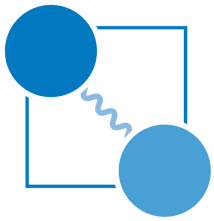

T30f
Lehrstuhl für Theoretische Teilchen- und Kernphysik
Prof. Dr. Nora Brambilla

Van der Waals Interaction in QCD

Bachelor Thesis

by

Steffen Biermann

Supervisor

-

Prof. Dr. Nora Brambilla

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Physik Department
Technische Universität München

Abstract

Gluonic van der Waals interaction between color singlet hadrons can be described in QCD, and is equivalent to electromagnetic interaction between two neutrally charged atoms. Due to this fact, this thesis presents a detailed calculation of the energy resulting of electromagnetic interaction between two hydrogen atoms. The calculations for gluonic van der Waals interactions are mentioned qualitatively to give a fundamental understanding, since they are expected to be important in hadron or nuclei interactions with heavy quarkonium states which are still not well understood. Without going into detail, we will discuss some experimental data of photo- and hadroproduction of quarkonia at low energies to motivate the analysis of gluonic interacting processes.

Zusammenfassung

Gluonische van der Waals Wechselwirkung zwischen farbneutralen Hadronen kann in der QCD beschrieben werden und ist in gewisser Weise ähnlich zu der elektromagnetischen Wechselwirkung von neutralen Atomen. Aufgrund dessen wird in dieser Arbeit eine detaillierte Berechnung der aus der elektromagnetischen Wechselwirkung resultierenden Energie in Abhängigkeit des Abstandes zweier Wasserstoff Atome im Rahmen der Quantenmechanik für kleine Abstände, sowie der QED für große Abstände präsentiert. Die Erweiterung auf QCD Prozesse ist rein qualitativ erläuert und soll prinzipiell das Interesse an solchen Prozessen begründen, die auf gluonischen van der Waals Kräften basieren. Ohne zu sehr ins Detail zu gehen, werden zudem einige experimentelle Ergebnisse diskutiert um die Motivation gluonische Prozesse zu untersuchen zu fundieren.

Contents

1 Introduction

The aim of this bachelor thesis is to give a description of forces between color singlet hadrons, i.e. colorless hadrons. The interaction treated in this thesis is the strong interaction, with its framework the Quantum Chromodynamics.

The study of neutral charged atoms is the equivalent QED process and will help to understand the behaviour of two color singlet hadrons. The first part of this thesis will be a description of the main principles of Quantum Chromodynamics, which contains a short mathematical view on strong interactions, as well as an introduction to two basic concepts of Quantum Chromodynamics - Asymptotic Freedom and Confinement. The introduction also contains a perspective on experiments and why it is important and interesting to study color singlet hadron interaction. The main focus has been set on quarkonia-nuclei scattering.

In chapter 2 the energy due to induced dipole-dipole interaction between two neutral charged hydrogen atoms separated by R is derived for two different distances [1]. We will see, that the R^{-6}-dependence [2] changes to a R^{-7}-dependence [3, 4] if we go from small to big distances. The transition is at a separation distance of approximately $c\tau = c\hbar/\Delta E$, for the atom specific excitation energy ΔE.

The last part is a qualitative sketch of what is possible in gluonic van der Waals interaction within a suitable framework the (p)NRQCD , and what we can learn from QED calculations. The appendix contains all calculations in details especially an accurate calculation of some essential tensor integrals in dimensional regularization. At this point it is mentioned that throughout this thesis natural units are used, i.e. $c = \hbar = 1$.

1.1 Quantum Chromodynamics: A Gauge Theory

Strong interaction is a fundamental interaction between quarks and gluons, as well as an interaction between gluons itself. Quarks as spin-$\frac{1}{2}$ particles obey the Dirac-equation.The starting point therefore is the Lagrangian for Dirac-fields

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi \quad . \tag{1}$$

Experiments have shown, that quarks carry a flavour quantum number and a strong interaction specific quantum number, called color. For each flavour three different colors exist. For simplicity the Lagrangian (1) contains only one flavour, which is represented by the color triplet Ψ. Quantum Chromodynamics is a gauge theory, which means it has to be invariant under local gauge transformations. From a particle physics point of view the force is mediated by particles, called gauge bosons or, in the case of QCD, gluons. Gluons are described in this theory by gauge fields [5].

To create an invariant Lagrangian under local gauge transformation let us assume the following transformation

$$\Psi \to U\Psi \quad , \tag{2}$$

with $U \in SU(3)$, which stands for the three dimensional special unitary group, i.e. $U^\dagger = U^{-1}$ and $\det(U) = 1$. Each of those matrices can be written as

$$U = e^{i\Theta_a T_a} \quad , \tag{3}$$

with parameters $\Theta^a \in \mathcal{R}$ and complex matrices T_a called generators.

Going further to local transformations, the parameters Θ are now dependent on the position in space $\Theta(x)$, so that

$$\Psi \to U(x)\Psi(x) = e^{i\Theta_a(x)T_a} \tag{4}$$

which leads to the problem that $\partial_\mu \Psi'(x) \neq U(x)\partial_\mu \Psi(x)$ therefore (1) is not invariant under this transformation. The problem can be solved if one replaces ∂_μ by the covariant derivative which transforms as

$$D'_\mu \Psi'(x) = U(x) D_\mu \Psi(x) \ . \tag{5}$$

The ansatz for this purpose is

$$D_\mu = \partial_\mu - ig\frac{\lambda_a}{2}G^a_\mu(x) \ , \tag{6}$$

introducing some additional gauge fields G^a_μ and a parameter g which will later be related to the coupling strength, as well as the Gell-Mann matrices λ_a, with $T_a = \frac{\lambda_a}{2}$ [1]. The additional part in this expression leads to a quark-gluon interaction.

If we assume an infinitesimal transformation $U = 1 + iT_a\Theta_a$ the gauge fields transform as

$$G'^a_\mu = G^a_\mu + \frac{1}{g}\partial_\mu \Theta^a + f_{abc}G^b_\mu \Theta^c \ . \tag{7}$$

To describe the dynamics of the newly introduced gauge fields, a new kinetic term has to be added to the Lagrangian. The ansatz for this interaction term is

$$\mathcal{F}_{\mu\nu} = \mathcal{D}_\mu T_a G^a_\nu - \mathcal{D}_\nu T_a G^a_\mu = \frac{i}{g}\left[\mathcal{D}_\mu, \mathcal{D}_\nu\right] \equiv F^a_{\mu\nu}T_a \ . \tag{8}$$

We can see by this definition that the trace of this expression is invariant under a transformation $\mathcal{F}_{\mu\nu} \to U\mathcal{F}_{\mu\nu}U^{-1}$ and is therefore a good candidate

[1] Appendix A.1

for the Lagrangian of G-fields

$$\mathcal{L}_W = -\frac{1}{2}Tr(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}) = -\frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu} \quad . \tag{9}$$

Inserting $F^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g f_{abc} G^b_\mu G^c_\nu$ we would see, that this Lagrangian now gives a self-coupling of the gluon fields.

The final invariant Lagrangian then is

$$\mathcal{L}_{QCD} = \bar{\Psi}\left(i\gamma^\mu D_\mu - m\right)\Psi - \frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu} \quad . \tag{10}$$

1.2 Running Coupling Constant

In the previous chapter 1.1 we mentioned the coupling constant g. This constant can be analogously defined as it is in QED. The QED-coupling constant in lowest order is $\alpha_{QED} = \frac{e^2}{4\pi}$. If we included higher order loop terms, we would notice a dependence on the internal momentum Q. Summing up all corrections with one loop accuracy, we end up with a geometric series and see that the coupling constant of QED

$$\alpha(Q^2) \equiv \frac{\alpha(Q_0^2)}{1 - \frac{\alpha(Q_0^2)}{4\pi}\beta_0 \log\frac{Q^2}{Q_0^2}} \tag{11}$$

has changed with respect to its primary definition, with $\beta_0 = \frac{4}{3}\sum_f Q_f^2 > 0$ and some experimental value $\alpha(Q_0^2)$. We see from the sign of the logarithm that for increasing Q^2 the coupling constant also increases. For small Q^2, however, the denominator increases, so the coupling constant is getting smaller.

Defining the coupling constant for strong interactions similarly to QED $\alpha_s = \frac{g_s^2}{4\pi}$ we get an expression for the coupling constant, which is based on Gluon-Quark interactions, but since in QCD gluon-gluon interactions are possible as well, we have another non-abelian term

$$\alpha_S(Q^2) = \frac{\alpha_s(Q_0^2)}{1 + \frac{\alpha_s(Q_0^2)}{4\pi}(11 - \frac{2}{3}n_f)\log\frac{Q^2}{Q_0^2}} \quad , \quad n_f : \#\text{ flavours} \ , \tag{12}$$

where we again included only loop corrections with one loop accuracy [6, 7]. In contrary to the QED coupling constant the sign in front of the logarithm is negative, which has serious consequences, as we will see consecutively. The fine structure constant, or coupling constant is the expansion parameter in QFT.

Asymptotic Freedom. Examining this result, we notice some interesting properties of the coupling constant. For $Q^2 \to \infty$, which is equivalent to the distance becoming smaller, one finds that $\alpha_s(Q^2) \to 0$. For small distances between quarks, they can be treated as "free", which is referred to as Asymptotic Freedom, in other words the coupling between quarks is getting weaker [8].

Confinement. For some $Q^2 = \Lambda_{QCD}^2 = Q_0^2 \exp\left(\frac{-12\pi}{(33-2n_f)\alpha_s(Q_0^2)}\right)$ we see that the coupling constant diverges and can be rewritten

$$\alpha_S(Q^2) = \frac{12\pi}{(33 - 2n_f)\log\frac{Q^2}{\Lambda_{QCD}^2}} \tag{13}$$

which clearly shows this divergent behaviour. Λ_{QCD} often refers to the QCD scale and has a value of $\Lambda_{QCD} \simeq (213 \pm 8)$MeV for 5 active quark flavors and an experimental input of $\alpha_S(M_Z^2) = 0.1184 \pm 0.0007$, with the mass of the Z-boson M_Z [9]. For $\alpha_S \ll 1$, or $|Q^2| \gg \Lambda_{QCD}^2$, it tells us, that perturbative QCD can be used [10, 11]. In the neighbourhood of this value and below, QCD becomes non-perturbative so that other methods are required to get a valid result for QCD processes. The strong interaction realm is thus given for distances around $1/\Lambda_{QCD}$. A descriptive explanation for Confinement can be given from a phenomenological point of view. In a quarkonium state the potential between the interacting quarks can be

described via the Cornell potential

$$V(r) = -\frac{a}{r} + br \ , \tag{14}$$

where r is the effective radius of the quark-antiquark pair, and a and b some parameters. The first part corresponds to a one-gluon exchange between the quark and anti-quark, whereas the second part is known as the Confinement part. We see from that Confinement part, that energy enormously increases with increasing r and makes a separation only possible, if the energy is high enough to create another quark-antiquark pair, which compensate the colors of the primary quarks. This idea states that every single object which freely appears is colorless, in other words a quark can never be separated and appear as an asymptotic state. The same form of the potential in (14) can be calculated in effective field theories which give the potential in form of Wilson loops as shown by W. Fischler [12]. Using such theories one would obtain a linear term dominating for large distances and a short distance determining term with$-\frac{4}{3}\frac{\alpha_S(R)}{R}$.

