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TESIS DOCTORAL

New approaches to the Hull-Strominger system: Futaki invariants and harmonic metrics

Nuevos enfoques para el sistema de Hull-Strominger: invariantes de Futaki y métricas armónicas

Memoria para optar al grado de Doctor presentada por

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Abstract

In this Thesis we address the existence problem for the Hull-Strominger system. Firstly, we carry out a systematic search for solutions induced by the invariant geometry of Lie groups on complex homogenenous manifolds, based on the study of a natural family of invariant holomorphic vector bundles on these manifolds, and discuss moduli aspects restricted to the invariant situation. Motivated by our results, we propose a refined version of a Conjecture by Yau for solutions to the Hull-Strominger, and find a new obstruction which goes beyond the balanced property of the Calabi-Yau manifold (X, Ω) and the Mumford-Takemoto slope stability of the bundle over it. The basic principle is the construction of a (possibly indefinite) Hermitian-Einstein metric $\mathbf{G}$ on the holomorphic string algebroid $\mathcal{Q}$ associated to a solution of the system, provided that the connection ∇ on the tangent bundle is Hermitian-Yang Mills. Using the construction of $(\mathcal{Q}, \mathbf{G})$, we define a family of Futaki invariants associated to an infinite dimensional moment map obstructing the existence of solutions in a given balanced class. The precise conditions for $\mathbf{G}$ to be Hermite-Einstein lead to study a new system of equations in Hermitian Geometry called the *coupled Hermite-Einstein system*, which is strictly weaker and can be solved, in principle, in any compact complex manifold. We then move on to investigate stability conditions on holomorphic Courant algebroids reminiscent of GIT, inspired by the picture provided by the Donaldson-Uhlenbeck-Yau theorem. At this point of the Thesis, our main development is a notion of harmonic metric for the Hull-Strominger system, motivated by an infinite-dimensional hyperKähler moment map and related to a numerical stability condition, which we expect to exist generically for families of solutions.

Resumen

En esta Tesis tratamos el problema de existencia para el sistema de Hull-Strominger. En primer lugar, buscamos sistemáticamente soluciones inducidas por la geometría invariante de grupos de Lie en variedades homogeneneas complejas, basada en el estudio de una familia natural de fibrados vectoriales holomorfos invariantes en estas variedades, y discutimos aspectos del espacio de moduli restringido a la situación invariante. Motivados por nuestros resultados, proponemos una versión refinada de una Conjetura de Yau para soluciones de Hull-Strominger, y encontramos una nueva obstrucción que va más allá de la propiedad balanceada de la variedad Calabi-Yau (X, Ω) y la estabilidad de pendiente de Mumford-Takemoto para el fibrado sobre ella. El principio básico es la construcción de una métrica Hermite-Einstein (posiblemente indefinida) $\mathbf{G}$ en el algebroide de cuerdas holomorfo $\mathcal{Q}$ asociado a una solución del sistema, suponiendo que la conexión ∇ en el fibrado tangente es Hermite-Yang-Mills. Usando la construcción de $(\mathcal{Q}, \mathbf{G})$, definimos una familia de invariantes de Futaki asociados a una aplicación momento en dimensión infinita obstruyendo la existencia de soluciones en una clase balanceada dada. Las condiciones precisas para que $\mathbf{G}$ sea Hermite-Einstein lleva a estudiar un nuevo sistema de ecuaciones en Geometría Hermítica llamado sistema Hermite-Einstein acoplado, que es estrictamente más débil que el sistema de Hull-Strominger y puede resolverse, en principio, en cualquier variedad compleja compacta. A continuación, investigamos condiciones de estabilidad en algebroides de Courant holomorfos reminiscentes de GIT inspirados por la imagen global dada por el Teorema de Donaldson-Uhlenbeck-Yau. Nuestra principal aportación en este punto de la Tesis es una noción de métrica armónica para el sistema de Hull-Strominger, motivada por una aplicación momento hyperKähler en dimensión infinita y relacionada con una condición de estabilidad numérica, que esperamos que exista genéricamente en familias de soluciones.

Introduction

The aim of the present Thesis is to address the existence problem for the Hull-Strominger system. This system first appeared in the physical literature [85, 127] as the consistency conditions for the compactification of the heterotic string to 4-dimensional space time with minimal supersymmetry. Mathematically, it is stated as a system of partial differential equations in the following terms. Let (X, Ω) be a compact complex manifold of dimension three endowed with a holomorphic volume form. Let V be a holomorphic vector bundle over X, and α be a real constant. Then a pair of hermitian metrics g on X and h on V satisfy the Hull-Strominger system if:

$$\begin{aligned} F_h \wedge \omega^2 &= 0, \\ d(\|\Omega\|_\omega \omega^2) &= 0, \\ dd^c\omega - \alpha(\mathrm{tr} R_\nabla \wedge R_\nabla - \mathrm{tr} F_h \wedge F_h) &= 0, \end{aligned} \tag{1}$$

where ω denotes the hermitian form of g. In the last equation, there is an ambiguity in the choice of the metric connection ∇ in the tangent bundle of the manifold, back to its origins in heterotic string theory.

In the past few decades, the Hull-Strominger system has generated a great deal of interest in mathematics, both for its applications to the study of non-Kähler Calabi-Yau manifolds [43, 62, 108] and its relation to a conjectural generalization of mirror symmetry [5, 138]. As originally proposed in the seminal work by Li-Yau [98] and Fu-Yau [57, 58] on these equations, it is expected that the Hull-Strominger system plays a key role on the geometrization of *Reid's fantasy* [27, 55], connecting complex threefolds with trivial canonical bundle via conifold transitions. This proposal has important implications in our understanding of the moduli space of projective Calabi-Yau manifolds in complex dimension three, and also physical applications to the *string landscape*.

The existence problem for the Hull-Strominger is currently widely open. The present work is motivated by a question about the existence of solutions by S.-T. Yau [139].

Conjecture (Yau [139])**.** *Let (X, Ω) be a compact Calabi-Yau threefold endowed with a balanced class $\mathfrak{b}_0$. Let V be a holomorphic vector bundle over X satisfying:*

$$\deg_{\mathfrak{b}_0}(V) = 0, \qquad ch_2(V) = ch_2(X) \in H^{2,2}_{BC}(X, \mathbb{R}). \tag{2}$$

If V is polystable with respect to $\mathfrak{b}_0$, then (X, Ω, V) admits a solution of (1)*.*

In order to make progress in this interesting question, about which we know very little at present, in this Thesis we strenghten the statement of this Conjecture in two ways. Firstly, it is natural to demand that the class $\mathfrak{b} = [||\Omega||_\omega \omega^2]$ associated to a solution coincides with $\mathfrak{b}_0$, so that one has control over the balanced class. Moreover, as originally formulated in [139], the connection ∇ in (1) is not specified in the statement of the Conjecture. Hence, we propose that ∇ is a hermitian connection (with respect to some fixed hermitian metric) satisfying the Hermite-Yang-Mills equations:

$$R_\nabla^{0,2} = 0, \qquad R_\nabla \wedge \omega^2 = 0. \tag{3}$$

This ansatz for ∇ seems to have strong physical and geometrical significance: a solution of (1) with this ansatz solves the *heterotic equations of motion* [47, 88] and furthermore has many desirable properties in perturbation theory [37, 84, 100]. As for the geometry, solutions of (1) satisfying (3) are *generalized Ricci flat* [61, 63] and have a moment map interpretation [13, 70], which leads to an interesting metric on its moduli space. Furthermore, there is currently strong evidence that these solutions play an important role in *(0,2) mirror symmetry* via T-duality and the theory of vertex algebras [5, 6, 64].

Motivated by the previous discussion, in this Thesis we address a Question which refines Yau's Conjecture taking into account these observations. Avoiding technical aspects that will be made precise, it is stated as follows:

Question. *Let (X, Ω) be a compact Calabi-Yau threefold, and let $\mathfrak{b}_0$ be a balanced class. Let V be a $\mathfrak{b}_0$-polystable holomorphic vector bundle over X satisfying* (2). *Let V_0 be a generic $\mathfrak{b}_0$-polystable holomorphic vector bundle structure on $T^{1,0}$. Does (X, Ω, V) admit a solution (ω, h) of the Hull-Strominger system* (1) *such that $[||\Omega||_\omega \omega^2] = \mathfrak{b}_0$ and ∇ is the Chern connection of a Hermite-Einstein metric h_0 on V_0?*

Observe that an affirmative answer to this Question provides, in particular, a solution of Yau's Conjecture with the ansatz (3).

To gain some insight into this Question, in this Thesis we explore the geometric situation provided by complex locally homogenenous manifolds with hermitian structure induced by the invariant geometry of Lie groups [51, 47, 105]. On these manifolds, there is a natural class of holomorphic vector bundles of invariant type that can always be considered. In Chapter 4, we charaterise them using the representation theory of Lie algebras and use this to develop a systematic approach to finding invariant solutions to the Hull-Strominger system. Furthermore, this approach allows to carry out simplified analyses on the moduli space of solutions to the Hull-Strominger system [70, 13].

With the insights obtained from this invariant setup, we then provide compelling evidence that this refined version of Yau's Conjecture has a negative answer. In order to do this, we will exploit the special features of the solutions of the Hull-Strominger system with the ansatz (3). More precisely, we will be able to use generalized geometry and to apply the theory of metrics on holomorphic string algebroids introduced in [68, 69]. In few words, let (X, Ω) be a (possibly non-Kähler) Calabi-Yau manifold and P a holomorphic principal bundle satisfying

$$p_1(P) = 0 \in H^{2,2}_{BC}(X, \mathbb{R}). \tag{4}$$

To link with the above discussion, one can take P to be the bundle of split frames of $V_0 \oplus V$. Using (4), one can canonically associate to P a family of holomorphic vector bundle extensions of the form

$$0 \longrightarrow T^*_{1,0} \longrightarrow \mathcal{Q} \longrightarrow A_P \longrightarrow 0, \tag{5}$$

where A_P denotes the holomorphic Atiyah algebroid of P. These are a particular class of holomorphic Courant algebroids, called *string*. Then, for a fixed balanced class $\mathfrak{b}_0$, we are able to construct a family of *Futaki invariants*:

$$\langle \mathcal{F}_{\mathfrak{s}}, \mathfrak{b}_0 \rangle : \mathfrak{H}_{\mathfrak{s}} \to \mathbb{C}, \tag{6}$$

where $\mathfrak{H}_{\mathfrak{s}} = H^0(X, \mathcal{Q}_{\mathfrak{s}})$ and $\mathfrak{s}$ parametrizes extensions of the form (5). Crucially, we prove that a solution to the Hull-Strominger system with the ansatz (3) determines a string algebroid $\mathcal{Q}_{\mathfrak{s}}$ such that its Futaki invariant vanishes for the balanced class determined by the solution. This construction provides a new obstruction to the existence of solutions which goes beyond the balanced property of the Calabi-Yau manifold (X, Ω) and the Mumford-Takemoto slope stability of the bundles V_0 and V_1.

As a consequence of our main result, in order to disprove the Question above, it suffices to find a tuple (X, Ω, V, V_0), for V_0 generic in moduli, and a balanced class $\mathfrak{b}_0 \in H^{2,2}_{BC}(X, \mathbb{R})$ as in the statement, such that V_0 and V are $\mathfrak{b}_0$-polystable and

$$\langle \mathcal{F}_{\mathfrak{s}}, \mathfrak{b}_0 \rangle \neq 0, \qquad \forall \mathfrak{s} \in \mathfrak{S}. \tag{7}$$

In the particular case X satisfies the $\partial\bar{\partial}$-Lemma, the family above reduces to a unique Futaki invariant $\mathcal{F}_0$. We expect that $\mathcal{F}_0$ provides an efficient tool to attack the posed Question, with potential interesting implications in the geometrization of Reid's fantasy and the string landscape.

Our method of proof has several interesting salient features. It is inspired by an important result by De La Ossa, Larfors, and Svanes [36], who showed that the Hull-Strominger system is equivalent to a suitable Hermite-Yang-Mills equation on a Courant algebroid to all orders in perturbation theory. Here we give a precise mathematical counterpart of their result characterizing the Hermite-Einstein condition:

$$F_{\mathbf{G}} \wedge \omega^{n-1} = 0, \tag{8}$$

for a generalized pseudo-hermitian metric $\mathbf{G}$ on a holomorphic string algebroid $\mathcal{Q}_{\mathfrak{s}}$ in terms of classical tensors. Using this, we prove that any solution of the Hull-Strominger system with the ansatz (3) induces a solution of (8), which allows us to construct Futaki invariants.

Interestingly, the hermitian conditions under which (8) holds motivate the definition of a new system of coupled equations in hermitian geometry, which we call the *coupled Hermite-Einstein system*. This system is more flexible than the Hull-Strominger system as it can be solved, in principle, in any compact complex manifold. Here we construct solutions on manifolds that do not carry balanced metrics, and whose canonical bundle is not trivial. Moreover, in Chapter 7, we prove that the coupled Hermite-Einstein system admits a natural interpretation as a dimensional reduction of hermitian metrics satisfying:

$$dd^c\omega = 0, \quad \rho_B = 0, \tag{9}$$

of central interest in the problem of finding canonical geometry for pluriclosed manifolds, and also as the fixed points of pluriclosed flow [126, 71].

We then go on to investigate GIT stability conditions for the holomorphic string algebroid $\mathcal{Q}$ associated to a solution of the Hull-Strominger system with the ansatz (3). Firstly, we recover a no-go result for solutions 'without the ∇ connection', which goes back to the seminal work of Candelas-Horowitz-Strominger-Witten [24]. From our point of view, this is a consequence of the slope stability of $\mathcal{Q}$ with respect to the balanced class $\mathfrak{b} = [||\Omega||_\omega \omega^2]$ of the solution, combined with the existence of the holomorphic volume form Ω. We argue that the naive guess of considering slope polystability of the Bott-Chern algebroid $\mathcal{Q}$ with respect to $\mathfrak{b}$ is too rigid. Motivated by this, we propose a refined stability condition based on hyperKähler moment maps. The basic idea is that a solution of the Hull-Strominger system should carry a positive definite *harmonic metric* $\mathbf{H}$ for $(\mathcal{Q}, \langle \cdot, \cdot \rangle, D^{\mathbf{G}})$, that is, satisfying

$$(\nabla^{\mathbf{H}})^* \Psi + i_{\theta_\omega^\sharp} \Psi = 0,$$

where $D^{\mathbf{G}} = \nabla^{\mathbf{H}} + \Psi$ is the unique decomposition of the Chern connection $D^{\mathbf{G}}$ into an $\mathbf{H}$-unitary connection and a *Higgs field*. Using a different decomposition of $D^{\mathbf{G}}$ *à la Hitchin* [83], we prove that the existence of a harmonic metric implies a numerical stability condition in the sense of GIT. Even though our picture is mostly conjectural, we expect that this stability condition will lead us to new obstructions to the existence of solutions in future studies. Our proposal is illustrated with a continuous family of solutions on the Iwasawa manifold.

The structure of this Thesis is as follows. The first Part, which is intended to serve as a background on non-Kähler geometry, is divided in three Chapters. In Chapter 1, we review metric and topological aspects of complex non-Kähler manifolds, and then we give some details on their gauge theory and its algebraic counterpart. Chapter 2 is devoted to explain the elements of Generalized Geometry that will be used in the sequel. We give a brief account on smooth and holomorphic Courant algebroids, including the study of generalized metrics, focusing on the exact and string cases. Next, in Chapter 3 we introduce the Hull-Strominger system, its potential applications and the known solutions in the literature. We then make precise the Conjecture by Yau about the existence problem. The second Part of this Thesis presents the contents that are new in the literature, and forms the rest of this work. In Chapter 4, we study complex locally homogenenous manifolds and apply the results obtained to carry out a search for solutions of the Hull-Strominger system using an invariant ansatz. Moreover, we propose a refined version of Yau's Conjecture. Then, we go on to discuss metric aspects of the moduli space of solutions to the Hull-Strominger system. Chapter 5 is at the core of this Thesis. In this Chapter, based on [65], we prove the relation between solutions to the Hull-Strominger system and connections on Courant algebroids satisfying a Hermite-Yang-Mills condition. We then use the moment map interpretation of this equation to construct Futaki invariants, thus providing obstructions to the Hull-Strominger system. The Chapter finishes with an account of the computations of Futaki invariants we have carried out. Chapters 6 and 7 deal with the geometry of the coupled Hermite-Einstein system. In the first one, we prove basic properties of the geometric and topological properties it determines, and construct the first solutions that do not satsify the Hull-Strominger system. Moreover, we show its relation to other topics in geometry and physics. In the next Chapter, based on ongoing work by the author jointly with M. García-Fernández and J. Streets, we recast the coupled Hermite-Einstein system as a dimensional reduction of canonical pluriclosed metrics. In Chapter 8, based on [66], we discuss suitable stability conditions for string algebroids carrying a solution to the Hull-Strominger system. Moreover, we propose a notion of harmonic metric on string algebroids based on a hyperKähler moment map picture, which allows to relate to a numerical stability condition. Finally, in Chapter 9, we give some interesting directions related to the material covered in this Thesis that are currently ongoing.

Introducción

El objetivo de esta Tesis es estudiar el problema de existencia para el sistema de Hull-Strominger. Este sistema apareció por primera vez en la literatura física [86, 127] como las condiciones de coherencia para la compactificación de la cuerda heterótica a 4 dimensiones espacio-temporales con mínima supersimetría. Matemáticamente, se trata de un sistema de ecuaciones en derivadas parciales en los siguientes términos. Sea (X, Ω) una variedad compleja, compacta de dimensión tres, provista de una forma de volumen holomorfa. Sea V un fibrado vectorial holomorfo sobre X, y α una constante real. Ahora, un par de métricas hermíticas g en X y h en V satisface el sistema de Hull-Strominger si:

$$\begin{aligned} F_h \wedge \omega^2 &= 0, \\ d(\|\Omega\|_\omega \omega^2) &= 0, \\ dd^c\omega - \alpha(\mathrm{tr} R_\nabla \wedge R_\nabla - \mathrm{tr} F_h \wedge F_h) &= 0, \end{aligned} \tag{10}$$

donde ω es la forma hermítica de g. En la última ecuación, hay una ambigüedad en la elección de la conexión métrica ∇ en el fibrado tangente a la variedad, por sus orígenes en teoría heterótica de cuerdas.

En las últimas décadas, el sistema de Hull-Strominger ha generado un gran interés en matemáticas, tanto por sus aplicaciones al estudio de variedades Calabi-Yau no Kähler [43, 62, 108] como por su relación con una generalización conjetural de la simetría espejo [5, 138]. Tal como fue originalmente propuesto en el trabajo seminal de Li-Yau [98] y Fu-Yau [57, 58] acerca de estas ecuaciones, se espera que el sistema de Hull-Strominger juegue un papel clave en la geometrización de la *fantasía de Reid* [27, 55], conectando variedades en tres dimensiones con fibrado canónico trivial por medio de transiciones coniformes. Esta propuesta tiene importantes implicaciones en nuestro entendimiento del espacio de moduli de variedades Calabi-Yau proyectivas de dimensión tres, y también aplicaciones físicas al *paisaje de las cuerdas*.

El problema de existencia para el sistema de Hull-Strominger está actualmente muy abierto. Este trabajo está motivado por una pregunta acerca de la existencia de soluciones por S.-T. Yau [139].

Conjetura (Yau [139])**.** *Sea (X, Ω) una variedad Calabi-Yau compacta de dimensión tres provista de una clase balanceada $\mathfrak{b}_0$. Sea V un fibrado vectorial holomorfo sobre X que satisface:*

$$\deg_{\mathfrak{b}_0}(V) = 0, \qquad ch_2(V) = ch_2(X) \in H^{2,2}_{BC}(X, \mathbb{R}). \tag{11}$$

Si V es poliestable con respecto a $\mathfrak{b}_0$, entonces (X, V) admite una solución a (10).

Para progresar en esta interesante pregunta, de la que sabemos bastante poco actualmente, en esta Tesis proponemos una pregunta más fuerte que esta Conjetura en dos maneras. En primer lugar, es natural pedir que la clase $\mathfrak{b} = [||\Omega||_\omega \omega^2]$ asociada a una solución coincida con $\mathfrak{b}_0$, para tener control sobre la clase balanceada. Además, como fue formulado originalmente en [139], la conexión ∇ en (10) no se especifica en la Conjetura . Aquí, proponemos que ∇ sea una conexión hermítica (con respecto a una métrica hermítica fija) que satisface las ecuaciones de Hermite-Yang-Mills:

$$R_\nabla^{0,2} = 0, \qquad R_\nabla \wedge \omega^2 = 0. \tag{12}$$

Este ansatz para ∇ parece tener un importante significado físico y geométrico: una solución a (10) con este ansatz satisface las *ecuaciones heteróticas del movimiento* [47, 88] y además tiene numerosas propiedades deseables en teoría de perturbaciones [37, 84, 100]. Para la geometría, soluciones de (10) que satisfacen (12) son *Ricci planas generalizadas* [61, 63] y tienen una interpretación de aplicación momento [13, 70], que proporciona una interesante métrica en el espacio de moduli. Más aún, actualmente hay una gran evidencia de que estas soluciones juegan un papel importante en simetría espejo $(0,2)$ por medio de T-dualidad y la teoría de álgebras de vértices [5, 6, 64].

Motivados por la discusión anterior, en esta Tesis tratamos una Pregunta que refina la Conjetura de Yau teniendo en cuenta estas observaciones. Evitando aspectos técnicos que serán precisados, se formula como sigue:

Pregunta. *Sea* (X, Ω) *una variedad Calabi-Yau compacta de dimensión tres, y sea* $\mathfrak{b}_0$ *una clase balanceada. Sea* V *un fibrado vectorial holomorfo* $\mathfrak{b}_0$*-poliestable que satisface* (11). *Sea* V_0 *una estructura holomorfa* $\mathfrak{b}_0$*-poliestable genérica sobre* $T^{1,0}$. *¿Admite* (X, Ω, V) *una solución* (ω, h) *al sistema de Hull-Strominger* (10) *tal que* $[||\Omega||_\omega \omega^2] = \mathfrak{b}_0$ *y* ∇ *es la conexión de Chern de una métrica Hermite-Einstein* h_0 *en* V_0*?*

Obsérvese que una respuesta afirmativa a esta pregunta proporciona, en particular, una solución a la Conjetura de Yau con el ansatz (12).

Para obtener alguna intuición acerca de esta pregunta, en esta Tesis exploramos la situación geométrica proporcionada por variedades complejas localmente homogéneas con una estructura hermítica inducida por la geometría invariante de grupos de Lie [51, 47, 105]. En estas variedades, hay una clase natural de fibrados vectoriales holomorfos que siempre puede ser considerada. En el Capítulo 4, los caracterizamos usando la teoría de representaciones de álgebras de Lie y usamos esto para desarrollar un método sistemático para encontrar soluciones invariantes al sistema de Hull-Strominger. Además, este método permite hacer análisis simplificados del espacio de moduli de soluciones al sistema de Hull-Strominger [70, 13].

Con estas intuiciones obtenidas de la situación invariante, a continuación proporcionamos evidencia importante de que esta versión refinada de la Conjetura de Yau tiene una respuesta negativa. Para esto, explotamos las características especiales de las soluciones al sistema de Hull-Strominger con el ansatz (12). En términos más precisos, usamos la geometría generalizada y la teoría de métricas en algebroides de cuerdas holomorfos introducidos en

[68, 69]. En pocas palabras, sea (X, Ω) una variedad Calabi-Yau (posiblemente no Kähler) y P un fibrado principal holomorfo que satisface

$$p_1(P) = 0 \in H^{2,2}_{BC}(X, \mathbb{R}). \tag{13}$$

Para relacionar esta construcción con la discusión anterior, basta elegir P el producto de los fibrados de referencias de V_0 y V. Usando (13), se asocia canónicamente a P una familia de fibrados vectoriales de la forma:

$$0 \longrightarrow T^*_{1,0} \longrightarrow \mathcal{Q} \longrightarrow A_P \longrightarrow 0, \tag{14}$$

donde A_P es el algebroide holomorfo de Atiyah de P. Estos son una clase particular de algebroides de Courant holomorfos, llamados *de cuerdas*. Ahora, para una clase balanceada $\mathfrak{b}_0$, construimos una familia de *invariantes de Futaki*:

$$\langle \mathcal{F}_{\mathfrak{s}}, \mathfrak{b}_0 \rangle : \mathfrak{H}_{\mathfrak{s}} \to \mathbb{C}, \tag{15}$$

donde $\mathfrak{H}_{\mathfrak{s}} = H^0(X, \mathcal{Q}_{\mathfrak{s}})$ y $\mathfrak{s}$ parametrizan extensiones de la forma (14). Crucialmente, probamos que una solución al sistema de Hull-Strominger con el ansatz (12) determina un algebroide string $\mathcal{Q}_{\mathfrak{s}}$ tal que su invariante de Futaki se anula para la clase balanceada de la solución. Esta construcción proporciona una obstrucción a la existencia de soluciones más allá de la existencia de métricas balanceadas en la variedad Calabi-Yau (X, Ω) y la estabilidad de pendiente de Mumford-Takemoto de los fibrados V_0 y V_1.

Como consecuencia de nuestro resultado principal, para refutar la pregunta anterior, es suficiente con encontrar una tupla (X, V, V_0), para V_0 genérico en moduli, y una clase balanceada $\mathfrak{b}_0 \in H^{2,2}_{BC}(X, \mathbb{R})$ como en el enunciado, tal que V_0 y V son $\mathfrak{b}_0$-poliestables y

$$\langle \mathcal{F}_{\mathfrak{s}}, \mathfrak{b}_0 \rangle \neq 0, \qquad \forall \mathfrak{s} \in \mathfrak{S}. \tag{16}$$

En el caso particular en que X satisface el Lema $\partial\overline{\partial}$, la familia anterior reduce a un único invariante de Futaki $\mathcal{F}_0$. Esperamos que $\mathcal{F}_0$ proporcione una herramienta eficaz para atacar el problema propuesto por la pregunta anterior, con potenciales importantes implicaciones en la geometrización de la fantasía de Reid o el paisaje de las cuerdas.

Nuestro método para probar el resultado tiene algunas propiedades sobresalientes. Está inspirado en un resultado de De la Ossa, Larfors, Svanes [36], que probaron que el sistema de Hull-Strominger es equivalente a una ecuación Hermite-Yang-Mills apropiada en un algebroide de Courant en todos los órdenes en teoría de perturbaciones. Aquí damos una contraparte matemática precisa caracterizando la condición Hermite-Einstein:

$$F_{\mathbf{G}} \wedge \omega^{n-1} = 0, \tag{17}$$

para una métrica generalizada pseudo-hermítica $\mathbf{G}$ en un algebroide de cuerdas holomorfo $\mathcal{Q}_{\mathfrak{s}}$ en términos de tensores clásicos. Usando esto, demostramos que una solución al sistema de Hull-Strominger con el ansatz (12) induce una solución a (17), que nos permite construir invariantes de Futaki.

Es interesante observar que las condiciones hermíticas bajo las que (17) se satisface motivan la definición de un nuevo sistema en geometría hermítica, que llamamos *sistema de*

Hermite-Einstein acoplado. Este sistema es más flexible que el sistema de Hull-Strominger y puede, en principio, ser resuelto en cualquier variedad compleja compacta. En esta Tesis construimos soluciones en variedades que no admiten métricas balanceadas y cuyo fibrado canónico no es trivial. Además, en el Capítulo 7, demostramos que el sistema de Hermite-Einstein acoplado admite una interpretación natural como una reducción dimensional de métricas hermíticas que satisfacen:

$$dd^c\omega = 0, \quad \rho_B = 0, \tag{18}$$

de interés central en el problema de encontrar geometría canónica para variedades pluricerradas, y también como los puntos fijos de el flujo pluricerrado [126, 71].

A continuación, investigamos condiciones de estabilidad GIT para algebroides de cuerdas holomorfos $\mathcal{Q}$ asociados a soluciones del sistema de Hull-Strominger con el ansatz (12). Primero, recuperamos un resultado de rigidez 'sin la conexión ∇', ya obtenido en el trabajo seminal de Candelas-Horowitz-Strominger-Witten [24]. Desde nuestro punto de vista, esto es una consecuencia de la estabilidad de pendiente de $\mathcal{Q}$ respecto de la clase balanceada $\mathfrak{b} = [||\Omega||_\omega \omega^2]$ de la solución, combinado con la existencia de una forma de volumen holomorfa Ω. Argumentamos que la noción ingenua de estabilidad de pendiente para el algebroide de Courant $\mathcal{Q}$ respecto de $\mathfrak{b}$ es demasiado rígida. Motivados por este resultado, proponemos una condición de estabilidad refinada basada en una aplicación momento hyperKähler. La idea básica es que una solución al sistema de Hull-Strominger debería admitir una *métrica armónica* definida positiva $\mathbf{H}$ para $(\mathcal{Q}, \langle\cdot,\cdot\rangle, D^{\mathbf{G}})$, esto es, satisfaciendo:

$$(\nabla^{\mathbf{H}})^*\Psi + i_{\theta_\omega^\sharp}\Psi = 0,$$

donde $D^{\mathbf{G}} = \nabla^{\mathbf{H}} + \Psi$ es la única descomposición de la conexión de Chern $D^{\mathbf{G}}$ en una conexión $\mathbf{H}$-unitaria y un *campo de Higgs*. Usando una descomposición diferente de $D^{\mathbf{G}}$ *à la Hitchin* [83], demostramos que la existencia de una métrica armónica implica una condición numérica de estabilidad en el sentido de GIT. Aunque esta construcción es mayoritariamente conjetural aún, esperamos que esta condición de estabilidad proporcione nuevas obstrucciones a la existencia de soluciones en futuros estudios. Nuestra propuesta está ilustrada en una familia continua de soluciones en la variedad de Iwasawa.

La estructura de esta Tesis es la siguiente. La primera Parte, prevista para servir de referencia para geometría no Kähler, está dividida en tres capítulos. En el Capítulo 1 revisamos aspectos métricos y topológicos de variedades complejas no Kähler, y damos algunos detalles de su teoría gauge y su contraparte algebraica. El Capítulo 2 está dedicado a explicar los elementos de Geometría Generalizada que se utilizarán más tarde. Explicamos brevemente los algebroides de Courant diferenciables y holomorfos, incluyendo el estudio de métricas generalizadas con atención especial a los casos exacto y de cuerdas . A continuación, en el Capítulo 3 introducimos el sistema de Hull-Strominger, sus potenciales aplicaciones y las soluciones conocidas en la literatura. Ahí precisamos la Conjetura de Yau sobre el problema de existencia. En la segunda Parte de esta Tesis presentamos los contenidos que son nuevos en la literatura, y forma el resto de este trabajo. En el Capítulo 4, estudiamos variedades complejas localmente homogéneas y aplicamos los resultados obtenidos para una búsqueda sistemática de soluciones al sistema de Hull-Strominger con un ansatz invariante. Además proponemos una versión refinada a la Conjetura de Yau. También discutimos aspectos métricos del espacio de moduli de soluciones al sistema de Hull-Strominger. El Capítulo 5 es el núcleo de esta Tesis. En este Capítulo, demostramos la relación entre soluciones al sistema de Hull-Strominger y conexiones en algebroides de Courant que satisfacen una condición Hermite-Yang-Mills. A continuación usamos la interpretación de aplicación momento de esta ecuación para construir invariantes de Futaki, proporcionando así obstrucciones al sistema de Hull-Strominger. Este Capítulo finaliza con un resumen acerca de los cálculos de invariantes de Futaki que hemos obtenido. Los Capítulos 6 y 7 tratan de la geometría del sistema Hermite-Einstein acoplado. En el primero, obtenemos propiedades métricas y topológicas básicas, y construimos las primeras soluciones que no satisfacen el sistema de Hull-Strominger. Además, exponemos su relación con otros temas en geometría y física. En el siguiente Capítulo, que está basado en trabajo en progreso junto con M. García-Fernández y J. Streets, reinterpretamos el sistema Hermite-Einstein acoplado como una reducción dimensional de métricas pluricerradas canónicas. En el Capítulo 8, basado en [66], discutimos condiciones de estabilidad apropiadas para algebroides de cuerdas que admiten solución al sistema de Hull-Strominger. Además, proponemos una noción de métrica armónica en algebroides de cuerdas basado en una construcción de aplicación momento hyperKähler, que permite relacionarla con una condición numérica de estabilidad. Finalmente, en el Capítulo 9, damos algunas direcciones interesantes relacionadas con el material expuesto en esta Tesis, que actualmente están en progreso.

Part I

Background

Chapter 1

Hermitian Geometry

In this Chapter, we review basic notions of Hermitian Geometry and set the notation and conventions that will be used throughout this Thesis. Intended to serve as background, we will limit ourselves to recall the relevant results that will be necessary in the sequel. We refer to [38, 79, 86, 142], for further details. References for the hermitian geometry of non-Kähler manifolds are not so abundant and will be given along the text. The author claims no originality for any of the results contained in this Chapter, except for Section 4.1.2.

1.1 Hermitian manifolds

1.1.1 Special metrics

In this Section we review fundamental notions of Hermitian Geometry that will be used throughout this Thesis. Hermitian manifolds are given by triples (M, J, g) where $X = (M, J)$ is a complex manifold and g is a riemannian metric satisfying:

$$g(JX, JY) = g(X, Y), \quad X, Y \in \Gamma(TM), \tag{1.1.1}$$

that is, it is hermitian. Frequently, by abuse of language we will also call *hermitian metric* the 2-form:

$$\omega = g(J\cdot, \cdot). \tag{1.1.2}$$

Let $\dim_{\mathbb{R}} M = 2n$. A compact hermitian manifold (M, J, g) admits a natural inner product on differential forms. Pointwise, given $x \in M$ and $\alpha_x, \beta_x \in \Lambda^k T^*_x M$:

$$\langle \alpha_x, \beta_x \rangle = \sum_{1 \leq i_1 < \ldots i_k \leq 2n} \alpha_x(e_1, ..., e_k)\beta_x(e_1, ..., e_k), \tag{1.1.3}$$

where $\{e_i\}$ stands for an orthonormal frame of $T_x M$ with respect to g. The pointwise inner product (1.1.3) defines implicitly a Hodge star operator:

$$\star : \Lambda^k T^* M \longrightarrow \Lambda^{2n-k} T^* M \tag{1.1.4}$$

given by declaring that:

$$\alpha_x \wedge \star\beta_x = \langle \alpha_x, \beta_x \rangle \tfrac{\omega^n}{n!} \tag{1.1.5}$$

holds for any α_x, β_x. Then, the inner product on global sections of $\alpha, \beta \in \Omega^\bullet_M$ is given by:

$$\langle\langle \alpha, \beta \rangle\rangle = \int_M \alpha \wedge \star\beta, \quad \alpha, \beta \in \Omega^\bullet_M, \tag{1.1.6}$$

inducing associated metric-adjoint operators:

$$\langle\langle d\alpha, \beta \rangle\rangle = \langle\langle \alpha, d^\star\beta \rangle\rangle, \quad \langle\langle \omega \wedge \alpha, \beta \rangle\rangle = \langle\langle \alpha, \Lambda_\omega \beta \rangle\rangle, \tag{1.1.7}$$

and a Lefschetz splitting of the space of differential forms:

$$\Omega^k_M = \bigoplus_i \omega^i \wedge P^{k-2i}, \quad P^r = \ker(\omega^{n-r+1} \wedge \cdot) \subset \Omega^r_M, \tag{1.1.8}$$

where sections of P^r are called *primitive*.

A broad goal of Hermitian Geometry is to find the canonical geometry of complex manifolds. To this end, several notions of special metrics are introduced:

Definition 1.1.1. *Let X be a complex manifold of complex dimension n. A hermitian metric is called*

1. *Kähler if $d\omega = 0$.*
2. *Balanced if $d\omega^{n-1} = 0$*
3. *Conformally balanced if $e^f\omega$ is balanced, for some smooth function f.*
4. *Gauduchon if $dd^c\omega^{n-1} = 0$.*
5. *Pluriclosed (or SKT) if $dd^c\omega = 0$.*

From the definitions, the following scheme of relations follows:

$$\text{pluriclosed} \Leftarrow \text{Kähler} \Rightarrow \text{balanced} \Rightarrow \text{Gauduchon}. \tag{1.1.9}$$

Currently, determining the existence of special non-Kähler metrics is, in general, a difficult task. Balanced metrics where introduced in [102] and shown to be topologically obstructed. Since then, a number of constructions [2], and further Examples have appeared in [1, 55]. For Gauduchon metrics, there is the following general existence result:

Theorem 1.1.2 ([72])**.** *Let X be a compact complex manifold, and let ω be a hermitian metric. Then, there exists a unique real function f such that*

1. *$e^f\omega$ is a Gauduchon metric.*
2. *The function f satisfies the normalization*

$$\int_X e^f \mathrm{dvol}_\omega = 1.$$

In particular, every compact complex surface admits pluriclosed metrics. Further examples of pluriclosed manifolds in higher dimension are constructed in [76, 112, 140].

There is also interest in determining to what extent the existence of special hermitian metrics implies *kählerianity*. The following is the prototype of these *no-go* results.

Theorem 1.1.3 ([90]). *Let (X, ω) be a hermitian manifold and suppose that ω is pluriclosed and conformally balanced. Then ω is Kähler.*

Interestingly, in Section 8.1, we use Generalized Geometry (see Chapter 2) as a novel approach to recover an instance of the above Theorem.

1.1.2 Torsion of hermitian metrics

To make progress in the geometrization of non-Kähler manifolds, the following tensors are introduced to measure the failure of a hermitian metric to be Kähler:

Definition 1.1.4. *Let (X, ω) be a hermitian manifold.*

1. *The* torsion *of ω is given by $-d^c\omega$.*
2. *The Lee 1-form of ω is given by*

$$\theta_\omega = Jd^\star\omega. \tag{1.1.10}$$

Remark 1.1.5. *In the literature, several other quantities such as $d^c\omega, d\omega, i\partial\omega$, etc. are often referred to as the torsion of ω too. Our choice of $-d^c\omega$ is motivated by* (1.2.3).

In complex dimension 2, both forms contain the same information, as ([71]):

$$\theta_\omega = \star(-d^c\omega). \tag{1.1.11}$$

In the sequel, we use the following properties of the Lee form:

Proposition 1.1.6. *Let (X, ω) be a hermitian manifold of complex dimension n.*

1. *θ_ω is the unique 1-form that satisfies*

$$d\omega^{n-1} = \theta_\omega \wedge \omega^{n-1}. \tag{1.1.12}$$

In particular, the following formula holds:

$$\theta_\omega = \Lambda_\omega d\omega \tag{1.1.13}$$

2. *Let $\tilde{\omega} = e^f\omega$, where f is a real function. Then:*

$$\theta_{\tilde{\omega}} = \theta_\omega + (n-1)df \tag{1.1.14}$$

From the previous Proposition, ω is balanced if and only if $\theta_\omega = 0$. Similarly, ω is (locally) conformally balanced if θ_ω is exact (resp. closed). An account of further properties and the geometry of the Lee 1-form can be found in [73].

1.1.3 Topology of non-Kähler manifolds

The complex cohomology of a Kähler manifold (X, ω) is characterised by the following consequence of the Hodge decomposition Theorem:

$$H^k_{dR}(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}_{\overline{\partial}}(X). \tag{1.1.15}$$

where the isomorphism above is canonical, but the cohomology of non-Kähler complex manifolds, or more precisely, of manifolds that do not satisfy the $\partial\overline{\partial}$-Lemma, do not satisfy this result, and can be substantially more difficult to study. Bott-Chern and Aeppli cohomology groups are introduced as tools to tackle this subtlety:

$$H^{p,q}_{BC}(X) = \frac{\ker(d : \Omega^{p,q}_X \longrightarrow \Omega^{p+q+1}_X)}{\operatorname{im}(dd^c : \Omega^{p-1,q-1}_X \longrightarrow \Omega^{p,q}_X)} \tag{1.1.16}$$

$$H^{p,q}_{A}(X) = \frac{\ker(dd^c : \Omega^{p,q}_X \longrightarrow \Omega^{p+1,q+1}_X)}{\operatorname{im}(d \oplus d^c : \Omega^{p+q-1}_X \longrightarrow \Omega^{p+q}_X) \cap \Omega^{p,q}_X}. \tag{1.1.17}$$

These complex cohomologies are related to one another fitting in the following diagram:

$$\begin{array}{ccccc} & & H^{\bullet,\bullet}_{BC} & & \\ & \swarrow & \downarrow & \searrow & \\ H^{\bullet,\bullet}_{\overline{\partial}} & & H^{\bullet}_{dR} \otimes \mathbb{C} & & H^{\bullet,\bullet}_{\partial} \\ & \searrow & \downarrow & \swarrow & \\ & & H^{\bullet,\bullet}_{A} & & \end{array} \tag{1.1.18}$$

where the maps are induced by identity at the level of forms. Importantly, on a compact complex manifold X^n, these cohomologies are related by duality:

$$H^{p,q}_{BC}(X) \xrightarrow{\cong} H^{n-p,n-q}_{A}(X)^*, \quad [\alpha] \mapsto \int_X \alpha \wedge \cdot \tag{1.1.19}$$

For a survey on these complex cohomologies, see [121].

While the isomorphism (1.1.15) does not hold for a general complex manifold, the Frölicher spectral sequence [54] relates Dolbeault and de Rham cohomology:

$$E^{p,q}(X) \Rightarrow H^{p+q}(X, \mathbb{C}), \quad E^{p,q}_1(X) = H^{p,q}_{\overline{\partial}}(X), \tag{1.1.20}$$

inducing further complex cohomology groups. The existence of special metrics on compact complex manifolds often has topological consequences in the complex cohomologies or in the Frölicher sequence (see [114, 113]). We will see an instance of this phenomenon in Proposition 6.2.2.

As a generalization of Kähler classes, special metrics (see Section 1.1.1) define suitable cohomology classes: if ω is balanced, then it defines naturally a cohomology class

$[\omega^{n-1}] \in H^{2n-2}_{dR}(X,\mathbb{R})$ or in $H^{n-1,n-1}_{BC}(X,\mathbb{R})$. Similarly, if ω is Gauduchon, then $[\omega^{n-1}] \in H^{n-1,n-1}_{A}(X,\mathbb{R})$, and if ω is pluriclosed, $[\omega] \in H^{1,1}_{A}(X,\mathbb{R})$. We also have generalizations of the Kähler cone of X. Given a compact complex manifold X, the balanced and Gauduchon cones ([56, 114]) are:

$$\mathcal{B}_X = \{[\omega^{n-1}] \in H^{n-1,n-1}_{BC}(X) \mid \omega \text{ balanced}\} \tag{1.1.21}$$

$$\mathcal{GC}_X = \{[\omega^{n-1}] \in H^{n-1,n-1}_{A}(X) \mid \omega \text{ Gauduchon}\}. \tag{1.1.22}$$

and cohomology classes in this cones are said to be balanced and Gauduchon, accordingly.

1.2 Hermitian connections

1.2.1 Distinguished linear connections

Let (M, J, g) be a hermitian manifold, and let ∇^g denote the Levi-Civita connection of g. The following formula describes a 2-parameter family of linear connections that have attracted attention in non-Kähler geometry [19, 72, 76, 105, 135, 141], among many others. For real parameters r, s:

$$g(\nabla^{t,s}_X Y, Z) = g(\nabla^g_X Y, Z) - t d^c\omega(X,Y,Z) - s d\omega(JX, Y, Z), \quad X, Y, Z \in \Gamma(TM). \tag{1.2.1}$$

Among this family of connections there are the Levi-Civita connection $\nabla^g = \nabla^{0,0}$, the Chern connection $\nabla^C = \nabla^{0,1/2}$ and the Bismut (also Strominger) connection $\nabla^B = \nabla^{1/2,0}$. The Bismut connection is the unique linear connection that is unitary:

$$\nabla^B J = \nabla^B g = 0, \tag{1.2.2}$$

and has totally skew-symmetric torsion. It is given by:

$$g(T_{\nabla^B}(X,Y), Z) = -d^c\omega(X,Y,Z), \quad X, Y, Z \in \Gamma(TM). \tag{1.2.3}$$

Finally, the connection $\nabla^- = \nabla^{-1/2,0}$ is an orthogonal connection with totally skew-symmetric torsion $T_{\nabla^-} = g^{-1}d^c\omega$ sometimes called Hull connection in the literature.

When ω is a Kähler metric, the above family of linear connections collapses to a single point. Otherwise, it defines a set of mutually distinct connections, among which the line joining ∇^C and ∇^B are the unitary connections. Hence, they induce complex connections on $T^{1,0}$ and is often called the *canonical line of unitary connections* [72].

For future reference, we include here Koszul formula for the ∇^g, which will be useful in computations:

$$\begin{aligned} g(\nabla^g_X Y, Z) = \tfrac{1}{2}\,(& X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) \\ & + g([X,Y],Z) + g([Z,X],Y) + g(X,[Z,Y]) \end{aligned} \tag{1.2.4}$$

for arbitrary vector fields X, Y, Z.

1.2.2 Curvature and holonomy

Let (M, J, g) be a hermitian manifold, and let $(E, \nabla) \to M$ be a real or complex vector bundle with connection. Our convention for the curvature tensor of ∇ is:

$$R_\nabla(X,Y)s = ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})s, \quad X, Y \in \Gamma(TM),\ s \in \Gamma(E). \tag{1.2.5}$$

Now, we restrict to the case $E = TM$. Then, the Ricci curvature of ∇ is:

$$\mathrm{Ric}_\nabla(X,Y) = \mathrm{tr}(Z \mapsto R_\nabla(Z,X)Y), \quad X, Y \in \Gamma(TM). \tag{1.2.6}$$

If ∇ is a *complex* connection (*i.e.* $\nabla J = 0$), and F_∇ denotes the curvature of the induced connection on $T^{1,0}X$, then the *Ricci form* of ∇ is given by:

$$\rho_\nabla = i\mathrm{tr}\, F_\nabla. \tag{1.2.7}$$

Similarly, we will denote by ρ_B, ρ_C the Bismut or Chern-Ricci forms. By definition, $\rho_\nabla \in 2\pi c_1(X)$ for any connection, hence in particular it is d-closed. The Ricci form is in general a real 2-form. However, if $\nabla = \nabla^C$, it is of type $(1,1)$ with respect to J.

In the sequel we will use the following identity for the Bismut and Hull curvature tensors (see the Proof of [71, Proposition 3.21]):

$$g(R_{\nabla^-}(X_1,X_2)X_3,X_4) - g(R_{\nabla^B}(X_3,X_4)X_1,X_2) = \tfrac{1}{2}dd^c\omega(X_1,X_2,X_3,X_4), \quad X_i \in \Gamma(TM). \tag{1.2.8}$$

Related to curvature is the notion of holonomy. For a linear connection ∇, we will denote by $\mathrm{hol}^0(\nabla)$ and $\mathrm{hol}(\nabla)$ for the restricted and general holonomy respectively. If ∇ is unitary, then $\mathrm{hol}(\nabla) \subset U(n)$, for n the complex dimension of X. Moreover, we have:

Proposition 1.2.1. *Let ∇ be a linear unitary connection. Then:*

- $\mathrm{hol}^0(\nabla) \subset SU(n)$ *if and only if* $\rho_\nabla = 0$.
- $\mathrm{hol}(\nabla) \subset SU(n)$ *if and only if there exists a ∇-parallel global section.*

Definition 1.2.2. *A hermitian metric ω such that*

$$\rho^B = 0 \tag{1.2.9}$$

is called Calabi-Yau with torsion (CYT).

CYT metrics can be regarded as a non-Kähler replacement for Kähler-Ricci flat metrics and have been the subject of much interest in the non-Kähler geometry literature [49, 67, 71, 76].

Definition 1.2.3. *A Kähler-Calabi-Yau manifold is tuple (M, J, g, Ω) where (M, J, g) is Kähler and $\Omega \in \Gamma(K_X)$ is holomorphic and $\nabla^g\Omega = 0$. Equivalently,* $\mathrm{hol}(\nabla^g) \subset SU(n)$.

1.3 Gauge theory of principal bundles

1.3.1 Differential geometry of principal bundles

Here we recall fundamental notions of the geometry of principal bundles. A particularly useful reference for this Section is [94].

Let G be a real or complex Lie group with quadratic Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$, that is, satisfying:

$$\langle[\xi,\eta],\gamma\rangle + \langle\eta,[\xi,\gamma]\rangle = 0, \quad \xi,\eta,\gamma \in \mathfrak{g}. \tag{1.3.1}$$

A quadratic structure exists if G is compact or admits a compact real form ([102]).

Moreover, let $P \xrightarrow{p} M$ be a principal G-bundle with a right G-action, *i.e.* a smooth manifold P with a free and transitive right G-action, for which $P/G \cong M$. If the structure group G is real, associated to P, we have the Atiyah-Lie short-sequence:

$$0 \longrightarrow \text{ad } P \longrightarrow A_P \longrightarrow TM \longrightarrow 0, \tag{1.3.2}$$

where ad $P = P \times_G \mathfrak{g}$ is the adjoint bundle of P, and $A_P = TP/G \to M$ is the Atiyah algebroid of P, with the bracket on right invariant vector fields on TP. Moreover, ad P is naturally a bundle of quadratic Lie algebras with the structure induced by $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$. In case G is complex, we will consider the *complex* Atiyah-Lie algebroid A^c_P fitting in the short exact sequence:

$$0 \longrightarrow \text{ad } P \longrightarrow A^c_P \longrightarrow TM \otimes \mathbb{C}, \tag{1.3.3}$$

that, is $A^c_P = TP \otimes \mathbb{C}/(\text{ad } P)^{0,1}$.

Example 1.3.1. *Let $E \to M$ be an either real or complex vector bundle, and let $P = \mathrm{Fr}\ E$ be the frame bundle of E, with structure group $G = GL(\mathrm{rk}(E))$. This group has a natural quadratic Lie algebra given by the pairing:*

$$\langle A, B\rangle = \mathrm{tr}(AB^t) \tag{1.3.4}$$

The fibration p gives rise to an involutive vertical distribution $VP = \ker dp$ which fits into the exact sequence

$$0 \longrightarrow VP \longrightarrow TP \longrightarrow p^*TM \longrightarrow 0. \tag{1.3.5}$$

The bundle VP admits a global trivialization $P \times \mathfrak{g} \cong VP$ given by the infinitesimal action:

$$(p,\xi) \mapsto X^\xi(p) = \frac{d}{dt}(p \cdot e^{t\xi}), \quad \xi \in \mathfrak{g}. \tag{1.3.6}$$

Such vector fields generated by a Lie algebra element are called *canonical.* Note, however that these vector fields are *not* G-invariant. Rather, vertical G-invariant vector fields are described by G-equivariant maps

$$\Phi : P \longrightarrow \mathfrak{g}, \quad \Phi(p \cdot g) = \mathrm{Ad}_g(\Phi(p)), \tag{1.3.7}$$

or, equivalently, by sections of the adjoint bundle ad $P = P \times_G \mathfrak{g} \to M$. These are the infinitesimal symmetries of P. More precisely, given the gauge group of P

$$\mathcal{G}(P) = \{\varphi : P \xrightarrow{G} P \mid p \circ \varphi = p\}, \tag{1.3.8}$$

then the variation of a one-parameter family $\partial_t \varphi_t$ of gauge transformations is naturally a section of ad P. Then, $\langle \cdot, \cdot \rangle$ induces a pairing on sections of ad P, which we denote again by $\langle \cdot, \cdot \rangle$. In a local trivialization of P, canonical and G-invariant vertical vector fields are given, respectively, by fibrewise left-invariant and right-invariant fields.

To describe the horizontal geometry, we make use of connections. In this context, a connection is given by an equivariant 1-form $A \in \Omega^1_P(\mathfrak{g})^G$ which restricts to the identity on VP. Given A, there is a G-equivariant splitting

$$TP = VP \oplus \ker A. \tag{1.3.9}$$

We call $H = \ker A$ the horizontal distribution with respect to A, and we have associated projection maps

$$p_V : TP \longrightarrow VP, \quad p_H : TP \longrightarrow H. \tag{1.3.10}$$

Using the above, for any basic vector field $X \in \Gamma(TM)$, we define a lifted horizontal vector field $X^A = p_H(\tilde{X})$, where $\tilde{X}$ is any lift of X. The resulting field X^A is G-invariant.

The connection A induces parallel transport and covariant derivatives in the usual manner in all vector bundles associated to P, in particular on ad P. The pullback of forms gives a natural embedding:

$$\Omega^k_M(\text{ad } P) \hookrightarrow \Omega^k_P(\mathfrak{g})^G, \tag{1.3.11}$$

whose image are basic forms. Here, $\Omega^k_P(\mathfrak{g})^G$ stand for k-forms α that are G-equivariant, that is:

$$R_g^* \alpha = \text{Ad}_{g^{-1}} \circ \alpha, \quad g \in G. \tag{1.3.12}$$

Through this embedding, the covariant derivative of a section $\beta \in \Omega^k_M(\text{ad } P)$ is given by:

$$d_A \beta = d\beta \circ p_H = d\beta + [A \wedge \beta]. \tag{1.3.13}$$

Moreover, the curvature of A, which measures the non-involutivity of H, is given by:

$$F_A = dA \circ p_H = dA + \tfrac{1}{2}[A \wedge A], \tag{1.3.14}$$

and is naturally a section of $\Omega^2_M(\text{ad } P)$, which satisfies the Bianchi identity $d_A F_A = 0$. In case (E, ∇) is associated to a principal bundle with connection (P, A) with representation map ρ and $\nabla = \nabla^A = d_{\rho_* A}$, then $R_\nabla = \rho_* F_A$, as defined in Section 1.2.2. Furthermore, the gauge group $\mathcal{G}(P)$ acts on the connection A by

$$\varphi \cdot A = d\varphi \circ A \circ (d\varphi)^{-1}. \tag{1.3.15}$$

At the infinitesimal level, given a section $s \in \Gamma(\mathrm{ad}\ P)$ identified with the G-invariant field $X^s \in \Gamma(VP)$, we have

$$\mathcal{L}_{X^s} A = d_A s. \tag{1.3.16}$$

where $A = \mathrm{pr}_H \in \Gamma(\mathrm{End}\, TP)$ is understood as a tensor on P to define the Lie derivative.

On the total space of the principal bundle P, associated to a principal connection A, we recall here the *Chern-Simons* 3-form $CS(A) \in \Omega^3_P$ given by:

$$CS(A) = -\tfrac{1}{6}\langle [A \wedge A] \wedge A\rangle + \langle F_A \wedge A\rangle. \tag{1.3.17}$$

satisfying the key property:

$$dCS(A) = \langle F_A \wedge F_A\rangle. \tag{1.3.18}$$

Given two connections A, A' on P, the following combination of 3-forms is basic:

$$CS(A') - CS(A) - d\langle A' \wedge A\rangle = p^*(2\langle a \wedge F_A\rangle + \langle a \wedge d_A a\rangle + \tfrac{1}{3}\langle a \wedge [a \wedge a]\rangle). \tag{1.3.19}$$

In the case the structure group G is a complex reductive group and P is a holomorphic principal bundle for G, there is a notion of hermitian reduction. Let $K \subset G$ be a maximal compact subgroup. Then, a section $h \in \Gamma(P/K)$ determines a K-principal bundle $P_h = h^{-1}([K]) \subset P$. The Chern correspondence in this context asserts that there is a unique connection A_h compatible with the holomorphic structure of P and restricting to a connection on P_h ([120]). Conversely, given a principal bundle P_K for a compact Lie group K, we will denote $P_K^c = P \times_K K^c$, where K^c stands for the complexification of K (see *e.g.* [26, Ch. 12]). P_K^c is naturally a K^c-principal bundle, but carries no natural holomorphic structure.

Moreover, complex reductive groups satisfy a polar decomposition: given a maximal compact subgroup $K \subset G$, then:

$$G = \exp(i\mathfrak{k}) \cdot K, \tag{1.3.20}$$

and the decomposition of any element $g \in G$ is unique with respect to (1.3.20). Therefore, any left K-coset is expressed as $e^{is}K$ for a unique $s \in \mathfrak{k}$. A global version of this fact is that on a G-principal bundle P, given two hermitian reductions h_0, h, there exists a section $\sigma \in \Gamma(i\mathrm{ad}\ P_{h_0})$ such that:

$$h = \exp(i\sigma)h_0. \tag{1.3.21}$$

In particular, the set of hermitian reductions is path-connected.

It is well-known by Chern-Weil theory that Chern classes of complex principal bundles are well-defined characteristic classes in de Rham cohomology. In Bott-Chern cohomology, these are well-defined using hermitian reductions if one specifies a holomorphic structure. Since this is not completely standard, we provide a proof of this fact here:

Proposition 1.3.2. *Let X be a complex manifold, and let $P \to X$ be a holomorphic principal G-bundle, where G is a complex reductive Lie group. Assume there is a bi-invariant product:*

$$\langle \cdot, \ldots, \cdot \rangle_k : \underbrace{\mathfrak{g} \times \cdots \times \mathfrak{g}}_{k} \longrightarrow \mathbb{C} \tag{1.3.22}$$

Then, the k^{th}-Chern character form given by:

$$\mathrm{ch}_k(\mathcal{E}, h) = \langle F_h \wedge \cdots \wedge F_h \rangle_k \in \Omega_X^{k,k} \tag{1.3.23}$$

where h is a hermitian reduction on P to a maximal compact subrogrup induces a well-defined characteristic class $\mathrm{ch}_k(\mathcal{E})_{BC} = [\mathrm{ch}_k(\mathcal{E}, h)] \in H_{BC}^{k,k}(X)$ that does not depend on the hermitian reduction h. If $P = \mathrm{Fr}\,\mathcal{E}$ is the frame bundle of a holomorphic vector bundle, then the total Chern class $c(\mathcal{E})_{BC} \in \oplus_j H_{BC}^{j,j}(X, \mathbb{R})$ given by:

$$c(\mathcal{E})_{BC} = [\det\left(\mathrm{Id}_{\mathcal{E}} + \tfrac{i}{2\pi} F_h\right)] \tag{1.3.24}$$

is independent of h too.

Proof. Let h_0 and h_1 be any two hermitian reductions on P. Then, since G is a reductive Lie group, by the polar decomposition, there exists a path $\sigma_t \in \mathrm{ad}(P_{h_0})$ such that $h_t = e^{i\sigma_t} h_0$ is a smooth path joining h_0 and h_1. Moreover, by the computation in the proof of [68, Lemma 3.24] (see also [40, Section 1]), we have:

$$\tfrac{d}{dt} F_{h_t} = \overline{\partial}\partial^{h_t} \sigma_t. \tag{1.3.25}$$

Now, the variation of the Chern character form is given by:

$$\begin{aligned}
\tfrac{d}{dt}\mathrm{ch}_k(\mathcal{E}, h_t) &= \tfrac{d}{dt} \langle F_{h_t} \wedge \ldots F_{h_t} \rangle_k \\
&= k \langle \tfrac{d}{dt} F_{h_t} \wedge \ldots F_{h_t} \rangle_k \\
&= k \langle \overline{\partial}\partial^{h_t} \sigma_t \wedge \ldots F_{h_t} \rangle_k \\
&= \overline{\partial}\partial k \langle \sigma_t, \cdots \wedge F_{h_t} \rangle_k,
\end{aligned}$$

where in the last step we use the Bianchi identity $d^{h_t} F_{h_t} = 0$. Since this result is Bott-Chern exact at any time t, we obtain that $[\mathrm{ch}_k(\mathcal{E}, h_t)] \in H_{BC}^{k,k}(X)$ is constant along the path, hence $[\mathrm{ch}_k(\mathcal{E}, h_0)] = [\mathrm{ch}_k(\mathcal{E}, h_1)]$. The last part of the statement follows for the particular case of $G = GL(r, \mathbb{C})$, where $r = \mathrm{rk}\,\mathcal{E}$, and the Ad-invariant matrix polynomials $\langle \cdot, \ldots, \cdot \rangle_k$ given implicitly by (see *e.g.* [95, Sections XII.1-3]):

$$\det(I + A) = 1 + \textstyle\sum_{i=1}^r \langle A, \ldots, A \rangle_i, \quad A \in \mathfrak{gl}(r, \mathbb{C}). \tag{1.3.26}$$

□

We finish this Section by recalling the notion of instanton connection. Although these are defined in multiple geometric contexts, in this Thesis we will be mainly interested in the ones arising in Hermitian Geometry:

Definition 1.3.3. *Let $P \to (M^{2n}, J, g)$ be a principal bundle over a hermitian manifold. Then:*

1. *a connection A on P is Hermite-Yang-Mills if:*

$$F_A \wedge \omega^{n-1} = 0, \quad F_A^{0,2} = 0. \tag{1.3.27}$$

2. *if P is holomorphic and the Chern connection of a hermitian reduction A_h is Hermite-Yang-Mills, h is called Hermite-Einstein metric.*

1.3.2 Slope stability in non-Kähler manifolds

Here, we recall briefly the fundamental notions of slope-stability in the non-Kähler setting. For details, we refer to [93, Chapter 5] for the general theory, and to [67, Section 4.1] for the application to complex non-Kähler manifolds. Let X be a compact complex manifold of complex dimension n, and let $\sigma = [\omega^{n-1}] \in H_A^{n-1,n-1}(X, \mathbb{R})$ be a Gauduchon class. Moreover, let $\mathcal{F}$ be a coherent, torsion-free $\mathcal{O}_X$-module. Then, it has a well defined rank rk $\mathcal{F} = r$. The determinant sheaf of $\mathcal{F}$ given by:

$$\det \mathcal{F} = ((\Lambda^r F)^*)^* \tag{1.3.28}$$

is a free sheaf, hence it is the sheaf of sections of a holomorphic line bundle, which we denote again by $\det \mathcal{F}$.

Definition 1.3.4. 1. *The* degree *of $\mathcal{F}$ with respect to σ is:*

$$\deg_\sigma \mathcal{F} = c_1(\det \mathcal{F})_{BC} \cdot \sigma \tag{1.3.29}$$

in the natural duality pairing:

$$H_{BC}^{1,1}(X, \mathbb{R}) \times H_A^{n-1,n-1}(X, \mathbb{R}) \longrightarrow \mathbb{R}. \tag{1.3.30}$$

2. *The* slope *of $\mathcal{F}$ with respect to σ is given by:*

$$\mu_\sigma(\mathcal{F}) = \frac{\deg_\sigma \mathcal{F}}{r}. \tag{1.3.31}$$

Definition 1.3.5. *The sheaf $\mathcal{F}$ is:*

1. *σ-stable if for any non-trivial coherent subsheaf $\mathcal{F}' \subset \mathcal{F}$:*

$$\mu_\sigma(\mathcal{F}') < \mu_\sigma(\mathcal{F}). \tag{1.3.32}$$

2. *σ-semistable if for any non-trivial coherent subsheaf $\mathcal{F}' \subset \mathcal{F}$:*

$$\mu_\sigma(\mathcal{F}') \leq \mu_\sigma(\mathcal{F}), \tag{1.3.33}$$

and if equality holds, then:

$$\mathcal{F} \cong \mathcal{F}' \oplus \mathcal{F}/\mathcal{F}'. \tag{1.3.34}$$

3. *σ-unstable if it is not semistable.*

4. *σ-polystable if it is isomorphic to a direct sum of σ-stable sheaves of the same degree.*

The fundamental connection between slope-stability and gauge theory is given by the Donaldson-Uhlenbeck-Yau theorem and its extension to general hermitian manifolds:

Theorem 1.3.6 ([99])**.** *Let $\mathcal{Q}$ be a holomorphic vector bundle and $\sigma = [\omega^{n-1}]$ a Gauduchon class. Then $\mathcal{Q}$ is σ-polystable if and only if there exists a Hermite-Einstein metric h on $\mathcal{Q}$ such that:*

$$F_h \wedge \omega^{n-1} = \frac{\lambda \omega^n}{n} \otimes \mathrm{Id}, \tag{1.3.35}$$

where λ is a topological constant determined by $\mathcal{Q}$ and σ. Moreover, in such case, h is unique up to holomorphic automorphism of $\mathcal{Q}$.

Remark 1.3.7. *In the Kähler setting, a result analogous to Theorem 1.3.6 for holomorphic principal bundles was proved in [7].*

Chapter 2

Generalized Geometry

In this Chapter we introduce the fundamental notions of Generalized Geometry that will be necessary in the sequel. In broad terms, Generalized Geometry studies Courant algebroids, which in its most elementary form correspond to the geometry of $T \oplus T^*$, where a number of non-trivial geometric structures arise. Some useful references for this Chapter are [71, 80, 81].

Here, in the first Section, we recall the notion of Courant algebroids, the two main families of *exact* and *string* Courant algebroids that will be of interest for this Thesis, and introduce generalized metrics. The second Section deals with the interaction of Generalized Geometry with Complex Geometry. Further references for these topics are given along the text. The author claims no originality for the contents of this chapter.

2.1 Courant algebroids

2.1.1 Definition

Throughout, let M be a real smooth manifold of dimension n. We will also denote $T = TM$ and $T^* = T^*M$ when the manifold is understood.

Definition 2.1.1. *A real, smooth Courant algebroid over M is a tuple $(E, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$, where:*

1. *$E \longrightarrow M$ is a vector bundle.*
2. *There is a symmetric, non-degenerate pairing on sections:*

$$\langle\cdot,\cdot\rangle : \Gamma(E) \times \Gamma(E) \longrightarrow \mathcal{C}^\infty_M \tag{2.1.1}$$

   bilinear over smooth functions.

3. *There is a bracket on sections:*

$$[\cdot,\cdot] : \Gamma(E) \times \Gamma(E) \longrightarrow \Gamma(E) \tag{2.1.2}$$

   bilinear over constants.

4. *There is an* anchor *bundle map $E \xrightarrow{\rho} T$.*

satisfying the following compatibility axioms. For sections $a, b, c \in \Gamma(E)$ and $f \in \mathcal{C}^\infty_M$:

$$1.\ [a,[b,c]] = [[a,b],c] + [b,[a,c]] \tag{2.1.3}$$
$$2.\ \rho(a)(\langle b,c\rangle) = \langle [a,b],c\rangle + \langle b,[a,c]\rangle \tag{2.1.4}$$
$$3.\ [a,fb] = f[a,b] + \rho(a)(f)b \tag{2.1.5}$$
$$4.\ \rho([a,b]) = [\rho(a),\rho(b)] \tag{2.1.6}$$
$$5.\ [a,b] + [b,a] = 2\rho^* d\langle a,b\rangle, \tag{2.1.7}$$

where, in the last relation, the pairing $\langle\cdot,\cdot\rangle$ is used to identify $T^* \stackrel{\rho^*}{\to} E^* \stackrel{\langle\cdot,\cdot\rangle}{\to} E$.

Remark 2.1.2. *1. The axioms of Definition 2.1.1 are not completely independent, and some of them can be derived from the others. However, we have kept them for clarity.*

2. In the literature, sometimes the Axioms (2.1.3) *and* (2.1.7) *are replaced by asking that $[\cdot,\cdot]$ is skew-symmetric. Here, with the above axioms, it is usually named* Dorfman *bracket, as opposed to other conventions.*

3. From the axioms above, it follows that E fits in the complex:

$$T^* \xrightarrow{\rho^*} E \xrightarrow{\rho} T. \tag{2.1.8}$$

We stress that this complex need not *be exact.*

4. The analogous notion of Courant algebroid is defined over the complex numbers.

A complex Courant algebroid is a smooth complex vector bundle $E_{\mathbb{C}}$ for which $\langle\cdot,\cdot\rangle$, $[\cdot,\cdot]$ are analogous morphisms of sheaves of smooth complex sections, and the anchor is defined as $\rho : E_{\mathbb{C}} \to T\otimes\mathbb{C}$, satisfying the analogous complex Axioms of (2.1.3)-(2.1.7). *If E is a real Courant algebroid, then $E\otimes\mathbb{C}$ is naturally a complex Courant algebroid.*

When attempting to classify Courant algebroids, there are several inequivalent notions of morphisms in the literature. We will not study each one of them and their differences here, but we will make precise which morphisms we take into account for the relevant families of Courant algebroids for this Thesis, in Section 2.1.2.

Associated to any Courant algebroid E, there are a number of distinguished bundles with additional structure induced from that of E. First, there are natural subbundles given by:

$$T^* \subset (\ker\rho)^\perp \subset \ker\rho \subset E,$$

where the inclusions follow from the Axioms (2.1.3)-(2.1.7). Then, we define:

$$A_E = \frac{E}{(\ker\rho)^\perp}, \quad \mathrm{ad}_E = \frac{\ker\rho}{(\ker\rho)^\perp}. \tag{2.1.9}$$

The bundle A_E is naturally a Lie algebroid with the bracket inherited from E. This follows from the fact that $(\ker\rho)^\perp$ is a two-sided ideal for $(E,[\cdot,\cdot])$. Since $\ker\rho$ is also an ideal, this structure restricts to ad_E. Moreover, ad_E also inherits the pairing from E, hence it is a bundle of quadratic Lie algebras. The same applies for a complex Courant algebroid, yielding a complex Lie algebroid $A_{E_{\mathbb{C}}}$ and a quadratic Lie algebroid $\mathrm{ad}_{E_{\mathbb{C}}}$.

We finish this Section recalling that for any real Courant algebroid, there always exists an isotropic splitting $\sigma : T \to E$, *i.e.* satisfying

$$\rho \circ \sigma = \mathrm{id}, \quad \langle \sigma(X), \sigma(Y) \rangle = 0, \quad X, Y \in \Gamma(T).$$

and the same is true for complex Courant algebroids. As will become clear in the sequel, the importance of splittings stems from the fact that they produce explicit representations of Courant algebroids.

2.1.2 Exact and string Courant algebroids

The sequence (2.1.8) in which a Courant algebroid fits allows to consider different families of Courant algebroids depending on the specific details of this sequence. The simplest case is the following.

Definition 2.1.3. *A real smooth exact Courant algebroid $(E, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \rho)$ is a Courant algebroid such that the sequence:*

$$0 \longrightarrow T^* \xrightarrow{\rho^*} E \xrightarrow{\rho} T \longrightarrow 0 \tag{2.1.10}$$

is exact. Similarly, a complex smooth Courant algebroid is exact if the sequence

$$0 \longrightarrow T^* \otimes \mathbb{C} \xrightarrow{\rho^*} E_{\mathbb{C}} \xrightarrow{\rho} T \otimes \mathbb{C} \longrightarrow 0 \tag{2.1.11}$$

is exact.

Definition 2.1.4. *An isomorphism ϕ of real exact Courant algebroids E, E' is an orthogonal, bracket preserving invertible bundle map covering the identity on the manifold, such that the following diagram commutes:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T^* & \longrightarrow & E & \longrightarrow & T & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & T^* & \longrightarrow & E' & \longrightarrow & T & \longrightarrow & 0 \end{array} \tag{2.1.12}$$

Similarly, an isomorphism ϕ between complex smooth Courant algebroids $E_{\mathbb{C}}$, $E'_{\mathbb{C}}$ is an orthogonal, bracket preserving invertible bundle map such that:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T^* \otimes \mathbb{C} & \longrightarrow & E_{\mathbb{C}} & \longrightarrow & T \otimes \mathbb{C} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & T^* \otimes \mathbb{C} & \longrightarrow & E' & \longrightarrow & T \otimes \mathbb{C} & \longrightarrow & 0. \end{array} \tag{2.1.13}$$

Remark 2.1.5. *In the literature, a wider notion of exact Courant algebroid isomorphisms is considered (see e.g. [71, Definition 2.18], which does not require the fibre-preserving condition. For the purposes of this Thesis, we will however restrict to Definition 2.1.4 above.*

Exact Courant algebroids can be described via explicit models. The following is the prototypical Example.

Example 2.1.6. *Let $E = T \oplus T^*$ and $H \in \Omega^3_M$ such that $dH = 0$. Moreover, consider the structure:*

$$\begin{aligned} \langle \cdot, \cdot \rangle_0 &: \Gamma(E) \times \Gamma(E) \longrightarrow \mathcal{C}^\infty_M \\ [\cdot, \cdot]_H &: \Gamma(E) \times \Gamma(E) \longrightarrow \Gamma(E) \\ \rho_0 &: E \to T \end{aligned} \tag{2.1.14}$$

given by the following formulae:

$$\begin{aligned} \langle X + \xi, Y + \eta \rangle_0 &= \tfrac{1}{2}(\eta(X) + \xi(Y)) \\ [X + \xi, Y + \eta]_H &= [X, Y] + \mathcal{L}_X \eta - i_Y d\xi + i_Y i_X H \\ \rho_0(X + \xi) &= X, \end{aligned} \tag{2.1.15}$$

where $X, Y \in \Gamma(T)$ and $\xi, \eta \in \Gamma(T^*)$. Then $(E, \langle \cdot, \cdot \rangle_0, [\cdot, \cdot]_H, \rho_0)$ is an exact Courant algebroid. Conversely, if $(E, \langle \cdot, \cdot \rangle_0, [\cdot, \cdot]_H, \rho_0)$ is an exact Courant algebroid for a 3-form H, then $dH = 0$. For a proof see e.g. [71, Proposition 2.17]. The analogous result holds for complex exact Courant algebroids $E_{\mathbb{C}} = (T \oplus T^*) \otimes \mathbb{C}$ and complex 3-forms. In what follows we will denote:

$$E_H = (T \oplus T^*, \langle \cdot, \cdot \rangle_0, [\cdot, \cdot]_H, \rho_0). \tag{2.1.16}$$

In what follows, we characterize abstract exact Courant algebroids and describe their symmetries. We will write the results for real Courant algebroids, being their complex counterparts straightforward generalizations.

Proposition 2.1.7. *Let $(E, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \rho)$ be a smooth real Courant algebroid, and let $\sigma : T \to E$ be an isotropic splitting. Then, the map*

$$\phi_\sigma : E_H \longrightarrow E, \quad X + \xi \mapsto \sigma(X) + \tfrac{1}{2}\rho^*(\xi) \tag{2.1.17}$$

is an isomorphism of Courant algebroids for

$$H(X, Y, Z) = 2\langle [\sigma(X), \sigma(Y)], \sigma(Z) \rangle, \quad X, Y, Z \in \Gamma(T). \tag{2.1.18}$$

As a consequence, we obtain:

Theorem 2.1.8 (Ševera)**.** *There is a one-to-one correspondence between isomorphism classes of real exact Courant algebroids and $H^3_{dR}(M)$.*

Remark 2.1.9. *The above result can be reinterpreted in terms of sheaf cohomology by considering the sheaf complex:*

$$0 \longrightarrow \Omega^2_{\mathrm{cl.}} \xrightarrow{j} \Omega^2_M \xrightarrow{d} \Omega^3_M \xrightarrow{d} \Omega^4_M \xrightarrow{d} \ldots. \tag{2.1.19}$$

Then, $H^3_{dR}(M) \cong H^1(\Omega^2_M)$.

Proposition 2.1.7 provides explicit presentations for abstract exact Courant algebroids. However, it is not unique, and any two of them differ by a Courant algebroid automorphism. These are characterized in the following result:

Proposition 2.1.10. *Let E_H and $E_{H'}$ be exact Courant algebroids as in Example 2.1.6, and assume $[H'] = [H] \in H^3_{dR}(M)$. Moreover, let $\phi : E_{H'} \to E_H$ be an isomorphism. Then:*

$$\phi(X + \xi) = X + i_X b + \xi, \tag{2.1.20}$$

where b is a 2-form satisfying

$$H' = H + db. \tag{2.1.21}$$

In particular, exact Courant algebroid automorphisms of E_H are given by maps:

$$e^b : E_H \longrightarrow E_H, \quad e^b(X + \xi) = X + i_X b + \xi, \tag{2.1.22}$$

where $db = 0$.

Observe this result is a geometric realization of exact cocycles in Remark 2.1.9 preserving the algebroid isomorphism class. In the literature, the 2-form b in Proposition 2.1.10 is usually called the B-field, and the maps (2.1.22) are B-field transformations. As a consequence of Propositions 2.1.7, 2.1.10, we obtain a characterization of the exact Courant algebroid explicit models that are isomorphic to a given abstract exact Courant algebroid E.

Next, we introduce *string algebroids* following [69], also known as *heterotic* in the literature [32]. This is a generalization of exact Courant algebroids that incorporates naturally the geometry of principal bundles and will play a central role in this Thesis. Here, we will follow the notations and conventions of Section 1.3.1. Let K be a compact Lie group with quadratic Lie algebra $(\mathfrak{k}, \langle \cdot, \cdot \rangle)$ and let $P \to M$ be a principal K-bundle.

Definition 2.1.11. *1. A real, smooth string algebroid is a triple (E, P, ρ_P) such that E is a real, smooth Courant algebroid fitting in the short exact sequence:*

$$0 \longrightarrow T^* \longrightarrow E \xrightarrow{\rho_P} A_P \longrightarrow 0, \tag{2.1.23}$$

where A_P is the Atiyah algebroid of P, ρ_P is a bracket-preserving map, and the induced map $\rho_P : A_E \to A_P$ is an isomorphism of Lie algebroids restricting to an isomorphism of quadratic Lie algebroids $\mathrm{ad}_E \cong \mathrm{ad}\, P$.

2. A complex, smooth string algebroid is a triple $(E_{\mathbb{C}}, P_{\mathbb{C}}, \rho_{P_{\mathbb{C}}})$ such that $E_{\mathbb{C}}$ is a complex smooth Courant algebroid fitting in the short exact sequence:

$$0 \to T^* \otimes \mathbb{C} \longrightarrow E_{\mathbb{C}} \xrightarrow{\rho_{P_{\mathbb{C}}}} A^c_{P_{\mathbb{C}}}, \tag{2.1.24}$$

where $A^c_{P_{\mathbb{C}}}$ is the complex Atiyah-Lie algebroid of $P_{\mathbb{C}}$, and the induced map $\rho_{P_{\mathbb{C}}} : A_{E_{\mathbb{C}}} \to A^c_{P_{\mathbb{C}}}$ is an isomorphism of complex Lie algebroids restricting to an isomorphism of quadratic Lie algebroids $\mathrm{ad}_{\mathrm{E}_{\mathbb{C}}} \cong \mathrm{ad}\, P_{\mathbb{C}}$.

Definition 2.1.12 ([69])**.** *Let (E, P, ρ_P) and $(E', P', \rho_{P'})$ be real string algebroids. Then, a map $\phi : E \to E'$ is an isomorphism of string Courant algebroids if it is orthogonal, bracket-preserving, and the following diagram commutes:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T^* & \longrightarrow & E & \xrightarrow{\rho_P} & A_P & \longrightarrow & 0 \\ & & \Big\downarrow \mathrm{id} & & \Big\downarrow \phi & & \Big\downarrow g & & \\ 0 & \longrightarrow & T^* & \longrightarrow & E' & \xrightarrow{\rho_{P'}} & A_{P'} & \longrightarrow & 0 \end{array} \tag{2.1.25}$$

where $g : A_P \to A_{P'}$ is the isomorphism induced by a principal bundle isomorphism $g : P \to P'$ covering the identity on M. In case $(E, P, \rho_P) = (E', P', \rho_{P'})$, we say ϕ is a restricted automorphism if moreover $g = \mathrm{id}$. An isomorphism of complex string algebroids $E_{\mathbb{C}}$ and $E'_{\mathbb{C}}$ is analogously defined substituting the defining short exact sequences of E, E' in (2.1.23) *by the corresponding complex short exact sequences* (2.1.24).

