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# Attractive gravity probe surfaces in higher dimensions

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A generalization of the Riemannian Penrose inequality in $n$-dimensional space ($3 \leq n < 8$) is done. We introduce a parameter $\alpha$ ($-\frac{1}{n-1} < \alpha < \infty$) indicating the strength of the gravitational field, and define a refined attractive gravity probe surface (refined AGPS) with $\alpha$. Then, we show the area inequality for a refined AGPS, $A \leq \omega_{n-1}\left[(n+2(n-1)\alpha)Gm/(1+(n-1)\alpha)\right]^{\frac{n-1}{n-2}}$, where $A$ is the area of the refined AGPS, $\omega_{n-1}$ is the area of the standard unit $(n-1)$-sphere, $G$ is Newton's gravitational constant, and $m$ is the Arnowitt–Deser–Misner mass. The obtained inequality is applicable not only to surfaces in strong gravity regions such as a minimal surface (corresponding to the limit $\alpha \to \infty$), but also to those in weak gravity existing near infinity (corresponding to the limit $\alpha \to -\frac{1}{n-1}$).

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## 1. Introduction

Study of surfaces is one of the ways to investigate the gravitational theory. For instance, an event horizon, defined as a boundary of the causal past of future null infinity, shows the distinctive properties of the gravitational theory, such as its area-increasing law [1]. Especially, black hole thermodynamics is expected to be one of the keys to quantum gravity, and the area of the event horizon is interpreted as the entropy of the black hole [2,3]. The notion of entropy is extended to the Ryu–Takayanagi surface [4,5] in the AdS/CFT correspondence, and it is applied to the quantum information theory [6].

The Penrose inequality [7], which is the main topic of this paper, is a conjecture about apparent horizons. The conjecture states that the area of a trapped surface has the upper bound characterized by that of the Schwarzschild solution with the same Arnowitt–Deser–Misner (ADM) mass [8–10], that is $A_{\rm AH} \leq 4\pi(2Gm)^2$ in four-dimensional spacetime, where $A_{\rm AH}$ is the area of the apparent horizon, $G$ is Newton's gravitational constant, and $m$ is the ADM mass. It is expected to hold if the cosmic censorship conjecture is true. The Penrose inequality was proved in special situations: on a time-symmetric hypersurface (under the assumption of its existence) the inequality was shown [11–13]. This statement on a time-symmetric hypersurface is equivalent to that the area of an outermost minimal surface $A_{\mathrm{MS}}$ in asymptotically flat space with a nonnegative Ricci scalar is bounded as $A_{\mathrm{MS}} \leq 4\pi(2Gm)^2$, which is called the Riemannian Penrose inequality. The Geroch energy [14], which had been originally introduced to show the positivity of the energy including gravity, was used in the proof [11,12], and thus, the relation to the positive energy theorem [15,16] is one of the interesting subjects of study. Furthermore, the analysis with the Geroch energy was generalized to spaces with negative cosmological constants [17,18], and applied to the study of the Ryu–Takayanagi surface in the AdS/CFT correspondence [19].

Recently, generalizations of the Penrose inequality applicable to regions with weaker gravitational field have been achieved in three-dimensional spaces [20,21]. Its motivations are as follows. Since apparent horizons exist in black holes under the cosmic censorship conjecture, they are not observable objects for distant observers. Thus, the Penrose inequality can never be verified in observations. In order to improve this circumstance, the authors of this paper have introduced the generalization of a photon sphere, which is one of the important objects in black hole observations, named the loosely trapped surface (LTS) [20], and have shown its area inequality. The generalizations of the Penrose inequality are also important in the theoretical point of view. The outside regions of event horizons provide us sometimes a main stage of study in gravitational theory (especially, in quantum gravity including Hawking radiation), but the trapped surface is located inside black holes. The generalizations to a weak gravity region enable us to apply the inequality to such studies. The authors of this paper have given the further generalization of the Riemannian Penrose inequality [21]. They have introduced attractive gravity probe surfaces (AGPSs), which can exist near the infinity, and shown its area inequality. This result would permit us to understand the meaning of the area inequality in the Newtonian approximation, and to relate the inequality with the asymptotic structures of spacetime, such as the Bondi–Metzner–Sachs symmetry [22,23].

The higher-dimensional version of the Riemannian Penrose inequality was proved by Bray and Lee [24]. In this paper, we show the area inequality of AGPSs in spaces whose dimension is higher than or equal to three (but less than eight[1]). The outline of the proof follows that for three-dimensional cases in our previous paper [21], while some improvements are made. One is that the condition in the definition of the AGPS is refined, which makes the evaluation of the inequality difficult. Therefore, the construction of sequences of smooth manifolds requires major improvements. It is a mathematically challenging problem and physically gives us a new perspective on what kinds of contributions are more essential for the inequality. Another progress is relaxing the conditions of the theorem. The proof in the previous paper [21] requires conditions on each leaf of foliation of the local inverse mean curvature flow in the neighborhood of the AGPS, while in the proof of this paper we succeed to remove them.

This paper is organized as follows. In Sect. 2, we give the definition of refined AGPSs. The main theorem is shown in Sect. 3. The proof is presented in Sects. 4–7. In Sect. 4, we give the idea of the proof. Then, we find that two conditions are required to be shown: the existence of a sequence of smooth manifolds approaching the manifold constructed in Sect. 4 and nonexistence of the minimal surface anywhere but the boundary of the constructed manifold. The

[1]Since our proof relies on Bray and Lee's theorem [24] using Schoen and Yau's positive energy theorem [15], which is true only for $n$-dimensional space with $n < 8$, our theorem holds under the same condition. However, Schoen and Yau gave a report on the establishment of the positive energy theorem in any dimensions [25], which may enable our theorem to apply in any dimensions.

former is fixed in Sect. 5, while the latter is in Sect. 6. The proof is completed in Sect. 7. Sect. 8 is devoted to a summary and discussion. Appendix A is given to show the existence of functions introduced for the discussion in Sect. 5.

We here show our notations of *inward* and *outward*, which would be different from the usual definitions. A manifold $\Sigma$ with boundaries, one of which is the infinity and the others of which are (refined) AGPSs, is our interest of study. We define *inward* and *outward* of an AGPS as follows. If a vector on an AGPS directs into $\Sigma$, we call the direction of the vector "*outward*" because it matches with the physical intuition that the vector naively directs to the infinity. The opposite direction is called "*inward*." The word "*outside*" is used in a similar way, that is, the region that exists in the *outward* direction is called *outside*.

## 2. Refined AGPSs

In our previous paper [21], for proving an area inequality in three-dimensional spaces, the notion of AGPSs is introduced, which are defined as surfaces satisfying the positivity of the mean curvature $k > 0$ and

$$r^a D_a k \geq \alpha k^2, \tag{1}$$

where $r^a$ is the outward unit normal, $D_a$ is the covariant derivative of the three-dimensional space, and $k$ is defined by $k = D^a r_a$. Note that a foliation in neighborhood of the AGPS is required to define $r^a D_a k$. In the proof of our previous paper based on the conformal flow (Sect. 3 in [21]) the local mean curvature flow is taken. In this paper, we refine the definition of the AGPS such that it does not rely on a local foliation. The refined definition includes all original AGPSs defined with the local mean curvature flow and more. Hence, the inequality is applicable to more surfaces.

Let us investigate the geometric identity (for example, see Refs. [20,21]),

$$r^a D_a k + \frac{1}{2}{}^{(n)}R + \frac{1}{2}\mathsf{k}_{ab}^2 = \frac{1}{2}{}^{(n-1)}R - \varphi^{-1}\mathcal{D}^2\varphi - \frac{n}{2(n-1)}k^2, \tag{2}$$

where $\mathsf{k}_{ab}$ and $\varphi$ are the traceless part of the extrinsic curvature $k_{ab}$ and the lapse function, respectively, and $\mathsf{k}_{ab}^2$ means $\mathsf{k}_{ab}\mathsf{k}^{ab}$. ${}^{(n)}R$ is the Ricci scalar of $n$-dimensional space, and ${}^{(n-1)}R$ and $\mathcal{D}_a$ are the Ricci scalar and the covariant derivative of an $(n-1)$-dimensional hypersurface. Since manifolds with the nonnegative Ricci scalar ${}^{(n)}R \geq 0$ are our interest, the second term in the left-hand side is nonnegative. The third term is manifestly nonnegative. Thus, the condition of Eq. (1) with replacing $r^a D_a k$ by the left-hand side of Eq. (2) becomes weaker than that original one. If the local inverse mean curvature flow ($\varphi k = 1$) is assumed, the right-hand side of Eq. (2) is written as

$$\frac{1}{2}{}^{(n-1)}R + k^{-1}\mathcal{D}^2 k - 2k^{-2}(\mathcal{D}k)^2 - \frac{n}{2(n-1)}k^2. \tag{3}$$

This is expressed in terms of the geometrical quantities of the $(n-1)$-dimensional hypersurface and the mean curvature, that is, these quantities are uniquely fixed by how the $(n-1)$-dimensional hypersurface is embedded. Meanwhile, $r^a D_a k$ includes the second derivative of the induced metric, which requires more information than the embedding. Thus, through Eqs. (2) and (3), there is room to improve the condition in Eq. (1). This motivates us to refine the conditions of AGPSs as follows.

Definition 1. **Refined Attractive Gravity Probe Surfaces (Refined AGPSs):** Suppose $\Sigma$ to be a smooth $n$-dimensional manifold with a positive definite metric $g$. A smooth compact hypersurface $S_\alpha$ in $\Sigma$ is a refined attractive gravity probe surface (refined AGPS) with a parameter $\alpha$ ($\alpha > -\frac{1}{n-1}$) if the following conditions are satisfied everywhere on $S_\alpha$:

(i) The mean curvature $k$ is positive.
(ii) The inequality

$$^{(n-1)}R + 2k^{-1}\mathcal{D}^2 k - 4k^{-2}(\mathcal{D}k)^2 \geq \left(2\alpha + \frac{n}{n-1}\right)k^2 \tag{4}$$

is satisfied.[2]

Here, $^{(n-1)}R$, $k$, and $\mathcal{D}$ are the Ricci scalar, the mean curvature, and the covariant derivative of $S_\alpha$. Surface $S_\alpha$ is not required to be connected. It can be multiple.

