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Article

On the Hypothesis of Exact Conservation of Charged Weak Hadronic Vector Current in the Standard Model

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Abstract: We investigate the reliability of the conservation of the vector current (CVC) hypothesis in the neutron beta decay ($n\ \beta^-$ decay). We calculate the contribution of the phenomenological term, responsible for the CVC in the hadronic current of the $n\ \beta^-$ decay (or the CVC effect), to the neutron lifetime. We show that the CVC effect increases the neutron lifetime with a relative contribution of 8.684×10^{-2}. This leads to the increase of the neutron lifetime by 76.4 s with respect to the world averaged value $\tau_n = 880.2(1.0)$s from the Particle Data Group. We show that since in the Standard Model there are no interactions that are able to cancel such a huge increase in the neutron lifetime, we have to turn to the interactions beyond the Standard Model, the contribution of which to the neutron lifetime reduces to the Fierz interference term b_F only. Cancelling the CVC effect at the level of the experimental accuracy, we obtain $b_F = 0.1219(12)$. If this value cannot be accepted for the Fierz interference term, the CVC effect induces irresistible problems for description and understanding of the $n\ \beta^-$ decay.

Keywords: neutron beta decay; ECVC

PACS: 12.15.Ff; 13.15.+g; 23.40.Bw; 26.65.+t

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1. Introduction

The neutron lifetime with the account for the complete set of corrections of order 10^{-3}, caused by the weak magnetism, proton recoil, and electromagnetic interaction, has been calculated in [1]. The theoretical value $\tau_n = 879.6(1.1)$s agrees well with the world averaged one $\tau_n = 880.2(1.0)$s [2] and recent experimental value $\tau_n = 880.2(1.2)$s [3]. The theoretical uncertainty ± 1.1 is fully defined by the experimental uncertainties of the axial coupling constant $\lambda = -1.2750(9)$ [4] and the Cabibbo–Kobayashi–Maskawa (CKM) matrix element $|V_{ud}| = 0.97425(22)$ [5], which agrees well with a new value $|V_{ud}| = 0.97417(21)$ [2] reported by Hardy and Tower [6]. Both values of the CKM matrix elements have been extracted from the $0^+ \to 0^+$ transitions, with the errors dominated by the theoretical uncertainties caused by nuclear Coulomb distortion and radiative corrections [2,6].

According to recent analysis by Hardy and Tower [6] (see also [7]), the effect of conservation of the vector current (CVC) in the $0^+ \to 0^+$ transitions (or in the pure Fermi transitions) is being observed at the level of 1.2×10^{-4}. In turn, as has been found by Naviliat-Cuncic and Severijns [8], in the mirror decays of ^{19}Ne, ^{21}Na, ^{29}P, ^{35}Ar, and ^{17}K caused by the Gamow-Teller mirror transitions, a new independent test of the CVC effect may be performed at the level of 4×10^{-3} [7]. Recently, the CVC effect has been investigated by Ankowski [9] and Giunti [10] in the inverse β decay $\bar{\nu}_e + p \to n + e^+$.

The main controversy between M. Ankowski's work and C. Giunti's reply is given by formulations of current conservation in the context of the hadronic current in neutron beta decay. Ankowski employs a phenomenological approach to adapt conservation terms that are ordinarily used in virtual transitions, which Giunti argues is inconsistent. Giunti states that this implies a new re-assessment of all past results with the new current form with a set of parameters that were derived from the assumption of a more conventional current conservation approach. Here we try to follow this suggestion by analyzing the neutron lifetime, taking into account the broken isospin symmetry and resulting mass difference of the u and d quarks, respectively, proton and neutron.

2. Precision Analysis of Neutron Lifetime to Order 10^{-4}

This study aims to evaluate the reliability of the CVC hypothesis, or CVC effect, regarding neutron lifetime. The effective low-energy Lagrangian for $V - A$ weak interactions is formulated as [1,11]:

$$\mathcal{L}_{\mathrm{W}}(x) = -\frac{G_F}{\sqrt{2}} V_{ud} J_\mu(x) \left[\bar{\psi}_e(x) \gamma^\mu \left(1 - \gamma^5\right) \psi_{\nu_e}(x) \right] \tag{1}$$

where $G_F = 1.1664 \times 10^{-11}\,\mathrm{MeV}^{-2}$ and $V_{ud} = 0.97417(21)$ are the Fermi weak constant and the Cabibbo–Kobayashi–Maskawa (CKM) matrix element [2], respectively, $J_\mu(x)$ is the hadronic $V - A$ current, $\psi_e(x)$ and $\psi_{\nu_e}(x)$ represent the operators of the electron and electron neutrino (antineutrino) fields. The amplitude of the $n\ \beta^-$ decay $n \to p + e^- + \bar{\nu}_e$ is equal to (for more, details see Appendix A)

$$M(n \to p e^- \bar{\nu}_e) = -\frac{G_F}{\sqrt{2}} V_{ud} \left\langle p\left(\vec{k}_p, \sigma_p\right) \middle| J_\mu(0) \middle| n\left(\vec{k}_n, \sigma_n\right) \right\rangle \left[\bar{u}_e\left(\vec{k}_e, \sigma_e\right) \gamma^\mu \left(1 - \gamma^5\right) v_\nu\left(\vec{k}, +\frac{1}{2}\right) \right] \tag{2}$$

