

# Non-commutativity in gravity, topological gravity and cosmology

## Hugo García Compeán

Departamento de Física, Centro de Investigación y de Estudios Avanzados del IPN  
P.O. Box 14-740, 07000 México D.F., México

E-mail: [compean@fis.cinvestav.mx](mailto:compean@fis.cinvestav.mx)

## Octavio Obregón

Instituto de Física de la Universidad de Guanajuato  
P.O. Box E-143, 37150 León Gto., México

E-mail: [octavio@fisica.ugto.mx](mailto:octavio@fisica.ugto.mx)

## C. Ramírez

Facultad de Ciencias Físico Matemáticas

E-mail: [cramirez@fcfm.buap.mx](mailto:cramirez@fcfm.buap.mx)

## Miguel Sabido

Instituto de Física de la Universidad de Guanajuato  
P.O. Box E-143, 37150 León Gto., México

E-mail: [msabido@fisica.ugto.mx](mailto:msabido@fisica.ugto.mx)

**Abstract.** In this work we present different results obtained, in analyzing several aspects of gravity in non commutative space-time. We obtain generalized Euler and Pontrgajin topological invariants, and argue the possibility of defining new topological invariants in the same manner. Also a non commutative self dual gravity is constructed. Finally a proposal of non commutative quantum cosmology is also given.

## 1. Introduction

The idea of the noncommutative nature of space-time coordinates is quite old [1]. Many authors have extensively studied it from the mathematical [2], as well as field theoretic points of view (for a review, see for instance [3, 4]).

Recently, noncommutative gauge theory has attracted a lot of attention. This is a consequence of the developments in M(atric) theory [5] and string theory [6]. In particular, Seiberg and Witten have found noncommutativity in the description of the low energy excitations of open strings in the presence of a NS constant background  $B$ -field, they have observed that, depending on the regularization scheme of the two dimensional correlation functions, ordinary

and noncommutative gauge fields can be induced from the same worldsheet action. Thus, this procedure tells us that there is a relation of the resulting theory of noncommutative gauge fields, deformed by the Moyal star-product, with a gauge theory in terms of usual commutative fields. This is the Seiberg-Witten map. In string theory, gravity and gauge theories are realized in very different ways. The gravitational interaction is associated with a massless mode of closed strings, while Yang-Mills theories are more naturally described in open strings or in heterotic string theory. Furthermore, as mentioned, noncommutative Yang-Mills theories should arise from string theory. Thus the question emerges, whether a noncommutative description of gravity would arise from it. This is a difficult question and it will not be addressed here. There are several approaches to noncommutative gravity [7, 8], in this work we study different aspects of noncommutative gravity, in a similar manner as is done in noncommutative Yang Mills theory. First we present Topological Gravity, and its noncommutative generalization, obtaining noncommutative versions of the Euler and Signature topological terms. Then we do a noncommutative version of self-dual gravity, and finally we present a different approach of noncommutativity and apply it to quantum cosmology.

## 2. Noncommutative Gauge Symmetry and the Seiberg-Witten Map

In this section we review the basic properties of noncommutative field theory [3]. Noncommutative spaces can be understood as generalizations of the usual quantum mechanical commutation relations

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \quad (1)$$

where  $\hat{x}^\mu$  are linear operators acting on the Hilbert space  $L^2(\mathbf{R}^n)$  and  $\theta^{\mu\nu} = -\theta^{\nu\mu}$  are real numbers. The Weyl-Wigner-Moyal correspondence establishes (under certain conditions) an isomorphic relation between  $\mathcal{A}$  and the algebra of functions on  $\mathbf{R}^n$ , the last with an associative and noncommutative  $\star$ -product, the Moyal product, given by

$$f(x) \star g(x) \equiv \left[ \exp \left( \frac{i}{2} \theta^{\mu\nu} \frac{\partial}{\partial \varepsilon^\mu} \frac{\partial}{\partial \eta^\nu} \right) f(x + \varepsilon) g(x + \eta) \right]_{\varepsilon=\eta=0}. \quad (2)$$

In order to avoid causality problems we will take  $\theta^{0\nu} = 0$ .

For nonabelian groups, we must include also matrix multiplication, so a  $\star$ -product will be used as the matrix multiplication with  $\star$ -product. Inside integrals, this product has the property  $\text{Tr} \int f_1 \star f_2 \star f_3 \star \dots \star f_n = \text{Tr} \int f_n \star f_1 \star f_2 \star f_3 \star \dots \star f_{n-1}$ .

Let us consider a gauge theory with a hermitian connection, invariant under a symmetry Lie group  $G$ , with gauge fields  $A_\mu$ ,

$$\delta_\lambda A_\mu = \partial_\mu \lambda + i[\lambda, A_\mu], \quad (3)$$

where  $\lambda = \lambda^i T_i$ , and  $T_i$  are the generators of the Lie algebra  $\mathcal{G}$  of the group  $G$ , in the adjoint representation. These transformations are generalized for the noncommutative theory as,

$$\delta_\lambda \hat{A}_\mu = \partial_\mu \hat{\Lambda} + i[\hat{\Lambda} \star, \hat{A}_\mu], \quad (4)$$

where the noncommutative parameters  $\hat{\Lambda}$  have some dependence on  $\lambda$  and the connection  $A$ . The commutators  $[A \star, B] \equiv A \star B - B \star A$  have the correct derivative properties when acting on products of noncommutative fields.