The principle of colorless objects is shown in the following figure [13]

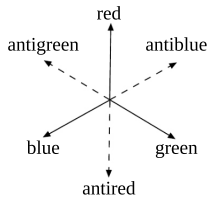

Figure 1: Hadron-colors

The colors represented by vectors have to add to zero (the summated arrows have to point to the origin). This means that for mesons the only possibility is a combination of a color and its anti-color and for baryons a combination of all colors, respectively anti-color.

1.3 Role and Importance of Gluonic van der Waals Interaction

Promising experiments to analyse gluonic van der Waals interaction are quarkonia-nuclei scattering [14]. Plenty of studies investigate the scattering of J/ψ at nuclei. The advantage of J/ψ over other quarkonia states is the long mean lifetime and since it is the first discovered stable state of quark-anti-quark interaction there is lots of experimental data available. This meson is a quarkonium state, consisting of a charm and an anti-charm quark, therefore called a charmonium state with a mass of 3097 MeV.

The interest in analysing $J\psi$-nucleon scattering is that it is expected to interact via meson exchange and via gluonic van der Waals interaction and therefore offers a good experimental setting to research the currently not fully understood theory of strong interactions [15]. S. Brodsky [16] discussed the importance of gluonic van der Waals interaction of $J\psi$ scattering at nuclei and arrived at the conclusion that the interaction via π meson exchange and $D\bar{D}$[2] intermediate state interaction can neglected compared to gluon exchange. He also proposed a promising method to study J/ψ scattering in the reaction $\pi^+ d \to J/\psi pp$. In comparison to the common production process of photoproduction $\gamma p \to \psi p$, the introduced method does not only have a non negligible total cross section compared to those of photonproduction but offers the possibility to measure scattering processes near threshold energies.

Another promising experiment at the FAIR facility at the GSI is the PANDA experiment. FAIR is an accelerator which provides the experiment with an antiproton beam. PANDA in particular has lots of different questions to solve but in this context we shall only discuss hadronic interaction studies.

Other studies are non-perturbative QCD in general, an analysis of the formfactors of nuclei as well as determination of in-medium properties[3]

[2] D-Meson: The lightest particle containing a charm quark

[3] Such as the origin of hadron masses in the context of spontaneous chiral symmetry

which can be achieved via different nuclear targets which provides a broad range of charm physics with nuclei. The hadronic interaction studies are based on D-meson interaction and the J/ψ dissociation. If the antiprotons have enough energy for higher mass charmonium states which decays into open charm we will get some insights about D-meson interaction with nuclei inside the target material.

The J/ψ dissociation in hadron collisions is mainly based on gluonic processes, hence new experimental data about gluon structure functions in nuclei can be gathered. The expected cross section for this experiment is little model dependent for antiproton momenta of approximately 4 GeV. At higher momenta such as 6.2 GeV which allow ψ' resonant production we can observe the inelastic process $\psi' N \to J/\psi N$.

As discussed above the gluonic interaction of $J\psi$ is supposed to dominate in elastic scattering processes. The PANDA experiment is therefore an up-and-coming opportunity to analyse these gluonic interactions, i.e. the still challenging description of strong interaction.

breaking in QCD

2 Van der Waals Forces Between Neutral Atoms

The interesting point of having two neutrally charged atoms with electric and magnetic polarizability α_E, respectively β_M separated by a distance R (Figure (2.1)), is that an electromagnetic interaction exists nonetheless, the van der Waals interaction. The polarizability describes the strength of dipole moments of an atom, due to external electromagnetic fields. We will see that this is the property responsible for van der Waals-interaction and is defined as follows [17]

$$\alpha_E = 2 \sum_{n,n\neq 0} \frac{|\langle n|e\vec{r}\cdot\hat{r}'_E|0\rangle|^2}{E_0 - E_n} \quad , \tag{15}$$

where the product of $\vec{r}$, the spatial vector for the electron displacement within the atom, and $\hat{r}'_E$, the direction of the electromagnetic field, is evaluated in the eigenstates of the atom, that is E_n are the energy eigenstates of the atom and $|n\rangle$ the corresponding states [18]. The magnetic polarizability can be defined analogously

$$\beta_M = 2 \sum_{n,n\neq 0} \frac{|\langle n|\vec{s}\cdot\hat{r}'_B|0\rangle|^2}{E_0 - E_n} \quad , \tag{16}$$

now being $\hat{r}'_B$ the direction of the magnetic field and $\vec{s}$ the spin of the atom.

In this chapter we will discuss two different approaches to the problem by focusing on large and small distances R which requires different methods. The easiest system of that kind consists of two hydrogen atoms and is studied below.

First the system is analysed quantum mechanically [19], which gives a small range behaviour for this system. On the contrary the long range behaviour is obtained by methods of quantum field theory.

2.1 London Force: Quantum Mechanical Calculation

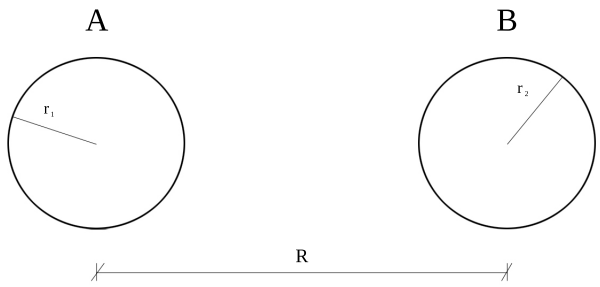

Figure 2: The set up

The vectors r_1 and r_2 refer to the separation of the electron from the proton of the first (A) and second (B) hydrogen atom. Electrons as well as nucleons are Dirac-particles, hence spin 1/2-particles, therefore the atoms can have spin as well, but for convenience it is neglected at this point.

The first part will be to calculate the energy of this system. The most basic object of this calculation is the Hamiltonian. The entire Hamiltonian then reads

$$H = -\frac{1}{2m}\left(\bigtriangledown_1^2 + \bigtriangledown_2^2\right) - \frac{e^2}{r_1} - \frac{e^2}{r_2} + \frac{e^2}{R} + \frac{e^2}{r_{12}} - \frac{e^2}{r_{1B}} - \frac{e^2}{r_{2A}} \ , \tag{17}$$

where we can identify

$$H_0 = -\frac{1}{2m}\left(\bigtriangledown_1^2 + \bigtriangledown_2^2\right) - \frac{e^2}{r_1} - \frac{e^2}{r_2} \tag{18}$$

which only contains the sum of internal electron-proton interaction of atom A and atom B, whereas

$$H' = \frac{e^2}{R} + \frac{e^2}{r_{12}} - \frac{e^2}{r_{1B}} - \frac{e^2}{r_{2A}} \tag{19}$$

represents the mixed interactions, i.e. the electron-electron, proton-proton and the opposite electron-proton interactions. r_{xy} stands for the absolute value of the vector from particle x to particle y, i.e. $R = |r_A - r_B|$, $r_{12} = |r_1 - r_2|$, $r_{1B} = |r_1 - r_b|$ and $r_{2A} = |r_2 - r_A|$ in which r_X goes from the origin to the center of atom (X).

For the sake of an argument let us assume that $R \gg a_0 = 4\pi\epsilon_o/m_e$, with a_0 being the Bohr-radius. Since we are dealing with hydrogen atoms for which $r_1, r_2 \sim a_0$, the H_0-interaction dominates, whereas H' can be treated as perturbation, as we will see later on when expanding the Hamiltonian in terms of the dimensionless parameter r_i/R. The following part will give a derivation of the energy $E(R)$ stored in this system in terms of the distance between both atoms.

We see that H_0 is composed of two independent hydrogen atom-Hamiltonians, hence the wavefunction u_0 is given by the product of isolated hydrogen atoms in the ground state [4]

$$u_0\left(\bar{r}_1, \bar{r}_2\right) = u_{100}\left(\bar{r}_1\right) \cdot u_{100}\left(\bar{r}_2\right) \quad . \tag{20}$$

Therefore the ground state energies of both isolated atoms add up to the total energy of the system

$$E_0 = -\frac{e^2}{a_0} \quad . \tag{21}$$

To tread the perturbation H' as easily as possible and still obtain a suitable result, we expand (19) to get

[4] Which each have a ground state energy of $E_0 = -\frac{e^2}{2a_0}$

$$
\begin{aligned}
H' &= e^2\left(\frac{1}{R}+\frac{1}{r_{12}}-\frac{1}{r_{1B}}-\frac{1}{r_{2A}}\right)= \\
&= \frac{e^2}{R}\left\{1-\left(1-\frac{2z_1}{R}+\frac{r_1^2}{R^2}\right)^{1/2}-\left(1+\frac{2z_2}{R}+\frac{r_2^2}{R^2}\right)^{1/2}\right. \\
&\left.+\left(1+\frac{2(z_1-z_2)}{R}+\frac{(x_1-x_2)^2+(y_2-y_1)^2+(z_2-z_1)^2}{R^2}\right)^{-1/2}\right\} \\
&\approx \frac{e^2}{R^3}\left(x_1x_2+y_1y_2-2z_1z_2\right) \quad . \qquad (22)
\end{aligned}
$$

Using this expansion of the first order in perturbation

$$\Delta E^{(1)} = \langle 0|H'|0\rangle = 0 \qquad (23)$$

vanishes. The ground state of this hydrogen system is only dependent on $|x|$, which means that this integral has to vanish, due to a parity transformation $\bar{x} \to -\bar{x}$.