The following are explicit Examples of string algebroids, ande are the counterpart of Example 2.1.6 for exact Courant algebroids.

Example 2.1.13. *Let $E = T \oplus \mathrm{ad}\, P \oplus T^*$ for a real principal bundle $P \to M$. Moreover, let $H \in \Omega^3_M$ and A be a principal connection for P such that:*

$$dH - \langle F_A \wedge F_A \rangle = 0. \tag{2.1.26}$$

We consider the structure:

$$\begin{aligned} \langle \cdot, \cdot \rangle_0 &: \Gamma(E) \times \Gamma(E) \longrightarrow \mathcal{C}^\infty_M \\ [\cdot, \cdot]_{H,A} &: \Gamma(E) \times \Gamma(E) \longrightarrow \Gamma(E) \\ \rho_0 &: E \to T \end{aligned} \tag{2.1.27}$$

given by the following formulae:

$$\begin{aligned} \langle X + r + \xi, Y + s + \eta \rangle_0 &= \tfrac{1}{2}(\xi(Y) + \eta(X)) + \langle r, s \rangle, \\ [X + r + \xi, Y + s + \eta]_{H,A} &= [X, Y] - F_A(X, Y) + i_X d_A r - i_Y d_A s - [r, s] + \mathcal{L}_X \eta + \\ &\quad + i_Y d\xi + i_Y i_X H + 2\langle d_A r, t \rangle + 2\langle i_X F_A, s \rangle - 2\langle i_Y F_A, r \rangle, \\ \rho(X + r + \xi)_0 &= X, \end{aligned} \tag{2.1.28}$$

where $X, Y \in \Gamma(T)$, $r, s \in \Gamma(\mathrm{ad}\, P)$ and $\xi, \eta \in \Gamma(T^*)$. Then $(E, \langle \cdot, \cdot \rangle_0, [\cdot, \cdot]_{H,A}, \rho_0)$ is a string algebroid. Conversely, if $(E, \langle \cdot, \cdot \rangle_0, [\cdot, \cdot]_{H,A}, \rho_0)$ is a string algebroid for a 3-form H and a principal connection A on P then $dH - \langle F_A \wedge F_A \rangle = 0$. The analogous result holds for complex exact Courant algebroids $E_{\mathbb{C}} = T \otimes \mathbb{C} \oplus \mathrm{ad}\, P_{\mathbb{C}} \oplus T^* \otimes \mathbb{C}$ and complex 3-forms and principal connections. The string algebroid described above will be denoted in the sequel by:

$$E_{P,H,A} = (E, \langle \cdot, \cdot \rangle_0, [\cdot, \cdot]_{H,A}, \rho_0). \tag{2.1.29}$$

The following results give the real string analogs of Propositions 2.1.7, 2.1.10. The complex versions are a straightforward generalization and are ommited.

Proposition 2.1.14. *Let (E, P, ρ_P) be a smooth real string algebroid, and let $\sigma_0 : T \to E$ be an isotropic splitting, and let $g_0 : P_0 \to P$ be a principal bundle isomorphism covering the identity. Then, the map*

$$\phi_{\sigma_0,g_0} : E_{P_0,H_0,A_0} \longrightarrow E, \quad X + \xi \mapsto \sigma_0(X) + \rho_P|^{-1}_{(\mathrm{im}\ \sigma_0)^\perp} \circ g_0(r) + \tfrac{1}{2}\rho^*(\xi) \tag{2.1.30}$$

is an isomorphism of Courant algebroids for

$$H_0(X, Y, Z) = 2\langle[\sigma_0(X), \sigma_0(Y)], \sigma_0(Z)\rangle, \quad X, Y, Z \in \Gamma(T). \tag{2.1.31}$$

and A_0 the principal connection on P_0 determined by $A_0^\perp = (g_0)^{-1} \circ \rho_P \circ \sigma_0$, where $A_0^\perp : T \to A_{P_0}$ is the horizontal lift of A_0.

The following results describe the relation between different models for string algebroids. In particular, we obtain the group of symmetries of a given string algebroid.

Proposition 2.1.15. *Let $E = E_{P,H,A}$ and $E' = E_{P',H',A'}$, and assume P and P' are isomorphic principal bundles. Then, the set of isomorphisms $\phi : E' \to E$ is one-to-one with pairs (g, b) of principal bundle isomorphisms covering the identity $g : P' \to P$ and 2-forms b such that:*

$$H' = H + CS(A') - CS(g^{-1}A) - d\langle A' \wedge g^{-1}A\rangle + db. \tag{2.1.32}$$

Explicitly, given (g, b) such that (2.1.32) holds, then $\phi = e^{(b,a)}$, where:

$$e^{(b,a)}(X + r + \xi) = X + g(a(X) + r) + i_X b + \langle a(X), a\rangle + 2\langle a, r\rangle + \xi, \tag{2.1.33}$$

where $a = g^{-1}A - A' \in \Omega^1(\mathrm{ad}\ P')$.

Corollary 2.1.16. *The automorphisms of $E_{P,H,A}$ are in correspondence with pairs $(g, b) \in \mathcal{G}_P \times \Omega^2_M$ such that*

$$db + CS(A) - CS(g^{-1}A) - d\langle A \wedge g^{-1}A\rangle = 0, \tag{2.1.34}$$

or, equivalently, such that:

$$db - 2\langle a \wedge F_A\rangle - \langle a \wedge d_A a\rangle - \tfrac{1}{3}\langle a \wedge [a \wedge a]\rangle = 0, \tag{2.1.35}$$

where $a = g^{-1}A - A \in \Omega^1(\mathrm{ad}\ P)$. The automorphism to a pair (g, b) satisfying (2.1.35) is given by the formula (2.1.33).

As a direct consequence of Propositions 2.1.14 and 2.1.15, the following result provides formulae for the change of presentation of a given abstract string algebroid (E, P, ρ_P).

Corollary 2.1.17. *Let $E = (E, P, \rho_P)$ be a string algebroid and suppose $\phi_{\sigma_0,g_0} : E_{P_0,H_0,A_0} \to E$ is an isomorphism. Then E_{P_1,H_1,A_1} is isomorphic to E if and only if $P_1 \cong P$ and there exist a pair (g, b) such that (2.1.33) holds, where $a = g^{-1}A_0 - A_1$. In that case, the induced isomorphism is $\phi_{\sigma_1,g_1} : E_{P_1,H_1,A_1} \to E$, where:*

$$g_1 = g_0 \circ g \tag{2.1.36}$$

$$\sigma_1(X) = \sigma_0(X) + \rho_P|^{-1}_{(\mathrm{im}\ \sigma_0)^\perp} \circ g_1(a(X)) - \tfrac{1}{2}\rho^*(i_X b + \langle a(X), a\rangle). \tag{2.1.37}$$

As in the exact case, the classification of isomorphism classes of smooth string algebroids can be described in terms of sheaf cohomology (see [69, Appendix A]). However, we will not use directly that classification and therefore we omit it here.

2.1.3 Generalized metrics

Generalized metrics are one of the fundamental geometric structures Courant algebroids can be endowed with. In broad terms, these play a similar role as riemannian metrics in standard differential geometry. In this Section, we recall basic properties of generalized metrics and particularize for the case of exact and string Courant algebroids. For further details we refer to [61]. Throughout, let $E \to M$ be a real, smooth Courant algebroid. Generalized metrics can be defined in various degrees of generality. In this Thesis, we will adopt the following:

Definition 2.1.18. *A generalized metric on E is a subbundle $V_+ \subset E$ such that:*

1. *The restriction $\langle \cdot, \cdot \rangle|_{V_+}$ is a positive-definite inner product.*
2. *$\rho|_{V_+} : V_+ \to T$ is an isomorphism.*

Given a pair (E, V_+), we define the complement $V_- = (V_+)^\perp$. Note that for general Courant algebroids, the restriction of the ambient pairing $\langle \cdot, \cdot \rangle|_{V_-}$ does not have a sign. However, by the condition of Definition 2.1.18(1), we have that:

$$E = V_+ \oplus V_-. \tag{2.1.38}$$

Moreover, by the condition of 2.1.18(2), we have a lifting induced by the generalized metric:

$$\sigma_+ = (\rho|_{V_+})^{-1} : T \xrightarrow{\cong} V_+, \tag{2.1.39}$$

and hence a riemannian metric $g = \langle \sigma_+, \sigma_+ \rangle$. Then, the lifting given by:

$$\sigma : T \to E, \quad \sigma(X) = \sigma_+(X) - \tfrac{1}{2}\rho^*(g(X)) \tag{2.1.40}$$

is an isotropic splitting. In this situation, we call σ the splitting induced or preferred by V_+. Conversely, given a pair (σ, g) of an isotropic splitting on E and a riemannian metric on M, the expression:

$$V_+(\sigma, g) = \{\sigma(X) + \tfrac{1}{2}\rho^*(g(X)) \mid X \in T\} \tag{2.1.41}$$

is a generalized metric on E. The next result further refines what is the geometric content of a generalized metric in case the Courant algebroid is of the types of Section 2.1.2.

Proposition 2.1.19.

1. *Let E be an exact Courant algebroid, and $V_+ \subset E$ a generalized metric. Let $\sigma : T \to E$ be the isotropic splitting preferred by V_+. Then:*

$$\phi_\sigma^{-1}(V_+) = e^g(T) := \{X + g(X) \mid X \in T\} \subset E_H, \tag{2.1.42}$$

   where E_H is the exact Courant algebroid of Example 2.1.6 for $H = 2\langle [\sigma, \sigma], \sigma \rangle$.

2. *Let (E, P, ρ_P) be a string algebroid and let $V_+ \subset E$ a generalized metric. Let σ be the isotropic splitting preferred by V_+. Then:*

$$\phi_{\sigma,\mathrm{id}}^{-1}(V_+) = e^g(T) = \{X + g(X) \mid X \in T\} \subset E_{P,H,A}, \tag{2.1.43}$$

   where $E_{P,H,A}$ is the exact Courant algebroid of Example 2.1.13 for $H = 2\langle [\sigma, \sigma], \sigma \rangle$ and A is the principal connection determined by $A^\perp = \rho_P \circ \sigma$.

Hence, a generalized metric determines a preferred presentation of an exact or string Courant algebroid. Alternatively, if E is an exact Courant algebroid, a generalized metric is equivalent to a triple (σ, g, H). Moreover, in this case:

$$\phi_\sigma^{-1}(V_-) = e^{-g}(T) \subset E_H. \tag{2.1.44}$$

If E is a string algebroid, a generalized metric is equivalent to a quadruple (σ, g, H, A). Moreover, then:

$$\phi_{\sigma,\mathrm{id}}^{-1}(V_-) = e^{-g}(T) \oplus \mathrm{ad}\, P \subset E_{P,H,A}. \tag{2.1.45}$$

2.2 Holomorphic Courant algebroids

In the previous Section we have reviewed the smooth theory of Courant algebroids and particularized to exact and string cases. Now, we use it to induce holomorphic structures on them. The relation of smooth and holomorphic Courant algebroids allows to study the interaction with the hermitian geometry of the manifold, leading to the notions of Bott-Chern algebroids and generalized hermitian metrics. These will play a central role in this Thesis. The references for this Section are [69, 68] and [81, Appendix A].

We start by introducing holomorphic Courant algebroids. Throughout, let $X = (M, J)$ a complex manifold. We will denote $T^{1,0} = T^{1,0}X$, and $T^*_{1,0} = (T^*X)^{1,0}$.

Definition 2.2.1. *A holomorphic Courant algebroid is a tuple $(\mathcal{Q}, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \rho)$, where:*

1. *$\mathcal{Q} \to X$ is a holomorphic vector bundle.*
2. *The* pairing *$\langle\cdot,\cdot\rangle$ is a symmetric, non-degenerate morphism of sheaves:*
$$\langle\cdot,\cdot\rangle : \mathcal{O}(\mathcal{Q}) \otimes_{\mathcal{O}_X} \mathcal{O}(\mathcal{Q}) \to \mathcal{O}_X. \tag{2.2.1}$$
3. *The* Dorfman bracket *$[\cdot,\cdot]$ is a morphism of sheaves:*
$$[\cdot,\cdot] : \mathcal{O}(\mathcal{Q}) \otimes_{\underline{\mathbb{C}}} \mathcal{O}(\mathcal{Q}) \to \mathcal{O}(\mathcal{Q}). \tag{2.2.2}$$
4. *The* anchor *is a holomorphic vector bundle map:*
$$\rho : \mathcal{Q} \to T^{1,0}. \tag{2.2.3}$$

such that the following hold for $a, b, c \in \mathcal{O}(\mathcal{Q})$ and $f \in \mathcal{O}_X$:

$$1.\ [a,[b,c]] = [[a,b],c] + [b,[a,c]] \tag{2.2.4}$$
$$2.\ \rho(a)(\langle b,c\rangle) = \langle [a,b],c\rangle + \langle b,[a,c]\rangle \tag{2.2.5}$$
$$3.\ [a,fb] = f[a,b] + \rho(a)(f)b \tag{2.2.6}$$
$$4.\ \rho([a,b]) = [\rho(a),\rho(b)] \tag{2.2.7}$$
$$5.\ [a,b] + [b,a] = 2\rho^*\partial\langle a,b\rangle, \tag{2.2.8}$$

where in (2.2.8) *we have used the identification $T^*_{1,0} \xrightarrow{\rho^*} \mathcal{Q}^* \xrightarrow{\langle\cdot,\cdot\rangle} \mathcal{Q}$.*

Associated to a holomorphic Courant algebroid $\mathcal{Q}$, we consider the following bundles:

$$A_{\mathcal{Q}} = \frac{\mathcal{Q}}{(\ker\rho)^\perp}, \quad \mathrm{ad}_{\mathcal{Q}} = \frac{\ker\rho}{(\ker\rho)^\perp}. \tag{2.2.9}$$

Analogous to the smooth case, $(A_{\mathcal{Q}}, [\cdot,\cdot])$ is a Lie algebroid, and $(\mathrm{ad}_{\mathcal{Q}}, \langle\cdot,\cdot\rangle, [\cdot,\cdot])$ is a quadratic Lie algebroid, where the structure is inherited from $\mathcal{Q}$.

2.2.1 Liftings and reduction

Given a complex, smooth Courant algebroid, one can produce a holomorphic Courant algebroid through a procedure of choosing a lifting followed by reduction. This was observed in [81, Appendix A], and we recall it here briefly. Then, we particularize to smooth and string algebroids.

Definition 2.2.2. *Let $E_{\mathbb{C}}$ be a smooth, complex Courant algebroid. Then, a lifting $\ell \subset E_{\mathbb{C}}$ is a complex subbundle that satisfies:*

1. *ℓ is isotropic: $\langle \ell, \ell \rangle = 0$.*
2. *ℓ is an involutive distribution of $(E_{\mathbb{C}}, [\cdot,\cdot])$: $[\ell, \ell] \subset \ell$.*
3. *$\rho_{\mathbb{C}}|_{\ell} : \ell \to T^{0,1}$ is an isomorphism.*

By property of Definition 2.2.2(3), given a lifting ℓ we have a vector bundle map:

$$\sigma_\ell = (\rho_{\mathbb{C}}|_\ell)^{-1} : T^{0,1} \xrightarrow{\cong} \ell. \tag{2.2.10}$$

Proposition 2.2.3. *Let $E_{\mathbb{C}}$ be a smooth, complex Courant algebroid, and let $\ell \subset E_{\mathbb{C}}$ be a lifting. Moreover, let $\mathcal{Q}_\ell = \ell^\perp/\ell$. Then, the following defines an integrable Dolbeault operator $\overline{\partial}_{\mathcal{Q}_\ell}$ such that $\mathcal{Q}_\ell$ is a holomorphic Courant algebroid:*

$$i_{X^{0,1}}\overline{\partial}_{\mathcal{Q}_\ell}[a] = [\sigma_\ell(X^{0,1}), a] \mod \ell. \tag{2.2.11}$$

Remark 2.2.4. *In Proposition 2.2.3, the definition of $\mathcal{Q}_\ell$ is chosen such that $\overline{\partial}_{\mathcal{Q}_\ell}$ is well defined independent of choices. Then, the integrability is a formal consequence of Axiom* (2.2.4).

Next, we detail what the above result amounts to in the case $E_{\mathbb{C}}$ is exact or string. For this, it is convenient to extend the formalism to the holomorphic category:

Definition 2.2.5.

1. *A holomorphic Courant algebroid $\mathcal{Q}$ is exact if the sequence:*

$$0 \longrightarrow T^*_{1,0} \xrightarrow{\rho^*} \mathcal{Q} \xrightarrow{\rho} T^{1,0} \longrightarrow 0 \tag{2.2.12}$$

   is exact.

2. *Let G be a complex Lie group and let $P \to X$ be a holomorphic principal bundle. Then, $(\mathcal{Q}, P, \rho_P)$ is a holomorphic string algebroid if the sequence:*

$$0 \longrightarrow T^*_{1,0} \xrightarrow{\rho^*} \mathcal{Q} \xrightarrow{\rho_P} A_P \longrightarrow 0 \tag{2.2.13}$$

   is exact, where $A_P = T^{1,0}P/G \to T^{1,0}$ is the holomorphic Atiyah algebroid of P, and ρ_P is a bracket preserving map inducing isomorphism of holomorphic Lie algebroids $A_{\mathcal{Q}} \cong A_P$, and of holomorphic quadratic Lie algebrois $\mathrm{ad}_{\mathcal{Q}} \cong \mathrm{ad}\, P$.

Example 2.2.6 ([69]). *1. Let $Q = T^{1,0} \oplus T^*_{1,0}$ as a smooth vector bundle, and $\tau \in \Omega_X^{3,0+2,1}$ such that $d\tau = 0$. Moreover, consider the structure:*

$$
\begin{aligned}
\langle \cdot, \cdot \rangle_0 &: \Gamma(Q) \times \Gamma(Q) \longrightarrow \mathcal{C}^\infty_X \\
\overline{\partial}_{\mathcal{Q}} &: \Gamma(T^{0,1}) \times \Gamma(Q) \to \Gamma(Q) \\
[\cdot, \cdot]_\tau &: \Gamma(Q) \times \Gamma(Q) \longrightarrow \Gamma(Q) \\
\rho_0 &: Q \to T^{1,0}
\end{aligned}
\tag{2.2.14}
$$

given by the following formulae:

$$
\begin{aligned}
\langle X + \xi, Y + \eta \rangle_0 &= \tfrac{1}{2}(\eta(X) + \xi(Y)) \\
i_{V^{0,1}} \overline{\partial}_{\mathcal{Q}}(X + \xi) &= \overline{\partial}_{V^{0,1}} X + \overline{\partial}_{V^{0,1}} \xi + \tau^{2,1}(V^{0,1}, X, \cdot) \\
[X + \xi, Y + \eta]_\tau &= [X, Y] + i_X \partial \eta + \partial(i_X \eta) - i_Y d\xi + i_Y i_X \tau^{3,0} \\
\rho_0(X + \xi) &= X,
\end{aligned}
\tag{2.2.15}
$$

where $X, Y \in \Gamma(T^{1,0})$, $V^{0,1} \in \Gamma(T^{0,1})$ and $\xi, \eta \in \Gamma(T^*_{1,0})$. Then, $\mathcal{Q}_\tau = (Q, \overline{\partial}_{\mathcal{Q}})$ is a holomorphic vector bundle and $(\mathcal{Q}_\tau, \langle \cdot, \cdot \rangle_0, [\cdot, \cdot]_\tau, \rho_0)$ is a holomorphic exact Courant algebroid. Conversely, if $(\mathcal{Q}_\tau, \langle \cdot, \cdot \rangle_0, [\cdot, \cdot]_\tau, \rho_0)$ is a holomorphic exact Courant algebroid for a 3-form $\tau \in \Omega_X^{3,0+2,1}$, then $d\tau = 0$.

2. Let G be a complex Lie group with quadratic Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$, and P a holomorphic principal G-bundle. Moreover, let $Q = T^{1,0} \oplus \mathrm{ad}\ P \oplus T^*_{1,0}$ as a smooth vector bundle, and $\tau \in \Omega_X^{3,0+2,1}$ and A principal connection on P compatible with the holomorphic structure, such that:

$$
d\tau - \langle F_A \wedge F_A \rangle = 0. \tag{2.2.16}
$$

Furthermore, consider the structure as in (2.2.14) given by the following formulae:

$$
\begin{aligned}
\langle X + + r + \xi, Y + s + \eta \rangle_0 &= \tfrac{1}{2}(\eta(X) + \xi(Y)) + \langle r, s \rangle \\
i_{V^{0,1}} \overline{\partial}_{\mathcal{Q}_\tau}(X + r + \xi) &= \overline{\partial}_{V^{0,1}} X + \overline{\partial}_A r - F_A(V^{0,1}, X) + \\
&\quad + \overline{\partial}_{V^{0,1}} \xi + \tau^{2,1}(V^{0,1}, X, \cdot) + 2\langle i_{V^{0,1}} F_A, r \rangle \\
[X + r + \xi, Y + s + \eta]_{\tau, A} &= [X, Y] - F_A^{2,0}(X, Y) + i_X \partial_A s - i_Y \partial_A r - [r, s] + \\
&\quad + i_X \partial \eta + \partial(i_X \eta) - i_Y d\xi + i_Y i_X \tau^{3,0} + 2\langle \partial_A r, s \rangle + \\
&\quad + 2\langle i_X F_A^{2,0}, t \rangle - 2\langle i_Y F_A^{2,0}, r \rangle \\
\rho_0(X + r + \xi) &= X,
\end{aligned}
\tag{2.2.17}
$$

where $X, Y \in \Gamma(T^{1,0})$, $V^{0,1} \in \Gamma(T^{0,1})$, $r, s \in \Gamma(\mathrm{ad}\ P)$ and $\xi, \eta \in \Gamma(T^*_{1,0})$. Then, $\mathcal{Q}_{P,\tau,A} = (Q, \overline{\partial}_{\mathcal{Q}})$ is a holomorphic vector bundle and $(\mathcal{Q}_{P,\tau,A}, \langle \cdot, \cdot \rangle_0, [\cdot, \cdot]_\tau, \rho_0)$ is a holomorphic string algebroid. Conversely, if $(\mathcal{Q}_{P,\tau,A}, \langle \cdot, \cdot \rangle_0, [\cdot, \cdot]_{\tau,A}, \rho_0)$ is a holomorphic string algebroid for a 3-form $\tau \in \Omega_X^{3,0+2,1}$ and a compatible principal connection A in P, then (2.2.16) holds.

The next two results characterize the space of liftings for exact and string algebroids, and identify what is the explicit holomorphic model of the type of Example 2.2.6 of the resulting holomorphic reduction. Although this procedure is purely complex, for the purposes of this Thesis it will be enough to apply it to complexified real algebroids. These have, by definition, a natural real form $E \subset E \otimes \mathbb{C}$. The description of liftings uses, in particular, this structure. The reader is directed to [70] for the general theory of holomorphic reduction of Courant algebroids.

Proposition 2.2.7. *The set of liftings on $E_H \otimes \mathbb{C}$ is one-to-one with pairs $(\psi, b) \in \Omega^{1,1}_{M,\mathbb{R}} \times \Omega^2_M$ such that:*

$$H + d^c\psi + db = 0. \tag{2.2.18}$$

Explicitly, given (ψ, b) satisfying (2.2.18), then:

$$\ell(\psi, b) = e^{-(b+i\psi)}(T^{0,1}) = \{X^{0,1} - i_{X^{0,1}}(b + i\psi) \mid X^{0,1} \in T^{0,1}\} \tag{2.2.19}$$

is a lifting. Moreover:

$$\phi_\ell : \mathcal{Q}_{2i\partial\psi} \to \mathcal{Q}_{\ell(\psi,b)}, \quad X + \xi \mapsto [e^{-(b+i\psi)}(X) + \xi] \tag{2.2.20}$$

is an isomorphism.

Proposition 2.2.8. *The set of liftings on $E_{P_K,H,A} \otimes \mathbb{C}$ is one-to-one with triples $(\psi, b, a) \in \Omega^{1,1}(M, \mathbb{R}) \times \Omega^2_M \times \Omega^1(\mathrm{ad}P_K)$ such that:*

$$\begin{aligned} H + d^c\psi + CS(A') - CS(A) - d\langle A' \wedge A\rangle + db = 0, \\ F^{0,2}_{A'} = 0, \end{aligned} \tag{2.2.21}$$

where $A' = A + a$. Explicitly, given (ψ, b, a) satisfying (2.2.21), then the associated lifting is:

$$\ell(\psi, b, a) = e^{(-(b+i\psi),-a)}(T^{0,1}), \tag{2.2.22}$$

where $e^{(-(b+i\psi),-a)}$ is as in (2.1.33). Moreover:

$$\phi_\ell : \mathcal{Q}_{P,2i\partial\psi,A+a} \to \mathcal{Q}_{\ell(\psi,b,a)}, \quad X + r + \xi \mapsto [e^{(-(b+i\psi),-a)}(X + r + \xi)] \tag{2.2.23}$$

is an isomorphism, where $P = (P^c_K, \overline{\partial}_A)$.

2.2.2 Generalized hermitian metrics

Throughout this Section, let $X = (M, J)$ be a complex manifold and E a real Courant algebroid. The fundamental compatibility condition between Generalized Geometry with the complex structure is the following:

Definition 2.2.9 ([67])**.** *A generalized metric $V_+ \subset E$ is compatible with J if:*

$$\ell = V_+ \otimes \mathbb{C} \cap \rho^{-1}(T^{0,1}) \subset E \otimes \mathbb{C} \tag{2.2.24}$$

is a lifting.

Proposition 2.2.10. *1. Let E be an exact Courant algebroid, and V_+ a generalized metric. Moreover, let (σ, g, H) be the data associated to V_+. Then, V_+ is compatible with J if and only if g is a hermitian metric and:*

$$H = -d^c\omega, \tag{2.2.25}$$

where $\omega = g(J\cdot, \cdot)$. In particular:

$$dd^c\omega = 0. \tag{2.2.26}$$

2. Let E be a string algebroid, and V_+ a generalized metric. Moreover, let (σ, g, H, A) be the data associated to V_+. Then, V_+ is compatible with J if and only if g is a hermitian metric and:

$$H = -d^c\omega, \quad F_A^{0,2} = 0. \tag{2.2.27}$$

In particular:

$$dd^c\omega + \langle F_A \wedge F_A \rangle = 0. \tag{2.2.28}$$

From Propositions 2.2.7, 2.2.8 and Proposition 2.2.10, we obtain as an immediate consequence explicit models for holomorphic reductions in case these are induced by J-compatible generalized metrics:

Corollary 2.2.11. *1. Let E be an exact Courant algebroid, V_+ a J-compatible generalized metric, and $\mathcal{Q}_\ell$ its holomorphic reduction. Moreover, let ω be the hermitian metric given by Proposition 2.2.10*(1)*. Then:*

$$\phi_\ell : \mathcal{Q}_{2i\partial\omega} \to \mathcal{Q}_\ell \tag{2.2.29}$$

is an isomorphism.

2. Let (E, P_K, ρ_{P_K}) be a string algebroid, V_+ a J-compatible generalized metric, and $\mathcal{Q}_\ell$ its holomorphic reduction. Moreover, let (ω, A) be as given by Proposition 2.2.10(2)*. Then:*

$$\phi_\ell : \mathcal{Q}_{P,2i\partial\omega,A} \to \mathcal{Q}_\ell \tag{2.2.30}$$

is an isomorphism, where $P = (P_K^c, \overline{\partial}_A)$.

Remark 2.2.12. *The Courant algebroids obtained in Corollary 2.2.11 do not fully exhaust the set of all holomorphic Courant algebroids, as the condition of $E \otimes \mathbb{C}$ admitting a real form and the lifting being induced by a generalized metric are not vacuous. In the exact case, one can readily see that algebroids in the family $\mathcal{Q}_{2i\partial\omega}$ for ω a hermitian metric are rather special within the family of Example 2.2.6*(1)*. Similarly, the string algebroids admitting a presentation as in Corollary 2.2.11*(2)* form a special family within its category, which is encoded in the following notion.*

Definition 2.2.13 ([68])**.** *Let $(\mathcal{Q}, P, \rho_P)$ a holomorphic string algebroid. Then, $\mathcal{Q}$ is called a Bott-Chern algebroid if there exists an isomorphism:*

$$\phi : \mathcal{Q}_{P,2i\partial\omega,A_h} \to \mathcal{Q}, \tag{2.2.31}$$

where ω is a $(1,1)$-form and A_h is the Chern connection of $P_h \subset P$ for a hermitian reduction to a maximal compact subgroup (see Section 1.3.1). If ω is moreover a hermitian metric, then $\mathcal{Q}$ is said to be positive.

Remark 2.2.14. *The existence of Bott-Chern structures on string algebroids is in general a difficult question. Moreover, how to determine if a Bott-Chern algebroid is positive is an open problem in general. For a discussion on these issues and particular known results see [68, Section 3].*

Now we introduce the notion that gives title to this Section. Let (E, V_+) be a Courant algebroid endowed with a J-compatible generalized metric, as in Definition 2.2.9. Then, it follows that:

$$\ell^{\perp} = V_- \otimes \mathbb{C} \oplus \ell \;\; \Rightarrow \;\; \mathcal{Q}_\ell = \ell^{\perp}/\ell \; \cong \; V_- \otimes \mathbb{C}, \tag{2.2.32}$$

by elementary considerations, where the isomorphism above is induced by the projection $E \xrightarrow{\pi_-} V_-$.

Definition 2.2.15. *Let E be a Courant algebroid and $V_+ \subset E$ a J-compatible generalized metric, and let $\mathcal{Q}_\ell$ the its holomorphic reduction. Then, we call*

$$\mathbf{G}([a],[b]) = -\langle \pi_-(a), \overline{\pi_-(b)} \rangle \tag{2.2.33}$$

the generalized hermitian metric on $\mathcal{Q}_\ell$.

Remark 2.2.16. 1. *The generalized hermitian metric $\mathbf{G}$ defined above has not a sign in general. For the case of interest for this Thesis, where E is a real string algebroid, let (σ, g, H, A) be induced by a generalized metric V_+ compatible with J. Then, in the given isomorphism $\mathcal{Q}_\ell \cong \mathcal{Q}_{P,2i\partial\omega,A}$ of Corollary 2.2.11(2), explicitly:*

$$\mathbf{G} = \left(\begin{pmatrix} g(\cdot,\bar{\cdot}) & 0 & 0 \\ 0 & -\langle \cdot,\bar{\cdot}\rangle & 0 \\ 0 & 0 & \frac{1}{4}g^{-1}(\cdot,\bar{\cdot}) \end{pmatrix} \right) \tag{2.2.34}$$

with respect to the natural smooth splitting $\mathcal{Q}_{P,2i\partial\omega,A} \overset{\mathcal{C}^\infty}{=} T^{1,0} \oplus \mathrm{ad}\ P \oplus T^*_{1,0}$ given by construction of $\mathcal{Q}_{P,2i\partial\omega,A}$ (see Example 2.2.6). Its indefinite signature is a fundamental feature that will be recurrent in subsequent Chapters. However, if E is exact, the analogous generalized hermitian metric $\mathbf{G}$ of (2.2.34) *is:*

$$\left(\begin{pmatrix} g(\cdot,\bar{\cdot}) & 0 \\ 0 & \frac{1}{4}g^{-1}(\cdot,\bar{\cdot}) \end{pmatrix} \right) \tag{2.2.35}$$

in the smooth splitting $\mathcal{Q}_{2i\partial\omega} \overset{\mathcal{C}^\infty}{=} T^{1,0} \oplus T^*_{1,0}$, and therefore, is positive definite. The string case will be dealt with in detail in Chapter 4.

2. *As given in Definition 2.2.15, generalized hermitian metrics are hermitian metrics on Courant algebroids in the standard sense of Differential Geometry. However, the name stems from the fact that these are induced by a choice of generalized data, and in some contexts it can be useful to adopt this more abstract point of view (see [67, Definition 3.15] for the exact case).*

2.2.3 Classification of holomorphic Courant algebroids

In Section 2.1.2, the classification of exact Courant algebroids was sketched (see Proposition 2.1.7 and Corollary 2.1.8). Here, we refine this classification for *holomorphic* Courant algebroids, and include the string case to the picture. Firstly, we provide the notion of isomorphism in this category.

Definition 2.2.17. *1. Let $\mathcal{Q}$, $\mathcal{Q}'$ be exact holomorphic Courant algebroids. Then a vector bundle map $\phi : \mathcal{Q} \to \mathcal{Q}'$ is an isomorphism if it preserves the Courant algebroid pairing and bracket, and makes the following diagram commute:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T^*_{1,0} & \longrightarrow & \mathcal{Q} & \longrightarrow & T^{1,0} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & T^*_{1,0} & \longrightarrow & \mathcal{Q}' & \longrightarrow & T^{1,0} & \longrightarrow & 0 \end{array}$$

2. *Let $(\mathcal{Q}, P, \rho_P)$, $(\mathcal{Q}', P', \rho_{P'})$ be string algebroids. Then a vector bundle map $\phi : \mathcal{Q} \to \mathcal{Q}'$ is an isomorphism if it preserves the Courant algebroid pairing and bracket, and makes the following diagram commute:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T^*_{1,0} & \longrightarrow & \mathcal{Q} & \longrightarrow & A_P & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & T^*_{1,0} & \longrightarrow & \mathcal{Q}' & \longrightarrow & A_{P'} & \longrightarrow & 0 \end{array}$$

where $g : A_P \to A'_P$ is an isomorphism induced from a principal bundle map $g : P \to P'$. If $P = P'$ and $g = \mathrm{id}$, we will call this isomorphism restricted.

The following result give a explicit characterization of isomorphisms in terms of explicit presentations:

Proposition 2.2.18 ([69]). *Let $\mathcal{Q}_{P,\tau,A}$, $\mathcal{Q}_{P',\tau',A'}$ be holomorphic string algebroids as described in Example* (2.2.6)*. Then, the set of isomorphisms $\phi : \mathcal{Q}_{P,\tau,A} \to \mathcal{Q}_{P',\tau',A'}$ is one-to one with the set of pairs (g, β) where $g : P \to P'$ is an isomorphism of principal bundles and $\beta \in \Omega^{2,0}$ such that:*

$$\tau' = \tau + CS(A') - CS(gA) - d\langle A' \wedge gA \rangle - d\beta. \tag{2.2.36}$$

We now recall the classification result obtained in [68] for Bott-Chern algebroids. For the exact case, the classification for positive algebroids follows formally by taking the structure group G to be trivial. For this, we introduce the sheaf complex $\Omega^{\leq 2}$ given by:

$$0 \longrightarrow \Omega^{2,0}_{\mathrm{cl.}} \xrightarrow{j} \Omega^{2,0} \xrightarrow{d} \Omega^{3,0+2,1} \xrightarrow{d} \Omega^{4,0+3,1+2,2} \xrightarrow{d} \dots \tag{2.2.37}$$

Proposition 2.2.19 ([68]). *Let G be a complex Lie group and P a holomorphic principal G-bundle. Then, the set of isomorphism classes of Bott-Chern algebroids $(\mathcal{Q}, P, \rho_P)$ with respect to restricted isomorphism is naturally a (possibly empty) affine space modelled on the image of the map:*

$$H_A^{1,1}(X, \mathbb{R}) \xrightarrow{\partial} H^1(\Omega^{\leq 2}), \quad [\psi] \mapsto [2i\partial\psi]. \tag{2.2.38}$$

Explicitly, given a Bott-Chern algebroid $\mathcal{Q} \cong \mathcal{Q}_{P,2i\partial\omega,A}$, to a class $[2i\partial\psi]$ in the image of (2.2.38) *we associate the class of $\mathcal{Q}_{2i\partial(\omega+\psi),A}$.*

Remark 2.2.20.

1. *The statement above does not make any claim about the positivity of the members of the family. Typically, once a positive Bott-Chern algebroid is fixed, only elements in a neighbourhood admit a positive structure.*

2. *In case X satisfies the $\partial\bar{\partial}$-lemma, it is straightforward to check that 2.2.38 is constant. Hence, there is at most one Bott-Chern algebroid over P up to restricted isomorphism.*

One can check that Proposition 2.2.19 above indeed agrees with Proposition 2.2.18: the Bott-Chern algebroids $\mathcal{Q}_{P,2i\partial\omega,P}$ and $\mathcal{Q}_{P,2i\partial(\omega+\psi),A}$ are isomorphic when $[2i\partial\psi] = 0 \in H^1(\Omega^{\leq 2})$, that is, when there exists $\beta \in \Omega^{2,0}$ such that $2i\partial\psi + d\beta = 0$. Then, it is clear that:

$$2i\partial(\omega + \psi) = 2i\partial\omega - d\beta \tag{2.2.39}$$

showing a particular case of the isomorphism provided by (2.2.36), where $(P', A') = (P, A)$.

Chapter 3

Background and Yau's Conjecture for the Hull-Strominger system

This Chapter introduces the main object of study of this Thesis: the Hull-Strominger system [85, 127]. See [62] for a survey and references therein. Here, we give an account of several aspects of its geometry.

In the first Section, we review its origin and how it is regarded currently in the mathematical literature, including the promising potential application to the Calabi-Yau web. We then give an overview of the known exact solutions in the literature. In the second Section, we address the existence problem for the Hull-Strominger system following a conjecture by S. T. Yau [138], and discuss the current state of the problem.

3.1 The Hull-Strominger system

3.1.1 Physical origin and mathematical formulation

The Hull-Strominger system is a system of geometric PDEs on a complex manifold. It first appeared independently in the physics literature in the works of A. Strominger [127] and C. Hull [85]. Here, we sketch how this system is obtained.

The starting point is the aim to describe the low energy limit of supersymmetric heterotic string theory from the spacetime point of view, that is, as a sigma model of maps $\mathcal{C}^\infty(\Sigma, T)$, where the space-time T is required to be 10-dimensional and assumed to be of the form:

$$T = \mathbb{R}^{3,1} \times M, \tag{3.1.1}$$

where the first factor stands for Minkowski 4-dim. space-time, and the second is called the *internal space*. With a suitable product ansatz for the matter content, the theory is reduced to M and the relevant fields are given by tuples (g, ϕ, H, A) of a riemannian metric g, a smooth function ϕ (dilaton), a 3-form H and a connection A on a principal bundle $P_K \to M$ with compact structure group K. We assume Lie K is endowed with an inner product, that we denote in this Section by tr. The low-energy theory is governed by the action:

$$S[g, \phi, H, A] = \int_M e^{-2\phi}(\mathrm{scal}^g + 4|d\phi|^2 - \tfrac{1}{12}|H|^2 + \tfrac{\alpha}{2}(|R_\nabla|^2 - |F_A|^2))d\mathrm{vol}_g, \tag{3.1.2}$$

where ∇ is a linear connection introduced to produce an anomaly cancellation, *i.e.* forcing a classical symmetry to carry over the quantized theory, and α is a physical parameter required to be positive. This requirement is expressed locally via the existence of a 2-form b such that:

$$H - \alpha CS(\nabla) + \alpha CS(A) + db = 0. \tag{3.1.3}$$

As a consequence, the following constraint arises for an anomaly-free theory:

$$dH - \alpha \mathrm{tr} R_\nabla \wedge R_\nabla + \alpha \mathrm{tr}\, F_A \wedge F_A = 0, \tag{3.1.4}$$

which is known in the literature as the Bianchi identity. The stationary points of S given the ansatz (3.1.3) satisfy the *heterotic equations of motion*:

$$\begin{aligned} \mathrm{Ric}^g + 2\nabla^g(d\phi) - \tfrac{1}{4}H^2 - \alpha \textstyle\sum_i \mathrm{tr}(i_{e_i}R_\nabla \otimes i_{e_i}R_\nabla) + \alpha \sum_i \mathrm{tr}(i_{e_i}F_A \otimes i_{e_i}F_A) &= 0 \\ d^\star(e^{-2\phi}H) &= 0 \\ d^\star_A(e^{-2\phi}F_A) + e^{-2\phi} \star (F_A \wedge \star H) &= 0 \\ \mathrm{scal}^g - 4\Delta_g \phi - 4|d\phi|^2 - \tfrac{1}{12}|H|^2 - \tfrac{\alpha}{2}|R_\nabla|^2 + \tfrac{\alpha}{2}|F_A|^2 &= 0, \end{aligned} \tag{3.1.5}$$

where $\{e_i\}$ is an orthonormal frame for g, and $\Delta_g = dd^\star + d^\star d$ is the Hodge Laplacian.

A special class of solutions satisfy suitable lower order equations, and are related to supersymmetry. In precise terms, this amounts to the existence of spinor field ϵ (Majorana spinor) in an irreducible real spin module for $\mathrm{Cl}(TM, g)$ with respect to the Hull connection $\nabla^- = \nabla^g - \frac{1}{2}g^{-1}H$ that satisfies the *Killing spinor equations*:

$$\begin{aligned} F_A \cdot \epsilon &= 0 \\ \nabla^- \epsilon &= 0 \\ (H + 2d\phi) \cdot \epsilon &= 0. \end{aligned} \tag{3.1.6}$$

Then, the main result is the following:

Theorem 3.1.1 ([85, 127])**.** *Let $(g, \phi, H, \nabla, A, \epsilon)$ be the geometric structure defined above. Then,* (3.1.4) *and* (3.1.6) *hold if and only if there is a (possibily non-Kähler) Calabi-Yau structure on M, (J, Ω) such that:*

$$\begin{aligned} F_A \wedge \omega^2 = 0, \quad F_A^{0,2} &= 0 \\ d^c \log ||\Omega||_\omega - d^\star \omega &= 0 \\ dd^c\omega - \alpha \mathrm{tr}\, R_\nabla \wedge R_\nabla + \alpha \mathrm{tr}\, F_A \wedge F_A &= 0, \end{aligned} \tag{3.1.7}$$

where $\omega = g(J\cdot, \cdot)$, and:

$$H = -d^c\omega, \quad d\phi = -\tfrac{1}{2} d\log ||\Omega||_\omega. \tag{3.1.8}$$

The geometry determined by the system becomes more transparent once the second equation is rewritten as a *conformally balanced* condition [98], see also [74]:

$$d^c \log ||\Omega||_\omega - d^\star \omega = 0 \quad \Leftrightarrow \quad d(||\Omega||_\omega \omega^2) = 0. \tag{3.1.9}$$

Moreover, it is useful to fix a holomorphic structure $P = (P_K^c, \overline{\partial}_P)$. Then, compatible connections A satisfying the first line of (3.1.7) are equivalent to hermitian metrics h on P satisfying:

$$F_h \wedge \omega^2 = 0, \tag{3.1.10}$$

by the Chern correspondence. Hence, we obtain the *Hull-Strominger system* as it is usually written in the mathematical literature:

$$\begin{aligned} F_h \wedge \omega^2 &= 0 \\ d(||\Omega||_\omega \omega^2) &= 0 \\ dd^c\omega - \alpha \mathrm{tr}\, R_\nabla \wedge R_\nabla + \alpha \mathrm{tr}\, F_h \wedge F_h &= 0 \end{aligned} \tag{3.1.11}$$

as a system in the unknowns (ω, h) on a Calabi-Yau manifold endowed with a holomorphic principal bundle (X, Ω, P). Moreover, in the mathematical literature the constant α is often regarded as a real parameter, and it is natural to study (3.1.11) also for non-positive values. The indefinition of the connection ∇ is an original feature of the system and has been the subject of much debate in the physical as well as in the mathematical community (see [37]). Since the Hull-Strominger system appeared in the mathematical literature with the remarkable work of [98], and further [58], to a considerable extent ∇ has been identified with the Chern connection of ω, yielding a strongly coupled system. However, other choices for ∇ have been considered, singularly the instanton ansatz. From the point of view of physics, this choice is supported by the following result.

Theorem 3.1.2 ([88])**.** *Let (ω, A) satisfy the Hull-Strominger system 3.1.7. Then, the heterotic equations of motion 3.1.5 hold if and only if ∇ satisfies:*

$$R_\nabla \wedge \omega^2 = 0, \quad R_\nabla^{0,2} = 0. \tag{3.1.12}$$

Mathematically, this choice is also at the core of a good amount of recent theory of the Hull-Strominger system [5, 13, 70]. Moreover, note that with this choice, ∇ also satisfies its own equation of motion [47, 63]:

$$d_\nabla^\star(e^{-2\phi} R_\nabla) + e^{-2\phi} \star (R_\nabla \wedge \star H) = 0. \tag{3.1.13}$$

In this Thesis, we will prove a slightly stronger result using purely hermitian geometry in Proposition 6.4.1. It is because of this evidence that throughout this Thesis, in the sequel we advocate for the ansatz (3.1.12). Further, for virtually all of the methods and techniques that we develop, it is worth to embrace an abstract formulation of the system. Hence by Hull-Strominger system, we will mean the following:

Definition 3.1.3. *Let (X, Ω) be a (possibly non-Kähler) compact Calabi-Yau manifold of complex dimension n, and let $P \to X$ be a holomorphic principal bundle for a complex reductive group. Then, a pair (ω, h) of a hermitian metric and a reduction of P to a maximal compact subgroup satisfies the Hull-Strominger system if:*

$$\begin{aligned} F_h \wedge \omega^{n-1} &= 0 \\ d(||\Omega||_\omega \omega^{n-1}) &= 0 \\ dd^c\omega + \langle F_h \wedge F_h \rangle &= 0. \end{aligned} \tag{3.1.14}$$

To recover (3.1.11) from the system in Definition (3.1.14) with the instanton ansatz for ∇, one considers Fr $TX \times_X P$ as the principal bundle and restricts to split solutions.

3.1.2 Application to the Calabi-Yau web

This Section provides a motivation for studying the Hull-Strominger system from the point of view of algebraic geometry, for the problem of classification of complex-algebraic Calabi-Yau threefolds.

This proposal stems from the contrast with the situation in complex dimension 2. For $K3$-surfaces, back to the work of Kodaira [96], as an application of the deformation theory of complex manifolds, a moduli space of analytic $K3$-surfaces is constructed, with a countable set of codimension 1 subvarieties of algebraic surfaces sitting inside. In particular, the moduli of smooth complex $K3$-surfaces is connected and every two of them are shown to be diffeomorphic.

The situation for Calabi-Yau 3-folds is dramatically different. The following construction yields examples of Calabi-Yau 3-folds with different topologies [23, 78]: we consider complete intersection Calabi-Yau manifolds (CICYs) defined as:

$$X = \{p_1 = \cdots = p_k = 0\} \subset \mathbb{C}P^{r_1} \times \cdots \times \mathbb{C}P^{r_l}, \tag{3.1.15}$$

where p_i are polynomial expressions that are homogeneous of a certain degree d_{ij} in each of the $\mathbb{C}P^{r_j}$ factors. Then, the elementary combinatorial considerations:

$$\begin{aligned} \textstyle\sum_i r_i &= k+3 \\ \textstyle\sum_i d_{ij} &= r_j + 1 \end{aligned} \tag{3.1.16}$$

place constraints on the degrees of p_i and dimensions r_i such that X is a Calabi-Yau manifold. Moreover, the non-trivial Hodge numbers $h^{1,1}(X), h^{2,1}(X)$ are obtained- in many cases, from this data too, resulting in a vast 10.000 topologically distinct CICYs, as proved by Friedman [53] building on results of Smale. To emulate the picture of surfaces, in [115] it was proposed to use complex transitions to relate threefolds with different topology, conjecturing that there could be a unique moduli of smooth Calabi-Yau threefolds in the birational sense. This expectation is now known as *Reid's fantasy.* In [53], conifold transitions are used to produce changes in the Hodge numbers of X. This is a two-step process of contracting a set of suitable disjoint rational curves $\sqcup_{i=1}^k C_i \subset X$ producing a singular space $\underline{X}$ with double point singularities. The typical element in the smooth resolution X_t has different Hodge numbers than X, depending on the analytical properties of X and of the transition (see [117, Theorem 3.2] and references therein). It can be the case that the resulting manifold X_t is a non-Kähler manifold *e.g.* because of this cohomological result, or for other reasons. Hence non-Kähler geometry enters naturally in the picture. An extreme case that appears is that of threefolds diffeomorphic to $\sharp_k S^3 \times S^3$ ([21]). Due to the lack of canonical geometry in non-Kähler manifolds, in [138], it was proposed that the Hull-Strominger system should serve as a tool to geometrize conifold transitions. The following results are successful instances of this program:

Theorem 3.1.4 ([55]). *Let X be a Kähler Calabi-Yau threefold and let $X \to \underline{X} \rightsquigarrow X_t$ be a conifold transition, with deformation parameter $t \in \Delta \subset \mathbb{C}$. Then, for sufficiently small t, X_t admits balanced metrics.*

Theorem 3.1.5 ([28]). *The tangent bundle $T^{1,0}X_t$ is slope-polystable with respect to the balanced class given by the metric in Theorem 3.1.4.*

The same result of Theorem 3.1.5 holds for the manifolds $\sharp_k S^3 \times S^3$ above [21]. A complete answer to the problem of classification of Calabi-Yau threefolds following this proposal should, in particular, provide sufficient conditions to pass a solution to the Hull-Strominger system across conifold transitions or obstructions that prevent it. In this Thesis, we study the existence problem for the Hull-Strominger system to obtain new insights on the viability of this program.

3.1.3 Solutions to the Hull-Strominger system

In a prior analysis of heterotic compactifications [24], it was proposed that the internal space M should be a complex manifold carrying a Kähler-Ricci flat metric. These geometries without torsion trivially satisfy the Hull-Strominger system and are known as *standard embeddings*. Later, when the Hull-Strominger system was first addressed in the mathematical literature in [98], the authors deformed the holomorphic structure of $TX \oplus \mathcal{O}_X^{\oplus r}$ providing solutions around the large volume limit ($\alpha = 0$). Moreover, large families of deformation solutions around the large volume limit were further obtained in [8, 9]. Recently, a general existence result for the Hull-Strominger system in Kähler backgrounds has been obtained, compatible with both the Chern and instanton choices for ∇ [28].

The first solutions in non-Kähler manifolds were obtained in [58] on elliptic fibrations over $K3$ surfaces, and have been. In [64], new solutions are constructed on the same manifolds with the instanton ansatz (3.1.12) and are compatible with T-duality. Recently, more solutions have been constructed on more general fibrations admitting orbifold bases [50]. Other non-Kähler backgrounds that admit solutions are twistor spaces for hyperKähler manifolds. In these geometries, called generalized Calabi-Gray, solutions are constructed in [42, 43, 44]. Moreover, homogeneous complex manifolds as described in Section 4.1 have invariant solutions with different choices for the connection ∇ [25, 45, 46, 47, 77, 105, 132, 133]. Apart from these known explicit solutions, there is also interest in reaching exact solutions by means of geometric flows, and study long-time existence, covergence and stability properties of the flows themselves. To this end, a family of *anomaly flows* is introduced and studied in [107, 109, 110], showing a diversity of behaviours.

3.2 Existence conjecture of Yau for the Hull-Strominger system

This Section introduces the problem of existence of solutions to the Hull-Strominger system that we will address in the following Chapters, following a suitable reformulation of a conjecture by Yau [137]. Before we state it, we comment on some aspects of the geometry determined by the system. Throughout, let (X, Ω) be a compact Calabi-Yau manifold. We do not assume that X supports a Kähler structure. Moreover, let V be a holomorphic vector bundle, and $V_0 = (T^{1,0}, \overline{\partial}_{V_0})$ a holomorphic structure on the smooth bundle $T^{1,0}X$. In the following discussion, principal bundles are substituted by vector bundles for clarity and to link to classical theory more directly.

3.2.1 Preliminary remarks and the conjecture by Yau

We discuss some cohomological and algebraic conditions that are implied directly by a solution (3.1.11) on (X, Ω, V). Firstly, the existence of a holomorphic volume form Ω and the conformally balanced equation:

$$d(||\Omega||_\omega \omega^2) = 0 \tag{3.2.1}$$

are non-trivial conditions for the complex geometry and topology of X (see Section 1.1.1). In particular, the form $||\Omega||_\omega \omega^2 \in \Omega^{2,2}_X$ determines a balanced class $\mathfrak{b} \in H^{2,2}_{BC}(X, \mathbb{R})$. Further, the Hermite-Einstein equation:

$$F_h \wedge \omega^2 = 0 \tag{3.2.2}$$

implies the vector bundle V satisfies:

$$\deg_{\mathfrak{b}} V = c_1(V) \cdot \mathfrak{b} = 0, \tag{3.2.3}$$

where $\mathfrak{b}$ is identified with its de Rham class through (1.1.18), and importantly V is slope-polystable in the sense of Mumford-Takemoto with respect to the balanced class $\mathfrak{b}$. This is a consequence of the Donaldson-Uhlenbeck-Yau theorem [40, 134] and its extension to hermitian manifolds [97] (see also [64, Section 4], [93, Chapter 5]). Finally, the Bianchi identity implies the cohomological condition (see Proposition 1.3.2):

$$[ch_2(TX, \nabla)] = ch_2(V) \in H^{2,2}_{BC}(X, \mathbb{R}) \tag{3.2.4}$$

In particular, if ∇ is taken to be the Chern connection of ω on $T^{1,0}$, then the previous condition reads:

$$ch_2(X) = ch_2(V) \in H^{2,2}_{BC}(X, \mathbb{R}). \tag{3.2.5}$$

Then, the existence conjecture by Yau states that the above necessary conditions are actually sufficient:

Conjecture 3.2.1 ([139])**.** *Let (X, Ω) be a compact Calabi-Yau manifold. Moreover, let $\mathfrak{b}_0$ be a balanced class and $V \to X$ a holomorphic vector bundle that is $\mathfrak{b}_0$-stable and satisfying* (3.2.3), (3.2.5)*. Then, there exists a solution of* (3.1.11)*, where ∇ is the Chern connection of ω.*

Note that the above Conjecture does not demand that the balanced class $\mathfrak{b} = [||\Omega||_\omega \omega^2]$ bears any relation with the given balanced class $\mathfrak{b}_0$ in the statement.

3.2.2 State of the problem

The Conjecture 3.2.1 was addressed in [8, 9] using a method inspired in the seminal article [98] to produce non-Kähler solutions deforming Kähler Calabi-Yau metrics. The result provides solutions to the Hull-Strominger system (3.1.11) with the ansatz (3.1.12) for arbitrary polystable vector bundles stable with respect to the initial Kähler class , by simultaneously deforming the hermitian metric and the Hermite-Einstein bundle metric by means of analytic techniques, but does not fix the balanced class. Recently, in [28] the authors solve the Hull-Strominger system with a control of the balanced class, in the large volume limit, which is in turn equivalent to deforming standard embedding solutions. Their existence result is moreover compatible with the choice of (3.1.12) assuming the holomorphic tangent bundle is stable (see [28, Section 3.2]), thus providing evidence for Question 4.3.1.

Despite the above positive results supporting Conjecture 3.2.1 and Question 4.3.1, recent advances in the theory of the Hull-Strominger system also suggest there may be non-trivial obstructions beyond the cohomological and algebraic necessary conditions stated in Section 3.2.1. In [13, 70] appearing in the physical and mathematical literature respectively, the Hull-Strominger system is recasted as the set of equations for a moment map, suggesting the existence of invariants reminiscent of GIT that prevent solutions. However, these may not be straightforward to interpret and compute. Hence, in this Thesis, we take a distinct approach to produce new moment map invariants for the Hull-Strominger system exploiting special features of the system amenable to the use of techniques in Generalized Geometry. This will be addressed in Chapter 5.

Part II

Results

Chapter 4

The Hull-Strominger system on locally homogeneous manifolds

This Chapter begins the investigation of the Hull-Strominger system in this Thesis. By first studying the geometry and gauge theory of complex locally homogenenous manifolds, we develop a systematic approach to obtain solutions to the Hull-Strominger system using an invariant ansatz. Motivated by these results, in the next Section, we suggest to tackle a refinement of the existence Conjecture by Yau (see Section 3.2), leading to a proposal that continues in subsequent Chapters. Then, we move on to study metric aspects of the moduli space of solutions to the system constructed in [70], restricted to invariant solutions, and illustrated through Examples.

4.1 Locally homogeneous manifolds

This Section describes families of locally homogeneous spaces of Lie groups admitting invariant hermitian structure. Hermitian manifolds of this type are particularly well suited for exterior algebra computations, and under certain assumptions, the determination of their complex cohomology groups is also completely explicit. Thus, they have been regarded as a pool of manageable geometries to look for special metrics [51, 131, 132], or for solutions to geometric PDE systems [105]. In the present Chapter, locally homogeneous manifolds are used to construct families of solutions to the Hull-Strominger system.

Throughout, let G be a real Lie group with Lie algebra $\mathfrak{g}$, and let $\Gamma \subset G$ a discrete subgroup with compact quotient $M = \Gamma\backslash G$. A left-invariant complex structure J on G is induced from a linear complex structure on $\mathfrak{g}$ satisfying the integrability condition $N_J = 0$, where:

$$N_J(\xi, \eta) = [\xi, \eta] + J[J\xi, \eta] + J[\xi, J\eta] - [J\xi, J\eta] = 0, \quad \xi, \eta \in \mathfrak{g}. \tag{4.1.1}$$

In particular, left-invariance implies it is Γ-invariant and descends to M. By abuse of language, we still call such a complex structure invariant. This motivates the following definition:

Definition 4.1.1. *A complex locally homogenenous manifold is a quotient $M = \Gamma\backslash G$ endowed with a left-invariant, integrable complex structure J.*

Remark 4.1.2. *While in the literature one can find more general notions of locally homogeneous manifolds, for the objectives of this Thesis it will be enough to consider Definition 4.1.1.*

Similarly, linear tensor fields on $\mathfrak{g}$ induce left-invariant fields on M. In particular, we have natural embeddings:

$$\mathfrak{g} \hookrightarrow \Gamma(TM), \quad \Lambda^k \mathfrak{g}^* \hookrightarrow \Omega^k_M, \quad \Lambda^{p,q}\mathfrak{g}^* \hookrightarrow \Omega^{p,q}_X \tag{4.1.2}$$

These fields are similarly, by abuse of language, still called left-invariant. If $\mathfrak{g}^*_{1,0} = \langle \omega_i \rangle$, then:

$$d\omega_i = \sum_{j,k} \alpha_{ijk}\omega_{j\bar{k}} + \sum_{j<k} \beta_{ijk}\omega_{jk}, \quad \alpha_{ijk}, \beta_{ijk} \in \mathbb{C}, \; i = 1, ..., \dim \mathfrak{g}^{1,0}. \tag{4.1.3}$$

These complex *structure equations* determine the Lie algebra structure and the complex structure J. Observe that (G, J) is not in general a complex Lie group. This is the case precisely when the structure constants $\alpha_{ijk} = 0$ in (4.1.3) above, as this is equivalent to $\mathfrak{g}^{1,0}$ being involutive.

4.1.1 Cohomology of locally homogeneous manifolds

Let $X = (M, J)$ be a complex locally homogenenous manifold. The set of invariant differential forms inherits a chain complex structure induced by Chevalley-Eilenberg differential:

$$0 \to \mathfrak{g}^* \xrightarrow{d} \Lambda^2\mathfrak{g}^* \xrightarrow{d} \Lambda^3\mathfrak{g}^* \xrightarrow{d} ... \tag{4.1.4}$$

determined by:

$$d\gamma(\xi, \eta) = -\gamma([\xi, \eta]), \quad \xi, \eta \in \mathfrak{g}, \; \gamma \in \mathfrak{g}^*. \tag{4.1.5}$$

and extending to higher exterior products by:

$$d(\gamma_1 \wedge \gamma_2) = d\gamma_1 \wedge \gamma_2 + (-1)^{|\gamma_1|}\gamma_1 \wedge d\gamma_2. \tag{4.1.6}$$

Observe that (4.1.5) makes (4.1.2) an embedding of chain complexes, hence inducing maps in cohomology:

$$H^\bullet(\mathfrak{g}) \longrightarrow H^\bullet_{dR}(M). \tag{4.1.7}$$

This map need not be injective nor surjective in general. However, for some families of Lie groups this is the case. To cite the result of interest here, we recall the following definition:

Definition 4.1.3 ([92])**.** *1. A Lie algebra $\mathfrak{g}$ is completely solvable (or split-solvable) if there is an ascending chain of ideals:*

$$0 \subset \mathfrak{a}_1 \subset ... \subset \mathfrak{a}_{n-1} \subset \mathfrak{g}, \tag{4.1.8}$$

where dim. $\mathfrak{a}_i = i$.

2. A Lie group G is completely solvable if Lie $\mathfrak{g}$ *is.*

Then, building on the work of [104], we have:

Theorem 4.1.4 ([82]). *Let G be a completely solvable Lie group, then* (4.1.7) *is a ring isomorphism.*

In particular, this result applies for nilpotent Lie groups. There is interest in extending the situation of Theorem 4.1.4 to the complex Dolbeault, Bott-Chern and Aeppli cohomologies (1.1.16), (1.1.17). Results in this direction can be found in [10, 30, 29, 116, 118] . In this Thesis, the following results will be sufficient for our purposes:

Theorem 4.1.5 ([10]). *Let G be a nilpotent Lie group and suppose J is a left-invariant complex structure satisfying one of the following conditions:*

1. *$(\mathfrak{g}, J)$ is a complex Lie algebra*
2. *$\mathfrak{g}^{1,0}$ is abelian.*
3. *$\mathfrak{g} = \mathfrak{g}_{\mathbb{Q}} \otimes \mathbb{R}$ and $J(\mathfrak{g}_{\mathbb{Q}}) \subset \mathfrak{g}_{\mathbb{Q}}$ for some rational Lie algebra $\mathfrak{g}_{\mathbb{Q}}$.*

Then, the Dolbeault cohomology map

$$H^{\bullet,\bullet}_{\overline{\partial}}(\mathfrak{g}, J) \to H^{\bullet,\bullet}_{\overline{\partial}}(X) \tag{4.1.9}$$

is a ring isomorphism.

Theorem 4.1.6 ([10]). *Let (M, J) be a complex locally homogenenous manifold and suppose that:*

$$H^{\bullet}(\mathfrak{g}) \xrightarrow{\cong} H^{\bullet}_{dR}(M), \quad H^{\bullet,\bullet}_{\overline{\partial}}(\mathfrak{g}, J) \xrightarrow{\cong} H^{\bullet,\bullet}_{\overline{\partial}}(X).$$

Then, also:

$$H^{\bullet,\bullet}_{BC}(\mathfrak{g}, J) \xrightarrow{\cong} H^{\bullet,\bullet}_{BC}(X).$$

Remark 4.1.7. *Under the hypothesis of Theorem 4.1.6, by the Bott-Chern and Aeppli cohomology duality* (1.1.19)*, then also the natural map from Chevalley-Eilenberg Aeppli cohomology maps isomorphically to Aeppli cohomology of (M, J).*

4.1.2 Vector bundles over locally homogeneous manifolds

In this Section we describe a natural class of holomorphic vector bundles that can be considered on locally homogeneous manifolds. First, we recall that given a compact complex manifold, isomorphism classes of holomorphic line bundles are classified by $H^1(X, \mathcal{O}^\times)$. This group fits in the exact sequence:

$$\cdots \longrightarrow H^1(X,\mathbb{Z}) \longrightarrow H^1(X,\mathcal{O}) \longrightarrow H^1(X,\mathcal{O}^\times) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \longrightarrow \dots \tag{4.1.10}$$

The topological type of a holomorphic line bundle $\mathcal{L}$ is determined by $c_1(\mathcal{L})$ while the space of holomorphic deformations is given by

$$\frac{H^1(X,\mathcal{O})}{H^1(X,\mathbb{Z})} \cong \frac{H^{0,1}_{\overline{\partial}}(X)}{H^1(X,\mathbb{Z})}, \tag{4.1.11}$$

where on the right hand side we quotient by the set:

$$\{[\alpha^{0,1}] \in H^{0,1}_{\overline{\partial}}(X) \mid \alpha \in H^1(X,\mathbb{Z})\}. \tag{4.1.12}$$

Here, we identify $\alpha \in H^1(X,\mathbb{Z})$ with its de Rham class in $H^1(X,\mathbb{Z}) \otimes \mathbb{R} \cong H^1_{dR}(X,\mathbb{R})$, up to torsion.

Observe that in the hypothesis of Theorems 4.1.5, 4.1.6, this quotient can be explicitly computed in terms of Lie algebra cohomologies.

In higher rank, the precise characterization of a holomorphic vector bundle is not available. However, as the first novel result of this Thesis, we now prove a correspondence for a specific family of vector bundles over locally homogeneous manifolds using representation theory and Lie group theory. Our results shall be compared with the analysis of holomorphic vector bundles over locally homogeneous manifolds in [4, 93].

Definition 4.1.8. *Let $X = (\Gamma\backslash G, J)$ be a complex locally homogenenous manifold and let $\mathcal{E} \to X$ be a holomorphic vector bundle. We say $\mathcal{E}$ is of homogeneous type if there exists a global frame $\{s_j\} \subset \Gamma(\mathcal{E})$ such that:*

$$\overline{\partial}_{\mathcal{E}} s_j = \textstyle\sum_k A_{kj} s_k, \tag{4.1.13}$$

where $\{A_{kj}\}$ are induced left-invariant $(0,1)$-forms on G.

The existence of a distinguished global frame in the previous definition implies in particular that the topological type of a holomorphic bundle $\mathcal{E}$ of homogeneous type is trivial, that is, it is diffeomorphic to $X \times \mathbb{C}^r$, for $r = \operatorname{rk} \mathcal{E}$. In particular $c_k(\mathcal{E}) = 0$ for $k \leq r$. The next Lemma shows we can associate a vector bundle of homogeneous type $\mathcal{E}_\rho$ of rank r and distinguished frame $\{s_i\}$ as in Definition 4.1.8 to any representation of $\mathfrak{g}^{0,1}$.

Lemma 4.1.9. *Let $\mathcal{E}_\rho = (X \times \mathbb{C}^r, \overline{\partial}_{\mathcal{E}_\rho})$ be the vector bundle with Dolbeault operator associated to the representation:*

$$\rho : \mathfrak{g}^{0,1} \longrightarrow \mathfrak{gl}(r,\mathbb{C}) \tag{4.1.14}$$

by declaring that:

$$i_{X^{0,1}} \overline{\partial}_{\mathcal{E}_\rho} e_i = \rho(X^{0,1})(e_i), \tag{4.1.15}$$

for the canonical basis $\{e_i\} \subset \mathbb{C}^r$. Then, $\mathcal{E}_\rho$ is holomorphic. Conversely, given a holomorphic vector bundle $\mathcal{E}$ of homogeneous type, we obtain an associated representation ρ.

Proof. Given a basis $\{X_i\}$ of $\mathfrak{g}^{1,0}$ and dual basis $\{\omega_i\}$:

$$\begin{aligned} \rho([\overline{X}_i, \overline{X}_j]) &= \rho\left(\textstyle\sum_k \omega_{\overline{k}}([\overline{X}_i, \overline{X}_j])\overline{X}_k\right) \\ &= -\textstyle\sum_k \overline{\partial}\omega_{\overline{k}}(\overline{X}_i, \overline{X}_j)\rho(\overline{X}_k) \\ &= -\textstyle\sum_k \overline{\partial}\omega_{\overline{k}}(\overline{X}_i, \overline{X}_j)A^k \\ [\rho(\overline{X}_i), \rho(\overline{X}_j)] &= [A^i, A^j]. \end{aligned}$$

Therefore:

$$
\begin{aligned}
& \rho\left([\overline{X}_i, \overline{X}_j]\right) = [\rho(\overline{X}_i), \rho(\overline{X}_j)], \ \ \forall\, i, j \\
\Leftrightarrow \ \ & [A^i, A^j] = -\textstyle\sum_k \overline{\partial}\omega_{\overline{k}}(\overline{X}_i, \overline{X}_j) A^k, \ \ \forall\, i, j \\
\Leftrightarrow \ \ & \overline{\partial}^2_{\mathcal{E}_\rho} = 0.
\end{aligned}
$$

The converse is obvious defining ρ by (4.1.15). □

The rest of this Section studies the relation between vector bundles of homogeneous type and their representations. Firstly, we obtain the following result, whose proof is straightforward from the construction above and Definition 4.1.8.

Proposition 4.1.10. *Let X be a homogeneous complex manifold. Then the map:*

$$\rho \mapsto \mathcal{E}_\rho \tag{4.1.16}$$

induces a surjective map between the set of $\mathfrak{gl}(r, \mathbb{C})$-representations of $\mathfrak{g}^{0,1}$ up to conjugation and isomorphism classes of holomorphic vectors bundles of homogeneous type of rank r.

Example 4.1.11. *The map $\rho \mapsto \mathcal{E}_\rho$ above is in general not injective, as the following Example shows. Let $X = \mathbb{C}/\mathbb{Z}\langle 1, i\rangle$, and consider 1-dim. representations:*

$$\rho : \mathbb{C} \longrightarrow \mathfrak{gl}(1, \mathbb{C}) \cong \mathbb{C}. \tag{4.1.17}$$

We denote $A = \rho(1)$. Then,

$$\Phi : \mathcal{O}_X \xrightarrow{\cong} \mathcal{E}_\rho \tag{4.1.18}$$

if and only if

$$\overline{\partial}\Phi + A\Phi = 0, \tag{4.1.19}$$

where Φ has been identified with a complex function in the smooth global trivializations of the bundles given by the constant section 1. Expanding in Fourier modes in X:

$$\Phi = \sum_{(n,m)\in\mathbb{Z}^2} c_{nm} e^{\pi i n(z+\bar{z})} e^{\pi m(z-\bar{z})}, \tag{4.1.20}$$

we rewrite the PDE (4.1.19) above as a $\mathbb{Z}^2$-indexed set of algebraic equations:

$$c_{nm}(\pi i n - \pi m + A) = 0, \quad (n, m) \in \mathbb{Z}^2. \tag{4.1.21}$$

Then, a non-trivial solution to (4.1.19) exists if and only if $A \in \pi\mathbb{Z}\langle 1, i\rangle$. Hence, the fibre of the map 4.1.16 at $\mathcal{O}_X$ is canonically identified with $H^1(X, \mathbb{Z})$ (compare with (4.1.11)).

Now, we extract some consequences of Proposition 4.1.10. These aim at understanding the *homogeneous locus* of the moduli of poystable bundles over homogeneous complex manifolds. We will need first a technical Lemma.

Lemma 4.1.12. *Let G be a Lie group and assume $\mathfrak{g}$ is unimodular. Moreover, assume G admits a discrete, cocompact subgroup $\Gamma \subset G$, and let Δ be a fundamental domain for Γ. Let:*

$$f : G \longrightarrow \mathbb{C}^n$$

be a Γ-invariant smooth function. Then,

$$f^0 : G \longrightarrow \mathbb{C}^n, \quad f^0(x) = \frac{1}{\mathrm{vol}(\Delta)} \int_\Delta (\ell_g^* f)(x) d\mathrm{vol}_G,$$

for $d\mathrm{vol}_G$ a bi-invariant measure, is constant.

Proof. First, observe that indeed there exists a bi-invariant measure $d\mathrm{vol}_G$: extend a linear volume form on $\mathfrak{g}$ to G by left-invariance. Then, unimodularity implies that it is moreover adjoint-invariant, and this is equivalent to bi-invariance. Now, we check that $f^0(h \cdot x) = f^0(x)$ for any $h, x \in G$. For this, first observe that since Γ is a lattice with compact quotient, a fundamental domain Δ for Γ has finite volume with respect to $d\mathrm{vol}_G$. Then, we compute:

$$\begin{aligned}
\mathrm{vol}(\Delta) f^0(h \cdot x) &= \int_\Delta (\ell_g^* f)(h \cdot x) d\mathrm{vol}_G|_\Delta \\
&= \int_{\Delta \cdot h} (\ell_{g'}^* f)(x) (r_{h^{-1}}^* d\mathrm{vol}_G)|_{\Delta \cdot h} \\
&= \int_{\Delta \cdot h} (\ell_{g'}^* f)(x) d\mathrm{vol}_G|_{\Delta \cdot h},
\end{aligned}$$

using the change of variable theorem applied to $r_h : \Delta \to \Delta \cdot h$ and integration variables $g \in \Delta$, $g' \in \Delta \cdot h$. Moreover, because of the finite volume of Δ and the fact that f is continuous, the above functions are indeed integrable. To complete the above computation, we introduce the measurable subsets:

$$U_\gamma = (\gamma^{-1} \cdot \Delta \cdot h) \cap \Delta, \quad \gamma \in \Gamma.$$

which satisfy:

1. $\Delta \cdot h = \bigsqcup_{\gamma \in \Gamma} \gamma \cdot U_\gamma$,
2. $\bigsqcup U_\gamma = \Delta$.

To see 1, first observe that $\{\gamma \cdot \Delta\}_{\gamma \in \Gamma}$ cover G by definition of Δ, hence $\{\gamma \cdot U_\gamma\}_{\gamma \in \Gamma}$ cover $\Delta \cdot h$. Moreover, they do not overlap as neither do $\{\gamma \cdot \Delta\}_{\gamma \in \Gamma}$. For 2, by definition of Δ, for any $x \in G$, there are unique $\gamma \in \Gamma$ and $\delta \in \Delta$ such that $x = \gamma(x) \cdot \delta(x)$. Applying this to xh^{-1}:

$$xh^{-1} = \gamma(xh^{-1}) \cdot \delta(xh^{-1}) \Rightarrow x = \gamma \delta h \in U_{\gamma^{-1}}.$$

Therefore, property 2 above follows considering $x \in \Delta$. Hence, finally:

$$\begin{aligned}
f^0(h \cdot x) &= \int_{\Delta \cdot h} (\ell_g^* f)(x) d\mathrm{vol}_G|_{\Delta \cdot h} \\
&= \textstyle\sum_{\gamma \in \Gamma} \int_{\gamma \cdot U_\gamma} (\ell_g^* f)(x) d\mathrm{vol}_G|_{\gamma \cdot U_\gamma} \\
&= \textstyle\sum_{\gamma \in \Gamma} \int_{U_\gamma} (\ell_{\gamma g}^* f)(x) (\ell_\gamma^* d\mathrm{vol}_G)|_{U_\gamma} \\
&= \int_\Delta (\ell_g^* f)(x) d\mathrm{vol}_G = f^0(x),
\end{aligned}$$

where in the last line we have used crucially that f is Γ-invariant. □

Now, we prove the main result of this Section. Here, following standard representation theory, a representation $\rho : \mathfrak{g}^{0,1} \to \mathfrak{gl}(r, \mathbb{C})$ is simple if it has no non-trivial subrepresentations, and it is semisimple if for any subrepresentation $V \subset \mathbb{C}^r$ there exists a complementary subrepresentation $W \subset \mathbb{C}^r$, that is, such that $V \oplus W = \mathbb{C}^r$.

Proposition 4.1.13. *Let $X = (\Gamma\backslash G, J)$, and assume $\mathfrak{g}$ is unimodular. Moreover, let $\mathcal{E} \to X$ be of homogeneous type with associated representation ρ, as given by* (4.1.15)*, and let σ be a Gauduchon class on X. If $\mathcal{E}$ is σ-slope polystable (resp. stable), then ρ is semisimple (resp. simple).*

Proof. We prove the polystable case. The stable case is then a formal consequence. First, observe that $c_1(\mathcal{E}) = 0$, since it is of homogeneous type. Now, assume $\mathcal{E}$ is σ-polystable. Then, given a coherent subsheaf $\mathcal{F} \subset \mathcal{E}$, either $\mu(\mathcal{F}) < 0$ or $\mu(\mathcal{F}) = 0$ and $\mathcal{E} \cong \mathcal{F} \oplus \mathcal{E}/\mathcal{F}$.

If $V \subset \mathbb{C}^r$ is a subrepresentation of ρ, then there is an associated holomorphic vector bundle $\mathcal{V} \subset \mathcal{E}$, since it is preserved by $\overline{\partial}_{\mathcal{E}}$. It is clear that $\mathcal{V}$ and $\mathcal{E}/\mathcal{V}$ are of homogeneous type, with Dolbeault operator given by restricting ρ to V, and inducing a quotient representation ρ mod V on $\mathbb{C}^r/V$, respectively. Moreover, let W be a linear complement of V and associated (smooth) vector bundle $\mathcal{W}$. Then, by polystability of $\mathcal{E}$ and given that $c_1(\mathcal{V}) = c_1(\mathcal{E}) = 0$, there exists a biholomorphic bundle map:

$$\Phi : \mathcal{E} \longrightarrow \mathcal{V} \oplus \mathcal{E}/\mathcal{V}, \tag{4.1.22}$$

where Φ has a matrix expression in terms of the smooth splitting $\mathcal{E} = \mathcal{V} \oplus \mathcal{W}$ given by:

$$\Phi = \begin{pmatrix} \mathrm{Id} & \Phi_{12} \\ 0 & \pi_{\mathcal{E}/\mathcal{V}} \circ \Phi|_{\mathcal{W}} \end{pmatrix} \tag{4.1.23}$$

To obtain the result, we argue that Φ can be chosen with constant coefficients with respect to a distinguished basis, hence $\Phi^{-1}(\mathcal{E}/\mathcal{V})$ is a holomorphic subbundle of $\mathcal{E}$ of homogeneous type, corresponding to a complementary subrepresentation of V.

We prove this claim by choosing basis $\{v_i\}$ and $\{w_j\}$ of $V, W \subset \mathbb{C}^r$ corresponding to distinguished frames of $\mathcal{V}, \mathcal{W}$. Hence $\{[w_i]\}$ is a frame for $\mathcal{E}/\mathcal{V}$. Then we can write explicitly Dolbeault operators in such frames as:

$$\overline{\partial}_{\mathcal{E}} = \overline{\partial} + \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \tag{4.1.24}$$

$$\overline{\partial}_{\mathcal{V}\oplus\mathcal{E}/\mathcal{V}} = \overline{\partial} + \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \tag{4.1.25}$$

Then, the condition

$$\overline{\partial}_{\mathcal{V}\oplus\mathcal{E}/\mathcal{V}} \circ \Phi = \Phi \circ \overline{\partial}_{\mathcal{E}} \tag{4.1.26}$$

of being biholomorphic translates to the PDE system on X:

$$\overline{\partial}\Phi_{12} - A_{12} + A_{11}\Phi_{12} - \Phi_{12}A_{22} = 0. \tag{4.1.27}$$

We now consider the pullback of this equation to G, and observe that since J is left-invariant, and so are A_{ij}, if Φ_{12} solves (4.1.27), so does the translation $\ell_g^*\Phi_{12}$. Hence, the following average is a solution to (4.1.27) too:

$$\Phi_{12}^0 = \frac{1}{\int_\Delta d\mathrm{vol}_G} \int_\Delta \ell_g^* \Phi_{12} d\mathrm{vol}_G, \tag{4.1.28}$$

where Δ is a fundamental domain for $\Gamma \subset G$, and $d\mathrm{vol}_G$ is a bi-invariant measure on G, whose existence follows from unimodularity of $\mathfrak{g}$. Finally, by Lemma 4.1.12, Φ_{12}^0 has constant coefficients. □

Corollary 4.1.14. *Let $X = (\Gamma\backslash G, J)$ homogeneous with $\mathfrak{g}$ unimodular and assume $\mathfrak{g}^{0,1}$ is solvable. Let $\mathcal{E} \to X$ of homogeneous type (see Definition 4.1.8) and polystable with respect to some Gauduchon class. Then: $\mathcal{E} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r$, for $\mathcal{L}_i$ line bundle of homogeneous type for each i.*

Proof. Since $\mathcal{E}$ is polystable, by Proposition 4.1.13 the representation ρ is semisimple. By Lie's Theorem on solvable representations, we can choose a dim. 1 subrepresentation L_1 corresponding to a line bundle of homogeneous type $\mathcal{L}_1$ such that $\mathcal{E} = \mathcal{E}' \oplus \mathcal{L}_1$, with $\mathcal{E}'$ of homogeneous type. Then, applying inductively this argument, the result follows. □

The following result give sufficient conditions under which the hypothesis of the above Corollary 4.1.14 hold.