Note that the conditions are written only with the functions and operators of the $(n-1)$-dimensional hypersurface. The theorem which we have proved in our previous paper [21] requires conditions for leaves of the foliation of the local inverse mean curvature flow, whereas the current definition of refined AGPSs is written only with the variables fixed by the embedding and does not require any other details of geometrical information around AGPSs. We will prove the area inequality with this refined condition in $n$-dimensional ($n \geq 3$) asymptotically flat spaces.

## 3. Main theorem

Now, we present our main theorem.

Theorem 1. *Let $\Sigma$ be an asymptotically flat, n-dimensional ($3 \leq n < 8$), smooth manifold with a nonnegative Ricci scalar. The boundaries of $\Sigma$ are composed of an asymptotically flat end and a refined AGPS $S_\alpha$, which can have multiple components, with a parameter $\alpha$. Here, the unit normal $r^a$ to define the mean curvature is taken to be outward of $S_\alpha$. Suppose that there exists a finite-distance smooth extension of $\Sigma$ from $S_\alpha$ to the interior of $S_\alpha$ satisfying the nonnegativity of the Ricci scalar. Moreover, suppose that no minimal hypersurface satisfying either one of the following conditions exists:*

*(i) It encloses (at least) one component of $S_\alpha$.*
*(ii) It has boundaries on $S_\alpha$ and its area is less than $\omega_{n-1}(2Gm)^{\frac{n-1}{n-2}}$, where $\omega_{n-1}$ is the area of the standard unit $(n-1)$-sphere.*

*Then, the area of $S_\alpha$ has an upper bound,*

$$A_\alpha \leq \omega_{n-1}\left[\frac{n+2(n-1)\alpha}{1+(n-1)\alpha}Gm\right]^{\frac{n-1}{n-2}}, \tag{5}$$

*where m is the ADM mass of the manifold and G is Newton's gravitational constant. Equality holds if and only if $\Sigma$ is a time-symmetric hypersurface of a Schwarzschild spacetime and $S_\alpha$ is a spherically symmetric hypersurface with $r^a D_a k = \alpha k^2$.*

Note that any AGPS defined with the local inverse mean curvaure flow in the previous paper [21] satisfies the condition of the refined AGPS. Thus, Theorem 1 holds for any AGPS. The proof of the theorem is given in the following sections. Even if no finite-distance smooth extension of $\Sigma$ with the nonnegativie Ricci scalar exists, we can show the inequality from Eq. (5)

[2] It may be better to introduce a new parameter equal to $\left(2\alpha + \frac{n}{n-1}\right)$ for simplification. However, we use $\alpha$ because it makes the relation to our previous study in Ref. [21] clear.

if, in a neighborhood of $S_\alpha$ on $\Sigma$, there exists a smooth foliation of hypersurfaces all of which (but $S_\alpha$) satisfy conditions (*i*) and (*ii*). We do not show the proof because it can be done by following the procedure shown in our previous paper [21]. We emphasize again that the proof based on this paper does not rely on the local mean curvature flow in the neighborhood of $S_\alpha$, which is one of the major revisions from our previous paper [21].

## 4. Sketch of proof

We take a smooth coordinate near $S_\alpha$ as

$$ds^2 = \varphi^2 dr^2 + g_{ab} dx^a dx^b \tag{6}$$

such that $S_\alpha$ is located at $r = 0$, $\varphi k = 1$ is imposed only on $S_\alpha$, and the manifold $\Sigma$ exists in $r \geq 0$. Note that the condition $\varphi k = 1$ is imposed only on $S_\alpha$, and the local inverse mean curvature flow is not required to be taken. The positivity of $k$ on $S_\alpha$ implies that $\varphi$ is positive.

Let us introduce a manifold $\bar{\Sigma}$ with a metric in the range $r \leq 0$

$$d\bar{s}^2 = \frac{1 - \exp\left(-\frac{n-2}{n-1} r_0\right)}{1 - \exp\left[-\frac{n-2}{n-1}(r + r_0)\right]} \exp\left(\frac{2}{n-1} r\right) \varphi_0^2(x^a) dr^2 + \exp\left(\frac{2}{n-1} r\right) g_{0,ab}(x^a) dx^a dx^b, \tag{7}$$

where $g_{0,\,ab}$ is the metric of $S_\alpha$, $\varphi_0$ is taken as

$$\varphi_0(x^a) := \varphi|_{r=0} \left(= k^{-1}|_{S_\alpha}\right), \tag{8}$$

and $r_0$ is defined as a constant satisfying

$$\exp\left(\frac{n-2}{n-1} r_0\right) = \frac{n + 2(n-1)\alpha}{2[1 + (n-1)\alpha]}. \tag{9}$$

Note that, by the coordinate transformation

$$\mathscr{R} := \exp\left(\frac{1}{n-1} r\right), \tag{10}$$

the metric Eq. (7) becomes

$$d\bar{s}^2 = \frac{1 - \mathscr{R}_0^{n-2}}{1 - \left(\frac{\mathscr{R}_0}{\mathscr{R}}\right)^{n-2}} \varphi_0^2(x^a)(n-1)^2 d\mathscr{R}^2 + \mathscr{R}^2 g_{0,ab}(x^a) dx^a dx^b, \tag{11}$$

where

$$\mathscr{R}_0 := \exp\left(-\frac{1}{n-1} r_0\right). \tag{12}$$

The structure of "radial" direction ($\mathscr{R}$) is the same as that of the Schwarzschild solution, and thus the surface at $\mathscr{R} = \mathscr{R}_0$, that is $r = -r_0$, where the radial component of the metric diverges, is not at infinity and is a minimal surface. The manifold $\bar{\Sigma}$ is continuously glued to $\Sigma$ at $r = 0$, because all components of both induced metrics in Eqs. (6) and (7) on the gluing surface $S_\alpha$($r = 0$) are the same. However, their extrinsic curvatures do not necessarily match with each other and then the metric components are generally $C^0$-class there.

Since, for $\bar{\Sigma}$, the metric components for each $r$-constant hypersurface are written by the product of the $r$-dependent part and $x^a$-dependent part, each hypersurface becomes umbilical, that is, the extrinsic curvature is written as

$$\bar{k}_{ab} = \frac{1}{n-1} \bar{k} \bar{g}_{ab}. \tag{13}$$

Hereinafter, quantities with bars indicate that they are associated with the metric in Eq. (7). The mean curvature and its normal derivative of the metric in Eq. (7) are calculated as

$$\bar{k} = \varphi_0^{-1}\sqrt{\frac{1-\exp\left[-\frac{n-2}{n-1}(r+r_0)\right]}{1-\exp\left(-\frac{n-2}{n-1}r_0\right)}\exp\left(-\frac{2}{n-1}r\right)}, \tag{14}$$

$$\bar{r}^a\bar{D}_a\bar{k} = -\frac{1}{n-1}\bar{k}^2\frac{1-\frac{n}{2}\exp\left[-\frac{n-2}{n-1}(r+r_0)\right]}{1-\exp\left[-\frac{n-2}{n-1}(r+r_0)\right]}, \tag{15}$$

where $\bar{r}^a = \bar{\varphi}^{-1}(\partial/\partial r)^a$. Note that $\bar{k}|_{r=0} = \varphi_0^{-1} = k|_{S_\alpha}$ and $\bar{r}^a\bar{D}_a\bar{k}|_{r=0} = \alpha\bar{k}^2|_{r=0}$ are satisfied, where we use Eq. (9).

The $n$-dimensional Ricci scalar of $\bar{\Sigma}$ is expressed with the $(n-1)$-dimensional quantities as

$${}^{(n)}\bar{R} = {}^{(n-1)}\bar{R} - 2\bar{\varphi}^{-1}\bar{\mathcal{D}}^2\bar{\varphi} - 2\bar{r}^a\bar{D}_a\bar{k} - \frac{n}{n-1}\bar{k}^2, \tag{16}$$

where we define $\bar{\varphi}$ as the lapse function in the metric in Eq. (7), that is,

$$\bar{\varphi} := \varphi_0(x^a)\sqrt{\frac{1-\exp\left(-\frac{n-2}{n-1}r_0\right)}{1-\exp\left[-\frac{n-2}{n-1}(r+r_0)\right]}\exp\left(\frac{2}{n-1}r\right)}. \tag{17}$$

With the explicit form of the metric in Eq. (7), all terms in the right-hand side of Eq. (16) are written with those on $S_\alpha$,

$$\begin{aligned} &{}^{(n-1)}\bar{R} = \exp\left(-\frac{2}{n-1}r\right){}^{(n-1)}R_0, \qquad \bar{\varphi}^{-1}\bar{\mathcal{D}}^2\bar{\varphi} = \exp\left(-\frac{2}{n-1}r\right)\varphi_0^{-1}\mathcal{D}^2\varphi_0, \\ &2\bar{r}^a\bar{D}_a\bar{k} + \frac{n}{n-1}\bar{k}^2 = \exp\left(-\frac{2}{n-1}r\right)\left(2\alpha + \frac{n}{n-1}\right)\varphi_0^{-2}, \end{aligned} \tag{18}$$

where each index "0" indicates the quantity on the $r = 0$ hypersurface $S_\alpha$. They give

$${}^{(n)}\bar{R} = \exp\left(-\frac{2}{n-1}r\right)\left\{{}^{(n-1)}R_0 - 2\varphi_0^{-1}\mathcal{D}^2\varphi_0 - \left(2\alpha + \frac{n}{n-1}\right)\varphi_0^{-2}\right\}. \tag{19}$$

If $S_\alpha$ is a refined AGPS, Eq. (19) with Eqs. (4) and (8) results in the nonnegativity of ${}^{(n)}\bar{R}$.

