SLAC – PUB – 4070
August 1986
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TWISTED STRINGS AND ORBIFOLDS*

JONATHAN A. BAGGER

The Institute for Advanced Study, Princeton, New Jersey 08540
and
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305

Orbifold compactifications provide a practical approach to string symmetry breaking. They have the potential to bridge the gap between string theory and the physics of the standard model.

As is by now well-known, string theories contain an enormous number of symmetries. For example, in their simplest form, heterotic strings describe ten-dimensional supergravity coupled to ten-dimensional super-Yang-Mills theory, with gauge group $E_8 \times E_8$ or $\mathrm{Spin}(32)/Z_2$.

How can these symmetries be broken to SU(3) × SU(2) × U(1) in four dimensions? One powerful approach to string symmetry breaking was proposed in a beautiful paper by Candelas, Horowitz, Strominger and Witten [1]. This group advocated compactifying the heterotic string on $M_4 \times K$, where M_4 is four-dimensional Minkowski space, and K is a compact six-dimensional manifold of SU(3) holonomy, a so-called Calabi-Yau space. Topological methods were used to show that compactifications on $M_4 \times K$ give rise to chiral fermions in four dimensions.

The problem with Calabi-Yau spaces is that they are very complicated. They are usually described as algebraic varieties in complex projective space. Their metrics are hard to find, and it is very difficult to compute the masses and mixings of the physical spectrum [2].

An alternative approach to string symmetry breaking is provided by *orbifolds* [3, 4, 5]. Orbifolds can be used to describe:

- toroidal compactification of strings on $M_{10-d} \times T^d$,
- a singular limit of Calabi-Yau compactification, and
- gauge symmetry breaking by Wilson lines and their generalizations.

As we shall see, orbifolds are very practical spaces for string compactification. The cases we consider give *exact* solutions to the classical string equations of motion. This is in striking contrast to Calabi-Yau spaces, which are solutions only if their metrics are adjusted order-by-order in the string tension α' [6].

*Work supported by the Department of Energy, contract numbers DE-AC02-76ER02220 and DE-AC03-76SF00515.

In the rest of this talk I will give a simple introduction to orbifolds. What I have to say is well-known to string experts, but it is time to explain orbifolds to the community at large. I will try to do this by stepping through a series of four examples, of gradually increasing complexity. I hope to show that – despite their name – orbifolds are, in fact, very simple objects.

To begin, let us define an *orbifold* $\mathcal{O}$ to be the quotient space formed by dividing a manifold $\mathcal{M}$ by the action of a discrete group $\mathcal{G}$: $\mathcal{O} = \mathcal{M}/\mathcal{G}$. For our purposes, we will take $\mathcal{M}$ to be flat, either R^d or T^d. If $\mathcal{G}$ acts freely on $\mathcal{M}$, the resulting orbifold $\mathcal{O} = \mathcal{M}/\mathcal{G}$ is a smooth manifold. If the action of $\mathcal{G}$ has fixed points, $\mathcal{O}$ is an orbifold, with singularities located at the fixed point sets.

For our first example, I would like to consider the orbifold $\mathcal{O} = R^2/Z \times Z$. The group $Z \times Z$ is generated by the lattice translations

$$g_1 = e^{2\pi i P_1 R_1}, \qquad g_2 = e^{2\pi i P_2 R_2}, \qquad (1)$$

where R_1 and R_2 are two vectors on the plane. The group action has no fixed points, so the orbifold $\mathcal{O}$ is a smooth manifold. In this case, it is obvious that the orbifold $\mathcal{O}$ is the torus T^2 (see Figure 1).

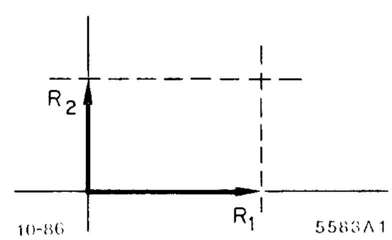

Fig. 1. The torus T^2 can be viewed as the orbifold $R^2/Z \times Z$.

Let us now consider the propagation of closed strings on this space. Clearly, closed strings can propagate consistently on the covering space R^2.

Talk presented at the 23rd International Conference on High Energy Physics, Berkeley, California, July 16-23, 1986

However, not all string configurations on R^2 are legal string configurations on T^2. The only legal configurations on the torus are the *translationally invariant* configurations on the plane. In the language of quantum mechanics, the physical states must be invariant under g_1 and g_2:

$$e^{2\pi i P_i R_i} \, | \, phys \, \rangle \; = \; | \, phys \, \rangle \; , \qquad (2)$$

for $i = 1$ or 2. The condition (2) forces the momenta to be quantized, with eigenvalues $P_i = M_i / R_i$, for $M_i \in Z$.