Proposition 4.1.15. *1. Let $(\mathfrak{g}, J)$ be a solvable Lie algebra endowed with a left-invariant complex structure. Then $\mathfrak{g}^{0,1}$ is solvable.*

2. If $\mathfrak{g}$ is a nilpotent Lie algebra, then it is unimodular.

In particular, Corollary 4.1.14 holds if Lie G *is nilpotent.*

Proof. Let $\mathfrak{g}^{(0)} = \mathfrak{g}$ and inductively define $\mathfrak{g}^{(k+1)}$ as the vector space generated by $[\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}]$. By induction, we claim that:

$$(\mathfrak{g}^{0,1})^{(k)} \subset \mathfrak{g}^{(k)} \otimes \mathbb{C}. \tag{4.1.29}$$

Indeed, for $(k) = 0$, this is trivial, and assuming this holds for $k \in \mathbb{N}$:

$$[(\mathfrak{g}^{0,1})^{(k)}, (\mathfrak{g}^{0,1})^{(k)}] \subset [(\mathfrak{g}_\mathbb{C})^{(k)}, (\mathfrak{g}_\mathbb{C})^{(k)}] \subset [\mathfrak{g}_{(k)}, \mathfrak{g}^{(k)}] \otimes \mathbb{C} \subset \mathfrak{g}^{(k+1)} \otimes \mathbb{C},$$

therefore $(\mathfrak{g}^{0,1})^{(k+1)} \subset \mathfrak{g}^{(k+1)} \otimes \mathbb{C}$ and the claim follows. By assumption $\mathfrak{g}$ is solvable, so $\mathfrak{g}^{(N)} = \{0\}$ for some $N >> 0$. Therefore $(\mathfrak{g}^{0,1})^{(N)} = \{0\}$, hence the first item follows. For the second item, see *e.g.* the Corollary to [103, Proposition 25]. □

We finish this Section with some results that will be useful in the sequel. The first of these, which holds for any compact complex manifold, allows to consider holomorphic line bundles endowed with hermitian metrics directly from their Chern curvature forms:

Lemma 4.1.16. *Let X be a compact complex manifold and let $F \in \Omega^{1,1}_X$ be a closed, purely imaginary form such that the cohomology class:*

$$[\tfrac{i}{2\pi}F] \in \operatorname{im}\left(H^2(X,\mathbb{Z}) \hookrightarrow H^2(X,\mathbb{R})\right). \tag{4.1.30}$$

Then, there exists a holomorphic line bundle L and a hermitian metric h on L such that the curvature of the Chern connection $F_h = F$.

Proof. By general theory (see *e.g.* [136]), under the integrality hypothesis (4.1.30), to the cohomology class $[\frac{i}{2\pi}F]$ corresponds a smooth line bundle L_0 such that $c_1(L_0) = [\frac{i}{2\pi}F]$. Let ∇_0 be an arbitrary connection on L_0. By considering the space of connections $\nabla + a$, for $a \in \Omega^1(X,\mathbb{C})$, the curvature:

$$F_{\nabla_0 + a} = F_{\nabla_0} + da \tag{4.1.31}$$

can be taken to agree with F. Let $\nabla = \nabla_0 + a$ such that $F_\nabla = F$. Then, in particular $(L_0, \nabla^{0,1})$ is a holomorphic line bundle. We consider the family of holomorphic line bundles $L_\alpha = (L_0, \nabla^{0,1} + \alpha)$, where $\alpha \in \Omega^{0,1}_X$ satisfying $\overline{\partial}\alpha = 0$, and let h be an arbitrary hermitian metric on L_0. We denote $D^{h,\alpha}$ the Chern connection of h on L_α. Observe that:

$$D^{h,0} = \nabla + \beta, \tag{4.1.32}$$

where $\beta \in \Omega^{1,0}_X$, and comparing the curvatures:

$$F_{D^{h,0}} = F_\nabla + d\beta, \tag{4.1.33}$$

we obtain that $d\beta \in i\Omega^{1,1}(X,\mathbb{R})$, in particular $\partial\beta = 0$. It follows that we can choose $\alpha = \frac{1}{2}\overline{\beta}$, and $L = L_\alpha$ with this choice. Then:

$$F_{D^{h,\alpha}} = F_{D^{h,0}} + d(\alpha - \overline{\alpha}) = F_\nabla + d(\alpha - \overline{\alpha}) + d\beta = F, \tag{4.1.34}$$

where in the last step we observe that $d\alpha = \frac{1}{2}d\overline{\beta} = -\frac{1}{2}d\beta$, as $d\beta$ is pure imaginary. $\square$

The next result holds for complex locally homogenenous manifolds. Let $X = (\Gamma\backslash G, J)$ be complex homogeneous. Then, $T^{1,0}X$ admits a global smooth frame given left-invariant sections induced by elements in $\mathfrak{g}^{1,0}$. Therefore, we can consider holomorphic structures on the smooth tangent bundle $T^{1,0}$ of homogeneous type, as in Definition 4.1.8.

Lemma 4.1.17. *Let $X = (\Gamma\backslash G, J)$ be a complex locally homogenenous manifold, and assume* Lie $\mathfrak{g}$ *is solvable and unimodular. Let $\mathcal{E} = (T^{1,0}, \overline{\partial}_{\mathcal{E}})$ be a holomorphic vector bundle of homogeneous type on X with distinguished global frame inducing a bundle diffeomorphism:*

$$s : \mathcal{E} \longrightarrow X \times \mathbb{C}^n \tag{4.1.35}$$

for $n = \dim.\ X$. Let $\mathfrak{b} \in H^{n-1,n-1}_{BC}(X,\mathbb{R})$ be a balanced class admitting an invariant metric $\omega \in \Omega^{1,1}(X,\mathbb{R})$ such that $[\omega^{n-1}] = \mathfrak{b}$. Moreover, assume $\mathcal{E}$ is $\mathfrak{b}$-polystable. Then, there exists a hermitian matrix $h \in M_{n\times n}(\mathbb{C})$ such that:

$$F_{s^*h} \wedge \omega^{n-1} = 0, \tag{4.1.36}$$

where s^*h is the hermitian metric on $\mathcal{E}$ obtained pulling back by (4.1.35) the constant hermitian metric induced by h on $X \times \mathbb{C}^n$.

Proof. Under the hypothesis, Corollary 4.1.14 implies that $\mathcal{E} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n$, where $\mathcal{L}_i$ are holomorphic line bundles of homogeneous type. Moreover, the change of frame from s to the distinguished split frame $(s_1, \ldots, s_n)$ of $\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n$ is given by a constant gauge transformation $g : s \mapsto (s_i)$. In the new frame:

$$(s_1, ..., s_n) : \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n \longrightarrow X \times \mathbb{C}^n \tag{4.1.37}$$

the standard hermitian hermitian matrix $h_0 = \mathrm{id}_n$ pulls back to a split hermitian metric $(s_i)^* h_0$. As the Dolbeault operator $\overline{\partial}_{\mathcal{E}}$ is also split, so is the Chern connection $D^{(s_i)^* h_0}$ and the Chern curvature is written as:

$$F_{(s_i)^* h_0} = \begin{pmatrix} F_1 & & 0 \\ & \ddots & \\ 0 & & F_n \end{pmatrix} \tag{4.1.38}$$

Setting $h = g^* h_0$, since g is holomorphic, we have that:

$$F_{s^* h} = g^{-1} \circ F_{(s_i)^* h_0} \circ g. \tag{4.1.39}$$

Therefore, it is enough to prove:

$$F_i \wedge \omega^{n-1} = 0. \tag{4.1.40}$$

Given that $\overline{\partial}_{\mathcal{E}}$ is of homogeneous type and h_0 is constant in the distinguished frame (s_i), it follows that $F_i \in \Omega^{1,1}_X$ is induced by an invariant form on G. Since ω is also invariant by hypothesis, there exists a constant λ_i such that:

$$F_i \wedge \omega^{n-1} = \lambda_i \omega^n, \tag{4.1.41}$$

and upon integration over X, we obtain that $\lambda_i = 0$ if and only if $\deg_{\mathfrak{b}} \mathcal{L}_i = 0$. Finally, indeed the degree of $\mathcal{L}_i$ vanishes as a consequence of polystability of $\mathcal{E}$ and $c_1(\mathcal{E}) = c_1(X \times \mathbb{C}^n) = 0$. □

4.2 Invariant solutions to the Hull-Strominger system

Here, we use the results on holomorphic vector bundles of homogeneous type obtained in Section 4.1.2 to look for solutions to the Hull-Strominger system with the instanton condition (3.1.12) on complex locally homogenenous manifolds systematically. Our solutions will be obtained using an invariant ansatz, that is, where hermitian metrics, connection and curvature components with respect to a distinguished frame are induced by invariant forms on Lie groups. In this analysis, we recover solutions already constructed in the literature using the instanton ansatz (3.1.12), but moreover, we construct new solutions by a careful determination of balanced classes and instantons.

4.2.1 Solutions on the Iwasawa manifold

The *Iwasawa manifold* $X = \Gamma\backslash H$ is a complex nilmanifold given by the quotient of the complex Heisenberg Lie group $H = H_3(\mathbb{C})$, given by:

$$H_3(\mathbb{C}) = \left\{ \begin{pmatrix} 1 & z_2 & z_3 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix} \mid z_i \in \mathbb{C} \right\} \tag{4.2.1}$$

by the lattice $\Gamma \subset H$ of matrices with entries in Gaussian integers $\mathbb{Z}[i]$. We briefly describe its geometry: the 6-dimensional real Lie algebra underlying Lie H is $\mathfrak{h}_5$, that is, Lie $H = (\mathfrak{h}_5, J_0)$, according to the classification in [132]. X is a holomorphic torus fibration given by the submersion to the standard complex torus:

$$p : X \to T^4 = \mathbb{Z}[i]^2\backslash\mathbb{C}^2, \quad [(z_1, z_2, z_3)] \mapsto [(z_1, z_2)]. \tag{4.2.2}$$

The 1-forms $\omega_i \in \Omega^{1,0}_H$ given by:

$$\omega_1 = dz_1 \ , \ \omega_2 = dz_2 \ , \ \omega_3 = dz_3 - z_2 dz_1 \tag{4.2.3}$$

are Γ-invariant and descend to X, defining a global frame of $T^*_{1,0}$ satisfying:

$$d\omega_1 = d\omega_2 = 0 \quad , \quad d\omega_3 = \omega_{12}. \tag{4.2.4}$$

The Iwasawa manifold admits an $SU(3)$ structure defined by:

$$\Omega_0 = \omega_{123} \quad , \quad \omega_0 = \tfrac{i}{2}(\omega_{1\bar{1}} + \omega_{2\bar{2}} + \omega_{3\bar{3}}). \tag{4.2.5}$$

A straightforward computation using (4.2.4) shows that ω_0 is a balanced hermitian metric and Ω is a holomorphic volume form.

Now we introduce line bundles over X that will play the role of the gauge bundle in our solutions to the Hull-Strominger system. For this, we first check that the vector space:

$$\langle [\omega_{1\bar{1}}],\ [\omega_{1\bar{2}}],\ [\omega_{2\bar{1}}],\ [\omega_{2\bar{2}}] \rangle_{\mathbb{C}} \tag{4.2.6}$$

is the subspace of $H^2_{dR}(X, \mathbb{C})$ of classes admitting representatives of bidegree $(1,1)$, where we use that the de Rham Lie cohomology computes the de Rham cohomology of X (see Theorem 4.1.4) Then, for any choice of:

$$(m, n, p, q) \in \mathbb{Z}^4\backslash\{0\} \tag{4.2.7}$$

we consider the following purely imaginary $(1,1)$-form on the base T^4:

$$F = \pi(m(\omega_{1\bar{1}} - \omega_{2\bar{2}}) + n(\omega_{1\bar{2}} + \omega_{2\bar{1}}) + ip(\omega_{1\bar{2}} - \omega_{2\bar{1}}) + q(\omega_{1\bar{1}} + \omega_{2\bar{2}})). \tag{4.2.8}$$

Note that $\frac{i}{2\pi}F$ has integral periods and hence, by Lemma 4.1.16, this is the curvature form of the Chern connection of a holomorphic hermitian line bundle $(L_N, h) \to T^4$, where $N = (m, n, p, q)$. In the sequel, we will identify (L_N, h) and $F = F_h$ with their corresponding pull-backs to X via p.

By the Donaldson-Uhlenbeck-Yau theorem (see Theorem 1.3.6), solutions to the Hull-Strominger system (3.1.11) with the instanton condition (3.1.12) involve, in particular, connections ∇ on $T^{1,0}$ such that $(T^{1,0}, \nabla^{0,1})$ is a holomorphic vector bundle that is polystable with respect to the balanced class of the solution. To produce these connections within the invariant ansatz of this Section, we demand that $(T^{1,0}, \nabla^{0,1})$ is also of homogeneous type (see Definition 4.1.8).

For this, we fix a balanced class $\sigma \in H^{2,2}_{BC}(X, \mathbb{R})$ admitting an invariant metric ω such that $\omega^2 \in \sigma$. By Theorem 4.1.5, σ admits an invariant representative. Therefore, given σ, it is enough to check if there is a *positive* $(2,2)$-form among its invariant representatives. For such a class, we obtain the following result:

Lemma 4.2.1. *Assume $\sigma \in H^{2,2}_{BC}(X, \mathbb{R})$ admits an invariant balanced metric ω, and let $\mathcal{E} = (T^{0,1}, \nabla^{0,1})$ be a σ-polystable bundle of homogeneous type. Then, any hermitian metric h on $\mathcal{E}$ satisfying the Hermite-Einstein equation:*

$$F_h \wedge \omega^{n-1} = 0 \tag{4.2.9}$$

is flat.

Proof. Since $(\mathfrak{h}_5, J_0)^{0,1}$ satisfies the hypothesis of Corollary 4.1.14 of solvability and unimodularity, assuming $(T^{1,0}, \nabla^{0,1})$ is σ-polystable implies that it splits as a sum of line bundles $V_0^1 \oplus V_0^2 \oplus V_0^3$, such that V_0^i are of homogeneous type, hence we may write their Dolbeault operator as:

$$\overline{\partial}_{V_0^i} = \overline{\partial} + a_i \tag{4.2.10}$$

with respect to its distinguished frame. Here $a_i \in \Omega^{0,1}_X$ are invariant forms satisfying $\overline{\partial} a_i = 0$. Using (4.2.4), this implies $a_i \in \langle \omega_{\overline{1}}, \omega_{\overline{2}} \rangle_{\mathbb{C}}$. Now, consider the hermitian metric h_0 on $\mathcal{E}$ satisfying:

$$F_{h_0} \wedge \omega^{n-1} = 0 \tag{4.2.11}$$

given by Lemma 4.1.17. Since h_0 is also split with respect to the splitting $V_0^1 \oplus V_0^2 \oplus V_0^3$, we may write $h_0 = (h_0^i)_{i=1,2,3}$. Then:

$$F_{h_0^i} = \partial a_i - \overline{\partial a_i} = 0, \tag{4.2.12}$$

as a simple consequence of the structure equations (4.2.4). Then, $F_{h_0} = 0$. Since the solution to the Hermite-Einstein equation is unique up to holomorphic gauge transformation, the result follows. □

Remark 4.2.2. *As X is a complex-parallelizable manifold, that is, the holomorphic tangent bundle $T^{1,0} \cong \mathcal{O}_X^3$, the Chern connection of any invariant hermitian metric is trivial, in particular, it is flat. Therefore, the previous result can be regarded as a generalization of this observation for different holomorphic structures on $T^{1,0}$.*

We are ready to state the family of solutions to the Hull-Strominger system that we obtain with this construction. For this, we introduce the following notation for the coefficients of an invariant hermitian form:

$$\begin{aligned}\omega = &\textstyle\sum_{j=1}^{3} is_j\omega_{j\bar{j}} + s_4(\omega_{1\bar{2}} - \omega_{2\bar{1}}) + is_5(\omega_{1\bar{2}} + \omega_{2\bar{1}}) \\ &+ s_6(\omega_{1\bar{3}} - \omega_{3\bar{1}}) + is_7(\omega_{1\bar{3}} + \omega_{3\bar{1}}) + s_8(\omega_{2\bar{3}} - \omega_{3\bar{2}}) + is_9(\omega_{2\bar{3}} + \omega_{3\bar{2}}).\end{aligned} \tag{4.2.13}$$

In particular, the constants $s_j \in \mathbb{R}$.

Proposition 4.2.3. *Let V_0 be a bundle of homogeneous type, and let $V_1 = \bigoplus_{i=1}^{k} L_{N_i}$ for tuples $N_i = (m_i, n_i, p_i, q_i) \in \mathbb{Z}^4\backslash\{0\}$. Moreover, let ω be an invariant hermitian metric on X. Then:*

1. *ω is balanced, and therefore defines a balanced class $\sigma(\omega) = [||\Omega_0||_\omega \omega^2] \in H^{2,2}_{BC}(X, \mathbb{R})$.*

Let h_0 be a (flat) hermitian metric on V_0 given by Lemma 4.1.17. Let h_1^i be the hermitian metric on L_{N_i} such that $F_{h_1^i} = F(N_i)$ (see (8.3.6)*), and let $h_1 = (h_1^i)$. Then:*

2. *(ω, h_0, h_1) is a solution to the Hull-Strominger system* (3.1.11) *with the instanton condition* (3.1.12) *if and only if:*

$$\alpha = \frac{-s_3}{\pi^2 \sum_i m_i^2 + n_i^2 + p_i^2 - q_i^2}, \tag{4.2.14}$$

is well-defined and:

$$\deg_{\sigma(\omega)} L_{N_i} = 0 \quad i = 1, ..., k. \tag{4.2.15}$$

Proof. The first item is a consequence of the fact (see [1]) that an invariant hermitian metric on a complex-parallelizable manifold is balanced, combined with Ω_0 being invariant, in particular $||\Omega_0||_\omega$ is constant. The conditions on the second item are equivalent to solving the rest of the equations of (3.1.11): the Hermite-Einstein equation for h_0 is automatic as h_0 is flat. Since h_1^i have invariant Chern curvature $F(N_i)$, the Hermite-Einstein equation for h_1 is actually equivalent to the cohomological condition (4.2.15). Finally, the straightforward computations:

$$dd^c\omega = 2s_3\omega_{12\overline{12}}, \tag{4.2.16}$$

$$\mathrm{tr}\, F_{h_0} \wedge F_{h_0} = 0, \tag{4.2.17}$$

$$F_{h_1^i} \wedge F_{h_1^i} = 2\pi^2(m_i^2 + n_i^2 + p_i^2 - q_i^2)\omega_{12\overline{12}} \tag{4.2.18}$$

imply that the condition (4.2.14) is equivalent to the Bianchi identity in (3.1.11). □

Remark 4.2.4. *Note that, by* (4.2.14)*, the solutions obtained in Proposition 4.2.3 need $\alpha < 0$.*

Remark 4.2.5. *This existence result shall be compared with [25, Section 4.1], where abelian instantons of the form* (8.3.6) *are used to solve the Hull-Strominger system, while the family of balanced classes of solutions in Proposition 4.2.3 and the study of instanton connections for bundles of homogeneous type in Lemma 4.1.17 are new of this Thesis.*

4.2.2 Further solutions on nilmanifolds

The search for solutions to the Hull-Stromiger system in other complex nilmanifolds emulates the case of the Iwasawa manifold.

We now turn to the Lie algebra $\mathfrak{h}_3$. Solutions to the Hull-Strominger system on nilmanifolds with this Lie algebra have been found in [46, 105, 132] with ∇ being an instanton or other ansatze. On $\mathfrak{h}_3$ there is only one complex structure J^- up to Lie isomorphism that supports balanced invariant metrics (see, [132]). Let $H = H_3$ be the associated connected, simply-connected Lie group. The corresponding bundle $T^*_{1,0}H$ has a global frame of invariant 1-forms subject to the structure equations:

$$d\omega_1 = d\omega_2 = 0, \quad d\omega_3 = \omega_{1\bar{1}} - \omega_{2\bar{2}}. \tag{4.2.19}$$

Note that, unlike the case of the Iwasawa manifold, here $\omega_i\}$ is not a holomorphic frame. Moreover, from (4.2.19), it follows that a linear lattice $\Lambda \subset \mathfrak{h}_3$ can be chosen such that if $\Gamma = \exp(\Lambda)$, then there is a well-defined holomorphic submersion:

$$p : X = \Gamma\backslash H \longrightarrow T = \mathbb{Z}[i]^2\backslash\mathbb{C}^2, \tag{4.2.20}$$

where a frame for T is given by $\{\omega_1, \omega_2\}$, similarly as in the case of the Iwasawa manifold. We fix a holomorphic volume form on X given by:

$$\Omega_0 = \omega_{123}. \tag{4.2.21}$$

Moreover, we consider the space:

$$\langle[\omega_{1\bar{1}} + \omega_{2\bar{2}}],\ [\omega_{1\bar{2}}],\ [\omega_{2\bar{1}}]\rangle_{\mathbb{C}} \tag{4.2.22}$$

spanning the classes in $H^2_{dR}(X, \mathbb{C})$ that admit a representative of bidegree $(1, 1)$. Now, we consider the 2-form:

$$F = \pi(n(\omega_{1\bar{2}} + \omega_{2\bar{1}}) + ip(\omega_{1\bar{2}} - \omega_{2\bar{1}}) + q(\omega_{1\bar{1}} + \omega_{2\bar{2}})). \tag{4.2.23}$$

By the same argument as for the Iwasawa manifold, F is the curvature of the Chern connection of a suitable hermitian metric on a holomorphic line bundle L_N, for $N = (n, p, q)$.

Over X, there exist non-flat instantons ∇ on $T^{1,0}$, as described in [47, 105]. Here, we classify instanton connections such that $\mathcal{E} = (T^{1,0}, \nabla^{0,1})$ is of homogeneous type. Since $\mathfrak{h}_3^{0,1}$ is abelian, the hypothesis of Corollary 4.1.14 are satisfied, and it is enough to classify abelian instantons of homogeneous type. Therefore, assume $\mathcal{E} = \bigoplus_{i=1}^k V_0^i$. With respect to a distinguished frame of V_0^i, we write:

$$\overline{\partial}_{V_0^i} = \overline{\partial} + a_i, \tag{4.2.24}$$

where $a_i \in \Omega^{0,1}_X$ is invariant and $\overline{\partial} a_i = 0$. Using Theorem 4.1.5 combined with the fact that $(\mathfrak{h}_3, J)$ is rational, we can compute $H^{p,q}_{\overline{\partial}}(X)$ via its Lie algebra cohomology. A quick computation shows:

$$H^{0,1}_{\overline{\partial}}(\mathfrak{h}_3, J) = \langle[\omega_{\bar{1}}], [\omega_{\bar{2}}], [\omega_{\bar{3}}]\rangle. \tag{4.2.25}$$

Therefore a_i is any invariant $(0,1)$-form, and we write:

$$a_i = \sum_j a_i^j \omega_{\bar{j}}. \tag{4.2.26}$$

Let h_0^i be the standard hermitian structure in V_0^i with respect to its distinguished frame. Then, using (4.2.19):

$$F_i = F_{h_1^i} = \partial a_i - \overline{\partial a_i} = -(a_i^3 + \overline{a_i^3})(\omega_{1\bar{1}} - \omega_{2\bar{2}}). \tag{4.2.27}$$

Here again assuming a balanced class $\sigma \in H^{2,2}_{BC}(X, \mathbb{R})$ that admits an invariant hermitian metric ω, the Hermite-Einstein equation:

$$F_i \wedge \omega^2 = 0 \tag{4.2.28}$$

is solved if and only if $\deg_\sigma V_0^i = 0$, which is obvious since $c_1(V_0^i) = 0$. Hence (4.2.27) describes, up to isomorphism, the curvature of all abelian instantons on line bundles of homogeneous type, and by Corollary 4.1.14 on bundles of homogeneous type and higher rank, it is enough to consider instantons that are isomorphic to split sums of these ones.

Remark 4.2.6. *With the aid of a mathematical software, one can check that if ω' is any invariant hermitian metric on X, then $\nabla^B(\omega')$ the Bismut connection of ω' satisfies:*

$$F_{\nabla^B(\omega')} \wedge \omega^2 = 0, \quad F^{0,2}_{\nabla^B(\omega')} = 0. \tag{4.2.29}$$

Therefore, this connections on $T^{1,0}$ fall inside the family of split sums of instantons just described.

We give now our existence result for solutions to the Hull-Strominger system on X:

Proposition 4.2.7. *Let V_0 be a bundle of homogeneous type, and let $V_1 = \bigoplus_{i=1}^k L_{N_i}$ for tuples $N_i = (n_i, p_i, q_i) \in \mathbb{Z}^3 \backslash \{0\}$. Moreover, let ω be an invariant hermitian metric on X. We parametrize ω as in* (4.2.13). *Then:*

1. *ω defines a balanced class $\sigma(\omega) = [||\Omega_0||_\omega \omega^2] \in H^{2,2}_{BC}(X, \mathbb{R})$ if and only if:*

$$(s_1 - s_2)s_3 - s_6^2 - s_7^2 + s_8^2 + s_9^2 = 0. \tag{4.2.30}$$

Let h_0^i be the hermitian metric on V_0^i of (4.2.27), *and let $h_0 = (h_0^i)$. Let h_1^i be the hermitian metric on L_{N_i} such that $F_{h_1^i} = F(N_i)$, and let $h_1 = (h_1^i)$. Then:*

2. *(ω, h_0, h_1) is a solution to the Hull-Strominger system* (3.1.11) *with the instanton condition* (3.1.12) *if and only if* (4.2.30) *holds and:*

$$\alpha = \frac{2s_3}{4\sum_{j=1}^3 \mathrm{Re}(a_j)^2 - \pi^2 \sum_i n_i^2 + p_i^2 - q_i^2}, \tag{4.2.31}$$

is well-defined and:

$$\deg_{\sigma(\omega)} L_{N_i} = 0 \quad i = 1, ..., k. \tag{4.2.32}$$

Proof. The first item is a consequence of the computation:

$$d\omega^2 = -2(s_6^2 + s_7^2 - s_8^2 + s_9^2 + (-s_1 + s_2)s_3)(\omega_{123\overline{12}} + \omega_{12\overline{123}}) \tag{4.2.33}$$

combined with Ω_0 being invariant, in particular $||\Omega_0||_\omega$ is constant. The conditions on the second item are equivalent to solving the rest of the equations of (3.1.11): the Hermite-Einstein equation for h_0 is automatic, using $c_1(V_0^i) = 0$ and the fact that F_i and ω are invariant forms. Since h_1^i also have invariant Chern curvature $F(N_i)$, the Hermite-Einstein equation for h_1 is actually equivalent to the cohomological condition (4.2.46). Finally, the straightforward computations:

$$dd^c\omega = 4s_3\omega_{12\overline{12}}, \tag{4.2.34}$$
$$F_i \wedge F_i = 8\mathrm{Re}(a_i)^2\omega_{12\overline{12}}, \tag{4.2.35}$$
$$F(N_i) \wedge F(N_i) = 2\pi^2(n_i^2 + p_i^2 - q_i^2)\omega_{12\overline{12}} \tag{4.2.36}$$

imply that the condition (4.2.45) is equivalent to the Bianchi identity in (3.1.11). □

Remark 4.2.8. *We note that here, unlike the case of the Iwasawa manifold, it is clear from* (4.2.45) *that there are solutions in Proposition 4.2.7 with coupling parameter* $\alpha > 0$.

Remark 4.2.9. *The solutions above can be modified to recover those of [47, Theorem 5.1.b), 5.2.b)] with* ∇_t^+, $\nabla_{t'}^+$ *and* A *as particular cases of the instantons of* (4.2.27), *and setting* $\omega = \omega_0$, *and similarly with the solutions in [105, Proposition 3.2]. Finally, the solutions in [132, Theorem 5.3] using the instantons* $\nabla^B(\omega)$ *also fall into Proposition 4.2.7 with* $\omega = \omega'$ *in Remark 4.2.6.*

By now, we have discussed the existence of solutions to the Hull-Strominger system with an invariant ansatz on two particular nilmanifolds. The reader can see that in both Examples the discussion about the different aspects of the discussion runs parallel, and this is the case for the rest of our solutions on nilmanifolds. Therefore, here we summarize the information for the solutions in the rest of Examples. These are based on the nilmanifolds described in [47, Sections 6-8]. Here, we approach systematically the problem of finding solutions to the Hull-Strominger system with the invariant ansatz of the previous Examples. In the following table we give first the structure equations in a global invariant frame, as in (4.2.4), (4.2.19). Then, we write the most general invariant form representing the curvature of a line bundle, like in the previous Examples in (4.2.8), (4.2.23), and then we give the curvature of abelian instantons of homogeneous type with respect to an arbitrary invariant balanced metric, as the flat instantons in the Iwasawa manifold of Lemma 4.2.1 and (4.2.27) for $\mathfrak{h}_3$. For completeness, we also include the computation of the relevant cohomology groups $H^{0,1}_{\overline{\partial}}(X)$ and the subspace $E \subset H^2_{dR}(X, \mathbb{R})$ of classes admitting a representative of bidegree $(1,1)$, which can be computed via the analogous Lie algebra cohomologies.

Lie algebra	$\mathfrak{h}_2, \mathfrak{h}_4, \mathfrak{h}_5 \quad (b \in \mathbb{Q})$	$\mathfrak{h}_6$	$\mathfrak{h}_{19}^-$
Structure equations	$d\omega_1 = 0, \;\; d\omega_2 = 0,$ $d\omega_3 = \omega_{12} + \omega_{1\bar{1}} + b\omega_{1\bar{2}} - \omega_{2\bar{2}}$	$d\omega_1 = 0, d\omega_2 = 0,$ $d\omega_3 = \omega_{12} - \omega_{2\bar{1}}$	$d\omega_1 = 0,$ $d\omega_2 = \omega_{13} + \omega_{1\bar{3}},$ $d\omega_3 = i(\omega_{1\bar{2}} - \omega_{2\bar{1}})$
E basis	$[i\omega_{1\bar{1}}]$, $[i\omega_{2\bar{2}}]$, $[\omega_{1\bar{2}} - \omega_{2\bar{1}}]$, $[i(\omega_{1\bar{2}} + \omega_{2\bar{1}})]$	$[i\omega_{1\bar{1}}]$, $[i\omega_{2\bar{2}}]$, $[\omega_{1\bar{2}} - \omega_{2\bar{1}}]$, $[i(\omega_{1\bar{2}} + \omega_{2\bar{1}})]$, $[\omega_{2\bar{3}} - \omega_{3\bar{2}}]$	$[i\omega_{1\bar{1}}]$
$H^{0,1}_{\bar{\partial}}(X)$ basis	$[\omega_{\bar{1}}]$, $[\omega_{\bar{2}}]$	$[\omega_{\bar{1}}]$, $[\omega_{\bar{2}}]$	$[\omega_{\bar{1}}]$, $[\omega_{\bar{3}}]$
Curvature of gauge line bundles	$N = (m, n, p, q)$, $F(N) = \pi(m(\omega_{1\bar{1}} - \omega_{2\bar{2}}) +$ $n(\omega_{1\bar{2}} + \omega_{2\bar{1}}) + ip(\omega_{1\bar{2}} - \omega_{2\bar{1}}) +$ $q(\omega_{1\bar{1}} + \omega_{2\bar{2}}))$	$N = (m, n, p, q, r)$, $F(N) = \pi(m(\omega_{1\bar{1}} - \omega_{2\bar{2}}) +$ $n(\omega_{1\bar{2}} + \omega_{2\bar{1}}) + ip(\omega_{1\bar{2}} - \omega_{2\bar{1}}) +$ $q(\omega_{1\bar{1}} + \omega_{2\bar{2}}) + ir(\omega_{2\bar{3}} - \omega_{3\bar{2}}))$	$N = m$, $F(N) = \pi m \omega_{1\bar{1}}$
Curvature of abelian instantons of homogeneous type	$F_i = 0$	$F_i = 0$	$F_i = \mathrm{Re}(a_i) i(\omega_{1\bar{2}} - \omega_{2\bar{1}})$

With this data, we now state our existence results, whose proofs are completely analogous to those of Propositions 4.2.3, 4.2.7.

For the first of these Examples, let $X = (\Gamma\backslash H, J)$, where Lie $H = \mathfrak{h}_2, \mathfrak{h}_4$ or $\mathfrak{h}_5$, depending on the parameter $b \in \mathbb{Q}$, corresponding to the structure equations:

$$d\omega_1 = 0, \;\; d\omega_2 = 0, \;\; d\omega_3 = \omega_{12} + \omega_{1\bar{1}} + b\omega_{1\bar{2}} - \omega_{2\bar{2}}. \tag{4.2.37}$$

Explicitly (see [47, Section 6]), $\mathfrak{h}_2$ for $b^2 < 1$, $\mathfrak{h}_4$ for $b^2 = 1$ and $\mathfrak{h}_5$ for $b^2 > 1$. Note, however, that the latter $(\mathfrak{h}_5, J)$ is not isomorphic to the complex structure of the Iwasawa manifold. Then, according to the second column of the table above, we have the following result:

Proposition 4.2.10. *Let $X = (\Gamma\backslash H, J)$, where* Lie $H = \mathfrak{h}_2, \mathfrak{h}_4$ *or* $\mathfrak{h}_5$, *and let $V_0 = \bigoplus_{i=1}^3 V_0^i$ be a direct sum of line bundles of homogeneous type, and $V_1 = \bigoplus_{i=1}^k L_{N_i}$ for tuples $N_i = (m_i, n_i, p_i, q_i) \in \mathbb{Z}^4\backslash\{0\}$. Moreover, let ω be an invariant hermitian metric on X. We parametrize ω as in* (4.2.13)*. Then:*

1. *ω defines a balanced class $\sigma(\omega) = [||\Omega_0||_\omega \omega^2] \in H^{2,2}_{BC}(X, \mathbb{R})$ if and only if:*

$$(s_1 - s_2)s_3 - s_6^2 - s_7^2 + s_8^2 + s_9^2 + b(s_3 s_5 - s_6 s_8 - s_7 s_9) = 0 \tag{4.2.38}$$

$$b(s_3 s_4 + s_7 s_8 - s_6 s_9) = 0. \tag{4.2.39}$$

Let h_0^i be the hermitian metric on V_0^i with curvature F_i, and let $h_0 = (h_0^i)$. Let h_1^i be the hermitian metric on L_{N_i} such that $F_{h_1^i} = F(N_i)$, and let $h_1 = (h_1^i)$. Then:

2. *(ω, h_0, h_1) is a solution to the Hull-Strominger system* (3.1.11) *with the instanton condition* (3.1.12) *if and only if* (4.2.38), (4.2.39) *hold and:*

$$\alpha = \frac{-(6 + b^2)s_3}{2\pi^2 \sum_i m_i^2 + n_i^2 + p_i^2 - q_i^2}, \tag{4.2.40}$$

is well-defined and:

$$\deg_{\sigma(\omega)} L_{N_i} = 0 \quad i = 1, ..., k. \tag{4.2.41}$$

Now, we consider the next column in the table above. Then:

Proposition 4.2.11. *Let $X = \Gamma \backslash H$ for* Lie $H = \mathfrak{h}_6$*, and let $V_0 = \bigoplus_{i=1}^{3} V_0^i$ be a sum of line bundles of homogeneous type, and $V_1 = \bigoplus_{i=1}^{k} L_{N_i}$ for tuples $N_i = (m_i, n_i, p_i, q_i, r_i) \in \mathbb{Z}^5 \backslash \{0\}$. Moreover, let ω be an invariant hermitian metric on X. We parametrize ω as in* (4.2.13)*. Then:*

1. *ω defines a balanced class $\sigma(\omega) = [||\Omega_0||_\omega \omega^2] \in H^{2,2}_{BC}(X, \mathbb{R})$ if and only if:*

$$s_3 s_4 - s_6 s_9 + s_7 s_8 = 0 \tag{4.2.42}$$

$$s_3 s_5 - s_6 s_8 - s_7 s_9 = 0. \tag{4.2.43}$$

Let h_0^i be the hermitian metric on V_0^i with curvature F_i, and let $h_0 = (h_0^i)$. Let h_1^i be the hermitian metric on L_{N_i} such that $F_{h_1^i} = F(N_i)$, and let $h_1 = (h_1^i)$. Then:

2. *(ω, h_0, h_1) is a solution to the Hull-Strominger system* (3.1.11) *with the instanton condition* (3.1.12) *if and only if* (4.2.42), (4.2.43) *hold,*

$$r_i = 0, \quad i = 1, ..., k, \tag{4.2.44}$$

and moreover:

$$\alpha = \frac{-s_3}{\pi^2 \sum_i (m_i^2 + n_i^2 + p_i^2 - q_i^2}, \tag{4.2.45}$$

is well-defined and:

$$\deg_{\sigma(\omega)} L_{N_i} = 0 \quad i = 1, ..., k. \tag{4.2.46}$$

Remark 4.2.12. *The solution to the Hull-Strominger system of Propositions 4.2.10, 4.2.11 require $\alpha < 0$, as the only instanton connections on $T^{1,0}$ are flat and therefore the same rigidity result as in the Iwasawa manifold applies (see Remark 4.2.4). However, if one does not require that ∇ is an instanton, then solutions with $\alpha > 0$ exist, and are described in [47, Theorem 6.1].*

When $X = \Gamma \backslash H$ for Lie $H = \mathfrak{h}_{19}^-$, it is easy to see that there are no solutions to the Hull-Strominger system such that ∇ is an instanton within our construction. Even more, it is not possible to solve the Bianchi identity:

$$dd^c \gamma - \alpha \mathrm{tr}\, F_{h_0} \wedge F_{h_0} + \alpha \mathrm{tr}\, F_{h_1} \wedge F_{h_1} = 0 \tag{4.2.47}$$

using an invariant $(1,1)$-form γ that is a hermitian metric (see Example 4.3.6). From the point of view of Generalized Geometry, the Bott-Chern algebroid associated to the solution (τ, h_0, h_1) of (4.2.47) does not admit an invariant positive structure (see Remark 2.2.14). Still, there are solutions (see [47, Theorem 8.2(ii)]) if one does not require (3.1.12). In this solution, however, Corollary 4.1.14 shows that holomorphic $T^{1,0}$ is not polystable with respect to any class admitting an invariant balanced metric.

4.2.3 Solutions on compact quotients of $SL(2,\mathbb{C})$

In this Section, we provide invariant solutions to the Hull-Strominger system (3.1.11) on the compact threefold given by the quotient $X = SL(2,\mathbb{Z}[i])\backslash SL(2,\mathbb{C})$. This complex manifold admits a global frame of $T^*_{1,0}$ induced by left-invariant forms on $SL(2,\mathbb{C})$, which satisfy the structure equations:

$$d\omega_1 = \omega_{23}, \quad d\omega_2 = -\omega_{13}, \quad d\omega_3 = \omega_{12}, \tag{4.2.48}$$

From the above equations, the frame $\{\omega_i\}$ is holomorphic. Therefore, X is a complex-parallelizable manifold. Explicitly, this frame is dual to the frame given by left-translation of the elements in $T_{[1]}X = \mathfrak{sl}(2,\mathbb{C})$:

$$X_1 = \begin{pmatrix} 0 & i/2 \\ i/2 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} i/2 & 0 \\ 0 & -i/2 \end{pmatrix} \tag{4.2.49}$$

Moreover, we fix the $SU(3)$ structure:

$$\Omega_0 = \omega_{123}, \quad \omega_0 = \tfrac{i}{2}(\omega_{1\bar{1}} + \omega_{2\bar{2}} + \omega_{3\bar{3}}), \tag{4.2.50}$$

where Ω_0 is a holomorphic volume form and ω_0 is a balanced metric.

Unlike the nilpotent Examples above, $\mathfrak{sl}(2,\mathbb{C})$ is a simple Lie algebra. Here, we use its irreducible representations to produce irreducible holomorphic vector bundles of homogeneous type and higher rank. Explicitly, using the notations of Section 4.1.2, the rank r representations:

$$\rho_r : \mathfrak{sl}(2,\mathbb{C})^{0,1} \longrightarrow \mathrm{End}(\mathbb{C}^r) \tag{4.2.51}$$

are obtained by conjugating the well-known irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$. In the (conjugated) basis of (4.2.48), ρ_r is given by:

$$\overline{X_1} \mapsto \frac{i}{2}\begin{pmatrix} 0 & \sqrt{1\cdot(r-1)} & 0 & \dots & 0 & 0 \\ \sqrt{1\cdot(r-1)} & 0 & \sqrt{2\cdot(r-2)} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \sqrt{(r-2)\cdot 2} & 0 \\ 0 & 0 & 0 & \dots & 0 & \sqrt{(r-1)\cdot 1} \\ 0 & 0 & 0 & \dots & \sqrt{1\cdot(r-1)} & 0 \end{pmatrix}, \tag{4.2.52}$$

$$\overline{X_2} \mapsto \frac{1}{2}\begin{pmatrix} 0 & \sqrt{1\cdot(r-1)} & 0 & \dots & 0 & 0 \\ -\sqrt{1\cdot(r-1)} & 0 & \sqrt{2\cdot(r-2)} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \sqrt{(r-2)\cdot 2} & 0 \\ 0 & 0 & 0 & \dots & 0 & \sqrt{(r-1)\cdot 1} \\ 0 & 0 & 0 & \dots & -\sqrt{1\cdot(r-1)} & 0 \end{pmatrix}, \tag{4.2.53}$$

$$\overline{X_3} \mapsto \frac{i}{2}\begin{pmatrix} r-1 & 0 & 0 & \dots & 0 & 0 \\ 0 & r-3 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & -(r-3)) & 0 \\ 0 & 0 & 0 & \dots & 0 & -(r-1) \end{pmatrix}. \tag{4.2.54}$$

We will denote by $\mathcal{E}_r$ the holomorphic vector bundle of homogeneous type given by the rank r representation above. In particular $\mathcal{E}_3$ is a holomorphic structure on $T^{1,0}$, given by the conjugated-adjoint representation ρ_3.

Incidentally, by uniqueness of irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$ in each dimension, the bundle of homogeneous type $(T^{1,0}, \nabla^B(\omega_0)^{0,1})$ is shown to be $[\omega_0^2]$-stable, as it is irreducible and the corresponding Hermite-Einstein equation holds (see [105, Section 4]), hence it is isomorphic to $\mathcal{E}_3$.

Now, we show that the bundles $\mathcal{E}_r$ are well-suited to solve the Hull-Strominger system. First, we prove the following:

Lemma 4.2.13. *Let $h_0^{(r)}$ be the hermitian metric on $\mathcal{E}_r$ induced by the standard hermitian metric on $\mathbb{C}^r$. Then, the following hold:*

$$F_{h_0^{(r)}} \wedge \omega_0^2 = 0, \tag{4.2.55}$$

$$\operatorname{tr} F_{h_0^{(r)}} \wedge F_{h_0^{(r)}} = \tfrac{r(r^2-1)}{6}(\omega_{12\overline{12}} + \omega_{13\overline{13}} + \omega_{23\overline{23}}). \tag{4.2.56}$$

Proof. For the first item, let A_r be the Chern connection of $h_0^{(r)}$. Then, observe that by functoriality of ρ_r:

$$\begin{aligned} F_{h_0^{(r)}} &= F_{\rho_{r*}A_2} \\ &= d\rho_r(A_2) + \rho_r(A_2) \wedge \rho_r(A_2) \\ &= \rho_r(dA_2 + A_2 \wedge A_2) \\ &= \rho_{r*} F_{h_0^{(2)}}. \end{aligned}$$

Therefore, it is enough to check that:

$$F_{h_0^{(2)}} \wedge \omega_0^2 = 0. \tag{4.2.57}$$

This can be checked directly, by computing first:

$$F_{h_0^{(2)}} = \begin{pmatrix} -\tfrac{i}{2}(\omega_{1\overline{2}} - \omega_{2\overline{1}}) & \tfrac{1}{2}(\omega_{1\overline{3}} - \omega_{3\overline{1}} - i(\omega_{2\overline{3}} - \omega_{3\overline{2}})) \\ \tfrac{1}{2}(-\omega_{1\overline{3}} + \omega_{3\overline{1}} - i(\omega_{2\overline{3}} - i\omega_{3\overline{2}})) & \tfrac{i}{2}(\omega_{1\overline{2}} - \omega_{2\overline{1}}) \end{pmatrix}\Bigg(\tag{4.2.58}$$

$$\omega_0^2 = 2(\omega_{12\overline{12}} + \omega_{13\overline{13}} + \omega_{23\overline{23}}). \tag{4.2.59}$$

While ω_0^2 is straightforward to compute, for $F_{h_0^{(2)}}$ we have used the formula for the curvature:

$$F_A = dA + A \wedge A \tag{4.2.60}$$

with respect to a global frame, and the facts that $\rho_2 = \mathrm{id}$, $A_2 = A^{0,1} - (A^{0,1})^\dagger$ with respect to the standard metric $h_0^{(2)}$ in $\mathbb{C}^2$.

For the second item, using (4.2.58), we first compute:

$$\operatorname{tr} F_{h_0^{(2)}} \wedge F_{h_0^{(2)}} = \omega_{12\overline{12}} + \omega_{13\overline{13}} + \omega_{23\overline{23}}. \tag{4.2.61}$$

To conclude, we use repeatedly the identity:

$$\mathrm{tr}(\rho_r(M)\rho_r(M)^\dagger) = \tfrac{r(r^2-1)}{6}\mathrm{tr}(MM^\dagger), \quad M \in \mathfrak{sl}(2,\mathbb{C})^{0,1} \tag{4.2.62}$$

applied to each matrix $M = F_{h_0^{(2)}}(X_i, \overline{X_j})$, for $1 \leq i, j \leq 3$. Finally, we argue that (4.2.62) holds by proving that ρ_r is a dilation of factor $\frac{r(r^2-1)}{6}$ with respect to the inner products $\mathrm{tr}(\cdot\cdot^\dagger)$. This follows by checking that $\{\rho_r(\overline{X_i})\}$ is an orthogonal basis of $\mathrm{im}\,\rho_r$ and:

$$\mathrm{tr}(\overline{X_i}\,\overline{X_i}^\dagger) = \tfrac{1}{2}, \tag{4.2.63}$$

$$\mathrm{tr}(\rho_r(\overline{X_i})\rho_r(\overline{X_i})^\dagger) = \tfrac{r(r^2-1)}{12}, \tag{4.2.64}$$

which the reader can check easily by induction on r. □

Now, we give the result of existence of solutions to Hull-Strominger on X. These come in a discrete family parametrized by r.

Proposition 4.2.14. *Let $V_0 = \mathcal{E}_3$, and $V_1 = \mathcal{E}_r$, for $r \neq 3$. Then, $(\omega_0, h_0^{(3)}, h_0^{(r)})$ is a solution to the Hull-Strominger system* (3.1.11) *with the instanton condition* (3.1.12) *if and only if:*

$$\alpha = \frac{12}{24 - r(r^2-1)}. \tag{4.2.65}$$

Proof. By [1], the invariant metric ω_0 is balanced as X is a complex-parallelizable manifold. Since Ω_0 is an invariant holomorphic volume form, $||\Omega_0||_{\omega_0}$ is constant. Hence we get:

$$d(||\Omega_0||_{\omega_0}\omega_0^2) = 0. \tag{4.2.66}$$

The Hermite-Einstein equation for $h_0^{(2)}, h_0^{(r)}$ follow from Lemma 4.2.13(1). Finally, the Bianchi identity is equivalent to (4.2.65) by Lemma 4.2.13(2) and:

$$dd^c\omega_0 = 2(\omega_{12\overline{12}} + \omega_{13\overline{13}} + \omega_{23\overline{23}}). \tag{4.2.67}$$

□

Remark 4.2.15. *The solutions above shall be compared with the family of solutions found in [45, Section 4]. There, non-flat instantons are produced on the trivial holomorphic bundle $\mathcal{O}_X^{\oplus r}$ on $SL(2,\mathbb{C})$ by a non-trivial ansatz for the hermitian metric. As this method is not compatible with taking compact quotients of $SL(2,\mathbb{C})$, here we rather consider the non-trivial holomorphic structure $\mathcal{E}_r$ while keeping the standard hermitian metric $h_0^{(r)}$.*

Remark 4.2.16. *With respect to the solutions found in [105], here we recover the one compatible with the instanton condition* (3.1.12) *(see [105, Theorem 4.3(i.2)]), as $(T^{1,0}, \nabla^B(\omega_0)^{0,1}) \cong \mathcal{E}_3$, as discussed above.*

Our solutions in Proposition 4.2.14 include the cases $\alpha > 0$ for $r < 3$ and $\alpha < 0$ for $r > 3$. Moreover, with independence of the sign of α, the instantons A_r for $r > 3$ above are, to the knowledge of the author, new in the literature.

4.2.4 Solutions on complex solvmanifolds

Here, we address the systematic approach to finding solutions to the Hull-Strominger system on solvmanifolds with the invariant ansatz used in the previous Sections.

The complex manifolds we examine are given by quotients $X = \Gamma\backslash G$, where G is a solvable real Lie group of even dimension, and $\Gamma \subset G$ is a discrete subgroup yielding a compact quotient. To support solutions to the Hull-Strominger system, X must admit in particular a holomorphic volume form and a balanced metric. Here, we use the classification results on solvable, unimodular Lie algebras admitting a (left)-invariant Calabi-Yau structure and special metrics in real dimension 6 in [51], see in particular Theorem 4.5. Moreover, the existence of lattices on the corresponding connected, simply-connected Lie groups is also guaranteed (see [51, Proposition 2.10]). Hence, we obtain explicit complex Calabi-Yau solvmanifolds supporting balanced metrics. These Examples are denoted by the underlying real Lie algebra as: $(\mathfrak{g}_1, J)$, $(\mathfrak{g}_2^\beta, J_\pm)$ for parameter value $\beta \in [0, \frac{\pi}{2})$, $(\mathfrak{g}_3, J_x)$ (for $x > 0$), $(\mathfrak{g}_5, J)$, $(\mathfrak{g}_7, J_\pm)$, and $(\mathfrak{g}_8, J_A)$ for $A \in \mathbb{C}\backslash\mathbb{R}$. While we have carried a systematic case-by-case examination of the previous solvmanifolds, here we report on the ones on which we find solutions to the Hull-Strominger system. These are $\mathfrak{g}_2^0$ and $\mathfrak{g}_7$.

We describe the invariant geometry of the solvmanifold $X = (\Gamma\backslash G, J)$, with Lie $G = \mathfrak{g}_2^0$ by the global frame $\{\omega_i\}$ of $T^*_{1,0}$ satisfying the structure equations:

$$d\omega_1 = i\omega_1 \wedge (\omega_3 + \omega_{\overline{3}}) \quad d\omega_2 = -i\omega_2 \wedge (\omega_3 + \omega_{\overline{3}}) \quad d\omega_3 = 0. \tag{4.2.68}$$

Moreover, the invariant form $\Omega_0 = \omega_{123}$ is a holomorphic volume form, and the family of hermitian metrics:

$$\omega = is_1\omega_{1\overline{1}} + is_2\omega_{2\overline{2}} + is_3\omega_{3\overline{3}}, \quad s_j > 0 \tag{4.2.69}$$

is in fact Kähler. It is easy to check that this is the most general invariant Kähler metric on X.

There is a one-parameter family of homogeneous line bundles of homogeneous type L_a, given by the Dolbeault operator:

$$\overline{\partial}_{L_a} = \overline{\partial} + a\omega_{\overline{3}}, \quad a \in \mathbb{C}, \tag{4.2.70}$$

with respect to a distinguished frame. We apply Corollary 4.1.14 to deduce that holomorphic structures on $T^{1,0}$ of homogeneous type that are poystable with respect to a class admitting an invariant balanced metric must be of the form $V_0 = \oplus_{i=1}^3 L_{a_i}$, and h_0 the hermitian metric on V_0 induced by the standard metric on $\mathbb{C}^3$. Then, using (4.2.68), it is immediate to check that h_0 is flat.

The most general invariant form $F \in \Omega^{1,1}$ that is d-closed and purely imaginary on X is given by:

$$F = (m\omega_{1\overline{1}} + n\omega_{2\overline{2}} + p\omega_{3\overline{3}}), \quad (m, n, p) \in \mathbb{R}^3. \tag{4.2.71}$$

If the cohomology class $[F]$ has integral periods, then by Lemma 4.1.16, it is the curvature of the Chern connection on some hermitian line bundle. We now state the existence result in this manifold. With the above ansatz, solutions are necessarily Kähler and flat:

Proposition 4.2.17. *Let V_0 and h_0 be as above. Assume $[F(N_i)]$ given by (4.2.71) with $N_i = (m_i, n_i, p_i) \in \mathbb{R}^3$ for $i = 1, ..., r$ have integral periods, and let (L_i, h_1^i) be the holomorphic line bundle and hermitian metric with Chern curvature $F(N_i)$, and $V_1 = \oplus_{i=1}^r L_i$, $h_1 = (h_1^i)$. Moreover, let ω be an invariant hermitian metric on X. Then, parametrizing ω as in (4.2.13), (ω, h_0, h_1) is a solution to the Hull-Strominger system if and only if ω is Kähler and h_0, h_1 are flat.*

Proof. Let ω be an arbitrary invariant hermitian metric, and $F(N_i)$ given by (4.2.71). Then, the terms in the Bianchi identity of the Hull-Strominger system are given by:

$$dd^c\omega = 8i(s_4 + is_5)\omega_{13\overline{23}} - 8i(s_4 - is_5)\omega_{23\overline{13}}, \tag{4.2.72}$$

$$\mathrm{tr}\, F_{h_0} \wedge F_{h_0} = 0, \tag{4.2.73}$$

$$\mathrm{tr}\, F_{h_1} \wedge F_{h_1} = -\sum_i \Big(m_i n_i \omega_{12\overline{12}} + m_i p_i \omega_{13\overline{13}} + n_i p_i \omega_{23\overline{23}}. \tag{4.2.74}$$

By inspection on the above computations, it is clear that for any solution to the Bianchi identity, ω must be pluriclosed, and two out of the three m_i, n_i, p_i must vanish for each i. Now, since $||\Omega_0||_\omega$ is a constant, the conformally balanced equation:

$$d(||\Omega_0||_\omega \omega^2) = 0 \tag{4.2.75}$$

implies that ω is balanced. Since it is also pluriclosed, it must be Kähler, hence it is given by (4.2.69). Then, the degree condition imposed by the Hermite-Einstein equations:

$$F_{h_0^i} \wedge \omega^2 = 0 \tag{4.2.76}$$

immediately implies that the remaining parameters in N_i must vanish, and the result follows. □

Remark 4.2.18. *The general rigidity result by which solutions to the Hull-Strominger system (ω, h_0, h_1) with h_0 flat and $\alpha > 0$ must have ω Kähler and h_1 flat (see [24]) holds here with reversed sign $\alpha < 0$ in the presence of non-flat h_1 and invariant ansatz. It is an interesting open question if one can find any non-Kähler solutions on this manifold.*

We now turn to the solvmanifold X with underlying real Lie algebra $\mathfrak{g}_7$, where non-Kähler solutions are already known in the literature [105, Section 5]. The structure equations are given with respect to an invariant frame $\{\omega_i\}$ of $T^*_{1,0}$ by:

$$d\omega_1 = i\omega_1 \wedge (\omega_3 + \omega_{\overline{3}}) \quad d\omega_2 = -i\omega_2 \wedge (\omega_3 + \omega_{\overline{3}}) \quad d\omega_3 = \pm(\omega_{1\overline{1}} - \omega_{2\overline{2}}), \tag{4.2.77}$$

where the choice of $\pm$ corresponds to the complex structure considered $J_\pm$. The complex manifolds $X_\pm = (\Gamma \backslash G, J_\pm)$ with Lie $G = \mathfrak{g}_7$ are Calabi-Yau with the invariant holomorphic volume form $\Omega_0 = \omega_{123}$. Moreover, since $\mathfrak{g}_7$ is unimodular and solvable, Corollary 4.1.14 applies. Hence, to look for instantons on bundles of homogeneous type to solve the Hull-Strominger system, we may restrict to the abelian case. The Dolbeault operator of homogeneous line bundles L are given, with respect to the distinguished frame, by:

$$\overline{\partial}_L = \overline{\partial} + a\omega_{\overline{3}}. \tag{4.2.78}$$

Let $V_0 = \oplus_{i=1}^3 L_i$. Using (4.2.77), we obtain the curvature of the Chern connection on V_0 with respect to the standard hermitian metric h_0:

$$F_{h_0} = \mp \left(\begin{pmatrix} 2\mathrm{Re}(a_1) & & \\ & 2\mathrm{Re}(a_1) & \\ & & 2\mathrm{Re}(a_3) \end{pmatrix} \right) (\omega_{1\bar{1}} - \omega_{2\bar{2}}). \tag{4.2.79}$$

The most general invariant form $F \in \Omega^{1,1}$ that is d-closed and purely imaginary is given, up to d-exact terms, by:

$$F = m(\omega_{1\bar{1}} + \omega_{2\bar{2}}), \quad m \in \mathbb{R}. \tag{4.2.80}$$

For any $[F]$ having integral periods, we denote L_m for a holomorphic line bundle with Chern curvature given by $F(m)$ as in (4.2.80) with respect to some hermitian metric h_m (see Lemma 4.1.16). Let $V_1 = \oplus_{i=1}^r L_{m_i}$ and let $h_1 = (h_{m_i})$. Then, we have the following:

Proposition 4.2.19. *Let (V_0, h_0), (V_1, h_1) be as above, and let ω be an invariant hermitian metric parametrized as in* (4.2.13). *Then, (ω, h_0, h_1) is a solution to the Hull-Strominger syste with ∇ satisfying* (3.1.12) *if and only:*

$$\omega = is_1(\omega_{1\bar{1}} + \omega_{2\bar{2}}) + is_3\omega_{3\bar{3}}, \quad s_j > 0, \tag{4.2.81}$$

and moreover h_1 is flat, and:

$$\alpha = \frac{s_3}{\sum_i \mathrm{Re}(a_i)^2} \tag{4.2.82}$$

is well-defined.

Proof. Let ω be an arbitrary invariant hermitian metric on X parametrized by (4.2.13). First, we observe that the Hermite-Einstein equation for L_{m_i} is given by:

$$F(m_i) \wedge \omega^2 = 2m_i(s_1s_3 - s_6^2 - s_7^2 + s_2s_3 - s_8^2 - s_9^2)\Omega_0 \wedge \overline{\Omega_0} = 0. \tag{4.2.83}$$

Given that ω is a positive $(1,1)$-form, we have that:

$$\omega^2(X_1, X_3, \overline{X_1}, \overline{X_3}) = s_1s_3 - s_6^2 - s_7^2 > 0, \tag{4.2.84}$$
$$\omega^2(X_2, X_3, \overline{X_2}, \overline{X_3}) = s_2s_3 - s_8^2 - s_9^2 > 0. \tag{4.2.85}$$

Therefore, equations (4.2.83) hold if and only if $m_i = 0$ for all i, and h_1 is flat.

Now, using that $||\Omega_0||_\omega$ is a constant and (4.2.77), the conformally balanced equation of (3.1.11) is equivalent to the system:

$$\begin{aligned} (s_1 - s_2)s_3 - s_6^2 - s_7^2 + s_8^2 + s_9^2 &= 0, \\ s_1s_8 + s_4s_7 - s_5s_6 &= 0, \\ s_1s_9 - s_4s_6 - s_5s_7 &= 0, \\ s_2s_6 - s_4s_9 - s_5s_8 &= 0, \\ s_2s_7 + s_4s_8 - s_5s_9 &= 0. \end{aligned} \tag{4.2.86}$$

Moreover, we compute the terms in the Bianchi identity:

$$dd^c\omega = 4s_3\omega_{12\overline{12}} + 8i((s_4 + is_5)\omega_{13\overline{23}} - (s_4 - is_5)\omega_{23\overline{13}}), \tag{4.2.87}$$
$$\mathrm{tr}\, F_{h_0} \wedge F_{h_0} = 4\sum_i \mathrm{Re}(a_i)^2(\omega_{12\overline{12}}), \tag{4.2.88}$$
$$\mathrm{tr}\, F_{h_1} \wedge F_{h_1} = 0. \tag{4.2.89}$$

By inspection of the terms appearing above, it is clear that we must have $s_4 = s_5 = 0$. Plugging this in the system (4.2.86), and using that $s_1, s_2, s_3 > 0$, we obtain (4.2.81). Finally, the value of α is also a consequence of the above computations. □

The solutions given by Proposition 4.2.19 coincide essentially with the ones found in [105, Section 5] that are compatible with the choice of (3.1.12). Here, moreover, we have shown that restricting to the invariant ansatz given by bundles of homogeneous type, it is unlikely that this family of solutions can be enlarged.

We finish this Section mentioning that the solutions to the Hull-Strominger system we have found in all of the preceding Examples give strong evidence for the Conjecture stated in [105, Introduction] (restricting to hermitian connections) about the existence of invariant solutions to the Hull-Strominger system. Indeed, if we impose the physically natural condition $\alpha > 0$, our solutions are given exactly in the manifolds X with underlying Lie algebra $\mathfrak{h}_3$, $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{g}_7$.

4.3 A refinement of the existence conjecture by Yau

In this Section, we propose to address a question based on Conjecture 3.2.1. In the refinement we make, we find it is natural to strenghten the statement of the conjecture by Yau in two ways. Firstly, Conjecture 3.2.1 does not specify the relation of $\mathfrak{b}_0$ with the balanced class of the solution $\mathfrak{b} = [||\Omega||_\omega \omega^2]$. Hence, it is desirable that a complete answer to Conjecture 3.2.1 has control on the balanced class, producing a solution of the Hull-Strominger with $\mathfrak{b}_0 = \mathfrak{b}$. Secondly, here, we propose to take ∇ satisfying the Hermite-Einstein equation as in (3.1.11), both for its physical and geometrical significance (see Section 3.1.1).

Our approach to the existence problem for the Hull-Strominger system, with ∇ satisfying (3.1.12), lead us to consider holomomorphic vector bundle structures V_0 on $T^{1,0}$ which are polystable with respect to the balanced class $\mathfrak{b}$. For special choices of V and V_0, however, one may find solutions of the Hermite-Einstein equation with special metric properties that obstruct the existence of solutions to the Bianchi identity when X does not admit Kähler structures. Some of these choices have to do with twisting V_0 by a holomorphic line bundle in the kernel of the natural map:

$$\mathrm{Pic}_X \xrightarrow{(c_1)_{BC}} H^{1,1}_{BC}(X,\mathbb{R}) \tag{4.3.1}$$

This motivates the statement of the following Question:

Question 4.3.1. *Let* (X,Ω) *be a compact Calabi-Yau threefold with* $\ker(c_1)_{BC} = 0$ *with respect to* (4.3.1)*, and endowed with a balanced class* $\mathfrak{b}$*. Let* V *be a* $\mathfrak{b}$*-polystable holomorphic* *vector bundle over X satisfying* (3.2.3), (3.2.5)*. Let V_0 be a generic $\mathfrak{b}$-polystable holomorphic vector bundle structure on $T^{1,0}$. Does (X,Ω,V) admit a solution (ω,h) of the Hull-Strominger system* (3.1.11) *and balanced class $[||\Omega||_\omega \omega^2] = \mathfrak{b}$, such that ∇ is the Chern connection of a Hermitian-Einstein metric h_0 on V_0?*

Remark 4.3.2. *Observe that an affirmative answer to Question 4.3.1 provides, in particular, a solution to Conjecture 3.2.1 with the ansatz* (3.1.12) *(see the discussion in Section 3.1.1). It is an open question whether, assuming that the holomorphic tangent bundle $T^{1,0}$ of X is $\mathfrak{b}$-polystable, one can reduce Yau's Conjecture 3.2.1 for ∇ the Chern connection of ω, as proposed in [58], to Question 4.3.1.*

To finish this Section, we justify the hypothesis in Question 4.3.1 on homogeneous complex manifolds. Firstly, the condition $\ker(c_1)_{BC} = 0$ is motivated by the following result.

Proposition 4.3.3. *Let (X,Ω) be a compact Calabi-Yau threefold endowed with a balanced class $\mathfrak{b}$. Assume that X does not admit any Kähler metric. Let $L \to X$ be a holomorphic line bundle on X with vanishing first Chern class $c_1(L) = 0 \in H^{1,1}_{BC}(X,\mathbb{R})$. Let V_0 be a holomorphic bundle structure on $T^{1,0}$ which is polystable with respect to $\mathfrak{b}$. Then, $(X,\Omega,V_0 \otimes L)$ does not admit a solution of the Hull-Strominger system* (3.1.11) *with balanced class $\mathfrak{b}$, such that ∇ is the Chern connection of a Hermitian-Einstein metric h_0 on V_0.*

Proof. Assume that (ω,h,∇) is such a solution, for ∇ the Chern connection of h_0. Since $c_1(L) = 0$ in Bott-Chern cohomology, there exists h_L be a flat metric on L. Then, $h_0 \otimes h_L$ is a Hermite-Einstein metric on $V_0 \otimes L$, and therefore there exists a holomorphic gauge transformation taking h to $h_0 \otimes h_L$. In particular, since h_L is flat, one has

$$\mathrm{tr}\, F_h \wedge F_h = \mathrm{tr}\, F_{h_0} \wedge F_{h_0}, \tag{4.3.2}$$

and therefore $dd^c\omega = 0$. From this, ω is both conformally balanced and pluriclosed, and hence it must be Kähler (see Theorem 1.1.3), contradicting the hypothesis. □

In a non-Kähler manifold X with non-trivial line bundles L such that $c_1(L)_{BC} = 0$, the previous result provides continuous families of pairs (V_0,V) for which there cannot be solutions of the Hull-Strominger system with the ansatz (3.1.12). In particular, one can always make the non-generic choice $V = V_0$, which obstructs the existence of solutions. For the sake of concreteness, we discuss an example below, which slightly generalize the previous situation. It considers the existence problem for the Hull-Strominger system on nilmanifolds, as in the seminal paper [47].

Example 4.3.4. *Let $X = (\Gamma\backslash G, J)$ be a non-Kähler compact balanced nilmanifold of complex dimension 3 with left-invariant complex structure J and trivial canonical bundle, as described in Sections 4.2.1, 4.2.2. The smooth tangent bundle is trivial, and we take the holomorphic structure on $T^{1,0} \cong X \times \mathbb{C}^3$ to be a direct sum of holomorphic line bundles*

$$V_0 = L_1^0 \oplus L_2^0 \oplus L_3^0 \tag{4.3.3}$$

with $c_1(L_j^0)_{BC} = 0$, which are straightforward to find given the structure equations of X and Theorems 4.1.5, 4.1.6. Consider the rank-r holomorphic vector bundle

$$V = \oplus_{j=1}^r L_j, \tag{4.3.4}$$

with L_j holomorphic line bundles with $c_1(L_j)_{BC} = 0$. Then, V_0 and V are both $\mathfrak{b}$-polystable with respect to any balanced class $\mathfrak{b} \in H^{2,2}_{BC}(X, \mathbb{R})$. Furthermore, the Hermite-Einstein metrics with respect to any balanced metric are flat. Arguing now as in the proof of Proposition 4.3.3, it follows that, for any given $\mathfrak{b}$, (X, Ω, V) does not admit a solution of the Hull-Strominger system (3.1.11) *with balanced class $\mathfrak{b}$, such that ∇ is the Chern connection of a Hermitian-Einstein metric h_0 on V_0.*

One can consider other non-generic choices of holomorphic vector bundles which do not admit solutions of the equations, as for instance $V = V_0^*$ or $V = W \oplus W'$ and $V_0 = W^* \oplus W'$, for some choice of polystable bundles W and W' on X. We consider an interesting explicit situation in the next example.

Example 4.3.5. *Let (X, Ω) be the Calabi-Yau compact threefold given by the quotient $X = SL(2, \mathbb{Z}[i]) \backslash SL(2, \mathbb{C})$. Recall from Section 4.2.3 that this complex manifold admits a global frame of $T^*_{1,0}$ induced by left-invariant forms on $SL(2, \mathbb{C})$, which satisfy the structure equations:*

$$d\omega_1 = \omega_{23}, \quad d\omega_2 = -\omega_{13}, \quad d\omega_3 = \omega_{12} \tag{4.3.5}$$

and a holomorphic volume form $\Omega = \omega_{123}$. Explicitly, this frame is dual to the frame given by left-translation of the elements in $T_{[1]}X = \mathfrak{sl}(2, \mathbb{C})$:

$$X_1 = \begin{pmatrix} 0 & i/2 \\ i/2 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} i/2 & 0 \\ 0 & -i/2 \end{pmatrix} \tag{4.3.6}$$

We fix the balanced class $\mathfrak{b}$ of the hermitian metric:

$$\omega_0 = \tfrac{i}{2}(\omega_{1\bar{1}} + \omega_{2\bar{2}} + \omega_{3\bar{3}}). \tag{4.3.7}$$

Let W be the holomorphic vector bundle on $X \times \mathbb{C}^2$ with Dolbeault operator given by:

$$\overline{\partial}_W = \overline{\partial} + \sum_{i=1}^{3} \omega_{\bar{i}} \otimes \overline{X_i} \tag{4.3.8}$$

and let $V = W^{\oplus 4}$. The integrability $\overline{\partial}_W^2 = 0$ boils down to the fact that we are using the standard representation of $\mathfrak{sl}(2, \mathbb{C})$ above for the matrix-valued $(0,1)$-forms of the operator. Similarly, let V_0 be the holomorphic vector bundle on $X \times \mathfrak{sl}(2, \mathbb{C})$ with Dolbeault operator induced by the adjoint representation of $\mathfrak{sl}(2, \mathbb{C})$:

$$\overline{\partial}_{V_0} = \overline{\partial} + \sum_{i=1}^{3} \omega_{\bar{i}} \otimes [\overline{X_i}, \cdot]. \tag{4.3.9}$$

By definition, V_0 is the associated bundle to the $SL(2, \mathbb{C})$-principal bundle of frames of W, via the adjoint representation:

$$\mathrm{Ad} : SL(2, \mathbb{C}) \to GL(\mathfrak{sl}(2, \mathbb{C})). \tag{4.3.10}$$

In other words, $V_0 \cong \mathrm{End}_0\, W$, where End_0 stands for null-trace endomorphisms. Interestingly, one can prove that V_0 is isomorphic to $(T^{1,0}, (\nabla^B)^{0,1})$, where ∇^B denotes the Bismut connection of ω_0 (cf. [45]).

Next, we observe that the standard Hermitian metric on $\mathbb{C}^2$ produces a Hermitian metric h_W on W whose Chern curvature is given by:

$$F_{h_W} = \begin{pmatrix} -\frac{i}{2}(\omega_{1\bar{2}} - \omega_{2\bar{1}}) & \frac{1}{2}((\omega_{1\bar{3}} - \omega_{3\bar{1}}) - i(\omega_{2\bar{3}} - \omega_{3\bar{2}})) \\ -\frac{1}{2}((\omega_{1\bar{3}} - \omega_{3\bar{1}}) + i(\omega_{2\bar{3}} - \omega_{3\bar{2}})) & \frac{i}{2}(\omega_{1\bar{2}} - \omega_{2\bar{1}}) \end{pmatrix} \tag{4.3.11}$$

It is a straightforward computation to check that:

$$F_{h_W} \wedge \omega_0^2 = 0. \tag{4.3.12}$$

Therefore, the bundle W is $\mathfrak{b}$-polystable, and so are V_0 and V.

We now consider the Hull-Strominger system (3.1.11) *with the ansatz* (3.1.12) *for the bundles V_0 and V as above. Suppose (ω', h_0, h) is a solution with balanced class $\mathfrak{b}$ (with ∇ the Chern connection of h_0 on V_0). Let $\tilde{h}_W$ be a Hermitian-Einstein metric on W with respect to ω'. By uniqueness of Hermitian-Einstein metrics, there is a holomorphic gauge transformation $u \in \mathrm{Aut}(V)$ such that $uh = \tilde{h}_W^{\oplus 4}$, and therefore $\mathrm{tr}_V F_h^2 = 4\mathrm{tr}_W F_{\tilde{h}_W}^2$. Similarly, the Chern connection of h_0 is related via a holomorphic gauge tranformation to the connection $\mathrm{Ad}_* D^{\tilde{h}_W}$, induced by the Chern connection $D^{\tilde{h}_W}$ of $\tilde{h}_W$ via the adjoint representation. Then, we have:*

$$\mathrm{tr}_{V_0} F_{h_0}^2 = \mathrm{tr}_{V_0}(\mathrm{Ad} \circ F_{\tilde{h}_W})^2 = 4\mathrm{tr}_W F_{\tilde{h}_W}^2 \tag{4.3.13}$$

where the last step boils down to checking:

$$\mathrm{tr}_{\mathfrak{sl}(2,\mathbb{C})}(\mathrm{Ad}(A)^2) = 4\mathrm{tr}_{\mathbb{C}^2}(A^2), \quad A \in \mathfrak{sl}(2,\mathbb{C}). \tag{4.3.14}$$

Therefore, arguing as in the previous examples, we conclude that ω' is Kähler, reaching a contradiction since X admits no Kähler metrics.

We finish this section with an example that illustrates a different potential obstruction to the existence of solutions for non-Kähler manifolds, related to the positivity of the solutions of the Bianchi identity. In fact, for this example we are not able to decide whether there exists a solution of the Hull-Strominger system with the ansatz (3.1.12), and speculate that it may yield a negative answer to Question 4.3.1

Example 4.3.6. *We go back to the situation of Example 4.3.4, for a compact Calabi-Yau nilmanifold (X, Ω) with underlying nilpotent Lie algebra $\mathfrak{h}_{19}^-$, considered in [47, Section 8]. This complex manifold admits a global frame of $T_{1,0}^*$ induced by left-invariant forms, satisfying the structure equations:*

$$d\omega_1 = 0, \quad d\omega_2 = \omega_{13} + \omega_{1\bar{3}}, \quad d\omega_3 = i(\omega_{1\bar{2}} - \omega_{2\bar{1}}), \tag{4.3.15}$$

and such that $\Omega = \omega_{123}$. The most general d-closed, purely imaginary, $(1,1)$-form on X induced by left-invariant forms is given by:

$$F = \pi(m\omega_{1\bar{1}} + ni(\omega_{1\bar{2}} - \omega_{2\bar{1}})), \tag{4.3.16}$$

for $m, n \in \mathbb{R}$. For a suitable choice of lattice, one can show that, for any $(m,n) \in \mathbb{Z}^2$, $\frac{i}{2\pi}F$ has integral periods and hence, by general theory, this is the curvature form of the Chern connection of a holomorphic hermitian line bundle $(L,h) \to X$.

We fix the balanced Hermitian form:

$$\omega = \tfrac{i}{2}(\omega_{1\bar{1}} + \omega_{2\bar{2}} + \omega_{3\bar{3}}) \tag{4.3.17}$$

and consider the associated balanced class $\mathfrak{b}$. With the previous notation, one can easily see that $c_1(L) \cdot \mathfrak{b} = 0$ when $m = 0$. Hence, for a choice of integers $n_j \in \mathbb{Z}\backslash\{0\}$, $j = 1, \dots, r$, with associated line bundles L_j as above, the holomorphic vector bundle:

$$V = \bigoplus_{j=1}^{r} L_j \tag{4.3.18}$$

is polystable with respect to $\mathfrak{b}$.

We take a holomorphic structure V_0 on $T^{1,0}$, with Dolbeault operator of the form:

$$\overline{\partial}^{\lambda}_{\overline{X_i}} X_j = \sum_{k=1}^{3} \lambda_{ijk} X_k, \tag{4.3.19}$$

where $\{X_i\}_{i=1,2,3}$ is the dual frame of $\{\omega_i\}_{i=1,2,3}$, and $\lambda_{i,j,k}$ are constant complex functions. Assuming that V_0 is polystable with respect to $\mathfrak{b}$, then by Proposition 4.1.13, we have that:

$$V_0 \cong L_1^0 \oplus L_2^0 \oplus L_3^0 \tag{4.3.20}$$

for L_i^0 line bundles with $c_1(L_i^0) = 0$.

With this setup, we consider the Hull-Strominger system with coupling constant $\alpha \in \mathbb{R}$ and the ansatz (3.1.12), that is, for triples $(\omega', \oplus_{j=1}^{3} \tilde{h}_j^0, \oplus_{j=1}^{r} \tilde{h}_j)$ with ω' a hermitian form on X and $\tilde{h}_j^0$ (resp. $\tilde{h}_j$) a hermitian metric on L_j^0 (resp. L_j). Provided that we have a solution, it is clear, in particular, that the Hermite-Einstein metrics $\tilde{h}_j^0$ with respect to ω' are flat. Assume first that $\alpha > 0$. Then, by [24] (we will give a different proof in Chapter 8), any solution must satisfy $d\omega' = 0$ and $F_{\tilde{h}_j} = 0$, in contradiction with our assumptions. If $\alpha = 0$, any solution is again Kähler by Theorem 1.1.3.

In the remaining case of $\alpha < 0$, we assume that our triples $(\omega', \oplus_{j=1}^{3} \tilde{h}_j^0, \oplus_{j=1}^{r} \tilde{h}_j)$ are such that ω', $F_{\tilde{h}_j}$ and $F_{\tilde{h}_j^0}$ are invariant $(1,1)$ forms on X. Then, it follows that $F_{\tilde{h}_j} = F_{h_j}$ and $F_{\tilde{h}_j^0} = 0$, and hence the Bianchi identity reduces to:

$$dd^c\omega' = -\alpha \sum_{j=1}^{r} F_{h_j}^2 = -2\alpha\pi^2 \sum_{j=1}^{r} n_j^2 \omega_{12\bar{1}\bar{2}}. \tag{4.3.21}$$

One can prove that the general solution of the previous equation is given by:

$$\omega' = -\frac{\alpha\pi^2}{2}\Big(\sum_{j=1}^{r} n_j^2 \omega_{3\bar{3}} + s_1 i\omega_{1\bar{1}} + s_2(\omega_{1\bar{2}} - \omega_{2\bar{1}}) + s_3 i(\omega_{1\bar{2}} + \omega_{2\bar{1}}) + \tag{4.3.22}$$

$$+ s_4(\omega_{1\bar{3}} - \omega_{3\bar{1}}) + s_5 i(\omega_{1\bar{3}} + \omega_{3\bar{1}}) + s_6(\omega_{2\bar{3}} - \omega_{3\bar{2}}) + s_7 i(\omega_{3\bar{2}} + \omega_{2\bar{1}})\Big) \tag{4.3.23}$$

where s_i are real constants and hence, since the component in $\omega_{2\bar{2}}$ vanishes, it follows that ω' is necessarily non-positive. This proves that, among the invariant solutions of the Bianchi identity for this choice of (V, V_0), there are no potential solutions of the Hull-Strominger system (3.1.11)*, since the corresponding ω' is not a Hermitian metric, as anticipated in Section 4.2.2.*

4.4 Some aspects of the moduli space of solutions for the Hull-Strominger system

4.4.1 The moduli space

In this Section, we recall the moment map interpretation for the Hull-Strominger system (we refer to [70] for details) and, reducing to the minimum the technicalities involved in this construction, we examine through homogeneous Examples the behaviour of the moduli metric in the locus of solutions with an invariant ansatz. We expect the insights provided by these explicit situations to carry over to the general picture.