Due to noncommutativity, commutators like  $[\hat{\Lambda} \star, \hat{A}_\mu]$  take values in the enveloping algebra of  $\mathcal{G}$  in the adjoint representation,  $\mathcal{U}(\mathcal{G}, \text{ad})$ . Therefore,  $\hat{\Lambda}$  and the gauge fields  $\hat{A}_\mu$  will also take values in this algebra. In general, for some representation  $R$ , we will denote  $\mathcal{U}(\mathcal{G}, R)$  the

corresponding section of the enveloping algebra  $\mathcal{U}(\mathcal{G})$ . Let us write for instance  $\widehat{\Lambda} = \widehat{\Lambda}^I T_I$  and  $\widehat{A} = \widehat{A}^I T_I$ , then,

$$[\widehat{\Lambda} * \widehat{A}_\mu] = \{\widehat{\Lambda}^I * \widehat{A}_\mu^J\} [T_I, T_J] + [\widehat{\Lambda}^I * \widehat{A}_\mu^J] \{T_I, T_J\}, \quad (5)$$

where  $\{A * B\} \equiv A * B + B * A$  is the noncommutative anti-commutator. Thus all the products of the generators  $T_I$  will be needed in order to close the algebra  $\mathcal{U}(\mathcal{G}, \text{ad})$ . Its structure can be obtained by successive computation of commutators and anti-commutators starting from the generators of  $\mathcal{G}$ , until it closes,

$$[T_I, T_J] = i f_{IJ}{}^K T_K, \quad \{T_I, T_J\} = d_{IJ}{}^K T_K. \quad (6)$$

The field strength is defined as  $\widehat{F}_{\mu\nu} = \partial_\mu \widehat{A}_\nu - \partial_\nu \widehat{A}_\mu - i[\widehat{A}_\mu * \widehat{A}_\nu]$ , hence it takes also values in  $\mathcal{U}(\mathcal{G}, \text{ad})$ . From Eq. (4) it turns out that,

$$\delta_\lambda \widehat{F}_{\mu\nu} = i \left( \widehat{\Lambda} * \widehat{F}_{\mu\nu} - \widehat{F}_{\mu\nu} * \widehat{\Lambda} \right). \quad (7)$$

We see that these transformation rules can be obtained from the commutative ones, just by replacing the ordinary product of smooth functions by the Moyal product, with a suitable product ordering. This allows constructing in a simple way invariant quantities.

The fact that the observed world is (up to the present experimental evidence) commutative, means that there must be possible to obtain it from the noncommutative one by taking the limit  $\theta \rightarrow 0$ . Thus the noncommutative fields  $\widehat{A}$  are given by a power series expansion on  $\theta$ , starting from the commutative ones  $A$ ,

$$\widehat{A} = A + \theta^{\mu\nu} A_{\mu\nu}^{(1)} + \theta^{\mu\nu} \theta^{\rho\sigma} A_{\mu\nu\rho\sigma}^{(2)} + \dots \quad (8)$$

The terms of this expansion are determined by the Seiberg-Witten map, which states that the symmetry transformations of (8), given by (4), are induced by the symmetry transformations of the commutative fields (3). In order that these transformations be consistent, the transformation parameter  $\widehat{\Lambda}$  must satisfy [10],

$$\delta_\lambda \widehat{\Lambda}(\eta) - \delta_\eta \widehat{\Lambda}(\lambda) - i[\widehat{\Lambda}(\lambda) * \widehat{\Lambda}(\eta)] = \widehat{\Lambda}(-i[\lambda, \eta]). \quad (9)$$

Similarly, the terms in Eq. (8) are functions of the commutative fields and their derivatives, and are determined by the requirement that  $\widehat{A}$  transforms as (4).

In order to obtain the Seiberg-Witten map to first order, the noncommutative parameters are first obtained from Eq. (9) [6, 9, 10, 11],

$$\widehat{\Lambda}(\lambda, A) = \lambda + \frac{1}{4} \theta^{\mu\nu} \{\partial_\mu \lambda, A_\nu\} + \mathcal{O}(\theta^2). \quad (10)$$

Then, from Eqs. (4) and (8), the following solution is given

$$\widehat{A}_\mu(A) = A_\mu - \frac{1}{4} \theta^{\rho\sigma} \{A_\rho, \partial_\sigma A_\mu + F_{\sigma\mu}\} + \mathcal{O}(\theta^2), \quad (11)$$

and for the field strength it turns out that,

$$\widehat{F}_{\mu\nu} = F_{\mu\nu} + \frac{1}{4} \theta^{\rho\sigma} \left( 2 \{F_{\mu\rho}, F_{\nu\sigma}\} - \{A_\rho, D_\sigma F_{\mu\nu} + \partial_\sigma F_{\mu\nu}\} \right) + \mathcal{O}(\theta^2). \quad (12)$$

The higher terms in Eq. (8) can be obtained from the observation that the Seiberg-Witten map preserves the operations of the commutative function algebra, hence the following differential equation can be written [6],

$$\delta\theta^{\mu\nu} \frac{\partial}{\partial\theta^{\mu\nu}} \widehat{A}(\theta) = \delta\theta^{\mu\nu} \widehat{A}_{\mu\nu}^{(1)}(\theta), \quad (13)$$

where  $\widehat{A}_{\mu\nu}^{(1)}$  is obtained from  $A_{\mu\nu}^{(1)}$  in Eq. (8), by substituting the commutative fields by the noncommutative ones under the  $*$ -product.