Here $\left| p\left(\vec{k}_p, \sigma_p\right) \right\rangle, \left| n\left(\vec{k}_n, \sigma_n\right) \right\rangle$ represent the wave functions of the free p and n with 3-momenta $\vec{k}_p$ and $\vec{k}_n = \vec{0}$ and polarizations $\sigma_p = \pm 1$ and $\sigma_n = \pm 1$, respectively. Then, $\bar{u}_e\left(\vec{k}_e, \sigma_e\right)$ and $v_\nu\left(\vec{k}_\nu, +\frac{1}{2}\right)$ represent the Dirac wave functions of the free e and electron antineutrino with 3-momenta $\vec{k}_e$ and $\vec{k}_\nu$ and polarizations $\sigma_e = \pm 1$ and $+\frac{1}{2}$ [1,11]. In the Standard Model, the matrix element of the hadronic current we take the form:

$$\begin{aligned} &\left\langle p\left(\vec{k}_p, \sigma_p\right) \middle| J_\mu(0) \middle| n\left(\vec{k}_n, \sigma_n\right) \right\rangle = \\ &\bar{u}_p\left(\vec{k}_p, \sigma_p\right) \left[\left(\gamma_\mu - \frac{q_\mu \hat{q}}{q^2} \right) + \frac{\kappa}{2M} i\sigma_{\mu\nu} q^\nu + \lambda \left(-\frac{2M q_\mu}{q^2 - m_\pi^2} + \gamma_\mu \right) \gamma^5 \right] u_n\left(\vec{k}_n, \sigma_n\right) \end{aligned} \tag{3}$$

This approach aligns with the methods used in [12,13], where $\bar{u}p\left(\vec{k}p, \sigma_p\right)$ and $u_n\left(\vec{k}n, \sigma n\right)$ represent the Dirac wave functions for a free p and n, respectively. The term $-q_\mu \hat{q}/q^2$, where $q = k_p - k_n$ represents the transferred 4-momentum, serves as the phenomenological component responsible for conserving the vector current (CVC) in $n\ \beta^-$ decay. The weak magnetism contribution is represented by the term $\frac{\kappa}{2M}$, where $2M = m_n + m_p$ with n and p masses $m_n = 939.5654\,\mathrm{MeV}$ and $m_p = 938.2720\,\mathrm{MeV}$. Here, $\kappa = \kappa_p - \kappa_n = 3.7058$ denotes the isovector anomalous magnetic moment of the nucleon, defined by the anomalous magnetic moments of the p ($\kappa_p = 1.7928$) and p ($\kappa_n = -1.9130$), measured in nuclear magnetons [2]. The axial current contribution is given by the last term in Equation (3), where $\lambda = -1.2750(9)$ represents the axial coupling constant [4] (also referenced in [1,11]), and m_π denotes the charged pion mass [2]. In the limit $m_\pi \to 0$ (or in the chiral limit), the axial current is also conserved [14,15]. Skipping standard calculations [1], we arrive at the rate of the $n\ \beta^-$ decay given by

$$\frac{1}{\tau_n} = \frac{1}{\tau_n^{(\mathrm{SM})}} \left(1 + \frac{f_n^{(\mathrm{CVC})}}{f_n} \right) \tag{4}$$

where $\tau_n^{(\mathrm{SM})} = 879.6(1.1)$s is the theoretical value of the neutron lifetime, calculated in [1] for $\lambda = -1.2750(9)$. It agrees perfectly well with the world averaged value $\tau_n = 880.2(1.0)$s [2] and the recent experimental one $\tau_n = 880.2(1.2)$s [3]. Then, the phase space factor f_n for the n, computed to order $O(1/M)$ and $O(\alpha/\pi)$ due to the contributions of weak magnetism, p recoil, and radiative corrections, respectively, is given by $f_n = 6.116 \times 10^{-2}\,\mathrm{MeV}^5$. The phase space factor associated with the n, denoted as $f_n^{(\mathrm{CVC})}$, due to the influence of the CVC effect, is represented by the following expression.