For point particles and open strings, that is the end of the story. The physical states on the torus are the translationally-invariant states on the plane. For closed strings, however, there is more to be done. Extra sectors must be added to the string Hilbert space. These sectors describe *shifted strings* – strings that are *open* on the plane but *closed* on the torus. The shifted strings obey the boundary conditions

$$X^i(\pi) \; = \; X^i(0) \; + \; N^i R_i \; , \qquad (3)$$

for $N^i \in Z$. The $N^i = 0$ sector contains to honest-to-God closed strings, on the plane and on the torus. The $N^i \neq 0$ states are open on the plane but closed on the torus. They are "soliton" states, and they are absolutely necessary for the modular invariance of the string. For $\mathcal{O} = R^2/Z \times Z$, there are an infinite number of soliton sectors, labelled by the winding numbers N^1 and N^2. In each sector of Hilbert space, the physical states must be invariant under g_1 and g_2.

Thus we have seen that string propagation on the torus can be identified with string propagation on the orbifold $R^2/Z \times Z$. For a less trivial example, let us now discuss the orbifold $\mathcal{O} = T^2/Z_2$, where T^2 is the torus generated by R_1 and R_2, and Z_2 acts on the torus by a π rotation about the origin. As shown in Figure 2, this rotation leaves four points invariant. At each fixed point, there is a conical singularity of deficit angle $\Delta = \pi$.

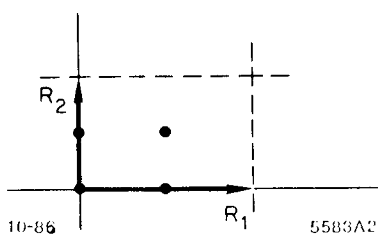

Fig. 2. The orbifold T^2/Z_2 has four fixed points.

How can strings propagate in the presence of these singularities? In the neighborhood of any one singularity, spacetime resembles a cone, with deficit angle $\Delta = \pi$ at the apex. For an arbitrary deficit angle, string propagation would probably be inconsistent, for a string encountering the singularity would develop a kink. However, for the special deficit angles $\Delta = 2\pi - 2\pi/N$, this is not so. For these special angles, N copies of the cone exactly cover the plane. Because of the symmetry restriction, the N-fold symmetric string configurations on the plane are legal string configurations on the cone.

To illustrate this, let us return to the case $N = 2$, or $\Delta = \pi$. Then two copies of the cone tile the plane, and rotationally-invariant string configurations of the plane are legal configurations on the cone. Because of the rotational symmetry, strings slip smoothly across the singularity, preserving the winding number about the singularity, modulo two. This is illustrated in Figure 3.

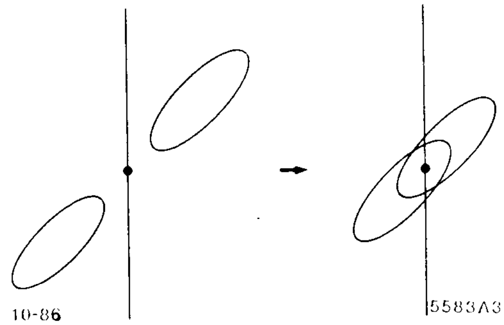

Fig. 3. Rotationally-invariant configurations on the plane are legal configurations on a cone of deficit angle $\Delta = \pi$. String propagation preserves winding number, modulo two.

As before, we must also consider *twisted sectors*. The twisted sectors on the cone are analogs of the soliton sectors on the torus. For the case at hand, the twisted sectors obey the boundary condition

$$X^i(\pi) \; = \; g \cdot X^i(0) \; , \qquad (4)$$

where g generates a rotation by $\Delta = \pi$. The boundary condition (4) fixes the center of mass of the string to lie at the apex of the cone. A typical twisted string is shown in Figure 4. Note that it has winding number one, modulo two. Twisted strings are open strings on the plane, but closed on the cone. For the orbifold $\mathcal{O} = T^2/Z_2$, there are twisted states located at each of the four fixed points of Figure 2.

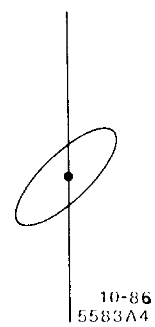

Fig. 4. Twisted strings wrap once around the singularity at the apex of the cone, modulo two.