If we have a group with real parameters and hermitian generators, with a hermitian connection, then the noncommutative connection and the noncommutative field strength will be also hermitian.

### 3. Noncommutative Topological Gravity

In this section we present a noncommutative formulation of topological gravity [12]. We briefly review four-dimensional topological gravity. We start from the following  $SO(3,1)$  invariant action

$$I_{TOP} = \frac{\Theta_G^P}{2\pi} \text{Tr} \int_X R \wedge R + i \frac{\Theta_G^E}{2\pi} \text{Tr} \int_X R \wedge \tilde{R}, \quad (14)$$

where  $R$  is the field strength, corresponding to a  $SO(3,1)$  connection  $\omega$

$$R_{\mu\nu}{}^{ab} = \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} + \omega_\mu{}^{ac} \omega_\nu{}^b{}_c - \omega_\mu{}^{bc} \omega_\nu{}^a{}_c, \quad (15)$$

$X$  is a four dimensional closed pseudo-Riemannian manifold and  $\tilde{R}_{\mu\nu}{}^{ab} = -\frac{i}{2} \varepsilon^{abcd} R_{\mu\nu}{}^{cd}$  is the dual with respect to the group.

In this action, the connection satisfies the first Cartan structure equation, which relates it to a given tetrad. This action can be written as the integral of a divergence, and a variation of it with respect to the tetrad vanishes, hence it is metric independent, and therefore topological.

The action (14) can be rewritten in terms of the self-dual and anti-self-dual parts (as in [13]),  $R^\pm = \frac{1}{2}(R \pm \tilde{R})$ , of the Riemann tensor as follows:

$$I_{TOP} = \text{Tr} \int_X (\tau R^+ \wedge R^+ + \bar{\tau} R^- \wedge R^-) = \text{Tr} \int_X (\tau R^+ \wedge R^+ + \bar{\tau} \overline{R^+} \wedge \overline{R^+}), \quad (16)$$

where  $\tau = \left(\frac{1}{2\pi}\right) (\Theta_G^E + i\Theta_G^P)$ , and the bar denotes complex conjugation. In local coordinates on  $X$ , this action can be rewritten as

$$I_{TOP} = 2\text{Re} \left( \tau \int_X d^4x \varepsilon^{\mu\nu\rho\sigma} R_{\mu\nu}{}^{ab} R_{\rho\sigma ab}^+ \right). \quad (17)$$

Therefore, it is enough to study the complex action,

$$I = \int_X d^4x \varepsilon^{\mu\nu\rho\sigma} R_{\mu\nu}{}^{ab} R_{\rho\sigma ab}^+, \quad (18)$$

after some manipulations it can be written as

$$I = \text{Tr} \int_X d^4x \varepsilon^{\mu\nu\rho\sigma} \mathcal{R}_{\mu\nu}(\omega) \mathcal{R}_{\rho\sigma}(\omega), \quad (19)$$

where,  $\mathcal{R}_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu - i[\omega_\mu, \omega_\nu]$  is the field strength. This action is invariant under the  $SL(2, \mathbf{C})$  transformations,  $\delta_\lambda \omega_\mu = \partial_\mu \lambda + i[\lambda, \omega_\mu]$ .

In the case of a Riemannian manifold  $X$ , the signature and the Euler topological invariants of  $X$ , are the real and imaginary parts of (19)

$$\sigma(X) = -\frac{1}{24\pi^2} \text{Re} \left( \text{Tr} \int_X d^4x \varepsilon^{\mu\nu\rho\sigma} \mathcal{R}_{\mu\nu}(\omega) \mathcal{R}_{\rho\sigma}(\omega) \right), \quad (20)$$

$$\chi(X) = \frac{1}{32\pi^2} \text{Im} \left( \text{Tr} \int_X d^4x \varepsilon^{\mu\nu\rho\sigma} \mathcal{R}_{\mu\nu}(\omega) \mathcal{R}_{\rho\sigma}(\omega) \right). \quad (21)$$

We wish to have a noncommutative formulation of the  $SO(3,1)$  action (14). Its first term, can be straightforwardly made noncommutative, in the same way as for usual Yang-Mills theory,

$$\text{Tr} \int_X \widehat{R} \wedge \widehat{R}. \quad (22)$$

Instead, for the second term of (14) such an action cannot be written, because it involves the Levi-Civita symbol, an invariant Lorentz tensor, but which is not invariant under the full enveloping algebra. However, as mentioned at the end of the preceding section, this term can be obtained from Eq. (19).