$$\begin{aligned} f_n^{(\mathrm{CVC})} =& \frac{1}{1+3\lambda^2}\int_{m_e}^{E_0} dE_e k_e (E_0 - E_e)^2 F(E_e, Z=1) \int \frac{d\Omega_{e\nu}}{4\pi}\Bigg\{-\frac{2m_e^2\Delta}{m_e^2 + 2E_e(E_0-E_e) - 2\vec{k}_e\cdot\vec{k}_\nu} \\ &+ \frac{m_e^2\Delta^2}{\left(m_e^2 + 2E_e(E_0-E_e) - 2\vec{k}_e\cdot\vec{k}_\nu\right)^2}\left(E_e - \frac{\vec{k}_e\cdot\vec{k}_\nu}{E_\nu}\right)\Bigg\} \end{aligned} \tag{5}$$

where $\Delta = m_n - m_p$, $E_0 = \left(m_n^2 - m_p^2 + m_e^2\right)/2m_n = 1.2927\,\mathrm{MeV}$ is the end-point energy of the electron energy spectrum of the n β^- decay [1], $k_e = \sqrt{E_e^2 - m_e^2}$ is an absolute value of the electron 3-momentum, $F(E_e, Z=1)$ is the relativistic Fermi function describing the proton–electron Coulomb final-state interaction [1]. Then, $d\Omega_{e\nu}$ is an element of the solid angle of the electron–antineutrino momentum correlations. To examine the primary impact of the Conserved Vector Current (CVC) effect, we compute the integrand of the phase space factor $f^{(CVC)}$ using a leading-order approach within the framework of a large nucleon mass expansion. Given the calculated value of $f_n^{(\mathrm{CVC})} = -4.887 \times 10^{-3}\,\mathrm{MeV}^5$, the CVC effect's impact on the n β^- decay rate is found to be $f_n^{(\mathrm{CVC})}/f_n = -7.991 \times 10^{-2}$. This corresponds to the relative correction to the neutron lifetime $\Delta\tau_n^{(\mathrm{CVC})}/\tau_n = 8.684 \times 10^{-2}$ that gives $\Delta\tau_n^{(\mathrm{CVC})} = 76.4$ s. Unfortunately, such a huge increase in the lifetime by the CVC effect cannot be accepted for the neutron and should be substantially suppressed for the correct agreement with recent experimental data $\tau_n = 880.2(1.2)$s [3] and world averaged value $\tau_n = 880.2(1.0)$s [2]. In this connection it is important to emphasize that in the Standard Model there are no contributions that are able to diminish such a huge increase of the neutron lifetime, induced by the phenomenological term $-q_\mu \hat{q}/q^2$ responsible for the CVC in the n β^- decay [16–18]. Indeed, the impact of the pseudoscalar term for a physical mass of the charged pion $m_\pi = 139.570\,\mathrm{MeV}$ decreases the neutron lifetime at the level of $\Delta\tau_n^{(\pi)}/\tau_n = 4.691 \times 10^{-6}$. However, in the chiral limit $m_\pi \to 0$ the contribution of the charged pion may only aggravate the problem. Hence, in order to reduce a huge contribution of the CVC effect at the level of $f_n^{(\mathrm{CVC})}/f_n = -7.991 \times 10^{-2}$ to the level of 1.2×10^{-4} one has to turn to interactions beyond the Standard Model.

3. Fierz Interference Term

It is widely recognized [19–22] (and discussed in review articles such as [4,23]) that the Fierz interference term, $b_F m_e/E_e$, which arises from tensor and scalar interactions beyond the Standard Model, represents the simplest contribution of such interactions to the n β^- decay. Below, for the analysis of the impact of the Fierz interference term, we use the results obtained in [1]. As has been shown in [1] all possible interactions beyond the Standard Model [21] give the contribution to the n lifetime only in the form of the Fierz interference term. As a result, the Fierz interference term changes the rate of the n β^- decay as follows:

$$\frac{1}{\tau_n} = \frac{1}{\tau_n^{(\mathrm{SM})}}\left(1 + b_F\left\langle\frac{m_e}{E_e}\right\rangle_{\mathrm{SM}}\right) \tag{6}$$

The value $\langle m_e/E_e\rangle_{\mathrm{SM}} = 0.6556$ is computed using the electron energy spectrum density as presented in Equation (D-59) of Ref. [1]. Considering the contribution from the CVC effect, we obtain the following result:

$$\frac{1}{\tau_n} = \frac{1}{\tau_n^{(\mathrm{SM})}}\left(1 + \frac{f_n^{(\mathrm{CVC})}}{f_n} + b_F\left\langle\frac{m_e}{E_e}\right\rangle_{\mathrm{SM}}\right) \tag{7}$$

where the right-hand side of Equation (7) contains a complete set of phenomenological contributions within the Standard Model and contributions beyond the Standard Model. In the obtained expression for the neutron lifetime, given by Equation (7), the contribution from the CVC effect and the Fierz interference term can be maintained at the required level of 1.2×10^{-4}, provided that the Fierz interference term is set to $b_F = 0.12189(12)$.