We have shown the nonnegativity of the $n$-dimensional Ricci scalar of $\bar{\Sigma}$, while that of $\Sigma$ is one of the assumptions of the theorem. Therefore, everywhere on the glued manifold $\Sigma \cup \bar{\Sigma}$ (but on $S_\alpha$) the $n$-dimensional Ricci scalar is nonnegative. Hence, one expects the application of Bray and Lee's proof to the Riemannian Penrose inequality [24] for $\Sigma \cup \bar{\Sigma}$. However, their theorem requires the smoothness of the manifold as an assumption, whereas our manifold $\Sigma \cup \bar{\Sigma}$ is generally $C^0$-class, not $C^\infty$-class on $S_\alpha$. This problem will be fixed in the following sections. In the rest of the present section, supposing that Bray and Lee's theorem is applicable to the manifold $\Sigma \cup \bar{\Sigma}$, we show the inequality in Eq. (5).

The extended manifold $\Sigma \cup \bar{\Sigma}$ has a minimal hypersurface at $r = -r_0$. Suppose that the $r = -r_0$ hypersurface is the outermost minimal hypersurface, which will be justified in Sect. 6. Since the metric of $\bar{\Sigma}$ is explicitly written in Eq. (7), we can relate the areas of $S_\alpha$ and of the minimal hypersurface $\mathscr{S}_0$ at $r = -r_0$,

$$A_0 = \exp(-r_0)A_\alpha = \left[\frac{2\{1+(n-1)\alpha\}}{n+2(n-1)\alpha}\right]^{\frac{n-1}{n-2}} A_\alpha, \tag{20}$$

where $A_0$ and $A_\alpha$ are the areas of $\mathscr{S}_0$ and $S_\alpha$, respectively. In the second equality, we used Eq. (9). If Bray and Lee's theorem is applicable for $\mathscr{S}_0$, the area of $\mathscr{S}_0$ is bounded as

$$A_0 \leq \omega_{n-1}(2Gm)^{\frac{n-1}{n-2}}. \tag{21}$$

This gives

$$A_\alpha \leq \omega_{n-1}\left[\frac{n+2(n-1)\alpha}{1+(n-1)\alpha}Gm\right]^{\frac{n-1}{n-2}}, \tag{22}$$

that is, the inequality in Eq. (5) is obtained.

When the equality holds in the inequality in Eq. (22), it also holds in the Riemannian Penrose inequality (21). Bray and Lee's theorem implies that the manifold has the metric of the time-symmetric slice of a (higher-dimensional) Schwarzschild spacetime. Then, the minimal hypersurface $\mathscr{S}_0$ is spherically symmetric, and thus $S_\alpha$ too because of the metric form of Eq. (7). Moreover, by construction of $\bar{\Sigma}$, $r^a D_a k = \alpha k^2$ holds on $S_\alpha$. As a result, the equality in the inequality in Eq. (22) holds if and only if $\Sigma$ is the time-symmetric hypersurface of a Schwarzschild spacetime, and then, $S_\alpha$ is the spherically symmetric hypersurface with $r^a D_a k = \alpha k^2$.

In the above discussion, it is reminded that two assumptions are imposed: $\mathscr{S}_0$ is the outermost minimal surface, and Bray and Lee's theorem is applicable for $\Sigma \cup \bar{\Sigma}$. The former is justified from the assumptions of the theorem, which we will see in Sect. 6. For the latter, in Sect. 5, we will show that $\Sigma \cup \bar{\Sigma}$ is achieved as a limit of a sequence of smooth manifolds for which Bray and Lee's theorem is applicable.

## 5. Smooth extension

In this section, we construct a sequence of smooth manifolds and show that $\Sigma \cup \bar{\Sigma}$ is obtained as a limit of it. We carry it out in the following steps. From the assumption of the theorem, there exists a smooth extension of $\Sigma$ from $S_\alpha$ with a nonnegative Ricci scalar. At first in Sect. 5.1, we deform this extended region of $\Sigma$ so that the Ricci scalar is strictly positive. We call the deformed manifold $\hat{\Sigma}$. Next, in Sect. 5.2, based on the manifold $\hat{\Sigma}$, we construct the $C^0$-class extension discussed in the previous section. This $C^0$-class extension is done on a surface slightly inward from $S_\alpha$, parameterized with a small positive value $\epsilon$. Due to the strict positivity of the Ricci scalar of $\hat{\Sigma}$, we can deform this $C^0$-class extension to be smooth. Then $\Sigma \cup \bar{\Sigma}$, constructed in the previous section, is achieved as the limit $\epsilon \to 0$.

Note that since the condition of the AGPS is refined, the straightforward application of the proof in our previous paper [21] does not work. A new evaluation is required to be introduced.

### 5.1. *The first step*

The existence of a smooth inward extension from $S_\alpha$ was imposed as one of the assumptions for the theorem. This means that the manifold $\Sigma$ with the metric in Eq. (6) is extended to the negative $r$ region slightly. One can take a smooth coordinate in Eq. (6) from $S_\alpha$ in the extended region $-\hat{\delta} < r < 0$, where $\hat{\delta}$ is a small positive constant. Note that we take the coordinate such that $\varphi k = 1$ is satisfied on $S_\alpha$, while it is not necessary for it to be imposed on other $r$-constant surfaces. The Ricci scalar in the slightly extended region is nonnegative by the assumption of the theorem. In this subsection, we deform $\Sigma$ in the region $-\hat{\delta} < r < 0$, and construct a manifold $\hat{\Sigma}$ where the Ricci scalar is strictly positive and which is glued to $\Sigma$ smoothly at $r = 0$.

Since the coordinate in Eq. (6) is smooth, we can expand the geometrical variables based on those on $S_\alpha$ in a sufficiently small region $-\hat{\delta} < r < 0$. Hence, since $k$ is positive on $S_\alpha$, $k$ can be positive in $-\hat{\delta} < r < 0$ with $\hat{\delta}$ being sufficiently small. Since $r^a D_a k$ and $k_{ab}$ have upper bounds due to the smoothness of $\Sigma$, there exists a positive constant $\beta$ satisfying

$$2r^a D_a k + k^2 + k_{ab}^2 < \beta k^2 \tag{23}$$

everywhere in $-\hat{\delta} < r < 0$. Similarly, due to the positivity of the lapse function $\varphi$ as commented after Eq. (6), there exists a positive constant $\hat{\beta}$ satisfying

$$\varphi^{-1} k > \hat{\beta} k^2. \tag{24}$$

With such sufficiently small $\hat{\delta}$, we introduce a metric in the region $-\hat{\delta} < r < 0$,

$$d\hat{s}^2 = \hat{\varphi}^2 dr^2 + g_{ab} dx^a dx^b, \tag{25}$$

$$\hat{\varphi} := u(r)\varphi, \tag{26}$$

$$u(r) := 1 - \exp\left(\frac{C\hat{\delta}}{r}\right), \tag{27}$$

where $C$ is a constant satisfying $C > \hat{\delta}\beta/\hat{\beta}$. We shall call this manifold $\hat{\Sigma}$. Note that $u(r) - 1$ and its any order derivatives approach zero in the limit where $r$ goes to zero from negative. Therefore, at $r = 0$, $\hat{\Sigma}$ is smoothly glued to $\Sigma$.

Let us focus on the Ricci scalar of $\hat{\Sigma}$. The $(n-1)$-dimensional geometrical variables are calculated as

$$\begin{aligned} &{}^{(n-1)}\hat{R} = {}^{(n-1)}R, \qquad \hat{\varphi}^{-1}\hat{\mathcal{D}}^2\hat{\varphi} = \varphi^{-1}\mathcal{D}^2\varphi, \\ &\hat{k}_{ab} = u^{-1}k_{ab}, \qquad \hat{k} = u^{-1}k, \qquad \hat{r}^a\hat{D}_a\hat{k} = -u^{-2}\varphi^{-1}k(\partial_r \log u) + u^{-2} r^a D_a k, \end{aligned} \tag{28}$$

where variables with a hat indicate those with respect to the metric in Eq. (25). Then the $n$-dimensional Ricci scalar becomes

$$\begin{aligned} {}^{(n)}\hat{R} &= {}^{(n-1)}\hat{R} - 2\hat{\varphi}^{-1}\hat{\mathcal{D}}^2\hat{\varphi} - 2\hat{r}^a\hat{D}_a\hat{k} - \hat{k}^2 - \hat{k}_{ab}^2 \\ &= {}^{(n)}R + \left(1 - u^{-2}\right)\left(2r^a D_a k + k^2 + k_{ab}^2\right) + 2u^{-2}\varphi^{-1}k\partial_r \log u. \end{aligned} \tag{29}$$

We can estimate the bound of the functions written in $u$,

$$\begin{aligned} 0 > 1 - u^{-2} &= u^{-2}\left[-2\exp\left(\frac{C\hat{\delta}}{r}\right) + \exp\left(\frac{2C\hat{\delta}}{r}\right)\right] \\ &> -2u^{-2}\exp\left(\frac{C\hat{\delta}}{r}\right), \end{aligned} \tag{30}$$

and

$$\partial_r \log u = u^{-1}\frac{C\hat{\delta}}{r^2}\exp\left(\frac{C\hat{\delta}}{r}\right) > \frac{C}{\hat{\delta}}\exp\left(\frac{C\hat{\delta}}{r}\right). \tag{31}$$

For the last inequality in Eq. (31), we used $0 < u < 1$ and $-\hat{\delta} < r < 0$. Through these estimates with Eqs. (23) and (24), Eq. (29) implies

$${}^{(n)}\hat{R} > {}^{(n)}R + 2\frac{k^2}{u^2}\left(\frac{C\hat{\beta}}{\hat{\delta}} - \beta\right)\exp\left(\frac{C\hat{\delta}}{r}\right). \tag{32}$$

Therefore, since the constant $C$ satisfies $C > \hat{\delta}\beta/\hat{\beta}$, ${}^{(n)}\hat{R}$ is strictly positive in $-\hat{\delta} < r < 0$.