Thus, in general we will consider as the noncommutative topological action of gravity, the  $SL(2, \mathbf{C})$  invariant action,

$$\widehat{I} = \text{Tr} \int_X d^4x \varepsilon^{\mu\nu\rho\sigma} \widehat{\mathcal{R}}_{\mu\nu} \widehat{\mathcal{R}}_{\rho\sigma}, \quad (23)$$

where  $\widehat{\mathcal{R}}_{\mu\nu} = \partial_\mu \widehat{\omega}_\nu - \partial_\nu \widehat{\omega}_\mu - i[\widehat{\omega}_\mu * \widehat{\omega}_\nu]$ , is the  $SL(2, \mathbf{C})$  noncommutative field strength. This action does not depend on the metric of  $X$ . Indeed, as well as the commutative one, it is given by a divergence,

$$\widehat{I} = \text{Tr} \int_X d^4x \varepsilon^{\mu\nu\rho\sigma} \partial_\mu \left( \widehat{\omega}_\nu * \partial_\rho \widehat{\omega}_\sigma + \frac{2}{3} \widehat{\omega}_\nu * \widehat{\omega}_\rho * \widehat{\omega}_\sigma \right). \quad (24)$$

Thus, a variation of (23) with respect to the noncommutative connection, will vanish identically because of the noncommutative Bianchi identities,

$$\delta_{\widehat{\omega}} \widehat{I} = 8 \text{Tr} \int \varepsilon^{\mu\nu\rho\sigma} \delta \widehat{\omega}_\mu * \widehat{D}_\mu \widehat{\mathcal{R}}_{\rho\sigma} \equiv 0, \quad (25)$$

where  $\widehat{D}_\mu$  is the noncommutative covariant derivative.

Further, from the first Cartan structure equation, the  $SO(3,1)$  connection, and thus its  $SL(2, \mathbf{C})$  projection  $\omega_\mu^i$ , can be written in terms of the tetrad and the torsion. Furthermore, from the Seiberg-Witten map, the noncommutative connection can be written as well as  $\widehat{\omega}(e)$ . Therefore, a variation of the action (23) with respect to the tetrad of the action, can be written as

$$\delta_e \widehat{I} = 8 \text{Tr} \int \varepsilon^{\mu\nu\rho\sigma} \delta_e \widehat{\omega}_\mu(e) * \widehat{D}_\mu \widehat{\mathcal{R}}_{\rho\sigma} \equiv 0, \quad (26)$$

hence it is topological, as the commutative one.

Thus, we see from (23) that, in a  $\theta$ -power expansion of the action, each one of the resulting terms will be independent of the metric, as well as they will be given by a divergence. Thus, unless these terms vanish identically, they will be topological. Furthermore, the whole noncommutative action, expressed in terms of the commutative fields by the Seiberg-Witten map, is invariant under the  $SO(3,1)$  transformations. Thus, each term of the expansion will be also invariant. Thus these terms will be topological invariants.

Action (23) is not real, as well as the limiting commutative action. Hence, it is not obvious that the signature (22) will be precisely its real part. In this case we could neither say that  $\widehat{\chi}(X)$  is given by its imaginary part. In fact we can only say that  $\widehat{\chi}(X)$  could be obtained from the difference of (23) and (22). However, the real and the imaginary parts of (25) are invariant under  $SL(2, \mathbf{C})$  and consequently under  $SO(3,1)$ , and thus they are the natural candidates for  $\widehat{\sigma}(X)$  and  $\widehat{\chi}(X)$ , as in (20) and (21).

In order to write down the  $\theta$  expansion of these noncommutative actions, we will take the fundamental representation of  $SL(2, \mathbf{C})$ , with generators given by the Pauli matrices. In this case, to second order in  $\theta$ , the Seiberg-Witten map for the Lie algebra valued commutative field strength  $\mathcal{R}_{\mu\nu} = \mathcal{R}_{\mu\nu}^i(\omega) \sigma_i$ , is given by

$$\widehat{\mathcal{R}}_{\mu\nu} = \mathcal{R}_{\mu\nu} + \theta^{\alpha\beta} \mathcal{R}_{\mu\nu\alpha\beta}^{(1)} + \theta^{\alpha\beta} \theta^{\gamma\delta} \mathcal{R}_{\mu\nu\alpha\beta\gamma\delta}^{(2)} + \dots, \quad (27)$$

therefore the action (23) will be given by,

$$\begin{aligned}\hat{I} &= \text{Tr} \int_X d^4x \varepsilon^{\mu\nu\rho\sigma} \left[ \mathcal{R}_{\mu\nu} \mathcal{R}_{\rho\sigma} + 2\theta^{\tau\vartheta} \mathcal{R}_{\mu\nu} \mathcal{R}_{\rho\sigma\tau\vartheta}^{(1)} \right. \\ &\quad \left. + \theta^{\tau\theta} \theta^{\vartheta\zeta} \left( 2\mathcal{R}_{\mu\nu} \mathcal{R}_{\rho\sigma\tau\theta\vartheta\zeta}^{(2)} + \mathcal{R}_{\mu\nu\tau\theta}^{(1)} \mathcal{R}_{\rho\sigma\vartheta\zeta}^{(1)} \right) \right] + \mathcal{O}(\theta^3).\end{aligned}\quad (28)$$