4. Conclusions

We have analyzed the CVC hypothesis in the $n\,\beta^-$ decay and calculated the impact of the CVC effect, i.e., the impact of the phenomenological term $-q_\mu\hat{q}/q^2$ responsible for the CVC of the hadronic weak current of the $n\,\beta^-$ decay. We have shown that the CVC effect gives a relative contribution to the rate of the $n\,\beta^-$ decay at the level of $f_n^{(\mathrm{CVC})}/f_n = -7.991 \times 10^{-2}$, which is one order of magnitude large compared with the level of 4×10^{-3} of the CVC test in the Gamow-Teller mirror transitions reported by Naviliat-Cuncic and Severijns [8]. As a result, the phenomenological term $-q_\mu\hat{q}/q^2$, providing the CVC of the hadronic current in the $n\,\beta^-$ decay, changes the lifetime of the n by $\Delta\tau_n = 76.4$ s. Because there are no interactions in the Standard Model, which are able to cancel such a large increase of the n lifetime, we have turned to interactions beyond the Standard Model. As has been shown in [1], the contributions of all possible interactions beyond the Standard Model [21], which may affect the energy spectra and angular distributions of the $n\,\beta^-$ decay, reduce themselves to the contribution to the n lifetimes in the form of the Fierz interference term only. Keeping the effective contribution, caused by the CVC effect and the Fierz interference term, at the level of the experimental uncertainty of the neutron lifetime 1.2×10^{-3}; we have obtained the Fierz interference term $b_F = 0.1219(12)$, which seems to be also huge in comparison with the results $b_F < 0.01$ Herczeg [24], $b_F = 0.0032(23)$ Faber et al. [25], $b_F = -0.0028(26)$ Hardy and Tower [6] (see also discussion below Equation (7) of Ref. [6]), and and $|b_F| < 0.03$, reported by H. Saul on behalf of the PERKEO III Collaboration [26]. If the value of the Fierz interference term $b_F = 0.1219(12)$ is not acceptable, it can be concluded that using the phenomenological form of the CVC hypothesis in $n\,\beta^-$ decay, represented by the term $-q_\mu\hat{q}/q^2$, introduces significant challenges for accurately describing and understanding the decay process.

Author Contributions: D.A.: Conceptualization, Methodology, Data Curation, Formal analysis, Investigation, Writing—Original draft preparation. R.H.: Conceptualization, Formal analysis, Data Curation, Investigation, Methodology, Software, Validation, Writing—Original draft preparation. M.W.: Conceptualization, Formal analysis, Data Curation, Investigation, Methodology, Software, Validation, Writing—Original draft preparation. The sole responsibility for the content of this publication lies with the authors. All authors have read and agreed to the published version of the manuscript.

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Data Availability Statement: The data and illustrations presented in this study can be obtained directly from the equations. All data are available on request from the corresponding author.

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Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A. The Amplitude of the Rate of the n Radiative β^- Decay

Figure A1 illustrates the Feynman diagrams for the amplitude of n radiative β^- decay. We describe this decay amplitude, as represented by the diagrams in Figure A1, using the following expression [1]:

$$M(n \to pe^-\bar{\nu}_e\gamma)_{\lambda'} = e\frac{G_F}{\sqrt{2}}V_{ud}\mathcal{M}(n \to pe^-\bar{\nu}_e\gamma)_{\lambda'} \quad \text{(A1)}$$

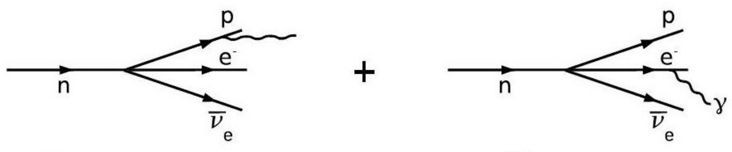

Figure A1. The amplitude of the n radiative β^- decay in the tree approximation is defined using Feynman diagrams.

Here, e represents the electric charge of the proton, G_F denotes the Fermi coupling constant, and V_{ud} is the Cabibbo–Kobayashi–Maskawa matrix element. The amplitude, expressed as $\mathcal{M}(n \to pe^-\bar{\nu}e\gamma)\lambda'$, is defined as follows:

$$\begin{aligned}\mathcal{M}(n \to pe^-\bar{\nu}_e\gamma)_{\lambda'} = &\left[\bar{u}_p\left(\vec{k}_p,\sigma_p\right)\hat{\varepsilon}^*_{\lambda'}(k)\frac{1}{m_p-\hat{k}_p-\hat{k}-i0}O_\mu u_n\left(\vec{k}_n,\sigma_n\right)\right]\left[\bar{u}_e\left(\vec{k}_e,\sigma_e\right)\gamma^\mu\left(1-\gamma^5\right)v_\nu\left(\vec{k},+\frac{1}{2}\right)\right] \\ &-\left[\bar{u}_p\left(\vec{k}_p,\sigma_p\right)O_\mu u_n\left(\vec{k}_n,\sigma_n\right)\right]\left[\bar{u}_e\left(\vec{k}_e,\sigma_e\right)\hat{\varepsilon}^*_{\lambda'}(k)\frac{1}{m_e-\hat{k}_e-\hat{k}-i0}\gamma^\mu\left(1-\gamma^5\right)v_\nu\left(\vec{k},+\frac{1}{2}\right)\right]\end{aligned} \quad \text{(A2)}$$