At second order in perturbation theory the correction to the energy is [2]

$$\Delta E^{(2)} = \sum_{n,n\neq 0} \frac{|\langle 0|H'|n\rangle|^2}{E_0 - E_n} \qquad (24)$$

This is difficult to calculate analytically because the eigenstates $|n\rangle$ contain Laguerre polynomials which cannot be generalized calculated in the matrix element $\langle 0|H'|n\rangle$ to eventually end up with an explicit form on an infinite series in n. As an alternative I present some approaches, which give a lower and an upper limit. At this point it is alluded that as $E_n > E_0$ the second order energy correction is always negative, or in physical expressions always attractive!

To estimate a lower limit of (24), we substitute E_n by $E^* = -e^2/(4a_0)$, which is the lowest energy of the combined hydrogen system not being equal to the

ground state energy to construct a constant, n-independent denominator:

$$\Delta E^{(2)} \geq \sum_{n,n\neq 0} \frac{\left|\langle 0|H'|n\rangle\right|^2}{E_0 - E^*} \quad . \tag{25}$$

This estimation is valid for a system for which the energy E_n in the state $|n\rangle$ is higher than in the ground state because by this substitution above the denominator gets smaller, thus the estimated value gives a lower bound. The next step would be to simplify the numerator to get rid of the summation over n:

$$\sum_{n,n\neq 0} \left|\langle 0|H'|n\rangle\right|^2 = \sum_{n} \left|\langle 0|H'|n\rangle\right|^2 - \left|\langle 0|H'|0\rangle\right|^2 = \langle 0|H'^2|0\rangle = 6 \cdot \frac{a_0^6 e^4}{R^6} \quad , \tag{26}$$

where we used the completeness relation and (22) to evaluate the expectation value of H'^2, which is

$$\langle 0|H'^2|0\rangle = \frac{e^4}{R^6} \langle 0| \left(x_1^2 x_2^2 + y_1^2 y_2^2 + 4 z_1^2 z_2^2 + \text{mixed terms}\right) |0\rangle \quad . \tag{27}$$

The mixed terms vanish again following the same parity argumentation as above. The expectation values are the same for $x-, y-$ and z-coordinates

$$\langle 0|H'^2|0\rangle = \frac{6e^4}{R^6} \langle 0|x_1^2 x_2^2|0\rangle \quad . \tag{28}$$

If we use this result we will end up with

$$\Delta E^{(2)} \geq -\frac{8 a_0^5 e^2}{R^6} \quad . \tag{29}$$

To get an upper limit it is necessary to use the variation method, since there

are no easy estimations we could use as we did for the lower limit. A trial function for this method is given by [20]

$$\psi = u_{100}\left(\bar{r}_1\right) \cdot u_{100}\left(\bar{r}_2\right)\left(1 + AH'\right) \quad , \tag{30}$$

with a parameter A which has to be deduced.

After some analytical calculations [5] we get

$$\Delta E^{(2)} \leq \frac{E_0 + 2\left\langle 0|AH'^2|0\right\rangle}{1 + \left\langle 0|A^2H'^2|0\right\rangle} \quad . \tag{31}$$

Using the Taylor-expansion in lowest order

$$H'^2 = \frac{e^4}{R^6}\left(x_1^2x_2^2 + y_1^2y_2^2 + 2z_1^2z_2^2 + \text{mixed Terms}\right) \quad , \tag{32}$$

and dropping the mixed terms, which vanish due to parity anyway. After finding that (31) reaches a minimum at $A = E_0^{-1}$ the final result is

$$\Delta E^{(2)} \leq -\frac{6e^2a_0^5}{R^6} \quad . \tag{33}$$

In summary the second-order energy correction is bounded by

$$-\frac{8a_0^5e^2}{R^6} \leq \Delta E^{(2)} \leq -\frac{6e^2a_0^5}{R^6} \quad . \tag{34}$$

2.1.1 Explanation

At the basis of this calculations are some assumptions we will discuss later on in chapter (2.3). To get a feeling why this correction occurs we connect a classical interpretation to the quantum mechanical perturbation theory. The first order energy correction corresponds to the constant dipole-dipole interaction of both atoms and is the zero energy of this system. The constant dipole of the first atom b develops an electromagnetic field $\sim \frac{d_b}{r^3}$ proportional

[5] Appendix A.2.1

to its dipole moment d_b which interacts with the constant dipole $d_a = -ex_a$ of the second atom a, where x_a is the charge separation. Calculating the dipole moments $\langle d \rangle = \langle -ex \rangle = 0$[6] in the ground state of the atoms, we can verify in a simple way that the energy due to a constant dipole-dipole interaction is equal to zero

$$\Delta E^{(1)} \sim \langle \frac{d_a d_b}{r^3} \rangle = 0 \quad . \tag{35}$$

The second order in perturbation corresponds to the self-interaction of both atoms. The first atom develops an instantaneous dipole moment, which induces a dipole in the second atom which is proportional to the electric field $\vec{d} = 4\pi\alpha_E^b \vec{E}$[7] and the arising field acts back on the first atom. Since the electric field of the first dipole d_b is proportional to $\sim \frac{d_b}{r^3}$, the induced dipole moment of the second atom a will show the same dependence on r. Transferring this considerations to the electric field of the second atom b, it will show due to the induced dipole a proportionality $\sim \frac{d_b d_a}{r^6}$. These considerations finally result in an overall energy of

$$\Delta E^{(2)} \sim -\frac{\alpha_E^a \alpha_E^b}{r^6} \quad . \tag{36}$$

The induced dipole orients towards the instantaneous dipole in exactly the opposite orientation, which explains the minus sign corresponding to an attractive force. Nonetheless it is remarkable that two neutral charged atoms separated by a distance R develop an attractive force, which is called van der Waals force. For this separation distances, for which the interaction can be assumed to be instantaneous, the van der Waals force is also referred to as London force, named after F. London who calculated this force first [21, 17].

[6] Parity $x \to -x$: $\langle x \rangle = -\langle x \rangle$ in the ground state

[7] The polarizability is assumed to be isotropic. This linear relation is therefore an approximation

2.2 Casimir-Polder force: Quantum Field Theoretical Calculation

In quantum field theory both atoms are represented by a scalar field Φ . p_1 is the momentum of atom A and p_2 the momentum of atom B. The primed quantities stand for the momenta after the photon absorption/emission. Each photon carries a momentum $q-k$, respectively k. As before we neglect the spin of interacting atoms.

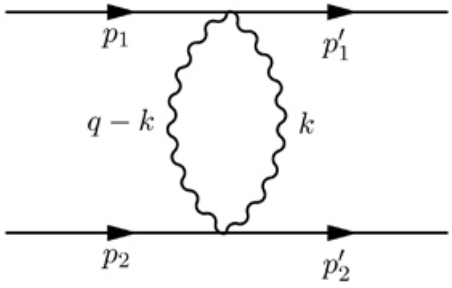

Figure 3: Feynman diagram for two photon exchange

Thinking about the energy in this system, it can be argued, that for any constant, weak electric field $\bar{E}$ the relations $\bar{E} = -\bar{\nabla}\Phi$ for some potential Φ, as well as for the polarization $\bar{P} = 4\pi\alpha_E\bar{E}$ with α_E being the polarizability of an atom holds. The energy change due to a change in the electric field is

$$\delta E_E^{(1)} = \int d^x\rho(x)\Phi = -\int d^3x\rho(x)\bar{x}\cdot\bar{E} = -\bar{P}\cdot\bar{E}$$
$$= -4\pi\alpha_E\bar{E}\cdot\bar{E} = -4\pi\alpha_E\frac{E^2}{2} \quad . \quad (37)$$

This calculation can be done analogously for the energy change due to the magnetic field $\delta E_B^{(1)} = -4\pi\beta_M\frac{\bar{B}^2}{2}$ and therefore the overall energy change is [3]

$$\delta E^{(1)} = -\frac{1}{2}\left(4\pi\alpha_E \bar{E}^2 + 4\pi\beta_M \bar{B}^2\right) \quad . \tag{38}$$

Equation (38) indirectly tells us the underlying Hamiltonian. Using $E^i = -F^{0i}$ and $B^i = -\frac{1}{2}\epsilon^{0i\mu\nu}F_{\mu\nu}$ the energy correction is equivalent to

$$\delta E^{(1)} = \left(-\frac{4\pi(\alpha_E + \beta_M)}{2}\right) F^{0i}F_{0i} + \left(-\frac{4\pi\beta_M}{4}\right) F^2 \tag{39}$$

which can be generalized to

$$\delta E^{(1)} = g_1 M v^a_\alpha v^b_\beta F^{\alpha\gamma}F^\beta_\gamma + g_2 F_{\mu\nu}F^{\mu\nu} \quad , \tag{40}$$

with $v^a_\alpha = v^b_\beta = (1,0,0,0)$ being the velocities of atom A and atom B and $g_1 = -4\pi\frac{\alpha_a+\alpha_b}{2M}$ and $g_2 = -4\pi\frac{\alpha_b}{4}$ the coupling constants and $M = m_a m_b$ the product of the atom masses. During the further calculations we use a heavy field approximation for Φ_i, which means $p_i \approx p_i'$. The momentum is given by $p_i = mv_i$ with the four velocity introduced in eq. (40). Heavy field approximation means that the particles are at rest compared to the photons.

This leads us to the Lagrangian

$$\mathcal{L}_{int} = g_1 \partial_\alpha\Phi\partial_\beta\Phi F^{\alpha\gamma}F^\beta_\gamma + g_2\Phi^2F^2 \quad , \tag{41}$$

since it is the only Lorentz-scalar, which is quadratic in $F_{\mu\nu}$ and Φ and reproduces the energy above.

Based on the given Feynman diagram, it is essential for further calculations to determine the corresponding amplitude, i.e. the vertex of photon annihilation and creation, which is given by [22]

$$(2\pi)^4\delta^4(p_1 - (q-k) - k - p_1')i\mathcal{M}_V = \langle k, q-k, p_1'|Te^{-i\int dt\mathcal{H}_{int}}|p_1\rangle \quad . \tag{42}$$

This expression contains the physical behaviour of the involved particles at those Feynman vertices and the pure interacting part is specified by the initial and final states (hence the corresponding bra's and ket's of the Fock space) and the interaction Hamiltonian. The delta-distribution on the left hand side represents a fundamental principle in physic namely four-momentum conservation, i.e. energy and momentum conservation at each vertex.