Taking into account (28) and calculating  $\mathcal{R}^{(1)}$  and  $\mathcal{R}^{(2)}$  using (12) we finally get,

$$\begin{aligned}\hat{I} &= \int_X d^4x \varepsilon^{\mu\nu\rho\sigma} \left\{ 2\mathcal{R}_{\mu\nu}^i \mathcal{R}_{\rho\sigma i} + \frac{1}{4} \theta^{\tau\theta} \theta^{\vartheta\zeta} \left[ -\varepsilon_{ijk} \mathcal{R}_{\mu\nu}^i \left[ \partial_{\vartheta} \mathcal{R}_{\rho\tau}^j \partial_{\zeta} \mathcal{R}_{\sigma\theta}^k \right. \right. \right. \\ &\quad \left. \left. - \partial_{\vartheta} \omega_{\tau}^j \partial_{\zeta} (\partial_{\theta} + D_{\theta}) \mathcal{R}_{\rho\sigma}^k \right] + [\mathcal{R}_{\mu\tau}^i \mathcal{R}_{\nu\theta i} - \frac{1}{2} \omega_{\tau}^i (\partial_{\theta} + D_{\theta}) \mathcal{R}_{i\mu\nu}] [\mathcal{R}_{\rho\vartheta}^j \mathcal{R}_{\sigma\zeta j} \right. \right. \\ &\quad \left. \left. - \frac{1}{2} \omega_{\vartheta}^j (\partial_{\zeta} + D_{\zeta}) \mathcal{R}_{\rho\sigma j} \right] + \mathcal{R}_{\mu\nu}^i \{ \mathcal{R}_{i\sigma\theta} [2\mathcal{R}_{\rho\vartheta}^j \mathcal{R}_{\tau\zeta j} - \omega_{\vartheta}^j (\partial_{\zeta} + D_{\zeta}) \mathcal{R}_{\rho\tau j}] \right. \\ &\quad \left. + \frac{1}{4} (\partial_{\theta} + D_{\theta}) \mathcal{R}_{\rho\sigma i} \omega_{\vartheta}^j (\partial_{\zeta} \omega_{\tau j} + \mathcal{R}_{\zeta\tau j}) + \omega_{\theta i} [\partial_{\tau} (\mathcal{R}_{\rho\vartheta}^j \mathcal{R}_{\sigma\zeta j}) - \frac{1}{2} \partial_{\tau} \omega_{\vartheta}^j (\partial_{\zeta} \right. \\ &\quad \left. + D_{\zeta}) \mathcal{R}_{\rho\sigma j}] \} - \frac{1}{2} \mathcal{R}_{\mu\nu}^i \omega_{\tau i} \partial_{\vartheta} \omega_{\theta}^j \partial_{\zeta} \mathcal{R}_{\rho\sigma j} \right\} + \mathcal{O}(\theta^4),\end{aligned}\quad (29)$$

where the second order correction does not identically vanish because  $\mathcal{R}^{(2)}$  is proportional to  $\sigma_i$ . Finally it seems that all its odd order terms vanish.

We could apply this procedure and deform other topological invariants, and see if they are noncommutative topological invariants.

#### 4. Noncommutative Self-dual Gravity

In this section we present a noncommutative version of gravity, we start by writing down, self dual gravity and finally writing down the noncommutative version [15]. Let us take the self-dual SO(3,1) BF action, defined on a (3 + 1)-dimensional pseudo-riemannian manifold  $(X, g_{\mu\nu})$ ,

$$I = i \text{Tr} \int_X \Sigma^+ \wedge R^+ = i \int_X \varepsilon^{\mu\nu\rho\sigma} \Sigma_{\mu\nu}^{+ab} R_{\rho\sigma ab}^+(\omega) d^4x, \quad (30)$$

where  $R_{\rho\sigma ab}^+ = \Pi_{ab}^{+cd} R_{\rho\sigma cd}$ , is the self-dual SO(3,1) field strength tensor. This action can be rewritten as

$$I = \frac{1}{2} \int_X \varepsilon^{\mu\nu\rho\sigma} \left( i \Sigma_{\mu\nu}^{ab} R_{\rho\sigma ab} + \frac{1}{2} \varepsilon_{abcd} \Sigma_{\mu\nu}^{ab} R_{\rho\sigma}^{cd} \right) d^4x. \quad (31)$$

If we take the solution of the constraints on  $\Sigma$ , which we can write as

$$\Sigma_{\mu\nu}^{ab} = e_{\mu}^a e_{\nu}^b - e_{\mu}^b e_{\nu}^a, \quad (32)$$

then

$$I = \int_X (\det e R + i \varepsilon^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}) d^4x. \quad (33)$$

The real and imaginary parts of this action must be varied independently because the fields are real. The first part represents Einstein action in the Palatini formalism, from which, after variation of the Lorentz connection, a vanishing torsion  $T_{\mu\nu}^a = 0$  turns out. As a consequence, the second term vanishes due to Bianchi identities.

From the decomposition  $\text{SO}(3,1) = \text{SL}(2, \mathbf{C}) \times \text{SL}(2, \mathbf{C})$ , it turns out that  $\omega_{\mu}^i = \omega_{\mu}^{+0i}$  is a  $\text{SL}(2, \mathbf{C})$  connection. Further, if we take into account self-duality,  $\varepsilon_{cd}^{ab} \omega_{\mu}^{+cd} = 2i \omega_{\mu}^{+ab}$ , we get  $\omega_{\mu}^{+ij} = -i \varepsilon^{ij}_k \omega_{\mu}^k$ . Then the action (30) can be written as a  $\text{SL}(2, \mathbf{C})$  BF-action

$$I = i \int_X \varepsilon^{\mu\nu\rho\sigma} \left[ \Sigma_{\mu\nu}^{+0i} R_{\rho\sigma 0i}(\omega^+) + \Sigma_{\mu\nu}^{+ij} R_{\rho\sigma ij}(\omega^+) \right] d^4x = -4i \int_X \varepsilon^{\mu\nu\rho\sigma} \Sigma_{\mu\nu}^i R_{\rho\sigma i}(\omega) d^4x. \quad (34)$$