### 5.2. *The second step*

In the previous subsection we have constructed the extended region $\hat{\Sigma}$, where the Ricci scalar ${}^{(n)}R$ is strictly positive. In this subsection, we construct a further extended manifold from $\hat{\Sigma}$. The strict positivity of the Ricci scalar ${}^{(n)}R$ enables the gluing discussed in Sect. 4 to be deformed into the smooth one as we will explain. The gluing is required to be done on a surface $\mathcal{S}_\epsilon$ in the extended region $\hat{\Sigma}$, which is different from $S_\alpha$. The manifold $\bar{\Sigma}$ is constructed from $S_\alpha$ in Sect. 4, but now we need to construct a manifold from $\hat{\Sigma}$ based on $\mathcal{S}_\epsilon$. Therefore, the obtained manifold (named $\mathring{\Sigma}$) is different from $\bar{\Sigma}$. In the limit where $\mathcal{S}_\epsilon$ approaches $S_\alpha$, $\mathring{\Sigma}$ approaches $\bar{\Sigma}$. In this subsection, we will take a region in $\hat{\Sigma}$, construct $\mathring{\Sigma}$ in the same way as that in Sect. 4, and make the gluing between $\hat{\Sigma}$ and $\mathring{\Sigma}$ smooth.

Let us introduce a positive constant $\epsilon$, such that $0 < 2\epsilon < \hat{\delta}$ is satisfied. We construct a manifold $\mathring{\Sigma}$ in the same way as that in Sect. 4, but based on the surface $\mathcal{S}_\epsilon$ at $r = -\epsilon$. In the region $-2\epsilon < r \leq -\epsilon$ we construct a smooth gluing between $\hat{\Sigma}$ and $\bar{\Sigma}$.

Since the metric components in Eq. (6) are smooth and the deformation of the metric in Eq. (25) is done with the manifold $\Sigma \cup \hat{\Sigma}$ keeping smooth, at $r = -\epsilon$ for enough small $\epsilon$ the modification from the condition in Eq. (4) is suppressed by order $\epsilon$, that is, there exists a constant $\gamma$ satisfying

$$ {}^{(n-1)}\hat{R} + 2\hat{k}^{-1}\hat{\mathcal{D}}^2\hat{k} - 4\hat{k}^{-2}\left(\hat{\mathcal{D}}\hat{k}\right)^2 \geq \left[2(\alpha + \gamma\epsilon) + \frac{n}{n-1}\right]\hat{k}^2 \tag{33} $$

at $r = -\epsilon$ on $\hat{\Sigma}$. We take another coordinate of $\hat{\Sigma}$ based on $S|_{r=-\epsilon} =: \mathcal{S}_\epsilon$,

$$ d\hat{\hat{s}}^2 = \hat{\hat{\varphi}}^2 d\rho^2 + \hat{\hat{g}}_{ab}dx^a dx^b. \tag{34} $$

Here, $\mathcal{S}_\epsilon$ is located at $\rho = 0$ where $\hat{\hat{g}}_{ab}|_{\rho=0} = \hat{g}_{ab}|_{r=-\epsilon}$. $\hat{\hat{\varphi}}$ is chosen so that $\hat{\hat{\varphi}} = \hat{\hat{k}}^{-1}(= \hat{k}^{-1})$ is satisfied. Note that the inequality in Eq. (33) still holds true for the variables with a double hat,

$$ {}^{(n-1)}\hat{\hat{R}} + 2\hat{\hat{k}}^{-1}\hat{\hat{\mathcal{D}}}^2\hat{\hat{k}} - 4\hat{\hat{k}}^{-2}\left(\hat{\hat{\mathcal{D}}}\hat{\hat{k}}\right)^2 \geq \left[2(\alpha + \gamma\epsilon) + \frac{n}{n-1}\right]\hat{\hat{k}}^2, \tag{35} $$

because it is written in $(n-1)$-dimensional geometrical variables.

Instead of $\bar{\Sigma}$ as given in Sect. 4, we introduce another manifold $\mathring{\Sigma}$, whose metric is written as

$$ d\mathring{s}^2 = \mathring{\varphi}^2 d\rho^2 + \mathring{g}_{ab}dx^a dx^b, \tag{36} $$

where

$$ \mathring{\varphi}^2 := \frac{1 - \exp\left(-\frac{n-2}{n-1}\rho_0\right)}{1 - \exp\left[-\frac{n-2}{n-1}(\rho + \rho_0)\right]} \exp\left(\frac{2}{n-1}\rho\right) \hat{\hat{\varphi}}^2\big|_{\rho=0}, \tag{37} $$

$$ \mathring{g}_{ab} := \exp\left(\frac{2}{n-1}\rho\right) \hat{\hat{g}}_{ab}(x^a)\big|_{\rho=0}, \tag{38} $$

and $\rho_0$ is defined by

$$ \exp\left(\frac{n-2}{n-1}\rho_0\right) = \frac{n + 2(n-1)\mathring{\alpha}}{2[1 + (n-1)\mathring{\alpha}]} \tag{39} $$

with a constant $\mathring{\alpha}$. We attempt to glue the manifold $\hat{\Sigma}$ to $\mathring{\Sigma}$ smoothly. At $\rho = 0$, the condition in Eq. (4) on $S_\alpha$ is modified into the inequality in Eq. (35), that is, $\alpha$ is shifted to $\alpha + \gamma\epsilon$. For later convenience, we consider a further shift of this parameter and set a constant $\mathring{\alpha}$ to be

$$\mathring{\alpha} := \alpha + \gamma\epsilon - \mathring{\gamma}\epsilon \tag{40}$$

where $\mathring{\gamma}$ is a positive constant. Introduction of $\mathring{\gamma}$ is one of the improvements from the previous proof [21]. This manifold $\mathring{\Sigma}$ depends on $\epsilon$ and converges to $\bar{\Sigma}$ in the limit $\epsilon \to 0$. Hereinafter, geometrical quantities with circles indicate that they are associated with the metric of Eq. (36) and quantities with subscripts $\epsilon$ show those evaluated on $\mathcal{S}_\epsilon(\rho = 0)$.

As discussed in Sect. 4, geometrical variables for the metric of Eq. (36) satisfy

$$\begin{aligned}
&\mathring{k}_{ab} = \frac{1}{n-1}\mathring{k}\mathring{g}_{ab}. \qquad \mathring{k}|_{\mathcal{S}_\epsilon} = \hat{\mathring{\varphi}}_\epsilon^{-1} = \hat{\mathring{k}}|_{\mathcal{S}_\epsilon} \qquad \mathring{r}^a\mathring{D}_a\mathring{k}|_{\mathcal{S}_\epsilon} = \mathring{\alpha}\mathring{k}^2|_{\mathcal{S}_\epsilon}\\
&{}^{(n)}\mathring{R} = {}^{(n-1)}\mathring{R} - 2\mathring{\varphi}^{-1}\mathring{\mathcal{D}}^2\mathring{\varphi} - 2\mathring{r}^a\mathring{D}_a\mathring{k} - \frac{n}{n-1}\mathring{k}^2,\\
&{}^{(n-1)}\mathring{R} = \exp\left(-\frac{2}{n-1}\rho\right){}^{(n-1)}\hat{\mathring{R}}_\epsilon, \qquad \mathring{\varphi}^{-1}\mathring{\mathcal{D}}^2\mathring{\varphi} = \exp\left(-\frac{2}{n-1}\rho\right)\hat{\mathring{\varphi}}_\epsilon^{-1}\hat{\mathring{\mathcal{D}}}^2\hat{\mathring{\varphi}}_\epsilon,\\
&2\mathring{r}^a\mathring{D}_a\mathring{k} + \frac{n}{n-1}\hat{\mathring{k}}^2 = \exp\left(-\frac{2}{n-1}\rho\right)\left(2\mathring{\alpha} + \frac{n}{n-1}\right)\hat{\mathring{\varphi}}_\epsilon^{-2}
\end{aligned} \tag{41}$$

and then, we have

$${}^{(n)}\mathring{R} = \exp\left(-\frac{2}{n-1}\rho\right)\left\{{}^{(n-1)}\hat{\mathring{R}}_\epsilon - 2\hat{\mathring{\varphi}}_\epsilon^{-1}\hat{\mathring{\mathcal{D}}}^2\hat{\mathring{\varphi}}_\epsilon - \left(2\mathring{\alpha} + \frac{n}{n-1}\right)\hat{\mathring{\varphi}}_\epsilon^{-2}\right\} \geq 0. \tag{42}$$

Since on $\mathcal{S}_\epsilon$ some geometrical quantities for the metrics of Eq. (6) and Eq. (36) coincide,

$$\hat{\mathring{\varphi}}|_{\mathcal{S}_\epsilon} = \mathring{\varphi}|_{\mathcal{S}_\epsilon}, \qquad \hat{\mathring{g}}_{ab}|_{\mathcal{S}_\epsilon} = \mathring{g}_{ab}|_{\mathcal{S}_\epsilon}, \qquad \hat{\mathring{k}}|_{\mathcal{S}_\epsilon} = \mathring{k}|_{\mathcal{S}_\epsilon}, \tag{43}$$

the difference between the metric components of Eq. (6) and those of Eq. (36) can be expressed as

$$\hat{\mathring{g}}_{ab} - \mathring{g}_{ab} = \rho^2 T\mathring{g}_{ab} + \rho h_{ab}, \qquad \hat{\mathring{\varphi}} - \mathring{\varphi} = 2\rho\mathring{\varphi}\Phi \tag{44}$$

with smooth functions $T$, $h_{ab}$, and $\Phi$. Here, $\rho h_{ab}$ shows the traceless part of $\hat{\mathring{g}}_{ab} - \mathring{g}_{ab}$, and therefore, $\mathring{g}^{ab}h_{ab} = 0$.