Let X be a compact complex manifold of dimension n, and let P be a smooth principal K-bundle, where K is a compact Lie group. We assume $(\mathrm{Lie}\, K, \langle\cdot,\cdot\rangle)$ is quadratic. Moreover, let $H_0 \in \Omega^3(X,\mathbb{R})$ and a principal connection A on P satisfy:

$$dH_0 - \langle F_{A_0} \wedge F_{A_0}\rangle = 0, \tag{4.4.1}$$

and let $E = E_{P,H_0,A_0}$ be the associated string algebroid (see Example 2.1.13). We denote by $\mathcal{L}$ the space of liftings $\ell \subset E \otimes \mathbb{C}$. By Proposition 2.2.8, there is an embedding:

$$\mathcal{L} \hookrightarrow \Omega^{1,1}(X,\mathbb{R}) \times \Omega^2_{\mathbb{R}} \times \Omega^1(\mathrm{ad}\, P). \tag{4.4.2}$$

$\mathcal{L}$ carries a natural (pseudo)Kähler structure $(\mathbf{g}, \mathbf{J}, \mathbf{\Omega})$. Explicitly, the metric is given by ([70, Equation 5.20]):

$$\begin{aligned}\mathbf{g}((\dot\omega,\dot b,\dot a),(\dot\omega,\dot b,\dot a)) = &-\tfrac{1}{M}\int_X \langle \dot a \wedge J\dot a\rangle \wedge ||\Omega||_\omega \tfrac{\omega^{n-1}}{(n-1)!} + \\ &+\tfrac{1}{2M}\int_X (|\dot\omega_0|^2 + |\dot b_0|^2)||\Omega||_\omega \tfrac{\omega^n}{n!} + \\ &+\tfrac{1}{2M}\left(\tfrac{1}{2} - \tfrac{n-1}{n}\right)\int_X (|\Lambda_\omega\dot\omega|^2 + |\Lambda_\omega \dot b|^2)||\Omega||_\omega \tfrac{\omega^n}{n!} + \\ &\tfrac{1}{4M^2}\quad \left(\left(\int_X \Lambda_\omega\dot\omega ||\Omega||_\omega \tfrac{\omega^3}{3!}\right)^2 + \left(\int_X \Lambda_\omega \dot b ||\Omega||_\omega \tfrac{\omega^3}{3!}\right)^2\right)\left(\right.\end{aligned} \tag{4.4.3}$$

where we have used the Lefschetz decomposition:

$$\dot\omega = \dot\omega_0 + \tfrac{1}{n}(\Lambda_\omega\dot\omega)\omega, \quad \dot b = \dot b_0 + \tfrac{1}{n}(\Lambda_\omega \dot b)\omega. \tag{4.4.4}$$

The group of automorphisms of E (see Definition 2.1.12) acts naturally on $(\mathcal{L}, \mathbf{\Omega})$ by symplectomorphisms (see [70, Section 5.2]). Let:

$$\mathcal{L}_+ = \{\ell \in \mathcal{L} \mid \omega(\cdot, J\cdot) > 0\} \tag{4.4.5}$$

Then, we have the following result:

Proposition 4.4.1 ([70]). *There exists a subgroup $\mathcal{H}(E) \subset \mathrm{Aut}(E)$ and a Hamiltonian action of $\mathcal{H}(E)$ on $(\mathcal{L}_+, \mathbf{\Omega})$ with moment map μ such that $\mu^{-1}(0)$ are given by liftings $\ell(\omega, b, a)$ such that:*

$$\begin{aligned} F_A \wedge \omega^{n-1} = 0, \quad F_A^{0,2} = 0, \\ d(||\Omega||_\omega \omega^{n-1}) = 0 \\ H_0 + d^c\omega + CS(A) - CS(A_0) - d\langle A \wedge A_0\rangle + db = 0. \end{aligned} \tag{4.4.6}$$

where $A = A_0 + a$.

Remark 4.4.2. *From the last equation, by derivating and using* (4.4.1)*, we obtain:*

$$dd^c\omega + \langle F_A \wedge F_A\rangle = 0, \tag{4.4.7}$$

hence we obtain a moment map interpretation of the solutions to (3.1.14) *restricted to the isomorphism class of E, and where moreover we vary the holomorphic structure on the bundle.*

Although many aspects of the global geometry of the moduli space of solutions:

$$\mathcal{M}_{HS}(E) = \mu^{-1}(0)/\mathcal{H}(E) \tag{4.4.8}$$

remain widely open, here we focus on the pointwise behaviour of the moduli metric, for which we first study the (formal) tangent space $T_{[\ell]}\mathcal{M}_{HS}(E)$. The linearization of (4.4.6) is given by:

$$\begin{aligned} d_A\dot{a} \wedge \omega^{n-1} + (n-1)F_A \wedge \dot{\omega} \wedge \omega^{n-2} = 0, \\ d(||\Omega||_\omega((n-1)\dot{\omega} \wedge \omega^{n-2} - \tfrac{1}{2}(\Lambda_\omega\dot{\omega})\omega^{n-1})) = 0, \\ \overline{\partial}_A\dot{a}^{0,1} = 0, \\ d^c\dot{\omega} + 2\langle\dot{a} \wedge F_A\rangle - d\dot{b} = 0. \end{aligned} \tag{4.4.9}$$

Now, under the technical assumption Condition A (see [70, Section 6.1]) each class in $T_{[\ell]}\mathcal{M}_{HS}(E)$ is gauge-fixed by a suitable harmonic representative $(\dot{\omega}, \dot{b}, \dot{a}) \in [\dot{\ell}]$, satisfying, moreover:

$$\begin{aligned} d(||\Omega||_\omega((n-1)\dot{b}^{1,1} \wedge \omega^{n-2} - \tfrac{1}{2}(\Lambda_\omega\dot{b})\omega^{n-1})) = 0 \\ d_AJ\dot{a} \wedge \omega^{n-1} - (n-1)F_A \wedge \dot{b} \wedge \omega^{n-2} = 0. \end{aligned} \tag{4.4.10}$$

The space of solutions of the joint systems (4.4.9), (4.4.10) is complex with respect to $\mathbf{J}$ and inherits the structure $(\mathbf{g}, \mathbf{J}, \mathbf{\Omega})$ [70, Theorem 5.18], but the metric $\mathbf{g}$ is possibly degenerate. Given that a solution to the Hull-Strominger system in particular determines a holomorphic principal bundle $(P^c, \overline{\partial}_A)$, Therefore there is a there is a fibration on moduli space:

$$\mathcal{M}_{HS}(E) \xrightarrow{p} H^1(X, \mathcal{O}_G^*), \tag{4.4.11}$$

where $G = K^c$. The fibers of this map are predicted to be Kähler, that is, the metric above, when restricted to the vertical space of this map is positive-definite (see the discussion of

[70, Section 6] and references therein). A vector tangent to a fibre of p at $T_{[\ell]}\mathcal{M}_{HS}$ is of the form:

$$(\dot{\omega}, \dot{b}, -Jd_A s + d_A s') \quad , \quad s, s' \in \Omega^0(\mathrm{ad}\, P) \tag{4.4.12}$$

Then, the following remarkable result express the moduli metric $\mathbf{g}$ restricted to the fibres of p in terms of cohomological data:

Proposition 4.4.3 ([70])**.** *Let:*

$$\dot{\mathfrak{a}} = [\dot{\omega} - 2\langle s, F_A\rangle] + i[\dot{b}^{1,1} - 2\langle s', F_A\rangle] \in H^{1,1}_A(X, \mathbb{C}) \tag{4.4.13}$$

$$\dot{\mathfrak{b}} = (n-1)||\Omega||_\omega \Big((\dot{\omega}_0 + i\dot{b}^{1,1}_0) \wedge \omega^{n-2} - \tfrac{1}{2}\Lambda_\omega(\dot{\omega} + i\dot{b}^{1,1})\omega^{n-1}\Big)\Big(\in H^{n-1,n-1}_{BC}(X, \mathbb{C}), \tag{4.4.14}$$

where ω_0, b_0 stand for the primitive components in:

$$\dot{\omega} = \dot{\omega}_0 + \tfrac{1}{n}(\Lambda_\omega\dot{\omega})\omega, \quad \dot{b}^{1,1} = \dot{b}^{1,1}_0 + \tfrac{1}{n}(\Lambda_\omega\dot{b}^{1,1})\omega, \tag{4.4.15}$$

and consider the dilaton functional:

$$M = \int_X ||\Omega||_\omega \tfrac{\omega^n}{n!}. \tag{4.4.16}$$

Moreover, assume Condition A holds, and $h^0(\mathrm{ad}\, P^c, \overline{\partial}_A) = 0$. Then, the moduli metric $\mathbf{g}$ restricted to a fibre F of the map (4.4.11) *is given by:*

$$\mathbf{g}|_F = \frac{1}{2M}\left(\frac{1}{2M}(\mathrm{Re}\,\dot{\mathfrak{a}}\cdot\mathfrak{b})^2 - \mathrm{Re}\,\dot{\mathfrak{a}}\cdot\mathrm{Re}\,\dot{\mathfrak{b}} + \frac{1}{2M}(\mathrm{Im}\,\dot{\mathfrak{a}}\cdot\mathfrak{b})^2 - \mathrm{Im}\,\dot{\mathfrak{a}}\cdot\mathrm{Im}\,\dot{\mathfrak{b}}\right)\Bigg(\tag{4.4.17}$$

where $\mathfrak{b} = \frac{1}{(n-1)!}[||\Omega||_\omega\omega^{n-1}] \in H^{n-1,n-1}_{BC}(X, \mathbb{R})$ is the balanced class determined by a solution to the Hull-Strominger system.

Observe that the fibre moduli metric $\mathbf{g}|_F$ is hermitian with respect to the natural real structure of the complexified classes (4.4.13),(4.4.14) .

4.4.2 The moduli space metric on locally homogeneous manifolds

In this Section we discuss some features of the moduli space metric. First, we prove the positivity of the fibre moduli metric $\mathbf{g}|_F$ in a simplified but non-trivial situation. For this, let $X = (\Gamma\backslash G, J, \Omega)$ be a complex locally homogenenous manifold endowed with an invariant holomorphic volume form, and let V_0, V_1 be holomorphic line bundles on X. Let (ω, h_0, h_1) be a solution to the Hull-Strominger system:

$$\begin{aligned} F_{h_i} \wedge \omega^{n-1} &= 0, \quad i = 0, 1, \\ d(||\Omega||_\omega\omega^{n-1}) &= 0, \\ dd^c\omega - \alpha F_{h_0} \wedge F_{h_0} + \alpha F_{h_1} \wedge F_{h_1} &= 0, \end{aligned} \tag{4.4.18}$$

for $\alpha \in \mathbb{R}$, where we have formally substituted $T^{1,0}$ by V_0, and we assume ω and F_{h_i} are invariant forms. Moreover, we consider deformations of V_i given by one-parameter families:

$$V_i^t = V_i \otimes L_t, \quad t \in \mathbb{R} \tag{4.4.19}$$

where $L_0 = \mathcal{O}_X$ and L_t is a line bundle of homogeneous type. We endow L_t with the standard hermitian metric, and therefore:

$$\overline{\partial}_{V_i^t} = \overline{\partial}_{V_i} + \alpha_i^t, \tag{4.4.20}$$

$$a_i^t = \alpha_i^t - \overline{\alpha_i^t}, \tag{4.4.21}$$

$$F_i^t = F_{h_i} + \partial\alpha_i^t - \overline{\partial\alpha_i^t}, \tag{4.4.22}$$

where a_i^t stand for the variations of the connection 1-forms, and $\alpha_i^t \in \Omega_X^{0,1}$ are invariant forms satisfying $\overline{\partial}\alpha_i^t = 0$.

Definition 4.4.4. *A tuple* $(\dot{\omega}, \dot{b}, \dot{a}_0, \dot{a}_1)$ *solving the lineralized systems* (4.4.9), (4.4.10) *on* (X, Ω, V_0, V_1) *is called* invariant *if* $\dot{\omega}, \dot{b} \in \Omega_X^{1,1}$ *are invariant forms, and* $\dot{a}_0, \dot{a}_1$ *generate a one-parameter deformation of the form* (4.4.19).

Under mild assumptions, it is easy to see that the metric $\mathbf{g}$ given by (4.4.3) is positive-definite restricted to invariant deformations that fix the holomorphic class of V_0, V_1. Indeed, assume the cohomological assumption:

$$H^{0,1}_{\overline{\partial}}(X) = H^{0,1}_{\overline{\partial}}(\mathfrak{g}, J), \tag{4.4.23}$$

then, an infinitesimal invariant deformation of V_i that fixes the holomorphic structure is given by:

$$\dot{a}_i^{0,1} = \overline{\partial}s_i, \tag{4.4.24}$$

for some smooth function s_i that is invariant, that is, constant. Hence $\dot{a}_i^{0,1} = 0$. Then, using the invariant assumption, the formula (4.4.3) simplifies to:

$$\begin{aligned}
\mathbf{g}((\dot{\omega}, \dot{b}, 0), (\dot{\omega}, \dot{b}, 0)) &= \tfrac{1}{2}(|\dot{\omega}_0|^2 + |\dot{b}_0|^2) \\
&\quad - \tfrac{1}{2}\left(\tfrac{1}{2} - \tfrac{n-1}{n}\right)\left(|\Lambda_{\omega_0}\dot{\omega}|^2 + |\Lambda_{\omega_0}\dot{b}|^2\right) + \tfrac{1}{4}(|\Lambda_{\omega_0}\dot{\omega}|^2 + |\Lambda_{\omega_0}\dot{b}|^2) \\
&= \tfrac{1}{2}(|\dot{\omega}_0|^2 + |\dot{b}_0|^2) + \tfrac{1}{2n}(|\Lambda_{\omega_0}\dot{\omega}|^2 + |\Lambda_{\omega_0}\dot{b}|^2) \\
&= \tfrac{1}{2}(|\dot{\omega}|^2 + |\dot{b}|^2) > 0.
\end{aligned}$$

In the next result, we give a conceptual interpretation of the positivity of the moduli metric in fibres of (4.4.11) in terms of the cohomological formula (4.4.17). One should take this rather heuristically, as some of the technical conditions involved in the construction of the moduli space for Hull-Strominger are not satisfied here (see Remark 4.4.6).

Proposition 4.4.5. *Assume* $H^{0,1}_{\overline{\partial}}(X) = H^{0,1}_{\overline{\partial}}(\mathfrak{g}, J)$. *Then, the fibre moduli metric* $\mathbf{g}|_F$ *given by* (4.4.17) *is positive definite restricted to invariant deformations of the system* (4.4.9),(4.4.10).

Proof. Let $(\dot{\omega}, \dot{b}, \dot{a}_0, \dot{a}_1)$ be an invariant infinitesimal deformation of the Hull-Strominger system. The condition of being tangent to the fibres implies that the isomorphism classes of V_i are fixed. Therefore:

$$\dot{a}_i^{0,1} = \dot{\alpha}_i = \overline{\partial}s_i, \tag{4.4.25}$$

where $s_i \in \Omega^0(\mathrm{End}\, V_i) \cong \mathcal{C}^\infty_X$. Using the assumption, we can choose s_i to be invariant, that is, constant, hence $\dot{a}_i = 0$. Therefore, without loss of generality, we may choose $s_0 = s_1 = 0$, so that the complexified variations of the cohomological classes read:

$$\dot{\mathfrak{a}} = [\dot{\omega} + i\dot{b}], \tag{4.4.26}$$

and:

$$\begin{aligned}
\dot{\mathfrak{b}} &= \tfrac{1}{(n-1)!}\left(-\tfrac{1}{2}||\Omega||_\omega(\Lambda_\omega\dot{\omega} + i\dot{b})\omega^{n-1} + (n-1)||\Omega||_\omega \omega^{n-2}\wedge\dot{\omega}\right)\Big(\\
&= -\tfrac{||\Omega||}{2}\frac{\int_X \frac{1}{(n-1)!}\omega^{n-1}\wedge(\dot{\omega}+i\dot{b})}{\int_X \frac{\omega^n}{n!}}\frac{\omega^{n-1}}{(n-1)!} + \tfrac{||\Omega||_\omega}{(n-2)!}\omega^{n-2}\wedge(\dot{\omega}+i\dot{b}).
\end{aligned}$$

As the Formula (4.4.17) for $\mathbf{g}|_F$ is hermitian, we only need to check positivity on the real part, and the imaginary part is analogous. Therefore, we compute:

$$\begin{aligned}
\tfrac{1}{2M}\left(\tfrac{1}{2M}(\mathrm{Re}\,\dot{\mathfrak{a}}\cdot\mathfrak{b}) - \mathrm{Re}\,\dot{\mathfrak{a}}\cdot\mathrm{Re}\,\dot{\mathfrak{b}}\right)\Big(&= \tfrac{1}{2M}\quad\tfrac{||\Omega||_\omega}{2}\left(\left(\int_X \tfrac{1}{(n-1)!}\omega^{n-1}\wedge\dot{\omega}\right)^2\right) \\
&+ \tfrac{1}{2M}\quad\tfrac{||\Omega||_\omega}{2}\left(\left(\int_X \tfrac{1}{(n-1)!}\omega^{n-1}\wedge\dot{\omega}\right)^2 - ||\Omega||_\omega\int_X \tfrac{1}{(n-2)!}\omega^{n-2}\wedge\dot{\omega}^2\right) \\
&= \frac{1}{2\left(\int_X \frac{\omega^n}{n!}\right)^2}\quad\left(\left(\int_X \tfrac{\omega^{n-1}\wedge\dot{\omega}}{(n-1)!}\right)^2 - \left(\int_X \tfrac{\omega^n}{n!}\right)\left(\int_X \tfrac{\omega^{n-2}\wedge\dot{\omega}^2}{(n-2)!}\right)\right)\Big(
\end{aligned}$$

What remains is a linear algebra argument. We show that the last expression is positive. With respect to an invariant frame $\{X_i\}$ of $T^{1,0}$, we consider the hermitian matrices H and T representing ω and $\dot{\omega}$:

$$H_{i\bar{j}} = \omega(X_i, X_{\bar{j}}), \quad T_{i\bar{j}} = \dot{\omega}(X_i, X_{\bar{j}}). \tag{4.4.27}$$

We write $\det(H + tT) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$. Then, by differentiating, we have:

$$\int_X \frac{\omega^n}{n!} = p_0 \int_X i\Omega\wedge\overline{\Omega} \tag{4.4.28}$$

$$\int_X \frac{\omega^{n-1}}{(n-1)!}\wedge\dot{\omega} = p_1 \int_X i\Omega\wedge\overline{\Omega} \tag{4.4.29}$$

$$\int_X \frac{\omega^{n-2}}{(n-2)!}\wedge\dot{\omega}^2 = 2p_2 \int_X i\Omega\wedge\overline{\Omega} \tag{4.4.30}$$

For simplicity, we assume Ω is rescaled so that $\int_X i\Omega\wedge\overline{\Omega} = 1$. Finally, let $H^{\frac{1}{2}}$ be a hermitian square root of H. Then:

$$\det(H + tT) = \det(H^{\frac{1}{2}})\det(I + tH^{-\frac{1}{2}}TH^{-\frac{1}{2}})\det(H^{\frac{1}{2}}) \tag{4.4.31}$$

where $H^{-\frac{1}{2}}TH^{-\frac{1}{2}}$ is a new hermitian matrix of real eigenvalues, say, λ_i, for $i = 1, \ldots, n$. Then, we get:

$$p_1^2 - 2p_0p_2 = \det(H)^2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) > 0. \tag{4.4.32}$$

□

Remark 4.4.6. *We stress that some technical conditions involved in the construction of* $\mathcal{M}_{HS}(E)$ *may not be satisfied in this situation, e.g.* $h^0(\mathrm{ad}\ P) = 0$*, however here* $\mathrm{ad}\ P \cong \mathcal{O}_X^{\oplus 2}$*. Nevertheless, we can still study the formal expression for the metric* (4.4.17)*. In Proposition 4.4.5, the result agrees with the physical prediction of positivity, suggesting a larger range of applicability of the construction in [70, Section 6] may be possible.*

The metric $\mathbf{g}$ on the moduli space $\mathcal{M}_{HS}(E)$ need not be Kähler along directions transversal to the fibre of (4.4.11). We now discuss about the degeneracy and signature of the moduli metric through explicit Examples.

Let X be the Iwasawa manifold (see Section 4.2.1). For simplicity, we will study the invariant deformations of the Hull-Strominger system with a single line bundle, that is, $P = \mathrm{Fr}\ L_0$, where L_0 is a smooth line bundle. Explicitly, let the pair (ω, A) of a hermitian metric on X and a principal connection on L_0 solve:

$$\begin{aligned} F_A \wedge \omega^2 = 0, \quad F_A^{0,2} = 0, \\ d(||\Omega_0||_\omega \omega^2) = 0, \\ dd^c\omega - \alpha F_A \wedge F_A = 0. \end{aligned} \tag{4.4.33}$$

We assume that $\omega, F_A \in \Omega_X^{1,1}$ are invariant forms. Then, consider the joint linearized systems (4.4.9),(4.4.10) on (X, Ω_0, L_0), and look for deformations consisting of triples $(\dot\omega, \dot b, \dot a)$ of invariant forms. We do not fix the holomorphic structure $(L_0, \overline{\partial}_A)$. First, we observe that the equations:

$$db^{2,0} = db^{0,2} = 0 \tag{4.4.34}$$

decouple from the system as a consequence of the structure equations (4.2.4), and we will not take them into account in what follows. Moreover, since any invariant hermitian metric is balanced [1], the equations corresponding to the linearization of the conformally balanced equation hold. Introducing $\gamma = \dot\omega + i\dot b^{1,1}$, the resulting equations are:

$$\begin{aligned} F_A \wedge \omega \wedge \gamma = 0, \\ i\overline{\partial}\gamma - 2\alpha F_A \wedge \dot a^{0,1} = 0, \\ \overline{\partial}\dot a^{0,1} = 0. \end{aligned} \tag{4.4.35}$$

We write the vector space V of (invariant) solutions to this system fitting in a short exact sequence:

$$0 \longrightarrow W \longrightarrow V \longrightarrow U \longrightarrow 0, \tag{4.4.36}$$

where W corresponds to the solutions that fix the holomorphic structure $(L_0, \overline{\partial}_A)$. By the same argument as in the preceeding Section, this corresponds, in the invariant ansatz, to $\dot a^{0,1} = 0$. We parametrize γ by the expression:

$$\gamma = \textstyle\sum_{ij} g_{ij}\omega_{i\bar{j}}, \tag{4.4.37}$$

where $g_{ij} \in \mathbb{C}$. In the next result, we study the deformations around a fixed solution:

Proposition 4.4.7. *Let $\omega = \omega_0$ be as in (4.2.5), and A such that $F_A = F(m,n,p,0)$, as in (4.2.8). Then:*

1. *The vector space W is given by:*

$$\dot{a}^{0,1} = 0, \tag{4.4.38}$$

$$\gamma \in \langle \omega_{i\bar{j}},\ j \neq 3 \mid m(g_{11} - g_{22}) + n(g_{12} + g_{21}) + ip(g_{21} - g_{12}) = 0 \rangle_{\mathbb{C}}. \tag{4.4.39}$$

2. *A splitting $V = W \oplus \tilde{U}$ of (4.4.36) is given by $\tilde{U}$ generated by the solutions:*

$$(\gamma_1, \dot{a}_1^{0,1}) = (2\pi\alpha((n+ip)\omega_{1\bar{3}} - m\omega_{2\bar{3}}), i\omega_{\bar{1}}), \tag{4.4.40}$$

$$(\gamma_2, \dot{a}_2^{0,1}) = (-2\pi\alpha(m\omega_{1\bar{3}} + (n-ip)\omega_{2\bar{3}}), i\omega_{\bar{2}}). \tag{4.4.41}$$

3. *The moduli space metric $\mathbf{g}$ on $\mathcal{M}_{HS}(E)$ restricted to V is positive semi-definite, and such that $\mathbf{g}|_{\tilde{U}} = 0$.*

Proof. For the first item, setting $\dot{a}^{0,1} = 0$, we obtain the system:

$$\begin{aligned} F_A \wedge \omega_0 \wedge \gamma &= 0, \\ \overline{\partial}\gamma &= 0. \end{aligned} \tag{4.4.42}$$

We obtain basis of solutions to the second equation directly from the structure equations (4.2.4), and is given by $\langle \omega_{i\bar{j}},\ j \neq 3 \rangle$. The linear equation in (4.4.39) comes from imposing the first equation in (4.4.42). For the second item, the equation $\overline{\partial}\dot{a}^{0,1}$ implies $\dot{a}^{0,1} \in \langle \omega_{\bar{1}}, \omega_{\bar{2}} \rangle_{\mathbb{C}}$. One can check then that $(\gamma_i, \dot{a}_i^{0,1})$ above solve the system (4.4.35). For the last item, first note that by Proposition 4.4.5, $\mathbf{g}|_W$ is positive-definite. Moreover, $\mathbf{g}(W, \tilde{U}) = 0$. To see this, we denote $(\dot{\omega} + i\dot{b}, 0)$ for element in W and $(\gamma_i, \dot{a}_i^{0,1})$ as above for an element in $\tilde{U}$, and use:

$$\Lambda_{\omega_0}\gamma_1 = \Lambda_{\omega_0}\gamma_2 = 0, \tag{4.4.43}$$

$$(\dot{\omega} \wedge \operatorname{Re}\gamma_i + \dot{b} \wedge \operatorname{Im}\gamma_i) \wedge \omega_0 = 0. \tag{4.4.44}$$

in the expression of $\mathbf{g}$ (4.4.3). It is easy to see that because of (4.4.43), only the second line is non-trivial, but it does vanish due to (4.4.44). Finally, we use:

$$\dot{a}_j = \dot{a}_j^{0,1} - \overline{\dot{a}_j^{0,1}} = i(\omega_j + \omega_{\bar{j}}), \quad j = 1,2 \tag{4.4.45}$$

to compute the expression for the metric on $\tilde{U}$:

$$\begin{aligned} \mathbf{g}|_{\tilde{U}} &= -\tfrac{1}{M}\int_X -\alpha\dot{a}_i \wedge J\dot{a}_j \wedge ||\Omega||_{\omega_0}\tfrac{\omega_0^2}{2} - \tfrac{1}{2M}\int_X (\operatorname{Re}\gamma_i \wedge \operatorname{Re}\gamma_j + \operatorname{Im}\gamma_i \wedge \operatorname{Im}\gamma_j) \wedge ||\Omega||_{\omega_0}\omega_0 \\ &= -4\alpha\delta_{ij} + 4\alpha\delta_{ij} = 0. \end{aligned}$$

$\square$

To finish this Section, we look at an Example with different behaviour, where the moduli metric $\mathbf{g}$ is not completely degenerate along transversal directions to the fibre of (4.4.11) (compare with Proposition 4.4.7(3). Let $X = (\Gamma\backslash H_3, J)$, as considered in Section 4.2.2. Let (ω_0, Ω_0) be as given in (4.2.21), and A is the Bismut connection of ω_0. Explicitly, in the frame $\{X_i\}$ the structure equations (4.2.19) are written, we have:

$$A = (\omega_3 - \omega_{\overline{3}}) \begin{pmatrix} -1 & & \\ & 1 & \\ & & 0 \end{pmatrix}. \tag{4.4.46}$$

Then, from Proposition (4.2.7), the pair (ω_0, A) solves the Hull-Strominger system:

$$\begin{aligned} F_A \wedge \omega_0^2 = 0, \quad F_A^{0,2} &= 0, \\ d(||\Omega_0||_{\omega_0}\omega_0^2) &= 0, \\ dd^c\omega - \alpha \mathrm{tr}\, F_A \wedge F_A &= 0, \end{aligned} \tag{4.4.47}$$

with $\alpha = \frac{1}{8}$. We consider the system of infinitesimal deformations of (4.4.47) given by (4.4.9), (4.4.10). Then, introducing $\gamma = \dot{\omega} + i\dot{b}^{1,1}$, the resulting system is:

$$\begin{aligned} \partial_A \dot{a}^{0,1} \wedge \omega_0^2 + 2F_A \wedge \omega_0 \wedge \gamma &= 0, \\ d(\gamma \wedge \omega_0 - \tfrac{1}{2}(\Lambda_{\omega_0}\gamma)\omega_0^2) &= 0, \\ \overline{\partial}_A \dot{a}^{0,1} &= 0, \\ i\overline{\partial}\gamma - \partial\dot{b}^{0,2} - 2\alpha \mathrm{tr}\, F_A \wedge \dot{a}^{0,1} &= 0. \end{aligned} \tag{4.4.48}$$

Note the difference with (4.4.35), due to the fact that here X is not complex-parallelizable. We now look for invariant solutions of this system. We will assume moreover that the infinitesimal deformation of A still preserves the standard $SU(3)$ structure of $\mathbb{C}^3$. These sit in an exact sequence:

$$0 \longrightarrow W \longrightarrow V \longrightarrow U \longrightarrow 0, \tag{4.4.49}$$

where W stand for the infinitesimal deformations that do not vary the holomorphic class of $(T^{1,0}, \overline{\partial}_A)$.

Proposition 4.4.8. *1. The vector space W is generated by:*

$$\dot{a}^{0,1} = 0, \tag{4.4.50}$$

$$\gamma + \dot{b}^{0,2} \in \langle \omega_{1\overline{1}} + \omega_{2\overline{2}}, \omega_{1\overline{2}}, \omega_{2\overline{1}}, \omega_{1\overline{3}}, \omega_{2\overline{3}}, \omega_{\overline{12}}, -i\omega_{3\overline{1}} + \omega_{\overline{13}}, -i\omega_{3\overline{2}} + \omega_{\overline{23}} \rangle_{\mathbb{C}}. \tag{4.4.51}$$

2. There is a splitting $V = W \oplus \tilde{U}$ of (4.4.36), where $\tilde{U}$ is generated by the solutions:

$$(\gamma, \dot{a}) \in \langle (2i\omega_{3\overline{1}}, e_1), (i\omega_{3\overline{1}}, e_2), (2i\omega_{3\overline{2}}, e_3), (i\omega_{3\overline{2}}, e_4), (2i\omega_{3\overline{3}}, e_5), (i\omega_{3\overline{3}}, e_6) \rangle_{\mathbb{C}}, \tag{4.4.52}$$

where e_j are given by (4.4.54).

3. The moduli space metric $\mathbf{g}$ on $\mathcal{M}_{HS}(E)$ is positive semi-definite.

Proof. First, we look at the equation $\overline{\partial}_A \dot{a}^{0,1} = 0$. Writing:

$$\dot{a}^{0,1} = \textstyle\sum_i \omega_{\bar{i}} a_i, \tag{4.4.53}$$

where $a_i \in M_{3\times 3}(\mathbb{C})$ are constant matrices. Then, the equation imposes the constraint:

$$\begin{aligned} 0 = \overline{\partial}_A \dot{a}^{0,1} &= \overline{\partial} \dot{a}^{0,1} + [A^{0,1} \wedge \dot{a}^{0,1}] \\ &= \textstyle\sum_i \omega_{\overline{3i}} [i_{\overline{X_3}} A, a_i]. \end{aligned}$$

Using (4.4.46), this is equivalent to a_1, a_2 being diagonal, and there is no contraint on a_3. Then, using A is diagonal too, we obtain:

$$\begin{aligned} \partial_A \dot{a} &= \partial \omega_{\bar{3}} a_3 - \omega_{\bar{3}} \wedge [A^{1,0}, a_3] \\ &= -(\omega_{1\bar{1}} - \omega_{2\bar{2}}) a_3 + \omega_{3\bar{3}} \left[\begin{pmatrix} -1 & & \\ & 1 & \\ & & 0 \end{pmatrix}, a_3 \right]. \end{aligned}$$

Then, by the first equation in (4.4.47) and using that F_A is diagonal, from the off-diagonal components, we get that a_3 is diagonal too.

Now, we introduce the notation:

$$e_{2j-1} = \omega_{\bar{j}} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_{2j} = \omega_{\bar{j}} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad j = 1, 2, 3 \tag{4.4.54}$$

Since we are assuming the infinitesimal deformations $\dot{a} = \dot{a}^{0,1} - (\dot{a}^{0,1})^*$ of A keep the natural $\mathfrak{su}(3)$ structure of $\mathbb{C}^3$ fixed, it is clear that $\dot{a}^{0,1}$ is a complex linear combination of the above elements. We claim that any infinitesimal deformation of this type varies the holomorphic class of $(T^{1,0}, \overline{\partial}_A)$. Indeed, otherwise there is a smooth section $s \in \Gamma(T^{1,0})$ such that $\dot{a}^{0,1} = \overline{\partial}_A s$. By an averaging process analogous to the one in the proof of Lemma 4.1.12, and using that A and $\dot{a}^{0,1}$ are invariant, we can assume that s is an invariant section, hence it is identified with a matrix $(s_{ij}) \in \Gamma(T^{1,0})$. But then:

$$\overline{\partial}_A s = [A^{0,1}, s] = \omega_{\bar{3}} \begin{pmatrix} 0 & 2s_{12} & s_{13} \\ -2s_{21} & 0 & -s_{23} \\ -s_{31} & s_{32} & 0 \end{pmatrix}, \tag{4.4.55}$$

which clearly is incompatible with the expression for $\dot{a}^{0,1}$. We now give a basis of the solutions to (4.4.47). From the argument above, a solution in W has $\dot{a}^{0,1} = 0$. Then, the system (4.4.48) can be solved explicitly and we have that $\gamma + \dot{b}^{0,2}$ is in:

$$\langle \omega_{1\bar{1}} + \omega_{2\bar{2}}, \omega_{1\bar{2}}, \omega_{2\bar{1}}, \omega_{1\bar{3}}, \omega_{2\bar{3}}, \omega_{\overline{12}}, -i\omega_{3\bar{1}} + \omega_{\overline{13}}, -i\omega_{3\bar{2}} + \omega_{\overline{23}} \rangle_{\mathbb{C}}, \tag{4.4.56}$$

and we get the first item. Now, we obtain further solutions index by e_j:

$$\langle (i\omega_{3\bar{1}}, e_1), (i\omega_{3\bar{1}}, 2e_2), (i\omega_{3\bar{2}}, e_3), (i\omega_{3\bar{2}}, 2e_4), (i\omega_{3\bar{3}}, e_5), (i\omega_{3\bar{3}}, 2e_6) \rangle_{\mathbb{C}}, \tag{4.4.57}$$

where we have used $\alpha = \frac{1}{8}$ for the solution (ω_0, A) to (4.4.47). These span a space complementary to W. Therefore the space generated $\tilde{U}$ is a splitting for (4.4.49).

Finally, we compute the moduli metric $\mathbf{g}$ restricted to the vector space V. Using the invariant ansatz, the formula (4.4.3) simplifies to:

$$\begin{aligned}\mathbf{g}((\gamma,\dot a),(\gamma,\dot a)) = &\; +\alpha\int_X \operatorname{tr}(\dot a\wedge J\dot a)\wedge\tfrac{\omega_0^2}{2}+\tfrac{1}{2}(|\dot\omega_0|^2+|\dot b_0|^2)\\ &-\tfrac{1}{12}(|\Lambda_{\omega_0}\dot\omega|^2+|\Lambda_{\omega_0}\dot b|^2)+\tfrac{1}{4}(|\Lambda_{\omega_0}\dot\omega|^2+|\Lambda_{\omega_0}\dot b|^2)\\ =&\;\tfrac{1}{8}\int_X \operatorname{tr}(\dot a_1\wedge J\dot a_2)\wedge\tfrac{\omega_0^2}{2}+\tfrac{1}{2}(|\Lambda_{\omega_0}\dot\omega|^2+|\Lambda_{\omega_0}\dot b|^2).\end{aligned}$$

This can be computed explicitly for the 14-dimensional vector space V. With respect to the basis obtained joining (4.4.56) and (4.4.57), the resulting metric is:

$$\mathbf{g}|_V = \begin{pmatrix} 4&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&2&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&2&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&2&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&2&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&2&0&-2&-2&0&0&0&0\\ 0&0&0&0&0&0&0&2&0&0&-2&-2&0&0\\ 0&0&0&0&0&0&-2&0&2&2&0&0&0&0\\ 0&0&0&0&0&0&-2&0&2&2&0&0&0&0\\ 0&0&0&0&0&0&0&-2&0&0&2&2&0&0\\ 0&0&0&0&0&0&0&-2&0&0&2&2&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&2&2\\ 0&0&0&0&0&0&0&0&0&0&0&0&2&2 \end{pmatrix}, \tag{4.4.58}$$

whose eigenvalues are: $\{6,6,4,4,2,2,2,2,0,0,0,0,0,0\}$, and the result follows.

□

Remark 4.4.9. *This Example shows a new feature with respect to the previous one. While in both cases the moduli metric is positive semi-definite, here there is a positive direction orthogonal to the fibre of* (4.4.11)*, given by the eigenvector* $(2i\omega_{3\bar 3}, e_5+2e_6)$ *with eigenvalue* $+4$*.*

Chapter 5

Hermite-Einstein metrics and Futaki invariants on Courant algebroids

The aim of this Chapter is to show how Generalized Geometry, as developed in Chapter 2, is a useful framework for studying the Hull-Strominger system. For this, here we retake the theory of generalized hermitian metrics applied to Bott-Chern algebroids. The study of the curvature properties of these metrics culminates in one of the main results of this Thesis, Theorem 5.3.7, recasting solutions to the Hull-Strominger system as special metrics on Courant algebroids satisfying a condition of Hermite-Einstein type. Building on it, we then use its moment map interpretation to produce new invariants that potentially obstruct the existence of solutions. This Chapter is based on [65], which we follow closely, building on previous work in [61, 67, 69].

5.1 Generalized hermitian metrics on Bott-Chern algebroids

In Section 2.2.2, we introduced generalized hermitian metrics for general Courant algebroids and discussed Bott-Chern algebroids (see Definitions 2.2.13, 2.2.15) as the family of string algebroids where these exist and behave properly with respect to the hermitian structure of the manifold. A fundamental feature we stress here is that generalized hermitian metrics are possibly of indefinite signature (see Remark 2.2.16).

Let $X = (M, J)$ be a complex manifold of complex dimension n, and let (E, P_K, ρ_{P_K}) be a real string algebroid, where P_K is a principal K-bundle for a Lie group K with quadratic Lie algebra $(\mathfrak{k}, \langle \cdot, \cdot \rangle)$. Recall that a generalized metric $V_+ \subset E$ compatible with J determines a representative $E_{P_K,H,A} \cong E$, where H, A satisfy (2.1.26), and a holomorphic string algebroid:

$$\ell^{\perp}/\ell = \mathcal{Q}_{\ell} \xrightarrow{\cong} V_- \otimes \mathbb{C} \subset E_{P_K,H,A} \otimes \mathbb{C} \tag{5.1.1}$$

induced by π_-. Here $\ell = V_+ \otimes \mathbb{C} \cap \rho^{-1}(T^{0,1})$ is a lifting. Moreover, explicitly:

$$V_+ = \{X + g(X),\ X \in T\}, \quad V_- = \{X - g(X) + r,\ X \in T,\ r \in \operatorname{ad} P_K\}. \tag{5.1.2}$$

Hence it inherits a generalized hermitian metric:

$$\mathbf{G}([a],[b]) = -\langle \pi_- a, \overline{\pi_- b}\rangle, \quad a,b \in \Gamma(\mathcal{Q}_\ell). \tag{5.1.3}$$

By abuse of notation, in the sequel we will identify $\mathbf{G}$ and other structures on $\mathcal{Q}_\ell$ with the ones induced in $V_- \otimes \mathbb{C}$ through the isomorphism (5.1.1). Then, the first result of this Section is a computation of the Chern connection of the generalized hermitian metric $\mathbf{G}$. Our result extends *Bismut's Identity* (see [19, Theorem 2.9]), interpreted recently in [67] in the language of exact holomorphic Courant algebroids.

Proposition 5.1.1. *The Chern connection of* $\mathbf{G}$ *induced on* $V_- \otimes \mathbb{C}$ *via the isomorphism* $\mathcal{Q}_\ell \cong V_- \otimes \mathbb{C}$ *is given by:*

$$D^{\mathbf{G}}_X s = \pi_-[\sigma_+ X, s]. \tag{5.1.4}$$

Here, $\sigma_+ X = X + g(X)$ *is the inverse of the isomorphism* $\pi_{|V_+}\colon V_+ \to T$*. More explicitly, via the identification* $V_- \cong T \oplus \operatorname{ad} P_K$*, we have*

$$D^{\mathbf{G}}_X(Y + r) = \nabla^-_X Y - g^{-1}\langle i_X F_A, r\rangle + i_X d_A r - F_A(X,Y), \tag{5.1.5}$$

where $\nabla^- = \nabla^g + \frac{1}{2} g^{-1} d^c\omega$*, for* ∇^g *the Levi-Civita connection of* g*.*

Proof. The right hand side of (5.1.4) defines an orthogonal connection on V_-:

$$\begin{aligned}
D^{\mathbf{G}}_X \mathbf{G}(s,t) &= -D^{\mathbf{G}}_X\langle s,t\rangle \\
&= -\rho(\sigma_+ X)\langle s,t\rangle \\
&= -\langle [\sigma_+ X, s], t\rangle - \langle s, [\sigma_+ X, t]\rangle \\
&= -\langle \pi_-[\sigma_+ X, s], t\rangle - \langle s, \pi_-[\sigma_+ X, t]\rangle \\
&= -\langle D^{\mathbf{G}}_X s, t\rangle - \langle D^{\mathbf{G}}_X t\rangle \\
&= \mathbf{G}(D^{\mathbf{G}}_X s, t) + \mathbf{G}(s, D^{\mathbf{G}}_X t).
\end{aligned}$$

Hence, it extends $\mathbb{C}$-linearly to a $\mathbf{G}$-unitary connection on $V_- \otimes \mathbb{C}$. By the abstract definition of the Dolbeault operator on $\mathcal{Q}_\ell$ given by Proposition 2.2.3 combined with the expression for ℓ above, we have that the $(0,1)$-part of the right hand side of (5.1.4) coincides with $\overline{\partial}_{\mathcal{Q}_\ell}$ of (2.2.11). Formula (5.1.5) follows from [69, Equation (5.10)]. □

Moreover, we recall by Section 2.2 that $\mathcal{Q}_\ell \cong \mathcal{Q}_{P,2i\partial\omega,A^h}$, for the holomorphic principal bundle $P = (P_K^c, \overline{\partial}_A)$, and where A^h stands for the Chern connection of the canonical hermitian reduction $P_K \subset P$. In our next result we compute an explicit formula for the generalized Hermitian metric $\mathbf{G}$ in terms of this isomorphism.

Lemma 5.1.2. *The hermitian isometry* $\psi\colon \mathcal{Q}_{P,2i\partial\omega,A^h} \to V_- \otimes \mathbb{C}$ *induced by Lemma 2.2.8 is given by*

$$\psi(X + r + \xi) = e^{i\omega} X + r - \tfrac{1}{2} e^{-i\omega} g^{-1}\xi.$$

Consequently,

$$\psi^*\mathbf{G}(X + r + \xi, X + r + \xi) = g(X, \overline{X}) + \tfrac{1}{4} g^{-1}(\xi, \overline{\xi}) - \langle r, \overline{r}\rangle,$$

where conjugation in $\operatorname{ad} P$ *is taken with respect to the hermitian reduction* $P_K \subset P$*.*

Proof. The formula for ψ follows by composing the isomorphisms:

$$\mathcal{Q}_{P,2i\partial\omega,A^h} \xrightarrow{e^{i\omega}} \mathcal{Q}_\ell \xrightarrow{\pi_-} V_- \otimes \mathbb{C}. \tag{5.1.6}$$

The first map is given, explicitly, by:

$$X + r + \xi \mapsto [e^{i\omega}(X + r + \xi)] \tag{5.1.7}$$

according to Proposition 2.2.8, for the lifting $\ell = \ell(\omega, 0, 0)$. Hence:

$$\begin{aligned}
\psi(X + r + \xi) &= \pi_-(e^{i\omega}(X + r + \xi)) \\
&= e^{i\omega}X + r + \pi_-(\xi) \\
&= e^{i\omega}X + r + \pi_-(\tfrac{1}{2}(-g^{-1}\xi + xi) + \tfrac{1}{2}(g^{-1}\xi + xi)) \\
&= e^{i\omega}X + r - \tfrac{1}{2}e^{-i\omega}g^{-1}\xi.
\end{aligned}$$

The pullback of $\mathbf{G}$ along ψ follows from:

$$\begin{aligned}
\psi^*\mathbf{G}(X + r + \xi, X + r + \xi) &= -\langle \psi(X + r + \xi), \overline{\psi(X + r + \xi)} \rangle \\
&= -\langle e^{i\omega}X + r - \tfrac{1}{2}e^{-i\omega}\xi, e^{-i\omega}\overline{X} + \overline{r} + \tfrac{1}{2}e^{i\omega}g^{-1}\overline{\xi} \rangle \\
&= -\langle e^{i\omega}X, e^{-i\omega}\overline{X} \rangle - \langle r, \overline{r} \rangle - \tfrac{1}{4}\langle e^{-i\omega}g^{-1}\xi, e^{i\omega}g^{-1}\overline{\xi} \rangle \\
&= -\langle e^{2i\omega}X, \overline{X} \rangle - \langle r, \overline{r} \rangle - \tfrac{1}{4}\langle e^{-2i\omega}g^{-1}\xi, g^{-1}\overline{\xi} \rangle \\
&= -i\omega(X, \overline{X}) - \langle r, \overline{r} \rangle + \tfrac{1}{4}i\omega(g^{-1}\xi, g^{-1}\overline{\xi}) \\
&= g(X, \overline{X}) - \langle r, \overline{r} \rangle + \tfrac{1}{4}g^{-1}(\xi, \overline{\xi}).
\end{aligned}$$

□

Remark 5.1.3. *By the previous lemma, the signature of* $\mathbf{G}$ *is* $(4n + 2l_2, 2l_1)$*, where* (l_1, l_2) *is the signature of* $\langle \cdot, \cdot \rangle : \mathfrak{k} \otimes \mathfrak{k} \to \mathbb{R}$ *and* $n = \dim_{\mathbb{C}} X$*.*

5.2 Curvature of generalized hermitian metrics

In this Section, we compute the full curvature tensor and second Ricci curvature (see Equation (5.2.11)) of the generalized Hermitian metric $\mathbf{G}$ in (5.1.3). The notations are the same as in the previous Section. We will systematically use the identifications $\mathcal{Q}_\ell \cong V_- \otimes \mathbb{C}$ and the isomorphism:

$$V_- \cong T \oplus \operatorname{ad} P_K, \tag{5.2.1}$$

where the isomorphism is given by the explicit expression in (5.1.2). Consider the (possibly) indefinite metric on V_- given by:

$$\langle X + r, X + r \rangle^0 := -\langle X - g(X) + r, X - g(X) + r \rangle = g(X, X) - \langle r, r \rangle. \tag{5.2.2}$$

Then, extending $\mathbb{C}$-linearly $\langle \cdot, \cdot \rangle^0$ to $V_- \otimes \mathbb{C}$, it follows from Definition 2.2.15 that $\mathbf{G}$ is given by

$$\mathbf{G}(s_1, s_2) = \langle s_1, \overline{s_2} \rangle^0. \tag{5.2.3}$$

By Proposition 5.1.1, the Chern connection $D^{\mathbf{G}}$ is the $\mathbb{C}$-linear extension of a $\langle\cdot,\cdot\rangle^0$-orthogonal connection

$$D\colon \Omega^0(V_-) \to \Omega^1(V_-), \tag{5.2.4}$$

and hence to calculate $F_{\mathbf{G}} := F_{D^{\mathbf{G}}}$ it suffices to give a formula for F_D. Explicitly, in terms of the decomposition (5.2.1) we have

$$D_X(Y+r) = \nabla^-_X Y - g^{-1}\langle i_X F_h, r\rangle + d^h_X r - F_h(X,Y). \tag{5.2.5}$$

For the computations, it will be useful to express D in matrix notation as

$$D = \begin{pmatrix} \nabla^- & \mathbb{F}^\dagger \\ -\mathbb{F} & d^\theta \end{pmatrix} \tag{5.2.6}$$

with respect to the splitting $V_- \cong T \oplus \mathrm{ad}\ P_K$, where $\mathbb{F} \in \Omega^1(\mathrm{Hom}(T, \mathrm{ad}\ P_K))$ is the $\mathrm{Hom}(T, \mathrm{ad}\ P_K)$-valued 1-form:

$$(i_X\mathbb{F})(Y) := F_h(X,Y). \tag{5.2.7}$$

and $\mathbb{F}^\dagger \in \Omega^1(\mathrm{Hom}(\mathrm{ad}\ P_K, T))$ is the $\langle\cdot,\cdot\rangle^0$-adjoint operator. It is straightforward to check that it is explicitly given by:

$$(i_X\mathbb{F}^\dagger)(r) = -g^{-1}\langle i_X F_h, r\rangle. \tag{5.2.8}$$

We will use the standard notation R_{∇^-} for the curvature of ∇^- and also $\nabla^{h,-}$ for the covariant derivative induced by A^h and ∇^- on $\Lambda^2 T^* \otimes \mathrm{ad}\ P_K$. In particular,

$$(\nabla^{h,-}_Z F_h)(X,Y) = d^h_Z(F_h(X,Y)) - F_h(\nabla^-_Z X, Y) - F_h(X, \nabla^-_Z Y) \tag{5.2.9}$$

for any triple of vector fields X, Y, Z on M.

Lemma 5.2.1. *The curvature of D is given by*

$$F_D = \begin{pmatrix} R_{\nabla^-} - \mathbb{F}^\dagger \wedge \mathbb{F} & -\mathbb{I}^\dagger \\ \mathbb{I} & [F_h,] - \mathbb{F} \wedge \mathbb{F}^\dagger \end{pmatrix}$$

where

$$\begin{aligned} i_Y i_X \mathbb{F}^\dagger \wedge \mathbb{F}(Z) &= g^{-1}\langle i_Y F_h, F_h(X,Z)\rangle - g^{-1}\langle i_X F_h, F_h(Y,Z)\rangle, \\ i_Y i_X \mathbb{I}(Z) &= (\nabla^{h,-}_Z F_h)(X,Y) - F_h(X, g^{-1} i_Z i_Y d^c\omega) + F_h(Y, g^{-1} i_Z i_X d^c\omega), \\ i_Y i_X \mathbb{F} \wedge \mathbb{F}^\dagger(r) &= F_h(Y, g^{-1}\langle i_X F_h, r\rangle) - F_h(X, g^{-1}\langle i_Y F_h, r\rangle). \end{aligned}$$

Proof. To compute the curvature, we write

$$D = D^0 + \begin{pmatrix} 0 & \mathbb{F}^\dagger \\ -\mathbb{F} & 0 \end{pmatrix}$$

where $D^0 = \nabla^- \oplus d^h$. Then, we have

$$
\begin{aligned}
F_D &= F_{D_0} + d^{D_0}\begin{pmatrix} 0 & \mathbb{F}^\dagger \\ -\mathbb{F} & 0 \end{pmatrix} + \begin{pmatrix} -\mathbb{F}^\dagger \wedge \mathbb{F} & 0 \\ 0 & -\mathbb{F} \wedge \mathbb{F}^\dagger \end{pmatrix} \\
&= \begin{pmatrix} R_{\nabla^-} - \mathbb{F}^\dagger \wedge \mathbb{F} & (d^{h,-}\mathbb{F})^\dagger \\ -d^{h,-}\mathbb{F} & [F_h,] - \mathbb{F} \wedge \mathbb{F}^\dagger \end{pmatrix}
\end{aligned}
\tag{5.2.10}
$$

where $d^{h,-} \colon \Omega^1(\mathrm{Hom}(T, \mathrm{ad}\ P_K)) \to \Omega^2(\mathrm{Hom}(T, \mathrm{ad}\ P_K))$ is the exterior covariant derivative induced by ∇^- and A^h. The explicit formulae for $\mathbb{F}^\dagger \wedge \mathbb{F}$ and $\mathbb{F} \wedge \mathbb{F}^\dagger$ above follow from:

$$
\begin{aligned}
i_Y i_X \mathbb{F}^\dagger \wedge \mathbb{F}(Z) &= i_X \mathbb{F}^\dagger (i_Y \mathbb{F})(Z) - i_Y \mathbb{F}^\dagger (i_X \mathbb{F})(Z) \\
&= -g^{-1}\langle i_X F_h, i_Y \mathbb{F}(Z)\rangle + g^{-1}\langle i_Y F_h, i_X \mathbb{F}(Z)\rangle \\
&= g^{-1}\langle i_Y F_h, F_h(X, Z)\rangle - g^{-1}\langle i_X F_h, F_h(Y, Z)\rangle \\
i_Y i_X \mathbb{F} \wedge \mathbb{F}^\dagger(r) &= i_X \mathbb{F}(i_Y \mathbb{F}^\dagger)(r) - i_Y \mathbb{F}(i_X \mathbb{F}^\dagger)(r) \\
&= F_h(Y, g^{-1}\langle i_X F_h, r\rangle) - F_h(X, g^{-1}\langle i_Y F_h, r\rangle).
\end{aligned}
$$

As for $d^{h,-}\mathbb{F}$, we have:

$$
\begin{aligned}
-i_Y i_X d^{h,-}\mathbb{F}(Z) &= -d^h_X(F_h(Y, Z)) + d^h_Y(F_h(X, Z)) + F_h([X, Y], Z) \\
&\quad + F_h(Y, \nabla^-_X Z) - F_h(X, \nabla^-_Y Z) \\
&= d^h_Z(F_h(X, Y)) + F_h([X, Z], Y) - F_h([Y, Z], X) \\
&\quad + F_h(Y, \nabla^-_X Z) - F_h(X, \nabla^-_Y Z) \\
&= (\nabla^{h,-}_Z F_h)(X, Y) + F_h(Y, T_{\nabla^-}(X, Z)) - F_h(X, T_{\nabla^-}(Y, Z))
\end{aligned}
$$

where T_{∇^-} denotes the torsion tensor of ∇^- and in the second equality we have used the Bianchi identity $d^h F_h = 0$. Our formula for $\mathbb{I}$ follows now from $T_{\nabla^-}(Y, Z) = g^{-1} i_Z i_Y d^c\omega$. $\square$

We next calculate the *second Ricci curvature* of the generalized Hermitian metric $\mathbf{G}$, defined by the expression

$$
S_{\mathbf{G}} \frac{\omega^n}{n} = F_{\mathbf{G}} \wedge \omega^{n-1} \tag{5.2.11}
$$

where ω is the hermitian metric in Proposition 2.2.8. Similarly as before, the skew-hermitian endomorphism $S_{\mathbf{G}} \in \Gamma(\mathrm{End}\ \mathcal{Q}_\ell)$ is given by the $\mathbb{C}$-linear extension of the second Ricci curvature S_D of the connection D. To calculate S_D below, we need the following technical lemma.

Lemma 5.2.2. *Let (M, g) be a Riemannian manifold of even dimension. Let $F \in \Omega^2$ and $H \in \Omega^3$ be differential forms. Then, the Hodge star operator satisfies:*

$$
i_X * (F \wedge *H) = \frac{1}{2}\sum_{i=1}^{m} F(e_i, g^{-1} i_X i_{e_i} H)
$$

for any vector field X and any choice of g-orthonormal frame $\{e_i\}$ of T.

Proof. For e^i the dual frame, one has

$$*(e^i \wedge *\psi) = i_{e_i}\psi$$

for $\psi \in \Omega^p$, and therefore

$$*(e^i \wedge e^j \wedge *\psi) = (-1)^p i_{e_j} i_{e_i}\psi.$$

By bilinearity, we get

$$*(F \wedge *H) = -\sum_{i<j} F(e_i, e_j)H(e_i, e_j, \cdot),$$

and therefore

$$i_X * (F \wedge *H) = -\sum_{i<j} F(e_i, e_j)H(e_i, e_j, X) = \tfrac{1}{2}\sum_i F(e_i, g^{-1} i_X i_{e_i} H).$$

□

Recall that the *Bismut connection* of the hermitian metric g in Proposition 2.2.8 is given by (cf. Proposition 5.1.1)

$$\nabla^B = \nabla - \tfrac{1}{2} g^{-1} d^c\omega. \tag{5.2.12}$$

for ∇^g the Levi-Civita connection of g, which is a unitary connection on $T^{1,0}$ (see Section 1.2.1). Recall also that it induces a well-defined curvature on the anti-canonical bundle $-i\rho_B$, where ρ_B is the *Bismut Ricci form* (1.2.7) . Explicitly, for a choice of g-orthonormal basis $\{e_i\}$ of T at a point, one has:

$$\rho_B(X, Y) = \tfrac{1}{2}\textstyle\sum_j g(R_{\nabla^B}(X, Y)Je_j, e_j). \tag{5.2.13}$$

Proposition 5.2.3. *The second Ricci form S_D of the connection D is given by*

$$S_D = \begin{pmatrix} -g^{-1}(\rho_B + \langle S_h, F_h\rangle) & -\mathbb{S}^\dagger \\ \mathbb{S} & [S_h,] \end{pmatrix}$$

where

$$\mathbb{S}(V) = i_{JV} \quad - d^{h\star}F_h - i_{\theta_\omega^\sharp}F_h + *(F_h \wedge *d^c\omega),$$

for $d^{h\star}$ the adjoint of d^h and $\theta_\omega = Jd^\star\omega$ the Lee form of g.

Proof. In terms of the g-orthonormal frame $\{e_i\}$, the second Ricci form is expressed as:

$$S_D = \tfrac{1}{2}\textstyle\sum_j F_D(e_j, Je_j). \tag{5.2.14}$$

Using this and applying Lemma 5.2.1, we have

$$S_D = \begin{pmatrix} S_{\nabla^-} - \frac{1}{2}\mathbb{F}^\dagger \wedge \mathbb{F}(e_i, Je_i) & -\frac{1}{2}\mathbb{I}^\dagger(e_i, Je_i) \\ \frac{1}{2}\mathbb{I}(e_i, Je_i) & [S_h,] - \frac{1}{2}\mathbb{F} \wedge \mathbb{F}^\dagger(e_i, Je_i) \end{pmatrix} \tag{5.2.15}$$

We first compute:

$$g(i_{Je_i}i_{e_i}\mathbb{F}^\dagger \wedge \mathbb{F}(V), W) = \langle F_h(Je_i, W), F_h(e_i, V)\rangle - \langle F_h(e_i, W), F_h(Je_i, V)\rangle. \qquad (5.2.16)$$

Combining this with the identity (1.2.8)and Proposition 2.2.8, we also obtain

$$\begin{aligned}
g(S_{\nabla^-}(X), Y) &= \tfrac{1}{2}g(R_{\nabla^-}(e_i, Je_i)X, Y) \\
&= \tfrac{1}{2}g(R_{\nabla^B}(X, Y)e_i, Je_i) + \tfrac{1}{4}dd^c\omega(e_i, Je_i, X, Y) \\
&= -\rho_B(X, Y) - \tfrac{1}{4}\langle F_h \wedge F_h\rangle(e_i, Je_i, X, Y) \\
&= -\rho_B(X, Y) - \tfrac{1}{2}\langle i_{e_i}F_h \wedge F_h\rangle(Je_i, X, Y) \\
&= -\rho_B(X, Y) - \langle S_h, F_h(X, Y)\rangle + \tfrac{1}{2}\langle i_{e_i}F_h \wedge i_{Je_i}F_h\rangle(X, Y) \\
&= -\rho_B(X, Y) - \langle S_h, F_h(X, Y)\rangle + \tfrac{1}{2}\langle F_h(e_i, X), F_h(Je_i, Y)\rangle \\
&\quad - \tfrac{1}{2}\langle F_h(e_i, Y), F_h(Je_i, X)\rangle \\
&= -\rho_B(X, Y) - \langle S_h, F_h(X, Y)\rangle + \tfrac{1}{2}g(i_{Je_i}i_{e_i}\mathbb{F}^\dagger \wedge \mathbb{F}(X), Y),
\end{aligned}$$

as claimed. Using again Proposition 2.2.8, in particular $F_h = F_h^{1,1}$, we also have

$$\begin{aligned}
i_{Je_i}i_{e_i}\mathbb{F} \wedge \mathbb{F}^\dagger(r) &= F_h(Je_i, g^{-1}\langle i_{e_i}F_h, r\rangle) - F_h(e_i, g^{-1}\langle i_{Je_i}F_h, r\rangle) \\
&= -F_h(e_i, Jg^{-1}\langle i_{e_i}F_h, r\rangle) - F_h(e_i, g^{-1}\langle F_h(Je_i,), r\rangle) \\
&= -2F_h(e_i, g^{-1}\langle F_h(Je_i,), r\rangle) \\
&= -2F_h(e_i, e_j)\langle F_h(Je_i, e_j), r\rangle.
\end{aligned}$$

Finally, the last expression vanishes using again $F_h = F_h^{1,1}$ and symmetry considerations.

In the computation of the remaining term, we will use the following standard expressions for the covariant derivative of the almost complex structure J, the adjoint of d^h, and the Lee form:

$$(\nabla^g_X J)Y = \frac{1}{2}g^{-1}(d\omega(X, Y, \cdot) - d^c\omega(JX, Y, \cdot)), \qquad (5.2.17)$$

$$d^{h\star}F_h = -i_{e_i}\nabla^{h,g}_{e_i}F_h, \qquad (5.2.18)$$

$$\theta_\omega(X) = \tfrac{1}{2}d\omega(e_i, Je_i, X). \qquad (5.2.19)$$

where $\nabla^{h,g}$ is the covariant derivative with respect to the Levi-Civita connection ∇ and A^h. Combining this with (5.2.10), we conclude that:

$$\begin{aligned}
i_{Je_i}i_{e_i}\mathbb{I}(X) &= -i_{Je_i}i_{e_i}d^{h,-}\mathbb{F}(X) \\
&= i_{Je_i}i_{e_i}d^h(i_X F_h) - F_h(e_i, \nabla^-_{Je_i}X) + F_h(Je_i, \nabla^-_{e_i}X) \\
&= 2d^h_{e_i}(F_h(e_i, JX)) + F_h([e_i, Je_i], X) + 2F_h(Je_i, \nabla^-_{e_i}X) \\
&= 2(\nabla^{h,g}_{e_i}F_h)(e_i, JX) + 2F_h(\nabla^g_{e_i}e_i, JX) + 2F_h(e_i, \nabla^g_{e_i}JX) \\
&\quad + 2F_h(\nabla^g_{e_i}Je_i, X) + 2F_h(Je_i, (\nabla^- - \nabla^g)_{e_i}X) + 2F_h(Je_i, \nabla^g_{e_i}X) \\
&= -2d^{h\star}F_h(JX) + 2F_h((\nabla^g_{e_i}J)e_i, X) + 2F_h(Je_i, (\nabla^- - \nabla^g)_{e_i}X) \\
&\quad + 2F_h(e_i, (\nabla^g_{e_i}J)X) \\
&= -2d^{h\star}F_h(JX) - 2F_h(\theta^\sharp_\omega, JX) + F_h(e_i, g^{-1}i_{JX}i_{e_i}d^c\omega).
\end{aligned}$$

The statement follows from Lemma 5.2.2. □

5.3 Hermite-Einstein metrics on Bott-Chern algebroids

5.3.1 Coupled Hermite-Einstein metrics

This section is devoted to the study of basic structural properties of coupled Hermite-Einstein metrics on Bott-Chern algebroids. Continuing with the notation of the previous Sections, let $X = (M, J)$ a complex manifold, and let E be a string algebroid. Moreover, let V_+ a generalized metric compatible with J, and we denote by $\mathcal{Q}_\ell$ the holomorphic reduction of E.

Definition 5.3.1. *Let* $\mathbf{G}$ *be the generalized hermitian metric on* $\mathcal{Q}_\ell$ *induced by* V_+. *Then,* $\mathbf{G}$ *is called a coupled Hermite-Einstein metric if:*

$$F_{\mathbf{G}} \wedge \omega^{n-1} = 0, \tag{5.3.1}$$

where $g = \omega(\cdot, J\cdot)$ *is the hermitian metric on* X *determined by* V_+.

Our first goal is to obtain the conditions for $\mathbf{G}$ on a Bott-Chern algebroid to be a coupled the Hermitian-Einstein metric (5.3.1).

The following Lemma can be compared with the classical result which states that a Hermite-Yang-Mills connection is Yang-Mills, provided that the background metric is Kähler. The analogue in Hermitian Geometry is apparently well-known to experts but, since we have not been able to find it in the literature, we shall provide a complete proof here.

Lemma 5.3.2. *Let* (X, ω) *be a hermitian manifold of complex dimension* n *endowed with a holomorphic principal* G*-bundle* P*, and let* h *be a hermitian reduction on* P *to a maximal compact subgroup. Then:*

$$J d^h(\Lambda_\omega F_h) = -d^{h\star} F_h - i_{\theta_\omega^\sharp} F_h + \star(F_h \wedge \star d^c \omega). \tag{5.3.2}$$

In particular, if h *is Hermite-Einstein, that is, satisfying:*

$$F_h \wedge \omega^{n-1} = \tfrac{z}{n} \omega^n \tag{5.3.3}$$

for z *a central element in the Lie algebra* $\mathfrak{k}$*, then:*

$$d^h(\Lambda_\omega F_h) = 0, \tag{5.3.4}$$

or, equivalently:

$$d^{h\star} F_h + i_{\theta_\omega^\sharp} F_h - \star(F_h \wedge \star d^c \omega) = 0. \tag{5.3.5}$$

Proof. By general theory, the curvature form of a Chern connection F_h satisfies:

$$F_h = F_h^{1,1}, \quad d^h F_h = 0, \tag{5.3.6}$$

Using the conditions above, we obtain:

$$\begin{aligned} d^{h*}F_h(V) =& -(\nabla^{h,g}_{e_i}F_h)(e_i,V)\\ =& -d^h_{e_i}(F_h(Je_i,JV)) + F_h(\nabla_{e_i}e_i,V) + F_h(e_i,\nabla_{e_i}V)\\ =& -d^hF_h(e_i,Je_i,JV) - d^h_{Je_i}(F_h(e_i,JV)) + d^h_{JV}(F_h(e_i,Je_i))\\ & - F_h([e_i,Je_i],JV) + F_h([e_i,JV],Je_i) - F_h([Je_i,JV],e_i)\\ & + F_h(\nabla_{e_i}e_i,V) + F_h(e_i,\nabla_{e_i}V)\\ =& (\nabla^{h,g}_{Je_i}F_h)(Je_i,V)) + F_h(\nabla_{Je_i}Je_i,V) + F_h(Je_i,\nabla_{Je_i}V)\\ & - F_h([e_i,Je_i],JV) + F_h([e_i,JV],Je_i) - F_h([Je_i,JV],e_i)\\ & + F_h(\nabla_{e_i}e_i,V) + F_h(e_i,\nabla_{e_i}V)\\ =& -d^{h*}F_h(V) + 2F_h(\nabla_{e_i}e_i,V) + 2F_h(e_i,\nabla_{e_i}V) + 2d^h_{JV}(\Lambda_\omega F_h)\\ & - 2F_h(\nabla_{e_i}Je_i,JV) + 2F_h(\nabla_{e_i}JV,Je_i) + 2F_h(\nabla_{JV}Je_i,e_i)\\ & + 2d^h_{JV}(\Lambda_\omega F_h). \end{aligned}$$

Collecting the terms $d^{h*}F_h(V)$ and using again that $F_h = F_h^{1,1}$, we have:

$$d^{h*}F_h(V) = -F_h((\nabla_{e_i}J)e_i,JV) + F_h((\nabla_{e_i}J)V,Je_i) + F_h(\nabla_{JV}Je_i,e_i) + d^h(\Lambda_\omega F_h)(JV).$$

Using elementary symmetry properties, which imply

$$F_h(\nabla_{JV}Je_i,e_i) = F_h(e_j,e_i)g(\nabla_{JV}Je_i,e_j) = 0,$$

combined with the formulae for ∇J and θ_ω in the proof of Proposition 5.2.3, which imply:

$$F_h((\nabla_{e_i}J)e_i,JV) = F_h(\theta^\sharp_\omega,V), \tag{5.3.7}$$

$$F_h((\nabla_{e_i}J)V,Je_i) = \tfrac{1}{2}F_h(e_i,g^{-1}i_Vi_{e_i}d^c\omega), \tag{5.3.8}$$

then (5.3.2) now follows from Lemma 5.2.2. For the second part of the Lemma, note that the Hermite-Einstein equation (5.3.3) implies:

$$\Lambda_\omega F_h = z, \tag{5.3.9}$$

for a central element $z \in \mathfrak{k}$, therefore $d^h(\Lambda_\omega F_h) = 0$. By (5.3.2), this is equivalent to:

$$d^{h\star}F_h + i_{\theta^\sharp_\omega}F_h - \star(F_h \wedge \star d^c\omega) = 0. \tag{5.3.10}$$

□

In the next result we characterize the generalized hermitian metrics on $\mathcal{Q}_\ell$ that are coupled Hermite-Einstein equation, in the sense of Definition (5.3.1).

Lemma 5.3.3. *Let X be a complex manifold endowed with a holomorphic principal G-bundle P. Assume that a pair (ω,h) satisfies the Bianchi identity:*

$$dd^c\omega + \langle F_h \wedge F_h\rangle = 0. \tag{5.3.11}$$

Consider the holomorphic vector bundle underlying the Bott-Chern algebroid $\mathcal{Q}_{P,2i\partial\omega,A^h}$ (see Example 2.2.6) endowed with the (possibly) indefinite hermitian metric $\mathbf{G}$ *in Lemma 5.1.2. Then,* $\mathbf{G}$ *solves the coupled Hermite-Einstein equation* (5.3.1) *if and only if the following conditions hold:*

$$\begin{aligned} [S_h, \cdot] &= 0, \\ d^h(\Lambda_\omega F_h) &= 0, \\ \rho_B + \langle S_h, F_h\rangle &= 0, \end{aligned} \tag{5.3.12}$$

where S_h denotes the second Ricci curvature of h.

Proof. By construction, the Chern connection of $\mathbf{G}$ is the $\mathbb{C}$-linear extension of the $\langle\cdot,\cdot\rangle^0$-orthogonal connection D in Section 5.2. Then, the proof is straightforward from Proposition 5.2.3. □

As one can directly see from (5.3.12), the coupled Hermite-Einstein condition (5.3.1) for a generalized Hermitian metric is very sensitive to the choice of quadratic Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$. For example, when $\mathfrak{g}$ is abelian the first condition is trivially satisfied. In particular, when $\mathfrak{g} = \{0\}$, Lemma 5.3.3 recovers [67, Proposition 4.4] for pluriclosed hermitian metrics. On the other extreme, when $\mathfrak{g}$ is semisimple, the first equation implies that $S_h = 0$ and hence the second equation is satisfied by Lemma 5.3.2. Furthermore, in this case one has $\rho_B = 0$. We will return to the geometry of this system in Chapter 5 under the name of *coupled Hermite-Einstein system*. We are ready to prove the first main result of this section.

Proposition 5.3.4. *Let X be a complex manifold endowed with a holomorphic principal G-bundle P. Assume that (ω, h) solves* (5.3.11) *and* (5.3.12). *Consider the holomorphic vector bundle $\mathcal{Q}_{P,2i\partial\omega,A^h}$ endowed with the (possibly) indefinite Hermitian metric* $\mathbf{G}$ *in Lemma 5.1.2. Then,* $\mathbf{G}$ *solves the Hermitian-Einstein equation* (5.3.1).

Proof. The proof is straightforward from (5.3.12) with Lemma 5.3.2 and Lemma 5.3.3. □

5.3.2 Relation to the Hull-Strominger system

We study next the relation between coupled Hermite-Einstein metrics on Bott-Chern algebroids (see Definition 5.3.1) and the Hull-Strominger system (3.1.11). We recall that the construction in the previous Section requires the ansatz (3.1.12) for the connection ∇, and hence in our discussion we will always assume this condition. In this Section, first we will embrace an abstract definition of the Hull-Strominger system, as considered in (3.1.14), and then particularize to the more familiar situation of (3.1.11).

Let (X, Ω) be a compact Calabi-Yau manifold of dimension n, and let P be a holomorphic principal G-bundle, for complex Lie group G with quadratic Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$, saitsfying:

$$p_1(P) = 0 \in H^{2,2}_{BC}(X, \mathbb{R}), \tag{5.3.13}$$

where its Chern-Weyl representative is taken with respect to the pairing $\langle\cdot,\cdot\rangle$ induced in ad P. The next result yields the fundamental relation between the Hull-Strominger system and coupled Hermite-Einstein metrics (5.3.1).

Proposition 5.3.5. *Let (ω, h) be a hermitian metric on X and a hermitian reduction of P satisfying the Hull-Strominger system* (3.1.14)*. Then, (ω, h) solve* (5.3.12)*.*

Proof. By the first equation of (3.1.14), it is clear that $S_h = 0$, hence also the first equation of (5.3.12) holds. Moreover, by Lemma 5.3.2 with $z = 0$, the second equation of (5.3.12) holds too. Therefore, to conclude, it is enough to show that the conformally balanced equation:

$$d(||\Omega||_\omega \omega^{n-1}) = 0 \tag{5.3.14}$$

implies that $\rho_B = 0$. To see this, we first use (6.1.8), which implies (see [62, Proposition 3.6])

$$\nabla^B(\|\Omega\|_\omega^{-1}\Omega) = 0. \tag{5.3.15}$$

In particular, the connection induced by ∇^B in the (anti)-canonical bundle is flat. Then, since ρ_B is proportional to the curvature of this connection, it vanishes. □

Corollary 5.3.6. *Let (ω, h) be a hermitian metric on X and a hermitian reduction of P satisfying the Hull-Strominger system* (3.1.14)*. Then, the generalized hermitian metric* $\mathbf{G}$ *given by Lemma 5.1.2 on $\mathcal{Q}_{P,2i\partial\omega,A^h}$ is a coupled Hermite-Einstein metric.*

Proof. This is a direct consequence of Propositions 5.3.5 and 5.3.4. □

The previous results apply to the Hull-Strominger system (3.1.11) straightforwardly in the following manner. Let V_0 and V_1 denote holomorphic vector bundles over X, where V_0 is a holomorphic structure on $T^{1,0}$, and satisfying

$$ch_2(V_0) = ch_2(V_1) \in H^{2,2}_{BC}(X, \mathbb{R}). \tag{5.3.16}$$

Let P be the holomorphic principal bundle with structure group given by:

$$P = \text{Fr } V_0 \times_X \text{Fr } V_1, \quad G = GL(r_0) \times GL(r_1), \tag{5.3.17}$$

for $r_i = \text{rk } V_i$. Its Lie algebra is endowed with the quadratic structure:

$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathfrak{gl}(r_0) \times \mathfrak{gl}(r_1) &\longrightarrow \mathbb{C} \\ ((s_0, s_1), (t_0, t_1)) &\mapsto -\alpha \text{tr}_0(s_0 t_0) + \alpha \text{tr}_1(s_1 t_1) \end{aligned} \tag{5.3.18}$$

depending on the parameter $\alpha \in \mathbb{R}$. Then, Corollary 5.3.6 reads:

Theorem 5.3.7. *Let (X, Ω) be a compact Calabi-Yau manifold and V_0, V_1 holomorphic vector bundles. Assume that V_0 has as underlying smooth vector bundle $T^{1,0}$. Moreover, let (ω, h) be a hermitian metric on X and a hermitian metric on V_1 solving the Hull-Strominger system* (3.1.11) *for ∇ the Chern connection on V_0 of a hermitian metric h_0 satisfying*

$$F_{h_0} \wedge \omega^2 = 0. \tag{5.3.19}$$

Then, on the Bott-Chern algebroid $\mathcal{Q}_{P,2i\partial\omega,A^{h_0,h}}$, the generalized hermitian metric $\mathbf{G}$ *of Lemma 5.1.2 is coupled Hermite-Einstein, where P is as in* (5.3.17)*, and $A^{h_0,h}$ is the product Chern connection of (h_0, h) on P.*

As we discussed in Section 3.1.1, solutions to the Hull-Strominger system (3.1.11) with the Hermite-Yang-Mills ansatz (3.1.12) for ∇ are not equivalent to solutions with the choice of ∇ being the Chern connection of g on the holomorphic tangent bundle $T^{1,0}$. However, these ansatze are related if one frees the holomorphic structure of $T^{1,0}$, as we observe in the following Remark.

Remark 5.3.8. *Any solution of* (3.1.11) *with the Hermite-Yang-Mills ansatz* (3.1.12) *determines a solution to* (3.1.11) *with* $V_0 = (T^{1,0}, \nabla^{0,1})$. *Hence* $\nabla = D^{h_0}$ *where* h_0 *is a hermitian metric on* V_0. *Moreover, by pulling-back the Chern connection* D^{h_0} *via a complex gauge transformation on* $T^{1,0}$ *taking* h_0 *to* g, *we obtain a solution to* (3.1.11) *for* ∇ *the Chern connection of* ω *on a vector bundle isomorphic to* V_0. *Observe that the equations* (3.1.11) *are invariant under this change.*

Remark 5.3.9. *Lemma 5.3.3 and Theorem 5.3.7 shall be compared with the original result by De la Ossa, Larfors, and Svanes in [36, Corollary 1], who observed that the Hull-Strominger system is equivalent to* (5.3.1) *to all orders in perturbation theory.*

5.4 Futaki invariants for the Hull-Strominger system

In this Section, we prove a moment map interpretation for the coupled Hermite-Einstein metric equation (5.3.1). Since the case of (possibly) indefinite signature of the metric is not completely standard in the literature, we will give here the details. Then, we will exploit this to provide families of non-Kähler Futaki invariants for the Hull-Strominger system. These are potentially non-trivial further obstructions to the existence of solutions to the system beyond balanced metrics and slope-stability of the bundles.

5.4.1 Finite-dimensional picture

The construction of our Futaki invariants stems from a general formalism that associates an invariant to any equation with a moment map interpretation, which we call Futaki invariant by analogy with the classical invariant obstructing the existence of Kähler-Einstein metrics on a Kähler manifold [60]. To draw parallels with the picture here, we give the following finite-dimensional abstraction of the Futaki invariant.

Lemma 5.4.1. *Let* (M, J, ω) *be a compact Kähler manifold, and let* $K \to \mathrm{Aut}(M, J, \omega)$ *be a hamiltonian action, for a real, connected, compact Lie group* K *with infinitesimal action:*

$$\mathfrak{k} \longrightarrow \Gamma(TM), \quad s \mapsto X^s, \tag{5.4.1}$$

and moment map:

$$\mu : M \longrightarrow \mathfrak{k}^*. \tag{5.4.2}$$

Let $G = K^c$ *be the complexification of* K. *Then, for any* $x \in M$ *and* $\xi \in \mathrm{Lie}\, G_x$ *the isotropy group of* x, *the map:*

$$\mathcal{F}_{x,\xi} : G \to \mathbb{C}, \quad g \mapsto \langle \mu(g \cdot x), \mathrm{Ad}_g(\xi) \rangle \tag{5.4.3}$$

is constant.