Therefore, if we choose the algebra  $sl(2, \mathbf{C})$  to satisfy  $[T_i, T_j] = -2\varepsilon_{ij}^k T_k$  and  $Tr(T_i T_j) = -2\delta_{ij}$ , we have that (30) can be rewritten as the self-dual action [14],

$$I = 2i \text{Tr} \int_X \Sigma \wedge R, \quad (35)$$

which is invariant under the  $SL(2, \mathbf{C})$  transformations  $\delta_\lambda \omega_\mu = \partial_\mu \lambda + i[\lambda, \omega_\mu]$  and  $\delta_\lambda \Sigma_{\mu\nu} = i[\lambda, \Sigma_{\mu\nu}]$ .

If the variation of this action with respect to the  $SL(2, \mathbf{C})$  connection  $\omega$  is set to zero, we get the equations

$$\Psi^{\mu i} = \varepsilon^{\mu\nu\rho\sigma} D_\nu \Sigma_{\rho\sigma}{}^i = \varepsilon^{\mu\nu\rho\sigma} \left( \partial_\nu \Sigma_{\rho\sigma}{}^i + 2i\varepsilon^i{}_{jk} \omega_\nu{}^j \Sigma_{\rho\sigma}{}^k \right) = 0, \quad (36)$$

which after putting in terms of the  $SO(3,1)$  connection and using (32), can be written as

$$\varepsilon^{\mu\nu\rho\sigma} (\partial_\nu e_\rho{}^a e_\sigma{}^b - \partial_\nu e_\rho{}^b e_\sigma{}^a + \omega_\nu{}^{ac} e_{\rho c} e_\sigma{}^b - \omega_\nu{}^{bc} e_{\rho c} e_\sigma{}^a) = \varepsilon^{\mu\nu\rho\sigma} (T_{\nu\rho}{}^a e_\sigma{}^b - T_{\nu\rho}{}^b e_\sigma{}^a) = 0. \quad (37)$$

From which the vanishing torsion condition once more turns out.

For the noncommutative case we start with  $SL(2, \mathbf{C})$  invariant action (35). From it, the noncommutative action can be obtained straightforwardly as

$$\hat{I} = 2i \text{Tr} \int_X \hat{\Sigma} \wedge \hat{R}. \quad (38)$$

This action is invariant under the noncommutative  $SL(2, \mathbf{C})$  transformations

$$\delta_{\hat{\lambda}} \hat{\omega}_\mu = \partial_\mu \hat{\lambda} + i[\hat{\lambda} * \hat{\omega}_\mu], \quad (39)$$

$$\delta_{\hat{\lambda}} \hat{\Sigma}_{\mu\nu} = i[\hat{\lambda} * \hat{\Sigma}_{\mu\nu}]. \quad (40)$$

Actually, in order to obtain the noncommutative generalization of the Einstein equation, we could consider the real part of (38),

$$\hat{I}_E = -i \text{Tr} \int_X \left[ \hat{\Sigma} \wedge \hat{R} - (\hat{\Sigma} \wedge \hat{R})^\dagger \right], \quad (41)$$

which is also invariant under (39) and (40).

In order to obtain the corresponding to the torsion condition, a  $\omega$  variation of (38) must be done. Thus we write,

$$\delta_\omega \hat{I} = 8i \text{Tr} \int_X \varepsilon^{\mu\nu\rho\sigma} \left( \partial_\rho \hat{\Sigma}_{\mu\nu} - i[\hat{\omega}_\rho * \hat{\Sigma}_{\mu\nu}] \right) * \delta \hat{\omega}_\sigma = 0, \quad (42)$$

from which we obtain the noncommutative version of (36)

$$\hat{\Psi}^\mu = \varepsilon^{\mu\nu\rho\sigma} \hat{D}_\nu \hat{\Sigma}_{\rho\sigma} = 0. \quad (43)$$

These equations are covariant under the noncommutative transformations (39) and (40), which means that their Seiberg-Witten expansion should be similar to the one of a matter field in the adjoint representation. Thus, we could expect that a solution to Eq. (43) would be given by the solution of the commutative equation  $\Psi^\mu = 0$ . Making use of the ambiguity of the Seiberg-Witten map, we make the following choice for  $\hat{\Sigma}$ ,

$$\hat{\Sigma}_{\mu\nu} = \Sigma_{\mu\nu} - \frac{1}{4} \theta^{\rho\sigma} (\{\omega_\rho, (D_\sigma + \partial_\sigma) \Sigma_{\mu\nu}\} - \{R_{\mu\nu}, \Sigma_{\rho\sigma}\}) + \mathcal{O}(\theta^2), \quad (44)$$

from which it turns out that,

$$\widehat{\Psi}^\mu = \Psi^\mu - \frac{1}{4}\theta^{\nu\rho} \left( \{\omega_\nu, (D_\rho + \partial_\rho)\Psi^\mu\} - \{R_{\nu\rho}, \Psi^\mu\} - 2\delta_\nu^\mu \varepsilon^{\sigma\tau\theta\zeta} D_\sigma \{R_{\tau\theta}, \Sigma_{\rho\zeta}\} \right) + \mathcal{O}(\theta^2). \quad (45)$$