In this notation the T operator stands for the time ordered product of what it is acting on. Using the Taylor expansion of the exponential function in lowest order of the coupling constants g_1 and g_2 and applying Wick's theorem which states that the time-ordered product of some expression is equivalent to the normal-ordered sum of the original expression and all its contractions, we first have to think about the contractions of all appearing operators.

Expressing the field in terms of creation and annihilation operators $a_p^\dagger$ and a_p the scalar field is [8]

$$\Phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x} \right) \quad . \tag{43}$$

These operators are used to create a particle with momentum p, respectively destroying a particle with momentum p, i.e. $a_p^\dagger |0\rangle = \sqrt{2E_p}\, |p\rangle$ with $|0\rangle$ being the vacuum state.

With this we can determine the contraction of the scalar fields

$$\Phi(x)\, |p\rangle = e^{ip\cdot x}\, |0\rangle \ , \ \langle p|\, \Phi(x) = \langle 0|\, e^{-ip\cdot x} \quad , \tag{44}$$

and its derivatives

[8] All operators in this chapter are given in the interaction representation

$$\partial_\mu \Phi(x) \left|p\right\rangle = i p_\mu e^{ip\cdot x} \left|0\right\rangle \ , \ \left\langle p\right| \partial_\mu \Phi(x) = \left\langle 0\right| (-) p_\mu e^{-ip\cdot x} \ . \tag{45}$$

So far we have dealt with the action of field operators acting on elements of a momentum Hilbert space. Since we are assuming photon exchange, it is necessary to work out how photon operators

$$A^\mu(x) = \sum_k \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_k^\dagger \epsilon_k^\mu e^{ik\cdot x} + a_k \epsilon_k^{\mu *} e^{-ik\cdot x} \right) \ , \tag{46}$$

and its derivatives act on the photon space. In this notation a_k and $a_k^\dagger$ are still annihilation and creation operators now operating on the photon space and ϵ_k^μ being the polarization.

Then we find

$$A^\mu(x) \left|k\right\rangle = \epsilon_k^{\mu *} e^{-ik\cdot x} \left|0\right\rangle \quad \text{and} \quad \partial^\nu A^\mu \left|k\right\rangle = -i \epsilon_k^{\mu *} k^\nu e^{-ik\cdot x} \left|0\right\rangle \ . \tag{47}$$

The same argumentation could be done with the analogue bra's while keeping in mind that the sign will change in the exponent of the exponential function and the polarization will change over to its Hermitian conjugate.

Looking back at the Hamiltonian in eq. (41) we are interested in $F^{\mu\nu}$

$$F^{\mu\nu} \left|k\right\rangle = \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) \left|k\right\rangle = -i \left(\epsilon^{\nu *} k^\mu - \epsilon^{\mu *} k^\nu\right) e^{-ik\cdot x} \left|0\right\rangle \ . \tag{48}$$

Now that we know how to deal with those operators we are able to calculate all required contractions and due to Wick's theorem it is the only part which contributes to this calculation, as one essential property of the normal ordered product is that it vanishes in the expectation value.

Applying all contractions the careful reader would have noticed, that both the Φ and F contractions give a symmetry factor of 2! so overall a factor of 4. Substituting in the contractions the amplitude reads (Notice another $\frac{1}{2!}$ of the exponential's Taylor-expansion)

$$\begin{aligned}(2\pi)^4\delta^4(p_1-(q-k)-k-p_1')i\mathcal{M}_V = 2\int d^4x\Big(&- g_1 p_{1\alpha}p_{1\beta}\epsilon_b^{\gamma *}(q-k)^\alpha - \\ &- \epsilon_b^{\alpha *}(q-k)^\gamma\left(\epsilon_\alpha^a k^\beta - \epsilon_\beta^a k^\alpha\right)e^{i\left(p_{1\beta}-p_{1\alpha}+(k-q)+k\right)x} - \\ &- g_2\left(\epsilon_\nu^{b*}(q-k)_\mu - \epsilon_\mu^{b*}(q-k)_\nu\right)\left(\epsilon_a^{\nu *}k^\mu - \epsilon_a^{\mu *}k^\nu\right)e^{i(k+q-k)x}\Big)\quad , \end{aligned} \tag{49}$$

and using the integral representation of the delta-distribution which cancels out on both sides we finally get

$$\begin{aligned}i\mathcal{M}_V = -2\Big(&g_1\Big(p_1\cdot(q-k)(p_1\cdot k)(\epsilon_b^*\cdot\epsilon_a) - (p_1\cdot\epsilon_b^*)(p_1\cdot k)((q-k)\cdot\epsilon_a) - \\ &- (p_1\cdot\epsilon_a)(p_1\cdot(q-k))(\epsilon_b^*\cdot k) + (p_1\cdot\epsilon_b^*)(p_1\cdot\epsilon_a)((q-k)\cdot k)\Big) + \\ &+ g_2\Big((\epsilon_a^*\cdot\epsilon_b^*)(k\cdot(q-k)) - (k\cdot\epsilon_b^*)(\epsilon_a^*\cdot(k-q)) - \\ &- (\epsilon_a^*\cdot(q-k))(k\cdot\epsilon_b^*) + (\epsilon_a^*\cdot\epsilon_b^*)(k\cdot(q-k))\Big)\Big)\quad . \end{aligned} \tag{50}$$

Now reinserting the coupling constants $g_1 = -4\pi\frac{\alpha_E+\beta_M}{2M}$ and $g_2 = -4\pi\frac{\beta_M}{4}$ and simplifying, i.e. combine the terms proportional to β_M and α_E where we have used the four-vector notation $p_i^2 = m_i^2$

$$
\begin{aligned}
i\mathcal{M}_V = -2\Bigg(4\pi\frac{\alpha_E}{2m_i^2}\Big\{\Big(p_1\cdot(q-k)\epsilon_b^{\gamma *} - (p_1\dot{\epsilon}_b^*)(q-k)^\gamma\Big)\times \\
\times\Big((p_1\cdot k)\epsilon_{a,\gamma} - (p_1\cdot\epsilon_a)k_\gamma\Big)\Big\} + 4\pi\frac{\alpha_B}{2m_i^2}\Big\{(\epsilon_a^*\cdot\epsilon_b^*)\Big((p_1\cdot(q-k))(p_1\cdot k) - \\
- p_1^2(k\cdot(q-k))\Big) + (k\cdot\epsilon_b^*)\Big(p_1^2(\epsilon_a^*\cdot(q-k)) - (p_1\cdot\epsilon_a)(p_1\cdot(q-k))\Big) + \\
+ (p_1\cdot\epsilon_b^*)\Big((p_1\cdot\epsilon_a(k\cdot(k-q)) - (p_1\cdot k)(\epsilon_a\cdot(q-k))\Big\}\Bigg)\ . \qquad (51)
\end{aligned}
$$

If we used some algebra, we would see that the expression proportional to α_B is equal to [9]

$$
(\epsilon^{\alpha\beta\gamma\delta}\epsilon^*_{\alpha\beta}k_\gamma p_{1,\delta})(\epsilon_{\alpha\rho\sigma\lambda}\epsilon_b^{\rho *}(q-k)^\sigma p_1^\lambda)\ . \qquad (52)
$$

The $\epsilon^{\alpha\beta\gamma\delta}$ is the Epsilon-Tensor, or Levi-Civita-Tensor, whereas the ϵ's with two indices remain the polarization vectors.

Let us now recall the primary problem and its corresponding Feynman diagram. Since we are still interested in some physical expressions we need to transfer the ideas of this section in some measurable quantities, or let's say some quantities which fully determine the physical behaviour of this system. In other words we need to compute the matrix-element of the 2 photon-exchange process

$$
\mathcal{M}(q) = \frac{1}{2!}\int\frac{d^4k}{(2\pi)^4}\frac{-i\eta^{\alpha\gamma}}{k^2}\cdot\frac{-i\eta^{\beta\delta}}{(k-q)^2}\times\text{Vertex a}\times\text{Vertex b} \qquad (53)
$$

which contains a symmetry factor, as well as both photon-propagators.

Both vertices are given by eq. (51), where we have to take care of the appearing momenta, by replacing p_1 with p_2 as well as k with $-k$ and $k-q$

[9] Appendix A.2.2

with $q-k$ if we are going from Vertex a to Vertex b. For convenience in the calculation of eq. (53) we factor out the polarization vectors, since $\epsilon_i^\mu \epsilon_{i,\mu} = \epsilon_i^{\mu *} \epsilon_{i,\mu}^* = 1$ so that we find

$$\begin{aligned} -\frac{i4\pi}{m_a^2}\epsilon_a^{\alpha *}\epsilon_b^{\beta *}\Big(\alpha_E^a\Big\{ & k\cdot p_1((q-k)\cdot p_1)\eta_{\alpha\beta} - (q-k)_\alpha p_{1,\beta} k\cdot p_1 \\ & - p_{1,\alpha}k_\beta (q-k)\cdot p_1 + k\cdot(q-k)p_{1,\alpha}p_{1,\beta}\Big\} \\ & - \beta_M^a\Big\{\epsilon_\alpha^{\lambda\gamma\delta}k_\gamma p_{1,\delta})(\epsilon_{\lambda\beta\sigma\tau}(q-k)^\sigma p_1^\tau\Big\}\Big) \end{aligned} \tag{54}$$

for Vertex a, and

$$\begin{aligned} -\frac{i4\pi}{m_b^2}\epsilon_{a,\alpha}^*\epsilon_{b,\beta}^*\Big(\alpha_E^b\Big\{ & (-k)\cdot p_2((k-q)\cdot p_2)\eta^{\alpha\beta} - (-k)^\alpha p_2^\beta (k-q)\cdot p_2 \\ & - p_2^\alpha (k-q)^\beta (-k)\cdot p_2 + (-k)\cdot(k-q)p_2^\alpha p_2^\beta\Big\} \\ & - \beta_M^b\Big\{\epsilon^{\alpha\beta\gamma\delta}(-k)_\gamma p_{2,\delta})(\epsilon_{\alpha\rho\sigma\lambda}(k-q)^\sigma p_2^\lambda \eta^{\rho\alpha}\Big\}\Big) \end{aligned} \tag{55}$$

for vertex b. As we can see we get some covariant expressions for the four momenta after factoring out the polarization.
Evaluating eq. (53) with this information we get a pretty longish integral to deal with

$$\begin{aligned} \mathcal{M}(q) = \frac{1}{2!}\frac{(4\pi)^2}{m_a^2 m_b^2}\int \frac{d^4k}{(2\pi)^4}\cdot\frac{1}{k^2(k-q)^2}\cdot\Big(\alpha_E^b\Big\{ & k\cdot p_2((k-q)\cdot p_2)\eta^{\alpha\beta} \\ & - k^\beta p_2^\alpha (k-q)\cdot p_2 - p_2^\beta (k-q)^\alpha k\cdot p_2 + k\cdot(k-q)p_2^\alpha p_2^\beta\Big\} + \\ & + \beta_M^b(\epsilon^{\lambda\alpha\gamma\delta}k_\gamma p_{2,\delta})(\epsilon_\lambda^{\beta\kappa\mu}(k-q)_\kappa p_{2,\mu}) + \alpha_E^a\Big\{k\cdot p_1((k-q)\cdot p_1)\eta_{\alpha\beta} \\ & - (k-q)_\alpha p_{1,\beta}k\cdot p_1 - p_{1,\alpha}k_\beta(k-q)\cdot p_1 \\ & + k\cdot(k-q)p_{1,\alpha}p_{1,\beta}\Big\} - \beta_M^a(\epsilon_{\alpha\sigma\tau}^\lambda k^\sigma p_1^\tau)(\epsilon_{\lambda\beta\kappa\mu}(k-q)^\kappa p_1^\mu)\Big)\ . \end{aligned} \tag{56}$$