Let $\check{\delta}$ be a positive constant such that the whole region of $-\check{\delta} \leq \rho \leq 0$ on $\hat{\Sigma}$ is included in $-2\epsilon < r < \epsilon$. Then, for an arbitrary $\check{\epsilon}$ satisfying $0 < \check{\epsilon} < \check{\delta}$, the whole region of $-\check{\epsilon} \leq \rho \leq 0$ is also included in $-2\epsilon < r < -\epsilon$. We introduce a manifold $\check{\Sigma}$ in $-\check{\epsilon} \leq \rho \leq 0$ with the following metric:

$$d\check{s}^2 = \check{\varphi}^2 dr^2 + \check{g}_{ab}dx^a dx^b, \tag{45}$$

where

$$\check{\varphi} = \mathring{\varphi}\left[1 + \left(\frac{d}{d\rho}F_1^2\right)\Phi\right], \tag{46}$$

$$\check{g}_{ab} = \mathring{g}_{ab}\left(1 + TF_1^2\right) + F_2 h_{ab}. \tag{47}$$

$F_1$ and $F_2$ are functions of $\rho$. Note that this form of a smoothed metric is different from one appearing in the proof of the previous version [21]. Hereinafter, the geometric functions with check marks are those with respect to the metric in Eq. (45). For $F_1 = F_2 = \rho$, the metric in Eq. (45) is reduced to Eq. (25), while for $F_1 = F_2 = 0$ it becomes Eq. (36). Hence, if both $F_1$ and $F_2$ are smoothly glued to $F := \rho(\rho > 0)$ at $\rho = 0$ and $\tilde{F} := 0(\rho < -\check{\epsilon})$ at $\rho = -\check{\epsilon}$, $\check{\Sigma}$ is smoothly glued to $\hat{\Sigma}$ at $\rho = 0$ and $\mathring{\Sigma}$ at $\rho = -\check{\epsilon}$. Suppose also that $\check{\Sigma}$ has a nonnegative Ricci scalar. In the limit $\epsilon \to 0$, the sequence of these smooth manifolds (with a nonnegative Ricci scalar) converges to $\Sigma \cup \bar{\Sigma}$. Then, the smoothing is achieved. Therefore, what we should do is the proof of the existence of $F_1$ and $F_2$ smoothly glued to $F := \rho$ at $\rho = 0$ and $\tilde{F} := 0$ at $\rho = -\check{\epsilon}$ with $\check{\Sigma}$ satisfying the nonnegativity of the Ricci scalar.

Let us examine the Ricci scalar of the metric in Eq. (45). The metric components and their derivatives are expanded as

$$\begin{aligned}
&\check{\varphi} = \mathring{\varphi}|_{\rho=0} + \mathcal{O}(\rho), \qquad \check{g}_{ab} = \mathring{g}_{ab}|_{\rho=0} + \mathcal{O}(\rho),\\
&\check{g}^{ab} = \mathring{g}^{ab}|_{\rho=0} - F_2 h^{ab} + \mathcal{O}\left(\rho^2\right),\\
&\partial_\rho \log\check{\varphi} = (\partial_\rho \log\mathring{\varphi})|_{\rho=0} + (F_1^2)''\Phi + \mathcal{O}(\rho),\\
&\partial_\rho \check{g}_{ab} = (\partial_\rho \mathring{g}_{ab})|_{\rho=0} + (F_2)' h_{ab} + \mathcal{O}(\rho),\\
&\partial_\rho^2 \check{g}_{ab} = \left(\partial_\rho^2 \mathring{g}_{ab}\right)\Big|_{\rho=0} + \mathring{g}_{ab}\left(F_1^2\right)'' T + (F_2)'' h_{ab} + 2(F_2)'\partial_\rho h_{ab} + \mathcal{O}(\rho),
\end{aligned} \tag{48}$$

where the prime means the derivative with respect to $\rho$, that is $(F_1)' = dF_1/d\rho$, and $h^{ab} := \mathring{g}^{ac} h_{cd} \mathring{g}^{db}$. The geometrical quantities are expressed as

$$\begin{aligned}
&\check{k}_{ab} = \mathring{k}_{ab}|_{\rho=0} + \frac{(F_2)'}{2\mathring{\varphi}|_{\rho=0}} h_{ab} + \mathcal{O}(\rho), \qquad \check{k} = \mathring{k}|_{\rho=0} + \mathcal{O}(\rho),\\
&\check{r}^a \check{D}_a \check{k} = (\mathring{r}^a \mathring{D}_a \mathring{k})|_{\rho=0} + (F_1^2)''\left[\frac{2(n-1)T - 4\mathring{\varphi}\Phi\mathring{k}}{4\mathring{\varphi}^2}\right] - (F_2^2)''\left(\frac{h_{ab}^2}{4\mathring{\varphi}^2}\right)\Bigg|_{\rho=0} + \mathcal{O}(\rho),\\
&{}^{(n-1)}\check{R} = {}^{(n-1)}\mathring{R}|_{\rho=0} + \mathcal{O}(\rho), \qquad \check{\varphi}^{-1}\check{\mathcal{D}}^2\check{\varphi} = \left(\mathring{\varphi}^{-1}\mathring{\mathcal{D}}^2\mathring{\varphi}\right)\Big|_{\rho=0} + \mathcal{O}(\rho),\\
&{}^{(n)}\check{R} = {}^{(n)}\mathring{R}|_{\rho=0} + \left[2(F_2^2)'' - (F_2{}')^2\right]\left(\frac{1}{2\mathring{\varphi}} h_{ab}\right)^2\Bigg|_{\rho=0}\\
&\qquad - \left(F_1^2\right)''\left[\frac{2(n-1)T - 4\mathring{\varphi}\Phi\mathring{k}}{2\mathring{\varphi}^2}\right]\Bigg|_{\rho=0} + \mathcal{O}(\rho),
\end{aligned} \tag{49}$$

where we used $\partial_\rho \check{g}^{ab} = -\check{g}^{ac}\check{g}^{bd}\partial_\rho \check{g}_{cd}$, $\partial_\rho \mathring{g}^{ab} = -\mathring{g}^{ac}\mathring{g}^{bd}\partial_\rho \mathring{g}_{cd}$, the traceless condition of $h_{ab}$, i.e. $\mathring{g}^{ab} h_{ab} = 0$, and $h^{ab}\partial_\rho \mathring{g}_{ab} = \mathring{g}^{ab}\partial_\rho h_{ab}$ that is derived from the former three formulas. Note also that we used $h_{ab}\mathring{k}^{ab} = 0$, which holds from $\mathring{k}_{ab} \propto \mathring{g}_{ab}$. One can obtain the relations between each geometric variable associated with the metrics in Eqs. (34) and (36) by setting $F_1 = F_2 = \rho$ in the above equalities as follows:

$$ {}^{(n)}\hat{\mathring{R}}|_{\rho=0} = {}^{(n)}\mathring{R}|_{\rho=0} - \left(\frac{1}{2\mathring{\varphi}} h_{ab}\right)^2\Bigg|_{\rho=0} - \left[\frac{2(n-1)T - 4\mathring{\varphi}\Phi\mathring{k} - h_{ab}^2}{\mathring{\varphi}^2}\right]\Bigg|_{\rho=0} \tag{50}$$

and

$$\hat{\mathring{r}}^a \hat{\mathring{D}}_a \hat{\mathring{k}}|_{\rho=0} = (\mathring{r}^a \mathring{D}_a \mathring{k})|_{\rho=0} + \left[\frac{2(n-1)T - 4\mathring{\varphi}\Phi\mathring{k} - h_{ab}^2}{2\mathring{\varphi}^2}\right]\Bigg|_{\rho=0}. \tag{51}$$

On $\mathcal{S}_\epsilon$, $\hat{\mathring{r}}^a \hat{\mathring{D}}_a \hat{\mathring{k}}$ is bounded from below as

$$\begin{aligned}
\hat{\mathring{r}}^a \hat{\mathring{D}}_a \hat{\mathring{k}}|_{\rho=0} &= \left[-\frac{1}{2}{}^{(n)}\hat{\mathring{R}} - \frac{1}{2}\hat{\mathring{k}}_{ab}^2 + \frac{1}{2}{}^{(n-1)}\hat{\mathring{R}} - \hat{\mathring{\varphi}}^{-1}\hat{\mathring{\mathcal{D}}}^2\hat{\mathring{\varphi}} - \frac{n}{2(n-1)}\hat{\mathring{k}}^2\right]\Bigg|_{\rho=0}\\
&\geq -\frac{1}{2}{}^{(n)}\hat{\mathring{R}}|_{\rho=0} - \frac{1}{2}\left(\frac{h_{ab}^2}{2\hat{\mathring{\varphi}}|_{\rho=0}}\right)^2 + (\alpha + \gamma\epsilon)\hat{\mathring{k}}^2|_{\rho=0},
\end{aligned} \tag{52}$$

where $\mathring{\hat{\mathsf{k}}}_{ab}$ is the traceless part of $\mathring{\hat{k}}_{ab}$ and we use the inequality in Eq. (35). Using the second and third equations in Eq. (41) for Eqs. (51) and (52), we have

$$\left[\frac{2(n-1)T - 4\mathring{\varphi}\Phi\mathring{k} - h_{ab}^2}{\mathring{\varphi}^2}\right]\Bigg|_{\rho=0} \geq -{}^{(n)}\mathring{\hat{R}}|_{\rho=0} - \left(\frac{h_{ab}^2}{2\mathring{\hat{\varphi}}|_{\rho=0}}\right)^2 + 2\mathring{\gamma}\epsilon\mathring{\hat{k}}^2|_{\rho=0}\,. \tag{53}$$