Hence if the zeroth order terms vanish,  $\Psi^\mu = 0$ , then the first two terms in (45) will vanish. These equations  $\Psi^\mu = 0$  are equivalent to set the commutative torsion equal to zero, that is, after the substitution  $\Sigma_{\mu\nu}^{ab} = e_\mu^a e_\nu^b - e_\nu^a e_\mu^b$ , their solution is given by,

$$\omega_\mu^{ab} = -\frac{1}{2}e^{a\nu}e^{b\rho} \left[ e_{\mu c}(\partial_\nu e_\rho^c - \partial_\rho e_\nu^c) - e_{\nu c}(\partial_\rho e_\mu^c - \partial_\mu e_\rho^c) - e_{\rho c}(\partial_\mu e_\nu^c - \partial_\nu e_\mu^c) \right]. \quad (46)$$

Furthermore, at first order, a computation of the last term in (45) shows that it is proportional to  $\theta^{\mu\nu}\partial_\rho(e^{-1}G_\nu^\rho)$ , where  $G_{\mu\nu}$  is the Einstein tensor. If we now substitute (46) back into the action (41), the equations of motion to zeroth order will give the vanishing of the Einstein tensor, and the last term in (45) will be automatically fulfilled. With these in mind, the corrections to the noncommutative action (41) can be computed as follows. First we write the Seiberg-Witten expansion of the  $\text{SL}(2, \mathbf{C})$  fields  $\widehat{\Sigma}$  and  $\widehat{\omega}$ . Furthermore, the commutative  $\text{SL}(2, \mathbf{C})$  fields are written by means of the self-dual  $\text{SO}(3,1)$  fields,  $\omega_\mu^i = \omega_\mu^{+0i}$  and  $\Sigma_{\mu\nu}^i = \Sigma_{\mu\nu}^{+0i}$ . Then, decompose these self-dual fields into the real ones  $\omega_\mu^{ab}$  and  $\Sigma_{\mu\nu}^{ab}$ , and then substitute  $\Sigma_{\mu\nu}^{ab} = e_\mu^a e_\nu^b - e_\nu^a e_\mu^b$  and write the connection as in (46). In this case we will have a noncommutative action, which will depend only on the tetrad.

If we consider the real part, as in (41), the first order correction vanishes, and, after a lengthy calculation, the second order one turns out to be, already written in terms of commutative  $\text{SO}(3,1)$  fields,

$$\begin{aligned} \widehat{I}_{\theta^2} = & \frac{1}{2^4}\theta^{\gamma\delta}\theta^{\tau\xi} \int dx^4 \left\{ 4e \left[ 4R_\delta{}^\rho (R_{\rho\tau}{}^{ab} R_{\gamma\xi ab} - \omega_\tau^{ab} \partial_\xi R_{\rho\gamma ab}) + \omega_\gamma^{\rho\sigma} \partial_\tau \omega_\delta^{ab} \partial_\xi R_{\rho\sigma ab} \right. \right. \\ & + R\partial_\delta(\omega_\tau^{ab}(\partial_\xi \omega_{\gamma ab} + R_{\gamma\xi ab})) + 2\omega_\gamma^{\rho\sigma} \partial_\delta(R_{\rho\tau}{}^{ab} R_{\sigma\xi ab} - \omega_\tau^{ab} \partial_\xi R_{\rho\sigma ab}) \left. \right] \\ & + \epsilon^{\mu\nu\rho\sigma} \left\{ 4e \left[ \epsilon_{\gamma\delta\alpha\beta} R_{\rho\sigma}{}^{\alpha\beta} (R_{\mu\tau}{}^{ab} R_{\nu\xi ab} - \omega_\tau^{ab} \partial_\xi R_{\mu\nu ab}) + 2\epsilon_{\tau\xi\alpha\beta} R_{\mu\nu}{}^{ab} R_{\rho\sigma ab} R_{\gamma\delta}{}^{\alpha\beta} \right] \right. \\ & + \epsilon_{abcd} \left[ 4R_{\rho\sigma\gamma\delta} (R_{\mu\tau}{}^{ab} R_{\nu\xi}{}^{cd} - \omega_\tau^{ab} \partial_\xi R_{\mu\nu}{}^{cd}) + 4R_{\mu\nu}{}^{ab} R_{\rho\sigma}{}^{cd} (2R_{\gamma\delta\tau\xi} - \omega_{\tau ef} \partial_\xi (e_\gamma^e e_\delta^f)) \right. \\ & - 2\omega_{\gamma\mu\nu} \partial_\delta (R_{\rho\tau}{}^{ab} R_{\sigma\xi}{}^{cd} - \omega_\tau^{ab} \partial_\xi R_{\rho\sigma}{}^{cd}) - 2\omega_\gamma^{ef} R_{\rho\sigma ef} \partial_\delta (2R_{\mu\nu}{}^{ab} e_\tau^c e_\xi^d - \omega_\tau^{ab} \partial_\xi (e_\mu^c e_\nu^d)) \\ & - 2\omega_\gamma^{ab} R_{\rho\sigma}{}^{cd} \partial_\delta (2R_{\mu\nu\tau\xi} - \omega_{\tau ef} \partial_\xi (e_\mu^e e_\nu^f)) - \omega_{\gamma\mu\nu} \partial_\tau \omega_\delta^{ab} \partial_\xi R_{\rho\sigma}{}^{cd} - \omega_\gamma^{ef} R_{\rho\sigma ef} \partial_\tau \omega_\delta^{ab} \partial_\xi (e_\mu^c e_\nu^d) \\ & \left. \left. - \omega_\gamma^{ab} R_{\rho\sigma}{}^{cd} \partial_\tau \omega_{\delta ef} \partial_\xi (e_\mu^e e_\nu^f) - 4R_{\mu\nu}{}^{ef} R_{\rho\sigma ef} \omega_\tau^{ab} \partial_\xi (e_\gamma^c e_\delta^d) \right] \right\}, \quad (47) \end{aligned}$$