After some calculations, i.e. expand eq. (56) and using the introduced

loop integrals (Appendix C.3) where we write the products appearing in the matrix element as $p_i \cdot k = p_i^\mu k_\mu$, as well as $k^2 = g^{\mu\nu} k_\nu k_\mu$ the final result is then

$$\mathcal{M}(q) = -\frac{Lq^4}{240}\left[23\left(\alpha_E^a \alpha_E^b + \beta_M^a \beta_M^b\right) - 7\left(\alpha_E^a \beta_M^b + \alpha_E^b \beta_M^a\right)\right] \quad . \tag{57}$$

This result only contains the logarithmic dependency. There would be additional constant terms, which are of no interest at this point. In order to determine the potential, which we need to compare the result to the quantum mechanical calculations of chapter 2.1, we Fourier-transform the matrix-element via

$$\begin{aligned} V(R) = -\int \frac{d^3q}{(2\pi)^3}\mathcal{M}(q)e^{-i\bar{q}\cdot\bar{R}} = \\ = \frac{-23\left(\alpha_E^a \alpha_E^b + \beta_M^a \beta_M^b\right) + 7\left(\alpha_E^a \beta_M^b + \alpha_E^b \beta_M^a\right)}{4\pi R^7} \quad , \end{aligned} \tag{58}$$

where we used the Fourier-Integral

$$\int \frac{d^3q}{(2\pi)^3} q^4 \ln(q^2) e^{-i\bar{q}\cdot\bar{R}} = -\frac{60}{\pi R^7} \quad . \tag{59}$$

In order to calculate the energy variation to the interaction described by this potential we use quantum mechanical perturbation theory in first order.

$$\Delta E^1 = \langle 0|V(R)|0\rangle = \frac{-23\left(\alpha_E^a \alpha_E^b + \beta_M^a \beta_M^b\right) + 7\left(\alpha_E^a \beta_M^b + \alpha_E^b \beta_M^a\right)}{4\pi R^7} \quad . \tag{60}$$

2.3 Explanation and Comparison to London Forces

We see that relativistic calculations lead to a different potential than the non-relativistic. This effect is known as retardation and is due to the finite speed of light. Using the Coulomb potential in non-relativistic quantum mechanical calculations of chapter (2.1), the interaction between both atoms is assumed to be instantaneous. This approximation is legitimate for small distances, i.e. the time photons need to travel between both atoms is very small.

In the quantum field theoretical treatment however, the finite speed of light plays a major role, since the atoms are separated too much to assume an instantaneous interaction. The result is the long distance behaviour of van der Waals forces between two neutral hydrogen atoms, first calculated by H. B. G. Casimir and D. Polder [23], which is why it is also called Casimir-Polder force.

To understand the reason behind this, we analyse the photon propagator $\frac{i\eta_{\mu\nu}}{p^2+i\epsilon}$ in more detail. If we integrated over the p^0 component, we would notice two poles at $p^0 = \pm\sqrt{\vec{p}^2 - i\epsilon}$ in the complex plane. The residue theorem states that the complex integral is completely determined by the enclosed residues, assumed that in the limit $R \to \infty$, with R being the radius of the semi-circle shaped integration path, the fraction of the imaginary part vanishes. Doing so we would obtain two results depending on which residue we choose. The two propagators obtained are called the retarded and advanced photon propagator. The retarded propagator describes a signal which travels forward in time. The result of quantum field theory calculations which therefore contains a time dependant effect due to the finite speed of the signal.

Until last we have used the assumption of spinless interacting atoms. In this context the question might arise what would happen to the result if

non-zero spin particles interacted. The answer justifies the overall treatment of spin zero atoms, since we would find that the potential still shows a $1/R^7$ dependency. The coefficient of eq. (58) suggests that we would obtain another spin based constant in the numerator which is indeed the case.

A result valid for all discussed distances both results have to be matched. The important question is where the transition takes place. One finds that retardation effects become noticeable at $\tau = 1/\Delta E$, with τ being the retardation time and ΔE a fraction of the ionisation energy of one atom [24].

3 Extension to QCD and Conclusion

The approach to electromagnetic interaction computations are based on a two photon exchange leading us to a long range formula. A gluonic interaction of two color singlet hadrons is based on a gluon exchange. Since both hadrons are colorless and remain colorless the exchange of a single color gluon is forbidden. The exchange of two gluons, however, which can be together in a color singlet state is not forbidden.
The arising question is how are the calculations done for electromagnetically interacting atoms related to the case of strongly interacting hadrons and are there finally gluonic van der Waals forces? [25]

To reveal some similar phenomenona of electromagnetic and strong interacting processes we will have a look on the Yukawa potential which was primary developed to describe an interaction due to the exchange of massive scalar fields[26]

$$V_{str}(R) \sim \frac{g^2}{4\pi}\frac{e^{-Rm}}{R} \quad , \tag{61}$$

with a constant g and the mass m of the mediated particle. His concept was to construct a potential which describes the observed finite range of strong interaction. He finally came up with a result, which is based on the assumption of meson exchange - at this time the pion exchange.

In fact we notice a similarity to the electromagnetic Coulomb potential which can be fully recovered in the photon mass limit $m \to 0$. However, as we have stated in chapter 1.3 the force conditioned by meson exchange is much smaller than forces of gluon exchange which is consistent with Yukawas model due to a $\exp(-R)$ dependency.

Most of the theoretical theories are based on a long range potential in the form of [27]

$$V_{str}(R) \sim -\frac{C}{R^N} \quad , \tag{62}$$

with a model and framework dependent exponent N and some constant C [28]. Y. Fujii and K.Mima [29] derived a static potential for a gluon exchange between two color singlet hadrons. They used an effective Hamiltonian H for baryon-baryon scattering

$$H \sim \bar{\psi}\psi F^a_{\mu\nu} F^{a,\mu\nu} \quad , \tag{63}$$

with baryon fields ψ and gluon fields $F^{\mu\nu}$ as introduced in chapter (1.1). Wit h this Hamiltonian they computed the amplitudes for two, three and four gluon exchange between baryons via dimensional regularisation. Combining all amplitudes their calculated potential is

$$V(R) \sim -R^{-7} \quad , \tag{64}$$

and has the same distance dependence like the electromagnetic potential and is referred to as the gluonic van der Waals potential. The effective Hamiltonian in (63) looks also similar to the Hamiltonian in QED calculations but due to non-commutativity of the gluon fields there are more Feynman diagrams, i.e. the already mentioned two, three and four gluon exchange.

4 Outlook

Through effective field theories the theoretical understanding of heavy quark-antiquark systems near threshold witnessed a significant progress [30]. Due to a small velocity $v \ll 1$, this system develops a hierarchy of widely spread scales, m the hard scale, mv the soft scale, mv^2 the ultrasoft scale, ...

By integrating out the scales above the energies we want to describe we obtain a suitable framework for theoretical work. The effective theory after integrating out the hard scale is referred to as non-relativistic QCD. Integrating out the next scale, the soft scale, the theory obtained is called potential NRQCD, or (p)NRQCD.

As we have seen perturbative methods are dependent on which scale the system is treated. The matching of these theories must therefore be treated very carefully. By definition of heavy quarks, with $m \gg \Lambda_{QCD}$, the matching from QCD to NRQCD can always be done perturbatively. The perturbative matching from NRQCD to (p)NRQCD is only possible if $mv \gg \Lambda_{QCD}$ [31, 32]. These effective field theories therefore provide the appropriate framework for specific energy settings, e.g. quarkonium-nuclei scattering near threshold energies. [33]

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A Detailed Calculations and Descriptions

A.1 Gell-Mann Matrices and Structure Constants

The generators of SU(3) Lie group transformation matrices are the Gell-Mann matrices λ_a, which are connected to the generators

$$T_a = \frac{1}{2}\lambda_a \ . \tag{65}$$

The matrices λ_a are definied as follows

$$\lambda_a = \begin{pmatrix} \sigma_a & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad a = 1,2,3 \quad \text{and Pauli matrices} \quad \sigma_a \tag{66}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \tag{67}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_1 \end{pmatrix} \quad , \quad \lambda_7 = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_2 \end{pmatrix} \quad , \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \ . \tag{68}$$

These matrices are traceless, Hermitian and obey the normalization relation

$$tr(\lambda_i \lambda_j) = 2\delta_{ij} \ . \tag{69}$$

Linked to those matrices are the structure constants which are defined over Lie-Algebra properties of Gell-Mann matrices

$$[\lambda_i, \lambda_j] = i2f_{ijk}\lambda_k \ . \tag{70}$$

The structure constants have the values

$$f^{123} = 1 \ , \ ^{458} = f^{678} = \frac{\sqrt{3}}{2} \ ,$$
$$f^{147} = f^{165} = f^{246} = f^{257} = f^{345} = f^{376} = \frac{1}{2} \ . \quad (71)$$

A.2 London and Casimir forces

A.2.1 Upper Limit via Variation Method

The trial function is given by

$$\psi(\bar{r_1}, \bar{r_2}) = u_{100}(\bar{r}_1) u_{100}(\bar{r}_2) \left(1 + AH'\right) \quad , \quad (72)$$

for some parameter A. The variation method provides a possibility to give some upper limit.

$$E' \leq \frac{\iint u_0 \left(1 + AH'\right) \left(H_0 + H'\right) u_0 \left(1 + AH'\right) d^3 r_1 dr_2^3}{\iint u_0^2 \left(1 + AH'\right)^2 d^3 r_1 d^3 r_2} \ , \quad (73)$$

with a shorter notation $u_0 = u_{100}(\bar{r_1}) u_{100}$ representing the ground state of the system. The nominator of this expression can be rewritten in this way

$$\iint u_0^2 \left(1 + 2AH' + A^2 H'^2\right) d^3 r_1 d^3 r_2 = 1 + A^2 \left\langle 0 | H'^2 | 0 \right\rangle \quad (74)$$

because $\langle 0|H'|0\rangle = 0$, as we have already determined.

