the effect of the Stark effect and ID in the presence of motion of atoms with larger values of ID and the Stark effect parameter. The QE sudden death and birth is not suppressed at smaller or larger values of the ID and the Stark parameter as compared to the larger values. Therefore, it is concluded that the ID with the Stark effect is suitable to sustain and maintain the QE in the presence of atomic motion for a three-level atomic system. These results show the strong dependence of QFI and VNE on the Stark effect and ID.

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