Other orbifolds $\mathcal{O}$ are constructed by dividing a torus T^d by a group $\mathcal{G}$ of automorphisms of T^d. The group $\mathcal{G}$ is a *point group* of the torus, and its action typically leaves fixed points or even fixed tori. By appropriately choosing the torus T^d and the point group $\mathcal{G}$, many interesting compactifications can be studied. All one has to do is follow the general procedure, valid for all orbifolds $\mathcal{O} = \mathcal{M}/\mathcal{G}$:

(1) First, pass to the covering space $\mathcal{M}$.

(2) Then construct all strings that obey the boundary conditions $X^i(\pi) = g \cdot X^i(0)$, for each element $g \in \mathcal{G}$.

(3) Finally, project onto the $\mathcal{G}$-invariant subspace of states.

The twisted sectors are necessary for the modular invariance of the string.

Let us now move on to discuss our third example, the orbifold $\mathcal{O} = T^6/Z_3$. This space is known as the Z-orbifold [3]. When $M_4 \times \mathcal{O}$ is used as a background for the heterotic string, both gauge *and* spacetime symmetries are broken. The Z-orbifold produces a quasi-realistic spectrum, with $N = 1$ supersymmetry in four dimensions, and chiral fermions in 27-dimensional representations of E_6.

We shall begin by taking the six-torus T^6 to be the direct product of three identical two-tori. One of the two-tori is shown in Figure 5. We choose to describe T^6 by three complex coordinates, $(z_1,\ z_2,\ z_3)$. In terms of these coordinates, the Z_3 generator g takes the following form:

$$g \;=\; \mathrm{diag}\;(e^{2\pi i/3},\ e^{2\pi i/3},\ e^{2\pi i/3})\ . \qquad (6)$$

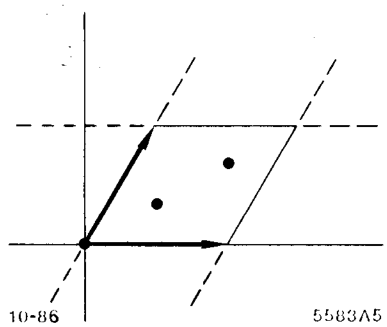

Fig. 5. The orbifold T^6/Z_3 has three fixed points in each plane.

The action of g leaves three fixed points in each plane, so there are a total of 27 fixed points. Each fixed point gives rise to its own twisted sector. Note that g is an element of SU(3), so the orbifold $\mathcal{O} = T^6/Z_3$ has discrete SU(3) holonomy. Therefore T^6/Z_3 is a singular limit of a Calabi-Yau space. It produces a tachyon-free spectrum, with unbroken $N = 1$ supersymmetry in four dimensions [3].

To describe gauge symmetry breaking, we associate an E_8 transformation $h \in E_8$ with each element $g \in \mathcal{G}$, and we project onto states invariant under $g' = gh$. For the case at hand, we choose h to lie in the center of the SU(3) subgroup defined by $E_8 \to E_6 \times SU(3)$. This breaks the gauge symmetry to $E_6 \times SU(3)$, and is the orbifold analog of symmetry breaking by Wilson lines.

The massless spectrum for the Z-orbifold is collected in Table 1. As expected, the states form $N = 1$ supersymmetry multiplets. The untwisted states contain the spin $(\frac{3}{2},\ 2)$ gravitational multiplet and the spin $(\frac{1}{2}, 1)$ gauge field multiplets, with unbroken gauge group $E_6\times$ SU(3) $\times$ E_8'. There are also spin $(0,\ \frac{1}{2})$ matter multiplets, in various representations of the gauge group. The twisted states are localized at each of the 27 fixed points in the internal space. They also form $N = 1$ supersymmetry multiplets. As seen in Table 1, this simple example gives 36 generations of ordinary quarks and leptons – plus lots of extra particles. This spectrum is not ideal, but neither is it absurd. One might hope that more complicated orbifolds will give more realistic results.

For our final example, we investigate string propagation on the orbifold $\mathcal{O} = T^8/Z_6$. This is a particularly interesting example, because $M_2 \times \mathcal{O}$ describes a four-dimensional cosmic string embedded in a Z-orbifold background [7]. The question of strings propagating on a cosmic string background is of interest for its own sake, and also because it gives rise to various subtle issues relating to compactification on manifolds of SU(4) holonomy.

To describe the orbifold $\mathcal{O}$, we use complex coordinates (z_1, z_2, z_3, w). The z_i are as above, and w describes the xy-plane of four-dimensional spacetime. In terms of these coordinates, the Z_6 element g is taken to be

$$g \;=\; \mathrm{diag}\;(e^{i\pi/3},\ e^{i\pi/3},\ e^{i\pi/3},\ -1)\ . \qquad (7)$$

Table 1

Z-Orbifold: The massless physical spectrum.