Expanding the denominator and keeping all non-vanishing terms it looks like

$$\iint u_0 \left(1 + AH'\right) \left(H_0 + H'\right) u_0 \left(1 + AH'\right) d^3 r_1 dr_2^3 = E_0 + 2A \left\langle 0|H'^2|0\right\rangle \quad . \tag{75}$$

Combining these two results

$$E' \leq \frac{E_0 + 2A \left\langle 0|H'^2|0\right\rangle}{1 + A^2 \left\langle 0|H'^2|0\right\rangle} \quad , \tag{76}$$

and since $\langle 0|H'|0\rangle \ll 1$ we can expand the denominator in a way

$$\left(1 + A^2 \left\langle 0|H'^2|0\right\rangle\right)^{-1} = 1 - A^2 \left\langle 0|H'^2|0\right\rangle + \mathcal{O}\left(\left\langle 0|H'^2|0\right\rangle^2\right) \quad , \tag{77}$$

so we have than a fairly simplified term

$$E' \leq E_0 + \left(2A - E_0 A^2\right) \left\langle 0|H'^2|0\right\rangle \quad , \tag{78}$$

so that the unknown parameter A can be determined via

$$\frac{\partial E'}{\partial A} \stackrel{!}{=} 0 \Rightarrow A = \frac{1}{E_0} \quad . \tag{79}$$

The upper limit is then

$$E' \leq -\frac{6e^2 a_0^5}{R^6} \quad . \tag{80}$$

A.2.2 Levi-Civita-Tensor Representation in Matrix-Element

The Levi-Civita Tensor is in n-dimensions defined over the permutation of its indices, i.e.

$$\epsilon_{ijkl\ldots} = \epsilon^{ijkl\ldots} = \begin{cases} +1 & \text{if (ijkl}\ldots\text{) even permuation of (1,2,3,4}\ldots\text{)} \\ -1 & \text{if (ijkl}\ldots\text{) odd permuation of (1,2,3,4}\ldots\text{)} \\ 0 & \text{otherwise} \end{cases} . \tag{81}$$

Hence it is equal to $+1$ if the indices are an even permutation of $(1, 2, 3, 4\ldots)$, -1 if it is an odd permutation and zero otherwise.

The expression in eq. (52) contains the product of two epsilon tensors. Using the definition above one finds that

$$\epsilon_{ijkl}\epsilon_{mnop} = \det \begin{pmatrix} \delta_{im} & \delta_{in} & \delta_{io} & \delta_{ip} \\ \delta_{jm} & \delta_{jn} & \delta_{jo} & \delta_{jp} \\ \delta_{km} & \delta_{kn} & \delta_{ko} & \delta_{kp} \\ \delta_{lm} & \delta_{ln} & \delta_{lo} & \delta_{lp} \end{pmatrix} . \tag{82}$$

Applied to our problem

$$(\epsilon^{\alpha\beta\gamma\delta}\epsilon^*_{\alpha\beta}k_\gamma p_{1,\delta})(\epsilon_{\alpha\rho\sigma\lambda}\epsilon_b^{\rho *}(q-k)^\sigma p_1^\lambda) \tag{83}$$

the determinant looks like

$$\delta^\lambda_\beta\delta^\sigma_\gamma\delta^\rho_\delta + \delta^\lambda_\delta\delta^\sigma_\beta\delta^\rho_\gamma + \delta^\lambda_\gamma\delta^\sigma_\delta\delta^\rho_\beta - \delta^\lambda_\delta\delta^\sigma_\gamma\delta^\rho_\beta - \delta^\lambda_\gamma\delta^\sigma_\beta\delta^\rho_\delta - \delta^\lambda_\beta\delta^\sigma_\delta\delta^\rho_\gamma \ , \tag{84}$$

using Laplace's formula. Contracting this expression with the remaining parts

$$\begin{aligned} &(\epsilon^*_a\cdot p_1)((q-k)\cdot k)(\epsilon^*_b\cdot p_1) + (\epsilon^*_a\cdot(q-k))(k\cdot\epsilon^*_b)p_1^2 + \\ &+ (\epsilon^*_a\cdot\epsilon^*_b)\,(k\cdot p_1)\,(p_1\cdot(q-k)) - (\epsilon^*_a\cdot\epsilon^*_b)\,(k\cdot(q-k))\,p_1^2 - \\ &- (\epsilon^*_a\cdot(q-k))\,(k\cdot p_1)\,(p_1\cdot\epsilon^*_b) - (\epsilon^*_a\cdot p_1)\,(k\cdot\epsilon^*_b)\,(p_1\cdot(q-k)) \ , \end{aligned} \tag{85}$$

or simplified

$$\begin{aligned}
&(\epsilon_a^* \cdot \epsilon_b^*) \Big((k \cdot p_1)\,(p_1 \cdot (q-k)) - (k \cdot (q-k)) \Big) p_1^2 + \\
&+ (\epsilon_b^* \cdot k) \Big((\epsilon_a^* \cdot (q-k))\, p_1^2 - (\epsilon_a^* \cdot p_1)\,(p_1 \cdot (q-k)) \Big) + \\
&+ (\epsilon_b^* \cdot p_1) \Big((\epsilon_a^* \cdot p_1)\,(k \cdot (q-k)) - (\epsilon_a^* \cdot (q-k))\,(k \cdot p_1) \Big) \quad . \qquad (86)
\end{aligned}$$

Comparing this to eq. (51) it is exactly the result obtained during this calculations. Since the epsilon-representation is much easier to handle and work with, it is kept for further calculations.

B Rules for Feynman Diagrams

As customary these rules are used to create the corresponding matrix element for some process in each of this theories. It is mentioned, apart from the following rules, that each undetermined loop momentum gives an additional integral $\int d^4p/(2\pi)^4$, as well as fermion loops a factor of (-1) and a process specific symmetry factor.

The underlying interaction Lagrangian for processes studied during quantum field theoretical treatment of the primary problem is

$$\mathcal{L}_{int} = g_1 \partial_\alpha \Phi \partial_\beta \Phi F^{\alpha\gamma} F_\gamma^\beta + g_2 \Phi^2 F^2 \quad . \qquad (87)$$

The following listing shows a summary of the rules which we used for the Feynman diagram 2.2.

$$\begin{aligned}
\text{Photon propagator} \quad & \frac{-i\eta^{\mu\nu}}{q^2+i\epsilon} \\
\text{Vertex} \quad & -\frac{i4\pi}{m_b^2}\epsilon^*_{a,\alpha}\epsilon^*_{b,\beta}\Big(\alpha_E^b\Big\{(-k)\cdot p((k-q)\cdot p)\eta^{\alpha\beta} - \\
& -(-k)^\alpha p^\beta (k-q)\cdot p - p^\alpha (k-q)^\beta (-k)\cdot p + \\
& +(-k)\cdot(k-q)p^\alpha p^\beta\Big\} - \beta_M^b\Big\{(\epsilon^{\alpha\beta\gamma\delta}(-k)_\gamma p_{2,\delta}) \\
& (\epsilon_{\alpha\rho\sigma\lambda}(k-q)^\sigma p^\lambda \eta^{\rho\alpha})\Big\}\Big) \\
\text{External scalar} \quad & 1 \ .
\end{aligned}$$

C Tensor-Integrals in Dimensional Regularization

C.1 General integrals

In further calculations there appear some essential integrals which are shortly listed in this chapter. The following integral vanishes in dimensional regularization, since it is scaleless.

$$\int \frac{d^dk}{(2\pi)^d} k^{2\alpha} = 0 \quad , \quad \alpha \in \mathbb{Z} \ . \tag{88}$$

There also appear some integrals which vanish, because of symmetric argumentation.

$$\int \frac{d^dk}{(2\pi)^d}\frac{k\cdot q}{k^2} = -(-1)^d \int \frac{d^dk}{(2\pi)^d}\frac{k\cdot q}{k^2} = 0 \quad \text{, for} \quad d=4 \ . \tag{89}$$

C.2 Master-Integral

The most important integral to evaluate the integral 56 is

$$I_0 \equiv \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2(k-q)^2} \quad . \tag{90}$$

To get an integral, which is only dependant on the magnitude of the momentum k, we introduce Feynman-parameters [22]

$$\frac{1}{AB} = \int_0^1 dxdy \frac{\delta(x+y-1)}{(xA^2+yB^2)^2} \quad . \tag{91}$$

Applied to our problem it reads

$$\int \frac{d^4k}{(2\pi)^4} \int_0^1 dxdy \frac{\delta(x+y-1)}{(xk^2+y(k-q)^2)^2} \quad . \tag{92}$$

Due to the delta-distribution x and y sum up to 1, so the denominator can be completed to a full square

$$xk^2 + y(k-q)^2 = (k-yq)^2 + (y-y^2)q^2 \quad , \tag{93}$$

and finally because of the translation invariance of the integral, the momentum can be shifted $(k-yq) \to k$, so the integral looks like

$$\int \frac{d^4k}{(2\pi)^4} \int_0^1 dxdy \frac{\delta(x+y-1)}{(k^2+(y-y^2)q^2)^2} \quad . \tag{94}$$

The integration over the momentum can be first calculated, so the interesting part is

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+(y-y^2)q^2)^2} \tag{95}$$

which is an integral with a solely dependence on the momentum magnitude.

This is the point where the method of dimensional regularization comes

into play, since this integral obviously diverges in four dimensions. The dimensional regularization act on the assumption that we deal with a d-dimensional integral like

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - \Delta^2)^2} \quad , \tag{96}$$

where $\Delta^2 = -(y - y^2)q^2$.