In this expression, $\bar{u}p\left(\vec{k}p,\sigma_p\right)$, $u_n\left(\vec{k}n,\sigma n\right)$, $\bar{u}e\left(\vec{k}e,\sigma_e\right)$, and $v_\nu\left(\vec{k}\nu,+\frac{1}{2}\right)$ represent the Dirac wave functions for the proton, neutron, electron, and electron antineutrino, respectively, with respective 3-momenta $\vec{k}p$, $\vec{k}n = \vec{0}$, $\vec{k}e$, and $\vec{k}\nu$, and polarizations $\sigma p = \pm 1$, $\sigma_n = \pm 1$, $\sigma_e = \pm 1$, and $+\frac{1}{2}$ [1,11]. Here, $\varepsilon^{*\alpha}_{\lambda'}(k)$ is the photon's polarization vector in state $\lambda' = 1,2$ with 4-momentum k, subject to the constraint $\varepsilon^*_{\lambda'}(k)\cdot k = 0$. Notably, amplitude Equation (A2) maintains gauge invariance, as substituting $\varepsilon^{\alpha *}_{\lambda'}(k) \to k^\alpha$ and applying the Dirac equations for the free p and e renders Equation (A2) null. The matrix O_μ is then specified by.

$$O_\mu = \left(\gamma_\mu - \frac{q_\mu\hat{q}}{q^2}\right) + \lambda\gamma_\mu\gamma^5 + i\frac{\kappa}{2M}\sigma_{\mu\nu}(k_p-k_n)^\nu \quad \text{(A3)}$$

where the matrix O_μ, used in [5,8], is modified by the term $-q_\mu\hat{q}/q^2$, which is introduced according to the CVC hypothesis (see [9,12,13]) with $q = k_n - k_p$, λ is the axial coupling, which we set $\lambda = -1.2750(9)$ (see [1,4,27]). In this case, the isovector anomalous magnetic moment of the nucleon is $\kappa = \kappa_p - \kappa_n = 3.7058$, which is obtained from the anomalous magnetic moments of the p ($\kappa_p = 1.7928$) and n ($\kappa_n = -1.9130$), measured in nuclear magnetons [2]. When analyzed using the baryon non-relativistic approximation and leading-order expansion for high baryon mass, the term $-q_\mu\hat{q}/q^2$ contributes to the n radiative β^- decay amplitude in the tree level approximation, producing a result equivalent to

$$\delta\mathcal{M}(n \to pe^-\bar{\nu}_e\gamma)_{\lambda'} = \frac{2m_n\Delta}{(k_n-k_p)^2}\left[\varphi^\dagger_p\varphi_n\right]\left[\bar{u}_e\left(\vec{k}_e,\sigma_e\right)\frac{1}{2k_e\cdot k}Q_{e\lambda'}\left(1-\gamma^5\right)v_\nu\left(\vec{k},+\frac{1}{2}\right)\right] \quad \text{(A4)}$$

where $\Delta = m_n - m_p$, $Q_{e\lambda'} = 2m_e k_e\cdot\varepsilon^*_{\lambda'} + 2k_e\cdot k\varepsilon^*_{\lambda'} + m_e\hat{\varepsilon}^*_{\lambda'}\hat{k}$ and we employed the Dirac equations for the free p, n and e antineutrino (see the Appendix of Ref. [27]). We have to emphasize that following [1,27,28] we have kept only the contribution of the photon emitted by the electron, which survives for the physical degrees of freedom of the p The contribution of the CVC effect to the branching ratio of the n radiative β^- decay is given

$$\mathrm{BR}^{(\mathrm{CVC})}_{\beta\gamma} = \tau_n \frac{\alpha}{\pi}\frac{G^2|V_{ud}|^2}{\pi^3}\int_{\omega_{\min}}^{\omega_{\max}} d\omega \int_{m_e}^{E_0-\omega} dE_e \sqrt{E_e^2 - m_e^2} F(E_e, Z=1)(E_0 - E_e - \omega)^2 \int \frac{d\Omega_{\nu e}}{4\pi}\int \frac{d\Omega_{\nu\gamma}}{4\pi}$$
$$\frac{E_0}{E_0^2 - \left(\vec{k}_e + \omega\vec{n}_{\vec{k}} + \vec{k}_\nu\right)^2}\left\{\frac{m_e^2}{\omega}\frac{k_e^2 - \left(\vec{k}_e\cdot\vec{n}_{\vec{k}}\right)^2}{\left(E_e - \vec{k}_e\cdot\vec{n}_{\vec{k}}\right)^2} + \frac{1}{E_e - \vec{k}_e\cdot\vec{n}_{\vec{k}}}\left[\left(k_e^2 - \left(\vec{k}_e\cdot\vec{n}_{\vec{k}}\right)^2\right)\right.\right.$$
$$\left.\left. + 2E_e\frac{\vec{k}_e\cdot\vec{k}_\nu}{E_\nu} - E_e(E_e+\omega)\frac{\vec{n}_{\vec{k}}\cdot\vec{k}_\nu}{E_\nu} - E_e(E_e-\omega)\right] + (E_e+\omega)\frac{\vec{n}_{\vec{k}}\cdot\vec{k}_\nu}{E_\nu} + \omega\right\}, \tag{A5}$$