Now, for the moment, we assume that $F_1$ and $F_2$ satisfy

$$\mathcal{O}(\check{\epsilon}) < \left(F_1^2\right)'' \leq 2, \qquad 4\left(F_2^2\right)'' - 2\left(F_2'\right)^2 - 3\left(F_1^2\right)'' \geq \mathcal{O}(\check{\epsilon})\,. \tag{54}$$

The required conditions for the functions $F_1$ and $F_2$ are stronger than those in the proof of the previous version [21]. The existence of $F_1$ and $F_2$ is shown in Appendix A. Then the inequality in Eq. (53) with Eq. (50) and the last equation of Eq. (49) give the lower bound of ${}^{(n)}\check{R}$,

$$\begin{aligned}{}^{(n)}\check{R} \geq{}& \frac{\left(F_1^2\right)''}{2}\,{}^{(n)}\mathring{\hat{R}}|_{\rho=0} + \left[2\left(F_2^2\right)'' - \left(F_2'\right)^2 - \frac{3}{2}\left(F_1^2\right)''\right]\left(\frac{h_{ab}^2}{2\mathring{\hat{\varphi}}|_{\rho=0}}\right)^2 \\ &+ \left[2 - \left(F_1^2\right)''\right]\mathring{\gamma}\epsilon\mathring{\hat{k}}^2|_{\rho=0} + \mathcal{O}(\rho).\end{aligned} \tag{55}$$

Since we are considering the region $-\check{\epsilon} < \rho < 0$, the inequality in Eq. (55) can be estimated as

$${}^{(n)}\check{R} \geq \min\left({}^{(n)}\mathring{\hat{R}}|_{\rho=0},\ 2\mathring{\gamma}\epsilon\mathring{\hat{k}}^2|_{\rho=0}\right) + \mathcal{O}(\check{\epsilon}). \tag{56}$$

Note that both ${}^{(n)}\mathring{\hat{R}}|_{\rho=0}$ and $2\mathring{\gamma}\epsilon\mathring{\hat{k}}^2$ are strictly positive in the region $-\check{\delta} \leq \rho \leq 0$ and do not depend on $\check{\epsilon}$. Hence, there exists a positive constant $\check{\epsilon}(< \check{\delta})$ such that, for any $\rho$ in $-\check{\epsilon} \leq \rho \leq 0$, ${}^{(n)}\check{R} \geq 0$ holds.

In sum, if there exist functions $F_1$ and $F_2$ of $\rho$ such that both of them are smoothly glued to the functions $F := \rho$ at $\rho = 0$ and $\tilde{F} := 0$ at $\rho = -\check{\epsilon}$ and satisfy inequalities of Eq. (54), the smooth gluing of $\Sigma$ with $\bar{\Sigma}$ is achieved as the limit of our sequence ($\epsilon \to 0$). The existence of such functions is shown in Appendix A.

## 6. No existence of minimal hypersurfaces outside $\mathscr{S}_0$

In application of Bray and Lee's theorem, the minimal hypersurface $\mathscr{S}_0$ should be the outermost one in the smooth manifold which we have constructed. The proof of no existence of minimal hypersurfaces outside $\mathscr{S}_0$ is basically the same as that written in the previous paper [21]. We briefly describe the point of the proof here.

If the outermost minimal hypersurface exists outside $\mathscr{S}_0$, it is classified into three cases, that is, it encloses $S_\alpha$, it exists in $r < 0$, and it has intersection with $S_\alpha$. The first case is prohibited due to the assumption of the theorem. The last case is also prohibited for the following reason. If the outermost minimal hypersurface intersects $S_\alpha$, its area of the part existing in $r > 0$ should be larger than $\omega_{n-1}(2Gm)^{\frac{n-1}{n-2}}$ by the assumption of the theorem. This results in the fact that the area of the outermost minimal hypersurface is larger than $\omega_{n-1}(2Gm)^{\frac{n-1}{n-2}}$, but it is inconsistent with Bray and Lee's theorem. Therefore, the first and the last cases never occur.

Let us investigate the second case. We take foliations characterized by $r =$ constant in the region $-\epsilon < r < 0$ on $\hat{\Sigma}$, $\rho =$ constant in the region $-\check{\epsilon} < \rho < 0$ on $\check{\Sigma}$, and $\rho_0 < \rho < -\check{\epsilon}$ on $\mathring{\Sigma}$. They fill all the region between $S_\alpha$ and $\mathscr{S}_0$ without any overlap. On each foliation, $k$ is strictly positive. Then, as we have shown in the previous paper [21], no minimal hypersurfaces enclosing $\mathscr{S}_0$ exist.

Therefore, $\mathscr{S}_0$ is the outermost minimal hypersurface.

## 7. Completion of the proof

Now, we can apply Bray and Lee's theorem to the smooth manifold which we have constructed in Sect. 5. The area of $\mathscr{S}_0$ is bounded as

$$A[\mathscr{S}_0] \leq \omega_{n-1}(2Gm)^{\frac{n-1}{n-2}}. \tag{57}$$

Since we have the explicit form of the metric in Eq. (45), the area of $\mathcal{S}_\epsilon$ is related to that of $\mathscr{S}_0$,

$$A\left[\mathscr{S}_0\right] = \exp(-\rho_0)A\left[\mathcal{S}_\epsilon\right] = \left[\frac{2(1+(n-1)\mathring{\alpha})}{n+2(n-1)\mathring{\alpha}}\right]^{\frac{n-1}{n-2}} A\left[\mathcal{S}_\epsilon\right], \tag{58}$$

where Eq. (39) is used in the second equality. Note that, by the construction of $\check{\Sigma}$, the area of $\mathcal{S}_\epsilon$ becomes the same in $\check{\Sigma}$ and $\hat{\Sigma}$. Since $\hat{\Sigma}$ is constructed by smooth deformation of $\Sigma$ and $\Sigma$ is a smooth manifold, the difference between the areas of $S_\alpha$ and $\mathcal{S}_\epsilon$ is the order $\epsilon$, that is

$$A\left[\mathcal{S}_\epsilon\right] = A\left[S_\alpha\right] + \mathcal{O}\left(\epsilon\right). \tag{59}$$

Therefore,

$$A\left[S_\alpha\right] \leq \omega_{n-1}\left[\frac{n+2(n-1)\alpha}{1+(n-1)\alpha}Gm\right]^{\frac{n-1}{n-2}} + \mathcal{O}\left(\epsilon\right) \tag{60}$$

is obtained. Taking the limit $\epsilon \to 0$, we have the inequality in Eq. (5).

## 8. Summary

In this paper, we have shown the inequality for refined AGPSs in asymptotically flat space with a nonnegative Ricci scalar whose dimension is higher than or equal to three but less than eight. The definition of the refined AGPS and the statement of the main theorem are shown in Sects. 2 and 3, respectively. The inequality is a generalization of the Riemannian Penrose inequality in higher dimensions [19], and the higher-dimensional generalization of our previous work [21].

In addition to the higher-dimensional generalization, the refinement of AGPSs has been done. The advantage of the refinement is that, wheares the definition of the original AGPSs requires the information around the surface due to the $r^a D_a k$ term [21], the refined AGPSs are defined with the induced metric and the mean curvature fixed only from the embedding of the surface. Moreover, since any original AGPS is a refined AGPS, the theorem in this paper is more widely applicable than the original. The condition in Eq. (4) looks somewhat factitious. If the terms with the derivative of $k$ are ignored, the condition in Eq. (4) shows the competition between the intrinsic and extrinsic curvatures. Since the Geroch energy

$$E := \frac{A^{1/2}}{64\pi^{3/2}G}\int_S \left(2^{(2)}R - k^2\right) dA, \tag{61}$$

used in the proof of the four-dimensional Penrose inequality involves both the intrinsic and extrinsic curvatures, they might be suitable quantites to evaluate quasilocal properties. This is worth investigating and left for future works.

There can exist an AGPS near spatial infinity. As we discussed in the previous paper [21], $\alpha \to \infty$ corresponds to the limit where the AGPS approaches the outermost minimal hypersurface (i.e. $k = 0$), and then our inequality is reduced to the Riemannian Penrose inequality [11–13,19]. Another limit $\alpha \to -\frac{1}{n-1}$ is achieved as the $S^{n-1}$ surface at spatial infinity or an $(n-1)$-dimensional sphere in the $n$-dimensional flat space, that is, it corresponds to the region without gravitational field. Therefore, our inequality gives a relation between strong and weak gravity regions through the setting of the parameter $\alpha$. The properties of surfaces in strong gravity regions such as an event horizon, on the one hand, are expected to give information on quantum gravity through lack hole thermodynamics. On the other hand, in weak gravity regions, we can use the Newtonian approximation, which may give us an intuitive understanding of the area inequality.

Furthermore, it would be interesting to explore the refined AGPS in terms of geodesics. In black hole observation, a photon sphere, which is a set of the circular photon orbits in static and spherically symmetric spacetime, is an important surface [26]. It is defined by the behavior of null geodesics and its generalizations are introduced [27–32], and in some of them the relation to the LTS, which is the version of the AGPS with $\alpha = 0$, is discussed. Recently, a nontrivial behavior of null geodesics near infinity has been reported [33–35]. Circular photon orbits can be realized in an asymptotic region of spacetime in a short interval of time. The relation between weak and strong gravitational regions in our inequality may give a hint of the understanding of these temporary circular photon orbits.

There are versions of the Riemannian Penrose inequality including the effect of the electric charge [36–38] and the angular momenta [39–44]. Studies of AGPSs with such contributions are partially done based on the inverse mean curvature flow [45–47]. Analysis with the conformal flow may give further interesting results, which is left for future works.