Proof. First, observe that the K-action can be regarded, in particular, as a map:

$$K \to \mathrm{Aut}(M, J), \tag{5.4.4}$$

where the group of biholomorphisms is a (finite-dimensional) complex Lie group ([20]). We now use the universal property of the complexification of Lie groups. Namely, given any real Lie group map:

$$K \to C, \tag{5.4.5}$$

where C is a complex Lie group, there exists a complex Lie group map:

$$G = K^{\mathbb{C}} \to C \tag{5.4.6}$$

that restricts to the previous real map on $K \subset K^{\mathbb{C}}$. Then, the universal property of G extends (5.4.4) to a complex Lie group map:

$$G \to \mathrm{Aut}(M, J), \tag{5.4.7}$$

extending the previous action, though G does not act symplectically on (M, ω). Now, fix $x \in X$. The complexification of the moment map μ satisfies:

$$d_x\langle \mu, \alpha\rangle(X^\beta) = \omega(X^{\mathrm{Re}\,\alpha}, X^\beta) + i\omega(X^{\mathrm{Im}\,\alpha}, X^\beta), \quad \alpha, \beta \in \mathfrak{g}. \tag{5.4.8}$$

Let $s \in \mathrm{Lie}\, G = \mathrm{Lie}\, K \otimes \mathbb{C}$, and $\xi \in \mathrm{Lie}\, G_x$. We write $s = s_0 + is_1$ and $\xi = \xi_0 + i\xi_1$ with $s_i, \xi_i \in \mathfrak{k}$. Then, using (5.4.7), (5.4.8), and the K-equivariance of μ:

$$\begin{aligned}
d\mathcal{F}_{x,\xi}\left(\frac{d}{dt}_{|t=0} e^{st}\right) &= \langle d_x\mu(X^s), \xi\rangle + \langle \mu(x), [s, \xi]\rangle \\
&= \omega(X^{\xi_0}, X^s) + i\omega(X^{\xi_1}, X^s) - \langle \mu(x), [\xi_0, s] + i[\xi_1, s]\rangle \\
&= \omega(X^{\xi_0}, X^{s_0} + JX^{s_1}) + i\omega(X^{\xi_1}, X^{s_0} + JX^{s_1}) + d_x\langle \mu, s\rangle(X^{\xi_0} + iX^{\xi_1}) \\
&= \omega(X^{\xi_0}, X^{s_0} + JX^{s_1}) + i\omega(X^{\xi_1}, X^{s_0} + JX^{s_1}) + \\
&\quad + \omega(X^{s_0} + iX^{s_1}, X^{\xi_0}) + i\omega(X^{s_0} + iX^{s_1}, X^{\xi_1}) \\
&= -\omega(JX^{s_1}, X^{\xi}) + i\omega(X^{s_1}, X^{\xi}) = 0,
\end{aligned}$$

the last step following from the hypothesis $\xi \in \mathrm{Lie}\, G_x$. Now, it follows that:

$$\frac{d}{dt}_{|t=t_0}\left(\mathcal{F}_{x,\xi}(e^{ts})\right) = d\mathcal{F}_{e^{t_0 s}\cdot x, \mathrm{Ad}_{e^{t_0 s}}\xi}\left(\frac{d}{dt}_{|t=0}(e^{ts})\right) = 0, \tag{5.4.9}$$

where the last step follows from the above computation substituting x by $e^{t_0 s}\cdot x$ and ξ by $\mathrm{Ad}_{e^{t_0 s}}\xi$, using that indeed $\mathrm{Ad}_{e^{t_0 s}}\xi \in \mathrm{Lie}\, G_{e^{t_0 s}x}$. Therefore, $\mathcal{F}(\cdot, \xi)$ is constant along the curve $t \mapsto e^{ts}$, for any $s \in \mathrm{Lie}\, G$.

Now by standard theory of Lie groups, the exponential map $\mathrm{Lie}\, G \stackrel{\exp}{\to} G$ is a local diffeomorphism around 0. Moreover, using that G is connected, since K is, any open set around $1 \in G$ generates G. Combining these facts with the above result that $\mathcal{F}(\cdot, \xi)$ is constant in exponential curves, the Lemma follows. □

Remark 5.4.2. *The interpretation of the Futaki invariant in [60] along the lines of the above Lemma follows from realizing the scalar curvature as a moment map [41, 59] (see also [128, Section 6.1]). However, in this infinite-dimensional situation, technical problems arise and one should think of Lemma 5.4.1 rather formally.*

5.4.2 Aeppli classes and Futaki invariants

In this Section we introduce a family of characters which obstructs the existence of solutions to the Hull-Strominger system in the form of (3.1.14). The construction of our Futaki invariants can be regarded as a formal infinite-dimensional analogous of the picture described in the previous Section. To see this, let X be a compact complex manifold and fix a balanced class $\mathfrak{b} \in H^{n-1,n-1}(X,\mathbb{R})$. Moreover, let $(Q,\mathbf{H})$ be a hermitian vector bundle, and let:

$$\mathcal{G}(Q,\mathbf{H}) \subset \mathcal{G}(Q) \tag{5.4.10}$$

be its unitary and complex gauge groups. Consider the space:

$$\mathcal{A}^{1,1}_{\mathbf{H}} = \{\nabla \text{ is } \mathbf{H}-\textit{unitary}\ ,\ F_\nabla \in \Omega^{1,1}(Q,\mathbf{H})\}, \tag{5.4.11}$$

where $[\omega^{n-1}] \in \mathfrak{b}$, carrying the Atiyah-Bott symplectic structure [14], and consider the moment map :

$$\langle \mu(\nabla), s\rangle = \int_X \mathrm{tr}(sF_\nabla)\wedge\omega^{n-1}, \quad s \in \mathrm{Lie}\,\mathcal{G}(Q,\mathbf{H}) \tag{5.4.12}$$

for the action of $\mathcal{G}(Q,\mathbf{H})$ by conjugation (see [41, Section 1.1] for details). The space of connections $\mathcal{A}^{1,1}_{\mathbf{H}}$ admits a complex structure induced by the natural complex structure on the space of integrable Dolbeault operators on Q via Chern correspondence. Moreover, the action is extended to a complex action of $\mathcal{G}(Q)$ on $\mathcal{A}^{1,1}_{\mathbf{H}}$. Explicitly, on the Chern connection associated to a holomorphic structure $\mathcal{Q} = (Q,\overline{\partial}_{\mathcal{Q}})$, the action is:

$$g\cdot D^C(\overline{\partial}_{\mathcal{Q}},\mathbf{H}) = D^C(g\cdot\overline{\partial}_{\mathcal{Q}},\mathbf{H}). \tag{5.4.13}$$

Therefore, the isotropy Lie algebra at this connection is naturally identified with $H^0(\mathrm{End}\,\mathcal{Q})$. Now, applying formally Lemma 5.4.1 to this infite-dimensional situation, we obtain:

$$\begin{aligned}\int_X \mathrm{tr}(\phi F_{\mathbf{H}})\wedge\omega^{n-1} &= \int_X \mathrm{tr}(\mathrm{Ad}_g\phi\ F_{D^C(g\cdot\overline{\partial}_{\mathcal{Q}},\mathbf{H})})\wedge\omega^{n-1}\\ &= \int_X \mathrm{tr}(\mathrm{Ad}_g\phi\ F_{g\circ D^C(\overline{\partial}_{\mathcal{Q}},g^*\mathbf{H})\circ g^{-1}})\wedge\omega^{n-1}\\ &= \int_X \mathrm{tr}(\mathrm{Ad}_g\phi\ \mathrm{Ad}_g F_{D^C(\overline{\partial}_{\mathcal{Q}},g^*\mathbf{H})})\wedge\omega^{n-1}\\ &= \int_X \mathrm{tr}(\phi F_{g^*\mathbf{H}})\wedge\omega^{n-1}.\end{aligned}$$

For the applications in this Thesis, we will use a stronger version of this result with the novelty that we allow (non-degenerate) hermitian metrics $\mathbf{G}$ on Q with arbitrary signature. Since this is not completely standard in the literature, and passing from the finite-dimensional picture to our situation is not completely straightforward, we give now the details that apply in our setting.

Lemma 5.4.3. *Let X be a compact complex manifold, $\mathcal{Q}$ a holomorphic vector bundle over X, and $\mathfrak{b} \in H^{n-1,n-1}_{BC}(X,\mathbb{R})$ a Bott-Chern class. Then, the map*

$$\begin{aligned}\mathcal{F}_{\mathfrak{b}} : H^0(X, \operatorname{End} \mathcal{Q}) &\longrightarrow \mathbb{C}\\ \varphi &\mapsto \int_X \operatorname{tr}(\varphi F_{\mathbf{G}}) \wedge \nu\end{aligned} \tag{5.4.14}$$

defines a character of the Lie algebra $H^0(X, \operatorname{End} \mathcal{Q})$, which does not depend on the representative ν of $\mathfrak{b} = [\nu]$ and neither on the choice of a pseudo-Hermitian metric $\mathbf{G}$ on $\mathcal{Q}$. In particular, $\mathcal{F}_{\mathfrak{b}} = 0$ if there exists a pseudo-Hermitian metric $\mathbf{G}$ on $\mathcal{Q}$ and a balanced Hermitian metric ω on X, with $\mathfrak{b} = [\omega^{n-1}]$, solving the Hermitian-Einstein equation

$$F_{\mathbf{G}} \wedge \omega^{n-1} = 0.$$

Proof. Let $\tilde{\nu}, \nu \in \Omega^{n-1,n-1}$ be d-closed forms on X, such that

$$\tilde{\nu} - \nu = \overline{\partial}\partial\alpha$$

for some $\alpha \in \Omega^{n-2,n-2}$. Then, by type decomposition and the Bianchi identity for $D^{\mathbf{G}}$, we have

$$\int_X \operatorname{tr}(\varphi F_{\mathbf{G}}) \wedge \overline{\partial}\partial\alpha = \int_X d(\operatorname{tr}(\varphi F_{\mathbf{G}}) \wedge \partial\alpha) - \int_X \operatorname{tr}(\overline{\partial}\varphi F_{\mathbf{G}}) \wedge \partial\alpha = 0,$$

where the two summands vanish independently by hypothesis. Now, let $\mathbf{G}$ and $\mathbf{G}'$ be arbitrary pseudo-Hermitian metrics on $\mathcal{Q}$. Since $\mathbf{G}$ and $\mathbf{G}'$ are both non-degenerate, there exists a smooth complex gauge transformation g on $\mathcal{Q}$ such that $\mathbf{G}'(\cdot,\cdot) = \mathbf{G}(g\cdot,\cdot)$. Then, their Chern curvatures are related by

$$F_{\mathbf{G}'} = F_{\mathbf{G}} + \overline{\partial}(g^{-1}\partial^{\mathbf{G}} g)$$

and it follows that, again by type decomposition and the holomorphicity of φ,

$$\begin{aligned}\int_X \operatorname{tr}(\varphi(F_{\mathbf{G}'} - F_{\mathbf{G}})) \wedge \nu &= \int_X \operatorname{tr}(\varphi\overline{\partial}(g^{-1}\partial^{\mathbf{G}} g)) \wedge \nu\\ &= \int_X d(\operatorname{tr}(\varphi(g^{-1}\partial^{\mathbf{G}} g)) \wedge \nu) - \int_X \operatorname{tr}(\overline{\partial}\varphi \wedge (g^{-1}\partial^{\mathbf{G}} g)) \wedge \nu\\ &+ \int_X \operatorname{tr}(\varphi(g^{-1}\partial^{\mathbf{G}} g)) \wedge \overline{\partial}\nu\\ &= 0.\end{aligned}$$

Finally, for $\varphi, \varphi' \in H^0(X, \operatorname{End} \mathcal{Q})$, using that $[F_{\mathbf{G}}, \varphi'] = \overline{\partial}\partial^{\mathbf{G}}\varphi'$, one has

$$\int_X \operatorname{tr}([\varphi,\varphi']F_{\mathbf{G}}) \wedge \nu = -\int_X \operatorname{tr}(\varphi\overline{\partial}\partial^{\mathbf{G}}\varphi') \wedge \nu = -\int_X d(\operatorname{tr}(\varphi\partial^{\mathbf{G}}\varphi')) \wedge \nu = 0.$$

□

Remark 5.4.4. *Lemma 5.4.3 can be regarded as a formal infinite-dimensional of the Futaki invariant as described in Lemma 5.4.1, where the complex gauge group of $\mathcal{Q}$ acts on the space of connections with curvature of bidegree* $(1,1)$ *(see [40] for details). Accordingly, the isotropy Lie subalgebra at $D^{\mathbf{G}}$ is precisely* $\operatorname{End}(\mathcal{Q})$.

Using the duality isomorphism $H^{n-1,n-1}_{BC}(X)^* \cong H^{1,1}_A(X)$, the *Futaki invariants* in Lemma 5.4.3, with $\mathfrak{b}$ varying along $H^{n-1,n-1}_{BC}(X)$, can be written more elegantly as a $H^{1,1}_A(X)$-valued character

$$\mathcal{F} : H^0(X, \operatorname{End} \mathcal{Q}) \to H^{1,1}_A(X) \colon \varphi \mapsto [\operatorname{tr}(\varphi F_{\mathbf{G}})].$$

In order to apply Lemma 5.4.3 to the Hull-Strominger system (3.1.14), we assume that the compact complex manifold X is endowed with a holomorphic volume form Ω. Let P be a holomorphic principal bundle for a complex reductive Lie group G with quadratic Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$. Moreover, we assume:

$$p_1(P) = 0 \in H^{2,2}_{BC}(X, \mathbb{R}), \tag{5.4.15}$$

with respect to the pairing $\langle\cdot,\cdot\rangle$. By Proposition 2.2.19, the set of equivalence classes of Bott-Chern algebroids over X with principal bundle P and bundle of quadratic Lie algebras $(\operatorname{ad} P, \langle\cdot,\cdot\rangle)$ is a non-empty affine space $\mathfrak{S}$ modelled on the image of

$$\partial \colon H^{1,1}_A(X, \mathbb{R}) \to H^1(\Omega^{2,0}_{cl}). \tag{5.4.16}$$

Our aim is to construct families of Futaki invariants indexed by isomorphism classes of Bott-Chern algebroids, as prescribed by (5.4.16). For this, we consider the family of finite-dimensional complex Lie algebras:

$$\mathfrak{H} \to \mathfrak{S}, \tag{5.4.17}$$

where the fibre over $\mathfrak{s} \in \mathfrak{S}$ is given by the Lie algebra of the group of holomorphic gauge transformations of the vector bundle $\mathcal{Q}_{\mathfrak{s}}$

$$\mathfrak{H}_{\mathfrak{s}} := H^0(X, \operatorname{End}\mathcal{Q}_{\mathfrak{s}}). \tag{5.4.18}$$

Then, by application of Lemma 5.4.3, there is a family of $H^{1,1}_A(X)$-valued characters

$$\mathcal{F}_{\mathfrak{s}} : \mathfrak{H}_{\mathfrak{s}} \longrightarrow H^{1,1}_A(X). \tag{5.4.19}$$

Theorem 5.4.5. *Assume that (X, Ω, P) admits a solution (ω, h) of the Hull-Strominger system* (3.1.14) *and balanced class*

$$\mathfrak{b} = [\|\Omega\|_\omega \omega^{n-1}] \in H^{n-1,n-1}_{BC}(X, \mathbb{R}). \tag{5.4.20}$$

Then, there exists $\mathfrak{s} \in \mathfrak{S}^\alpha$ such that $\langle \mathcal{F}_{\mathfrak{s}}, \mathfrak{b}\rangle = 0$.

Proof. Consider the Bott-Chern algebroid $\mathcal{Q}_{P,2i\partial\omega,A^h}$ associated to the solution (ω, h), defined as in Example 2.2.6. Denote by $\mathfrak{s} = [\mathcal{Q}_{P,2i\partial\omega,A^h}] \in \mathfrak{S}$ its isomorphism class. Then, by Proposition 5.3.5, the (possibly) indefinite Hermitian metric $\mathbf{G}$ in Lemma 5.1.2 solves the Hermitian-Einstein equation (5.3.1), and hence $\langle \mathcal{F}_{\mathfrak{s}}, \mathfrak{b}\rangle = 0$ by application of Lemma 5.4.3. □

Remark 5.4.6. *Following [70], we expect that the family of Lie algebras $\mathfrak{H}$ depends holomorphically on parameters, upon restriction to any locus $\mathfrak{S}_\sigma \subset \mathfrak{S}^\alpha$ with fixed* real string class σ *(see [115] and [69, Proposition 3.11]).*

Observe from Lemma 5.4.3, that the algebroid structure of $\mathcal{Q}$ is superfluous to the Futaki invariant. Rather, only the holomorphic bundle underlying $\mathcal{Q}$ determines the value of $\mathcal{F}$. These are classified by the image of the composition:

$$H^{1,1}_A(X,\mathbb{R}) \xrightarrow{\partial} H^1(\Omega^{2,0}_{cl}) \longrightarrow H^{2,1}_{\overline{\partial}}(X). \tag{5.4.21}$$

Hence, Lemma 5.4.3 implies $\mathcal{F}_\mathfrak{s}$ is constant in the fibres of the second map of (5.4.21). This is the main motivation to introduce the more flexible anchored endomorphisms in Section 5.4.3 to compute Futaki invariants. It is an open question whether it is possible to refine Lemma 5.4.3 to produce a genuine Bott-Chern algebroid invariant.

As a direct application of Theorem 5.4.5, we obtain:

Theorem 5.4.7. *Let (X,Ω) be a Calabi-Yau threefold endowed with a pair of holomorphic vector bundles V_0 and V, where the underlying smooth bundle of V_0 is isomorphic to $T^{1,0}$ and satisfying* (3.2.5)*. Assume that (X,Ω,V) admits a solution (ω,h) of the Hull-Strominger system* (3.1.14) *with balanced class $\mathfrak{b} \in H^{2,2}_{BC}(X,\mathbb{R})$, such that ∇ is the Chern connection on V_0 of a Hermite-Einstein metric h_0 satisfying:*

$$F_{h_0} \wedge \omega^{n-1} = 0. \tag{5.4.22}$$

Then, there exists $\mathfrak{s} \in \mathfrak{S}$ such that $\langle \mathcal{F}_\mathfrak{s}, \mathfrak{b}\rangle = 0$.

In the case that the holomorphic tangent bundle $T^{1,0}$ (with the standard holomorphic structure) is polystable with respect to some balanced class $\mathfrak{b} \in H^{2,2}_{BC}(X,\mathbb{R})$, we expect that Theorem 5.4.5 provides also an obstruction to the existence of solutions to (3.1.11) with $\nabla = D^g$.

As a consequence of Theorem 5.4.5, in order to disprove the strong version of Yau's Conjecture in Question 4.3.1 for the case of Calabi-Yau threefolds, it suffices to find a tuple (X,Ω,V), $\alpha \in \mathbb{R}$ and a balanced class $\mathfrak{b} \in H^{2,2}_{BC}(X,\mathbb{R})$, such that V is $\mathfrak{b}$-polystable and

$$\langle \mathcal{F}_\mathfrak{s}, \mathfrak{b}\rangle \neq 0, \quad \forall \mathfrak{s} \in \mathfrak{U}^0,$$

where $\mathfrak{U}^0$ denotes the restriction of the relative family of string algebroid extensions over a dense open subset of the moduli space for V_0.

When the Calabi-Yau manifold X satisfies the $\partial\overline{\partial}$-Lemma the space $\mathfrak{S}$ reduces to a point (see Proposition 2.2.19). In this case we obtain a unique invariant $\mathcal{F}_0$ obstructing the existence of solutions, which can be regarded as a *stringy version* of the classical Futaki invariant for the holomorphic bundle P. Based on this, we expect that $\mathcal{F}_0$ provides a useful tool to address Question 4.3.1 in the case of Calabi-Yau manifolds obtained via conifold transitions and flops.

Remark 5.4.8. *As a consequence of Proposition 5.3.4 and Lemma 5.4.3, we obtain a stronger version of Theorem 5.4.5. For instance, let P be a holomorphic principal bundle over a compact complex manifold X which admits a solution (g, h) of* (5.3.12). *Let $\tilde{g}$ be a Gauduchon metric in the conformal class of g. Then, if $\tilde{g}$ is balanced, then $\mathcal{F}_{\mathfrak{b}} = 0$ where $\mathfrak{b} = [\tilde{\omega}^{n-1}]$.*

Remark 5.4.9. *While Theorem 5.4.5 provides a new method to tackle Question 4.3.1, as discussed in this Section, it may also be regarded as a tool to study canonical geometry on string algebroids. This question was addressed in [68]. In the light of the results of this Section, can be rephrased as the possibility of constructing a conjectural map:*

$$\Sigma_{\mathcal{Q}}(\mathbb{R}) \supset U \longrightarrow \{\mathfrak{b} \in H^{n-1,n-1}_{BC}(X, \mathbb{R}) \mid \langle \mathcal{F}, \mathfrak{b} \rangle = 0\} \tag{5.4.23}$$

that associates to an Aeppli class σ in $\mathcal{Q}$ (see [68, Definition 3.20]), the balanced class of the *solution to the Hull-Strominger system* (3.1.14) *in σ.*

5.4.3 Anchored endomorphisms of Bott-Chern algebroids

Our first goal is to define a new family of holomorphic endomorphisms of Bott-Chern algebroids. This notion, much wider than Definition 2.1.12, is actually enough to define the Futaki invariants in this setting, and allows to test a larger number of obstructions.

Let (X, Ω) a compact Calabi-Yau manfiold and we fix P a holomorphic principal bundle, for a complex Lie group G, with quadratic Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ as in the previous Sections. Consider the Bott-Chern algebroid $\mathcal{Q} = \mathcal{Q}_{P,2i\partial\omega,A^h}$ (see Example 2.2.6) associated to a solution of

$$dd^c\omega + \langle F_h \wedge F_h \rangle = 0. \tag{5.4.24}$$

Let End $\mathcal{Q}$ be the bundle of endomorphisms of the holomorphic vector bundle underlying $\mathcal{Q}$, that is, *a priori* no compatibility is required with the algebroid structure. We will denote by $\Lambda^2\mathcal{Q} \subset \text{End}\,\mathcal{Q}$ the bundle of orthogonal endomorphisms of $\mathcal{Q}$ with respect to the ambient pairing $\langle \cdot, \cdot \rangle_0$, that is, of sections satisfying

$$\langle \varphi \cdot, \cdot \rangle_0 + \langle \cdot, \varphi \cdot \rangle_0 = 0. \tag{5.4.25}$$

Definition 5.4.10. *An element $\varphi \in \Gamma(\Lambda^2\mathcal{Q})$ is called an* anchored endomorphism *of $\mathcal{Q}$ if there exists $\phi \in \Gamma(\text{End}\,T^{1,0})$ such that:*

$$\pi \circ \varphi = \phi \circ \pi. \tag{5.4.26}$$

In our next result we provide an explicit characterization of anchored endomorphisms, via the identification of the smooth complex vector bundle underlying $\mathcal{Q}$ with $T^{1,0} \oplus \text{ad}P \oplus T^*_{1,0}$ (see Example 2.2.6).

Lemma 5.4.11. *Let $\mathcal{Q} = \mathcal{Q}_{P,2i\partial\omega,A^h}$ be the Bott-Chern algebroid associated to a solution of (5.4.24). Let φ be a smooth anchored endomorphism of $\mathcal{Q}$. Then, there exists*

$\phi \in \Gamma(\mathrm{End}T^{1,0})$, $b \in \Omega^{2,0}$, $\sigma \in \Gamma(\Lambda^2 \mathrm{ad}P)$ skew-orthogonal, and $\alpha \in \Omega^{1,0}(\mathrm{ad}\, P)$, uniquely determined by φ, such that

$$\varphi = \varphi(\phi, \alpha, \sigma, b) := \begin{pmatrix} \phi & 0 & 0 \\ \alpha & \sigma & 0 \\ b & -2\langle \alpha, \cdot \rangle & -\phi^* \end{pmatrix}. \tag{5.4.27}$$

Conversely, any tuple $(\phi, b, \sigma, \alpha)$ as above defines a smooth anchored endomorphism φ of $\mathcal{Q}$ via formula (5.4.27).

Proof. The proof follows directly from [61, Section 3.1]. □

In our next result we characterize the holomorphicity condition $\overline{\partial}_{\mathcal{Q}}\varphi = 0$, for φ in (5.4.27), where $\overline{\partial}_{\mathcal{Q}}$ denotes the Dolbeault operator in Example 2.2.6.

Lemma 5.4.12. *Let $\mathcal{Q} \cong \mathcal{Q}_{P,2i\partial\omega,A^h}$ be the Bott-Chern algebroid associated to a solution of (5.4.24). Let $\varphi = \varphi(\phi, \alpha, \sigma, b)$ be a smooth anchored endomorphism of $\mathcal{Q}$. Then φ is holomorphic if and only if the following conditions are satisfied*

$$\begin{aligned} \overline{\partial}\phi &= 0 \\ \overline{\partial}\sigma &= 0 \\ \overline{\partial}\alpha + \sigma(F_h) - \phi \lrcorner F_h &= 0 \\ \overline{\partial}b + \phi \lrcorner (2i\partial\omega) - 2\langle \alpha \wedge F_h \rangle &= 0 \end{aligned} \tag{5.4.28}$$

where:

$$i_{X^{1,0}}(\phi \lrcorner F_h) = i_{\phi(X^{1,0})} F_h \tag{5.4.29}$$

$$i_{Y^{1,0}} i_{X^{1,0}}(\phi \lrcorner (2i\partial\omega)) = i_{Y^{1,0}} i_{\phi(X^{1,0})}(2i\partial\omega) + i_{\phi(Y^{1,0})} i_{X^{1,0}}(2i\partial\omega) \tag{5.4.30}$$

Proof. With the notation in Lemma 5.4.11, the proof follows from

$$(\overline{\partial}_{\mathcal{Q}}\varphi)(X + \xi + r) = \overline{\partial}_{\mathcal{Q}}(\varphi(X + \xi + r)) - \varphi(\overline{\partial}_{\mathcal{Q}}(X + \xi + r))$$

using the expression for $\overline{\partial}_{\mathcal{Q}}$ given by (2.2.17). Imposing that this expression vanishes for any X, ξ and r, a lengthy but straightforward computation shows it is equivalent to the equations above. □

The system of holomorphicity equations (5.4.28) can be, in general, difficult to solve completely. However, we now observe there exist natural families of solutions.

Remark 5.4.13. *The subspace of solutions to* (5.4.28) *with $\phi = \sigma = 0$ is given by*

$$S_0 = \{(b, \alpha) \mid \overline{\partial}b - 2\langle \alpha \wedge F_h \rangle = 0\ , b \in \Omega^{2,0},\ \alpha \in H^0(X, \Omega^{1,0}(\mathrm{ad}\, P))\}. \tag{5.4.31}$$

If we define (cf. [70, Proposition 4.6])

$$\delta_P : H^0(X, \Omega^{1,0}(\mathrm{ad}\, P)) \longrightarrow H^{2,1}_{\overline{\partial}}(X) \colon \alpha \mapsto [2\langle \alpha \wedge F_h \rangle], \tag{5.4.32}$$

the space S_0 fits in the short exact sequence

$$0 \to H^{2,0}_{\overline{\partial}}(X) \to S_0 \to \ker \delta_P \to 0 \tag{5.4.33}$$

In particular if $h^{2,0}_{\overline{\partial}}(X) > 0$ or if $h^0(\Omega^{1,0}(\mathrm{ad}\, P)) > h^{2,1}_{\overline{\partial}}(X)$, then $\mathcal{Q}$ has a holomorphic anchored endomorphism.

Remark 5.4.14. *Suppose that X satisfies the $\partial\overline{\partial}$-Lemma. Then, for any $s \in H^0(X, \mathrm{ad}\, P)$, we can construct holomorphic anchored endomorphisms $\varphi = \varphi(\phi, \alpha, \sigma, b)$, defined by* (5.4.27)*, as follows: set*

$$\phi = 0 \ , \ \ \alpha = \partial^h s \ , \ \ \sigma = [s, \cdot]. \tag{5.4.34}$$

Then, the first three equations of (5.4.28) *hold. Now, we have:*

$$\overline{\partial}\langle \partial^h s \wedge F_h \rangle = \langle \overline{\partial}\partial^h s \wedge F_h \rangle = \langle [F_h, s] \wedge F_h \rangle = -\langle s \wedge [F_h \wedge F_h] \rangle = 0, \tag{5.4.35}$$

and hence, by the $\partial\overline{\partial}$-Lemma, there exists $b \in \Omega^{2,0}$ such that

$$\overline{\partial} b = 2\langle \partial^h s \wedge F_h \rangle, \tag{5.4.36}$$

since the right hand side is ∂-exact and d-closed.

Next, we address the computation of the Futaki invariants in Theorem 5.4.5 for holomorphic anchored endomorphism of $\mathcal{Q}$. For this, given a pair of Hermitian metrics g and g_0 on X, $\gamma \in \Gamma(\mathrm{End}\, T \otimes \mathbb{C})$, and $\tau \in \Omega^2$, we denote

$$\mathrm{tr}_{g,g_0}\gamma = \frac{1}{2} g(\gamma J e_j^0, e_j^0), \qquad \Lambda_{\omega_0}\tau = \frac{1}{2}\tau(e_j^0, J e_j^0), \tag{5.4.37}$$

for any choice of g_0-orthonormal basis $e_1^0, \ldots, e_{2n}^0$ of T, where we use Einstein's convention to sum over repeated indices.

Proposition 5.4.15. *Consider the Bott-Chern algebroid $\mathcal{Q} = \mathcal{Q}_{P,2i\partial\omega,A^h}$ associated to a solution (ω, h) of (5.4.24), with ω positive. Let $\varphi = \varphi(\phi, \alpha, \sigma, b)$ be a holomorphic anchored endomorphism of $\mathcal{Q}$ and let $\mathfrak{b} = [\omega_0^{n-1}] \in H^{n-1,n-1}_{BC}(X, \mathbb{R})$ be a balanced class. Then, the evaluation of the Futaki character in Lemma 5.4.3 is given by*

$$\begin{aligned} \langle \mathcal{F}(\varphi), \mathfrak{b} \rangle = & -\int_X (\mathrm{tr}_{g,g_0} R_{\nabla^B} + \langle \Lambda_{\omega_0} F_h, F_h \rangle)(e_k, \phi^{*_g} e_k^{0,1} - \phi e_k^{1,0}) \frac{\omega_0^n}{n} \\ & - \int_X \langle \mathrm{tr}_{g,g_0} R^{0,2}_{\nabla^B}, b \rangle_g \frac{\omega_0^n}{n} \\ & + \int_X \left(\mathrm{tr}_{\mathrm{ad}\, P}(\sigma[\Lambda_{\omega_0} F_h, \cdot]) + \langle \sigma F_h(e_j^0, e_k), F_h(J e_j^0, e_k) \rangle \right) \frac{\omega_0^n}{n} \\ & + 2\int_X \langle \alpha(e_k), \Lambda_{\omega_0} \nabla^{h,-}_{e_k} F_h + F_h(J e_j^0, g^{-1} d^c \omega(e_j^0, e_k, \cdot)) \rangle \frac{\omega_0^n}{n} \end{aligned} \tag{5.4.38}$$

for any choices of g-orthonormal basis $e_1, \ldots, e_{2n}$ and g_0-orthonormal basis $e_1^0, \ldots, e_{2n}^0$ of T.

Proof. Consider the isomorphism $\psi \colon \mathcal{Q}_{P,2i\partial\omega,A^h} \to T \otimes \mathbb{C} \oplus \mathrm{ad}\, P$ defined by Lemma 5.1.2, that is,

$$\psi(X + r + \xi) = X - \tfrac{1}{2} g^{-1}\xi + r.$$

Then $\tilde{\varphi} := \psi \circ \varphi \circ \psi^{-1}$ is given by

$$\tilde{\varphi}(X + r) = \phi(X^{1,0}) - \phi^{*_g}(X^{0,1}) - \tfrac{1}{2} g^{-1} i_{X^{1,0}} b + g^{-1}\langle \alpha, r \rangle + \sigma(r) + i_{X^{1,0}}\alpha$$

where $\phi^{*_g}(X^{0,1}) = g^{-1}g(X^{0,1}, \phi\cdot)$. By Lemma 5.4.3 and formula (5.2.11), it suffices to compute:

$$\operatorname{tr}(\tilde{\varphi} S_{\mathbf{G}}) = \frac{1}{2}\operatorname{tr}(\tilde{\varphi} F_{\mathbf{G}}(e_j^0, Je_j^0)).$$

For this, using that $\varphi = \varphi(\phi, \alpha, \sigma, b)$ depends linearly on ϕ, α, σ, and b, we can decompose uniquely

$$\tilde{\varphi} = \tilde{\varphi}_\phi + \tilde{\varphi}_b + \tilde{\varphi}_\alpha + \tilde{\varphi}_\sigma,$$

so that $\tilde{\varphi}_\phi$ only depends on ϕ, and similarly for the rest. Now, denoting $\pi_{1,0}\colon T\otimes\mathbb{C} \to T^{1,0}$ the natural projection, by Lemma 5.2.1 we have

$$\begin{aligned}
\operatorname{tr}(\tilde{\varphi}_\phi F_{\mathbf{G}}(e_j^0, Je_j^0)) &= \operatorname{tr}_{T\otimes\mathbb{C}}(\phi\circ\pi_{1,0}(R_{\nabla^-}(e_j^0, Je_j^0) - \mathbb{F}^\dagger\wedge\mathbb{F}(e_j^0, Je_j^0)))\\
&\quad - \operatorname{tr}_{T\otimes\mathbb{C}}(\phi^{*_g}\circ\pi_{0,1}(R_{\nabla^-}(e_j^0, Je_j^0) - \mathbb{F}^\dagger\wedge\mathbb{F}(e_j^0, Je_j^0)))\\
&= g(R_{\nabla^-}(e_j^0, Je_j^0)e_k, \phi^{*_g}e_k^{0,1} - \phi e_k^{1,0})\\
&\quad - \langle F_h(Je_j^0, \phi^{*_g}e_k^{0,1} - \phi e_k^{1,0}), F_h(e_j^0, e_k)\rangle\\
&\quad + \langle F_h(e_j^0, \phi^{*_g}e_k^{0,1} - \phi e_k^{1,0}), F_h(Je_j^0, e_k)\rangle\\
&= \tfrac{1}{2}dd^c\omega(e_j^0, Je_j^0, e_k, \phi^{*_g}e_k^{0,1} - \phi e_k^{1,0})\\
&\quad + g(R_{\nabla^B}(e_k, \phi^{*_g}e_k^{0,1} - \phi e_k^{1,0})e_j^0, Je_j^0)\\
&\quad - \langle F_h(Je_j^0, \phi^{*_g}e_k^{0,1} - \phi e_k^{1,0}), F_h(e_j^0, e_k)\rangle\\
&\quad + \langle F_h(e_j^0, \phi^{*_g}e_k^{0,1} - \phi e_k^{1,0}), F_h(Je_j^0, e_k)\rangle\\
&= -2\operatorname{tr}_{g,g_0}(R_{\nabla^B}(e_k, \phi^{*_g}e_k^{0,1} - \phi e_k^{1,0}))\\
&\quad - 2\langle\Lambda_{\omega_0}F_h, F_h(e_k, \phi^{*_g}e_k^{0,1} - \phi e_k^{1,0})\rangle,
\end{aligned}$$

where in the third and fourth equalities we have used (1.2.8) and (5.4.24), respectively. Similarly,

$$\begin{aligned}
\operatorname{tr}(\tilde{\varphi}_b F_{\mathbf{G}}(e_j^0, Je_j^0)) &= -\tfrac{1}{2}\operatorname{tr}_{T\otimes\mathbb{C}}(g^{-1}b\circ\pi_{1,0}(R_{\nabla^-}(e_j^0, Je_j^0) - \mathbb{F}^\dagger\wedge\mathbb{F}(e_j^0, Je_j^0)))\\
&= -\tfrac{1}{2}b(R_{\nabla^-}(e_j^0, Je_j^0)e_k, e_k) + \tfrac{1}{2}b(g^{-1}\langle F_h(Je_j^0, \cdot), F_h(e_j^0, e_k)\rangle, e_k)\\
&\quad - \tfrac{1}{2}b(g^{-1}\langle F_h(e_j^0, \cdot), F_h(Je_j^0, e_k)\rangle, e_k)\\
&= -\tfrac{1}{2}g(R_{\nabla^B}(e_k, e_m)e_j^0, Je_j^0)b(e_m, e_k)\\
&\quad + \tfrac{1}{2}b(g^{-1}\langle F_h(e_j^0, \cdot), F_h(e_j^0, e_k)\rangle, Je_k)\\
&\quad - \tfrac{1}{2}b(g^{-1}\langle F_h(e_j^0, \cdot), F_h(Je_j^0, e_k)\rangle, e_k)\\
&= -2\langle\operatorname{tr}_{g,g_0}R^{0,2}_{\nabla^B}, b\rangle_g,
\end{aligned}$$

and, using the notation in Lemma 5.2.1,

$$\begin{aligned}
\operatorname{tr}\tilde{\varphi}_\alpha F_{\mathbf{G}}(e_j^0, Je_j^0) &= 2\langle\alpha(e_k), \mathbb{I}(e_j^0, Je_j^0)e_k\rangle\\
&= 4\langle\alpha(e_k), \Lambda_{\omega_0}\nabla^{h,-}_{e_k}F_h\rangle + 4\langle\alpha(e_k), F_h(Je_j^0, g^{-1}d^c\omega(e_j^0, e_k, \cdot))\rangle
\end{aligned}$$

Finally, taking basis $\{r_k\}$ and $\{\tilde{r}_k\}$ of ad P such that $\langle r_k, \tilde{r}_j\rangle = \delta_{kj}$, we have:

$$\begin{aligned}
\mathrm{tr}(\tilde{\varphi}_\sigma F_{\mathbf{G}}(e^0_j, Je^0_j)) &= \mathrm{tr}_{\mathrm{ad}\,P}(\sigma\circ([F_h(e^0_j, Je^0_j),\cdot] - \mathbb{F}\wedge\mathbb{F}^\dagger(e^0_j, Je^0_j))) \\
&= \langle \tilde{r}_k, \sigma([F_h(e^0_j, Je^0_j), r_k])\rangle \\
&\quad - \langle \tilde{r}_k, \sigma F_h(Je^0_j, g^{-1}\langle F_h(e^0_j,\cdot), r_k\rangle)\rangle \\
&\quad + \langle \tilde{r}_k, \sigma F_h(e^0_j, g^{-1}\langle F_h(Je^0_j,\cdot), r_k\rangle)\rangle \\
&= 2\mathrm{tr}_{\mathrm{ad}\,P}(\sigma([\Lambda_{\omega_0}F_h,\cdot]) \\
&\quad - \langle \tilde{r}_k, \sigma F_h(Je^0_j, e_k)\rangle\langle F_h(e^0_j, e_k), r_k\rangle \\
&\quad + \langle \tilde{r}_k, \sigma F_h(e^0_j, e_k)\rangle\langle F_h(Je^0_j, e_k), r_k\rangle \\
&= 2\mathrm{tr}_{\mathrm{ad}\,P}(\sigma([\Lambda_{\omega_0}F_h,\cdot]) + 2\langle \sigma F_h(e^0_j, e_k), F_h(Je^0_j, e_k)\rangle.
\end{aligned}$$

□

5.4.4 Computations of Futaki invariants

In this Section we give an account of several computations of the Futaki invariant on families of Bott-Chern algebroids (5.4.19). While for the moment being our results are not conclusive to give an answer to the Question 4.3.1, they rule out some families of manifolds and hint towards some others where potential non-trivial obstructions may appear.

We begin by giving the details of a slight generalization to of a comment that was anticipated in [28, Introduction].

Proposition 5.4.16. *Let (X,Ω) be a compact Kähler Calabi-Yau threefold, and let V_0 and V_1 be holomorphic vector bundles satisfying* (3.2.3), (3.2.5)*. We assume V_0 is diffeomorphic to $T^{1,0}$. Moreover, suppose there exists a Kähler class $\kappa\in H^{1,1}(X)$ such that V_0 and V_1 are κ-stable. Then:*

1. *There exists a unique Bott-Chern algebroid $(\mathcal{Q},P,\rho)$, up to isomorphism, where $P =$ Fr $V_0\times_X$ Fr V_1 with Lie algebra pairing given by:*

$$\langle (r_0,r_1),(s_0,s_1)\rangle = -\alpha\mathrm{tr}_{V_0}(r_0s_0) + \alpha\mathrm{tr}_{V_1}(r_1s_1), \quad \alpha\in\mathbb{R}. \tag{5.4.39}$$

2. *For any balanced class $\mathfrak{b}\in H^{2,2}(X)$, the Futaki invariant:*

$$\langle\mathcal{F},\mathfrak{b}\rangle:\ \mathrm{End}(\mathcal{Q})\longrightarrow\mathbb{C} \tag{5.4.40}$$

given by pairing (5.4.19) *against $\mathfrak{b}$ vanishes.*

Proof. Given that (3.2.5) holds, by assumption for any h_i hermitian metrics on V_i, there exists $\tau\in\Omega^{1,1}(X,\mathbb{R})$ such that:

$$dd^c\tau - \alpha\mathrm{tr}_{V_0}\,F_{h_0}\wedge F_{h_0} + \alpha\mathrm{tr}_{V_1}\,F_{h_1}\wedge F_{h_1} = 0, \tag{5.4.41}$$

which defines a Bott-Chern algebroid $\mathcal{Q} = \mathcal{Q}_{P,2i\partial\tau,A_{(h_0,h_1)}}$ (see Definition 2.2.13). Since X is Kähler, in particular it satisfies the $\partial\overline{\partial}$-Lemma, hence the deformation map (2.2.38) is constant. Then, the first item follows by Proposition 2.2.19.

For the second item, first recall that the stability condition for V_i is open in the Kähler cone of X. Therefore there exists an open set $U^\kappa \subset H^{1,1}(X)$ within the stability locus of V_0 and V_1. Let $\kappa' \in U^\kappa$. Then, by the existence result in [28, Proposition 3.2], there exists a solution to the Hull-Strominger system (3.1.11) with the condition (3.1.12) and $V_0 = (T^{1,0}, \nabla^{0,1})$, and with balanced class $\lambda\kappa'^2$, for $\lambda >> 0$. Therefore, by Theorem 5.4.7, the Futaki invariant $\langle \mathcal{F}, \lambda\kappa'^2\rangle = 0$. Using the linearity of $\mathcal{F}$, we see that it vanishes on any balanced class given by the square of a Kähler class in U^κ. Now, we claim this is actually an open set in the balanced cone of X. Hence the Futaki invariant vanishes identically on all the balanced cone, using again that it is linear in the balanced class. Finally, to prove the claim, we just use that the map (see [56]):

$$\mathbf{b} : H^{1,1}(X) \longrightarrow H^{2,2}(X), \quad \sigma \mapsto \sigma^2 \tag{5.4.42}$$

has invertible differential at Kähler classes, as a consequence of Hard Lefschetz Theorem. □

Our next aim is to exhibit non-trivial Futaki invariants that capture slope unstability of the bundles. Observe that, in general, for a holomorphic string algebroid $(\mathcal{Q}, P, \rho)$, the set $H^0(\text{End}(\text{ad } P))$ does not embed in $H^0(\text{End } \mathcal{Q})$, hence (5.4.19) does not accomodate in a straightforward manner the classical Futaki invariant of Lemma 5.4.14 for the bundle ad P. Therefore, in the following Example we consider a particular situation:

Example 5.4.17. *Let X be a compact complex manifold of dimension n, and let V_0, V_1 be holomorphic vector bundles. Moreover, assume $V_0 = U \oplus W$ is a split sum. Let (ω, h_0, h_1) a solution to the Bianchi identity:*

$$dd^c\omega - \alpha \text{tr } F_{h_0} \wedge F_{h_0} + \alpha \text{tr } F_{h_1} \wedge F_{h_1}, \quad \alpha \in \mathbb{R}\backslash\{0\}, \tag{5.4.43}$$

where h_0 is a split sum (h_U, h_W). Consider the Bott-Chern algebroid $\mathcal{Q} = \mathcal{Q}_{P,2i\partial\omega,A_{(h_0,h_1)}}$, where $P = \text{Fr } V_0 \times \text{Fr } V_1$. Then:

$$\begin{aligned} \text{ad } P &\cong \text{End } V_0 \oplus \text{End } V_1 \\ &\cong \text{End } U \oplus \text{End } W \oplus \text{Hom}(U, W) \oplus \text{Hom}(W, U) \oplus \text{End } V_1. \end{aligned} \tag{5.4.44}$$

Then, we define a section $\sigma \in \Gamma(\text{End}(\text{ad } P))$ given by:

$$\sigma = \text{id}_{\text{Hom}(U,W)} - \text{id}_{\text{Hom}(W,U)}, \tag{5.4.45}$$

extending by zero on the rest of components of End(ad P)*. and vanishes on traceless endomorphisms of U, W, and has zero component in the rest of the terms of* End(ad P)*. Next, check that σ is orthogonal as a section of $(\mathcal{Q}, \langle\cdot,\cdot\rangle)$, where the pairing is the same as in* (5.4.39)*. For this, we write for a section $r \in \Gamma(\text{ad } P)$:*

$$r = r_U + r_W + r_{UW} + r_{WU} + r_1 \tag{5.4.46}$$

to denote each of the terms in the decomposition (5.4.44)*, with obvious notations. Then, for* *any two sections $r, s \in \mathrm{ad}\, P$ we have:*

$$\begin{aligned}
\langle\sigma(r), s\rangle + \langle r, \sigma(s)\rangle &= \langle r_{UW} - r_{WU}, s\rangle + \langle r, s_{UW} - s_{WU}\rangle \\
&\quad - \alpha \mathrm{tr}\left(\begin{pmatrix} 0 & -r_{WU} \\ r_{UW} & 0 \end{pmatrix}\begin{pmatrix} s_U & s_{WU} \\ s_{UW} & s_W \end{pmatrix}\right) + \alpha \mathrm{tr}(0 \cdot s_1) \\
&\quad - \alpha \mathrm{tr}\left(\begin{pmatrix} r_U & r_{WU} \\ r_{UW} & r_W \end{pmatrix}\begin{pmatrix} 0 & -s_{WU} \\ s_{UW} & 0 \end{pmatrix}\right) + \alpha \mathrm{tr}(r_1 \cdot 0) \\
&= -\alpha\left(-\mathrm{tr}(r_{WU}s_{UW}) + \mathrm{tr}(r_{UW}s_{WU})\right) - \alpha\left(\mathrm{tr}(r_{WU}s_{UW}) - \mathrm{tr}(r_{UW}s_{WU})\right) \\
&= 0.
\end{aligned}$$

Moreover, by Lemma 5.4.12, $\overline{\partial}_Q\sigma = 0$ if and only if:

$$\begin{aligned}
\overline{\partial}_{\mathrm{End}(\mathrm{ad}\, P)}\sigma &= 0, \\
\sigma(F_{(h_0,h_1)}) &= 0.
\end{aligned} \tag{5.4.47}$$

The first of this equations holds clearly by the expression of σ. For the second one, we use that σ has vanishing components in $\mathrm{End}(\mathrm{End}(U))$, $\mathrm{End}(\mathrm{End}(W))$ and $\mathrm{End}(\mathrm{End}(V_1))$, while, by assumption on h_0, $F_{(h_0,h_1)}$ has non-trivial components precisely in these terms. Then, we can evaluate the Futaki invariant on σ for a balanced class $\mathfrak{b}_0 = [\omega_0^{n-1}] \in H^{n-1,n-1}_{BC}(X, \mathbb{R})$. By (5.4.38), and using the second equation in (5.4.47):

$$\begin{aligned}
\langle \mathcal{F}(\sigma), \mathfrak{b}_0\rangle &= 2\int_X \mathrm{tr}_{\mathrm{ad}\, P}(\sigma \circ [\Lambda_{\omega_0}F_{(h_0,h_1)}, \cdot])\frac{\omega_0^n}{n} \\
&= 2\int_X \mathrm{tr}_{\mathrm{End}\, V_0}(\sigma \circ [\Lambda_{\omega_0}F_{h_0}, \cdot])\frac{\omega_0^n}{n}
\end{aligned}$$

To compute this, first note that if $r \in \Gamma(\mathrm{End}\, V_0)$, then:

$$\begin{aligned}
[\Lambda_{\omega_0}F_{h_0}, r] &= \begin{pmatrix} 0 & (\Lambda_{\omega_0}F_{h_U}) \circ r_{WU} - r_{WU} \circ (\Lambda_{\omega_0}F_{h_W}) \\ (\Lambda_{\omega_0}F_{h_W}) \circ r_{UW} - r_{UW} \circ (\Lambda_{\omega_0}F_{h_U}) & 0 \end{pmatrix} \\
\sigma \circ [\Lambda_{\omega_0}F_{h_0}, r] &= \begin{pmatrix} 0 & -(\Lambda_{\omega_0}F_{h_U}) \circ r_{WU} + r_{WU} \circ (\Lambda_{\omega_0}F_{h_W}) \\ (\Lambda_{\omega_0}F_{h_W}) \circ r_{UW} - r_{UW} \circ (\Lambda_{\omega_0}F_{h_U}) & 0 \end{pmatrix}
\end{aligned}$$

from where we obtain:

$$\mathrm{tr}_{\mathrm{End}\, V_0}\left(\sigma \circ [\Lambda_{\omega_0}F_{h_0}, \cdot]\right) = 2\mathrm{tr}_W(\Lambda_{\omega_0}F_{h_W}) - 2\mathrm{tr}_U(\Lambda_{\omega_0}F_{h_U}). \tag{5.4.48}$$

Therefore, we can finish computing the Futaki invariant:

$$\begin{aligned}
\langle \mathcal{F}(\sigma), \mathfrak{b}_0\rangle &= 4\int_X (\mathrm{tr}_W(\Lambda_{\omega_0}F_{h_W}) - \mathrm{tr}_U(\Lambda_{\omega_0}F_{h_U}))\frac{\omega_0^n}{n} \\
&= -8\pi i(n-1)!(\mathrm{rk}\, W \cdot \mu_{\mathfrak{b}_0}W - \mathrm{rk}\, U \cdot \mu_{\mathfrak{b}_0}U).
\end{aligned}$$

If a solution to the Hull-Strominger (3.1.14) exists in $\mathfrak{b}_0$ implies, in particular that $\mu_{\mathfrak{b}_0}V_0 = 0$, we conclude that $\langle \mathcal{F}(\sigma), \mathfrak{b}_0\rangle$ measures whether the splitting $V_0 = U \oplus W$ destabilizes V_0 with respect to $\mathfrak{b}_0$.

Now, we give some Examples about the computation of Futaki invariants for the Hull-Strominger system on complex homogeneous manifolds. Next, we exhibit a situation in which one can abstractly argue the vanishing of the Futaki invariant, while at the same time we show that it is computationally very involved, even for homogeneous manifolds.

Example 5.4.18. *Let $X = (\Gamma\backslash G, J)$ be the one-parameter family of complex locally homogenous threefolds determined by the structure relations:*

$$d\omega_1 = d\omega_2 = 0, \quad d\omega_3 = \omega_{12} + \omega_{1\bar{1}} + b\omega_{1\bar{2}} - \omega_{2\bar{2}}, \tag{5.4.49}$$

in a suitable invariant frame $\{\omega_i\}$ of $T^*_{1,0}$, corresponding to the real Lie algebras Lie $G = \mathfrak{h}_2$, *$\mathfrak{h}_4$ or $\mathfrak{h}_5$ depending on whether the parameter $b \in \mathbb{Q}$ satisfies $b^2 < 1$, $b^2 = 1$ or $b^2 > 1$ respectively (see Section 5.4.28), and let $\alpha < 0$ be a fixed constant. Then, we obtain a family of solutions to the Bianchi identity:*

$$dd^c\omega - \alpha \mathrm{tr}\, F_{h_0} \wedge F_{h_0} + \alpha \mathrm{tr}\, F_{h_1} \wedge F_{h_1}, \tag{5.4.50}$$

by Proposition 4.2.10 setting rk $V_1 = 1$. *Here, the parameter s_3 of ω as in* (4.2.13) *is determined by α and F_{h_1}. In particular, we also obtain a family of balanced classes $\{\sigma(\omega)\}$ indexed by ω.*

Let $\mathcal{Q}_{P,2i\partial\omega,A_{(h_0,h_1)}}$ be the associated family of Bott-Chern algebroids, where $P =$ Fr $V_0 \times_X$ Fr V_1*, and the pairing $\langle\cdot,\cdot\rangle$ is as in* (5.4.39)*. In the manifold X, the cohomology group $H^{1,1}_A((\mathrm{Lie}\ G, J), \mathbb{R})$ computes $H^{1,1}_A(X, \mathbb{R})$ as a direct application of Theorems 4.1.5 ,4.1.6 and Remark 4.1.7. It is then straightforward to check that the map* (2.2.38) *is constant. Therefore, by Proposition* (2.2.19)*, every Bott-Chern algebroid in the family $\mathcal{Q}_{P,2i\partial\omega,A_{(h_0,h_1)}}$ shares the same isomorphism class, that we denote by $\mathcal{Q}$.*

Now, again by Proposition 4.2.10, $\mathcal{Q}$ admits solutions to the Hull-Strominger system in any of the balanced classes $\sigma(\omega)$ in the above family. Then, by Theorem 5.4.5 the Futaki invariants for $\mathcal{Q}$:

$$\langle \mathcal{F}, \sigma(\omega) \rangle = 0. \tag{5.4.51}$$

Since $\mathcal{F}$ is linear in the balanced class, then even:

$$\langle \mathcal{F}, \mathfrak{b} \rangle = 0, \tag{5.4.52}$$

where $\mathfrak{b}$ is any balanced class represented by an invariant metric. Therefore, we do not find non-trivial invariants for this Example. Even though, for completeness we now give some details on how an explicit computation of (5.4.52) *is performed exploiting the invariant ansatz. This can be extrapolated to other locally homogenenous Examples, where an abstract argument to determine $\mathcal{F}$ may not be available.*

The computation of the Futaki invariant requires first knowing the holomorphic sections of End $\mathcal{Q}$*. While determining all of them may not be feasible, a distinguished family is given by anchored endomorphisms (see Section 5.4.3). Recall that an anchored endomorphism is a tuple $(\phi, \alpha, \sigma, b)$ as in Lemma* (5.4.11)*, where $\phi \in \Gamma(\mathrm{End}\ T^{1,0})$, $\alpha = (\alpha_0, \alpha_1) \in \Omega^{1,0}(\mathrm{End}\ V_0 \oplus \mathrm{End}\ V_1)$, $\sigma \in \Gamma(\mathrm{End}(\mathrm{ad}\ P))$ and $b \in \Omega^{2,0}_X$. To obtain manageable formulae, we will restrict* *here to* $V_0 = \mathcal{O}_X^{\oplus 3}$*,* $\sigma \in \Gamma(\mathrm{End}(\mathrm{End}\ V_0))$ *and* $F_{h_1} = F(0,n,p,0)$ *(see Proposition 4.2.10). Then, using the global invariant frame* $\{X_i\}$ *for* $T^{1,0}$ *and dual frame* $\{\omega_i\}$ *for* $T^*_{1,0}$*, each of these tensors admit a natural invariant ansatz. Therefore, we can write them in the following form:*

$$\begin{aligned} \phi &= (\phi_{ij}), \quad \phi_{ij} \in \mathbb{C}, \quad i,j = 1,2,3. \\ \alpha_0 &= \textstyle\sum_{i=1}^3 \omega_i(\alpha^i_{0jk}), \quad \alpha^i_{0jk} \in \mathbb{C}, \quad j,k = 1,2,3. \\ \alpha_1 &= \textstyle\sum_{i=1}^3 \omega_i \alpha^i_1, \quad \alpha^i_1 \in \mathbb{C}. \\ \sigma &= (\sigma_{ijkl}), \quad \sigma_{ijkl} \in \mathbb{C}, \quad i,j,k,l = 1,2,3. \\ b &= \textstyle\sum_{1 \leq i < j \leq 3} \omega_{ij} b_{ij}, \quad b_{ij} \in \mathbb{C}. \end{aligned} \tag{5.4.53}$$

With the aid of a mathematical software, one can check that the invariant solutions to the holomorphicity equations (5.4.28) *for the presentation* $\mathcal{Q} = \mathcal{Q}_{P,2i\partial\omega_0,A_{(h_0,h_1)}}$*, where:*

$$\omega_0 = \tfrac{i}{2}(\omega_{1\bar{1}} + \omega_{2\bar{2}} + \omega_{3\bar{3}}), \tag{5.4.54}$$

and coupling constant α *as in* (4.2.40)*, are given by:*

$$\begin{aligned} \phi &= \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \phi_{31} & \frac{2(3+b^2)(im_2\alpha_1^1 - m_1\alpha_1^2)}{m_2^2} - b_{23} + b\phi_{31} & 0 \end{pmatrix} \right), \\ \alpha_0 &\ \text{free}, \\ \alpha_1 &= \omega_1\alpha_1^1 + \omega_2\alpha_1^2, \\ \sigma &\in \mathrm{End}(\mathrm{End}\ V_0, \langle\cdot,\cdot\rangle) \ \text{ free}, \\ b &= b_{12}\omega_{12} + \left(\frac{2i(3+b^2)\alpha_1^2}{m_2} - bb_{23} - \phi_{31}\right)\omega_{13} + b_{23}\omega_{23}. \end{aligned} \tag{5.4.55}$$

We give now the result of evaluating each of the terms of the expression (5.4.38)*. Although our computations are general, here because of limitations of space, we have restricted to the family of balanced classes* $\mathfrak{b} = [\omega^2]$*, where (cf.* (4.2.13)*):*

$$\omega = i(\omega_{1\bar{1}} + s_2\omega_{2\bar{2}} + s_3\omega_{3\bar{3}}) + s_4(\omega_{1\bar{2}} - \omega_{2\bar{1}}) + s_6(\omega_{1\bar{3}} - \omega_{3\bar{1}}) + is_9(\omega_{2\bar{3}} + \omega_{3\bar{2}}). \tag{5.4.56}$$

Then, we compute separately each of the integrals in the expression for the Futaki invariant of Proposition 5.4.15 associated to each of the classical tensors for anchored endomorphisms. We provide here the explicit expressions:

$$\begin{aligned} I_\phi(\mathfrak{b}) &= \frac{2i(2(3+b^2)s_6\alpha_1^1 + (m_2+im_3)(is_6b_{23} + s_9\phi_{31}))}{(m_2+im_3)(s_3-s_6^2)} \int_X \frac{\omega^3}{3}, \\ I_\alpha(\mathfrak{b}) &= \frac{4(3+b^2)(-is_6\alpha_1^1(m_2-im_3) + s_9\alpha_1^2(m_2+im_3)}{(m_2^2+m_3^2)(s_3-s_6^2))} \int_X \frac{\omega^3}{3}, \\ I_\sigma(\mathfrak{b}) &= 0, \\ I_b(\mathfrak{b}) &= \frac{2s_6b_{23}(m_2-im_3) - 2s_9(2(3+b^2)\alpha_1^2 + i(m_2-im_3)\phi_{31})}{(m_2-im_3)(s_3-s_6^2)} \int_X \frac{\omega^3}{3}. \end{aligned} \tag{5.4.57}$$

We stress that each of the terms φ_ϕ, φ_α, φ_b is not holomorphic and only the sum of these Futaki terms is a sensible invariant. One can readily check that this sum indeed vanishes, as expected. However, these computations show that the Futaki invariant constructed can be computationally very involved.

A number of issues from the above Example should be avoided to find non-trivial invariants for the Hull-Strominger system. In the first place, we can test Futaki invariants for families of Bott-Chern algebroids modelled on the image of the map (2.2.38), which was trivial in Example 5.4.18. Moreover, from Section 5.4.2, it is clear that only the holomorphic structure of Bott-Chern algebroids is used to define Futaki invariants, hence the restriction to evaluating on anchored endomorphisms is not necessary. In the following Example, we embrace this more general setting to compute invariants.

Example 5.4.19. *Let $X = (\Gamma\backslash G, J)$ be the complex homogeneous manifold with underlying real Lie algebra* Lie $G = \mathfrak{h}_{19}^-$. *We refer to Section 4.2.2 and Example 4.3.6 for details. We have not been able to find solutions to the Hull-Strominger system with the instanton condition* (3.1.12) *on X, hence it is reasonable to look for obstructions on this manifold.*

Recall that $T^*_{1,0}$ has global frame $\{\omega_i\}$ satisfying the structure equations:

$$d\omega_1 = 0, \quad d\omega_2 = \omega_{13} + \omega_{1\bar{3}}, \quad d\omega_3 = i(\omega_{1\bar{2}} - \omega_{2\bar{1}}). \tag{5.4.58}$$

For simplicity, we will consider the Hull-Strominger system (3.1.14)*, where the principal bundle is $P = \mathrm{Fr}\, L_0 \times_X \mathrm{Fr} L_1$, where L_0 and L_1 are the holomorphic line bundles admitting hermitian metrics h_0, h_1 respectively satisfying:*

$$F_{h_j} = im_j(\omega_{1\bar{2}} - \omega_{2\bar{1}}), \quad m_j \in \mathbb{R}. \tag{5.4.59}$$

Then, a solution to the Bianchi identity with coupling constant $\alpha \in \mathbb{R}$:

$$dd^c\gamma - \alpha F_{h_0} \wedge F_{h_0} + \alpha F_{h_1} \wedge F_{h_1} = 0 \tag{5.4.60}$$

is given by:

$$\gamma = \tfrac{\alpha}{8}(m_0^2 - m_1^2)\omega_{3\bar{3}}. \tag{5.4.61}$$

This determines a Bott-Chern algebroid $\mathcal{Q}_{P,2i\partial\gamma,A}$ (see Example 2.2.6), where we denote $A = A_{(h_0,h_1)}$ for the Chern connection of of (h_0, h_1) on P.

On X, by Theorems 4.1.5, 4.1.6 and Remark 4.1.7, the Chevalley-Eilenberg cohomology group $H^{1,1}_A((\mathfrak{h}_{19}^-, J), \mathbb{R})$ computes the group $H^{1,1}_A(X, \mathbb{R})$. Therefore, it is easy to check that the image of the map (2.2.38) *parametrizing isomorphism classes of Bott-Chern algebroids extending P is parametrized by:*

$$\tau = it(\omega_{1\bar{2}} + \omega_{2\bar{1}}), \quad t \in \mathbb{R}. \tag{5.4.62}$$

determining an associated one-parameter family $\mathcal{Q}_\tau = \mathcal{Q}_{P,2i\partial(\gamma+\tau),A}$ of Bott-Chern algebroids. Moreover, using the composition map (5.4.21)*, one can see the holomorphic vector bundles underlying the algebroids $\mathcal{Q}_\tau$ and $\mathcal{Q}_{\tau'}$ are also non-isomorphic if $\tau \neq \tau'$. The holomorphic* *bundle $\mathcal{Q}_\tau$ has by construction a natural smooth splitting $\mathcal{Q}_\tau = (T^{1,0} \oplus \mathrm{ad}\ P \oplus T^*_{1,0}, \overline{\partial}_{\mathcal{Q}_\tau})$. Moreover, here* ad $P \cong \mathcal{O}_X^{\oplus 2}$. *Therefore, using the smooth global frames $\{X_i\}$ for $T^{1,0}$ and $\{\omega_i\}$ for $T^*_{1,0}$, we have a fibre-preserving bundle diffeomorphism $\mathcal{Q}_\tau \to X \times \mathbb{C}^8$. Explicitly:*

$$X_i \mapsto e_i, \quad \mathrm{id}_{\mathrm{End}\, V_j} \mapsto e_{j+4}, \quad \omega_i \mapsto e_{i+5}, \quad i = 1,2,3,\ j = 0,1, \tag{5.4.63}$$

where e_i are the constant sections induced by the standard basis of $\mathbb{C}^8$. With respect to this frame, we compute the Dolbeault operator of $\mathcal{Q}_\tau$, $\overline{\partial}_{\mathcal{Q}_\tau} = \overline{\partial} + \delta$, where $\delta \in M_{8\times 8}(\Omega^{0,1}_X)$ is given by:

$$\delta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \omega_{\overline{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ i\omega_{\overline{2}} & -i\omega_{\overline{1}} & 0 & 0 & 0 & 0 & 0 & 0 \\ im_0\omega_{\overline{2}} & -im_0\omega_{\overline{1}} & 0 & 0 & 0 & 0 & 0 & 0 \\ im_1\omega_{\overline{2}} & -im_1\omega_{\overline{1}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4t\omega_{\overline{1}} - \frac{i\alpha(m_0^2-m_1^2)}{4}\omega_{\overline{2}} & 2i\alpha m_0\omega_{\overline{2}} & -2i\alpha m_1\omega_{\overline{2}} & 0 & -\omega_{\overline{3}} & -i\omega_{\overline{2}} \\ 0 & 0 & \frac{i\alpha(m_0^2-m_1^2)}{4}\omega_{\overline{1}} & -2i\alpha m_0\omega_{\overline{1}} & 2i\alpha m_1\omega_{\overline{1}} & 0 & 0 & i\omega_{\overline{1}} \\ -4t\omega_{\overline{1}} + \frac{i\alpha(m_0^2-m_1^2)}{4}\omega_{\overline{2}} & \frac{-i\alpha(m_0^2-m_1^2)}{4}\omega_{\overline{1}} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.4.64}$$

Moreover, with respect to this frame, invariant endomorphisms are identified with constant sections with values in $M_{8\times 8}(\mathbb{C})$. Moreover, they are holomorphic if they commute with δ, that is, it commutes with $i_{\overline{X}_j}\delta$ for $j = 1,2,3$. For generic $t \in \mathbb{R}$, there are 12 *independent holomorphic endomorphisms. For illustration, we give here one of them:*

$$\varphi = (\varphi_{ij}), \quad \varphi_{41} = m_1,\ \varphi_{51} = m_0, \tag{5.4.65}$$

and the rest of components vanish.

We endow $\mathcal{Q}_\tau$ with the hermitian metric $\mathbf{H}_0$ induced by the standard hermitian inner product on $\mathbb{C}^8$. Then, the Futaki invariants with values in $H^{1,1}_A(\mathbb{X},\mathbb{R})$ with respect to the above holomorphic endomorphisms are given by $[\mathrm{tr}_{\mathcal{Q}_\tau}(\varphi F_{\mathbf{H_0}})]$. For instance, for φ in (5.4.65) *we obtain:*

$$[\mathrm{tr}_{\mathcal{Q}_\tau}(\varphi F_{\mathbf{H_0}})] = 2im_0m_1[\omega_{1\overline{3}}] = 2im_0m_1[\overline{\partial}\omega_2] = 0 \in H^{1,1}_A(X,\mathbb{C}). \tag{5.4.66}$$

For the other elements in the basis of holomorphic endomorphisms of $\mathcal{Q}_\tau$, we also obtain the zero Aeppli class. Therefore, we do not find non-trivial Futaki invariants for this Example.

The computations in Example 5.4.19 can be performed analogously on other complex homogeneous manifolds, with several choices for bundles (including higher rank), and for metrics solving the Bianchi identity. Using the ICMAT cluster, we have performed a large amount of computations applying the above method of Examples 5.4.18 and 5.4.19 systematically on every complex manifold described in Section 4.2, and also on the rest of homogenous manifolds in [51], with bundles V_0 and V_1 of ranks 1 to 5, and obtained Futaki invariants symbolically whenever possible or recurring to random sampling when the set of free parameters is too large. However, despite our efforts, we have not been able to find non-vanishing Futaki invariants with the property that the bundles are polystable with respect to the balanced class. It is highly unlikely that there is a non-trivial invariant on these manifolds using an invariant ansatz. Nevertheless, it is still completely plausible that there are non-vanishing Futaki invariants in higher cohomogeneity or outside the homogeneous realm.

Chapter 6

The coupled Hermite-Einstein system

The aim of this Chapter is to introduce a new system of differential equations in hermitian geometry called the *coupled Hermite-Eistein system*, first appearing in [65] as the set of conditions to construct a coupled Hermite-Einstein metric, in the sense of Chapter 4. Here, we retake the study of the geometry of this system, construct explicit solutions and discuss some interesting relations to other systems of equations in hermitian geometry and heterotic supergravity.

6.1 Introduction

Let $X = (M, J)$ be a complex manifold of complex dimension n, and let P be a holomorphic principal bundle with structure group G. We will assume that G is a complex reductive Lie group, with $K \subset G$ maximal compact subgroup. Moreover, we will assume $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is a quadratic Lie algebra, with the pairing restricting to real values on $\mathfrak{k} = \mathrm{Lie}\, K$.

Definition 6.1.1 ([65]). *Let (ω, h) be a pair of a hermitian metric on X and a reduction of structure of P to K, and let $z \in \mathrm{Lie}\, Z(K)$. Then (ω, h) solves the* coupled Hermite-Einstein system *with degree z in (X, P) if:*

$$\begin{aligned} F_h \wedge \omega^{n-1} &= \frac{z}{n}\omega^n, \\ \rho_B + \langle z, F_h \rangle &= 0, \\ dd^c\omega + \langle F_h \wedge F_h \rangle &= 0. \end{aligned} \tag{6.1.1}$$

From the last equation, the Bianchi identity, a solution to the coupled Hermite-Einstein system determines a positive Bott-Chern algebroid $\mathcal{Q} = \mathcal{Q}_{P,2i\partial\omega,A^h}$ as described in Example 2.2.6 (see also Definition 2.2.13). In fact, the coupled Hermite-Einstein system is closely related to the existence of coupled Hermite-Einstein metrics on $\mathcal{Q}$, as the next result shows:

Proposition 6.1.2. *Let (ω, h) be a solution to the Bianchi identity:*

$$dd^c\omega + \langle F_h \wedge F_h \rangle = 0. \tag{6.1.2}$$

Moreover, assume that (ω, h) solves the coupled Hermite-Einstein system (6.1.1) *on (X, P) with degree z. Then, the generalized hermitian metric $\mathbf{G}$ in Lemma 5.1.2 is a coupled Hermite-Einstein metric in $\mathcal{Q}$. The converse holds if K is connected.*

Proof. Let (ω, h) solve the coupled Hermite-Einstein system. Then, by Lemma 5.3.3, it is enough to show that the system (5.3.12) holds. By the first equation of (6.1.1), it follows that:

$$S_h = \Lambda_\omega F_h = z \tag{6.1.3}$$

is central, hence $[S_h, \cdot] = 0$. Moreover, combining the first equation of (6.1.1) with Lemma 5.3.2, we get that the second equation of (5.3.12) holds too, hence (5.3.12) is satisfied.

Conversely, assume (ω, h) satisfies the Bianchi identity and (5.3.12), and K is connected. By the first equation of (5.3.12), $\Lambda_\omega F_h$ is a section of $P_h \times_K \mathfrak{z}(\mathfrak{k}) \subset \operatorname{ad} P_h$. Using that K is connected, the adjoint action of K on $\mathfrak{z}(\mathfrak{k})$ is trivial, hence the bundle $P_h \times_K \mathfrak{z}(\mathfrak{k})$ admits a global trivialization by a basis of $\mathfrak{z}(\mathfrak{k})$. Therefore, we obtain that:

$$\Lambda_\omega F_h \in \Gamma(X \times \mathfrak{z}(\mathfrak{k})). \tag{6.1.4}$$

By the computation in the proof of Lemma 5.3.2, it follows that:

$$d^{h\star} F_h + i_{\theta_\omega^\sharp} F_h - \star(F_h \wedge \star d^c\omega) = -J d^h(\Lambda_\omega F_h), \tag{6.1.5}$$

which vanishes by the first equation of (5.3.12). But then, combined with (6.1.4), we conclude that d^h restricted to the trivial subbundle $X \times \mathfrak{z}(\mathfrak{k}) \subset \operatorname{ad} P_h$ is the trivial connection and $\Lambda_\omega F_h$ is constant. Therefore, we obtain that:

$$\Lambda_\omega F_h = z, \tag{6.1.6}$$

for some central element z. Using the last equation of (5.3.12) and the Bianchi identity, we conclude that the coupled Hermite-Einstein system (6.1.1) holds with degree z. □

Remark 6.1.3. *The condition of K being connected does not suppose a strong loss of generality as the principal application of the results in this Thesis is the study of Hull-Strominger system* (3.1.11) *with the ansatz* (3.1.12), *for which $K = SU(3) \times SU(r)$ is indeed a connected Lie group. Therefore, in the sequel, we will implicitly assume this and identify coupled Hermite-Einstein metrics as in Definition 5.3.1 with equivalently solutions to* (5.3.12) *or* (6.1.1).

As suggested in the previous Remark, the coupled Hermite-Einstein system is related to the Hull-Strominger system via the following result, which should be regarded as a slight strengthening of Proposition 5.3.5, in the light of Proposition 6.1.2 above.

Proposition 6.1.4. *Let (X, Ω) be a Calabi-Yau manifold. Let P a holomorphic principal G-bundle as stated in this Section. Assume (ω, h) solves the Hull-Strominger system* (3.1.14). *Then (ω, h) is a solution to the coupled Hermite-Einstein system on (X, Ω, P) with $z = 0$.*

Proof. The Hermite-Einstein equation of the Hull-Strominger system (3.1.14) forces $z = 0$. Now, by the argument of the proof of Proposition 5.3.5, we obtain that:

$$\rho_B = 0, \tag{6.1.7}$$

hence the coupled Hermite-Einstein system (6.1.1) holds with $z = 0$. □

Observe that in contrast to the Hull-Strominger system (3.1.14), the more flexible coupled Hermite-Einstein system can be, in principle, posed for any complex manifold with no topological or holomorphic requirements. However, if X satisfies some additional conditions, one has the next partial converse to Proposition 6.1.4:

Proposition 6.1.5. *Let (X, Ω) be a compact, simply-connected Calabi-Yau manifold and P a holomorphic principal G-bundle as in this Section. Then a solution (ω, h) to the coupled Hermite-Einstein system* (6.1.1) *with $z = 0$ solves the Hull-Strominger system* (3.1.14).

Proof. It is enough to prove that under the assumptions, $\rho_B = 0$ actually implies the conformally balanced condition:

$$d(||\Omega||_\omega \omega^{n-1}) = 0. \tag{6.1.8}$$

To see this, observe that by Proposition 1.2.1(1) and the fact that X is simply connected, then the global holonomy of ∇^B induced in K_X is trivial. Then, since X is compact, by [62, Proposition 3.6], the equation (6.1.8) holds.

□

We finish this Section with a discussion about the quantity z appearing in the coupled Hermite-Einstein system. To give an interpretation of z as a degree, we need to regard:

$$F_h \wedge \omega^{n-1} = \frac{z}{n}\omega^n \tag{6.1.9}$$

as the Hermite-Einstein equation with respect to a hermitian metric on X defining a cohomology class. For this, let $\tilde{\omega} = e^f \omega$ be the Gauduchon representative of the conformal class of ω (see Theorem 1.1.2). Recall that f is unique up to an additive constant, which for the moment we do not fix. Then, (6.1.9) is equivalent to:

$$F_h \wedge \tilde{\omega}^{n-1} = \frac{ze^{-f}}{n}\tilde{\omega}^n. \tag{6.1.10}$$

Where now $[\tilde{\omega}] \in H_A^{n-1,n-1}(X, \mathbb{R})$. To recover a Hermite-Einstein equation, we now describe a conformal change in h.

Using the polar decomposition of a complex reductive group, we have that $G = \exp(i\mathfrak{k})K$. Hence, a reduction to a maximal compact subgroup K given by $h \in \Gamma(P/K)$ is equivalent to an equivariant map:

$$h : P \longrightarrow \exp(i\mathfrak{k}). \tag{6.1.11}$$

Then, we set $h' = e^{iu}h$, for a section u with central values. Under the hypothesis that K is connected, u is identified with a section of the trivial bundle $u \in \Gamma(X \times \mathfrak{k})$ (see the proof of Proposition 6.1.2. Then, one can prove that (see e.g. the proof of [68, Lemma 3.23]):

$$F_{h'} = F_h + 2i\overline{\partial}\partial u. \tag{6.1.12}$$

Therefore, the Hermite-Einstein equation for h':

$$F_{h'} \wedge \tilde{\omega}^{n-1} = \frac{z}{n}\tilde{\omega}^n \tag{6.1.13}$$

is equivalent to the scalar Poisson equation:

$$\Delta_{\tilde{\omega}} u = z(1 - e^{-f}). \tag{6.1.14}$$

Then, the normalization of f may be chosen such that (6.1.14) admits a solution. The upshot is that (6.1.9) is actually equivalent to a Hermite-Einstein equation that measures a topological degree, and therefore links directly to the GIT theory of stability for principal bundles (see [7], in a Kähler setting). Of course, if in the previous discussion the structure group is taken to be $G = GL(r, \mathbb{C})$ acting on the frames of a vector bundle $P = \mathrm{Fr}\, V$, then, the conformal rescaling described above is equivalent to the rescaling of a hermitian metric on V.

Once the interpretation of z as a degree is justified, one can ask about its uniqueness. In the vector bundle case, it is well-known the degree is determined by its topological type. For principal bundles, this result carries over with minor assumptions:

Proposition 6.1.6. *Let (X, P) be a compact complex manifold and a holomorphic principal bundle as in this Section. Moreover, assume that:*

$$[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{z}(\mathfrak{k}) = \{0\}. \tag{6.1.15}$$

Moreover, let ω be a Gauduchon metric and h a hermitian metric on P solving the Hermite-Einstein equation with degree $z \in \mathfrak{k}$:

$$F_h \wedge \omega^{n-1} = \frac{z}{n}\omega^n, \tag{6.1.16}$$

then z is uniquely determined.