where the branching ratio is defined for the photon emitted with an energy from the interval $\omega_{\min} \le \omega \le \omega_{\max}$. Then, $E_0 = 1.2927\,\mathrm{MeV}$ is the end-point energy of the electron energy spectrum of the n β^- decay [1], $F(E_e, Z=1)$ is the relativistic Fermi function taking into account the Coulomb proton-electron final-state interaction [1,27], $d\Omega_{\nu e} = \sin\vartheta_{\nu e} d\vartheta_{\nu e} d\varphi_{\nu e}$ and $d\Omega_{\nu gamma} = \sin\vartheta_{\nu\gamma} d\vartheta_{\nu\gamma} d\varphi_{\nu\gamma}$ are elements of solid angles of antineutrino-electron and antineutrino-photon momentum correlations, respectively, such as $\vec{k}_e\cdot\vec{k}_\nu = k_e(E_0 - E_e - \omega)\cos\vartheta_{\nu e}, \vec{n}_{\vec{k}}\cdot\vec{k}_\nu = (E_0 - E_e - \omega)\cos\vartheta_{\nu\gamma}$ and $\vec{k}_e\cdot\vec{n}_{\vec{k}} = k_e(\cos\vartheta_{\nu e}\cos\vartheta_{\nu\gamma} + \sin\vartheta_{\nu e}\sin\vartheta_{\nu\gamma}\cos(\varphi_{\nu e} - \varphi_{\nu\gamma})$. For the numerical analysis of the branching ratio $\mathrm{B}^{(\mathrm{CVC})}_{\beta\gamma}$ we use the theoretical value of the neutron lifetime $\tau_n = 879.6(1.1)\mathrm{s}$, which aligns precisely with the globally averaged value $\tau_n = 880.2(1.0)\mathrm{s}$ [2]. The numerical values of the branching ratio we adduce in Table A1, more details on the numerical solution of Equation (A5) find in Appendix B.

Table A1. The impact of the CVC effect on the branching ratio of neutron radiative β^- decay is analyzed across three photon energy regions. The final column presents the total theoretical branching ratio for n radiative β^- decay, calculated for 3 photon energy ranges.

ω [keV]	$\mathrm{BR}_{\beta\gamma}$ (Experiment)		$\mathrm{BR}_{\beta\gamma}{}^{(\mathrm{CVC})}$ (Theory)	$\mathrm{BR}_{\beta\gamma}$ (Theory)
$15 \le \omega \le 340$	$(3.09 \pm 0.32) \times 10^{-3}$	[4]	3.57×10^{-4}	3.25×10^{-3}
$14 \le \omega \le 782$	$(3.35 \pm 0.05\,[\mathrm{stat}] \pm 0.15\,[\mathrm{syst}]) \times 10^{-3}$	[1]	3.78×10^{-4}	3.42×10^{-3}
$0.4 \le \omega \le 14$	$(5.82 \pm 0.23\,[\mathrm{stat}] \pm 0.62\,[\mathrm{syst}]) \times 10^{-3}$	[1]	8.07×10^{-4}	5.89×10^{-3}

The CVC effect contributes to the amplitude of the n β^- decay with a value of

$$\mathcal{M}(n \to pe^-\bar{\nu}_e) = \left[\bar{u}_p\left(\vec{k}_p, \sigma_p\right)\left(\gamma_\mu - \frac{q_\mu\hat{q}}{q^2} + \lambda\gamma_\mu\gamma^5\right)u_n\left(\vec{k}_n, \sigma_n\right)\right]\left[\bar{u}_e\left(\vec{k}_e, \sigma_e\right)\gamma^\mu\left(1-\gamma^5\right)v_\nu\left(\vec{k}, +\frac{1}{2}\right)\right]. \tag{A6}$$

This changes the neutron lifetime as follows (see also Appendix C):

$$\Delta\tau_n^{(\mathrm{CVC})} = -\tau_n^2\frac{G_F^2|V_{ud}|^2}{2\pi^3}\int_{m_e}^{E_0} dE_e k_e(E_0 - E_e)F(E_e, Z=1)\int\frac{d\Omega_{e\nu}}{4\pi}$$
$$\frac{E_0 m_e^2}{\left(m_e^2 + 2E_e(E_0 - E_e) - 2\vec{k}_e\cdot\vec{k}_\nu\right)^2}\left\{(4E_e - 3E_0)\left[E_e(E_0 - E_e) - \vec{k}_e\cdot\vec{k}_\nu\right] - 2m_e^2(E_0 - E_e)\right\}. \tag{A7}$$