This integral hast to be computed in Minkowski space, but for further calculations we need a Euclidean representation. To put things right we transform this integral via Wick rotation, which gives us an additional factor of i. In general a d-dimensional integration can be split up into an angle-dependent and a magnitude-dependent part

$$\int d^d k = \int d\Omega_d \int dk k^{d-1} \quad . \tag{97}$$

Since the integral we deal with is only k-dependent the angle-integration can be done separately. Using the definition of a d-dimensional Gaussian integral

$$\sqrt{\pi}^d = \int_{-\infty}^{\infty} dx_1 \ldots \int_{-\infty}^{\infty} dx_d exp\left(-\sum_{i=1}^{d} x_i^2\right) \tag{98}$$

which can be rewritten in a way like

$$\sqrt{\pi}^d = \int d\Omega_d \int_0^{\infty} dx x^{d-1} e^{-x^2} \tag{99}$$

since its only dependence is x^2.

Doing some transformations one gets for the integration over x

$$\int_0^\infty dx x^{d-1} e^{-x^2} = \int_0^\infty dx (x^2)^{\frac{d-1}{2}} e^{-x^2} = \int_0^\infty d(x^2)(x^2)^{\frac{d}{2}-1} e^{-x^2}$$
$$= \int_0^\infty dy (y)^{\frac{d}{2}-1} e^{-y} = \frac{1}{2}\Gamma\left(\frac{d}{2}\right) \quad , \tag{100}$$

where it is used that $d(x^2) = 2x \cdot dx$ and the definition of the Gamma function. For some properties of the Gamma-function first read the next section of the appendix, where I summarized the most important properties for the following calculations.

Comparing both sides of equation (97) one can determine the representation of the angle-integration

$$\int d\Omega_d = \frac{2\sqrt{\pi}^d}{\Gamma\left(\frac{d}{2}\right)} \quad . \tag{101}$$

Substituting this into the primary integral we get

$$\int \frac{d^d k}{(2\pi)^d} \frac{i}{(k^2 - \Delta^2)^2} = \frac{2i\sqrt{\pi}^d}{\Gamma\left(\frac{d}{2}\right)} \int dk \frac{k^{d-1}}{(k^2 - \Delta^2)^2} \quad . \tag{102}$$

The next task would be to calculate the simplified remaining integral which after a transform into an integral over k^2 and a substitution $p = \frac{\Delta^2}{k^2 - \Delta^2}$ looks like

$$\begin{aligned} \frac{1}{2}\int_0^1 dp \Delta^{-2} \left(\frac{\Delta^2}{p} + \Delta^2\right)^{d/2-1} &= \frac{1}{2}\Delta^{d-4} \int_0^1 dp \, (1+p)^{d/2-1} \, p^{-d/2+1} \\ &= \frac{1}{2}\Delta^{d-4} \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(2 - \frac{d}{2}\right)}{\Gamma(2)} \quad , \end{aligned} \tag{103}$$

where the definition of the Beta-function (eq. 147) is used.

The preliminary result is therefore

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2-\Delta^2)^2} = \frac{i}{(4\pi)^{d/2}} \Gamma\left(2-\frac{d}{2}\right) \Delta^{d-4} \quad . \tag{104}$$

Reinserting $\Delta^2 = -(y-y^2)q^2$ the remaining part to calculate is

$$\frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{d/2}} \int_0^1 dx \left(-(x-x^2)q^2)\right)^{d/2-2} \quad , \tag{105}$$

where we have inserted the afore calculated integral and substituted $y-y^2 = x-x^2$.

As we can see the integral has been transformed by dimensional regularization to continuous function of the dimension d, with a pole at $d=4$. To avoid the divergence at $d=4$ we set $d=4-2\epsilon$, which gives us a finite value and let us identify the term responsible for the divergent behaviour. Since we can now get rid of that term by renormalization methods, we are still interested in converging terms. That is why we focus on terms which are finite when taking the limit $\epsilon \to 0$.

$$\frac{i\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}} \int_0^1 dx \left(-(x-x^2)q^2)\right)^{-\epsilon} \qquad \text{, for } \; d=4-2\epsilon \quad . \tag{106}$$

The integrand can be rewritten in $\exp\left(-\epsilon \cdot \ln(-q^2(x-x^2))\right)$ and taylor expanded $\exp(x) \approx 1+x+O(x^2)$ so that it should look like

$$\int_0^1 dx \left(1-\epsilon \cdot \ln(-q^2(x-x^2))\right) + \mathcal{O}(\epsilon^2) \quad . \tag{107}$$

With the expansion of the Gamma-function inserted and using $(4\pi)^\epsilon = 1+\epsilon \ln(4\pi) + \mathcal{O}(\epsilon^2)$ we get (apart from a factor $1/(4\pi)^2$)

$$\lim_{\epsilon\to 0}\left(\frac{1}{\epsilon}+\gamma_E\right)\int_0^1 dx\left(1-\epsilon\cdot\ln(-q^2(x-x^2))\right)(1+\epsilon\ln(4\pi))+\mathcal{O}(\epsilon^2)\ . \tag{108}$$

As we can see here, the first part gives us the divergent part, which we could eliminate and therefore drop at this point.

Since we are still interested in four dimensions, we take the limit $\epsilon \to 0$

$$\boxed{I_0=\lim_{\epsilon\to 0}\frac{-i\epsilon\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}}\ln(-q^2)+const.=\frac{-i}{32\pi^2}\cdot 2L+const.} \tag{109}$$

with $L=\ln(-q^2)$.

C.3 Loop Integrals

Referring to the previous chapter one can express the following integrals with the master-integral

$$I_0\equiv\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2(k-q)^2}\ . \tag{110}$$

The keyword is tensorial decomposition which assumes that the tensorintegrals are proportional to combinations of the metric tensor and the parameters which are not integrated over, i.e. the momentum q which results in identification of proportionalities. The general form of the decomposition is then contracted with different momenta and metric tensors to get actual calculable integrals and finally a solvable system of equations.

The calculations are based on dimensional regularization of the master integral and since the integral is given up to the order $\mathcal{O}(\epsilon^2)$ the following are

as well.[10]

$$\boxed{\int \frac{d^d k}{(2\pi)^d} \frac{k_\mu}{k^2 (k-q)^2} = \frac{1}{2} q_\mu I_0} \tag{111}$$

Proof. Using the principle of tensorial decomposition the integral is split up like

$$\int \frac{d^d k}{(2\pi)^d} \frac{k_\mu}{k^2 (k-q)^2} = q_\mu \cdot I_{11} \ , \tag{112}$$

where I_{11} is an integral proportional to I_0.

Contracting with q^μ and using $k \cdot q = \frac{1}{2}(k^2 + q^2 - (k-q)^2)$ as well as each scaleless integral is equal to zero (compare appendix A.1)

$$\begin{aligned} \int \frac{d^d k}{(2\pi)^d} \frac{k \cdot q}{k^2 (k-q)^2} &= \int \frac{d^d k}{(2\pi)^d} \frac{k^2 + q^2 - (k-q)^2}{2 \cdot k^2 (k-q)^2} \\ &= \frac{q^2}{2} I_0 \\ &= q^2 \cdot I_{11} \ . \end{aligned} \tag{113}$$

Comparing the last two expressions one can see that

$$I_{11} = \frac{1}{2} I_0 \tag{114}$$

□

Plugging this into equation (112) we get the required result

$$\int \frac{d^4 k}{(2\pi)^4} \frac{k_\mu}{k^2 (k-q)^2} = \frac{i}{32\pi^2} q_\mu \mathrm{L} + \dots \ . \tag{115}$$

[10] The dots in the framed equations refer to the constant terms of the master integral

$$\boxed{\int \frac{d^dk}{(2\pi)^d}\frac{k_\mu k^\nu}{k^2(k-q)^2} = \frac{d}{4(d-1)}q_\mu q_\nu I_0 - \frac{q^2}{4(d-1)}\eta_{\mu\nu}I_0} \tag{116}$$

Proof.

$$\int \frac{d^dk}{(2\pi)^d}\frac{k_\mu k_\nu}{k^2(k-q)^2} = q_\mu q_\nu I_{21} + \eta_{\mu\nu}I_{22} \ . \tag{117}$$

Contracting with $q^\mu q^\nu$ and $\eta^{\mu\nu}$

$$\int \frac{d^dk}{(2\pi)^d}\frac{(k\cdot q)^2}{k^2(k-q)^2} = q^4 I_{21} + q^2 I_{22} = \frac{q^4}{4}I_0 \ , \tag{118}$$

$$\int \frac{d^dk}{(2\pi)^d}\frac{k^2}{k^2(k-q)^2} = q^2 I_{21} + dI_{22} = 0 \ . \tag{119}$$

Solving this system of equation finally gives us

$$I_{21} = \frac{d}{4(d-1)}I_0 \quad , \quad I_{22} = -\frac{q^2}{4(d-1)}I_0 \ . \tag{120}$$

□

Substituting $d = 4-2\epsilon$, expanding around $\epsilon = 0$ and reinserting I_0 leads to

$$I_{21} = \lim_{\epsilon\to 0}\frac{-i}{(4\pi)^2}\left\{\frac{1}{3} + \epsilon\left(\frac{1}{3}\ln(4\pi) + \frac{1}{2} + \frac{1}{3}\gamma\right) + \right.$$
$$\left. + \mathcal{O}(\epsilon^2)\right\}\ln(-q^2) + \ldots = \frac{-i}{32\pi^2}\frac{2}{3}L + \ldots \ , \tag{121}$$

and

$$I_{22} = \lim_{\epsilon\to 0} \frac{iq^2}{4(4\pi)^2}\left\{\frac{1}{3} + \epsilon\left(\frac{1}{3}\ln(4\pi) + \frac{2}{3} + \frac{1}{3}\gamma\right) + \right.$$
$$\left. + \mathcal{O}(\epsilon^2)\right\}\ln(-q^2) + \ldots = \frac{iq^2}{32\pi^2}\frac{1}{6}L + \ldots \quad (122)$$

finally gives

$$\int \frac{d^4k}{(2\pi)^4}\frac{k_\mu k^\nu}{k^2(k-q)^2} = \frac{-i}{32\pi^2}\left(q_\mu q_\nu \frac{2}{3}\mathrm{L} - q^2\eta_{\mu\nu}\frac{1}{6}\mathrm{L}\right) + \ldots \quad . \quad (123)$$

$$\boxed{\int \frac{d^dk}{(2\pi)^d}\frac{k_\mu k_\nu k_\alpha}{k^2(k-q)^2} = \frac{2+d}{8(d-1)}q_\mu q_\nu q_\alpha I_0 - \\ - \frac{q^2}{8(d-1)}\left(\eta_{\mu\nu}q_\alpha + \eta_{\mu\alpha}q_\nu + \eta_{\nu\alpha}q_\mu\right) I_0} \quad (124)$$