Proof. The characters of the Lie algebra $\mathfrak{g}$ are one-to-one with elements in $(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$. Therefore, under condition (6.1.15), there are no non-trivial central elements in the kernel of every character. Now, let χ be a character. Observe that by definition, it induces a well-defined map:

$$\chi : \Gamma(\mathrm{ad}\ P) \longrightarrow \mathcal{C}^\infty_{\mathbb{C}}. \tag{6.1.17}$$

Hence, by evaluation of (6.1.16), we obtain complex numbers:

$$c_1(P, \chi) = \int_X \chi(F_h) \wedge \omega^{n-1}, \tag{6.1.18}$$

which may be regarded as an abstract first Chern class for P. We claim that these are invariants of P: indeed, for a different metric h' on P, given the polar decomposition of G, we can write $h' = e^{iu}h$, where $u \in \Gamma(\mathrm{ad}\ P_h)$. Then, we consider the path $h_t = e^{itu}h$ joining h and h'. Using the formulae in the proof of [68, Lemma 3.23]:

$$\frac{d}{dt}_{|t=t_0} \chi(F_{h_t}) = \chi(-2\overline{\partial}\partial^{h_t}((\tfrac{d}{dt}_{|t=t_0} h_t)h_{t_0}^{-1})) = -2\overline{\partial}\partial\chi(\tfrac{d}{dt}_{|t=t_0} h_t)h_{t_0}^{-1}),$$

hence, we obtain:

$$\frac{d}{dt}_{|t=t_0} c_1(P, \chi) = -2\int_X \overline{\partial}\partial\chi(\tfrac{d}{dt}_{|t=t_0} h_t)h_{t_0}^{-1}) \wedge \omega^{n-1} = 0,$$

since ω is Gauduchon. Finally, by the first argument in this proof, if any two solutions of (6.1.16) exist on (X, P), they share the same degree z, as it is uniquely determined by P using all characters. □

6.2 Obstructions the existence of solutions

6.2.1 Topological obstructions

In this Section, we exploit the cohomological features of the coupled Hermite-Einstein system 6.1.1 to produce obstructions depending on the complex topology of X and the signature of the pairing on $\mathfrak{g}$. Throughout this Section, we make the same assumptions on X and P as in the previous Section. First, we need the following definition:

Definition 6.2.1. *Let X be a compact complex manifold of complex dimension n and $\sigma \in H^{1,1}_{BC}(X,\mathbb{R})$. Then σ is called:*

1. *Bott-Chern numerically positive (negative) if for any cohomology class $\mathfrak{a} \in H^{n-1,n-1}_A(X,\mathbb{R})$ represented by a Gauduchon metric,*

$$\sigma \cdot \mathfrak{a} > 0 \ \ (\text{resp. } < 0) \tag{6.2.1}$$

2. *Bott-Chern numerically semipositive (seminegative) if for any cohomology class $\mathfrak{a} \in H^{n-1,n-1}_A(X,\mathbb{R})$ represented by a Gauduchon metric,*

$$\sigma \cdot \mathfrak{a} \geq 0 \ \ (\text{resp. } \leq 0) \tag{6.2.2}$$

Observe that any class represented by a pointwise (semi)positive $(1,1)$-form is Bott-Chern numerically (semi)positive, and analogously in the (semi)negative case. Moreover, if X is Kähler, then any positive (negative) class is Bott-Chern numerically positive (resp. negative) in this sense too. However, the converse is not true, even if X is Kähler. In that case, integral Bott-Chern numerically semipositive classes are naturally identified with nef line bundles, for which hermitian metrics with positive curvature need not exist (see e.g. [39, Example 1.7]).

Before introducing the topological obstructions, we recall that for a smooth vector bundle, while $E \to X$, $c_1(E)$ is well-defined in $H^2_{dR}(X,\mathbb{R})$, to define a class in Bott-Chern cohomology, a holomorphic structure $\mathcal{E}$ on E is needed (see Section 1.3.1), so that:

$$c_1(\mathcal{E}) = [c_1(E, D^h)] = [\tfrac{i}{2\pi} F_{D^h}] \in H^{1,1}_{BC}(X,\mathbb{R}) \tag{6.2.3}$$

where D^h is the Chern connection of any hermitian metric h on E. In what follows, we reserve the notation $c_1(T^{1,0}) \in H^{1,1}_{BC}(X,\mathbb{R})$ for the first Chern class of the holomorphic tangent bundle of X. This discussion motivates the introduction of the vector space:

$$K_{BC}(X) = \ker(H^2_{BC}(X,\mathbb{R}) \to H^2_{dR}(X,\mathbb{R})), \tag{6.2.4}$$

where we denote:

$$H^2_{BC}(X) = \frac{\ker(d : \Omega^2_X \longrightarrow \Omega^3_X)}{\mathrm{im}(\mathrm{dd^c} : \Omega^0_X \longrightarrow \Omega^2_X)}. \tag{6.2.5}$$

Note that:

$$H^2_{BC}(X) \supset H^{2,0}_{BC}(X) \oplus H^{1,1}_{BC}(X) \oplus H^{0,2}_{BC}(X), \tag{6.2.6}$$

but they are not equal, in general. Here, $H^2_{BC}(X,\mathbb{R}) \subset H^2_{BC}(X)$ is the set of fixed classes under conjugation. Then:

$$c_1(E) \in H^2_{BC}(X,\mathbb{R})/K_{BC} \tag{6.2.7}$$

is well-defined, irrespective of holomorphic structures. We also denote:

$$K^{1,1}_{BC}(X,\mathbb{R}) = K_{BC}(X) \cap H^{1,1}_{BC}(X,\mathbb{R}). \tag{6.2.8}$$

In the sequel we will use the following observation connecting the topology and hermitian geometry of complex manifolds, which is interesting on its own.

Proposition 6.2.2. *Let X be a compact complex manifold, and assume $K_{BC}(X) = \{0\}$. Then, any Gauduchon metric is balanced. Conversely, if any Gauduchon metric is balanced, then $K^{1,1}_{BC}(X) = 0$.*

Proof. Suppose $K_{BC}(X) = \{0\}$ and let ω be a Gauduchon metric. We denote $\mathfrak{a} = [\omega^{n-1}] \in H^{n-1,n-1}_A(X,\mathbb{R})$. Since $[dd^\star\omega] = 0 \in H^2_{dR}(X,\mathbb{R})$, using the hypothesis, we have that $[dd^\star\omega] = 0 \in H^2_{BC}(X,\mathbb{R})$. Therefore, we may write $dd^\star\omega = dd^c f$, for some smooth function f. Then :

$$\int_X dd^\star\omega \wedge \omega^{n-1} = \int_X dd^c f \wedge \omega^{n-1} = -\int_X f dd^c\omega^{n-1} = 0.$$

Using this result, integrating by parts we obtain:

$$\begin{aligned} 0 = \int_X dd^\star\omega \wedge \omega^{n-1} &= \int_X dd^\star\omega \wedge (n-1)! \star\omega = (n-1)! \int_X d^\star\omega \wedge d(\star\omega) \\ &= (n-1)! ||d^\star\omega||^2_{L^2(\omega)} \end{aligned}$$

which holds if and only if $d^*\omega = 0$, that is, ω is balanced.

Conversely, let $\mathfrak{b} \in K^{1,1}_{BC}(X)$. Then, using the duality pairing $H^{1,1}_{BC}(X) \cong H^{n-1,n-1}_A(X)^*$, the class $\mathfrak{b}$ vanishes on the Gauduchon cone, since $\mathfrak{b}$ is represented by a d-exact form and any Gauduchon metric is balanced, hence d-closed. Given that the Gauduchon cone is a non-empty open set inside $H^{n-1,n-1}_A(X)$, it follows that $\mathfrak{b} = 0$. □

Remark 6.2.3. *Observe that the condition $K_{BC}(X) = \{0\}$ is equivalent to the condition that the $\partial\bar{\partial}$-Lemma holds on 2-forms. Hence Proposition 6.2.2 can be regarded as a strengthening of the fact that Gauduchon metrics are balanced on $\partial\bar{\partial}$-manifolds.*

Now, building on Proposition 6.2.2, we obtain topological obstructions to solving the coupled Hermite-Einstein system (6.1.1). Interestingly, these restrictions depend on the signature of the pairing in the quadratic Lie algebra $(\mathfrak{k}, \langle\cdot,\cdot\rangle)$.

Proposition 6.2.4. *Let X be a compact complex manifold, P a principal G-bundle as in this Section. Consider the coupled Hermite-Einstein system on (X, P) with degree z. Then:*

1. *If $\langle \cdot, \cdot \rangle$ is negative definite and $c_1(T^{1,0})$ is Bott-Chern seminegative, then for any solution (ω, h), ω is Kähler and h is flat.*

2. *If $\langle \cdot, \cdot \rangle$ is seminegative definite and $c_1(T^{1,0})$ is Bott-Chern negative, then there exist no solutions.*

Moreover, if $K_{BC}(X) = \{0\}$:

3. *If $\langle \cdot, \cdot \rangle$ is positive definite and $c_1(T^{1,0})$ is Bott-Chern semipositive, then any solution is conformally balanced and has degree $z = 0$.*

4. *If $\langle \cdot, \cdot \rangle$ is semipositive definite and $c_1(T^{1,0})$ is Bott-Chern positive, then there exist no solutions.*

Proof. Suppose (ω, h) is a solution to the coupled Hermite-Einstein system with degree z, and let $\tilde{\omega} = e^u \omega$ be the Gauduchon metric in the conformal class of ω, for some smooth real function u, and let $\mathfrak{a} = [\tilde{\omega}^{n-1}]$. Then, by the transformation rule for conformal rescaling (see e.g. [17]):

$$\rho_B(\tilde{\omega}) = \rho_B(\omega) + (\tfrac{n}{2} - 1) dd^c u \tag{6.2.9}$$

and the relation to the Chern-Ricci form ([3]):

$$\rho_B = \rho_C - dd^*(\omega) \tag{6.2.10}$$

which is valid for any hermitian metric. Then,

$$\begin{aligned} 0 &= \rho_B(\omega) \wedge \tilde{\omega}^{n-1} + \langle z, F_h \wedge \tilde{\omega}^{n-1} \rangle \\ &= (\rho_C(\tilde{\omega}) - dd^{\tilde{*}} \tilde{\omega} - (\tfrac{n}{2} - 1) dd^c u) \wedge \tilde{\omega}^{n-1} + \langle z, z \rangle e^{(n-1)u} \tfrac{\omega^n}{n} \end{aligned}$$

where we have used the above formulas and the Hermite-Einstein equation for h with respect to ω.

Then, integrating on X the above equation:

$$2\pi c_1(T^{1,0}) \cdot \mathfrak{a} - ||d^{\tilde{*}} \tilde{\omega}||_{L^2(\tilde{\omega})} + |z|^2 \int_X e^{(n-1)u} \frac{\omega^n}{n} = 0 \tag{6.2.11}$$

In the hypothesis of (1), since $e^{(n-1)u}$ is a positive function, all the terms in (6.2.11) are non-positive. Therefore any solution must be, in particular, conformally balanced and with degree $z = 0$. Hence, in particular:

$$F_h \wedge \tilde{\omega}^{n-1} = 0 \tag{6.2.12}$$

$$d\tilde{\omega}^{n-1} = 0. \tag{6.2.13}$$

Using these conditions, it follows from a computation that:

$$\int_X \Lambda_{\tilde{\omega}} (dd^c \omega + \langle F_h \wedge F_h \rangle) \wedge \tilde{\omega}^{n-2} = 2(n-1)!(\langle\langle e^{-u} d^c \tilde{\omega}, d^c \tilde{\omega} \rangle\rangle_{\tilde{\omega}} - ||F_h||^2_{\tilde{\omega}}) \tag{6.2.14}$$

related to already existing formulas in the literature with no bundle term [52, Formula (18)], [64, Lemma 5.3] (see also [101]). Since the left hand side vanishes and the pairing is negative-definite, h is flat. It follows that ω is pluriclosed and conformally balanced. Then, by Theorem 1.1.3 it is a Kähler metric. In the hypothesis of (2), the first term in (6.2.11) is negative and the rest are non-positive, reaching a contradiction.

If $K_{BC}(X) = \{0\}$, by Proposition 6.2.2, the second term vanishes hence ω is conformally balanced. Moreover, in the hypothesis of (3), the remaining terms are non-negative. Therefore, $z = 0$. In the case of (4) any solution yields a contradiction. $\square$

Remark 6.2.5. *We observe that combining Proposition 6.1.4 (see also [65, Section 5.2]) and Proposition 6.2.4*(1)*, we recover in particular the no-go result in [24] for solutions on a (a priori non-Kähler) Calabi-Yau manifold* (X, Ω) *endowed with a holomorphic vector bundle, to the system*

$$\begin{aligned} F_h \wedge \omega^{n-1} &= 0 \\ d(||\Omega||_\omega \omega^{n-1}) &= 0 \\ dd^c\omega + \alpha \mathrm{tr}\, F_h \wedge F_h &= 0 \end{aligned}$$

where Ω *is a holomorphic volume form, and with coupling constant* $\alpha > 0$*, which corresponds to the Hull-Strominger system 3.1.11 dropping the* ∇ *connection on the tangent bundle.*

Remark 6.2.6. *As a by-product of Proposition 6.2.4, observe in particular that taking the principal bundle* P *to be trivial,* (2) *and* (4) *provide topological obstructions to existence of hermitian metrics satisfying:*

$$dd^c\omega = 0, \quad \rho_B = 0. \tag{6.2.15}$$

We will return to the study of these metrics in Chapter 7.

The rest of this Section is devoted to illustrate in explicit Examples applications of the topological obstructions of Proposition 6.2.4 in a variety of contexts.

Example 6.2.7. *Let* $X \subset \mathbb{C}P^N$ *be a projective manifold defined by the vanishing of homogenenous polinomials*

$$X = \{P_1 = ... = P_r = 0\}$$

and let $d_i = \deg\, P_i$*. In particular,* X *is Kähler and therefore* $K_{BC}(X) = \{0\}$*. By the adjunction formula,* $c_1(T^{1,0})$ *is positive, negative or vanishing if* $\sum_{i=1}^r d_i$ *is less, greater or equal to* $N+1$ *respectively.*

Let $P \to X$ *be a principal* G*-bundle with the assumptions of this Section, and suppose* (ω, h) *is a solution to the coupled Hermite-Einstein with degree* z*. By Proposition 6.2.4, in case* $\sum_i d_i < N+1$*,* X *is Fano and no solutions exist for semipositive-definite pairing. Analogously, in case* $\sum_i d_i > N+1$*, no solutions exist for seminegative-definite pairing.*

In the case $\sum_i d_i = N+1$*,* X *is a projective Calabi-Yau manifold and only solutions of degree* $z = 0$ *can exist when* $\langle \cdot, \cdot \rangle$ *has a sign. Observe that these do in fact exist e.g. for* ω *Kähler Calabi-Yau and* h *flat.*

Example 6.2.8. *Let $X = \Gamma\backslash H$ be the Iwasawa manifold (see Example 4.2.1), where $H_{\mathbb{C}}$ is the complex Heisenberg group:*

$$H = \left\{ \begin{pmatrix} 1 & z_2 & z_3 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix} \mid z_i \in \mathbb{C} \right\} \tag{6.2.16}$$

and Γ is the lattice of matrices with entries in Gaussian integers $\mathbb{Z}[i]$. Recall the left-invariant $(1,0)$-forms that descend to global frame of $T^*_{1,0}X$

$$\omega_1 = dz_1, \;\; \omega_2 = dz_2, \;\; \omega_3 = dz_3 - z_2 dz_1 \tag{6.2.17}$$

and the real Lie algebra of X is determined by the structure equations:

$$d\omega_1 = d\omega_2 = 0, \;\; d\omega_3 = \omega_{12}. \tag{6.2.18}$$

Then, it is straightforward that $\Omega = \omega_{123}$ is a holomorphic volume form, hence $c_1(T^{1,0}) = 0 \in H^{1,1}_{BC}(X, \mathbb{R})$. Moreover, the d-exact form $d\omega_3 \in \Omega^{2,0}$ is not dd^c-exact, X does not satisfy the $\partial\bar{\partial}$-lemma. In particular X does not admit Kähler metrics.

Let $P \to X$ be a principal G-bundle as in this Section and consider the coupled Hermite-Einstein system on (X, P) with negative-definite pairing. By Proposition 6.2.4, supposing a solution (ω, h) exists, in particular ω is a Kähler metric. Since X does not support Kähler structures, we reach a contradiction.

The following is an interesing Example where we illustrate how Proposition 6.2.4 carries over to complex manifolds which do not admit a Kähler structure and neither carry a holomorphic volume form.

Example 6.2.9. *Consider the non-Kähler family of Inoue surfaces S_M ([87]), which we briefly describe here for the benefit of the reader. Let $M = (m_{ij}) \in SL(\mathbb{Z}, 3)$ with real eigenvalue $r > 1$ and complex eigenvalues $w, \overline{w}$, and consider the action*

$$\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}^3), \quad n \cdot (k_1, k_2, k_3) \mapsto (k_1, k_2, k_3)M^n. \tag{6.2.19}$$

We denote $(r_1, r_2, r_3) \in \mathbb{R}^3$ and $(w_1, w_2, w_3) \in \mathbb{C}^3$ its r and w-eigenvectors respectively. Then, $\{(r_i, w_i)\}_{i=1}^3$ is an $\mathbb{R}$-basis of $\mathbb{R} \times \mathbb{C}$. Consider the group action $\Gamma = \mathbb{Z} \ltimes \mathbb{Z}^3$ on $\tilde{S} = \mathbb{H} \times \mathbb{C}$, where $\mathbb{H}$ denotes the complex upper-half plane, given by the biholomorphisms:

$$\begin{aligned} f_{(k_1,k_2,k_3)}(z_1, z_2) &= (z_1 + \Sigma_i\, k_i r_i, z_2 + \Sigma_i\, k_i w_i), \;\; (k_1, k_2, k_3) \in \mathbb{Z}^3 \\ f_n(z_1, z_2) &= (r^n z_1, w^n z_2), \;\; n \in \mathbb{Z}. \end{aligned} \tag{6.2.20}$$

which is free and properly discontinuous due to the relations:

$$\textstyle\sum_j m_{ij}(r_j, w_j) = (r r_i, w w_i). \tag{6.2.21}$$

The Inoue surface is $S_M = \Gamma\backslash\tilde{S}$. Now, following [130], the $(1,1)$-forms on $\tilde{S}$

$$\alpha_1 = i\frac{dz_1 \wedge d\bar{z}_1}{\mathrm{Re}(z_1)^2}, \quad \alpha_2 = i\mathrm{Re}(z_1) dz_2 \wedge d\bar{z}_2 \tag{6.2.22}$$

are Γ-invariant and hence they descend to S_M. Moreover,

$$\rho_C(\alpha_1 + \alpha_2) = -\frac{\alpha_1}{4} \tag{6.2.23}$$

so we conclude $c_1(T^{1,0}S_M) \in H^{1,1}_{BC}(S_M, \mathbb{R})$ is Bott-Chern numerically seminegative in the sense of Definition 6.2.1.

Now, let $P \to S_M$ be a principal G-bundle as in this Section. Then the coupled Hermite-Einstein system on (S_M, P) with negative-definite pairing admits no solutions. Indeed, by Proposition 6.2.4, any such solution is Kähler on S_M, reaching a contradiction.

6.2.2 Algebraic obstructions

There is another source of obstructions to the coupled Hermite-Einstein system 6.1.1 related to slope-stability in the sense of Mumford-Takemoto. This theory has been recently studied in [67] to produce algebraic obstructions with no bundle P. We now generalize this picture to accomodate the gauge bundle.

Next, we apply this theory to the coupled Hermite-Einstein system. As usual, we denote by X a compact complex manifold of complex dimension n, and P a holomorphic principal bundle for a complex reductive Lie group G. The Lie algebra of G is quadratic with pairing $\langle \cdot, \cdot \rangle$ restricting to real values on Lie K, where $K \subset G$ is a fixed maximal compact subgroup. First, as a consequence of the above result we obtain the following:

Proposition 6.2.10. *Let X be a compact complex manifold and P a principal bundle as above. Assume the pairing $\langle \cdot, \cdot \rangle$ is negative definite. Moreover, assume (ω, h) is a solution to the coupled Hermite-Einstein system* (6.1.1) *on (X, P) for some degree z. Then, the Bott-Chern algebroid $\mathcal{Q}_{P,2i\partial\omega,A^h}$ in Example 2.2.6 is polystable with respect to the Gauduchon class $[\tilde{\omega}^{n-1}]$, with $\tilde{\omega}$ in the conformal class of ω.*

Proof. As a consequence of Proposition 6.1.2, the generalized hermitian metric $\mathbf{G}$ of Lemma 5.1.2 in $\mathcal{Q} = \mathcal{Q}_{P,2i\partial\omega,A^h}$ is positive-definite, as $\langle \cdot, \cdot \rangle$ is negative-definite, by assumption, and moreover satisfies:

$$F_{\mathbf{G}} \wedge \omega^{n-1} = 0. \tag{6.2.24}$$

Then, Theorem 1.3.6 applies with $h = \mathbf{G}$ and $\sigma = [\tilde{\omega}^{n-1}]$, hence $\mathcal{Q}$ is σ-polystable. □

As a consequence of this result, we extract algebraic obstructions to the existence of solutions to the coupled Hermite-Einstein system in the case the pairing $\langle \cdot, \cdot \rangle$ is negative-definite, that go beyond the topological obstructions of Section 6.2.2. In particular, when the bundle P is trivial these obstructions recover the ones introduced in [67, Section 4.3].

Corollary 6.2.11. *Let X be a compact complex manifold and P a principal bundle as above, such that the structure group G is unimodular. Assume the pairing $\langle \cdot, \cdot \rangle$ is negative definite, and consider the coupled Hermite-Einstein system* (6.1.1) *on (X, P):*

1. *If $c_1(T^{1,0})$ is Bott-Chern numerically seminegative, then for any solution (ω, h), ω is Kähler and h is flat.*

2. *If there exists a subsheaf $\mathcal{F} \subset T^*_{1,0}$ with $c_1(\mathcal{F})$ Bott-Chern numerically positive, then there exist no solutions.*

Proof. Let (ω, h) be a solution to the coupled Hermite-Einstein system under the hypothesis. Using Proposition 6.2.10, the bundle $\mathcal{Q} = \mathcal{Q}_{P,2i\partial\omega,A^h}$ is σ-polystable, for σ the Gauduchon class determined by the solution. Now, observe that by construction of $\mathcal{Q}$:

$$\begin{aligned} c_1(\mathcal{Q}) &= c_1(T^*_{1,0}) + c_1(A_P) \\ &= c_1(T^*_{1,0}) + c_1(T^{1,0}) + c_1(\mathrm{ad}\ P) = 0, \end{aligned}$$

where we have used $c_1(T^*_{1,0}) = -c_1(T^{1,0})$, and $c_1(\mathrm{ad}\ P) = 0$, as its curvature form $[F_A, \cdot]$ for A a principal connection on P, is traceless, since by assumption G is unimodular. Now consider $T^*_{1,0} \overset{\rho^*}{\hookrightarrow} \mathcal{Q}$ as a holomorphic subbundle. In the hypothesis of (1):

$$\deg_\sigma T^*_{1,0} = -\deg_\sigma T^{1,0} \geq 0,$$

but by polystability of $\mathcal{Q}$ it must actually vanish and $\mathcal{Q}$ splits. Since $\mathbf{G}$ is a Hermite-Einstein metric for $\mathcal{Q}$, it provides a holomorphic and metric splitting:

$$\mathcal{Q} = T^*_{1,0} \underset{\perp_{\mathbf{G}}}{\oplus} A_P. \tag{6.2.25}$$

However, then by the expression of $\overline{\partial}_{\mathcal{Q}}$ in Example 2.2.6, this means that:

$$\partial\omega = 0, \quad \langle F_h, \cdot \rangle = 0. \tag{6.2.26}$$

It follows that ω is Kähler, and since $\langle\cdot,\cdot\rangle$ is non-degenerate, that h is flat, and the first item follows. For the second, the existence of a solution to the coupled Hermite-Einstein system implies the polystability of $\mathcal{Q}$ as before, but then, by assumption:

$$\deg_\sigma \mathcal{F} > 0 = \deg_\sigma \mathcal{Q},$$

reaching a contradiction. □

Example 6.2.12. *Let $X \overset{p}{\to} Z$ be a generically holomorphic submersion, where Z is Kähler manifold with $c_1(Z) < 0$, and let $P \to X$ as in this Section, with unimodular structure group and negative-definite pairing. Then, the coupled Hermite-Einstein system* (6.1.1) *does not admit solutions on (X, P). Indeed, by the Aubin-Yau theorem [15, 137], there exists a hermitian metric ω_Z in Z such that:*

$$\rho(\omega_Z) = -\omega_Z. \tag{6.2.27}$$

Then:

$$c_1(p^*T^*_{1,0}Z) = -c_1(p^*T^{1,0}Z) = 2\pi[p^*\omega_Z]$$

is Bott-Chern numerically positive, as for any Gauduchon class $[\omega_0^{n-1}]$:

$$\int_X p^*\omega_Z \wedge \omega_0^{n-1} = \int_{X\setminus Y} p|^*\omega_Z \wedge \omega_0^{n-1} = \int_{X\setminus Y} (\Lambda_{\omega_0} p|^*\omega_Z)\frac{\omega_0^n}{n} > 0,$$

where we have restricted to the locus $X\setminus Y$ where p is submersive. Then, Corollary 6.2.11(2) *applies for the subsheaf $p^*T^*_{1,0}Z \subset T^*_{1,0}X$.*

6.3 Exact solutions

The Hull-Strominger system (3.1.14) is a natural source of solutions to the coupled Hermite-Einstein system. However, the latter is more flexible and does not require any extra topological or holomorphic condition. In particular, it admits solutions on complex manifolds that do not support a holomorphic volume form or balanced metrics. The purpose of this Section is to describe some solutions of the coupled Hermite-Einstein system on these manifolds.

6.3.1 Solutions in low dimensions

In this Section, we construct solutions to the coupled Hermite-Einstein system in low dimensions. Throughout, we denote by C a Riemann surface, S a compact complex surface, and in either case P is a holomorphic principal bundle with the same assumptions as in previous Sections. In the case of Riemann surfaces, the solutions are completely classified.

Proposition 6.3.1. *Let C be a compact Riemann surface of genus g, and let P be a holomorphic principal bundle as in this Section. Let (ω, h) be a solution to the coupled Hermite-Einstein system* (6.1.1) *on (C, P) with degree z. Then, ω is a constant scalar curvature metric and h is Hermite-Einstein with respect to ω. Moreover:*

1. *$g = 1$ if and only if $|z|^2 = 0$.*
2. *If $g \neq 1$, then the volume of C is fixed by:*

$$\mathrm{vol}(C, \omega) = \frac{2\pi(2g-2)}{|z|^2}. \tag{6.3.1}$$

Proof. Let (ω, h) be a solution. Then, combining the first and second equations of (6.1.1), we obtain:

$$F_h = z\omega \tag{6.3.2}$$

$$\rho = -|z|^2\omega, \tag{6.3.3}$$

hence the first part of the result follows. Now, recall that:

$$\int_C \rho = 2\pi\chi(C) = 2\pi(2-2g). \tag{6.3.4}$$

Therefore, integrating (6.3.3):

$$2\pi(2-2g) + |z|^2\mathrm{vol}(C, \omega) = 0. \tag{6.3.5}$$

Therefore (1) and (2) follow. □

From complex dimension 2 onwards, the situation is richer and more subtle. This shall be compared with the case of the Hull-Strominger system, for which solutions in surfaces are completely classified (see e.g. [62]). This is due to the fact that there, in dimension 2 solutions are still (conformally) Kähler, while this is not the case for the coupled Hermite-Einstein system. For the case of trivial principal bundle, the system reduces to finding hermitian metrics satisfying the following conditions:

Definition 6.3.2. *[71] Let X be a complex manifold. Then a hermitian metric ω is* Bismut-Hermite-Einstein *if:*

$$dd^c\omega = 0, \quad \rho_B = 0. \tag{6.3.6}$$

In dimension 2, solutions are already classified by the following result.

Proposition 6.3.3 ([71])**.** *Let S be a compact complex surface. Let ω be a Bismut-Hermite-Einstein metric on S. Then either:*

1. *ω is Kähler Calabi-Yau and S is biholomorphic to a complex torus or a $K3$ surface.*
2. *S is a finite quotient of a diagonal Hopf surface, which is given by*

   $$(\mathbb{C}^2\backslash\{0,0\})/\mathbb{Z}, \quad n\cdot(z_1,z_2) = (t_1^n z_1, t_2^n z_2), \quad |t_1| = |t_2| > 1.$$

   with hermitian metric induced on S by

   $$\omega = \frac{idz_1\wedge d\overline{z}_1 + idz_2\wedge d\overline{z}_2}{|z_1|^2+|z_2|^2}$$

   on $\mathbb{C}^2\backslash\{0,0\}$.

In the following results, we give sufficient conditions under which a similar rigidity result holds. Note that we do not claim that these exhaust the solutions of the coupled Hermite-Einstein system in complex dimension 2.

Proposition 6.3.4. *Let (S,P) be a compact complex surface and a holomorphic principal bundle as in this Section. Moreover, assume the pairing $\langle\cdot,\cdot\rangle$ has definite signature. Consider the coupled Hermite-Einstein system on (S,P). If (ω,h) is a solution with degree $z=0$, then ω is Bismut-Hermite-Einstein and h is flat.*

Proof. Let (ω,h) be a solution to (6.1.1). Then, (ω,h) solves in particular the Bianchi identity:

$$dd^c\omega + \langle F_h\wedge F_h\rangle = 0. \tag{6.3.7}$$

Now, using that h satisfies the Hermite-Eistein equation with respect to ω, integrating (6.3.7) over S we obtain:

$$0 = \int_S dd^c\omega + \langle F_h\wedge F_h\rangle = -\int_S \langle F_h\wedge *F_h\rangle = -||F_h||^2_\omega \tag{6.3.8}$$

Since the pairing has definite-signature, we conclude that h is flat, and in consequence, that ω is Bismut-Hermite-Einstein. □

Corollary 6.3.5. *Let S be a compact Kähler surface with $c_1(S)=0$, and let P be a holomorphic principal bundle as in this Section. Moreover, assume the pairing $\langle\cdot,\cdot\rangle$ has a definite signature. Consider the coupled Hermite-Einstein system on (S,P). If (ω,h) is a solution, then ω is Kähler Calabi-Yau and h is flat.*

Proof. Since $\langle \cdot, \cdot \rangle$ is positive-definite or negative definite, by Proposition 6.2.4, in either case we have (ω, h) solves the coupled Hermite-Einstein system with degree $z = 0$. Then by Proposition 6.3.4, h is flat and ω is Bismut-Hermite-Einstein. Then by Proposition 6.3.3 we conclude that ω is in fact Kähler Calabi-Yau. □

Remark 6.3.6. *The conditions of S being of dimension 2 in Propositions 6.3.3, 6.3.4, and S being Kähler in dim. 2 in Corollary 6.3.5 are necessary. Otherwise, in Section 6.3.2 we construct solutions to the coupled Hermite-Einstein system which escape these results.*

6.3.2 Examples in higher dimension

The purpose of this section is to construct families of solutions to the coupled Hermite-Einstein system (6.1.1) in any dimension. The construction in this Section follows closely the torus bundle geometries described in [75, 76]. We briefly recall their fundamentals here.

Throughout, let (Z, ω_Z) be a compact Kähler manifold of complex dimension n, and let L_i, for $i = 1, ..., 2k$ be holomorphic line bundles. We denote by h_i the Hermite-Einstein metric on L_i with respect to ω_Z, and $P_i \subset \mathrm{Fr}\, L_i$ the hermitian reduction with respect to h_i. Hence, P_i are naturally $U(1)$-principal bundles endowed with a Chern connection $A_i = A^{h_i}$ such that:

$$F_i \wedge \omega^{n-1} = \tfrac{z_i}{n} \omega_Z^n, \tag{6.3.9}$$

where z_i are the degrees of L_i.

Now, consider the fibered product:

$$T^{2k} \hookrightarrow X = P_1 \times_Z \cdots \times_Z P_{2k} \xrightarrow{p} Z, \quad A = \textstyle\bigoplus_{i=1}^{2k} p_i^* A_i, \tag{6.3.10}$$

where $p_i : P_i \to Z$ are the canonical projections. Using the horizontal lift given by A, we define the complex structure on X given by

$$J = A^{\perp} J_Z + J_F \circ A \tag{6.3.11}$$

where J_Z is the complex structure of Z, and J_F is an invariant complex structure on the fibre. The integrability of J follows from the integrability of J_0 and J_F, together with $F_i^{0,2} = 0$ (see Lemma 7.1.6). Moreover, we endow X with the 2-form:

$$\omega_X = p^* \omega_Z + \frac{b}{2} \sum_{i=1}^{2k} \Big(J A_i \wedge A_i. \tag{6.3.12}$$

This form is of bidegree $(1,1)$ with respect to J and $g_X = \omega_X(\cdot, J\cdot)$ is positive-definite if $b > 0$, hence defines a hermitian metric. Similarly we denote $g_Z = \omega_Z(\cdot, J_Z \cdot)$. Then, building on the results of [76] we have:

Lemma 6.3.7. *The following formulas hold.*

$$\rho_B(\omega_X) = p^* \rho(\omega_Z) + \tfrac{b}{2} \textstyle\sum_j \Big(z_j F_j + (\Lambda_{\omega_Z} d^c A_j) d^c A_j \tag{6.3.13}$$

$$dd^c \omega_X = \tfrac{b}{2} \textstyle\sum_j \Big(F_j \wedge F_j + \Big(d^c A_j \wedge d^c A_j \tag{6.3.14}$$

Proof. For the first item, first we compute $d^\star\omega_X$. For this, let $N = \dim_{\mathbb{C}} X$, $n = \dim_{\mathbb{C}} Z$, and let $\{e_j\}_{j=1}^N$ be a g_X-orthonormal frame, where $\{e_j\}_{j=1}^n$ is p^*g_Z-orthonormal restricted to the horizontal lift of TZ with respect to the connection A. Then:

$$d^\star\omega_X(V) = -\textstyle\sum_{j=1}^N(\nabla^{g_X}_{e_j}\omega_X)(e_j,V) = \frac{1}{2}d\omega_X(e_j,Je_j,JV)$$

where the first step is standard in riemannian geometry and for the second we use the identity valid for any hermitian manifold:

$$(\nabla^{g_X}_U\omega)(V,W) = \frac{1}{2}(d\omega_X(U,V,W) - d\omega_X(U,JV,JW)). \tag{6.3.15}$$

Next, we observe that dA_i is basic, and since $JA_i \in \langle A_1,\ldots,A_{2k}\rangle_{\mathbb{R}}$, then so is $d(JA_i)$. It follows that:

$$d\omega_X \in \mathrm{Ann}\ \Lambda^2\mathrm{ker}\ dp \tag{6.3.16}$$

Therefore, we have:

$$\begin{aligned} d^\star\omega_X(V) &= \tfrac{b}{4}\textstyle\sum_{j=1}^n\sum_{k=1}^{2k}(dJA_k\wedge A_k - JA_k\wedge F_k)(e_j,Je_j,JV)\\ &= \tfrac{b}{2}\textstyle\sum_{k=1}^{2k}\left((\Lambda_{\omega_Z}d^cA_k)JA_k - z_kA_k\right)(V). \end{aligned}$$

Now, recall the formula relating the Ricci forms of the canonical connections in the Gauduchon line $\{\nabla^t\}_{t\in\mathbb{R}}$ (see Section 1.2.1) for which $t=-1$ is the Bismut connection and $t=+1$ is Chern:

$$\rho_t(\omega) - \rho_s(\omega) = \tfrac{t-s}{2}dd^\star\omega. \tag{6.3.17}$$

Using this together with $\rho_C(\omega_X) = p^*\rho(\omega_Z)$ ([76, Lemma 3]), we compute:

$$\rho_B(\omega_X) = p^*\rho(\omega_Z) - dd^\star\omega_X. \tag{6.3.18}$$

Finally, A_i are Hermitian-Yang-Mills connections over Z with respect to ω_Z, so we have

$$d(\Lambda_{\omega_Z}dA_i) = d(\Lambda_{\omega_Z}d^cA_i) = 0, \quad i=1,2. \tag{6.3.19}$$

Combining this with the above formula for $d^\star\omega_X$ we obtain the result. The second item follows using ω_Z Kähler and:

$$dd^c(JA_i\wedge A_i) = dA_i\wedge dA_i + d^cA_i\wedge d^cA_i. \tag{6.3.20}$$

□

Calabi-Eckmann threefolds

With the previous computations at hand, now we specify to the first case of our interest. In order to obtain infinite families of solutions to the coupled Hermite-Einstein system, we introduce several real and complex parameters to be determined. Let $Z = Z_1\times Z_2$ where

Z_i are compact complex manifolds of dimension n_i carrying Kähler-Einstein metrics ω_{Z_i} of positive scalar curvature, with the normalization

$$\rho(\omega_{Z_i}) = \omega_{Z_i}. \tag{6.3.21}$$

In the sequel, pullback through the canonical projections $Z \to Z_i$ is implicitly understood, and we omit it in the formulae. Then we define the Kähler metric:

$$\omega_Z = a_1\omega_{Z_1} + a_2\omega_{Z_2}. \tag{6.3.22}$$

for positive constants $a_i \in \mathbb{R}$. Moreover, consider the holomorphic line bundles $L_i = K_{Z_i}^{-k_i}$, where $k_i \in \mathbb{Z}$, $i = 1, 2$. These are naturally endowed with hermitian metrics h_i satisfying:

$$iF_{h_j} = k_j\omega_{Z_j} \tag{6.3.23}$$

In particular, a simple computation shows they are Hermite-Einstein with respect to ω_Z:

$$F_{h_j} \wedge \omega_Z^{n_1+n_2-1} = \tfrac{-in_jk_j}{a_j(n_1+n_2)}\omega_Z^{n_1+n_2} \tag{6.3.24}$$

Let P_i be the $U(1)$-principal bundle determined by the hermitian reduction of (L_i, h_i), and $X = P_1 \times_Z P_2$. Then X is a T^2-principal bundle over Z and denote by ∂_i the infinitesimal generators of the action of P_i sitting inside X. We consider a family of T^2-invariant complex structures J_F on the fibre F of $X \to Z$ via the isomorphism:

$$TF \to F \times \mathbb{C}, \ \ \partial_1 \mapsto 1, \ \partial_2 \mapsto \beta, \quad \beta \in \mathbb{C}\backslash\mathbb{R}. \tag{6.3.25}$$

Alternatively, $T^2 = (F, J_F) \cong \mathbb{C}/\mathbb{Z}\langle 1, \beta\rangle$ is an elliptic curve with complex structure parametrized by β. It will be useful for computations to express J_F in the basis $\{\partial_1, \partial_2\}$:

$$J_F = \begin{pmatrix} -\frac{\operatorname{Re}\beta}{\operatorname{Im}\beta} & -\left(\operatorname{Im}\beta + \frac{(\operatorname{Re}\beta)^2}{\operatorname{Im}\beta}\right) \\ \frac{1}{\operatorname{Im}\beta} & \frac{\operatorname{Re}\beta}{\operatorname{Im}\beta} \end{pmatrix} \tag{6.3.26}$$

By the discussion above, we get a family of hermitian manifolds (X, ω_X) depending on complex parameter β and real parameters a_1, a_2, b.

To construct solutions to the coupled Hermite-Einstein system, we consider $P = \operatorname{Fr}\mathcal{E}$ where $\mathcal{E} \to X$ is a holomorphic line bundle of the family:

$$\mathcal{E} = K_{Z_1}^{-\ell_1} \otimes K_{Z_2}^{-\ell_2}, \tag{6.3.27}$$

for integers $\ell_i \in \mathbb{Z}$. We endow P with the hermitian metric with Chern curvature:

$$iF_h = \ell_1\omega_{Z_1} + \ell_2\omega_{Z_2}, \tag{6.3.28}$$

and further endow the bundle P with the bi-invariant pairing

$$\langle r, s\rangle = -\alpha r \cdot s, \quad r, s \in \mathfrak{u}(1), \ \alpha \in \mathbb{R}. \tag{6.3.29}$$

Then, we consider the coupled Hermite-Einstein system on (X, P) with degree $z \in \mathfrak{u}(1)$:

$$\begin{aligned} F_h \wedge \omega_X^{N-1} &= \tfrac{z}{N}\omega_X^N \\ \rho^B(\omega_X) - \alpha z F_h &= 0 \\ dd^c\omega_X - \alpha F_h \wedge F_h &= 0. \end{aligned} \tag{6.3.30}$$

in the variables $a_1, a_2, b, \beta, k_1, k_2, \ell_1, \ell_2, z$ and α. To avoid cumbersome notation, we denote $x = \mathrm{Re}\,\beta$, $y = \mathrm{Im}\,\beta \neq 0$, and we will omit pullback notation when the map is understood.

Proposition 6.3.8. *Let (X, P) as constructed in this Section. Then, the solutions of the coupled Hermite-Einstein system* (6.3.30) *are given, up to complex and hermitian isometry, by*

1. $Z_1 = \{*\}$, $Z_2 \cong \mathbb{C}P^1$, *and:*

$$a_2 = \frac{bk_2^2(1+|\beta|^2)|\beta|^2}{2y^2} - \alpha\ell_2^2, \tag{6.3.31}$$

where b, β, k_2, ℓ_2 and α are subject to $a_2 > 0$. Furthermore, the degree is given by:

$$iz = \frac{\ell_2}{a_2}. \tag{6.3.32}$$

2. $Z_1, Z_2 \cong \mathbb{C}P^1$, *and either:*

   (a) $\alpha\ell_1\ell_2 = 0$, $k_1k_2x = 0$, *and:*

$$a_1 = \frac{bk_1^2(1+|\beta|^2)}{2y^2} - \alpha\ell_1^2,\ a_2 = \frac{bk_2^2(1+|\beta|^2)|\beta|^2}{2y^2} - \alpha\ell_2^2 \tag{6.3.33}$$

   where $b, \beta, k_1, k_2, \ell_1, \ell_2$ and α are moreover subject to $a_1, a_2 > 0$.

   (b) $\alpha, \ell_1\ell_2, k_1, k_2, x \neq 0$, *and:*

$$a_1 = \alpha\ell_1\left(\frac{k_1\ell_2}{k_2x} - \ell_1\right),\ a_2 = \alpha\ell_2\left(\frac{k_2\ell_1|\beta|^2}{k_1x} - \ell_2\right),\ b = \frac{2\alpha\ell_1\ell_2y^2}{k_1k_2x(1+|\beta|^2)} \tag{6.3.34}$$

   where $\beta, k_1, k_2, \ell_1, \ell_2$ and α are moreover subject to $a_1, a_2, b > 0$.

Further, in either (2)(a), (2)(b), *the degree is given by:*

$$iz = \frac{\ell_1}{a_1} + \frac{\ell_2}{a_2}. \tag{6.3.35}$$

Remark 6.3.9. *The above solutions provide infinite families of Hopf surfaces and Calabi-Eckmann threefolds [22] supporting solutions to the coupled Hermite-Einstein system. In the case of threefolds, these manifolds do not support either a holomorphic volume form, nor balanced metrics.*

Before proving this result, we will need some preliminary computations.

Lemma 6.3.10. *The system of equations* (6.3.30) *is equivalent to:*

$$\left(\frac{n_1\ell_1}{a_1}+\frac{n_2\ell_2}{a_2}\right) = iz \tag{6.3.36}$$

$$\left(1+i\alpha z\ell_1-\frac{bk_1(1+|\beta|^2)}{2y^2}\left(\tfrac{n_1k_1}{a_1}+x\tfrac{n_2k_2}{a_2}\right)\right)\omega_{Z_1}=0 \tag{6.3.37}$$

$$\left(1+i\alpha z\ell_2-\frac{bk_2(1+|\beta|^2)}{2y^2}\left(x\tfrac{n_1k_1}{a_1}+|\beta|^2\tfrac{k_2n_2}{a_2}\right)\right)\omega_{Z_2}=0 \tag{6.3.38}$$

$$\left(\frac{bk_1^2(1+|\beta|^2)}{2y^2}-\alpha\ell_1^2\right)\omega_{Z_1}^2=0 \tag{6.3.39}$$

$$\left(\frac{bk_1k_2x(1+|\beta|^2)}{2y^2}-\alpha\ell_1\ell_2\right)\omega_{Z_1}\wedge\omega_{Z_2}=0 \tag{6.3.40}$$

$$\left(\frac{bk_2^2|\beta|^2(1+|\beta|^2)}{2y^2}-\alpha\ell_2^2\right)\omega_{Z_2}^2=0 \tag{6.3.41}$$

Proof. The Hermite-Einstein equation (6.3.30) is rewritten as:

$$\begin{aligned} F_h\wedge\omega_X^{N-1} =& -i(\ell_1\omega_{Z_1}+\ell_2\omega_{Z_2})\wedge(n_1+n_2)\tfrac{b}{2}(\textstyle\sum_j JA_j\wedge A_j)a_1^{n_1}a_2^{n_2}\wedge \\ &\wedge\omega_{Z_1}^{n_1-1}\wedge\omega_{Z_2}^{n_2-1}\wedge\left(\binom{n_1+n_2-1}{n_1}\frac{\omega_{Z_1}}{a_2}+\binom{n_1+n_2-1}{n_2}\frac{\omega_{Z_2}}{a_1}\right) \\ =& \tfrac{-i(n_1+n_2)}{n_1+n_2+1}\left(\frac{\binom{n_1+n_2-1}{n_2}}{\binom{n_1+n_2}{n_1}}\frac{\ell_1}{a_1}+\frac{\binom{n_1+n_2-1}{n_1}}{\binom{n_1+n_2}{n_2}}\frac{\ell_2}{a_2}\right)\omega_X^N \\ =& \tfrac{-i}{N}\left(\tfrac{n_1\ell_1}{a_1}+\tfrac{n_2\ell_2}{a_2}\right)\omega_X^N \end{aligned}.$$

from which we can read (6.3.36). To rewrite in terms of the variables the second equation in (6.3.30), we first observe that:

$$\rho_C(\omega_Z)=\omega_{Z_1}+\omega_{Z_2}. \tag{6.3.42}$$

Moreover, using the matrix expression for J_F (6.3.26), we obtain the following formulas.

$$\begin{aligned} id^cA_1 =& -\tfrac{x}{y}k_1\omega_{Z_1}-(y+\tfrac{x^2}{y})k_2\omega_{Z_2} \\ id^cA_2 =& \tfrac{1}{y}k_1\omega_{Z_1}+\tfrac{x}{y}k_2\omega_{Z_2} \\ (\Lambda_{\omega_Z}idA_1)idA_1 =& \tfrac{k_1^2n_1}{a_1}\omega_{Z_1} \\ (\Lambda_{\omega_Z}idA_2)idA_2 =& \tfrac{k_2^2n_2}{a_2}\omega_{Z_2} \\ (\Lambda_{\omega_Z}id^cA_1)id^cA_1 =& \left(\tfrac{x^2}{y^2}\tfrac{k_1^2n_1}{a_1}+(x+\tfrac{x^3}{y^2})\tfrac{k_1k_2n_2}{a_2}\right)\omega_{Z_1} \\ &+\left((x+\tfrac{x^3}{y^2})\tfrac{k_1k_2n_1}{a_1}+(y+\tfrac{x^2}{y})^2\tfrac{k_2^2n_2}{a_2}\right)\omega_{Z_2} \\ (\Lambda_{\omega_Z}id^cA_2)id^cA_2 =& \left(\tfrac{1}{y^2}\tfrac{k_1^2n_1}{a_1}+\tfrac{x}{y^2}\tfrac{k_1k_2n_2}{a_2}\right)\omega_{Z_1} \\ &+\left(\tfrac{x}{y^2}\tfrac{k_1k_2n_1}{a_1}+\tfrac{x^2}{y^2}\tfrac{k_2^2n_2}{a_2}\right)\omega_{Z_2}. \end{aligned}$$

Then, we use Lemma 6.3.7 combined with the above formulae to express the left hand side of the second equation in system (6.3.30) in terms of ω_{Z_i}, which must vanish. Hence we get (6.3.37), (6.3.38). Finally, using Lemma 6.3.7 together with the above formulas, we can rewrite the components of the Bianchi identity in the terms $\omega_{Z_1}^2$, $\omega_{Z_1} \wedge \omega_{Z_2}$ and $\omega_{Z_2}^2$. These result in the equations (6.3.39), (6.3.40), (6.3.41). □

Proof of Proposition 6.3.8. By exchanging the role of Z_1 and Z_2 if necessary, we will assume dim. $Z_1 \leq$ dim. Z_2. If dim. $Z_1 = 0$, then the only equations of Lemma 6.3.10 that are non-trivial are (6.3.36) and (6.3.38), and also (6.3.41) if dim. $Z_2 \geq 2$. In this case, it is easy to see that there are no solutions. Indeed, by plugging the value of z given by (6.3.36) in equation (6.3.38), the resulting equation is in contradiction with (6.3.41). On the contrary, if dim. $Z_2 = 1$, then $Z_2 \cong \mathbb{C}P^1$, as we have assumed Z_i admit a positive curvature Kähler-Einstein metric. The two relevant equations combined give us

$$a_2 + \ell_2^2 \alpha = \frac{b k_2^2 |\beta|^2 (1 + |\beta|^2)}{2y^2},$$

from which a_2 is determined from the rest of the parameters. The degree z of the solution is given by (6.3.36) with $n_1 = 0$, $n_2 = 1$.

Now, we move on to the case dim. $Z_1 \geq 1$. Then dim. $Z_2 \geq 1$ too. Here every equation in Lemma 6.3.10 is non-trivial, except (6.3.37) if dim. $Z_1 = 1$, and (6.3.41) if dim. $Z_2 = 1$. Using (6.3.36) and (6.3.40) in equations (6.3.37) and (6.3.38), we obtain

$$\begin{aligned} a_1 &= n_1 \left(\frac{b k_1^2 (1 + |\beta|^2)}{2y^2} - \alpha \ell_1^2 \right) \\ a_2 &= n_2 \left(\frac{b k_2^2 (1 + |\beta|^2) |\beta|^2}{2y^2} - \alpha \ell_2^2 \right) \end{aligned}$$

which are assumed to be positive. Observe the terms in brackets match the ones in equations (6.3.39) and (6.3.41). Therefore, there is no solution to the system (6.3.36)-(6.3.41) unless dim. $Z_1 =$ dim. $Z_2 = 1$. Finally, we focus on this case. We must have $Z_1, Z_2 \cong \mathbb{C}P^1$ by the assumption that Z_i are positive curvature Kähler-Einstein manifolds. We have already observed that a_1, a_2 must be given by the formulas above. Further, equation (6.3.40) must be satisfied. Then, either both terms vanish, in which case we obtain (2)(a), or they do not, in which case $k_1, k_2, x, \ell_1, \ell_2, \alpha \neq 0$, from where

$$b = \frac{2\alpha \ell_1 \ell_2 y^2}{k_1 k_2 x (1 + |\beta|^2)}.$$

Substituting this in the expressions for a_i we obtain (2)(b). In either case (2)(a) or (2)(b) the degree z of the solution is then given by (6.3.36) with $n_1 = n_2 = 1$. □

Torus bundles over Calabi-Yau manifolds

Here, we construct solutions to the coupled Hermite-Einstein system in every dimension. Let Z be a Kähler Calabi-Yau manifold of complex dimension n, and let L_i, $1, ..., 2k$ be holomorphic line bundles on Z, and let:

$$P = \text{Fr } L_1 \times_Z \cdots \times_Z \text{Fr } L_{2k} \tag{6.3.43}$$

the principal fibered product. Then, P is a $(\mathbb{C}^\times)^{2k}$-principal bundle, with a natural family of Lie algebra pairings given by:

$$\langle (r_i), (s_i) \rangle = -\alpha \textstyle\sum_{i=1}^{2k} r_i \cdot s_i, \quad (r_i), (s_i) \in \mathbb{C}^{\oplus 2k}. \tag{6.3.44}$$

where α is a real constant.

Proposition 6.3.11. *Let $\sigma \in \mathcal{K}_Z$ be a Kähler class, and $\omega_Z \in \sigma$ be its Kähler Ricci flat metric. Let h_i be Hermite-Einstein metrics on L_i with respect to ω_Z, A_i their Chern connections and P_i the associated $U(1)$ reductions. Moreover, define:*

$$M = P_1 \times_Z \cdots \times_Z P_{2k}. \tag{6.3.45}$$

Then:

1. *The complex structure on M given by:*

$$J = A^\perp J_Z + J_0 \circ A \tag{6.3.46}$$

is integrable, where $A = (A_i)$ is the connection on the bundle $M \xrightarrow{p} Z$ induced by A_i in each factor, and J_0 is the standard fibre complex structure given by:

$$J_0 \partial_{2i-1} = \partial_{2i}, \quad J_0 \partial_{2i} = -\partial_{2i-1}, \quad i = 1, ..., k. \tag{6.3.47}$$

We denote $X = (M, J)$.

2. *Let:*

$$\omega_X = \omega_Z + \tfrac{\alpha}{2} \textstyle\sum_{i=1}^{2k} JA_i \wedge A_i. \tag{6.3.48}$$

Then, if $\alpha > 0$, ω_X is a hermitian metric and $(\omega_X, (p^* h_i))$ solves the coupled Hermite-Einstein system on $(X, p^* P)$, where P is given by (6.3.43).

Remark 6.3.12. *In the cases $\alpha \leq 0$, the tuple $(\omega_X, (p^* h_i))$ formally solve the coupled Hermite-Einstein equations too. However, ω_X is not a hermitian metric anymore, since it becomes degenerate $\alpha = 0$ or indefinite $\alpha < 0$ in the bundle directions.*

Proof. The first item is already known in the literature (see e.g. [76], or Lemma 7.1.6). Note moreover that with this choice of J_0, we have:

$$JA_{2i-1} = A_{2i}, \quad JA_{2i} = -A_{2i-1}, \quad i = 1, ..., k. \tag{6.3.49}$$

and the formulas in Lemma 6.3.7 simplify to:

$$\rho_B(\omega_X) = \rho(\omega_Z) + \alpha \textstyle\sum_{i=1}^{2k} z_i F_i \tag{6.3.50}$$

$$dd^c \omega_X = \alpha \textstyle\sum_{i=1}^{2k} F_i \wedge F_i \tag{6.3.51}$$

using that h_i are Hermite-Einstein with respect to ω_Z, where we have denoted $F_i = dA_i$ and $z_i = \Lambda_{\omega_Z} F_i$.

Now, to check that $(\omega_X, (p^*h_i))$ solve the coupled Hermite-Einstein system, we first check that h_i are Hermite-Einstein with respect to ω_X with the same degree as h_i:

$$\begin{aligned} F_i \wedge \omega_X^n &= nF_i \wedge \omega_Z^{n-1} \wedge \tfrac{\alpha}{2} \textstyle\sum_{i=1}^{2k} JA_i \wedge A_i \\ &= z_i \omega_Z^n \wedge \tfrac{\alpha}{2} \textstyle\sum_{i=1}^{2k} JA_i \wedge A_i = \tfrac{z_i}{n+1} \omega_X^{n+1}. \end{aligned}$$

Moreover, since ω_Z is Kähler Ricci flat, $\rho(\omega_Z) = 0$. Thus, using the above formulas (6.3.50), (6.3.51), we obtain:

$$\rho_B(\omega_X) - \alpha \sum_{i=1}^{2k} z_i F_i = 0 \tag{6.3.52}$$

$$dd^c\omega_X - \alpha \sum_{i=1}^{2k} F_i \wedge F_i = 0, \tag{6.3.53}$$

and the result follows. □

Remark 6.3.13. *We stress that (p^*h_i) are a solution to the Hermite-Einstein equation for P on X, even if the line bundles L_i have different slopes, as the structure group is the abelian split group $(\mathbb{C}^\times)^{2k}$, rather than $GL(2k, \mathbb{C})$, hence the tuple of degrees (z_i) is a central element.*

6.4 Relation with Heterotic Supergravity and Vertex algebras

The purpose of this Section is to discuss the position of the coupled Hermite-Einstein system in relation to some other systems of equations or constructions relevant to Hermitian and Generalized Geometry, and Physics.

In the next Chapter, we will investigate the geometry determined by the coupled Hermite-Einstein system via a systematic study of the equivariant geometry of suitable total spaces of principal bundles, yielding a non-abelian generalization of some aspects that have already implicitly appeared in the solutions provided in Section 6.3.2. The coupled Hermite-Einstein system will then be regarded as a reduction of natural geometry for these manifolds.

For the time being, here we provide a riemannian characterization of the coupled Hermitian-Einstein system. In particular, we will see that the solutions of (6.1.1) correspond to a natural class of generalized Ricci flat metrics on string algebroids and exhibit an interesting relation to heterotic supergravity, giving further motivation for their study.

Proposition 6.4.1. *Let X be a complex manifold endowed with a holomorphic principal G-bundle P. Assume that (ω, h) solves* (5.3.12) *and the Bianchi identity:*

$$dd^c\omega + \langle F_h \wedge F_h \rangle = 0. \tag{6.4.1}$$

Then, (g, h) solves the equations

$$\begin{aligned} \mathrm{Ric}_g - \tfrac{1}{4}H^2 + F_A^2 + \tfrac{1}{2}\mathcal{L}_{\varphi^\sharp} g &= 0, \\ d^\star H - d\varphi + i_{\varphi^\sharp} H &= 0, \\ d_A^\star F_A + i_{\varphi^\sharp} F_A + \star(F_A \wedge \star H) &= 0, \end{aligned} \tag{6.4.2}$$

where Ric_g *is the Riemannian Ricci tensor,* $F_A^2 = \sum_i \langle i_{e_i} F_A, i_{e_i} F_A \rangle$ *and*

$$H = -d^c\omega, \qquad A = A^h, \qquad \varphi = \theta_\omega. \tag{6.4.3}$$

Proof. We have already seen that (5.3.12) implies the last equation in (6.4.2) (see Lemma 5.3.2). Therefore, it is enough to prove that (5.3.12) implies:

$$\rho_B^{1,1}(\cdot, J\cdot) = \mathrm{Ric}_g - \tfrac{1}{4}H^2 + \langle i_{e_i}F_h, i_{e_i}F_h\rangle + \langle S_h, F_h(J,)\rangle + \tfrac{1}{2}\mathcal{L}_{\varphi^\sharp}g, \tag{6.4.4}$$

$$\rho_B^{2,0+0,2}(\cdot, J\cdot) = -\tfrac{1}{2}(d^\star H - d\varphi + i_{\varphi^\sharp}H). \tag{6.4.5}$$

To check this, we will use the following formulae, valid on any Hermitian manifold (see [90, Proposition 3.1]):

$$\mathrm{Ric}_g(X,Y) = \mathrm{Ric}_B(X,Y) - \tfrac{1}{2}d^*d^c\omega(X,Y) + \tfrac{1}{4}g(d^c\omega(X,e_i), d^c\omega(Y,e_i)), \tag{6.4.6}$$

$$\rho_B(X,Y) = -\mathrm{Ric}_B(X,JY) - (\nabla^B_X\theta_\omega)JY + \tfrac{1}{4}dd^c\omega(X,Y,e_i,Je_i), \tag{6.4.7}$$

where Ric_B denotes the Ricci tensor of ∇^B, and $\{e_i\}$ is g-orthonormal frame. To prove the first identity in (6.4.4), we now compute:

$$\begin{aligned}
\rho_B^{1,1}(X,JY) &= \tfrac{1}{2}\left(\rho_B(X,JY) - \rho_B(JX,Y)\right)\\
&= \tfrac{1}{2}(\mathrm{Ric}_B(X,Y) + \mathrm{Ric}_B(JX,JY) + (\nabla^B_X\theta_\omega)Y + (\nabla^B_{JX}\theta_\omega)JY\\
&\quad + \tfrac{1}{4}dd^c\omega(X,JY,e_i,Je_i) - \tfrac{1}{4}dd^c\omega(JX,Y,e_i,Je_i))\\
&= \tfrac{1}{2}(2\mathrm{Ric}_g(X,Y) - \tfrac{1}{2}g(d^c\omega(X,e_i,), d^c\omega(Y,e_i,))\\
&\quad - \tfrac{1}{2}\langle F_h\wedge F_h\rangle(X,JY,e_i,Je_i) + L_{\theta_\omega^\sharp}g(X,Y))\\
&= \Big(\mathrm{Ric}_g - \tfrac{1}{4}H^2 + \langle i_{e_i}F_h, i_{e_i}F_h\rangle + \langle S_h, F_h(J\cdot,\cdot)\rangle + \tfrac{1}{2}\mathcal{L}_{\theta_\omega^\sharp}g\Big)(X,Y),
\end{aligned}$$

where we denote:

$$H^2 = \textstyle\sum_{i,j} H(e_i,e_j,\cdot)H(e_i,e_j,\cdot) \tag{6.4.8}$$

and for the third equality we have used the identity (see [90, Equation 3.23]):

$$\mathrm{Ric}_B(Y,JX) = -\mathrm{Ric}_B(X,JY) - (\nabla^B_X\theta_\omega)JY - (\nabla^B_Y\theta_\omega)JX \tag{6.4.9}$$

combined with:

$$\begin{aligned}
(\nabla^B_X\theta_\omega)Y + (\nabla^B_Y\theta_\omega)X &= (\nabla^g_X\theta_\omega)Y + (\nabla^g_Y\theta_\omega)X\\
&= g(\nabla^g_X\theta^\sharp_\omega, Y) + g(\nabla^g_Y\theta^\sharp_\omega, X)\\
&= g(\nabla^g_{\theta^\sharp_\omega}X + [X,\theta^\sharp_\omega], Y) + g(\nabla^g_{\theta^\sharp_\omega}Y + [Y,\theta^\sharp_\omega], X)\\
&= \theta^\sharp_\omega(g(X,Y)) - g(\mathcal{L}_{\theta^\sharp_\omega}X, Y) - g(X, \mathcal{L}_{\theta^\sharp_\omega}Y)\\
&= (\mathcal{L}_{\theta^\sharp_\omega}g)(X,Y).
\end{aligned}$$

Similarly, the second identity (6.4.5) follows from:

$$\begin{aligned}
\rho_B^{2,0+0,2}(X,JY) &= \tfrac{1}{2}(\rho_B(X,JY) + \rho_B(JX,Y))\\
&= \tfrac{1}{2}(\mathrm{Ric}_B(X,Y) + (\nabla^B_X\theta_\omega)Y + \tfrac{1}{4}dd^c\omega(X,JY,e_i,Je_i)\\
&\quad - \mathrm{Ric}_B(JX,JY) - (\nabla^B_{JX}\theta_\omega)JY + \tfrac{1}{4}dd^c\omega(JX,Y,e_i,Je_i))\\
&= \tfrac{1}{2}(\mathrm{Ric}_B(X,Y) - \mathrm{Ric}_B(Y,X) + (\nabla^B_X\theta_\omega)Y - (\nabla^B_Y\theta_\omega)X)\\
&= \tfrac{1}{2}(d^*d^c\omega(X,Y) + d\theta_\omega(X,Y) + \theta_\omega(g^{-1}d^c\omega(X,Y)))\\
&= -\tfrac{1}{2}(d^*H - d\theta_\omega + i_{\theta^\sharp_\omega}H)(X,Y).
\end{aligned}$$

□

Remark 6.4.2. *Consider the smooth string algebroid in Example 2.1.13. Applying [63, Lemma 7.1], equations* (6.4.2) *correspond to the vanishing of the* generalized Ricci tensor Ric^+ *of the generalized metric V_+ (see* (5.1.2)*), for a suitable choice of divergence operator determined by φ. Thus, by the previous result, any solution to* (5.3.12) *is generalized Ricci flat. This applies in particular to solutions to the coupled Hermite-Einstein system* (6.1.1)*, by Proposition 6.1.2 and Lemma 5.3.3.*

In the next result, we make the connection between the Generalized Ricci flat equations (6.4.2) and the equations of motion of the heterotic supergravity in the mathematical physics literature (3.1.5) (see *e.g.* [47]).

Proposition 6.4.3. *Let M be a smooth manifold, and let $P \to M$ be a principal K-bundle. Let (g, A, H, ϕ) be a tuple consisting of a riemannian metric, a principal connection on P, a real* 3*-form and a smooth function, satisfying* (6.4.2)*, where $\varphi = 2d\phi$. Then, the following system holds:*

$$\begin{aligned} \mathrm{Ric}^g + 2\nabla^g(d\phi) - \tfrac{1}{4}H^2 + F_A^2 &= 0 \\ d^\star(e^{-2\phi}H) &= 0 \\ d_A^\star(e^{-2\phi}F_A) + e^{-2\phi} \star (F_A \wedge \star H) &= 0. \end{aligned} \tag{6.4.10}$$

Proof. To obtain the first equation of (6.4.10), we combine the first equation of (6.4.2) with the computations of the symmetric and skew components of the term $\nabla^g(d\phi)$. These are well-known in riemannian geometry, but we provide them for the benefit of the reader:

$$\begin{aligned} \Lambda^2\nabla^g(d\phi) &= \tfrac{1}{2}(\nabla^g_X(d\phi)(Y) - \nabla^g_Y(d\phi)(X)) \\ &= \tfrac{1}{2}(X(d\phi(Y)) - Y(d\phi(X)) - d\phi(\nabla^g_X Y - \nabla^g_Y X)) \\ &= \tfrac{1}{2}d^2\phi(X, Y) \\ &= 0, \\ S^2\nabla^g(d\phi)(X, Y) &= \tfrac{1}{2}(\nabla^g_X(d\phi)(Y) + \nabla^g_Y(d\phi)(X)) \\ &= \tfrac{1}{2}\left(X(g(\nabla\phi, Y)) + Y(g(\nabla\phi, X)) - g(\nabla\phi, \nabla^g_X Y) - g(\nabla\phi, \nabla^g_X Y)\right) \\ &= \tfrac{1}{2}\left(g(\nabla^g_X(\nabla\phi), Y) + g(\nabla^g_Y(\nabla\phi), X)\right) \\ &= \tfrac{1}{2}\left(g(\nabla^g_{\nabla\phi}X + [X, \nabla\phi], Y) + g(X, \nabla^g_{\nabla\phi}Y + [Y, \nabla\phi])\right) \\ &= \tfrac{1}{2}\left(\nabla\phi(g(X, Y)) - g([\nabla\phi, X], Y) - g(X, [\nabla\phi, Y])\right) \\ &= \tfrac{1}{2}(\mathcal{L}_{\nabla\phi}g)(X, Y) \\ &= \tfrac{1}{4}(\mathcal{L}_{\varphi^\sharp}g)(X, Y). \end{aligned}$$

Therefore, we obtain: $2\nabla^g(d\phi) = \frac{1}{2}\mathcal{L}_{\varphi^\sharp}g$, and consequently, the first equation in (6.4.10) holds. Now, for the second equation in (6.4.10), we compute:

$$\begin{aligned} d^\star(e^{-2\phi}H) &= - \star d \star (e^{-2\phi}H) \\ &= e^{-2\phi}d^\star H + 2e^{-2\phi} \star (d\phi \wedge \star H) \\ &= e^{-2\phi}d^\star H + 2e^{-2\phi}i_{\nabla\phi}H \\ &= e^{-2\phi}(d^\star H + i_{\varphi^\sharp}H) \\ &= 0, \end{aligned}$$

where in the last step we have used the second equation in (6.4.2) combined with $d\varphi = 2d^2\phi = 0$. Similarly, for the third equation in (6.4.10):

$$\begin{aligned} d_A^\star(e^{-2\phi}F_A) &= -\star d_A \star (e^{-2\phi}F_A) \\ &= e^{2\phi}d_A^\star F_A + 2e^{-2\phi}\star(d\phi\wedge\star F_A) \\ &= e^{-2\phi}d_A^\star F_A + 2e^{-2\phi} i_{\nabla\phi}F_A. \end{aligned}$$

Then:

$$\begin{aligned} d_A^\star(e^{-2\phi}F_A) + e^{-2\phi}\star(F_A\wedge\star H) &= e^{-2\phi}(d_A^\star F_A + 2i_{\nabla\phi}F_A + \star(F_A\wedge\star H)) \\ &= e^{-2\phi}(d_A^\star F_A + i_{\varphi^\sharp}F_A + \star(F_A\wedge\star H)) \\ &= 0, \end{aligned}$$

where in the last step we have used the last equation in (6.4.2). □

The system (6.4.10) matches the equations of motion of heterotic supergravity (3.1.5) (with the choices outlined in that Section) except for the last equation of the system, the dilaton equation. We now prove that, under natural assumptions, it holds up to a constant, generalizing the already known result in the case of trivial principal bundle [71, Proposition 4.33]. For this, we will need some technical computations.

Lemma 6.4.4. *Let M be a smooth manifold, and let $P\to M$ be a principal K-bundle, where K has a quadratic Lie algebra $(\mathfrak{k},\langle\cdot,\cdot\rangle)$. Let (g,A,H) be a tuple consisting of a riemannian metric, a principal connection on P and a real 3-form. Moreover, let $X_p\in T_pM$ at a point $p\in M$. Then:*

1. *There exists a smooth vector field X extending X_p such that $(\nabla^g X)_p = 0$.*
2. *The following formulas hold at p:*

$$\begin{aligned} X(\operatorname{div} Y) &= \operatorname{div}(\nabla^g_X Y) - \operatorname{Ric}^g(X,Y) &(6.4.11)\\ d^\star(i_X\operatorname{Ric}^g) &= -\tfrac{1}{2}X(\operatorname{scal}^g) &(6.4.12)\\ d^\star(i_X H^2) &= -\tfrac{1}{6}X(|H|^2) + \tfrac{1}{3}\textstyle\sum_{i,j,k} dH(X,e_i,e_j,e_k)H(e_i,e_j,e_k) + \\ &\quad + \textstyle\sum_{j,k} d^\star H(e_j,e_k)H(X,e_j,e_k) &(6.4.13)\\ d^\star(i_X F_A^2) &= -\tfrac{1}{4}X(|F_A|^2) + \textstyle\sum_i\langle d_A^\star F_A(e_i), F_A(X,e_i)\rangle, &(6.4.14)\end{aligned}$$

where X is any vector field extending X_p such that $(\nabla^g X)_p = 0$, Y is any vector field and $\{e_i\}$ is a g-orthonormal basis.

Proof. For the first item, we work in normal coordinates around $p\in M$. In particular, there is a basis of vector fields $\{e_i\}$ such that:

$$(\nabla^g e_i)_p = 0, \quad g_p(e_i,e_j) = \delta_{ij}. \tag{6.4.15}$$

Then, we choose:

$$X = \textstyle\sum_i g_p(X_p,e_i)e_i. \tag{6.4.16}$$

Now, let Y be any vector field and X as above. Then, at p:

$$\begin{aligned} X(\mathrm{div}\, Y) &= \textstyle\sum_i \left(g(\nabla^g_X \nabla^g_{e_i} Y, e_i) + g(\nabla^g_{e_i} Y, \nabla^g_X e_i) \right) \\ &= \textstyle\sum_i \left(g((R^g(X, e_i) + \nabla^g_{e_i}\nabla^g_X + \nabla^g_{[X,e_i]})Y, e_i) \right) \\ &= -\mathrm{Ric}^g(X, Y) + \mathrm{div}(\nabla^g_X Y). \end{aligned}$$

For (6.4.12), using the normal coordinates conditions and the assumption for X, we compute at p:

$$\begin{aligned} d^\star(i_X \mathrm{Ric}^g) &= -\textstyle\sum_i \left(\nabla^g_{e_i} i_X \mathrm{Ric}^g\right)(e_i) \\ &= -\textstyle\sum_i e_i(\mathrm{Ric}^g(X, e_i)) + \sum_i \mathrm{Ric}^g(X, \nabla^g_{e_i} e_i) \\ &= -\textstyle\sum_{i,j} e_i \left(g(R^g(e_j, X)e_i, e_j)\right) \\ &= \textstyle\sum_{i,j} g(\nabla^g_{e_i} R^g(e_j, X)e_i, e_j) \\ &= \textstyle\sum_{i,j} g((\nabla^g_{e_j} R^g(X, e_i) + \nabla^g_X R^g(e_i, e_j))e_i, e_j) \\ &= \textstyle\sum_{i,j} g(\nabla^g_{e_j} R^g(e_i, X)e_j, e_i) + g(\nabla^g_X R^g(e_j, e_i)e_i, e_j) \\ &= -\tfrac{1}{2}\textstyle\sum_{i,j} g(\nabla^g_X R^g(e_j, e_i)e_i, e_j) \\ &= -\tfrac{1}{2} X\left(\textstyle\sum_{i,j} R^g(e_j, e_i)e_i, e_j)\right) \\ &= -\tfrac{1}{2} X(\mathrm{scal}^g), \end{aligned}$$

where we have used throughout the symmetries and the Bianchi identity for the Riemann curvature tensor R^g.

The third formula follows from the same computation as in [71, Lemma 3.19]. Note however the extra term in (6.4.13) since we are not assuming H is closed. The last formula (6.4.14) follows again from analogous computations, using here that $d_A F_A = 0$. $\square$

Proposition 6.4.5. *Let M be a smooth manifold, and let $P \to M$ be a principal K-bundle, where K has a quadratic Lie algebra $(\mathfrak{k}, \langle \cdot, \cdot \rangle)$. Let (g, A, H, ϕ) be a tuple consisting of a riemannian metric, a principal connection on P, a real 3-form and a smooth function, satisfying* (6.4.2)*, where $\varphi = 2d\phi$ and the Bianchi identity:*

$$dH - \langle F_A \wedge F_A \rangle = 0. \tag{6.4.17}$$

Then:

$$d\left(\mathrm{scal}^g - 4\Delta_g \phi - 4|d\phi|^2 - \tfrac{1}{12}|H|^2 + \tfrac{1}{2}|F_A|^2\right) = 0. \tag{6.4.18}$$

Proof. First, taking the trace of the first equation in (6.4.10), we get:

$$\mathrm{scal}^g - \tfrac{1}{4}|H|^2 + |F_A|^2 - 2\Delta_g \phi = 0, \tag{6.4.19}$$

where $\Delta_g \phi = d^\star d\phi$ is the Hodge Laplacian of ϕ. In the sequel, let X be a vector field such that $(\nabla^g X)_p = 0$ for some $p \in M$. Then, using (6.4.19), at the point p we have:

$$\begin{aligned} X(\mathrm{scal}^g) &= X(\tfrac{1}{4}|H|^2 - |F_A|^2) + 2X(\mathrm{div}(\nabla\phi)) \\ &= X(\tfrac{1}{4}|H|^2 - |F_A|^2) - 2\mathrm{Ric}^g(X, \nabla\phi) + 2\mathrm{div}(\nabla^g_X \nabla\phi) \\ &= X(\tfrac{1}{4}|H|^2 - |F_A|^2) - 2\mathrm{Ric}^g(X, \nabla\phi) - d^\star\left(i_X(\mathrm{Ric}^g - \tfrac{1}{4}H^2 + F_A^2)\right) \end{aligned}$$

where in the last step we have used again the first equation in (6.4.10). Then, using the formulae in Lemma 6.4.4, we continue the above computation to get:

$$
\begin{aligned}
X(\mathrm{scal}^g) &= X\left(\tfrac{1}{4}|H|^2 - |F_A|^2\right) - 2\mathrm{Ric}^g(X,\nabla\phi) + \tfrac{1}{2}X(\mathrm{scal}^g) + d^\star\left(\tfrac{1}{4}H^2 - F_A^2\right)(X) \\
&= X\left(\tfrac{1}{2}|H|^2 - 2|F_A|^2\right) - 4\mathrm{Ric}^g(X,\nabla\phi) + d^\star\left(\tfrac{1}{2}H^2 - 2F_A^2\right)(X) \\
&= X\left(\tfrac{1}{2}|H|^2 - 2|F_A|^2\right) - 4\mathrm{Ric}^g(X,\nabla\phi) - \tfrac{1}{12}X(|H|^2) \\
&\quad + \tfrac{1}{6}\textstyle\sum_{i,j,k} dH(X,e_i,e_j,e_k)H(e_i,e_j,e_k) + \tfrac{1}{2}\sum_{j,k} d^\star H(e_j,e_k)H(X,e_j,e_k) \\
&\quad + \tfrac{1}{2}X(|F_A|^2) - 2\textstyle\sum_i \langle d_A^\star F_A(e_i), F_A(X,e_i)\rangle \\
&= X\left(\tfrac{5}{12}|H|^2 - \tfrac{3}{2}|F_A|^2\right) - 4\mathrm{Ric}^g(X,\nabla\phi) + \textstyle\sum_{j,k} H(\nabla\phi,e_j,e_k)H(X,e_j,e_k) + \\
&\quad + \tfrac{1}{6}\textstyle\sum_{i,j,k}\langle F_A\wedge F_A\rangle(X,e_i,e_j,e_k)H(e_i,e_j,e_k) - 2\sum_i\langle d_A^\star F_A(e_i), F_A(X,e_i)\rangle \\
&= X\left(\tfrac{5}{12}|H|^2 - \tfrac{3}{2}|F_A|^2\right) - 4\left(\mathrm{Ric}^g - \tfrac{1}{4}H^2\right)(X,\nabla\phi) \\
&\quad - 2\langle (d_A^\star F_A + \star(F_A\wedge\star H))(e_i), F_A(X,e_i)\rangle \\
&= X\left(\tfrac{5}{12}|H|^2 - \tfrac{3}{2}|F_A|^2\right) - 4(\mathrm{Ric}^g - \tfrac{1}{4}H^2 + F_A^2)(X,\nabla\phi) \\
&= X\left(\tfrac{5}{12}|H|^2 - \tfrac{3}{2}|F_A|^2\right) - 8(\nabla^g_X d\phi)(\nabla\phi) \\
&= X\left(\tfrac{5}{12}|H|^2 - \tfrac{3}{2}|F_A|^2\right) - 4X(|d\phi|^2),
\end{aligned}
$$

where, in the second line we have collected the terms in $X(\mathrm{scal}^g)$, and from the sixth line onwards, we use the equations in (6.4.2) with $\varphi = 2d\phi$ and the Bianchi identity (6.4.17). Since the above computation is tensorial in X, it follows that:

$$
d\left(\mathrm{scal}^g - \tfrac{5}{12}|H|^2 + \tfrac{3}{2}|F_A|^2 + 4|d\phi|^2\right) = 0. \tag{6.4.20}
$$

Finally, the result follows substracting (6.4.20) from twice the expression obtained by taking exterior derivative in (6.4.19). □

Corollary 6.4.6. *Let $X = (M, J)$ be a complex manifold, and let $P \to X$ be a holomorphic principal G-bundle. Assume (ω, h) is a solution to the coupled Hermite-Einstein system* (6.1.1)*, and assume the Lee form is exact, $\theta_\omega = 2d\phi$. Moreover, let:*

$$
g = \omega(\cdot, J\cdot), \qquad A = A^h, \qquad H = -d^c\omega. \tag{6.4.21}
$$

Then (g, A, H, ϕ) solve (6.4.10).

Proof. This is a straightforward consequence of Propositions 6.4.1, 6.4.3. □

We finish this Section mentioning a further motivation for the study of the coupled Hermite-Einstein system. Recently, the system has also appeared remarkably in the context of SUSY vertex algebras. There, the existence of coupled Hermite-Einstein metrics on Bott-Chern algebroids (see Definition 5.3.1) is a fundamental structure to regard the chiral de Rham complex - of central interest in the study of superconformal field theory in the physics literature - as a representation of certain $N = 2$ superconformal vertex algebras [6, Theorem 4.18]. This proposal is then used to study certain aspects of $(0, 2)$-mirror symmetry, recasted as an involution of suitable vertex algebras on mirror spaces [5, 34]. While this theory is already well-known in the Kähler and Generalized Kähler case, the coupled Hermite-Einstein system seems to stand out as consistency conditions for such phenomena to be realised in the non-Kähler setting.

Chapter 7

Dimensional reduction of canonical pluriclosed metrics

In this Chapter, we revisit the coupled Hermite-Einstein system introduced in Chapter 5 from the point of view of hermitian reduction. The underlying principle, which was already implicit in Section 6.3.2, is to build on the results of [75, 76] to show that suitable invariant geometry on the total space of principal bundle fibrations induces the coupled Hermite-Einstein system on the base upon reduction. We make this precise in Theorem 7.2.3. Here we generalize this picture to non-abelian symmetries and find correspondences between both spaces. This Chapter is based on ongoing work jointly with M. García-Fernández and J. Streets (see Chapter 9).