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## A. An example of function for smooth extension

The proof requires the existence of functions $F_1$ and $F_2$ that satisfy Eq. (54) in the range $-\check{\epsilon} < x < 0$. (In this appendix, we use $x$ as the argument of function, instead of $\rho$.) We present an example. Note that the conditions for the functions are stronger than those in our previous paper [21], which makes the proof of their existence difficult and requires a different analysis. We decompose the range $-\check{\epsilon} < x < 0$ into two parts, $-\check{\epsilon}^2 < x < 0$ and $-\check{\epsilon} < x < -\check{\epsilon}^2$. In the first part, we set $F_2 = x$ and smoothly glue $F_1$ to $F_1 = x$ at $x = 0$ and to $F_1 = 0$ at $x = -\check{\epsilon}^2$. Then, in this region, Eq. (54) becomes

$$\mathcal{O}(\check{\epsilon}) < \left(F_1^2\right)'' \leq 2. \tag{A1}$$

In the second part, since $F_1$ is already glued to $F_1 = 0$, keeping this property of $F_1$, we smoothly glue $F_2$ to $F_2 = x$ at $x = -\check{\epsilon}^2$ and to $F_2 = 0$ at $x = -\check{\epsilon}$. Then, Eq. (54) becomes

$$2\left(F_2^2\right)'' - \left(F_2'\right)^2 \geq \mathcal{O}\left(\check{\epsilon}\right). \tag{A2}$$

We construct the smooth gluing for $F_1$ in $-\check{\epsilon}^2 < x < 0$ and for $F_2$ in $-\check{\epsilon} < x < -\check{\epsilon}^2$ in order, as shown in Sect. A.2 and in Sect. A.3, respectively. Before that, we first show a generic discussion to construct smooth gluing in Sect. A.1.

### *A.1. Smoothing at gluing point*

We consider a smooth gluing of functions $f_1(x)$ and $f_2(x)$ at $x = x_1$. Suppose that at $x = x_1$, these functions coincide with each other up to the first derivative,

$$f_1(x_1) = f_2(x_1), \qquad f_1'(x_1) = f_2'(x_1). \tag{A3}$$

We define $f_3$ as $f_3 := f_1 - f_2$, and then, the above conditions become

$$f_3(x_1) = 0, \qquad f_3'(x_1) = 0. \tag{A4}$$

The smooth gluing of functions $f_1(x)$ and $f_2(x)$ at $x = x_1$ is equivalent to that of $f_3$ and 0.

Let us construct a smooth function $f$ which is glued at $x = x_1$ to 0 defined in $x < x_1$ and at $x_1 + \Delta x$ to $f_3$ defined in $x > x_1 + \Delta x$. We shall introduce a function $w$,

$$w(y(x)) = \frac{1}{e^y + 1}, \tag{A5}$$

$$y(x) = \frac{\Delta x}{x - x_1} + \frac{\Delta x}{x - x_1 - \Delta x}. \tag{A6}$$

We can show by induction for $m \geq 1$ ($m \in \mathbb{N}$)

$$\frac{d^m w}{dy^m} \in \mathcal{F} := \{w(w-1)p(w) | p(w) \text{ is a polynomial of } w\}. \tag{A7}$$

For $m = 1$, since we have

$$\frac{dw}{dy} = w(w-1), \tag{A8}$$

we see that Eq. (A7) holds. Suppose that Eq. (A7) holds for $m = q$, that is, with a polynomial $p_q(w)$ the $q$-th order derivative of $w$ is written as

$$\frac{d^q w}{dy^q} = w(w-1)p_q(w). \tag{A9}$$

Differentiating it, we have

$$\frac{d^{q+1} w}{dy^{q+1}} = w(w-1)\frac{d}{dw}\left(w(w-1)p_q(w)\right). \tag{A10}$$

Since $p_q(w)$ is a polynomial, $\frac{d}{dw}\left(w(w-1)p_q(w)\right)$ is also. Therefore Eq. (A7) holds for $m = q + 1$.

Next, we show that $\frac{d^m w}{dx^m}$, for any $m \geq 1$, asymptotes to zero in both limits $x \to x_1$ and $x \to x_1 + \Delta x$. Let us investigate the limit $x \to x_1$ first. From Eq. (A5) we know

$$0 < w < 1. \tag{A11}$$

In the neighborhood of $x = x_1$, $w$ is estimated as

$$|w| = \frac{1}{e^y + 1} < e^{-y} = \exp\left(-\frac{\Delta x}{x - x_1} - \frac{\Delta x}{x - x_1 - \Delta x}\right) < \bar{c}\exp\left(-\frac{\Delta x}{x - x_1}\right), \quad \text{(A12)}$$

where $\bar{c}$ is a constant. Then we find that

$$\left|\frac{d^k w}{dy^k}\right| = |w(w-1)p_k(w)| < c_k \exp\left(-\frac{\Delta x}{x - x_1}\right), \quad \text{(A13)}$$

where $c_k$ is a constant, and we use $0 < w < 1$, i.e. $w$ is bounded. The $m$-th derivative of $w$ with respect to $x$ is expressed in terms of Faà di Bruno's formula

$$\frac{d^m w}{dx^m} = \sum_{k=1}^{m} \frac{d^k w}{dy^k} B_{m,k}\left(\frac{dy}{dx}, \frac{d^2 y}{dx^2}, \cdots, \frac{d^{m-k+1} y}{dx^{m-k+1}}\right), \quad \text{(A14)}$$

where $B_{m,k}\left(\frac{dy}{dx}, \frac{d^2 y}{dx^2}, \cdots, \frac{d^{m-k+1} y}{dx^{m-k+1}}\right)$ is the Bell polynomial. From the definition of $y$ (Eq. (A6)), we have

$$\frac{d^l y}{dx^l} = (-1)^l (l!)\left[\frac{\Delta x}{(x - x_1)^{l+1}} + \frac{\Delta x}{(x - x_1 - \Delta x)^{l+1}}\right] \quad \text{(A15)}$$

and thus, $\left|\frac{d^l y}{dx^l}\right|$ is bounded by a polynomial of $(x - x_1)^{-1}$ in the neighborhood of $x = x_1$. Since the right-hand side of Eq. (A14) is a finite sum, $\left|\frac{d^m w}{dx^m}\right|$ is bounded as

$$\left|\frac{d^m w}{dx^m}\right| < \left|p\left((x - x_1)^{-1}\right)\right| \exp\left(-\frac{\Delta x}{x - x_1}\right) \quad \text{(A16)}$$

where $p((x - x_1)^{-1})$ is a polynomial of $(x - x_1)^{-1}$. Since the right-hand side goes to zero in the limit $x \to x_1$, we have

$$\lim_{x \to x_1} \frac{d^m w}{dx^m} = 0. \quad \text{(A17)}$$

In a similar way, we can also show that, in the limit $x \to x_1 + \Delta x$, $\frac{d^m w}{dx^m}$ asymptotes to zero.

Here, we introduce a function $f := f_3 w$ in the range $x_1 < x < x_1 + \Delta x$. Taking the $m$-th order derivative, we have

$$\frac{d^m}{dx^m} f = \left(\frac{d^m}{dx^m} f_3\right) w + \sum_{k=1}^{m} \binom{m}{k} \left(\frac{d^{m-k}}{dx^{m-k}} f_3\right)\left(\frac{d^k}{dx^k} w\right). \quad \text{(A18)}$$

Since we find from Eq. (A17) that, in the right-hand side of the above equation, the terms except the first one go to zero in the limit $x \to x_1$, we obtain

$$\lim_{x \to x_1} \frac{d^m}{dx^m} f = 0, \quad \text{(A19)}$$

where we use $\lim_{x \to x_1} w = 0$. Hence, $f$ is smoothly glued to 0 at $x = x_1$. In a similar way, we can show the smooth gluing of $f$ and $f_3$ at $x = x_1 + \Delta x$.

Now, we estimate $f$ and its derivatives. The first derivative of $w$ becomes

$$\frac{dw}{dx} = -\frac{1}{\Delta x}\left[\left(\frac{\Delta x}{x - x_1}\right)^2 + \left(\frac{\Delta x}{x - x_1 - \Delta x}\right)^2\right] w(w-1). \quad \text{(A20)}$$

At first, we consider the range $x_1 < x < x_1 + \frac{\Delta x}{3}$. Using the estimate of $w$

$$(0<)w < e^{-y} = \exp\left(-\frac{\Delta x}{x - x_1} - \frac{\Delta x}{x - x_1 - \Delta x}\right)$$
$$= \exp\left[-\frac{\Delta x}{2(x - x_1)}\right]\exp\left[-\frac{\Delta x}{2(x - x_1)} - \frac{\Delta x}{x - x_1 - \Delta x}\right], \quad \text{(A21)}$$

we have

$$\left|\frac{dw}{dx}\right| < \left|\frac{1}{\Delta x}\left[\left(\frac{\Delta x}{x - x_1}\right)^2 + \frac{9}{4}\right]\exp\left[-\frac{\Delta x}{2(x - x_1)}\right]\right| < \frac{5}{\Delta x}, \quad \text{(A22)}$$

where we use $0 < w < 1$ and the fact that, in the range $x_1 < x < x_1 + \frac{\Delta x}{3}$,

$$\exp\left[-\frac{\Delta x}{2(x - x_1)} - \frac{\Delta x}{x - x_1 - \Delta x}\right] < 1 \quad \text{(A23)}$$

holds. In the range $x_1 + \frac{2\Delta x}{3} < x < x_1 + \Delta x$, we can also show $\frac{dw}{dx} = \mathcal{O}\left(\Delta x^{-1}\right)$ in a similar way. For the case with $x_1 + \frac{\Delta x}{3} \le x \le x_1 + \frac{2\Delta x}{3}$, using $0 < w < 1$ and

$$(0<)\left(\frac{\Delta x}{x - x_1}\right)^2 + \left(\frac{\Delta x}{x - x_1 - \Delta x}\right)^2 \le 9 + 9 = 18, \quad \text{(A24)}$$

we can show $\frac{dw}{dx} = \mathcal{O}\left(\Delta x^{-1}\right)$. Therefore, in all the range $x_1 < x < x_1 + \Delta x$, $\frac{dw}{dx} = \mathcal{O}\left(\Delta x^{-1}\right)$ holds. Similarly, we can also show $\frac{d^2w}{dx^2} = \mathcal{O}\left(\Delta x^{-2}\right)$.