The impact of the CVC effect to the rate of the n β^- decay is defined by

$$\lambda_n^{(\mathrm{CVC})} = \left(1 + 3\lambda^2\right)\frac{G_F^2|V_{ud}|^2}{2\pi^3}f_n^{(\mathrm{CVC})}(E_0) \tag{A8}$$

where $f_n^{(\mathrm{CVC})}(E_0)$ is given by

$$f_n^{(\mathrm{CVC})}(E_0) = \frac{1}{1+3\lambda^2}\int_{m_e}^{E_0} dE_e k_e (E_0-E_e)^2 F(E_e, Z=1) \int \frac{d\Omega_{e\nu}}{4\pi}\Bigg\{\Bigg[-\frac{2E_0 m_e^2}{m_e^2+2E_e(E_0-E_e)-2\vec{k}_e\cdot\vec{k}_\nu}$$
$$+\frac{E_0^2 m_e^2}{\big(m_e^2+2E_e(E_0-E_e)-2\vec{k}_e\cdot\vec{k}_\nu\big)^2}\left(E_e-\frac{\vec{k}_e\cdot\vec{k}_\nu}{E_\nu}\right)\Bigg]$$
$$+\lambda^2\Bigg[\frac{m_e^2}{m_e^2+2E_e(E_0-E_e)-2\vec{k}_e\cdot\vec{k}_\nu}\left(E_0-E_e+\frac{\vec{k}_e\cdot\vec{k}_\nu}{E_\nu}\right)$$
$$+\frac{m_e^2}{\big(m_e^2+2E_e(E_0-E_e)-2\vec{k}_e\cdot\vec{k}_\nu\big)^2}\Big((E_0-E_e)^2+k_e^2+2\vec{k}_e\cdot\vec{k}_\nu\Big)\left(E_e-\frac{\vec{k}_e\cdot\vec{k}_\nu}{E_\nu}\right)\Bigg]\Bigg\} \quad \text{(A9)}$$

Appendix B. Numerical Analysis of the Branching Ratio in Equation (A5)

The numerical calculations of the impact of the CVC effect on the branching ratio of the n radiative β^--decay reduce to the calculation of the following integral:

$$I(\omega_{\max},\omega_{\min}) = \frac{E_0}{8\pi}\int_{\omega_{\min}}^{\omega_{\max}} d\omega \int_{m_e}^{E_0-\omega} dE_e k_e F(E_e, Z=1)(E_0-E_e-\omega)^2 \int_{-1}^{+1} dx \int_{-1}^{+1} dy \int_0^{2\pi} d\varphi$$
$$\Big[E_0^2-k_e^2-\omega^2-(E_0-E_e-\omega)^2-2k_e(E_0-E_e-\omega)x-2\omega(E_0-E_e-\omega)y$$
$$-2k_e\omega\Big(xy+\sqrt{(1-x^2)(1-y^2)}\cos\varphi\Big)\Big]^{-1}\Bigg\{\frac{m_e^2}{\omega}\frac{k_e^2\Big[1-\Big(xy+\sqrt{(1-x^2)(1-y^2)}\cos\varphi\Big)^2\Big]}{\Big[E_e-k_e\Big(xy+\sqrt{(1-x^2)(1-y^2)}\cos\varphi\Big)\Big]^2}$$
$$+\frac{1}{E_e-k_e\Big(xy+\sqrt{(1-x^2)(1-y^2)}\cos\varphi\Big)}\Big\{k_e^2\Big[1-\Big(xy+\sqrt{(1-x^2)(1-y^2)}\cos\varphi\Big)^2\Big]$$
$$+2k_eE_ex-E_e(E_e+\omega)y-E_e(E_e-\omega)\Big\}+(E_e+\omega)y+\omega\Bigg\}, \quad \text{(A10)}$$

where $k_e = \sqrt{E_e^2-m_e^2}$. The integral should be calculated for three intervals (i) $15\times10^{-3}\,\mathrm{MeV} \le \omega \le 340\times10^{-3}\,\mathrm{MeV}$, (ii) $14\times10^{-3}\,\mathrm{MeV} \le \omega \le 782\times10^{-3}\,\mathrm{MeV}$, and (iii) $0.4\times10^{-3}\,\mathrm{MeV} \le \omega \le 14\times10^{-3}\,\mathrm{MeV}$ with $E_0 = 1.2927\,\mathrm{MeV}$ and $m_e = 0.511\,\mathrm{MeV}$. The Fermi function $F(E_e, Z=1)$ is equal to

$$F(E_e, Z=1) = \left(1+\frac{1}{2}\gamma\right)\frac{4(2r_p m_e\beta)^{2\gamma}}{\Gamma^2(3+2\gamma)}\frac{e^{\pi\alpha/\beta}}{(1-\beta^2)^\gamma}\left|\Gamma\left(1+\gamma+i\frac{\alpha}{\beta}\right)\right|^2 \quad \text{(A11)}$$

Here, $\beta = k_e/E_e = \sqrt{E_e^2-m_e^2}/E_e$ denotes the electron velocity, $\gamma = \sqrt{1-\alpha^2}-1$, $r_p = 4.262\times10^{-3}$, MeV^{-1} represents the proton's electric radius, and $\alpha = 1/137.036$ represents the fine-structure constant.