7.1 Equivariant geometry of principal bundle fibrations

7.1.1 Riemannian curvatures of equivariant metrics

Let M be a compact smooth manifold and let K be a real, compact Lie group. We assume K has a quadratic Lie algebra $(K, \langle\cdot,\cdot\rangle)$. Moreover, we fix a smooth principal bundle:

$$K \longrightarrow P \xrightarrow{p} M. \tag{7.1.1}$$

We will assume that P satisfies:

$$p_1(P) = 0 \in H^4_{dR}(M). \tag{7.1.2}$$

Next, we introduce the equivariant geometry of P. Let (g_M, H_M) be a riemannian metric and a 3-form on M, and let $A \in (\Omega^1_P)^K$ be a principal connection. Then, we consider the total space symmetric tensor and 3-form given by:

$$g = p^* g_M - \langle A \otimes A\rangle \tag{7.1.3}$$
$$H = p^* H_M - CS(A), \tag{7.1.4}$$

where $CS(A)$ stands for the Chern-Simons 3-form of A (1.3.17). Observe that when $\langle\cdot,\cdot\rangle$ is negative definite, g is a riemannian metric on P. In the sequel, if this is case, we will denote:

$$g_K(\cdot,\cdot) = -\langle\cdot,\cdot\rangle. \tag{7.1.5}$$

Moreover, the restriction of H to the fibres of P, identified with the Lie group K is the Cartan 3-form of K, given by the extension of:

$$H_K(\xi, \mu, \gamma) = \langle [\xi, \mu], \gamma \rangle, \quad \xi, \mu, \gamma \in \mathfrak{k}, \tag{7.1.6}$$

by left translations. This form satisfies the following key property:

Lemma 7.1.1. *Let (K, g_K, H_K) as in this Section, and let D be the Levi-Civita connection of g_K. Then:*

$$DH_K = 0. \tag{7.1.7}$$

In particular:

$$dH_K = d^\star H_K = 0. \tag{7.1.8}$$

Proof. We compute DH_K evaluating at left-invariant vector fields. For clarity in notation, only in this proof we denote them by A, B, C and E. First, we note that:

$$D_A B = \tfrac{1}{2}[A, B] \tag{7.1.9}$$

as a straightforward consequence of the Koszul formula for D (1.2.4), using bi-invariance of g_K. Then, we have:

$$\begin{aligned}(D_A H_K)(B, C, E) &= A(H_K(B, C, E)) - \tfrac{1}{2}H_K([A, B], C, E) - \tfrac{1}{2}H_K(B, [A, C], E) \\ &\quad - \tfrac{1}{2}H_K(B, C, [A, E]) \\ &= \tfrac{1}{2}\langle [[B, A], C], E\rangle - \tfrac{1}{2}\langle [B, [A, C]], E\rangle - \tfrac{1}{2}\langle [B, C], [A, E]\rangle \\ &= 0,\end{aligned}$$

where in the last step we use the Jacobi indentity combined with the fact that $\langle \cdot, \cdot \rangle$ is adjoint-invariant. The second part of the statement follows inmediately using that dH_K is the complete skew-symmetrization of DH_K, and the standard formula in Riemannian Geometry:

$$d^\star H_K = -\textstyle\sum_j (D_{U_j} H_K)(U_j, \cdot, \cdot), \tag{7.1.10}$$

where $\{U_j\}$ is an orthonormal basis for g_K. □

We now compute the Levi-Civita and Bismut curvatures of the Ricci tensors of g (resp. of (g, H)) in (7.1.3),(7.1.4), in the case it is riemannian. While these results seem to be known by experts, we have not been able to find the proofs, hence we give some details about the former and fully spell out the latter. For efficiency in the next computations, we adopt the following notation and use it sistematically: we will denote by X, Y, Z (and possibly primed or with some other decoration) for horizontal lifts with respect to A of basic vector fields, and abusing of notation, we will identify them with their basic projections. Similarly, $U, V, W, ...$ will denote vertical vector fields. Whenever they are canonical (see (1.3.6)) and we want to make explicit reference to the Lie algebra generator, we will prefer the notation X^ξ for $\xi \in \mathfrak{k}$. Furthermore, $\{X_j\}$ will denote an orthonormal frame of p^*TX (resp. $\{U_j\}$ of the vertical distribution $VP = \ker dp \subset TP$).

To obtain the Ricci tensor of g on P, we adapt the general computations for riemannian submersions in [18, Chapter 9] to the case of interest.

Definition 7.1.2. *[18, Definition 9.25] Assume g in* (7.1.3) *is a riemannian metric, and let D be the Levi-Civita connection of g. Consider the tensors $T, B \in \Gamma(\operatorname{End} TP)$ given by:*

$$T = p_H D_{p_V} p_V + p_V D_{p_V} p_H \tag{7.1.11}$$
$$B = p_H D_{p_H} p_V + p_V D_{p_H} p_H. \tag{7.1.12}$$

Equivalently, D is decomposed as:

$$D_U V = D_U^{g_K} V + T_U V \tag{7.1.13}$$
$$D_U X = T_U X + p_H D_U X \tag{7.1.14}$$
$$D_X U = p_V D_X U + B_X U \tag{7.1.15}$$
$$D_X Y = B_X Y + p_H D_X Y, \tag{7.1.16}$$

where D^{g_K} is the Levi-Civita connection of g_K, and p_H, p_V are the projections (1.3.10).

Now, we make some preliminary observations that will be applied to the case of principal bundle fibrations (7.1.1) throughout without further mention.

Lemma 7.1.3. *With the previous notations, the following hold:*

1. $[U, V] \in VP$.
2. $[X^\xi, R] = 0$, *for R a K-invariant vector field. In particular, for R lifted horizontal, or induced by a section of* ad P.
3. $B_X Y = \frac{1}{2} p_V([X, Y]) = -\frac{1}{2} F_A(X, Y)$ *under the identification $VP \cong P \times \mathfrak{k}$.*
4. *The tensor T in* (7.1.11) *vanishes.*
5. $p_V D_X X^\xi = 0$, *for any lifted horizontal X, and canonical X^ξ.*

Proof. The first item follows from the involutivity of VP. The second follows from the fact that the flow for canonical fields is given by right translations. The third item follows from [18, Proposition 9.24] and the definition of F_A. For the fourth, we prove that:

$$T_U V = T_U X = 0 \tag{7.1.17}$$

for any choice of vertical fields U, V and horizontal X. First, we show that $g(D_U V, X)$ vanishes. Using the Koszul formula for D (1.2.4), we write it in terms of:

$$X(g(U, V)), \quad g(X, [U, V]), \tag{7.1.18}$$

and their cyclic permutations. Since T is a tensor, this computation is pointwise and extensions of U, V are irrelevant, hence we may use canonical fields, and we choose X to be a lifted horizontal field, similarly. Then, it is straightforward to check that all these terms vanish by our elections of fields combined with [18, Proposition 9.18]. $T_U X = 0$ is similar. For the last item, $g(D_X X^\xi, V) = 0$ similarly, but note however that it is not tensorial in X^ξ. □

Proposition 7.1.4. *[18, Proposition 9.36] With the previous notations, the following formulas hold:*

$$\mathrm{Ric}_g(U,V) = \mathrm{Ric}_{g_K}(U,V) + \tfrac{1}{4}\textstyle\sum_{j,k} g_K(F_A(X_j,X_k),U)g_K(F_A(X_j,X_k),V), \tag{7.1.19}$$

$$\mathrm{Ric}_g(U,X) = \tfrac{1}{2}g_K(d_A^\star F_A(X),U), \tag{7.1.20}$$

$$\mathrm{Ric}_g(X,Y) = \left(\mathrm{Ric}_{g_M} - \tfrac{1}{2}F_A^2\right)(X,Y), \tag{7.1.21}$$

where the notation F_A^2 stands for:

$$F_A^2(X,Y) = \textstyle\sum_j g_K(F_A(X_j,X),F_A(X_j,Y)). \tag{7.1.22}$$

Proof. These formulas follow combining [18, Proposition 9.36] in the case the riemannian submersion is as in this Section and Lemma 7.1.3. The explicit computations are straightforward and hence ommited. □

Now, we move on to compute the Ricci tensor of the Bismut connection of the pair (g,H) as in (7.1.3),(7.1.4). By this, we mean the connection given by:

$$D^{g,H} = D + \tfrac{1}{2}g^{-1}H, \tag{7.1.23}$$

where D is the Levi-Civita connection of g, and H is given by (7.1.4). The dimensional reduction of Bismut curvature quantities on torus bundles has already appeared in the literature with a view towards relating different geometric flows [124, 125] and Generalized Geometry and T-duality [123]. Here, we provide the formulae for the Bismut-Ricci tensor of a general principal bundle fibration as in (7.1.1), which will be used in the sequel:

Proposition 7.1.5. *The Ricci tensor of $D^{g,H}$ satisfies:*

$$\begin{aligned}
\mathrm{Ric}_{g,H}(X,Y) &= \left(\mathrm{Ric}_{g_M} - \tfrac{1}{4}H_M^2 - F_A^2 - \tfrac{1}{2}d^\star H_M\right)(X,Y),\\
\mathrm{Ric}_{g,H}(U,X) &= g_K(U, i_X(d_A^\star F_A + \star(F_A\wedge\star H_M))),\\
\mathrm{Ric}_{g,H}(X,U) &= 0,\\
\mathrm{Ric}_{g,H}(U,V) &= 0,
\end{aligned} \tag{7.1.24}$$

where we use the notation (6.4.8) *for H_M^2.*

Proof. The proof is a combination of Proposition 7.1.4 and the relation between Bismut and Levi-Civita Ricci tensor (e.g. [89, Proposition 3.1]):

$$\mathrm{Ric}_{g,H} = \mathrm{Ric}_g - \tfrac{1}{2}d^\star H - \tfrac{1}{4}H^2,$$

that holds on any smooth manifold. Now, using the notations of this section, we compute explicitly the components of each of these extra terms for (P,g,H). The vertical fields involved are assumed to be canonical throughout the computations.

$$
\begin{aligned}
d^{\star}H(U,V) &= -\sum_j (D_{X_j}H)(X_j,U,V) - \sum_k \big(D_{U_k}H)(U_k,U,V) \\
&= \sum_j \big(-X_j(H(X_j,U,V)) + H(D_{X_j}X_j,U,V) \\
&\quad + H(X_j,D_{X_j}U,V) + H(X_j,U,D_{X_j}V) + d^{\star g_K} H_K(U,V) \\
&= d^{\star g_K} H_K(U,V) + \sum_j \big(H(X_j,B_{X_j}U,V) + H(X_j,U,B_{X_j}V) \\
&= d^{\star g_K} H_K(U,V) - \sum_{j,k} H(X_j,X_k,V) g_K(B_{X_j}X_k,U) \\
&\quad - \sum_{j,k} H(X_j,U,X_k) g_K(V,B_{X_j}X_k) \\
&= d^{\star g_K} H_K(U,V) + \tfrac{1}{2}\sum_{j,k} g_K(F_A(X_j,X_k),U) g_K(F_A(X_j,X_k),V) \\
&\quad - \tfrac{1}{2}\sum_{j,k} g_K(F_A(X_j,X_k),V) g_K(F_A(X_j,X_k),U) \\
&= d^{\star g_K} H_K(U,V) = 0.
\end{aligned}
$$

where in the last step we apply (7.1.8). The next component of this term is:

$$
\begin{aligned}
d^{\star}H(U,X) = &\ \sum_j \big(-X_j(H(X_j,U,X)) + H(D_{X_j}X_j,U,X) + \\
&+ \sum_j H(X_j,D_{X_j}U,X) + H(X_j,U,D_{X_j}X) + \\
&+ \sum_k \big(-U_k(H(U_k,U,X)) + H(D_{U_k}U_k,U,X) + \\
&+ \sum_j \big(H(U_k,D_{U_k}U,X) + H(U_k,U,D_{U_k}X) \\
= &\ \sum_j \big(X_j(g_K(F_A(X_j,X),U)) - g_K(F_A(D^{g_M}_{X_j}X_j,X),U) \\
&- g_K(F_A(X_j,D^{g_M}_{X_j}X),U) - H_M(X_j,X,B_{X_j}U) \\
= &\ \sum_j \big(g_K(i_{X_j}d_A(F_A(X_j,X)),U) - g_K(F_A(D^{g_M}_{X_j}X_j,X),U) + \\
&+ \sum_j \big(-g_K(F_A(X_j,D^{g_M}_{X_j}X),U) - H_M(X_j,X,B_{X_j}U) \\
= &\ -g_K\big(d^{\star}_A F_A(X),U) + \tfrac{1}{2}\sum_{j,k} H_M(X_j,X_k,X) g_K(F_A(X_j,X_k),U)
\end{aligned}
$$

where, in the last computation, we have used throughout Lemma 7.1.3 combined with:

$$
p_* D_{X'}Y' = D^{g_M}_{X'}Y', \tag{7.1.25}
$$

which can be easily proved combining the Koszul formulas (1.2.4) for g and g_M. Turning to the horizontal component of this term, we have:

$$
d^{\star}H(X,Y) = \sum_j -(D_{X_j}H)(X_j,X,Y) + \sum_k \big(-(D_{U_k}H)(U_k,X,Y). \tag{7.1.26}
$$

Each of these two terms require attention.

$$
\begin{aligned}
-(D_{X_j}H)(X_j,X,Y) = &\ -X_j(H(X_j,X,Y)) + H(D_{X_j}X_j,X,Y) + \\
&+ H(X_j,D_{X_j}X,Y) + H(X_j,X,D_{X_j}Y) \\
&= d^{\star g_M} H_M(X,Y) + \sum_j \big(H(X_j,B_{X_j}X,Y) + H(X_j,X,B_{X_j}Y) \\
&= d^{\star g} H_M(X,Y) + \tfrac{1}{2}\sum_{j,k} g_K(F_A(X_j,Y),F_A(X_j,X)) \\
&\quad - \tfrac{1}{2}\sum_{j,k} g_K(F_A(X_j,X),F_A(X_j,Y)) \\
&= d^{\star g_M} H_M(X,Y).
\end{aligned}
$$

Furthermore, we compute the second term in (7.1.26):

$$\begin{aligned}
-(D_{U_k}H)(U_k,X,Y) = &\textstyle\sum_k \big(-U_k(H(U_k,X,Y)) + H(D_{U_k}U_k,X,Y)+ \\
&+ H(U_k,D_{U_k}X,Y) + H(U_k,X,D_{U_k}Y\big) \\
=&\textstyle\sum_k \big(-U_k(g_K(F_A(X,Y),U_k)) + g_K(F_A(X,Y),D^{g_K}_{U_k}U_k\big) \\
&+ \textstyle\sum_{j,k} H(U_k,X_j,Y)g(D_{U_k}X,X_j) + H(U_k,X,X_j)g(D_{U_k}Y,X_j) \\
=&\textstyle\sum_k -g_K([U_k,F_A(X,Y)],U_k) \\
&- \tfrac{1}{4}\textstyle\sum_{j,k} g_K(F_A(X_j,Y),U_k)g_K(F_A(X_j,X),U_k) \\
&+ \tfrac{1}{4}\textstyle\sum_{j,k} g_K(F_A(X_j,X),U_k)g_K(F_A(X_j,Y),U_k) \\
=&\textstyle\sum_k -g_K([U_k,F_A(X,Y)],U_k) = 0.
\end{aligned}$$

Then, we move to the components of the remaining term.

$$\begin{aligned}
H^2(U,V) =&\textstyle\sum_{j,k} H(X_j,X_k,U)H(X_j,X_k,V) + H(U_j,U_k,U)H(U_j,U_k,V) \\
=&\textstyle\sum_{j,k} g_K(F_A(X_j,X_k),U)g_K(F_A(X_j,X_k),V)+ \\
&+\textstyle\sum_{j,k}([U_j,U_k],U)([U_j,U_k],V) \\
H^2(U,X) =&\textstyle\sum_{j,k} H(X_j,X_k,U)H(X_j,X_k,X) \\
=&\textstyle\sum_{j,k} H_M(X_j,X_k,X)g_K(F_A(X_j,X_k),U) \\
H^2(X,Y) =&\textstyle\sum_{j,k} H(X_j,X_k,X)H(X_j,X_k,Y) + 2H(X_j,U_k,X)H(X_j,U_k,Y) \\
=& H^2_M(X,Y) + 2\textstyle\sum_{j,k} g_K(F_A(X_j,X),U_k)g_K(F_A(X_j,Y),U_k) \\
=& (H^2_M + 2F^2_A)(X,Y).
\end{aligned}$$

Finally, combining all the previous computations and taking into account that $d^\star H$ is skew and H^2 is symmetric yields the result. The second formula uses also the fact that:

$$\textstyle\sum_{j,k} F_A(X_j,X_k)H_M(X_j,X_k,X) = -2\star(F_A\wedge\star H_M)(X). \tag{7.1.27}$$

For the last formula, we get:

$$\mathrm{Ric}_{g,H}(U,V) = \mathrm{Ric}_{g_K,H_K}(U,V) = 0, \tag{7.1.28}$$

where the last step follows from the fact that the ansatz we are using for the fibre metric and for H_K is actually Bismut-flat ([71, Proposition 3.53]). □

7.1.2 Equivariant hermitian geometry of principal bundles

Next, we introduce the complex structure in the picture of the previous Section. Let $X = (M, J_X)$ be a compact complex manifold of complex dimension n. Let (K, J_K) be a compact, real Lie group of even dimension endowed with a left-invariant, integrable complex structure. We will assume K has a quadratic Lie algebra $(\mathfrak{k}, \langle\cdot,\cdot\rangle)$ and that J_K is orthogonal for this structure, that is, $\langle J_K\cdot, J_K\cdot\rangle = \langle\cdot,\cdot\rangle$. Moreover, we fix a smooth principal bundle:

$$K \longrightarrow P \xrightarrow{p} X \tag{7.1.29}$$

We will assume that P satisfies:

$$p_1(P) = 0 \in H^4_{dR}(X, \mathbb{R}). \tag{7.1.30}$$

Furthermore, we will assume that P supports holomorphic complexifications. These are given by principal connections $A \in (\Omega^1_P)^K$ such that:

$$F_A^{0,2} = 0. \tag{7.1.31}$$

In such case, indeed $P^c = P \times_K K^c$ endowed with the Dolbeault operator $\overline{\partial}_A = d_A^{0,1}$ is naturally a holomorphic principal bundle over X, where K^c is the complexification of the real group K, inducing a canonical complex structure on P^c.

In the sequel, we adopt the notation of Section 1.3.1 to denote the canonical vector fields (1.3.6), and we identify basic vector fields $X, Y, ... \in \Gamma(TX)$ with the lifted horizontal fields on P when the connection A used is implicit. The following result yields a wealth of integrable complex structures on the total space of P:

Lemma 7.1.6. *Let A be a principal connection on P satisfying* (7.1.31)*, then:*

$$J = A^{\perp} J_X + J_K \circ A \tag{7.1.32}$$

is an integrable complex structure on P, where $A^{\perp}$ denotes the horizontal lift induced by X.

Proof. This result is well-known, but we give a sketch of the proof here. The integrability of J is equivalent to $T^{0,1}P$ being involutive. Then, the result follows from these computations:

$$\begin{aligned}
[(X^{\xi})^{0,1}, (X^{\mu})^{0,1}]^{1,0} &= \tfrac{1}{4} N_{J_K}(X^{\xi}, X^{\mu})^{1,0} = 0, \\
[X^{0,1}, (X^{\xi})^{0,1}]^{1,0} &= ([X, X^{\xi}] + i[JX, X^{\xi}] + i[X, JX^{\xi}] - [JX, JX^{\xi}])^{1,0} = 0, \\
[X^{0,1}, Y^{0,1}]_q^{1,0} &= (dp_{|\ker A})_q^{-1}([X^{0,1}, Y^{0,1}]^{1,0}_{p(q)}) - F_A(X^{0,1}, Y^{0,1})^{1,0} = 0.
\end{aligned}$$

For the first item, we use the integrability of J_K. For the second, we use that J as in (7.1.32) preserves the families of canonical and lifted horizontal fields, combined with the fact that lifted horizontal fields are K-invariant, hence invariant under the flow of canonical fields. For the last one we use integrability of J_X together with (7.1.31). □

By the previous result, given a principal connection A, the manifold (P, J) where J is as in (7.1.32) is complex. Hence, we may study its hermitian geometry. We consider the hermitian metric given by:

$$\omega = g(J\cdot, \cdot) = p^*\omega_X + \tfrac{1}{2}\langle JA \wedge A \rangle, \tag{7.1.33}$$

where g is as in (7.1.3). Moreover, in the hermitian case, there is a distinguished choice for the 3-form H_X given by:

$$H_X = -d^c \omega_X. \tag{7.1.34}$$

Then, we get the following compatibility result for the torsion of ω on the total space of (P, J):

Lemma 7.1.7. *Let* H_X *be as in* (7.1.34)*. Then:*

$$H = -d^c\omega, \tag{7.1.35}$$

where H *and* ω *are given by* (7.1.4) *and* (7.1.33)*.*

Proof. We compute:

$$\begin{aligned}
-d^c\omega &= p^*(-d^c\omega) + \tfrac{1}{2}Jd\langle J_K \circ A \wedge A\rangle \\
&= p^*H_X + J\langle J_K \circ (F_A - \tfrac{1}{2}[A \wedge A]) \wedge A\rangle \\
&= p^*H_X + \tfrac{1}{2}\langle [J_K \circ A \wedge J_K \circ A] \wedge A\rangle - \langle F_A \wedge A\rangle \\
&= p^*H_X + \tfrac{1}{6}\langle A \wedge [A \wedge A]\rangle - \langle F_A \wedge A\rangle \\
&= p^*H_X - CS(A) \\
&= H,
\end{aligned}$$

where we use throughout that $N_{J_K} = 0$, combined with the identity:

$$JA = -J_K \circ A. \tag{7.1.36}$$

□

We now provide a formula for the Lee 1-form θ_ω in terms of basic data, which will be useful in the sequel.

Lemma 7.1.8. *With the notations of this Section, the following formula holds:*

$$\theta_\omega = p^*\theta_{\omega_X} - \langle \Lambda_{\omega_X} F_A, JA\rangle - \langle \theta^\sharp_{\omega_K}, A\rangle. \tag{7.1.37}$$

Proof. We first obtain an equality that will be used in the main computation:

$$\begin{aligned}
\tfrac{1}{2}\Lambda_\omega \langle [A \wedge A] \wedge JA\rangle &= \tfrac{1}{4}\textstyle\sum_j \langle [A \wedge A] \wedge JA\rangle(U_j, JU_j) \\
&= \tfrac{1}{2}\textstyle\sum_j \langle [U_j, JU_j], JA\rangle \\
&= \textstyle\sum_j \langle D^{g_K}_{U_j} JU_j, JA\rangle \\
&= -\tfrac{1}{2}\textstyle\sum_j \langle g_K^{-1} d\omega_K(U_j, JU_j, J\cdot), JA\rangle \\
&= \langle J\theta^\sharp_{\omega_K}, JA\rangle \\
&= -\langle \theta^\sharp_{\omega_K}, A\rangle,
\end{aligned}$$

where we have used the standard identity in a hermitian manifold (M, g, J):

$$g((D^g_X J)Y, Z) = \tfrac{1}{2}d\omega(X, Y, Z) - \tfrac{1}{2}d\omega(X, JY, JZ). \tag{7.1.38}$$

Now, to compute the Lee form θ_ω we apply the general formula that holds in any Hermitian manifold (X, ω):

$$\theta_\omega = \Lambda_\omega d\omega. \tag{7.1.39}$$

Then, combining (7.1.33) with the above formulae, we have:

$$\begin{aligned}\theta_\omega &= \Lambda_\omega(d\omega_X + \tfrac{1}{2}\langle d(JA)\wedge A\rangle - \tfrac{1}{2}\langle JA\wedge dA\rangle \\ &= \Lambda_{\omega_X} d\omega_X - \tfrac{1}{2}\Lambda_\omega(\langle J_K \circ dA\wedge A\rangle + \langle JA\wedge dA\rangle) \\ &= \theta_{\omega_X} - \Lambda_\omega\langle J_K\circ(F_A - \tfrac{1}{2}[A\wedge A])\wedge A\rangle \\ &= \theta_{\omega_X} - \langle \Lambda_{\omega_X} F_A, JA\rangle + \langle J_K\theta^\sharp_{\omega_K}, JA\rangle \\ &= \theta_{\omega_X} - \langle \Lambda_{\omega_X} F_A, JA\rangle - \langle \theta^\sharp_{\omega_K}, A\rangle.\end{aligned}$$

□

Next, to prove that the situation described in this Section is not trivial, we provide a large family of real Lie groups endowed with integrable complex structures for which the construction of this Section applies.

Definition 7.1.9. *A tuple $(P, K, J_K, \langle\cdot,\cdot\rangle)$ is of split type if:*

1. *$K = K'\times K'$, where K' is a compact, real Lie group.*
2. *$P \cong P_1 \times_X P_2$, where P_j are (possibly non-isomorphic) principal K'-bundles.*
3. *One has:*
$$J_K\partial_i^{(1)} = \partial_i^{(2)}, \quad J_K\partial_i^{(2)} = -J_K\partial_i^{(1)}, \quad i = 1, ..., \dim K' \tag{7.1.40}$$
where $\{\partial_i^{(j)}\}$ runs over a basis of canonical vector fields on each P_j, $j = 1, 2$.
4. *The inner product $\langle\cdot,\cdot\rangle$ splits:*
$$\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_1 + \langle\cdot,\cdot\rangle_2, \tag{7.1.41}$$
where each $(\cdot,\cdot)_j$ is a (possibly different) pairing in the j^{th}-factor K', and this splitting is respected by the adjoint action of K.

When the structure is implicit, we will just say that P is of split type. It is obvious from (7.1.40) and (7.1.41) that split-type complex structures are orthogonal. The following result proves that they are moreover integrable.

Lemma 7.1.10. *The split-type almost complex structure J_K of (7.1.40) is integrable.*

Proof. It is enough to show that N_{J_K} vanishes on left-invariant vector fields at a point. We denote by $X_1^\xi + X_2^\eta$ the left invariant vector field which at the identity element of $K = K'\times K'$ is the Lie algebra element (ξ, η). We first compute:

$$N_{J_K}(X_1^\xi, X_1^\eta) = [X_1^\xi, X_1^\eta] + J_K([J_K X_1^\xi, X_1^\eta] + [X_1^\xi, J_K X_1^\eta]) - [J_K X_1^\xi, J_K X_1^\eta]. \tag{7.1.42}$$

Observe that in a fixed coordinate chart:

$$[J_K X_1^\xi, X_1^\eta] = J_K X_1^\xi(X_1^\eta) - X_1^\eta(J_K X_1^\xi) = -X_1^\eta(J_K X_1^\xi) = -J_K(X_1^\eta(X_1^\xi))$$

where, in the second equality, the first term vanishes given that $J_K X_1^\xi$ only differentiates along the second $K' \times K'$ factor. Similarly, we also get:

$$[X_1^\xi, J_K X_1^\eta] = J_K(X_1^\xi(X_1^\eta)), \quad [J_K X_1^\xi, J_K X_1^\eta] = 0.$$

Therefore, we conclude:

$$N_{J_K}(X_1^\xi, X_1^\eta) = [X_1^\xi, X_1^\eta] + (X_1^\eta(X_1^\xi) - X_1^\xi(X_1^\eta)) = [X_1^\xi, X_1^\eta] + [X_1^\eta, X_1^\xi] = 0.$$

Now, this result does not depend on coordinates as N_{J_K} is tensorial. Reasoning similarly,

$$\begin{aligned} N_{J_K}(X_1^\xi, X_2^\eta) &= [X_1^\xi, X_2^\eta] + J_K([J_K X_1^\xi, X_2^\eta] + [X_1^\xi, J_K X_2^\eta]) - [J_K X_1^\xi, J_K X_2^\eta] \\ &= J_K(J_K X_1^\xi(X_2^\eta) - J_K X_2^\eta(X_1^\xi)) - [J_K X_1^\xi, J_K X_2^\eta] \\ &= J_K X_1^\xi(J_K X_2^\eta) - J_K X_2^\eta(J_K X_1^\xi) - [J_K X_1^\xi, J_K X_2^\eta] = 0. \end{aligned}$$

The vanishing of $N_{J_K}(X_2^\xi, X_1^\eta)$ and X_2^ξ, X_2^η, are analogous. Hence extending by bilinearity, $N_{J_K} = 0$. □

Observe that among split-type examples, there is the principal bundle involved in the Hull-Strominger system 3.1.11, for which the structure group is $G = SU(3) \times SU(3)$, if the gauge bundle has rank 3.

In general, in [111] there is a systematic study of the moduli space of complex structures orthogonal with respect to a fixed bi-invariant pairing in real compact Lie groups. The cases of low rank are explicit and classified. These are described in detail in [17] for the groups $SU(3)$, $Spin(5)$ and G_2.

7.2 Hermitian metrics with non-abelian symmetries

The aim of this Section is to prove a correspondence result for Bismut-Hermite-Einstein metrics (6.3.2) and solutions to the coupled Hermite-Einstein system (6.1.1). To do this, we study the hermitian reduction of invariant hermitian metrics satisfying special metric properties. In this Section, we use the same notations and conventions as in the previous Section.

Proposition 7.2.1. *Let H be as in* (7.1.4)*. Then:*

$$dH = dH_X - \langle F_A \wedge F_A \rangle. \tag{7.2.1}$$

In particular, if (X, ω_X) is a hermitian manifold and H_X is given by (7.1.34)*, then:*

$$dd^c\omega = dd^c\omega_X + \langle F_A \wedge F_A \rangle. \tag{7.2.2}$$

Proof. The proof of (7.2.1) follows from (1.3.18). For the benefit of the reader, we give here computation. We use throughout the following expressions in the total space of P:

$$F_A = dA + \tfrac{1}{2}[A \wedge A], \quad d_A F_A = dA + [A \wedge F_A] = 0. \tag{7.2.3}$$

Moreover, the pairing $\langle \cdot, \cdot \rangle$ is ad-invariant. In particular, we have:

$$\langle [\cdot,\cdot],\cdot\rangle - \langle \cdot, [\cdot,\cdot]\rangle = 0. \tag{7.2.4}$$

Then, in the total space of P,

$$\begin{aligned} dCS(A) = & -\tfrac{1}{6} d\langle [A\wedge A]\wedge A\rangle + d\langle F_A \wedge A\rangle \\ = & -\tfrac{1}{2}\langle [dA\wedge A]\wedge A\rangle - \langle [A\wedge F_A]\wedge A\rangle + \langle F_A\wedge dA\rangle \\ = & -\tfrac{1}{2}\langle [(F_A - \tfrac{1}{2}[A\wedge A])\wedge A]\wedge A\rangle + \langle [F_A\wedge A]\wedge A\rangle + \langle F_A \wedge (F_A - \tfrac{1}{2}[A\wedge A])\rangle \\ = & \tfrac{1}{4}\langle [[A\wedge A]\wedge A]\wedge A\rangle + \langle F_A\wedge F_A\rangle \\ = & \langle F_A \wedge F_A\rangle. \end{aligned}$$

where the last step follows from elementary manipulations using (7.2.4) combined with the Jacobi identity for $[\cdot,\cdot]$. The last part of the statement is consequence of (7.2.1) and Lemma 7.1.7. □

The other piece of information we are interested in is the reduction of the Bismut-Ricci form ρ_B (see Section 1.2.2). We will also use the notation D^B, Ric_B, etc. for the Bismut connection whenever the hermitian metric involved is implicit.

Proposition 7.2.2. *Let ω be as in* (7.1.33)*. Then:*

$$\rho_B(\omega) = \rho_B(\omega_X) + \langle \Lambda_{\omega_X} F_A, F_A\rangle + \langle d_A(\Lambda_{\omega_X} F_A)\wedge A\rangle + \tfrac{1}{2}\langle [\Lambda_{\omega_X} F_A, A]\wedge A\rangle. \tag{7.2.5}$$

Proof. We use throughout Propositions 7.1.5, 7.2.1 and Lemma 7.1.8. Recall [89, Formula (3.16)] that on any hermitian manifold (M,J,g):

$$\rho_B(\omega)(E_1,E_2) = -\mathrm{Ric}_B(\omega)(E_1,JE_2) - (D^B_{E_1}\theta_\omega)(JE_2) + \tfrac{1}{4}\textstyle\sum_i dd^c\omega(E_1,E_2,e_i,Je_i), \tag{7.2.6}$$

where $\{e_i\}$ is a g-orthonormal frame. To prove (7.2.5), we pay attention to the number of vertical components on each term in the equation. First, we compute the purely horizontal part of $\rho_B(\omega)$. We examine separately each of the terms in (7.2.6) for the hermitian manifold (P,J,g) where J and g are given by (7.1.32), (7.1.3), and $E_1 = X$ and $E_2 = Y$ are lifted horizontal vectors with respect to A:

$$\begin{aligned} \mathrm{Ric}_B(X,JY) = & \mathrm{Ric}_{g_X}(X,JY) - \tfrac{1}{4}\textstyle\sum_{j,k} H_X(X_j,X_k,X)H_X(X_j,X_k,JY) \\ & - \textstyle\sum_j g_K(F_A(X_j,X),F_A(X_j,JY)) - \tfrac{1}{2} d^{\star_{g_X}} H_X(X,JY) \\ = & \mathrm{Ric}_{g_X,H_X}(X,JY) - \textstyle\sum_j g_K(F_A(X_j,X),F_A(X_j,JY)). \end{aligned}$$

Next we have:

$$\begin{aligned}
D^B_X\theta_\omega(JY) &= D^{g,H}_X\theta_{\omega_X}(JY) + D^{g,H}_X(g_K(\Lambda_{\omega_X}F_A, JA) + g_K(\theta^\sharp_{\omega_K}, A))(JY)\\
&= D^{g,H}_X\theta_{\omega_X}(JY) - (g_K(\Lambda_{\omega_X}F_A, JA) + g_K(\theta^\sharp_{\omega_K}, A))(p_V D^{g,H}_X JY)\\
&= D^{g,H}_X\theta_{\omega_X}(JY) - (g_K(\Lambda_{\omega_X}F_A, JA)\\
&\quad + (\theta^\sharp_{\omega_K}, A))(B_X JY + \tfrac{1}{2}F_A(X, JY))\\
&= D^{g,H}_X\theta_{\omega_X}(JY)\\
&= X(\theta_{\omega_X}(JY)) - \theta_{\omega_X}(D^{g,H}_X JY)\\
&= X(\theta_{\omega_X}(JY)) - \theta_{\omega_X}(D_X JY + \tfrac{1}{2}g^{-1}H(X, JY, \cdot))\\
&= X(\theta_{\omega_X}(JY)) - \theta_{\omega_X}(D^{g_X}_X JY + \tfrac{1}{2}g^{-1}_X H_X(X, JY, \cdot))\\
&= D^{g_X,H_X}_X\theta_{\omega_X}(JY).
\end{aligned}$$

Finally:

$$\begin{aligned}
\textstyle\sum_i dd^c\omega(X, Y, X_i, JX_i) &= \textstyle\sum_i (dd^c\omega_X - g_K(F_A\wedge F_A))(X, Y, X_i, JX_i)\\
&= \textstyle\sum_i dd^c\omega_X(X, Y, X_i, JX_i) - 4\sum_i g_K(F_A(X_i, X), F_A(X_i, JY))\\
&\quad - 4g_K(\Lambda_{\omega_X}F_A, F_A(X, Y)).
\end{aligned}$$

Plugging in these expressions in the above computation, we first get:

$$\begin{aligned}
\rho_B(\omega)(X, Y) &= -\operatorname{Ric}_{g_X,H_X}(X, JY) - D^{g_X,H_X}_X\theta_{\omega_X}(JY) + \tfrac{1}{4}\textstyle\sum_i dd^c\omega_X(X, Y, X_i, JX_i)\\
&\quad - g_K(\Lambda_{\omega_X}F_A, F_A(X, Y))\\
&= (\rho_B(\omega_X) - g_K(\Lambda_{\omega_X}F_A, F_A))(X, Y),
\end{aligned}$$

which indeed corresponds to the expected horizontal terms with the substitution $g_K = -\langle\cdot,\cdot\rangle$. Next we compute:

$$\begin{aligned}
\rho_B(\omega)(X, U) &= -\operatorname{Ric}_{g,H}(X, JU) - D^{g,H}_X\theta_\omega(JU) +\\
&\quad + \tfrac{1}{4}\textstyle\sum_i (dd^c\omega_X - g_K(F_A\wedge F_A))(X, U, X_i, JX_i)\\
&= -X(\theta_\omega(JU)) + \theta_\omega(D^{g,H}_X JU)\\
&= -X(g_K(\Lambda_{\omega_X}F_A, U) - g_K(J_K\theta^\sharp_{\omega_K}, U)) + \theta_\omega(D_X JU + \tfrac{1}{2}g^{-1}H(X, JU, \cdot))\\
&= -g_K([X, \Lambda_{\omega_X}F_A], U) + \theta_\omega(B_X JU - \tfrac{1}{2}g^{-1}(g_K(i_X F_A, JU)))\\
&= -g_K(i_X d_A(\Lambda_{\omega_X}F_A), U) + \theta_\omega(\tfrac{1}{2}g^{-1}(g_K(i_X F_A, JU)) - \tfrac{1}{2}g^{-1}(g_K(i_X F_A, JU)))\\
&= -g_K(d_A(\Lambda_{\omega_X}F_A)\wedge A)(X, U),
\end{aligned}$$

as claimed. Finally we have:

$$\begin{aligned}
\rho_B(\omega)(U, V) &= -\operatorname{Ric}_{g,H}(U, JV) - D^{g,H}_U\theta_\omega(JV) +\\
&\quad + \tfrac{1}{4}\textstyle\sum_i (dd^c\omega_X - g_K(F_A\wedge F_A))(U, V, X_i, JX_i)\\
&= -U(\theta_\omega(JV)) + \theta_\omega(D^{g,H}_U JV)\\
&= -U(g_K(\Lambda_{\omega_X}F_A, V) + g_K(\theta^\sharp_{\omega_K}, JV))\\
&= -g_K(\Lambda_{\omega_X}F_A, [U, V])\\
&= -\tfrac{1}{2}g_K([\Lambda_{\omega_X}F_A, A]\wedge A)(U, V),
\end{aligned}$$

which finishes the proof replacing $g_K = -\langle\cdot,\cdot\rangle$. □

We are now in position to prove the main result of this Section:

Theorem 7.2.3. *Assume the structure group K is connected and the pairing $\langle\cdot,\cdot\rangle$ on* Lie K *is negative definite. Then, with the notations of this Section:*

1. *The hermitian metric ω on P given by* (7.1.33) *is pluriclosed if and only if:*

$$dd^c\omega_X + \langle F_A \wedge F_A\rangle = 0. \tag{7.2.7}$$

2. *The hermitian metric ω on P given by* (7.1.33) *is Bismut-Hermite-Einstein if and only if:*

$$\begin{aligned} F_A \wedge \omega_X^{n-1} = \tfrac{z}{n}\omega_X^n, \quad F_A^{0,2} = 0, \\ \rho_B(\omega_X) + \langle z, F_A\rangle = 0, \\ dd^c\omega_X + \langle F_A \wedge F_A\rangle = 0, \end{aligned} \tag{7.2.8}$$

for a central element $z \in \mathfrak{k}$.

Proof. The first part of the result follows inmediately from Proposition 7.2.1. For the second part, by Proposition 7.2.2 and using the filtration by the number of basic components of $\Omega_P^\bullet$, the vanishing of ρ_B is equivalent to:

$$\rho_B(\omega_X) + \langle \Lambda_{\omega_X} F_A, F_A\rangle = 0, \tag{7.2.9}$$

$$d_A(\Lambda_{\omega_X} F_A) = 0, \tag{7.2.10}$$

$$[\Lambda_{\omega_X} F_A, \cdot] = 0. \tag{7.2.11}$$

Then, by the proof of Proposition 6.1.2, the equations (7.2.10),(7.2.11), are actually equivalent to:

$$F_A \wedge \omega_X^{n-1} = \tfrac{z}{n}\omega_X^n, \tag{7.2.12}$$

for some degree $z \in \mathfrak{k}$, and consequently $\Lambda_{\omega_X} F_A = z$. Hence, the result follows under the assumption (7.1.31) and the connectedness of K. □

Remark 7.2.4. *Observe that the system* (7.2.8) *is completely equivalent to the coupled Hermite-Einstein system using the one-to-one correspondence of Hermite-Einstein metrics on P^c and Hermite-Yang-Mills connections on P. Therefore, we call* (7.2.8) *the* coupled Hermite-Yang-Mills system.

Theorem 7.2.3 provides a further motivation to study the coupled Hermite-Einstein system (6.1.1), as it then prescribes a method to construct Bismut-Hermite-Einstein metrics (6.3.2), for which the only known non-Kähler examples are (up to finite quotients) given by local Samelson spaces [119, 135].

Chapter 8

Harmonic metrics for the Hull-Strominger system and stability

As explained in Chapter 4, the Hull-Strominger system is reinterpreted in the context of Hermite-Einstein metrics for Courant algebroids, and a moment map picture has been put forward. Guided by the general principles of Kempf-Ness and its extensions to infinite dimension, in this Chapter, we study the existence of an algebraic counterpart of this picture in the sense of Geometric Invariant Theory. However, a fundamental feature intrinsic to this theory is the indefiniteness of generalized hermitian metrics. This will be a recurrent issue leading to the introduction in this Chapter of harmonic metrics and Higgs fields, tailored to the needs of the Hull-Strominger system. We then propose a new notion of stability for holomorphic Courant algebroids reminiscent of GIT, which is sensible to study in this context. This Chapter is based on the article [66], which we follow closely.

8.1 The Hull-Strominger system and slope stability

In this Section we investigate the Mumford-Takemoto slope stability of holomorphic orthogonal bundles $(\mathcal{Q}, \langle\cdot,\cdot\rangle)$. Of special interest will be those that arise in relation to solutions to the Hull-Strominger system, with a focus on the Kähler property of the solution. In particular, we will recover a no-go result for the Hull-Strominger system, back to the seminal work of Candelas, Horowitz, Strominger, and Witten [24].

Let X be a compact complex manifold of dimension n. We assume that X admits a balanced hermitian metric ω_0. We denote by:

$$\mathfrak{b}_0 = [\omega_0^{n-1}] \in H^{n-1,n-1}_{BC}(X, \mathbb{R}) \tag{8.1.1}$$

the associated balanced class in Bott-Chern cohomology. Let $(\mathcal{Q}, \langle\cdot,\cdot\rangle)$ be a holomorphic orthogonal bundle over X. A positive definite hermitian metric $\mathbf{H}$ on $\mathcal{Q}$ is said to be compatible with the orthogonal structure $\langle\cdot,\cdot\rangle$ if there exists a $\mathbb{C}$-antilinear orthogonal involution $\sigma\colon \mathcal{Q} \to \mathcal{Q}$, that is, $\langle\sigma\cdot,\sigma\cdot\rangle = \overline{\langle\cdot,\cdot\rangle}$ and $\sigma^2 = id$, and such that:

$$\mathbf{H} = \langle\cdot,\sigma\rangle. \tag{8.1.2}$$

Given such a metric $\mathbf{H}$, we will denote by $D^{\mathbf{H}}$ and $F_{\mathbf{H}} := F_{D^{\mathbf{H}}}$ its Chern connection and Chern curvature, respectively.

Remark 8.1.1. *Observe that the (possibly indefinite) hermitian metric* $\mathbf{G}$ *in Lemma 5.1.2 is precisely of this form, for* $\sigma(s) = -\bar{s}$*. Here, the conjugation is obtained via the isomorphism* $\mathcal{Q} \cong (TX \oplus \mathrm{ad}\, P_h) \otimes \mathbb{C}$ *induced by Lemma 5.1.2 composed with* (5.2.1)*, where* P_h *denotes the bundle of unitary frames of* P*. In the sequel, we will reserve the notation* $\mathbf{H}$ *for hermitian metrics which are positive definite.*

The existence of a compatible hermitian metric $\mathbf{H}$ on $(\mathcal{Q}, \langle\cdot,\cdot\rangle)$ satisfying the Hermite-Einstein equation

$$F_{\mathbf{H}} \wedge \omega_0^{n-1} = 0 \tag{8.1.3}$$

can be characterized in terms of a slope stability criteria as in the Donaldson-Uhlenbeck-Yau Theorem [40, 134] and its extensions to hermitian manifolds (see [98]). Here, to accomodate the fact that $\mathcal{Q}$ is endowed with an orthogonal structure, we slightly refine the theory. Given a torsion-free coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules over X, we say that a subsheaf $\mathcal{F} \subset \mathcal{Q}$ is isotropic if $\langle \mathcal{F}, \mathcal{F} \rangle = 0$ (see e.g. [11, 12]).

Definition 8.1.2. *Let* X *be a compact complex manifold endowed with a balanced class* $\mathfrak{b}_0 \in H_{BC}^{n-1,n-1}(X, \mathbb{R})$*. A holomorphic orthogonal bundle* $(\mathcal{Q}, \langle\cdot,\cdot\rangle)$ *over* X *is:*

1. slope $\mathfrak{b}_0$-semistable *if for any isotropic coherent subsheaf* $\mathcal{F} \subset \mathcal{Q}$ *one has:*

$$\mu_{\mathfrak{b}_0}(\mathcal{F}) \leqslant 0, \tag{8.1.4}$$

2. slope $\mathfrak{b}_0$-stable *if for any proper isotropic coherent subsheaf* $\mathcal{F} \subset \mathcal{Q}$ *one has:*

$$\mu_{\mathfrak{b}_0}(\mathcal{F}) < 0, \tag{8.1.5}$$

3. slope $\mathfrak{b}_0$-polystable *if it is slope* $\mathfrak{b}_0$*-semistable and whenever* $\mathcal{F} \subset \mathcal{Q}$ *is an isotropic coherent subsheaf with* $\mu_{\mathfrak{b}_0}(\mathcal{F}) = 0$*, there is a coisotropic coherent subsheaf* $\mathcal{W} \subset \mathcal{Q}$ *such that:*

$$\mathcal{Q} = \mathcal{W} \oplus \mathcal{F}. \tag{8.1.6}$$

The relation between slope stability and the Hermite-Einstein equation (8.1.3) for compatible hermitian metrics is provided by the following version of the Donaldson-Uhlenbeck-Yau Theorem (see e.g. [12, 99]):

Theorem 8.1.3. *Let* X *be a compact complex manifold. Let* ω_0 *be a balanced hermitian metric on* X *with balanced class* $\mathfrak{b}_0 = [\omega_0^{n-1}] \in H_{BC}^{n-1,n-1}$*. A holomorphic orthogonal bundle* $(\mathcal{Q}, \langle\cdot,\cdot\rangle)$ *over* X *admits a compatible hermitian metric* $\mathbf{H}$ *solving the Hermite-Einstein equation* (8.1.3) *if and only if it is slope* $\mathfrak{b}_0$*-polystable.*

Let us turn next to the relation with the Hull-Strominger system. For the purposes of this Chapter, we will adopt the formalism of vector bundles, much more transparent for the aspects we will cover here. Moreover, we stress that the connection ∇ in this Thesis is assumed to be an instanton (3.1.12). In precise terms, we assume that X admits a holomorphic volume form Ω. We fix a pair of holomorphic vector bundles V_0 and V_1 over our compact complex manifold X satisfying (5.3.16). Although it will not be necessary in our methods, one can assume that V_0 has $T^{1,0}$ as its underlying smooth vector bundle. Then, we will assume throughout P is the holomorphic principal bundle of split frames of $V_0 \oplus V_1$, and accordingly $\mathrm{ad}\, P = \mathrm{End}\, V_0 \oplus \mathrm{End}\, V_1$ equipped with the pairing:

$$\langle (r_0, r_1), (s_0, s_1) \rangle = \alpha \mathrm{tr}_{V_0}(r_0 s_0) - \alpha \mathrm{tr}_{V_1}(r_1 s_1), \tag{8.1.7}$$

where $\alpha \in \mathbb{R}$. Then, the formulation of the Hull-Strominger system (3.1.14) with the above choices translates to the following definition.

Definition 8.1.4. *Let (X, Ω, V_0, V_1) be a compact Calabi-Yau manifold of complex dimension n endowed with a pair of holomorphic vector bundles satisfying* (5.3.16). *Then, a triple (ω, h_0, h_1) of a hermitian metric on X and hermitian metrics h_i on V_i is a* solution to the Hull-Strominger in (X, Ω, V_0, V_1) *with coupling constant α if:*

$$\begin{aligned} F_{h_0} \wedge \omega^{n-1} &= 0 \\ F_{h_1} \wedge \omega^{n-1} &= 0 \\ d(||\Omega||_\omega \omega^{n-1}) &= 0 \\ dd^c \omega - \alpha \mathrm{tr}_{V_0}\, F_{h_0} \wedge F_{h_0} + \alpha \mathrm{tr}_{V_1}\, F_{h_1} \wedge F_{h_1} &= 0. \end{aligned} \tag{8.1.8}$$

We stress that Definition 8.1.4 is a particular case of Definition 3.1.3, hence the theory developed in previous Chapter applies to this case. In the sequel, we will avoid confusion between both definitions by making explicit that a solution to the Hull-Strominger system is considered for the tuple (X, Ω, V_0, V_1) or by making reference to Definition 8.1.4.

Let (ω, h_0, h_1) be a solution of the Hull-Strominger system (8.1.8). Consider the associated holomorphic orthogonal bundle $(\mathcal{Q}, \langle \cdot, \cdot \rangle)$ as in Example 2.2.6. In our next results we investigate the relationship between the slope polystable of $(\mathcal{Q}, \langle \cdot, \cdot \rangle)$, in the sense of Definition 8.1.2, and the Kähler property of the solution. The key to our argument is the existence of a canonical isotropic subsheaf given by the holomorphic cotangent bundle

$$T^*_{1,0} \stackrel{\pi^*}{\hookrightarrow} \mathcal{Q}. \tag{8.1.9}$$

Lemma 8.1.5. *Let X be a compact Kähler manifold endowed with a holomorphic volume form Ω. Let V_0 and V_1 be holomorphic vector bundles over X satisfying* (5.3.16). *Let (ω, h_0, h_1) be a solution of the Hull-Strominger system* (8.1.8) *with $\alpha \in \mathbb{R}$ and consider the associated holomorphic orthogonal bundle $(\mathcal{Q}, \langle \cdot, \cdot \rangle)$. Suppose that X admits a balanced class $\mathfrak{b}_0 \in H^{n-1,n-1}(X, \mathbb{R})$ such that $(\mathcal{Q}, \langle \cdot, \cdot \rangle)$ is slope $\mathfrak{b}_0$-polystable, and Kähler classes $[\omega_i] \in H^{1,1}(X, \mathbb{R})$ such that V_i is $[\omega_i]$-polystable. Then g is a Kähler metric and h_0 and h_1 are flat.*

Proof. Assume first that $\alpha \neq 0$. We will prove that (ω, h_0, h_1) is also a solution of (8.1.8) with $\alpha = 0$. Consider the canonical isotropic subsheaf (8.1.9). The existence of a holomorphic volume form Ω implies that:

$$\mu_{\mathfrak{b}_0}(T^*_{1,0}) = 0, \tag{8.1.10}$$

for any given balanced class $\mathfrak{b}_0 \in H^{n-1,n-1}(X, \mathbb{R})$. Hence, assuming that $\mathcal{Q}$ is slope $\mathfrak{b}_0$-polystable, we have that:

$$\mathcal{Q} = \mathcal{W} \oplus T^*_{1,0} \tag{8.1.11}$$

for a holomorphic coisotropic subbundle $\mathcal{W} \subset \mathcal{Q}$. Note further that there are canonical isomorphisms of holomorphic vector bundles:

$$\mathcal{W} \cong \mathcal{Q}/T^*_{1,0} \cong A_P, \tag{8.1.12}$$

and therefore the class of the extension:

$$0 \longrightarrow T^*_{1,0} \xrightarrow{\pi^*} \mathcal{Q} \xrightarrow{\pi} A_P \longrightarrow 0 \tag{8.1.13}$$

vanishes. Note that there is a biholomorphism $A_P \cong (T^{1,0} \oplus \text{End } V_0 \oplus \text{End } V_1, \overline{\partial}_0)$ where the Dolbeault operator on the right hand-side is:

$$\overline{\partial}_0(V + r_0 + r_1) = \overline{\partial}V + i_V F_{h_0} + i_V F_{h_1} + \overline{\partial}^{V_0} r_0 + \overline{\partial}^{V_1} r_1 \tag{8.1.14}$$

and the class of (8.1.13) is represented by $\gamma \in \Omega^{0,1}(\text{Hom }(A_P, T^*_{1,0}))$, defined by:

$$i_W \gamma(V + r_0 + r_1) = -i_W i_V (2i\partial\omega) - 2\alpha \text{tr}_{V_0}(i_W F_{h_0} r_0) + 2\alpha \text{tr}_{V_1}(i_W F_{h_1} r_1) \tag{8.1.15}$$

for any $V + r_0 + r_1 \in T^{1,0} \oplus \text{End } V_0 \oplus \text{End } V_1$ and $W \in T^{0,1}$. Therefore, the condition:

$$[\gamma] = 0 \in H^1(\text{Hom }(A_P, T^*_{1,0})) \tag{8.1.16}$$

jointly with $\alpha \neq 0$ implies, in particular, the existence of $a_j \in \Omega^{1,0}(\text{End } V_j)$ such that:

$$\overline{\partial}^{V_0} a_0 = F_{h_0}, \qquad \overline{\partial}^{V_1} a_1 = F_{h_1}. \tag{8.1.17}$$

By hypothesis, there exists Kähler classes $[\omega_i] \in H^{1,1}(X, \mathbb{R})$ such that V_i is $[\omega_i]$-polystable. Let $\tilde{h}_j$ be a Hermite-Einstein metric on V_j with respect to ω_j. Then, we can use the standard identity in Kähler geometry:

$$-\frac{8\pi^2}{(n-2)!} ch_2(V_j) \cdot [\omega_j]^{n-2} = \|F_{\tilde{h}_j}\|^2_{L^2}, \tag{8.1.18}$$

where the L^2-norm of the curvature $F_{\tilde{h}_j}$ is calculated with respect to the metrics $\tilde{h}_j$ and ω_j using the volume form $\omega_j^n/n!$. Using that $\overline{\partial}^{V_j} a_j = F_{h_j}$, the left hand side of this expression vanishes by Chern-Weyl theory, and therefore $\tilde{h}_0$ and $\tilde{h}_1$ are flat. In particular,

$$F_{\tilde{h}_j} \wedge \omega^{n-1} = 0 \tag{8.1.19}$$

and hence, since h_j must be related to $\tilde{h}_j$ by a holomorphic gauge transformation, h_j are also flat. Therefore, (ω, h_0, h_1) solves (8.1.8) with $\alpha = 0$, and the Bianchi identity reads

$$dd^c\omega = 0. \tag{8.1.20}$$

By [98], the conformally balanced equation is equivalent to

$$d^c(\log\|\Omega\|_\omega) - d^*\omega = 0 \tag{8.1.21}$$

and, by 5.3.5, this implies

$$\nabla^B(\|\Omega\|_\omega^{-1}\Omega) = 0. \tag{8.1.22}$$

Then, since $-i\rho_B(\omega)$ is the induced curvature of ∇^B on the anti-canonical bundle K_X^{-1}, it follows that $\rho_B(\omega) = 0$. Thus, applying [67, Theorem 4.7] it follows from the existence of a holomorphic volume form Ω that g is Kähler.

□

Remark 8.1.6. *Notice that the proof of [67, Theorem 4.7], which we have used to conclude that g is Kähler, uses a slope stability argument via exact holomorphic Courant algebroids. Therefore, our proof of Lemma 8.1.5 reduces to Geometric Invariant Theory.*

Our next result provides an obstruction to the existence of non-Kähler solutions of the Hull-Strominger system (8.1.8). It is a direct consequence of Lemma 8.1.5.

Proposition 8.1.7. *Let X be a compact Kähler manifold endowed with a holomorphic volume form Ω. Let V_0 and V_1 be holomorphic vector bundles over X satisfying* (5.3.16). *Let (ω, h_0, h_1) be a solution of the Hull-Strominger system* (8.1.8) *with $\alpha \in \mathbb{R}$ and consider the associated holomorphic orthogonal bundle $(\mathcal{Q}, \langle\cdot,\cdot\rangle)$. Suppose that $(\mathcal{Q}, \langle\cdot,\cdot\rangle)$ is slope polystable with respect to $\mathfrak{b} := [\|\Omega\|_\omega \omega^{n-1}]$ and furthermore that $\mathfrak{b}$ is the $(n-1)^{\mathrm{th}}$-power of a Kähler class, then g is Kähler and h_0 and h_1 are flat.*

Remark 8.1.8. *Our previous result applies, in particular, to the solutions of the Hull-Strominger system found recently by Collins, Picard, and Yau in [28, Section 3.2]. These solutions are on a Kähler Calabi-Yau threefold, have V_0 isomorphic to $T^{1,0}$, and are constructed such that $\mathfrak{b}$ can be prescribed to be the square of any given Kähler class. Given that $T^{1,0}$ has non-trivial Chern classes (e.g., when X is simply connected), Proposition 8.1.7 proves that the associated $(\mathcal{Q}, \langle\cdot,\cdot\rangle)$ is not slope $\mathfrak{b}$-polystable in this case.*

Remark 8.1.9. *Observe that the proof of Proposition 8.1.7 via Lemma 8.1.5 uses crucially the Kähler hypothesis of the manifold. This poses the question of whether on a a priori general compact complex manifold, a solution to the Hull-Strominger system with $\mathfrak{b}$-polystable orthogonal bundle $(\mathcal{Q}, \langle\cdot,\cdot\rangle)$ must be Kähler.*

In virtue of this result, one is lead to think that although the generalized hermitian metric $\mathbf{G}$ constructed in Lemma 5.1.2 on $\mathcal{Q}$ is coupled Hermite-Einstein (see Definition 5.3.1, Corollary 5.3.6), its indefinite signature does not tie in properly with the notion of slope stability. To make future reference, we include here the explicit matrix expression of

$\mathbf{G}$ with respect to the (smooth) splitting $\mathcal{Q} = T \otimes \mathbb{C} \oplus \text{End } V_0 \oplus \text{End } V_1$. This follows inmediately from complexifying (5.2.2) and (8.1.7):

$$\mathbf{G} = \begin{pmatrix} g(\cdot, \bar{\cdot}) & 0 & 0 \\ 0 & \alpha \text{tr}_{V_0}(\cdot, \bar{\cdot}) & 0 \\ 0 & 0 & -\alpha \text{tr}_{V_1}(\cdot, \bar{\cdot}) \end{pmatrix}, \tag{8.1.23}$$

where conjugation in End V_i is taken with respect to the hermitian metrics h_i. Now, we show that even in the definite signature case, a similar rigidity result holds. We consider next the special case that the Hermite-Einstein metric $\mathbf{G}$ on $\mathcal{Q}$ associated to our solution is positive definite. Without loss of generality, we can assume that $\alpha > 0$. Thus, by looking at the expression of $\mathbf{G}$ in (8.1.23), the metric $\mathbf{G}$ is positive definite precisely when rk $V_0 = 0$. Specifying to the case of complex dimension three, we recover a no-go result for the original Hull-Strominger system back to the seminal work of Candelas, Horowitz, Strominger, and Witten [24]. This shows the necessity of introducing the connection ∇ for the existence of non-Kähler solutions.

Proposition 8.1.10. *Let X be compact complex manifold endowed with a holomorphic volume form Ω. Let V be holomorphic vector bundle over X satisfying $ch_2(V) = 0 \in H^{2,2}_{BC}(X, \mathbb{R})$. Let (ω, h) be a solution of the system*

$$\begin{aligned} F_h \wedge \omega^{n-1} &= 0, \\ d(\|\Omega\|_\omega \omega^{n-1}) &= 0, \\ dd^c\omega + \alpha \text{tr } F_h \wedge F_h &= 0, \end{aligned} \tag{8.1.24}$$

with $\alpha > 0$. Then, ω is Kähler and h is flat.

Proof. Consider the holomorphic orthogonal bundle $(\mathcal{Q}, \langle \cdot, \cdot \rangle)$ with the Hermite-Einstein metric $\mathbf{G}$ associated to the solution (ω, h) as in Corollary 5.3.6. Via the identification

$$\mathcal{Q} \cong TX \otimes \mathbb{C} \oplus \text{End } V \tag{8.1.25}$$

we have the explicit formula (cf. (8.1.23)):

$$\mathbf{G} = \begin{pmatrix} g & 0 \\ 0 & -\alpha \text{tr}_V \end{pmatrix} \tag{8.1.26}$$

and therefore $\mathbf{G}$ is defines a compatible, positive definite, Hermite-Einstein metric on $(\mathcal{Q}, \langle \cdot, \cdot \rangle)$ with respect to the balanced metric:

$$\omega' = \|\Omega\|_\omega^{\frac{1}{n-1}} \omega. \tag{8.1.27}$$

From Theorem 8.1.3, $(\mathcal{Q}, \langle \cdot, \cdot \rangle)$ is $\mathfrak{b}$-polystable for $\mathfrak{b} = [||\Omega||_\omega \omega^{n-1}]$. Hence, $\mu_{\mathfrak{b}}(T^*_{1,0}) = \mu_{\mathfrak{b}}(\mathcal{Q}) = 0$ implies that $\mathcal{Q} \cong T^*_{1,0} \oplus A_P$ holomorphically and metrically with respect to the metric $\mathbf{G}$. This means that the second fundamental form of the extension:

$$0 \to T^*_{1,0} \to \mathcal{Q} \to A_P \to 0 \tag{8.1.28}$$

given by $\gamma \in \Omega^{0,1}(\text{Hom}(A_P, T^*_{1,0}))$ as:

$$i_W \gamma(V + r) = -i_W i_V (2i\partial\omega) + 2\alpha \text{tr}_V(F_h r) \tag{8.1.29}$$

must vanish identically. Therefore, the result follows. □

8.2 Harmonic metrics, Higgs fields and stability

8.2.1 HyperKähler moment maps

Let (X, ω) be a compact complex manifold of dimension n endowed with a balanced metric ω, that is, satisfies $d\omega^{n-1} = 0$. We fix a smooth complex vector bundle Q over X of degree zero:

$$c_1(Q) \cdot [\omega^{n-1}] = 0, \tag{8.2.1}$$

where the product is considered in de Rham cohomology. We are interested in the geometry of the space of complex connections on Q, which we denote by $\mathcal{A}_Q$. In the application to Section 8.2.2, Q is a complex orthogonal bundle and $\mathcal{A}_Q$ is replaced by the space of orthogonal complex connections. Nonetheless, the setup discussed here applies with minor modifications and hence we stick to the simpler situation stated above.

The infinite-dimensional space of complex connections $\mathcal{A}_Q$ is affine, modelled on the complex vector space:

$$\Omega^1(\mathrm{End}\, Q). \tag{8.2.2}$$

It is endowed with a natural complex symplectic structure, defined by:

$$\Omega_{\mathbb{C}}(a_1^c, a_2^c) = -\int_X \mathrm{tr}\, a_1^c \wedge a_2^c \wedge \frac{\omega^{n-1}}{(n-1)!}. \tag{8.2.3}$$

The group of complex gauge transformations $\mathcal{G}(Q)$ of Q acts on $\mathcal{A}_Q$ by symplectomorphisms and, similarly as in the Atiyah-Bott-Donaldson picture, there is a complex moment map.

Lemma 8.2.1. *The $\mathcal{G}(Q)$-action on $(\mathcal{A}_Q, \Omega_{\mathbb{C}})$ is Hamiltonian with moment map:*

$$\langle \mu_{\mathbb{C}}(D), s^c \rangle = -\int_X \mathrm{tr}(s^c F_D) \wedge \frac{\omega^{n-1}}{(n-1)!}, \tag{8.2.4}$$

where $s^c \in \Gamma(\mathrm{End}\, Q) \cong \mathrm{Lie}\, \mathcal{G}(Q)$ and F_D denotes the curvature of D.

Proof. The action of $\mathcal{G}(Q)$ on the space of connections is given by conjugation:

$$g \cdot D = g \circ D \circ g^{-1}, \quad g \in \mathcal{G}(Q). \tag{8.2.5}$$

Now, consider the one-parameter family of gauge transformations $g_t \in \mathcal{G}(Q)$ such that $\frac{d}{dt}_{|t=0} g_t = s^c$. Then, by standard theory, we have:

$$\frac{d}{dt}_{|t=0}(g_t \cdot D) = -d_D s^c. \tag{8.2.6}$$

Applying this, we compute:

$$\begin{aligned}
d\langle \mu_{\mathbb{C}}(D), s^c\rangle(a^c) &= \tfrac{d}{dt}_{|t=0}\langle \mu_{\mathbb{C}}(D+ta^c), s^c\rangle \\
&= \tfrac{d}{dt}_{|t=0}\left(-\int_X \operatorname{tr}(s^c F_{D+ta^c}) \wedge \tfrac{\omega^{n-1}}{(n-1)!}\right) \\
&= -\int_X \operatorname{tr}(s^c d_D a^c) \wedge \tfrac{\omega^{n-1}}{(n-1)!} \\
&= \int_X \operatorname{tr}(d_D s^c \wedge a^c) \wedge \tfrac{\omega^{n-1}}{(n-1)!} \\
&= -\int_X \operatorname{tr}\left(\tfrac{d}{dt}_{|t=0}(g_t \cdot D) \wedge a^c\right) \wedge \tfrac{\omega^{n-1}}{(n-1)!} \\
&= \Omega_{\mathbb{C}}(X^{s^c}, a^c).
\end{aligned}$$

Therefore, the result follows. □

Observe that the zeros of the complex moment map are given by connections $D \in \mathcal{A}_Q$ satisfying:

$$F_D \wedge \omega^{n-1} = 0. \tag{8.2.7}$$

One can restrict the Hamiltonian $\mathcal{G}(Q)$-action to the complex subspace $\mathcal{A}_Q^{1,1} \subset \mathcal{A}_Q$ given by connections with $F_D^{0,2} = F_D^{2,0} = 0$, obtaining a complex analogue of the Hermite-Yang-Mills equations (see [91]).

To introduce the hyperKähler structure on $\mathcal{A}_Q$ of our interest, following [83] we fix a positive definite hermitian metric $\mathbf{H}$ on Q. Then, given $D \in \mathcal{A}_Q$ there is a unique decomposition:

$$D = \nabla^{\mathbf{H}} + \Psi, \tag{8.2.8}$$

where $\nabla^{\mathbf{H}}$ is an $\mathbf{H}$-unitary connection and $\Psi \in i\Omega^1(\mathrm{End}\,_{\mathbf{H}}Q)$, where:

$$\Omega^1(\mathrm{End}\,_{\mathbf{H}}Q) := \{a \in \Omega^1(\mathrm{End}\, Q) \mid a^{*_{\mathbf{H}}} = -a\}. \tag{8.2.9}$$

This induces an identification:

$$\mathcal{A}_Q = \mathcal{A}_{\mathbf{H}} \times i\Omega^1(\mathrm{End}\,_{\mathbf{H}}Q) \tag{8.2.10}$$

and a decomposition:

$$\Omega_{\mathbb{C}} = \Omega_{\mathbf{I}} + i\Omega_{\mathbf{J}} \tag{8.2.11}$$

where:

$$\Omega_{\mathbf{I}}(a_1^c, a_2^c) = -\int_X \operatorname{tr} a_1 \wedge a_2 \wedge \frac{\omega^{n-1}}{(n-1)!} - \int_X \operatorname{tr} \psi_1 \wedge \psi_2 \wedge \frac{\omega^{n-1}}{(n-1)!} \tag{8.2.12}$$

$$\Omega_{\mathbf{J}}(a_1^c, a_2^c) = i\int_X \operatorname{tr} \psi_1 \wedge a_2 \wedge \frac{\omega^{n-1}}{(n-1)!} + i\int_X \operatorname{tr} a_1 \wedge \psi_2 \wedge \frac{\omega^{n-1}}{(n-1)!} \tag{8.2.13}$$

for $a_j^c = a_j + \psi_j$. From this, using the fact that the base manifold has a complex structure J, one can infer a hyperKähler structure with metric:

$$g(a^c, a^c) = -\int_X \operatorname{tr} a \wedge *_\omega a + \int_X \operatorname{tr} \psi \wedge *_\omega \psi \tag{8.2.14}$$

and complex structures $\mathbf{I}, \mathbf{J}, \mathbf{K}$, satisfying $\mathbf{IJK} = \mathbf{I}^2 = \mathbf{J}^2 = \mathbf{K}^2 = -\mathrm{Id}$, defined by:

$$\mathbf{I}a^c = Ja - J\psi, \qquad \mathbf{J}a^c = -iJ\psi + iJa, \qquad \mathbf{K}a^c = i\psi + ia. \tag{8.2.15}$$

We are interested in the Hamiltonian action of the unitary gauge group $\mathcal{G}_{\mathbf{H}} \subset \mathcal{G}(Q)$ for the triple of symplectic structures $\Omega_{\mathbf{I}}, \Omega_{\mathbf{J}}, \Omega_{\mathbf{K}}$, where $\Omega_{\mathbf{K}} := g(\mathbf{K}\cdot, \cdot)$ is given by:

$$\Omega_{\mathbf{K}}(a_1^c, a_2^c) = -i\int_X \operatorname{tr} \psi_1 \wedge Ja_2 \wedge \frac{\omega^{n-1}}{(n-1)!} + i\int_X \operatorname{tr} a_1 \wedge J\psi_2 \wedge \frac{\omega^{n-1}}{(n-1)!}. \tag{8.2.16}$$

Proposition 8.2.2. *Assume that ω is balanced. Then, the $\mathcal{G}_{\mathbf{H}}$-action on $\mathcal{A}_Q$ is Hamiltonian for the three symplectic structures $\Omega_{\mathbf{I}}, \Omega_{\mathbf{J}}, \Omega_{\mathbf{K}}$, and there is a hyperKähler moment map:*

$$\mu = (\mu_{\mathbf{I}}, \mu_{\mathbf{J}}, \mu_{\mathbf{K}}) \tag{8.2.17}$$

where:

$$\langle \mu_{\mathbf{I}}(D), s\rangle = -\int_X \operatorname{tr} s(F_{\nabla^{\mathbf{H}}} + \tfrac{1}{2}[\Psi \wedge \Psi]) \wedge \frac{\omega^{n-1}}{(n-1)!}, \tag{8.2.18}$$

$$\langle \mu_{\mathbf{J}}(D), s\rangle = i\int_X \operatorname{tr} s\nabla^{\mathbf{H}}\Psi \wedge \frac{\omega^{n-1}}{(n-1)!}, \tag{8.2.19}$$

$$\langle \mu_{\mathbf{K}}(D), s\rangle = i\int_X \operatorname{tr} s\nabla^{\mathbf{H}}(J\Psi) \wedge \frac{\omega^{n-1}}{(n-1)!}, \tag{8.2.20}$$

and $s \in \Omega^0(\mathrm{End}_{\mathbf{H}} Q) \cong \mathrm{Lie}\, \mathcal{G}_{\mathbf{H}}$.

Proof. Equation (8.2.8) implies that

$$F_{\mathbf{G}} = F_{\nabla^{\mathbf{H}}} + \nabla^{\mathbf{H}}\Psi + \tfrac{1}{2}[\Psi \wedge \Psi]. \tag{8.2.21}$$

Then, the formula for $\mu_{\mathbf{I}}$ and $\mu_{\mathbf{J}}$ follow from Lemma 8.2.1 by taking real and imaginary parts in the formula for $\mu_{\mathbb{C}}$. The fact that $\mu_{\mathbf{K}}$ is a moment map follows easily from the explicit expression for $\Omega_{\mathbf{K}}$ above. □

To finish this section, we give a characterization of the hyperKähler moment map equations $\mu(D) = 0$ for a complex connection $D = \nabla^{\mathbf{H}} + \Psi$, given by

$$\begin{aligned} (F_{\nabla^{\mathbf{H}}} + \tfrac{1}{2}[\Psi \wedge \Psi]) \wedge \omega^{n-1} &= 0, \\ (\nabla^{\mathbf{H}}\Psi) \wedge \omega^{n-1} &= 0, \\ (\nabla^{\mathbf{H}} J\Psi) \wedge \omega^{n-1} &= 0. \end{aligned} \tag{8.2.22}$$

To link with the definition of a harmonic metric in Section 8.2.2, it is convenient to remove our assumption that the hermitian metric ω is balanced (in our applications, the hermitian metric is conformally balanced).

Lemma 8.2.3. *Let ω be an arbitrary hermitian form on X. Then, a complex connection $D = \nabla^{\mathbf{H}} + \Psi$ satisfies* (8.2.22) *if and only if*

$$\begin{aligned} (F_{\nabla^{\mathbf{H}}} + \tfrac{1}{2}[\Psi \wedge \Psi]) \wedge \omega^{n-1} &= 0, \\ (\nabla^{\mathbf{H}})^\star(J\Psi) - i_{J\theta_\omega^\sharp}\Psi &= 0, \\ (\nabla^{\mathbf{H}})^\star\Psi + i_{\theta_\omega^\sharp}\Psi &= 0. \end{aligned} \tag{8.2.23}$$

where $\theta_\omega = Jd^*\omega$ is the Lee form of ω and $(\nabla^{\mathbf{H}})^*$ denotes the adjoint operator with respect to the metric $g = \omega(\cdot, J\cdot)$.

Proof. The statement follows easily from the formula

$$(\nabla^{\mathbf{H}})^\star(J\Psi) = \tfrac{1}{(n-1)!} \star (\nabla^{\mathbf{H}}\Psi) \wedge \omega^{n-1} + i_{J\theta_\omega^\sharp}\Psi. \tag{8.2.24}$$

□

The third equation in (8.2.23), corresponding to the condition $\mu_{\mathbf{K}}(D) = 0$ when ω is conformally balanced, will be taken in Section 8.2.2 as the defining equation for our notion of harmonic metric for the Hull-Strominger system.

8.2.2 Harmonic metrics

We introduce next our notion of harmonic metric for the Hull-Strominger system, motivated by the hyperKähler moment map construction in the previous Section. We fix (X, Ω) and V_0, V_1 as in Section 8.1. Let (ω, h_0, h_1) be a solution of the Hull-Strominger system (8.1.8) and consider the associated holomorphic orthogonal bundle $(\mathcal{Q}, \langle\cdot,\cdot\rangle)$. We are mainly interested in non-Kähler solutions, and therefore we will assume that $\alpha > 0$ and $\mathrm{rk}\, V_0 > 0$ (see Corollary 8.1.10). Consequently, the generalized hermitian metric $\mathbf{G}$ associated to our solution will be indefinite (see (8.1.23)).

The fundamental object in our development is the orthogonal connection $D^{\mathbf{G}}$ on $(\mathcal{Q}, \langle\cdot,\cdot\rangle)$ in Proposition 5.1.1. Explicitly, in matrix notation in terms of the identification:

$$\mathcal{Q} \cong TX \otimes \mathbb{C} \oplus \mathrm{End}\, V_0 \oplus \mathrm{End}\, V_1, \tag{8.2.25}$$

for any vector field X the operator $D^{\mathbf{G}}_X$ is given by:

$$D^{\mathbf{G}}_X = \begin{pmatrix} \nabla^-_X & g^{-1}\alpha\mathrm{tr}\,(i_X F_{h_0}\cdot) & -g^{-1}\alpha\mathrm{tr}\,(i_X F_{h_1}\cdot) \\ -F_{h_0}(X,\cdot) & d^{h_0}_X & 0 \\ -F_{h_1}(X,\cdot) & 0 & d^{h_1}_X \end{pmatrix}, \tag{8.2.26}$$

where ∇^- denotes the $\mathbb{C}$-linear extension of the g-compatible connection with totally skew-symmetric torsion $d^c\omega$, that is,

$$\nabla^- = \nabla^g + \tfrac{1}{2}g^{-1}d^c\omega \tag{8.2.27}$$

for ∇^g the Levi-Civita connection of g, which we called Hull connection in Section 1.2.1. Given a compatible (positive definite) hermitian metric $\mathbf{H}$ on the holomorphic orthogonal bundle $(\mathcal{Q}, \langle\cdot,\cdot\rangle)$ there exists a unique decomposition:

$$D^{\mathbf{G}} = \nabla^{\mathbf{H}} + \Psi, \tag{8.2.28}$$

where $\nabla^{\mathbf{H}}$ is an $\mathbf{H}$-unitary connection and $\Psi \in \Omega^1(\mathrm{End}\, \mathcal{Q})$ satisfies:

$$\Psi^{\star_{\mathbf{H}}} = \Psi. \tag{8.2.29}$$

Lemma 8.2.4. *The pair* $(\nabla^{\mathbf{H}}, \Psi)$ *in* (8.2.28) *satisfies the equations*

$$\begin{aligned} (F_{\nabla^{\mathbf{H}}} + \tfrac{1}{2}[\Psi \wedge \Psi]) \wedge \omega^{n-1} &= 0, \\ (\nabla^{\mathbf{H}})^\star (J\Psi) - i_{J\theta_\omega^\sharp}\Psi &= 0, \\ F^{0,2}_{\nabla^{\mathbf{H}}} + (\nabla^{\mathbf{H}})^{0,1}\Psi^{0,1} + \tfrac{1}{2}[\Psi^{0,1} \wedge \Psi^{0,1}] &= 0. \end{aligned} \tag{8.2.30}$$

where $(\nabla^{\mathbf{H}})^\star$ *denotes the adjoint operator with respect to the metric* g.

Proof. By Corollary 5.3.6, $\mathbf{G}$ is Hermite-Einstein. Decomposing $F_{\mathbf{G}}$ into its hermitian and skew-hermitian components with respect to $\mathbf{H}$ as in the proof of Proposition 8.2.2, the proof follows easily from Equation (8.2.21) and the proof of Lemma 8.2.3. □

Given that the hermitian form ω is conformally balanced, the first and second equations in (8.2.30) correspond to the zeros of an infinite-dimensional complex moment map in the space of complex orthogonal connections on $(\mathcal{Q}, \langle\cdot,\cdot\rangle)$ (see Section 8.2.1). Similarly as in the theory of *Higgs bundles* [83], it is therefore very natural to supplement these conditions with an additional equation arising from a hyperKähler moment map (see Proposition 8.2.2 and Lemma 8.2.3).

Definition 8.2.5. *Let* $(\mathcal{Q}, \langle\cdot,\cdot\rangle, D)$ *be a holomorphic orthogonal bundle over a hermitian manifold* (X, ω) *endowed with an orthogonal connection* D*. We say that a compatible hermitian metric* $\mathbf{H}$ *on* $(\mathcal{Q}, \langle\cdot,\cdot\rangle)$ *is* harmonic *if:*

$$(\nabla^{\mathbf{H}})^\star \Psi + i_{\theta_\omega^\sharp}\Psi = 0, \tag{8.2.31}$$

where we use the decomposition (8.2.28).

Remark 8.2.6. *The notion of harmonicity for* $\mathbf{H}$ *we propose is well-known in the Kähler case. Indeed, for* (X, ω) *a Kähler manifold, by [31],* $\mathbf{H}$ *is harmonic in the sense of Definition 8.2.5 if and only if it is a critical point of the functional:*

$$E(H) = \int_X |\Psi|^2_{\mathbf{H},\omega} \frac{\omega^n}{n!}. \tag{8.2.32}$$

Adapting the proof to the case ω *has torsion, the Euler-Lagrange equation of this functional is precisely* (8.2.31).