Proof.

$$\int \frac{d^dk}{(2\pi)^d}\frac{k_\mu k_\nu k_\alpha}{k^2(k-q)^2} = q_\mu q_\nu q_\alpha I_{31} + \left(\eta_{\mu\nu}q_\alpha + \eta_{\mu\alpha}q_\nu + \eta_{\nu\alpha}q_\mu\right) I_{32} \quad . \quad (125)$$

After contracting with $q^\mu q^\nu q^\alpha$, $q^\mu \eta^{\nu\alpha}$ and calculating the arising integrals the following system has to be solved

$$\int \frac{d^dk}{(2\pi)^d}\frac{(k\cdot q)^3}{k^2(k-q)^2} = q^6 I_{31} + 3q^4 I_{32} = \frac{1}{8}q^6 I_0 \quad , \quad (126)$$
$$\int \frac{d^dk}{(2\pi)^d}\frac{(k\cdot q)}{(k-q)^2} = q^4 I_{31} + (2q^2 + d)q^4 I_{32} = 0 \quad (127)$$

which gives us in the end

$$I_{32} = -\frac{q^2}{8(d-1)} I_0 \quad , \quad I_{31} = \frac{2+d}{8(d-1)} I_0 \ . \tag{128}$$

□

Explicit for four dimensions

$$I_{31} = \lim_{\epsilon\to 0} \frac{iq^2}{8(4\pi)^2} \left\{ \frac{1}{3} + \epsilon \left(\frac{1}{3}\ln(4\pi) + \frac{2}{3} + \frac{1}{3}\gamma \right) + \right. \\ \left. + \mathcal{O}(\epsilon^2) \right\} \ln(-q^2) + \ldots = \frac{i}{32\pi^2} \frac{1}{12} L + \ldots \quad , \tag{129}$$

and

$$I_{32} = \lim_{\epsilon\to 0} \frac{i}{(4\pi)^2} \left\{ -\frac{1}{2} - \epsilon \left(\frac{2}{9}\ln(4\pi) + \frac{2}{9} + \frac{2}{9}\gamma \right) + \right. \\ \left. + \mathcal{O}(\epsilon^2) \right\} \ln(-q^2) + \ldots = \frac{-i}{32\pi^2} \frac{1}{2} L + \ldots \quad , \tag{130}$$

so the integral is then

$$\int \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu k_\alpha}{k^2(k-q)^2} = \frac{i}{32\pi^2} \left(-q_\mu q_\nu q_\alpha \frac{1}{2} \mathrm{L} + \right. \\ \left. (\eta_{\mu\nu} q_\alpha + \eta_{\mu\alpha} q_\nu + \eta_{\nu\alpha} q_\mu) \frac{1}{12} q^2 \mathrm{L} \right) + \ldots \quad . \tag{131}$$

$$\boxed{\begin{aligned} &\int \frac{d^dk}{(2\pi)^d} \frac{k_\mu k_\nu k_\alpha k_\beta}{k^2(k-q)^2} = \frac{d^2+6d+8}{16(d^2-1)} q_\mu q_\nu q_\alpha q_\beta I_0 + \frac{q^2(d+2)}{16(d^2-1)} \times \\ &\quad \times \Big(\eta_{\mu\nu} q_\alpha q_\beta + \eta_{\mu\alpha} q_\nu q_\beta + \eta_{\mu\beta} q_\nu q_\alpha + \eta_{\nu\alpha} q_\mu q_\beta + \eta_{\nu\beta} q_\mu q_\alpha + \\ &\quad + \eta_{\alpha\beta} q_\mu q_\nu \Big) I_0 + \frac{q^4}{16(d^2-1)} \left(\eta_{\mu\nu}\eta_{\alpha\beta} + \eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\nu\alpha}\eta_{\mu\beta} \right) I_0 \end{aligned}} \tag{132}$$

Proof.

$$\int \frac{d^dk}{(2\pi)^d}\frac{k_\mu k_\nu k_\alpha k_\beta}{k^2(k-q)^2} = q_\mu q_\nu q_\alpha q_\beta I_{41} + \Big(\eta_{\mu\nu}q_\alpha q_\beta + \eta_{\mu\alpha}q_\nu q_\beta + \\ + \eta_{\mu\beta}q_\nu q_\alpha + \eta_{\nu\alpha}q_\mu q_\beta + \eta_{\nu\beta}q_\mu q_\alpha + \eta_{\alpha\beta}q_\mu q_\nu\Big)I_{42} \\ + \Big(\eta_{\mu\nu}\eta_{\alpha\beta} + \eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\nu\alpha}\eta_{\mu\beta}\Big)I_{43} \ . \tag{133}$$

Contracting with $q^\mu q^\nu q^\alpha q^\beta$, $q^\mu q^\nu \eta^{\alpha\beta}$ and $\eta^{\mu\nu}\eta^{\alpha\beta}$ the system of equations

$$\int \frac{d^dk}{(2\pi)^d}\frac{(k\cdot q)^4}{k^2(k-q)^2} = q^8 I_{41} + 6q^6 I_{42} + 3q^4 I_{43} = \frac{q^8}{16}I_0 \ , \tag{134}$$

$$\int \frac{d^dk}{(2\pi)^d}\frac{(k\cdot q)^2 k^2}{k^2(k-q)^2} = q^6 I_{41} + (5+d)q^4 I_{42} + (2+d)q^2 I_{43} = 0 \ , \tag{135}$$

$$\int \frac{d^dk}{(2\pi)^d}\frac{k^4}{k^2(k-q)^2} = q^4 I_{41} + (2d+4)q^2) I_{42} + (d^2+2d) I_{43} = 0 \tag{136}$$

has the solution

$$I_{41} = \frac{d^2+6d+8}{16(d^2-1)}I_0 \quad , \quad I_{42} = \frac{q^2(d+2)}{16(d^2-1)}I_0 \quad , \quad I_{43} = \frac{q^4}{16(d^2-1)}I_0 \ . \tag{137}$$

□

For $d = 4-2\epsilon$ the expanded coefficients

$$I_{43} = \lim_{\epsilon\to 0}\frac{-i}{(4\pi)^2}q^4\left(\frac{1}{240} + \epsilon\left\{\frac{1}{240}\ln(4\pi) + \frac{1}{225} + \frac{1}{240}\gamma\right) + \right. \\ \left. + \mathcal{O}(\epsilon^2)\right\}\ln(-q^2) + \ldots = \frac{-i}{32\pi^2}\frac{q^4}{120}L + \ldots \ , \tag{138}$$

$$I_{42} = \lim_{\epsilon\to 0} \frac{-i}{(4\pi)^2} q^2 \Big\{ \frac{1}{40} + \epsilon \left(\frac{1}{40}\ln(4\pi) + \frac{22}{75} + \frac{1}{40}\gamma \right) + \\ + \mathcal{O}(\epsilon^2) \Big\} \ln(-q^2) + \ldots = \frac{-i}{32\pi^2}\frac{q^2}{40} + \ldots \quad , \quad (139)$$

$$I_{41} = \lim_{\epsilon\to 0} \frac{-i\epsilon\Gamma(\epsilon)}{(4\pi)^{4-\epsilon}} \Big\{ \frac{1}{5} + \epsilon \left(\frac{1}{5}\ln(4\pi) + \frac{29}{300} + \frac{1}{5}\gamma \right) + \\ + \mathcal{O}(\epsilon^2) \Big\} \ln(-q^2) + \ldots = \frac{-i}{32\pi^2}\frac{1}{5} L + \ldots \quad (140)$$

are reinserted in the primary equation to obtain

$$\int \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu k_\alpha k_\beta}{k^2(k-q)^2} = \frac{-i}{32\pi^2} \Big\{ q_\mu q_\nu q_\alpha q_\beta \frac{1}{5} L - (\eta_{\mu\nu} q_\alpha q_\beta + \eta_{\mu\alpha} q_\nu q_\beta + \\ + \eta_{\mu\beta} q_\nu q_\alpha + \eta_{\nu\alpha} q_\mu q_\beta + \eta_{\nu\beta} q_\mu q_\alpha + \eta_{\alpha\beta} q_\mu q_\nu \Big) q^2 \frac{1}{40} L \quad (141)$$

$$+ \Big(\eta_{\mu\nu}\eta_{\alpha\beta} + \eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\nu\alpha}\eta_{\mu\beta} \Big) q^4 \frac{1}{120} L \Big\} + \ldots \quad . \quad (142)$$

C.4 Gamma Function

The Gamma-function is formally defined via

$$\Gamma(x) = \int_0^\infty dt\; t^{x-1} e^{-t} \quad , \quad (143)$$

for $x \in \mathbb{R}$ which leads to the functional equation

$$\Gamma(x+1) = x \cdot \Gamma(x) \quad . \quad (144)$$

If $x \in \mathbb{Z}$ the Gamma function interpolates the factorial

$$\Gamma(x+1) = x! \ . \tag{145}$$

The asymptotic behaviour for $x \to 0$ is

$$\Gamma(x) = \frac{1}{x} - \gamma + \mathcal{O}(x^2) \ . \tag{146}$$

Using this definitions one can define the Beta function

$$B(\alpha,\gamma) \equiv \int_0^1 dy \; y^{\alpha-1}(1+y)^{-\alpha-\gamma} = \frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \ . \tag{147}$$

Proof.

$$\Gamma(\alpha)\Gamma(\gamma) = \int_0^\infty e^{-u}u^{\alpha-1}du \int_0^\infty e^{-v}v^{\gamma-1}dv = \int_0^\infty \int_0^\infty e^{-u-v}u^{\alpha-1}v^{\gamma-1}dudv \ . \tag{148}$$

Substituting $u = z \cdot t$ and $v = z(t-1)$

$$\int_{z=0}^\infty \int_{t=0}^1 e^{-z}(zt)^{\alpha-1}(z(1-t))^{\gamma-1}zdtdz = \int_0^\infty e^{-z}z^{\alpha+\gamma-1}dz \int_0^1 t^{\alpha-1}(1-t)^{\gamma-1}dt \ . \tag{149}$$

This shows

$$\Gamma(\alpha)\Gamma(\gamma) = \Gamma(\alpha+\gamma)B(\alpha,\gamma) \ . \tag{150}$$

□

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