Appendix C. Numerical Analysis of Equation (A7)

The numerical analysis of Equation (A7) reduces to the computing of the integration

$$
\begin{aligned}
I =& \frac{1}{2} m_e^2 E_0 \int_{m_e}^{E_0} dE_e \sqrt{E_e^2 - m_e^2} (E_0 - E_e)^2 F(E_e, Z = 1) \\
& \int_{-1}^{+1} dx \frac{1}{\left(m_e^2 + 2(E_0 - E_e)\left(E_e - x\sqrt{E_e^2 - m_e^2}\right)\right)^2} \left\{ (4E_e - 3E_0)\left(E_e - x\sqrt{E_e^2 - m_e^2}\right) - 2m_e^2 \right\}
\end{aligned} \quad \text{(A12)}
$$

for $E_0 = 1.2927\,\text{MeV}$ and $m_e = 0.511\,\text{MeV}$ with $k_e = \sqrt{E_e^2 - m_e^2}$. The Fermi function is represented by Equation (A11).

One has to compute the following integration:

$$
\begin{aligned}
f_n^{(\text{CVC})}(E_0) =& \frac{1}{1 + 3\lambda^2} \int_{m_e}^{E_0} dE_e k_e (E_0 - E_e)^2 F(E_e, Z = 1) \frac{1}{1} \int_{-1}^{+1} dx \\
& \left\{ k_1 \left[-\frac{2E_0 m_e^2}{m_e^2 + 2(E_0 - E_e)(E_e - k_e x)} + \frac{E_0^2 m_e^2}{(m_e^2 + 2(E_0 - E_e)(E_e - k_e x))^2} (E_e - k_e x) \right] \right. \\
& + k_2 \lambda^2 \left[\frac{m_e^2}{m_e^2 + 2(E_0 - E_e)(E_e - k_e x)} (E_0 - E_e + k_e x) \right. \\
& \left. \left. + \frac{m_e^2}{(m_e^2 + 2(E_0 - E_e)(E_e - k_e x))^2} \left((E_0 - E_e)^2 + k_e^2 + 2(E_0 - E_e) k_e x \right) (E_e - k_e x) \right] \right\}
\end{aligned} \quad \text{(A13)}
$$

where $\lambda = -1.2750$, $E_0 = 1.2927\,\text{MeV}$ and $m_e = 0.511\,\text{MeV}$ with $k_e = \sqrt{E_e^2 - m_e^2}$. The Fermi function is given by Equation (A11)). The calculation to perform for (i) $k_1 = k_2 = 1$, (ii) $k_1 = 1$ and $k_2 = 0$ and (iii) $k_1 = 0$ and $k_2 = 1$.

Taking into account the non-vanishing mass of charged pions one has to calculate the following integral:

$$
\begin{aligned}
f_n^{(\text{CVC})}(E_0) =& \frac{1}{1 + 3\lambda^2} \int_{m_e}^{E_0} dE_e k_e (E_0 - E_e)^2 F(E_e, Z = 1) \frac{1}{1} \int_{-1}^{+1} dx \\
& \left\{ k_1 \left[-\frac{2E_0 m_e^2}{m_e^2 + 2(E_0 - E_e)(E_e - k_e x)} + \frac{E_0^2 m_e^2}{(m_e^2 + 2(E_0 - E_e)(E_e - k_e x))^2} (E_e - k_e x) \right] \right. \\
& + k_2 \lambda^2 \left[\frac{m_e^2}{m_e^2 + 2(E_0 - E_e)(E_e - k_e x) - m_\pi^2} (E_0 - E_e + k_e x) \right. \\
& \left. \left. + \frac{m_e^2}{(m_e^2 + 2(E_0 - E_e)(E_e - k_e x) - m_\pi^2)^2} \left((E_0 - E_e)^2 + k_e^2 + 2(E_0 - E_e) k_e x \right) (E_e - k_e x) \right] \right\},
\end{aligned} \quad \text{(A14)}
$$

where $m_\pi = 139.570\,\text{MeV}$, $\lambda = -1.2750$, $E_0 = 1.2927\,\text{MeV}$ and $m_e = 0.511\,\text{MeV}$ with $k_e = \sqrt{E_e^2 - m_e^2}$. The Fermi function is given by Equation (A11). The calculation should be performed for $k_1 = 0$ and $k_2 = 1$.

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