We are interested in studying the existence of harmonic metrics under the hypothesis:

$$F_D \wedge \omega^{n-1} = 0, \quad F_D^{0,2} = 0. \tag{8.2.33}$$

Notice this connection D is, in particular, a *non-Hermite-Yang-Mills connection* in the sense of Kaledin and Verbitsky [91]. It would be interesting to find further relations between our picture and the theory proposed in this reference.

Our stability condition for the Hull-Strominger system is related to the existence of a harmonic metric on $(\mathcal{Q}, \langle\cdot,\cdot\rangle, D^{\mathbf{G}})$. We postpone its study to Section 8.2.3. Here, we propose to address the following problem:

Question. *Let* (ω, h_0, h_1) *be a solution of the Hull-Strominger system* (8.1.8) *and consider the associated holomorphic orthogonal bundle* $(\mathcal{Q}, \langle\cdot,\cdot\rangle)$ *and orthogonal connection* $D^{\mathbf{G}}$ *as in Proposition 5.1.1. Does* $(\mathcal{Q}, \langle\cdot,\cdot\rangle, D^{\mathbf{G}})$ *admit a harmonic metric* $\mathbf{H}$*?*

In order to provide a non-trivial example of harmonic metric for $(\mathcal{Q}, \langle\cdot,\cdot\rangle, D^{\mathbf{G}})$ in Section 8.3.2, next we compute the equation (8.2.31) for a particular choice of hermitian metric. Via the identification (8.2.25), we define (compare with (8.1.23)):

$$\mathbf{H} = \begin{pmatrix} g & 0 & 0 \\ 0 & -\alpha \mathrm{tr}_{V_0} & 0 \\ 0 & 0 & -\alpha \mathrm{tr}_{V_1} \end{pmatrix}. \tag{8.2.34}$$

It is not difficult to see that (8.2.34) defines a compatible hermitian metric on $(\mathcal{Q}, \langle\cdot,\cdot\rangle)$. In our next result we compute the decomposition (8.2.28) for this particular choice of hermitian metric.

Lemma 8.2.7. *Let* $(\nabla^{\mathbf{H}}, \Psi)$ *be the pair in* (8.2.28) *associated to the compatible hermitian metric* (8.2.34). *Then, in matrix notation in terms of the identification* (8.2.25)*, one has:*

$$\nabla^{\mathbf{H}} = \begin{pmatrix} \nabla^- & 0 & \mathbb{F}_{h_1}^\dagger \\ 0 & d^{h_0} & 0 \\ -\mathbb{F}_{h_1} & 0 & d^{h_1} \end{pmatrix}, \qquad \Psi = \begin{pmatrix} 0 & \mathbb{F}_{h_0}^\dagger & 0 \\ -\mathbb{F}_{h_0} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{8.2.35}$$

where $\mathbb{F}_{h_j} \in \Omega^1(\mathrm{Hom}\,(TX \otimes \mathbb{C}, \mathrm{End}\, V_j))$ *are the* $\mathrm{Hom}\,(TX \otimes \mathbb{C}, \mathrm{End}\, V_j)$*-valued* 1*-forms defined by*

$$(i_V \mathbb{F}_{h_j})(W) := F_{h_j}(V, W) \tag{8.2.36}$$

and $\mathbb{F}_{h_j}^\dagger$ *denote the corresponding adjoints with respect to* $\mathbf{G}$*, that is,*

$$i_V \mathbb{F}_{h_0}^\dagger(r_0) = g^{-1}\alpha \mathrm{tr}\,(i_V F_{h_0} r_0), \qquad i_V \mathbb{F}_{h_1}^\dagger(r_1) = -g^{-1}\alpha \mathrm{tr}\,(i_V F_{h_1} r_1) \tag{8.2.37}$$

Proof. It is not difficult to see that $\nabla^{\mathbf{H}}$, as defined above, is $\mathbf{H}$-unitary and furthermore that $\Psi^{*_{\mathbf{H}}} = \Psi$. The statement follows from formula (8.2.26). $\square$

The desired characterization of the harmonicity of (8.2.34) is as follows:

Lemma 8.2.8. *Let* $(\nabla^{\mathbf{H}}, \Psi)$ *be the pair in* (8.2.28) *associated to the compatible hermitian metric* $\mathbf{H}$ *defined by* (8.2.34). *Then,*

$$(\nabla^{\mathbf{H}})^\star \Psi + i_{\theta^\sharp_\omega} \Psi = \begin{pmatrix} 0 & -\mathbb{U}^\dagger & 0 \\ \mathbb{U} & 0 & -\mathbb{V}^\dagger \\ 0 & \mathbb{V} & 0 \end{pmatrix} \tag{8.2.38}$$

where

$$\mathbb{U}(V) = -i_V \left(d^{h_0\star} F_{h_0} + i_{\theta^\sharp_\omega} F_{h_0} + \star(F_{h_0} \wedge \star d^c\omega) \right), \tag{8.2.39}$$

$$\mathbb{V}(r_0) = \alpha F_{h_1}(e_i, e_j) \mathrm{tr}\, (F_{h_0}(e_i, e_j) r_0) \tag{8.2.40}$$

for any choice of g*-orthonormal frame* $e_1, \ldots, e_{2n}$ *of* T*, and* $\mathbb{U}^\dagger$ *and* $\mathbb{V}^\dagger$ *denote the corresponding adjoints with respect to* $\mathbf{G}$. *Consequently,* $\mathbf{H}$ *is harmonic if and only if the following conditions are satisfied*

$$F_{h_0} \wedge \star d^c\omega = 0, \qquad \alpha F_{h_1}(e_i, e_j) \mathrm{tr}\, (F_{h_0}(e_i, e_j)\cdot) = 0. \tag{8.2.41}$$

Proof. By the identity $\Psi^{\star_{\mathbf{H}}} = \Psi$, it suffices to calculate $\mathbb{U}$ and $\mathbb{V}^\dagger$. For this, we compute

$$\begin{aligned}
(\nabla^{\mathbf{H}})^\star \Psi(V) &= -i_{e_i}(\nabla^{\mathbf{H},\mathbf{g}}_{e_i} \Psi)(V) \\
&= -(\nabla^{\mathbf{H}}_{e_i}(i_{e_i}\Psi(V)) - i_{e_i}\Psi(\nabla^{\mathbf{H}}_{e_i} V) - i_{\nabla_{e_i} e_i}\Psi(V)) \\
&= d^{h_0}_{e_i}(F_{h_0}(e_i, V)) - F_{h_0}(e_i, \nabla^-_{e_i} V) - F_{h_0}(\nabla_{e_i} e_i, V) \\
&= -i_V d^{h_0\star} F_{h_0} - \tfrac{1}{2} F_{h_0}(e_i, g^{-1} i_V i_{e_i} d^c\omega) \\
&= -i_V (d^{h_0\star} F_{h_0} + \star(F_{h_0} \wedge \star d^c\omega)), \\
(\nabla^{\mathbf{H}})^\star \Psi(r_1) &= i_{e_i}\Psi(\nabla^{\mathbf{H}}_{e_i} r_1) = \alpha F_{h_0}(e_i, e_j) \mathrm{tr}\, (F_{h_1}(e_i, e_j) r_1).
\end{aligned}$$

Formula (8.2.38) follows now from the explicit formula for Ψ in Lemma 8.2.7. The last part of the statement follows from Lemma 5.3.2, which proves that the Hermite-Einstein equation for h_0:

$$F_{h_0} \wedge \omega^{n-1} = 0 \tag{8.2.42}$$

implies, in particular,

$$d^{h_0\star} F_{h_0} + i_{\theta^\sharp_\omega} F_{h_0} - \star(F_{h_0} \wedge \star d^c\omega) = 0. \tag{8.2.43}$$

□

Remark 8.2.9. *Geometrically, the condition* (8.2.41) *means that the two-form components of* F_{h_0} *are orthogonal to the two-form components of the torsion* $g^{-1}d^c\omega$ *and also to the two-form components of the curvature* F_{h_1}.

8.2.3 Stability and Higgs fields

The moment map constructions in [13, 70] suggest that the Hull-Strominger system is related to a stability condition in the sense of Geometric Invariant Theory. As we have seen in Proposition 8.1.7, the naive guess of considering slope polystability of the orthogonal bundle $(\mathcal{Q}, \langle\cdot,\cdot\rangle)$ with respect to the balanced class of the solution does not work. We propose next a refined stability condition based on the existence of harmonic metrics for the Hull-Strominger system. Even though our picture is mostly conjectural, we expect that this stability condition will lead us to new obstructions to the existence of solutions in future studies.

In order to relate the existence of a harmonic metric in the sense of Definition 8.2.5 with a numerical stability condition, we introduce the following technical definition.

Definition 8.2.10. *Let X be a complex manifold, $\mathcal{Q}$ a holomorphic vector bundle over X, and $\mathcal{F} \subset \mathcal{Q}$ a coherent subsheaf of $\mathcal{O}_X$-modules with singularity set $S \subset X$, i.e. S is minimal such that $\mathcal{F}|_{X\backslash S}$ is locally free. Then, given a (smooth complex) connection D on $\mathcal{Q}$, we say that $\mathcal{F}$ is preserved by D if:*

$$D(\mathcal{F}|_{X\backslash S}) \subset \Omega^1(X\backslash S, \mathcal{F}|_{X\backslash S}). \tag{8.2.44}$$

Remark 8.2.11. *Observe that any coherent subsheaf of a vector bundle is in particular torsion-free, so the notions of degree and slope introduced in Section 6.2.2 apply.*

The stability condition of our interest, is for tuples $(\mathcal{Q}, \langle\cdot,\cdot\rangle, D)$, where $(\mathcal{Q}, \langle\cdot,\cdot\rangle)$ is a holomorphic orthogonal bundle and D is a (smooth) orthogonal connection such that $D^{0,1} = \overline{\partial}_{\mathcal{Q}}$, as follows (cf. [91, Definition 8.3]).

Definition 8.2.12. *Let (X, ω) be a compact complex manifold X endowed with a balanced hermitian metric ω with balanced class $\mathfrak{b} \in H^{n-1,n-1}_{BC}(X, \mathbb{R})$. Let $(\mathcal{Q}, \langle\cdot,\cdot\rangle, D)$ be a holomorphic orthogonal bundle over X endowed with an orthogonal connection D such that $D^{0,1} = \overline{\partial}_{\mathcal{Q}}$. We say that $(\mathcal{Q}, \langle\cdot,\cdot\rangle, D)$ is:*

1. slope $\mathfrak{b}$-semistable *if for any isotropic coherent subsheaf $\mathcal{F} \subset \mathcal{Q}$ that is preserved by D one has:*

$$\mu_{\mathfrak{b}}(\mathcal{F}) \leqslant 0, \tag{8.2.45}$$

2. slope $\mathfrak{b}$-stable *if for any proper isotropic coherent subsheaf $\mathcal{F} \subset \mathcal{Q}$ that is preserved by D one has:*

$$\mu_{\mathfrak{b}}(\mathcal{F}) < 0, \tag{8.2.46}$$

3. slope $\mathfrak{b}$-polystable *if it is slope $\mathfrak{b}$-semistable and whenever $\mathcal{F} \subset \mathcal{Q}$ is a isotropic coherent subsheaf that is preserved by D with $\mu_{\mathfrak{b}}(\mathcal{F}) = 0$, there is a coisotropic subsheaf $\mathcal{W} \subset \mathcal{Q}$ that is D-preserved and:*

$$\mathcal{Q} = \mathcal{W} \oplus \mathcal{F}. \tag{8.2.47}$$

Remark 8.2.13. *The analogous notions of* $\mathfrak{b}$*-semistability (resp. stability, polystability) can be defined for pairs* $(\mathcal{Q}, D)$ *with no preferred orthogonal structure as in Definition 8.2.12, by ommiting the requirement that* $\mathcal{F}$ *and* $\mathcal{W}$ *be isotropic and coisotropic, respectively.*

The relation between the existence of harmonic metrics, in the sense of Definition 8.2.5, and slope stability is given in our next result.

Proposition 8.2.14. *Let* (X, ω) *be a compact complex manifold* X *endowed with a balanced hermitian metric* ω *with balanced class* $\mathfrak{b} \in H^{n-1,n-1}_{BC}(X, \mathbb{R})$*. Let* $(\mathcal{Q}, \langle\cdot,\cdot\rangle, D)$ *be a holomorphic orthogonal bundle over* X *endowed with an orthogonal connection* D *such that* $D^{0,1} = \overline{\partial}_{\mathcal{Q}}$ *and satisfying:*

$$F_D \wedge \omega^{n-1} = 0. \tag{8.2.48}$$

Assume that $(\mathcal{Q}, \langle\cdot,\cdot\rangle, D)$ *admits a harmonic metric* $\mathbf{H}$*. Then,* $(\mathcal{Q}, \langle\cdot,\cdot\rangle, D)$ *is slope* $\mathfrak{b}$*-polystable..*

Before proving Proposition 8.2.14, we make some observations.

Remark 8.2.15. *The analogous result of Proposition 8.2.14 holds also if* $(\mathcal{Q}, D)$ *does not have a preferred orthogonal structure, using the notion of slope stability given by Remark 8.2.13.*

Remark 8.2.16. *Proposition 8.2.14 shall be compared with the main result of [106], where the authors show, in a Kähler setting, that the existence of harmonic metrics on* $\mathcal{Q}$ *is equivalent to* $\mathcal{Q}$ *being semisimple. Moreover, the methods used in both results can be combined in the following result, which we expect to carry over to non-Kähler manifolds.*

Theorem 8.2.17. *Let* (X, ω) *a compact Kähler manifold and* $(\mathcal{Q}, D)$ *a holomorphic vector bundle endowed with a connection such that:*

$$F_D \wedge \omega^{n-1} = 0, \quad D^{0,1} = \overline{\partial}_{\mathcal{Q}}. \tag{8.2.49}$$

Assume that $(\mathcal{Q}, D)$ *admits a harmonic metric* $\mathbf{H}$*, and let* $\mathcal{F} \subset \mathcal{Q}$ *be a* D*-preserved subbundle. Then:*

$$\deg_{[\omega]}\mathcal{F} = \deg_{[\omega]}\mathcal{Q}. \tag{8.2.50}$$

The analogous result holds for orthogonal bundles $(\mathcal{Q}, \langle\cdot,\cdot\rangle, D)$ *and isotropic subbundles* $\mathcal{F} \subset \mathcal{Q}$*.*

Proof. Let $\mathcal{F}$ be a D-preserved subbundle of $\mathcal{Q}$. Then, by the main result of [106]:

$$\mathcal{Q} = \mathcal{F} \oplus \mathcal{F}^{\perp_{\mathbf{H}}}, \tag{8.2.51}$$

where $\mathcal{F}^{\perp_{\mathbf{H}}}$ is D-preserved. Then, combining (8.2.51) with Proposition 8.2.14 (ignoring the orthogonal structure, see Remark 8.2.15), we obtain:

$$\deg_{[\omega]}\mathcal{F} = \deg_{[\omega]}\mathcal{Q}. \tag{8.2.52}$$

If moreover $\mathcal{Q}$ has an orthogonal structure, the result of [106] also applies, as if $\mathcal{F}$ is isotropic, an elementary linear algebra argument shows $\mathcal{F}^{\perp_{\mathbf{H}}}$ is coisotropic, whenever $\mathbf{H}$ is compatible with $\langle\cdot,\cdot\rangle$. Then, the argument follows in this refined case too. □

In order to prove Proposition 8.2.14, we consider the following decomposition of our connection D. Given a compatible hermitian metric $\mathbf{H}$ on $(\mathcal{Q}, \langle\cdot,\cdot\rangle)$ we can uniquely write:

$$D = D^{\mathbf{H}} + \phi, \tag{8.2.53}$$

where $D^{\mathbf{H}}$ denotes the Chern connection of $\mathbf{H}$ and ϕ is a *Higgs field*:

$$\phi \in \Omega^{1,0}(\mathrm{End}\, \mathcal{Q}). \tag{8.2.54}$$

Lemma 8.2.18. *Let $(\mathcal{Q}, \langle\cdot,\cdot\rangle, D)$ be a holomorphic orthogonal bundle over X endowed with an orthogonal connection D such that $D^{0,1} = \overline{\partial}_{\mathcal{Q}}$ and satisfying:*

$$F_D \wedge \omega^{n-1} = 0. \tag{8.2.55}$$

Then, the pair $(D^{\mathbf{H}}, \phi)$ in (8.2.53) satisfies the equations

$$\begin{aligned} (F_{\mathbf{H}} + \tfrac{1}{2}\overline{\partial}_{\mathcal{Q}}\phi - \tfrac{1}{2}\partial^{\mathbf{H}}\phi^{\star_{\mathbf{H}}}) \wedge \omega^{n-1} &= 0, \\ (\overline{\partial}_{\mathcal{Q}}\phi + \partial^{\mathbf{H}}\phi^{\star_{\mathbf{H}}}) \wedge \omega^{n-1} &= 0, \\ \partial^{\mathbf{H}}\phi + \tfrac{1}{2}[\phi \wedge \phi] &= 0, \end{aligned} \tag{8.2.56}$$

where $F_{\mathbf{H}}$ denotes the Chern curvature of $\mathbf{H}$. Futhermore, $\mathbf{H}$ is harmonic if and only if:

$$(F_{\mathbf{H}} + \tfrac{1}{2}[\phi \wedge \phi^{\star_{\mathbf{H}}}]) \wedge \omega^{n-1} = 0. \tag{8.2.57}$$

Proof. Taking the $\mathbf{H}$-unitary part in the expression (8.2.53), one can easily see that:

$$\nabla^{\mathbf{H}} = D^{\mathbf{H}} + \tfrac{1}{2}(\phi - \phi^{\star_{\mathbf{H}}}), \qquad \Psi = \tfrac{1}{2}(\phi + \phi^{\star_{\mathbf{H}}}). \tag{8.2.58}$$

The first part of the statement follows from Lemma 8.2.4. As for the second part, we combine (8.2.24) with:

$$(\nabla^{\mathbf{H}} J\Psi) \wedge \omega^{n-1} = \tfrac{i}{2}(-\overline{\partial}_{\mathcal{Q}}\phi + \partial^{\mathbf{H}}\phi^{\star_{\mathbf{H}}} + [\phi \wedge \phi^{\star_{\mathbf{H}}}]) \wedge \omega^{n-1}. \tag{8.2.59}$$

□

We give next the proof of Proposition 8.2.14.

Proof of Proposition 8.2.14. Let $\mathbf{H}$ be a harmonic metric for D. Let $\mathcal{F} \subset \mathcal{Q}$ be an isotropic subsheaf preserved by D. By [93, Ch. V, Proposition 7.6], there exists a reflexive subsheaf $\mathcal{F}_1 \subset \mathcal{Q}$ such that $\mathcal{F} \subset \mathcal{F}_1$, $\mathcal{F}_1/\mathcal{F}$ is a torsion sheaf, and:

$$\mu_{\mathfrak{b}_0}(\mathcal{F}) \leqslant \mu_{\mathfrak{b}_0}(\mathcal{F}_1). \tag{8.2.60}$$

Since the singular (analytic) set $S \subset X$ of $\mathcal{F}$ has $\operatorname{codim} S \geq 2$, it follows by a density argument that $\mathcal{F}_1$ is also preserved by D. Hence, it suffices to assume that $\mathcal{F}$ is reflexive. In that case, there exists an analytic set $S \subset X$ of $\operatorname{codim} S \geq 3$ and a holomorphic vector bundle F defined on $X \backslash S$ such that $\mathcal{F}|_{X\backslash S} \cong \mathcal{O}(F)$. Denote by $E = \mathcal{Q}|_{X\backslash S}/F$. Using $\mathbf{H}$ we can make a smooth identification of E and $F^{\perp_{\mathbf{H}}}$ on $X\backslash S$. In the splitting $\mathcal{Q}|_{X\backslash S} = F \oplus F^{\perp_{\mathbf{H}}}$ we have:

$$D^{\mathbf{H}}|_{X\backslash S} = \begin{pmatrix} D_F^{\mathbf{H}} & \beta \\ -\beta^{\star_{\mathbf{H}}} & D_E^{\mathbf{H}} \end{pmatrix} \tag{8.2.61}$$

for $D_F^{\mathbf{H}}$, $D_E^{\mathbf{H}}$ the restricted Chern connections of F and E and some $\beta \in \Omega^{0,1}(\mathrm{Hom}(E, F))$. Similarly, for the (restricted) Higgs field we have:

$$\phi = \begin{pmatrix} \phi_F & \theta \\ \beta^{\star_{\mathbf{H}}} & \phi_E \end{pmatrix} \ , \ \phi^{\star_{\mathbf{H}}} = \begin{pmatrix} \phi_F^{\star_{\mathbf{H}}} & \beta \\ \theta^{\star_{\mathbf{H}}} & \phi_E^{\star_{\mathbf{H}}} \end{pmatrix} \tag{8.2.62}$$

since we assumed that $\mathcal{F}$ is D-preserved. Now, by Lemma 8.2.18,

$$\left(F_{\mathbf{H}} + \tfrac{1}{2}[\phi \wedge \phi^{\star_{\mathbf{H}}}]\right) \wedge \omega^{n-1} = 0. \tag{8.2.63}$$

and therefore:

$$0 = \left(F_{\mathbf{H}} + \tfrac{1}{2}[\phi \wedge \phi^{\star_{\mathbf{H}}}]\right)|_F \wedge \omega^{n-1} = (F_{D_F^{\mathbf{H}}} - \tfrac{1}{2}\beta \wedge \beta^{\star_{\mathbf{H}}} + \tfrac{1}{2}[\phi_F \wedge \phi_F^{\star_{\mathbf{H}}}] + \tfrac{1}{2}\theta \wedge \theta^{\star_{\mathbf{H}}}) \wedge \omega^{n-1}.$$

Note that $\mathrm{tr}\, F_{D_F^{\mathbf{H}}}$ is the restriction to $X\backslash S$ of a (smooth) representative of $-2\pi i c_1(\det \mathcal{F})$). Then, taking traces in the previous expression and integrating over X we get:

$$c_1(\mathcal{F}) \cdot \mathfrak{b} + \tfrac{(n-1)!}{2\pi} \int_X \tfrac{1}{2}\mathrm{tr}_F(\beta \wedge (\star\beta)^{\star_{\mathbf{H}}}) + \tfrac{1}{2}\mathrm{tr}_F(\theta \wedge (\star\theta)^{\star_{\mathbf{H}}}) = 0, \tag{8.2.64}$$

where we used that $\mathrm{tr}_F[\phi_F \wedge \phi_F^{\star_{\mathbf{H}}}] = 0$. Note that the integral in the previous expression is nonnegative, since for any $\varphi \in \mathrm{Hom}(E, F)$

$$\mathrm{tr}_F(\varphi \circ \varphi^{\star_{\mathbf{H}}}) = \mathrm{tr}_{F\oplus E}\left(\begin{pmatrix} 0 & \varphi \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \varphi^{\star_{\mathbf{H}}} & 0 \end{pmatrix}\right) \geq 0. \tag{8.2.65}$$

Thus, it follows that $\mu_{\mathfrak{b}}(\mathcal{F}) \leq \mu_{\mathfrak{b}}(\mathcal{Q}) = 0$ and hence $\mathcal{Q}$ is semistable. In case of equality, one has $\beta, \theta = 0$ and we get a holomorphic splitting $\mathcal{Q}|_{X\backslash S} = \mathcal{F}|_{X\backslash S} \oplus \mathcal{Q}/\mathcal{F}|_{X\backslash S}$ which is furthermore preserved by D. Then, since $\mathcal{F}$ is reflexive, so are Hom $(\mathcal{Q}, \mathcal{F})$ and Hom $(\mathcal{F}, \mathcal{F})$; in particular they are normal. Then we can extend uniquely the projection map $r : \mathcal{Q}|_{X\backslash S} \to \mathcal{F}|_{X\backslash S}$ to X. Moreover, the composition with $j : \mathcal{F} \to \mathcal{Q}$ is the unique extension to X of $\mathrm{Id}_{\mathcal{F}|_{X\backslash S}}$ and hence r is a retraction for the exact sequence:

$$0 \to \mathcal{F} \to \mathcal{Q} \to \mathcal{Q}/\mathcal{F} \to 0. \tag{8.2.66}$$

We conclude that the sequence is split. Then, $\mathcal{F}$ and $\mathcal{Q}/\mathcal{F}$ are locally free. Therefore, we identify $\mathcal{Q}/\mathcal{F} \cong \mathcal{F}^{\perp_{\mathbf{H}}}$ extending the identification over $X\backslash S$, and $\mathcal{F}^{\perp_{\mathbf{H}}}$ is D-preserved. Finally, it is a linear algebra exercise that $\mathcal{F}^{\perp_{\mathbf{H}}}$ is a coisotropic vector bundle. Hence, the result follows.

□

We are ready to prove the main result of this Section.

Theorem 8.2.19. *Let* (ω, h_0, h_1) *be a solution of the Hull-Strominger system* (8.1.8) *with balanced class* $\mathfrak{b} := [\|\Omega\|_\omega \omega^{n-1}]$*. Consider the associated triple* $(\mathcal{Q}, \langle\cdot,\cdot\rangle, D^{\mathbf{G}})$ *as in Proposition 5.1.1 and assume that* $(\mathcal{Q}, \langle\cdot,\cdot\rangle)$ *admits a harmonic metric* $\mathbf{H}$*. Then,* $(\mathcal{Q}, \langle\cdot,\cdot\rangle, D^{\mathbf{G}})$ *is slope* $\mathfrak{b}$*-polystable.*

Proof. Observe that the $\mathbf{H}$ is also harmonic for $\omega' = \|\Omega\|_\omega^{\frac{1}{n-1}} \omega$ (see Lemma 8.2.3), which is balanced $d\omega'^{n-1} = 0$. Consequently, the result follows as a direct consequence of Proposition 5.1.1 and Proposition 8.2.14. □

8.2.4 Non-holomorphic Higgs fields

We establish next a comparison between the equations in Lemma 8.2.18 and the *Hitchin's Equations* in the theory of Higgs bundles [83]. The main qualitative difference between these quations is that the Higgs field ϕ in our picture is very often not holomorphic, as we can see from the following result.

Lemma 8.2.20. *Let* X *be a compact Kähler manifold endowed with a holomorphic volume form* Ω*. Let* V_0 *and* V_1 *be holomorphic vector bundles over* X *satisfying* (5.3.16)*. Let* (ω, h_0, h_1) *be a solution of the Hull-Strominger system* (8.1.8) *and consider the associated triple* $(\mathcal{Q}, \langle\cdot,\cdot\rangle, D^{\mathbf{G}})$ *as in Proposition 5.1.1. Let* $\mathbf{H}$ *be a compatible hermitian metric on* $(\mathcal{Q}, \langle\cdot,\cdot\rangle)$ *such that the associated Higgs field in* (8.2.53) *satisfies*

$$\overline{\partial}_{\mathcal{Q}}\phi \wedge \omega^{n-1} = 0.$$

Assume that $\mathfrak{b} := [\|\Omega\|_\omega \omega^{n-1}]$ *is a* $(n-1)^{\text{th}}$*-power of a Kähler class. Then,* g *is Kähler and* h_0 *and* h_1 *are flat.*

Proof. By (8.2.53), we have that

$$F_{\mathbf{G}} \wedge \omega^{n-1} = (F_{\mathbf{H}} + \overline{\partial}_{\mathcal{Q}}\phi) \wedge \omega^{n-1} = 0.$$

By Theorem 8.1.3, $\overline{\partial}_{\mathcal{Q}}\phi \wedge \omega^{n-1} = 0$ implies that $(\mathcal{Q}, \langle\cdot,\cdot\rangle)$ is slope $\mathfrak{b}$-polystable and hence the statement follows from Proposition 8.1.7. □

Remark 8.2.21. *In order to relate our stability condition to a practical obstruction to the existence of solutions to* (8.1.8)*, it seems necessary to establish a more clear relation between the Dorfman bracket* $[,]$ *on* $\mathcal{Q}$ *(see Example 2.2.6) and the orthogonal connection* $D^{\mathbf{G}}$*, in a way that the slope inequality is formulated more naturally in terms of the triple* $(\mathcal{Q}, \langle\cdot,\cdot\rangle, [,])$*.*

In Section 8.3, we will see an explicit family of Examples over homogeneous manifolds where the Higgs field ϕ as in this picture is not holomorphic in these non-Kähler backgrounds as well.

8.3 Examples

8.3.1 A family of solutions on the Iwasawa manifold

In Section 8.2.3 we have proved that triples $(\mathcal{Q}, \langle\cdot,\cdot\rangle, D^{\mathbf{G}})$ associated to solutions of the Hull-Strominger system are polystable in the sense of Definition 8.2.12, provided that they admit a harmonic metric (see Definition 8.2.5). The aim of this section is to give some examples where one has a positive answer to Question 8.2.2.

Consider the *Iwasawa manifold* $X = \Gamma \backslash H$ (see Example 4.2.1), given by the quotient of the complex Heisenberg Lie group:

$$H = \left\{ \begin{pmatrix} 1 & z_2 & z_3 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix} \mid z_i \in \mathbb{C} \right\} \tag{8.3.1}$$

by the lattice $\Gamma \subset H$ of matrices with entries in Gaussian $\mathbb{Z}[i]$. Note that there is a holomorphic projection to the standard complex torus:

$$p : X \to T^4 = \mathbb{C}^2/\mathbb{Z}[i]^2, \tag{8.3.2}$$

which makes X a holomorphic torus fibration. Recall that the 1-forms:

$$\omega_1 = dz_1 \ , \ \omega_2 = dz_2 \ , \ \omega_3 = dz_3 - z_2 dz_1 \tag{8.3.3}$$

are Γ-invariant and descend to X, defining a global frame of $T^*_{1,0}$ satisfying:

$$d\omega_1 = d\omega_2 = 0 \quad , \quad d\omega_3 = \omega_{12}. \tag{8.3.4}$$

For any choice of:

$$(m, n, p) \in \mathbb{Z}^3 \backslash \{0\} \tag{8.3.5}$$

we consider the following purely imaginary $(1,1)$-form on the base T^4

$$F = \pi(m(\omega_{1\bar{1}} - \omega_{2\bar{2}}) + n(\omega_{1\bar{2}} + \omega_{2\bar{1}}) + ip(\omega_{1\bar{2}} - \omega_{2\bar{1}})). \tag{8.3.6}$$

Note that $\frac{i}{2\pi}F$ has integral periods and hence, by general theory, this is the curvature form of the Chern connection of a holomorphic hermitian line bundle $(L, h) \to T^4$. In the sequel, we will identify (L, h) and $F = F_h$ with their corresponding pull-backs to X via p.

Fix $(m_0, n_0, p_0), (m_1, n_1, p_1) \in \mathbb{Z}^3 \backslash \{0\}$ and consider the associated holomorphic hermitian bundles $(L_j, h_j) \to X$, for $j = 0, 1$. Consider the $SU(3)$ structure on X defined by:

$$\Omega = \omega_{123} \quad , \quad \omega_0 = \tfrac{i}{2}(\omega_{1\bar{1}} + \omega_{2\bar{2}} + \omega_{3\bar{3}}). \tag{8.3.7}$$

Note that ω_0 is a balanced hermitian metric and Ω is a holomorphic volume form.

Proposition 8.3.1. *With the notation above, the triple* (ω_0, h_0, h_1) *is a solution of the Hull-Strominger system* (8.1.8) *on* (X, Ω, L_0, L_1) *if and only if:*

$$\alpha = \frac{1}{2\pi^2(m_0^2 + n_0^2 + p_0^2 - m_1^2 - n_1^2 - p_1^2)}. \tag{8.3.8}$$

Proof. The first two equations of the system follow from the fact that L_0 and L_1 have degree zero combined with the fact that F_{h_0} and F_{h_1} are induced by left-invariant forms. The conformally balanced equation follows from $d\omega_0^2 = 0$ and the fact that $\|\Omega\|_{\omega_0}$ is constant. Finally,

$$\begin{aligned} dd^c\omega_0 &= \omega_{12\overline{12}}, \\ F_{h_0}^2 &= 2\pi^2(m_1^2 + n_1^2 + p_1^2)\omega_{12\overline{12}}, \\ F_{h_1}^2 &= 2\pi^2(m_2^2 + n_2^2 + p_2^2)\omega_{12\overline{12}}. \end{aligned}$$

and hence the Bianchi identity, given by the last equation in (8.1.8), is equivalent to (8.3.8). □

8.3.2 Existence of harmonic metrics

Recall that there is a fibration structure $p\colon X \to T^4$ and that L_i are pull-back from the base. Let $\underline{P}$ be the smooth complex principal bundle underlying the bundle of split frames of $L_0 \oplus L_1$. We consider the set of holomorphic structures on $\underline{P}$ given by the torsor for the abelian group:

$$B = \mathrm{Pic}_X^0 \times \mathrm{Pic}_X^0, \tag{8.3.9}$$

which is a complex 4-dimensional Abelian variety, since $H_{\overline{\partial}}^{0,1}(X) \cong \mathbb{C}^2$ (see [10]). We will denote by P_x the holomorphic bundle associated to $x \in B$. More precisely, we will identify P_x with the holomorphic bundle of split frames of a direct sum of line bundles $L_0^x \oplus L_1^x$, where $L_0^0 \oplus L_1^0 = L_0 \oplus L_1$.

To state the main result of this section, for each $x \in B$ we need to consider holomorphic orthogonal bundles $(\mathcal{Q}_x, \langle\cdot,\cdot\rangle)$ which may arise from solutions of the Hull-Strominger system (see Corollary 5.3.6 and Remark 8.3.3). As discussed in Section 5.4.2 around (5.4.21), these are parametrized by the image of the natural map

$$\partial\colon H_A^{1,1}(X,\mathbb{R}) \to H_{\overline{\partial}}^{2,1}(X). \tag{8.3.10}$$

The holomorphic deformations of $\mathcal{Q}_x$ given by the map above correspond to the observation here that the construction of the Dolbeault operator in Example 2.2.6 can be modified in the following way: given $[\tau] \in H_A^{1,1}(X,\mathbb{R})$ we can change $2i\partial\omega \to 2i\partial(\omega + \tau)$, which still defines an integrable Dolbeault operator. Observe furthermore that this induces a new holomorphic orthogonal bundle structure on $\mathcal{Q}$ with the same pairing.

Proposition 8.3.2. *Let $(m_i, n_i, p_i) \in \mathbb{Z}^3\backslash\{0\}$, $i = 0, 1$ such that*

$$c_1(L_0) \cdot c_1(L_1) = 0 \in H_{dR}^4(T^4, \mathbb{R}). \tag{8.3.11}$$

We fix the coupling constant α as in (8.3.8)*. Then, for any $x \in B$ there exists a holomorphic orthogonal bundle $(\mathcal{Q}_x^0, \langle\cdot,\cdot\rangle)$ induced by a solution of the Hull-Strominger system on $(X, \Omega, L_0^x, L_1^x)$ which admits a harmonic metric. Furthermore, for any small deformation $(\mathcal{Q}_x, \langle\cdot,\cdot\rangle)$ of $(\mathcal{Q}_x^0, \langle\cdot,\cdot\rangle)$ parametrized by an element in the image of* (8.3.10)*, there exists a solution to the Hull-Strominger system inducing $(\mathcal{Q}_x, \langle\cdot,\cdot\rangle)$ and a harmonic metric for this solution.*

Proof. We proceed in steps. Firstly, we prove the result for a single holomorphic orthogonal bundle as in the statement. Then, we check that this construction is stable with respect to deformations. The holomorphic pairing will remain constant along the family.

Step 1. Let $x = (x_0, x_1) \in B$ and choose lifts $\tilde{x}_i \in H^{0,1}_{\overline{\partial}}(X)$. We denote by L'_i the holomorphic line bundle corresponding to $\tilde{x}_i$. The Chern connections of h_i on L_i and L'_i are related by:

$$A'_i = A_i + a_i, \; i = 0, 1 \tag{8.3.12}$$

where $a_i^{0,1} \in \Omega^{0,1}_X$ is an invariant form representative of $\tilde{x}_i$. Observe that:

$$F_{h'_i} = F_{h_i} + da_i = F_{h_i} \tag{8.3.13}$$

since $F_{h'_i}, F_{h_i} \in \Omega^{1,1}_X$ and $da_i \in \Omega^{2,0}_X \oplus \Omega^{0,2}_X$ for a_i invariant. From this, (ω_0, A'_0, A'_1) is a solution of the Bianchi identity:

$$dd^c\omega - \alpha F_{A'_0} \wedge F_{A'_0} + \alpha F_{A'_1} \wedge F_{A'_1} = 0, \tag{8.3.14}$$

and we consider the associated Bott-Chern algebroid $\mathcal{Q}^0_x$. Then, $\mathcal{Q}^0_x$ admits a solution to the Hull-Strominger system (ω_0, h_0, h_1). Furthermore, we obtain a harmonic metric on $\mathcal{Q}^0_x$:

$$\mathbf{H}_0 = \begin{pmatrix} g_0 & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & -\alpha \end{pmatrix} \tag{8.3.15}$$

under the topological constraints (8.3.11) in the statement. To see this, by Lemma 8.2.8, it is equivalent to:

$$F_{h_0} \wedge \star d^c\omega_0 = 0 \;\; , \;\; \textstyle\sum_{i,j} F_{h_0}(e^0_i, e^0_j) F_{h_1}(e^0_i, e^0_j) = 0. \tag{8.3.16}$$

for any ω_0-orthonormal basis $\{e^0_i\}$. Using the expression (8.3.6) for the curvature F_{h_0} combined with:

$$\star d^c\omega_0 = \tfrac{i}{2}(\omega_{12\overline{3}} - \omega_{3\overline{12}}) \tag{8.3.17}$$

we get that the first of these equations holds for any value of the integers $(m_0, n_0, p_0) \in \mathbb{Z}^3 \backslash \{0\}$. Using that F_{h_i} are Hermite-Yang-Mills, the second equation may be rewritten as:

$$F_{h_0} \wedge F_{h_1} \wedge \omega_0 = 0 \tag{8.3.18}$$

or, in terms of the parameters,

$$m_0 m_1 + n_0 n_1 + p_0 p_1 = 0. \tag{8.3.19}$$

Finally, using that F_{h_i} are pull-back from the base torus T^4, one can easily see that this condition is equivalent to (8.3.11).

Step 2. It is easy to check that in this manifold, a basis of $H^{1,1}_A(X,\mathbb{R})/\ker\partial$ is given by the classes of the real $(1,1)$-forms:

$$\tau_1 = \omega_{1\bar{3}} - \omega_{3\bar{1}},\ \tau_2 = i(\omega_{1\bar{3}} + \omega_{3\bar{1}}),\ \tau_3 = \omega_{2\bar{3}} - \omega_{3\bar{2}},\ \tau_4 = i(\omega_{2\bar{3}} + \omega_{3\bar{2}}). \tag{8.3.20}$$

Then, any holomorphic orthogonal bundle in the deformation family of $(\mathcal{Q}^0_x, \langle\cdot,\cdot\rangle)$ (as in the statement) is isomorphic to $(\mathcal{Q}^\tau_x, \langle\cdot,\cdot\rangle)$ where $\tau = \sum_{i=1}^4 t_i\tau_i$, $t_i \in \mathbb{R}$, $\mathcal{Q}^\tau_x$ is the holomorphic vector bundle with Dolbeault operator as in Example (2.2.6) with $2i\partial\omega$ replaced by $2i\partial(\omega+\tau)$, and $\langle\cdot,\cdot\rangle$ is constant given by (8.1.7). Now, one can readily check that, for any small τ, $(\omega_0 + \tau, h'_0, h'_1)$ is a solution to the Hull-Strominger system which induces $(\mathcal{Q}^\tau_x, \langle\cdot,\cdot\rangle)$. Furthermore, applying Lemma 8.2.8 the harmonicity conditions for the metric:

$$\mathbf{H}_\tau = \begin{pmatrix} g_0 + \tau(\cdot, J\cdot) & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & -\alpha \end{pmatrix} \tag{8.3.21}$$

follow again from a straightforward calculation which implies:

$$F_{h_0} \wedge \star_{\omega_0+\tau} d^c(\omega_0 + \tau) = 0, \tag{8.3.22}$$

combined with condition (8.3.11). □

Remark 8.3.3. *More invariantly, in the formalism of Bott-Chern algebroids introduced in Chapter 2, the previous result can be stated as follows: for any $x \in B$ there exists a Bott-Chern algebroid with underlying bundle P_x and fixed Lie algebra bundle determined by the pairing* (8.1.7) *such that any small Bott-Chern algebroid deformation admits a solution to the Hull-Strominger system and a harmonic metric for this solution.*

As an immediate consequence of the previous result and Theorem 8.2.19, we obtain families of examples of holomorphic orthogonal bundles with connection associated to solutions of the Hull-Strominger system which satisfy the stability condition in Definition 8.2.12.

Corollary 8.3.4. *The orthogonal vector bundles with connection*

$$(\mathcal{Q}_x, \langle\cdot,\cdot\rangle, D^{\mathbf{G}})$$

given by Proposition 8.3.2 are $\mathfrak{b}$-polystable in the sense of Definition 8.2.12, where $\mathfrak{b}$ is the balanced class of the solution which induces $(\mathcal{Q}_x, \langle\cdot,\cdot\rangle)$.

Proof. This is immediate from Proposition 8.2.14 and Proposition 8.3.2. □

8.3.3 Higgs fields on the Iwasawa manifold

As we discussed in Section 8.2.4, we cannot expect the Higgs field defined by a harmonic metric $\mathbf{H}$ for the Hull-Strominger system to be holomorphic. Motivated by Lemma 8.2.20, the aim of this section is to provide a non-Kähler example which illustrates this phenomenon. The failure of the Higgs field ϕ to be holomorphic was computed in general in [66, Lemma 3.19]. Here, we avoid this somewhat involved expression by means of a direct computation building on the results of the previous Section.

We shall focus on the family of examples constructed in Proposition 8.3.2, with harmonic metrics $\mathbf{H}_\tau$ defined in (8.3.21).

Proposition 8.3.5. *Let* $(\omega_0+\tau, h_0, h_1)$ *be the solution of the Hull-Strominger system in* $(X, \Omega, L_0^x, L_1^x)$ *constructed in Proposition 8.3.2, for* α *given by* (8.3.8)*. Let* $\mathbf{H}_\tau$ *be the harmonic metric for this solution constructed in the proof of Proposition 8.3.2. Then, for any small* τ*, the Higgs field* ϕ *of* $\mathbf{H}_\tau$ *satisfies*

$$\overline{\partial}_{\mathcal{Q}}\phi \neq 0.$$

Proof. By continuity on the parameter τ, it is enough to check that $\overline{\partial}_{\mathcal{Q}}\phi \neq 0$ for $\tau = 0$. Notice that in our examples both V_0 and V_1 are line bundles, and therefore $\mathrm{End}\, V_i \cong \mathbb{C}$ canonically. The general expression for the Higgs field ϕ is obtained from (8.2.35) and (8.2.58). Using the identification of $\mathcal{Q} = T_{\mathbb{C}} \oplus \mathbb{C} \oplus \mathbb{C}$ (see (8.2.25)), we obtain:

$$\phi = \begin{pmatrix} 0 & 2(\mathbb{F}_{h_0}^\dagger)^{1,0} & 0 \\ -2\mathbb{F}_{h_0}^{1,0} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{8.3.23}$$

where $\mathbb{F}_{h_0}^{1,0} \in \Omega^{1,0}(\mathrm{Hom}(T_{\mathbb{C}}, \mathbb{C}))$ and $(\mathbb{F}_{h_0}^\dagger)^{1,0} \in \Omega^{1,0}(T_{\mathbb{C}})$ are given by the formulas:

$$i_{X^{1,0}}\mathbb{F}_{h_0}^{1,0} = i_{X^{1,0}}F_{h_0} \tag{8.3.24}$$

$$i_{X^{1,0}}(\mathbb{F}_{h_0}^\dagger)^{1,0} = 2\alpha g_0^{-1}(i_{X^{1,0}}F_{h_0}). \tag{8.3.25}$$

We can write $\overline{\partial}_{\mathcal{Q}}\phi$ in matrix form corresponding to this splitting:

$$\overline{\partial}_{\mathcal{Q}}\phi = \begin{pmatrix} (\overline{\partial}_{\mathcal{Q}}\phi)_{11} & (\overline{\partial}_{\mathcal{Q}}\phi)_{12} & 0 \\ (\overline{\partial}_{\mathcal{Q}}\phi)_{21} & (\overline{\partial}_{\mathcal{Q}}\phi)_{22} & (\overline{\partial}_{\mathcal{Q}}\phi)_{23} \\ 0 & (\overline{\partial}_{\mathcal{Q}}\phi)_{32} & 0 \end{pmatrix}. \tag{8.3.26}$$

whose components are $(1,1)$-forms with values in $\mathrm{End}\, T_{\mathbb{C}}$ for $(\overline{\partial}_{\mathcal{Q}}\phi)_{11}$, in $T_{\mathbb{C}}$ for $(\overline{\partial}_{\mathcal{Q}}\phi)_{12}$, in $T^*_{\mathbb{C}}$ for $(\overline{\partial}_{\mathcal{Q}}\phi)_{21}$, and are scalar for the rest. Now, we compute the term $(\overline{\partial}_{\mathcal{Q}}\phi)_{23}$ above. Let $s = (0,0,1) \in \Gamma(\mathcal{Q})$ corresponding to the constant section 1 in $\mathrm{End}\, V_1 = \mathcal{O}_X$. By the matrix expression above (8.3.26), it follows that:

$$(\overline{\partial}_{\mathcal{Q}}\phi)_{23} = (\overline{\partial}_{\mathcal{Q}}\phi)(s) = \overline{\partial}_{\mathcal{Q}}(\phi(s)) + \phi \wedge (\overline{\partial}_{\mathcal{Q}}(s)) = \phi \wedge (\overline{\partial}_{\mathcal{Q}}(s)).$$

Then, using the expression for F_{h_1} given by (8.3.6) and the Dolbeault operator given by the $(0,1)$-component of (8.2.26), we obtain:

$$\begin{aligned} \overline{\partial}_{\mathcal{Q}}(s) &= -\alpha g_0^{-1}\mathbb{F}_{h_1}^{0,1} + \overline{\partial}_{\mathrm{End}\, V_1}(1) \\ &= -2\pi\alpha(\omega_{\bar{1}} \otimes (m_1\overline{X}_1 - (n_1 - ip_1)\overline{X}_2) + \omega_{\bar{2}}(-m_1\overline{X}_1 - (n_1+ip_1)\overline{X}_2)) \end{aligned}$$

It then follows that:

$$\begin{aligned} (\overline{\partial}_{\mathcal{Q}}\phi)_{23} =& -\phi(\overline{\partial}_{\mathcal{Q}}(s)) \\ =& -4\pi^2\alpha((m_0m_1 + (n_0+ip_0)(n_1-ip_1))\omega_{1\bar{1}} + \\ & + (m_0(n_1+ip_1) - m_1(n_0+ip_0))\omega_{1\bar{2}} + \\ & + (-m_0(n_1-ip_1) + m_1(n_0-ip_0))\omega_{2\bar{1}} + \\ & + (m_0m_1 + (n_0-ip_0)(n_1+ip_1))\omega_{2\bar{2}}). \end{aligned}$$

Finally, subject to the condition (8.3.11) together with $(m_i, n_i, p_i) \in \mathbb{Z}^3\backslash\{0\}$, it is easy to check that $(\overline{\partial}_{\mathcal{Q}}\phi)_{23} \neq 0$.

□

Chapter 9

Outlook

Higher gauge theory

A promising approach for studying the geometry of Courant algebroids and constructing moduli of gauge-like equations is provided by higher gauge theory, in the sense of [16].

Roughly, in this context, one regards the space of sections of a Courant algebroid as infinitesimal symmetries of a higher gauge group [122]. Morally, this picture is equivalent to interpreting sections of the Atiyah bundle A_P of a principal bundle P over a smooth manifold M as the differential version of the extended gauge group of P:

$$1 \longrightarrow \mathcal{G}(P) \longrightarrow \tilde{\mathcal{G}}(P) \longrightarrow \mathrm{Diff}(M) \longrightarrow 1.$$

The ongoing work of R. Téllez-Domínguez in his PhD Thesis, under the supervision of L. Álvarez-Cónsul and M. García-Fernández make precise this correspondence. Furthermore, Téllez obtains a higher Chern correspondence in [129], recovering the Chern correspondence for string algebroids of [70]. Moreover, jointly with the above cited supervisors, they construct the space of connections for higher principal bundles up to gauge equivalence in the language of higher stacks, where gauge or holomorphicity equations can be studied.

This point of view provides a convenient mathematical framework to study stability conditions for holomorphic higher principal bundles, whose classical interpretation for Courant algebroids could clarify some aspects of the stability notions put forward in Chapter 8, particularly its (in)dependence on metric data. If such a program is successful, potentially there are higher analogous of a GIT theory that allow to consider a less singular space of stable objects. Then, one can speculate with the possibility of a higher Donaldson-Uhlenbeck-Yau result that regards this space as a symplectic reduction for suitable moment map equations *e.g.* the Hull-Strominger system, or else, as a higher analog of the moduli construction in [70].

Heterotic geometry of G_2 and $Spin(7)$ manifolds

The Hull-Strominger system has natural analogs in dimension 7 and 8, called the *heterotic G_2 system* and *heterotic $Spin(7)$ system*, respectively. These were first studied in physics as consistency conditions for the inner spaces of compactifications of the heterotic string with minimal supersymmetry to spacetimes with a different number of dimensions, hence the different dimensions, and later introduced in the mathematical literature in [48]. A solution to these systems determines, in particular, a G_2 or $Spin(7)$ structure with torsion on the manifold, respectively, and share further formal features with the Hull-Strominger system. It is therefore reasonable to expect that the results obtained in this Thesis can be exported to dimension 7 and 8.

In this direction, prior to this Thesis, in the physical literature [35, 36] obtain results analogous to the Hull-Strominger case regarding instanton connections and moduli of solutions. As a matter of fact, the perturbative result in [36, Corollary 1] inspired our Corollary 5.3.6. Recently, A. A. da Silva Junior has shown [33] that a solution to the heterotic G_2 system determines a G_2-instanton connection on a real Courant algebroid. Moreover, he obtains the precise G_2-instanton conditions analogous to the system (5.3.12).

On the other hand, in the G_2 and $Spin(7)$ cases, generalizing other aspects covered in this Thesis, such as the Futaki invariants in Chapter 5 or algebraic obstructions for real Courant algebroids related to GIT remain mysterious, mainly due to the lack of rigidity of complex manifolds.

New flows for the Hull-Strominger system

The dimensional reduction approach of Chapter 7 is based on ongoing work jointly with Mario García-Fernández and Jeffrey Streets. In a nutshell, Theorem 7.2.3 suggests a strong relation between the coupled Hermite-Einstein system and pluriclosed geometry on manifolds with non-abelian symmetries.

In this context, running pluriclosed flow [126], is a promising tool to look for canonical geometry. Moreover, under suitable invariant initial conditions, the flow preserves the symmetry and hence admits a natural reduction, leading to a new family of coupled flows worth exploring. These serve, at the same time, as a motivation and as a tool to study the coupled Hermite-Einstein system beyond the basic properties and solutions established in Chapter 6.

The analysis of these flows poses new challenges. For instance, when one regards them as instances of Generalized Ricci flow [71], typically, the indefinite signature of the Lie algebra $(\mathrm{Lie}\, K, \langle\cdot,\cdot\rangle)$ associated to the bundle means the techniques based on the flow of monotone quantities, which is at the core of a good number of results on Generalized Ricci flow, should now be handled very carefully. Nevertheless, we expect that studying the behaviour of these flows on non-Kähler Calabi-Yau threefolds provides new insights on the geometrization of Reid's fantasy [115].

Conclusión

El objetivo inicial marcado para esta Tesis ha sido estudiar la existencia de soluciones al sistema de Hull-Strominger [86, 127], y más concretamente, una versión refinada de la Conjetura de Yau propuesta en [139] (ver también Capítulo 3). Para este sistema de ecuaciones, cuya comprensión es aún muy incompleta, son conocidas algunas familias de soluciones (ver Sección 4.2) pero las herramientas teóricas que sirvan para abordar sistemáticamente este problema están ahora solamente empezando a ser construidas en la literatura matemática. Este trabajo propone enfoques novedosos que, según esperamos, puedan ser utilizados para progresar en el problema de existencia de soluciones y de sus posibles implicaciones en la construcción de un espacio de moduli de variedades Calabi-Yau proyectivas [115, 27, 55], y en teoría heterótica de cuerdas. A continuación se resumen las aportaciones originales que han sido desarrolladas en esta Tesis.

En los últimos años, la geometría generalizada se ha demostrado clave en la comprensión del sistema de Hull-Strominger [36, 61, 68, 70]. En este trabajo, siguiendo este principio como guía, en el Capítulo 5 hemos reinterpretado el sistema de Hull-Strominger, fijando los datos holomorfos de los fibrados, y cohomológicos de la clase balanceada en términos de un algebroide de Courant holomorfo $\mathcal{Q}$ provisto de una métrica generalizada $\mathbf{G}$ que satisface una ecuación de tipo Hermite-Einstein (Sección 5.3):

$$F_{\mathbf{G}} \wedge \omega^{n-1} = 0,$$

donde ω es la métrica hermítica dada por la solución al sistema. Este resultado es una contraparte en la literatura matemática que hace preciso el obtenido en teoría de perturbaciones en física [36]. Además generaliza para un tipo concreto algebroides de Courant transitivos, llamados *de cuerdas* [64], el resultado obtenido para algebroides de Courant holomorfos exactos en [67]. Más aún la interpretación de la ecuación Hermite-Einstein para $\mathbf{G}$ como aplicación momento nos permite construir invariantes de Futaki (ver Sección 5.4.2) en forma de caracteres holomorfos:

$$\mathcal{F} : H^0(\mathrm{End}\ \mathcal{Q}) \longrightarrow H^{1,1}_A,$$

que proporcionan un nuevo criterio de obstrucción para el sistema de Hull-Strominger más allá de la existencia de métricas balanceadas y estabilidad de pendiente para los fibrados. Esperamos que el cálculo de estos invariantes constituya, en futuros estudios, una técnica que permita decidir eficazmente sobre la posibilidad de resolver el sistema de Hull-Strominger en variedades Calabi-Yau no Kähler. De esta manera, obtenemos importante evidencia de que la pregunta que motiva esta Tesis (ver Sección 4.3) tiene una respuesta negativa.

Como resultado derivado de esta construcción, en los Capítulos 6 y 7 hemos estudiado las condiciones en geometría hermítica correspondientes a la existencia de métricas generalizadas

G Hermite-Einstein en el sentido anterior. Este nuevo sistema de ecuaciones, que llamamos *sistema Hermite-Einstein acoplado* (Definición 6.1.1) es más flexible que Hull-Strominger, y según hemos visto, tiene interesantes propiedades en Geometría Generalizada, y relación con las ecuaciones del movimiento de supergravedad heterótica y la teoría de álgebras de vértices (Sección 6.4, ver también [6]). Más aún, en el Capítulo 7 hemos visto que, vía reducción dimensional, el sistema Hermite-Einstein acoplado está también relacionado con la búsqueda de geometría canónica en variedades pluricerradas, dada por métricas hermíticas que satisfacen:

$$dd^c\omega = 0, \quad \rho_B = 0.$$

La segunda aportación principal en esta Tesis está relacionada con describir algunos aspectos de una conjetural correspondencia de Hitchin-Kobayashi [40, 99, 134] para algebroides de Courant que admiten soluciones al sistema de Hull-Strominger. En el Capítulo 8 hemos discutido la rigidez de la noción clásica de estabilidad de Mumford-Takemoto en este contexto y hemos propuesto una noción refinada de estabilidad inspirados por la construcción de Hitchin [83], y damos una definición de métrica armónica basada en una aplicación momento hyperKähler en dimensión infinita. Asimismo, demostramos su relación con una condición numérica de estabilidad. Este estudio es un paso importante para obtener, en el futuro, condiciones algebraicas para la existencia de soluciones de Hull-Strominger y para la construcción de espacios de moduli de algebroides de Courant holomorfos.

Además de los avances conceptuales descritos, una buena comprensión del problema de existencia pasa por estudiar ejemplos concretos. Así, en esta Tesis, nos hemos centrado en la geometría dada por variedades complejas localmente homogéneas (ver Capítulo 4), en la que los cálculos son particularmente explícitos, y hemos dado un procedimiento sistemático para la búsqueda de soluciones del sistema de Hull-Strominger con un ansatz de tipo invariante, recuperando muchas de las soluciones que se hayan ya en la literatura [25, 47, 105] y añadiendo nuevas familias. Estas geometrías también nos han servido para ilustrar discusiones sobre aspectos métricos del espacio de moduli de soluciones a Hull-Strominger (Sección 4.4), el cálculo de invariantes de Futaki explícitos (Sección 5.4.4), y una familia de métricas armónicas para algebroides de Courant (Sección 8.3.2).

Bibliography

[1] E. Abbena and A. Grassi. Hermitian left invariant metrics on complex Lie groups and cosymplectic Hermitian manifolds. *Boll. Un. Mat. Ital. A*, 5:371–379, 1986.

[2] L. Alessandrini and G. Bassanelli. Smooth proper modifications of compact Kähler manifolds. *Proc. Int. Work. Comp. Man., Comp. Anal. Asp. Math. E*, 17, 1990.

[3] B. Alexandrov and S. Ivanov. Vanishing theorems on hermitian manifolds. *Diff. Geom. Appl.*, 14(3):251–265, 2001.

[4] L. Álvarez-Cónsul. *Dimensional reduction and the Hitchin-Kobayashi correspondence.* PhD thesis, Universidad Autónoma de Madrid, 2000.

[5] L. Álvarez-Cónsul, A. De Arriba de La Hera, and M. Garcia-Fernandez. $(0, 2)$ Mirror Symmetry on homogeneous Hopf surfaces. *Int. Math. Res. Not., to appear.*

[6] L. Álvarez-Cónsul, A. De Arriba de La Hera, and M. Garcia-Fernandez. Vertex algebras from the Hull-Strominger system. *arXiv:2305.06836.*

[7] B. Anchouche and I. Biswas. Einstein-Hermitian connections on polystable principal bundles over a compact Kähler manifold. *Amer. Jour. Math.*, 123(2):207–228, 2001.

[8] B. Andreas and M. Garcia-Fernandez. Heterotic non-Kähler geometries via polystable bundles on Calabi-Yau threefolds. *Jour. Geom. Phys.*, 62(2):183–188, 2012.

[9] B. Andreas and M. Garcia-Fernandez. Solutions of the Strominger system via stable bundles on Calabi-Yau threefolds. *Comm. Math. Phys.*, 315(1):153–168, 2012.

[10] D. Angella. The cohomologies of the Iwasawa manifold and of its small deformations. *Jour. Geom. Anal.*, 23:1355–1378, 2013.

[11] M. Aparicio-Arroyo, S.B. Bradlow, B. Collier, O. García-Prada, P.B. Gothen, and A. Oliveira. $so(p, q)$-higgs bundles and higher Teichmüller components. *Inv. Math.*, 218:197–299, 2019.

[12] D. Araujo. Generalized quivers, orthogonal and symplectic representations, and Hitchin-Kobayashi correspondences. *Int. Jour. Math.*, 30(3):1–28, 2019.

[13] A. Ashmore, C. Strickland-Constable, D. Tennyson, and D. Waldram. Heterotic backgrounds via generalised geometry: moment maps and moduli. *JHEP*, 23, 2020.

[14] M. Atiyah and R. Bott. The Yang-Mills equations over Riemann surfaces. *Phil. Trans. R. Soc. Lond. A*, 308:523–615, 1983.

[15] T. Aubin. Equations du type Monge-Ampère sur les varietés kähleriennes compactes. *Bull. Sci. Math. Sér. AB*, 102:63–95, 1978.

[16] J. Baez and U. Schreiber. Higher gauge theory. *Davydov, A. et al. (eds) Cat. Alg. Geo. Math. Phys. Con. Math.*, 431, 2007.

[17] G. Barbaro. *The geometry of the Bismut connection.* PhD thesis, University of Rome, 2023.

[18] A. Besse. *Einstein manifolds.* Springer, 1987.

[19] J.M. Bismut. A local index theorem for non-Kähler manifolds. *Math. Ann.*, 284(4):681–699, 1989.

[20] S. Bochner and D. Montgomery. Groups on analytic manifolds. *Ann. Math*, 48:659–669, 1947.

[21] Y. D. Bozkhov. *Specific geometry of certain complex surfaces and three-folds.* PhD thesis, University of Warwick, 1992.

[22] E. Calabi and B. Eckmann. A class of compact complex manifolds which are not algebraic. *Ann. Math.*, 58:494–500, 1953.

[23] P Candelas, A. M. Dale, C. A. Lütken, and R. Schimmrigk. Complete intersection Calabi-Yau manifolds. *Nuc. Phys. B*, 298:493–525, 1988.

[24] P Candelas, G. Horowitz, A. Strominger, and E. Witten. Vacuum configurations for superstrings. *Nuc. Phys. B*, 258:46–74, 1985.

[25] G.L. Cardoso, G. Curio, G. Dall'Agata, D. Lust, P. Manousselis, and G. Zoupanos. Non-Kähler string backgrounds and their five torsion classes. *Nuc. Phys B*, 652:5–34, 2003.

[26] R. W. Carter, I. G. MacDonald, and G. B. Segal. *Lectures on Lie groups and Lie algebras.* Cambridge University Press, 2012.

[27] T. Collins, S. Picard, and S.T. Yau. Stability of the tangent bundle through conifold transitions. *arXiv:2102.11170*, 2021.

[28] T. Collins, S. Picard, and S.T. Yau. The Strominger system in the square of a Kähler class". *arXiv:2211.03784*, 2022.

[29] S. Console and A. Fino. Dolbeault cohomology of compact nilmanifolds. *Trans. Gro*, 6:111–124, 2001.

[30] L.A. Cordero, M. Fernández, A. Gray, and L. Ugarte. Compact nilmanifolds with nilpotent complex structures: Dolbeault cohomology. *Trans. Amer. Math. Soc.*, 352:5405–5433, 2000.

[31] K. Corlette. Flat g-bundles with canonical metrics. *Jour. Diff. Geom.*, 28(3):361–382, 1988.

[32] Baraglia D. and P. Hekmati. Transitive Courant algebroids, string structures and T-duality. *Adv. Theo. Math. Phys.*, 19(3):613–672, 2015.

[33] A. A. da Silva Junior. A curvatura generalizada de Ricci em 7-variedades Calabi-Yau de contato. *Relatorio Cientifico (BEPE)*, 2023.

[34] A. De Arriba de La Hera. *Supersymmetric Vertex Algebras and Killing Spinors*. PhD thesis, Universidad Complutense de Madrid, 2022.

[35] X. de la Ossa, M. Larfors, and E. Svanes. The infinitesimal moduli space of heterotic G_2 systems. *Comm. Math. Phys.*, 360:727–775, 2018.

[36] X. de la Ossa, M. Larfors, and E. Svanes. Restrictions of Heterotic G_2 structures and instanton connections. *Geom. Phys. Vol. II Fest. hon. Nigel Hitchin*, pages 503–518, 2018.

[37] X. de la Ossa and E. Svanes. Connections, field redefinitions and heterotic supergravity. *JHEP*, 1412, 2014.

[38] J.-P. Demailly. *Complex Analytic and Differential Geometry*. Universite de Grenoble, 2007.

[39] J.P. Demailly, T. Peternell, and M. Schneider. Compact complex manifolds with numerically effective tangent bundles, 1992.

[40] S. K. Donaldson. Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles. *Proc. Lon. Math. Soc*, s3-50(1):1–26, 1985.

[41] S. K. Donaldson. Remarks on gauge theory, complex geometry and 4-manifold topology. *World Sci. Ser. 20th Century Math., World Sci. Publ., River Edge, NJ*, 5:384–403, 1997.

[42] T. Fei. A construction of non-Kähler Calabi-Yau manifolds and new solutions to the Strominger system. *Adv. Math.*, 302:529–550, 2016.

[43] T. Fei. Generalized Calabi-Gray geometry and heterotic superstrings. *Proc. ICCM*, pages 261–282, 2017.

[44] T. Fei, Z. Huang, and S. Picard. A construction of infinitely many solutions to the Strominger system. *Jour. Diff. Geom.*, 117(1):23–39, 2021.

[45] T. Fei and S.T. Yau. Invariant solutions to the Strominger system on complex Lie groups and their quotients. *arXiv:1407.7641*, 2014.

[46] M. Fernandez, S. Ivanov, L. Ugarte, and D. Vassilev. Non-Kähler heterotic string solutions with non-zero fluxes and non-constant dilaton. *Jour. High Ener. Phys.*, 6, 2014.

[47] M. Fernandez, S. Ivanov, L. Ugarte, and R. Villacampa. Non-Kähler heterotic string solutions with non-zero fluxes and constant dilaton. *Comm. Math. Phys.*, 288:677–697, 2009.

[48] M. Fernandez, S. Ivanov, L. Ugarte, and R. Villacampa. Compact supersymmetric solutions of the heterotic equations of motion in dimensions 7 and 8. *Adv. Theor. Math. Phys.*, 15(2):245–284, 2011.

[49] A. Fino and G. Grantcharov. CYT and SKT metrics on compact semi-simple Lie groups. *SIGMA*, 19, 2023.

[50] A. Fino, G. Grantcharov, and L. Vezzoni. Solutions to the Hull-Strominger system with torus symmetry. *Comm. Math. Phys*, to appear.

[51] A. Fino, A. Otal, and L. Ugarte. Six dimensional solvmanifolds with holomorphically trivial canonical bundle. *Int. Math. Res. Not.*, 24:13757—13799, 2015.

[52] A. Fino and A. Tomassini. On astheno-Kähler metrics. *Jour. Lon. Math. Soc.*, 83(2):290–308, 2011.

[53] R. Friedman. On threefolds with trivial canonical bundle. *Proc. Symp. Pure Math.*, 53:103–134, 1991.

[54] A. Frölicher. Relations between the cohomolog groups of Dolbeault and topological invariants. *Proc. Nat. Aca. Sci.*, 41(9):641–644, 1955.

[55] J. Fu, J. Li, and S.T. Yau. Balanced metrics on non-Kähler Calabi-Yau threefolds. *Jour. Diff. Geom.*, 90:81–129, 2012.

[56] J. Fu and J. Xiao. Relations between the Kähler cone and the balanced cone of a Kähler manifold. *Adv. Math.*, 263:230–252, 2014.

[57] J. Fu and S.T. Yau. A Monge-Ampère type equation motivated by string theory. *Comm. Anal. Geom.*, 15:29–76, 2007.

[58] J. Fu and S.T. Yau. The theory of superstring with flux on non-Kähler manifolds and the complex Monge-Ampère equation. *Jour. Diff. Geom.*, 78:369–428, 2009.

[59] A. Fujiki. Moduli space of polarized algebraic manifolds and Kähler metrics. *Sug. Exp. 5*, 2:173–191, 1992.

[60] A. Futaki. An obstruction to the existence of Einstein-Kähler metrics. *Inv. Math.*, 73:437–444, 1983.

[61] M. Garcia-Fernandez. Torsion-free generalized connections and heterotic supergravity. *Comm. Math. Phys.*, 332:89–115, 2014.

[62] M. Garcia-Fernandez. Lectures on the Hull-Strominger system. *Trav. Math.*, XXIV:7–61, 2016.

[63] M. Garcia-Fernandez. Ricci flow, Killing spinors and T-duality in generalized geometry. *Adv. Math.*, 350:1059–1108, 2019.

[64] M. Garcia-Fernandez. T-dual solutions of the Hull-Strominger system on non-Kähler threefolds. *Jour. Reine Angew. Math.*, to appear.

[65] M. Garcia-Fernandez and R. Gonzalez Molina. Futaki invariants and Yau's conjecture for the hull-strominger system. *arXiv:2303.05274*, 2023.

[66] M. Garcia-Fernandez and R. Gonzalez Molina. Harmonic metrics for the hull-strominger system and stability. *arXiv:2301.08236*, 2023.

[67] M. Garcia-Fernandez, J. Jordan, and J. Streets. Non-Kähler Calabi-Yau geometry and pluriclosed flow. *Jour. Math. Pure Appl*, 177:329–367, 2023.

[68] M. Garcia-Fernandez, R. Rubio, C.S. Shahbazi, and C. Tipler. Canonical metrics on holomorphic Courant algebroids. *Proc. Lon. Math. Soc.*, 125(3):700–758, 2022.

[69] M. Garcia-Fernandez, R. Rubio, and C. Tipler. Holomorphic string algebroids. *Tran. Amer. Math. Soc.*, 373(10):7347–7382, 2020.

[70] M. Garcia-Fernandez, R. Rubio, and C. Tipler. Gauge theory for string algebroids. *Jour. Diff. Geom.*, to appear.

[71] M. Garcia-Fernandez and J. Streets. *Generalized Ricci Flow*. University Lecture Series. American Mathematical Society, 2021.

[72] P. Gauduchon. La 1-forme de torsion d'une varieté hermitienne compacte. *Math. Ann.*, 267:495–518, 1984.

[73] P. Gauduchon. Hermitian connections and Dirac operators. *Boll. U. M. I.*, (7)11-B, Supp. Fasc. 2:257–288, 1997.

[74] J. Gauntlett, D. Martelli, and D. Waldram. Superstrings with intrinsic torsion. *Phys. Rev. D*, 69, 2004.

[75] E. Goldstein and S. Prokushkin. Geometric models for complex non-Kaehler manifolds with $SU(3)$-structure. *Commun. Math. Phys.*, 251:65–78, 2004.

[76] D. Grantcharov, G. Grantcharov, and Y.S. Poon. Calabi-Yau connections with torsion on toric bundles. *Jour. Diff. Geom.*, 78(1):13–32, 2008.

[77] G. Grantcharov. Geometry of compact complex homogeneous spaces with vanishing first Chern class. *Adv. Math.*, 226(4):3136–3159, 2011.

[78] P. S. Green, T. Hübsch, and A. Lütken. All the Hodge numbers for all Calabi-Yau complete intersections. *Class. Quan. Grav.*, 6:105–124, 1989.

[79] P. Griffiths and J. Harris. *Principles of Algebraic Geometry*. Wiley Classics Library. Wiley-Interscience, 1994.

[80] M. Gualtieri. Generalized complex geometry. *Ann. Math*, 174:75–123, 2011.

[81] M. Gualtieri. Generalized Kähler geometry. *Comm. Math. Phys.*, 331:297–331, 2014.

[82] A. Hattori. Spectral sequence in the de Rham cohomology of fibre bundles. *J. Fac. Sci. Univ. Tokyo Sect. I*, 8:289–331, 1960.

[83] N. Hitchin. The self-duality equations on a Riemann surface. *Proc. Lon. Math. Soc.*, 55:59–126, 1987.

[84] C. Hull. Compactifications of the heterotic superstring. *Phys. Lett. B*, 178:357–364, 1986.

[85] C. Hull. Superstring compactifications with torsion and space-time supersymmetry. *Proc. 'Superunification and Extra Dimensions'*, pages 347–375, 1986.

[86] D. Huybrechts. *Complex Geometry: An Introduction*. Universitext. Springer Berlin, Heidelberg, 2005.

[87] M. Inoue. On surfaces of class VII_0. *Acta Math.*, 24:269–310, 1974.

[88] S. Ivanov. Heterotic supersymmetry, anomaly cancellation and equations of motion. *Phys. Lett. B.*, 685:190–196, 2010.

[89] S. Ivanov and G. Papadopoulos. Vanishing theorems and string backgrounds. *Class. Quan. Grav.*, 18:1089, 2001.

[90] S. Ivanov and G. Papadopoulos. Vanishing theorems on $(l|k)$-strong Kähler manifolds with torsion. *Adv. Math.*, 237:147–164, 2013.

[91] D. Kaledin and M. Verbitsky. Non-Hermitian Yang-Mills connections. *Sel. Math. New Ser.*, 4:279–320, 1998.

[92] A. Knapp. *Lie groups. Beyond an introduction*. Progress in Mathematics. Springer, 1996.

[93] S. Kobayashi. *Differential geometry of complex vector bundles*. Publications of the Mathematical Society of Japan. Iwanami Shoten, Publishers and Princeton University Press, 1987.

[94] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry, Volume I.* Wiley Classics Library. Wiley Interscience, 1996.

[95] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry, Volume II.* Wiley Classics Library. Wiley Interscience, 1996.

[96] K. Kodaira. On the structure of compact complex analytic surfaces I. *Amer. Jour. Math.*, 86:751–798, 1964.

[97] J. Li and S.T. Yau. Hermitian–Yang–Mills connections on non-Kähler manifolds. *Adv. Ser. Math. Phys., 1, World Sci. Publishing*, pages 560–573, 1986.

[98] J. Li and S.T. Yau. The existence of supersymmetric string theory with torsion. *Jour. Diff. Geom.*, 70(1):143–181, 2005.

[99] M. Lübke and A. Teleman. *The Kobayashi-Hitchin correspondence*. World Scientific Publishing Co Pte Ltd, 1995.

[100] D. Martelli and J. Sparks. Non Kähler heterotic rotations. *Adv. Theor. Math. Phys.*, 15(1):131–174, 2011.

[101] K. Matsuo and T. Takahashi. On compact astheno-Kähler manifolds. *Coll. Math.*, 89(2), 2001.

[102] M.L. Michelson. On the existence of special metrics in Complex Geometry. *Acta Math.*, 143:261–295, 1982.

[103] L. Nachbin. *The Haar integral*. Krieger Publishing Company, 1976.

[104] K. Nomizu. On the cohomology of compact homogeneous spaces of nilpotent Lie groups. *Ann. Math*, 59(2):531–538, 1954.

[105] A. Otal, L. Ugarte, and R. Villacampa. Invariant solutions to the Hull-Strominger system and the heterotic equations of motion. *Nuc. Phys. B.*, 920:442–474, 2017.

[106] C. Pan, Z. Shen, and X. Zhang. On the existence of harmonic metrics on non-Hermitian Yang-Mills bundles. *arXiv:2301.01428*, 2023.

[107] D.H. Phong, S. Picard, and X. Zhang. The anomaly flow on unimodular Lie groups. *Adv. Comp. Geom.*, 2017.

[108] D.H. Phong, S. Picard, and X. Zhang. New flows in complex geometry. *Sur. Diff. Geom.*, 22(1):331–364, 2017.

[109] D.H. Phong, S. Picard, and X. Zhang. The anomaly flow and the Fu-Yau equation. *Ann. PDE*, 4, 2018.

[110] D.H. Phong, S. Picard, and X. Zhang. Anomaly flows. *Comm. Anal. Geom.*, 26(4):955–1008, 2018.

[111] H. Pittie. The Dolbeault-cohomology ring of a compact, even-dimensional Lie group. *Proc. Indian Acad. Sci.*, 98(2-3):117–152, 1988.

[112] M. Poddar and A.S. Thakur. Group actions, non-Kähler complex manifolds and skt structures. *Comp. Man.*, 5(1):9–25, 2018.

[113] D. Popovici. Adiabatic limit and the Frölicher spectral sequence. *Pac. Jour. Math*, 300:121–158, 2019.

[114] D. Popovici. Non-Kähler Hodge theory and deformations of complex structures, 2020.

[115] C. Redden. String structures and canonical 3-forms. *Pac. Jour. Math.*, 249:447–484, 2011.

[116] S. Rollenske. Geometry of nilmanifolds with left-invariant complex structure and deformations in the large. *Proc. Lon. Math. Soc.*, 99(2):425–460, 2009.

[117] M. Rossi. Geometric transitions. *Jour. Geo. Phys.*, 56:1940–1983, 2006.

[118] Y. Sakane. On compact complex parallelisable solvmanifolds. *Osa. Jour. Math.*, 13:187–219, 1976.

[119] H. Samelson. A class of complex-analytic manifolds. *Portugal. Math.*, 12:129–132, 1953.

[120] S. Sandon. On the Chern correspondence for principal fibre bundles with complex reductive structure group, 2005.

[121] M. Schweitzer. Autour de la cohomologie de Bott-Chern. *arXiv:0709.3528*.

[122] Y. Sheng, X. Xu, and C. Zhu. String principal bundles and Courant algebroids. *Int. Math. Res. Not.*, 2021(7):5290–5348, 2021.

[123] J. Streets. Generalized geometry, T-duality, and renormalization group flow. *Jour. Geom. Phys.*, 114:506–522, 2017.

[124] J. Streets. Classification of solitons for pluriclosed flow on complex surfaces. *Math. Ann.*, 375:1555—1595, 2019.

[125] J. Streets. Ricci-yang-mills flow on surfaces and pluriclosed flow on elliptic fibrations. *Adv. Math.*, 394:1555—1595, 2022.

[126] J. Streets and G. Tian. A parabolic flow for pluriclosed metrics. *Int. Math. Res. Not.*, 16, 2010.

[127] A. Strominger. Superstrings with torsion. *Nuc. Phys. B*, 274(2):253–284, 1986.

[128] G. Székelyhidi. *An Introduction to extremal Kähler metrics.* Graduate Studies in Mathematics. American Mathematical Society, 2014.

[129] R. Téllez-Domínguez. Chern correspondence for higher principal bundles. *arXiv:2310.12738*, 2023.

[130] F. Tricerri. Some examples of locally conformal Kähler manifolds. *Rend. Sem. Mat. Uni. Pol. Torino*, 40(1):81–92, 1982.

[131] L. Ugarte. Hermitian structures on six-dimensional nilmanifolds. *Trans. Gro.*, 12:175–202, 2007.

[132] L. Ugarte and R. Villacampa. Balanced hermitian geometry of 6-dimensional nilmanifolds. *For. Math*, 27(2), 2013.

[133] L. Ugarte and R. Villacampa. Non-nilpotent complex geometry of nilmanifolds and heterotic supersymmetry. *Asi. Jour. Math.*, 18(2):229–246, 2014.

[134] K. Uhlenbeck and S.T. Yau. On the existence of Hermitian-Yang-Mills connections on stable vector bundles. *Comm. Pure App. Math.*, 39(1):257–293, 1986.

[135] Q. Wang, B. Yang, and F. Zheng. On Bismut Flat manifolds. *Trans. Amer. Math. Soc.*, 373:5747–5772, 2020.

[136] R. Wells. *Differential Analysis on Complex Manifolds*. Springer, 2008.

[137] S.T. Yau. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I. *Comm. Pure App. Math.*, 31(3):339–411, 1978.

[138] S.T. Yau. Complex geometry: Its brief history and its future. *Sci. Chi. Ser. A: Math.*, 48:47–60, 2005.

[139] S.T. Yau. Heterotic fluxes and non-Kähler geometry. *Talk at the fourth Simons Workshop in Mathematics and Physics*, 2006.

[140] Q. Zhao and F. Zheng. Strominger connection and pluriclosed metrics. *arXiv:1904.06604*, 2019.

[141] Q. Zhao and F. Zheng. On Gauduchon Kähler-like manifolds. *J. Geom. Anal.*, 4, 2022.

[142] F. Zheng. *Complex Differential Geometry*. Amer. Math. Soc., 2000.