Taking the range $x_1 < x < x_1 + \Delta x$ to be short enough, that is, $\Delta x$ to be sufficiently small, in the neighborhood of $x = x_1$, $f_3$ is bounded as

$$f_3 = \frac{1}{2}\left(\frac{d^2}{dx^2}f_3\right)\bigg|_{x=x_1}(x - x_1)^2 + \mathcal{O}\left((x - x_1)^3\right). \quad \text{(A25)}$$

Suppose that the absolute value of $\left(\frac{d^2}{dx^2}f_3\right)\Big|_{x=x_1}$ takes a small one $\varepsilon$ but $\Delta x$ is set to be smaller than it. Then, in the range $x_1 < x < x_1 + \Delta x$, we can estimate $f$ and its derivatives as

$$|f| = |f_3 w| < \varepsilon\mathcal{O}\left(\Delta x^2\right)\mathcal{O}\left(\Delta x^0\right) = \varepsilon\mathcal{O}\left(\Delta x^2\right), \quad \text{(A26)}$$

$$\left|\frac{d}{dx}f\right| = \left|\left(\frac{d}{dx}f_3\right)w + f_3\left(\frac{d}{dx}w\right)\right| < \varepsilon\mathcal{O}\left(\delta x^1\right)\mathcal{O}\left(\Delta x^0\right) + \varepsilon\mathcal{O}\left(\delta x^2\right)\mathcal{O}\left(\Delta x^{-1}\right) = \varepsilon\mathcal{O}\left(\Delta x\right), \quad \text{(A27)}$$

$$\left|\frac{d^2}{dx^2}f\right| = \left|\left(\frac{d^2}{dx^2}f_3\right)w + 2\left(\frac{d}{dx}f_3\right)\left(\frac{d}{dx}w\right) + f_3\left(\frac{d^2}{dx^2}w\right)\right|$$
$$< \varepsilon\mathcal{O}\left(\Delta x^0\right) + \varepsilon\mathcal{O}\left(\delta x^1\right)\mathcal{O}\left(\Delta x^{-1}\right) + \varepsilon\mathcal{O}\left(\delta x^2\right)\mathcal{O}\left(\Delta x^{-2}\right) = \varepsilon\mathcal{O}\left(\Delta x^0\right). \quad \text{(A28)}$$

Therefore, the smooth gluing is done, keeping $f$, $\frac{d}{dx}f$, and $\frac{d^2}{dx^2}f$ small.

### *A.2. Smoothing $F_1$ in $-\check{\epsilon}^2 < x < 0$*

In this subsection, we construct a smooth gluing of $F_1$ from $x$ at $x = 0$ to 0 at $x = -\check{\epsilon}^2$. As discussed in the beginning of this appendix, $F_1$ should satisfy Eq. (A1). Moreover, if a $C^2$ function $\tilde{F}_1$ satisfies Eq. (A1), in the method that we have shown in Sect. A.1, a $C^2$ function $F_1$ can be constructed with the difference up to the second-order derivatives being as small as possible. However, in Eq. (A1) the upper bound is exactly 2, and thus at the point with $\tilde{F}_1'' = 2$, we cannot use the method of Sect. A.1 because the deviation of function by the smoothing shown in Sect. A.1 is tiny but nonzero.

Now, we introduce $\tilde{F}_1$ as

$$\tilde{F}_1 = \begin{cases} x & (-\gamma_1 < x < 0) \\ -\sqrt{F(x)} & (x_0 < x < -\gamma_1) \\ -\sqrt{g_1(x)} & (-A_1 < x < x_0) \\ -\sqrt{B_1} & (-2\epsilon_1 < x < -A_1) \\ -\sqrt{B_1} w|_{x_1 = -3\epsilon_1,\, \Delta x = \epsilon_1} & (-3\epsilon_1 < x < -2\epsilon_1) \\ 0 & (-4\epsilon_1 < x < -3\epsilon_1) \end{cases} \tag{A29}$$

where

$$\begin{aligned} &F(x) := x^2 - \alpha_1 \exp\left(X(x)\right), \qquad g_1(x) = -\epsilon_1 (x + A_1)^2 + B_1, \\ &X(x) := \frac{\beta_1}{x + \gamma_1}, \qquad x_0 := \frac{\beta_1}{X_0} - \gamma_1, \qquad X_0 := -(3 + \sqrt{3}), \\ &\alpha_1 := \frac{2(1+\epsilon_1)\beta_1^2}{X_0^3 (2 + X_0) \exp(X_0)}, \qquad \beta_1 := \epsilon_1^3, \qquad \gamma_1 := \epsilon_1^2, \qquad \epsilon_1 = \frac{1}{4}\check{\epsilon}^4, \\ &A_1 := \frac{1+\epsilon_1}{\epsilon_1}\left(\gamma_1 - \frac{3 + X_0}{X_0(2 + X_0)}\beta_1\right) =: \frac{1+\epsilon_1}{\epsilon_1}\hat{\gamma}_1, \\ &B_1 := \frac{1+\epsilon_1}{\epsilon_1}\hat{\gamma}_1^2 - (1+\epsilon_1)\frac{4 + X_0}{X_0^3 (2 + X_0)^2}\beta_1^2. \end{aligned} \tag{A30}$$

Then, one can confirm that $\tilde{F}_1$ is $C^2$-class and satisfies Eq. (A1) everywhere in the range $(-\check{\epsilon}^4 =) - 4\epsilon_1 < x < 0$. At the gluing points of smooth segments in Eq. (A29) except $x = -\gamma_1$, the second derivative of $F_1$ is not equal to 2, but at $x = -\gamma_1$ it is. Therefore, we cannot use the method shown in Sect. A.1 only at $x = -\gamma_1$. However, $\tilde{F}_1$ is smooth at this point. Let us show it. Since $\tilde{F}_1$ is strictly negative at $x = -\gamma_1$, if $F_1^2$ is smooth, $F_1$ is also. We investigate

$$\tilde{F}_1^2 - x^2 = \begin{cases} 0 & (-\gamma_1 < x < 0) \\ -\alpha_1 \exp\left(X(x)\right) = -\alpha_1 \exp\left(\frac{\beta_1}{x+\gamma_1}\right) & (x_0 < x < -\gamma_1)\,. \end{cases} \tag{A31}$$

This function is well known to be smooth at $x = -\gamma_1$.

As a result, we can construct a smooth function $F_1$ constructed by smoothing $\tilde{F}_1$ satisfying Eq. (A1). The smoothing is done within the range $-\check{\epsilon}^2 < x < 0$.

### *A.3. Smoothing $F_2$ in $-\check{\epsilon} < x < -\check{\epsilon}^2$*

Let us consider a function $\tilde{F}_2$ defined as

$$\tilde{F}_2 = \begin{cases} x & (-\gamma_2 < x < 0) \\ -\left(\tilde{F}(x)\right)^{\frac{4}{7}} & (\tilde{x}_0 < x < -\gamma_2) \\ -\left(g_2(x)\right)^{\frac{4}{7}} & (-A_2 < x < \tilde{x}_0) \\ -B_2^{\frac{4}{7}} & (-\frac{1}{2}\sqrt{\epsilon_2} < x < -A_2) \\ -B_2^{\frac{4}{7}} w|_{x_1=-\sqrt{\epsilon_2},\, \Delta x=\frac{1}{2}\sqrt{\epsilon_2}} & (-\sqrt{\epsilon_2} < x < -\frac{1}{2}\sqrt{\epsilon_2}) \\ 0 & (x < -\sqrt{\epsilon_2}) \end{cases} \tag{A32}$$

where

$$\begin{aligned} &\tilde{F}(x) := (-x)^{\frac{7}{4}} - \alpha_2 \exp\left(\tilde{X}(x)\right), \qquad g_2(x) = (-\tilde{x}_0)^{-\frac{1}{4}}\left[-\epsilon_2(x+A_2)^2 + B_2\right], \\ &\tilde{X}(x) := \frac{\beta_2}{x+\gamma_2}, \qquad \tilde{x}_0 := \frac{\beta_2}{\tilde{X}_0} - \gamma_2, \qquad \tilde{X}_0 := -(3+\sqrt{3}), \\ &\alpha_2 := (-\tilde{x}_0)^{-\frac{1}{4}} \frac{\frac{21}{16} + 2\epsilon_2}{\tilde{X}_0^3(2+\tilde{X}_0)\exp(\tilde{X}_0)}\beta_2^2, \quad \beta_2 := \epsilon_2^3, \quad \gamma_2 := \epsilon_2^2, \quad \epsilon_2 = \check{\epsilon}^2, \\ &A_2 = \frac{1}{2\epsilon_2}\left[\left(\frac{7}{4} + 2\epsilon_2\right)\gamma_2 - \frac{\frac{7}{16}(11+4\tilde{X}_0) + 2\epsilon_2(3+\tilde{X}_0)}{\tilde{X}_0(2+\tilde{X}_0)}\beta_2\right], \\ &B_2 = \epsilon_2(\tilde{x}_0 + A_2)^2 + \tilde{x}_0^2 - \frac{\frac{21}{16} + 2\epsilon_2}{\tilde{X}_0^3(2+\tilde{X}_0)}\beta_2^2. \end{aligned} \tag{A33}$$

Then, one can confirm that $\tilde{F}_2$ is $C^2$-class and satisfies Eq. (A2). In a similar discussion to Sect. A.2, smoothing of $\tilde{F}_2$ is done and we obtain a smooth function satisfying Eq. (A2), which is done in the range $-\check{\epsilon} < x < -\check{\epsilon}^2$.

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