Sector	Number	Spin	$E_6\times$ SU(3) $\times$ E_8'
Untwisted	1	$(\frac{3}{2}, 2)$	(1, 1, 1)
	1	$(\frac{1}{2}, 1)$	(78, 1, 1)
	1	$(\frac{1}{2}, 1)$	(1, 8, 1)
	1	$(\frac{1}{2}, 1)$	(1, 1, 248)
	3	$(0, \frac{1}{2})$	(27, 3, 1)
	10	$(0, \frac{1}{2})$	(1, 1, 1)
Twisted g, g^2	27	$(0, \frac{1}{2})$	(27, 1, 1)
	81	$(0, \frac{1}{2})$	$(1, \bar{3}, 1)$

This group element gives rise to a conical singularity of deficit angle $\Delta = \pi$, located at the origin of the w-plane. If we take the tori in the z_i-directions to be tiny, and that in the w-direction to be huge, this background looks, for all intents and purposes, like the exterior spacetime surrounding an infinitesimally thin cosmic string source, of tension $\mu = 1/8G$. The cosmic string runs up and down the z-axis, and is located at the origin of the xy-plane in four-dimensional spacetime.

The group element g lies in SU(4), so this background is an eight-dimensional version of a Calabi-Yau space. As such, we expect it to be supersymmetric and tachyon-free. The complete string spectrum can be calculated as described above. The tachyon is indeed absent, so the cosmic orbifold is stable at tree level. Furthermore, the massive spectrum turns out to be supersymmetric, with unbroken gauge group O(10) × SU(3).

Table 2

Cosmic orbifold: the up-moving, massless physical spectrum.

Number	Spin	O(10) × SU(3) × E_8'	L_Z
		Untwisted Sector	
1	$(\frac{3}{2}, 2)$	(1, 1, 1)	(odd, even)
1	$(\frac{1}{2}, 1)$	(45, 1, 1)	(even, odd)
1	$(\frac{1}{2}, 1)$	(1, 8, 1)	(even, odd)
1	$(\frac{1}{2}, 1)$	(1, 1, 248)	(even, odd)
1	$(\frac{1}{2}, 1)$	(16, 1, 1)	(odd, even)
1	$(\frac{1}{2}, 1)$	$(\overline{16}, 1, 1)$	(odd, even)
1	$(\frac{1}{2}, 1)$	(1, 1, 1)	(even, odd)
3	$(0, \frac{1}{2})$	(16, 3, 1)	(even, odd)
3	$(0, \frac{1}{2})$	(10, 3, 1)	(odd, even)
3	$(0, \frac{1}{2})$	(1, 3, 1)	(odd, even)
10	$(0, \frac{1}{2})$	(1, 1, 1)	(even, odd)
		Twisted Sector g, g^5	
–	–	–	–
		Twisted Sector g^2, g^4	
27	$(0, \frac{1}{2})$	(16, 1, 1)	(even, odd)
27	$(0, \frac{1}{2})$	(10, 1, 1)	(odd, even)
27	$(0, \frac{1}{2})$	(1, 1, 1)	(odd, even)
81	$(0, \frac{1}{2})$	$(1, \bar{3}, 1)$	(even, odd)
		Twisted Sector g^3	
1	F	(10, 1, 1)	–

The computation of the massless spectrum is a little more subtle. This is because the massless spectrum is *different* for states moving up and down the cosmic string. The crucial point is that the cosmic string breaks four-dimensional Lorentz invariance. Massive states moving up the z-axis can be reversed by an unbroken Lorentz transformation, so the massive up- and down-moving spectra are identical. Massless states cannot be turned around, so the up- and down-moving spectra are free to differ – as indeed they do.

The massless physical spectrum for the cosmic orbifold is presented in Tables 2 and 3. The states are organized into representations of O(10) × SU(3), and their spins and multiplicities are indicated as well. Note that strings in sectors twisted an odd number of times have no coordinate zero modes. They are effectively bound to

Table 3

Cosmic string: the down-moving, massless physical spectrum.

Number	Spin	O(10) × SU(3) × E_8'	L_Z
		Untwisted Sector	
1	$(\frac{3}{2}, 2)$	(1, 1, 1)	(even, even)
1	$(\frac{1}{2}, 1)$	(45, 1, 1)	(odd, odd)
1	$(\frac{1}{2}, 1)$	(1, 8, 1)	(odd, odd)
1	$(\frac{1}{2}, 1)$	(1, 1, 248)	(odd, odd)
1	$(\frac{1}{2}, 1)$	(16, 1, 1)	(even, even)
1	$(\frac{1}{2}, 1)$	$(\overline{16}, 1, 1)$	(even, even)
1	$(\frac{1}{2}, 1)$	(1, 1, 1)	(odd, odd)
3	$(0, \frac{1}{2})$	(16, 3, 1)	(even, even)
3	$(0, \frac{1}{2})$	(10, 3, 1)	(odd, odd)
3	$(0, \frac{1}{2})$	(1, 3, 1)	(odd, odd)
10	$(0, \frac{1}{2})$	(1, 1, 1)	(even, even)
		Twisted Sector g, g^5	
1	(B, F)	(16, 1, 1)	–
3	(B, F)	(1, 3, 1)	–
12	(B, F)	(1, 1, 1)	–
		Twisted Sector g^2, g^4	
27	$(0, \frac{1}{2})$	(16, 1, 1)	(even, even)
27	$(0, \frac{1}{2})$	(10, 1, 1)	(odd, odd)
27	$(0, \frac{1}{2})$	(1, 1, 1)	(odd, odd)
81	$(0, \frac{1}{2})$	$(1, \bar{3}, 1)$	(even, even)
		Twisted Sector g^3	
–	–	–	–

the cosmic string, and behave like genuine two-dimensional objects. Therefore we do not indicate their spins, only whether they are bosons or fermions. On the other hand, strings in sectors twisted an even number of times *do* have coordinate zero modes in the xy plane. States in these sectors are ordinary four-dimensional massless particles. They are not bound to the string, so we are free to list their spins.

It is important to remember that the coordinate zero-mode wave functions transform under the holonomy group. This implies that there are different sets of states associated with even and odd angular momenta about the z-axis. In a compactification down to two dimensions, where the dimensions transverse to the string are "small," the states of non-zero angular momentum are viewed as having finite mass, and are not included in the massless spectrum. In a cosmic string interpretation, where two of the transverse dimensions are "large," all angular-momentum states are treated on the same footing.

In Tables 2 and 3, we have classified the states according to their spins and their $O(10) \times SU(3)$ representations. We see that the up-moving states are *not* supersymmetric, but that the down-moving states are. This is a generic feature of chiral strings compactified on manifolds of SU(4) holonomy. As discussed earlier, there is no problem with this, since the cosmic string breaks four-dimensional Lorentz invariance.

Since we are describing a cosmic string embedded in the Z-orbifold background, we expect states far from the string to be identified with those of the Z-orbifold. This suggests that states in the even-twist sectors should fall into multiplets of $N = 1$ supersymmetry, with gauge group $E_6 \times SU(3)$. A glance at the tables shows that if we ignore the distinction between even and odd orbital angular momenta, as is appropriate for states far from the string, the even-twist states do fall into $E_6 \times SU(3)$ representations. The states are precisely those of the Z-orbifold.

This spectrum as an interesting, almost realistic example of a cosmic string that can be built in string theory. It is very different from the type of string expected in grand unification models, for there is no topology to guarantee the stability of the solution. The fact that supersymmetry is broken for the massless up-movers can be shown to induce a non-vanishing contribution to the vacuum energy, once string loops are taken into consideration. This contribution is properly interpreted as a correction to the tension μ of the cosmic string. This correction acts as a line source for the dilaton field, and results in dilaton emission.

What then is the final fate of the cosmic string solution when string loop corrections are included? There are at least two possibilities. One is that the configuration decays by dilaton emission to a configuration with no deficit angle. Another is that there might be a solution to the string equations of motion with a renormalized but non-zero deficit angle, and a spatially varying dilaton field.

It would be very interesting to find such a solution. It might help develop an understanding of how the cosmological constant and dilaton vacuum expectation value are determined once supersymmetry is broken. In cosmic string compactifications, supersymmetry is broken in the most innocuous possible way – only the massless modes are not supersymmetric. Analyzing radiative corrections and their effect on the dilaton field should be much simpler here than in a string theory where supersymmetry breaking affects all string modes. In addition to providing a useful laboratory for addressing these purely string-theoretic questions, it is possible that the renormalized values of the string energy density and deficit angle might be such that these strings are of cosmological interest.

In this talk I have given a simple introduction to orbifolds. The orbifolds presented here are consistent, exact solutions to the classical string equations of motion. I have shown how the singularities in orbifolds can be thought of as cosmic-string-like singularities in spacetime. Much work needs to be done to more fully explore orbifold compactifications of string theory. As far as I know, there is still no acceptable orbifold compactification with gauge group $SU(3) \times SU(2) \times U(1)$ in four dimensions. It would be wonderful to arrive at a standard-model orbifold, in order to make some connection between string theory and the world in which we live.

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