where the connection  $\omega_\mu^{ab}$  is given by (46).

## 5. Noncommutative Quantum Cosmology

In the previous section we have defined noncommutative gravity, unfortunately because of the complexity of the resulting action it is difficult to analyze even the simplest cosmological models. So in this section we do an alternate proposal [16]. The noncommutativity we propose can be reformulated in terms of a Moyal deformation of the Wheeler-DeWitt equation, similar to the case of the noncommutative Schrödinger equation [17]. As an example we will consider the cosmological model of the Kantowski-Sachs metric. In the parametrization due to Misner, this metric looks like:

$$ds^2 = -N^2 dt^2 + e^{2\sqrt{3}\beta} dr^2 + e^{-2\sqrt{3}\beta} e^{-2\sqrt{3}\Omega} (d\theta^2 + \sin^2\theta d\varphi^2). \quad (48)$$

The corresponding Wheeler-DeWitt equation, in a particular factor ordering, can be written as

$$\exp(\sqrt{3}\beta + 2\sqrt{3}\Omega)[-P_\Omega^2 + P_\beta^2 - 48\exp(-2\sqrt{3}\Omega)]\psi(\Omega, \beta) = 0, \quad (49)$$

where  $P_\Omega = -i\frac{\partial}{\partial\Omega}$  and  $P_\beta = -i\frac{\partial}{\partial\beta}$ . Thus, in this parametrization the Wheeler-DeWitt equation has a simple form, which can be formally identified with usual quantum mechanics in cartesian coordinates.

The solutions to this Wheeler-DeWitt equation are given by

$$\psi_\nu^\pm(\beta, \Omega) = e^{\pm i\nu\sqrt{3}\beta} K_{i\nu}(4e^{-\sqrt{3}\Omega}), \quad (50)$$

where  $K_{i\nu}$  is the modified Bessel function. Wave packets of these solutions have been constructed as superpositions of these solutions. Summing over  $e^{i\nu\sqrt{3}\beta}$  and  $e^{-i\nu\sqrt{3}\beta}$  to make real trigonometric functions, the ‘‘Gaussian’’ state

$$\begin{aligned} \Psi(\beta, \Omega) &= 2i\mathcal{N} \int_0^\infty \nu [\psi_\nu^+(\beta, \Omega) - \psi_\nu^-(\beta, \Omega)] d\nu \\ &= \mathcal{N} e^{-\sqrt{3}\Omega} \sinh(\sqrt{3}\beta) \exp[-2\sqrt{3}e^{-\sqrt{3}\Omega} \cosh(\sqrt{3}\beta)], \end{aligned} \quad (51)$$

has been obtained [18]. For the noncommutative case, we will assume that the ‘‘cartesian coordinates’’  $\Omega$  and  $\beta$  of the Kantowski-Sachs minisuperspace obey a kind of commutation relation, like the ones in noncommutative quantum mechanics [17]

$$[\Omega, \beta] = i\theta. \quad (52)$$

The relation with other minisuperspace coordinates would follow in a similar way as in standard spacetime.

As usual, this deformation can be reformulated in terms of a noncommutativity of minisuperspace functions, with the Moyal product, now our noncommutative Wheeler-DeWitt equation will be

$$\exp(\sqrt{3}\beta + 2\sqrt{3}\Omega) * [-P_\Omega^2 + P_\beta^2 - 48\exp(-2\sqrt{3}\Omega)] * \psi(\Omega, \beta) = 0. \quad (53)$$

Then, as is well known in noncommutative quantum mechanics [17], the original phase-space, as well as its symplectic structure, is modified. It is possible to reformulate it in terms of commutative variables and the ordinary product of functions, if new variables are introduced,  $\Omega \rightarrow \Omega - \frac{1}{2}\theta P_\beta$  and  $\beta \rightarrow \beta - \frac{1}{2}\theta P_\Omega$ , the momenta remain the same. As a consequence, the original equation changes, with a potential modified due to these new coordinates.

$$V(\Omega, \beta) * \psi(\Omega, \beta) = V(\Omega - \frac{1}{2}\theta P_\beta, \beta - \frac{1}{2}\theta P_\Omega) \psi(\Omega, \beta). \quad (54)$$

Thus, we get

$$\left[-\frac{\partial^2}{\partial\Omega^2} + \frac{\partial^2}{\partial\beta^2} + 48\exp(-2\sqrt{3}\Omega + \sqrt{3}\theta P_\beta)\right]\psi(\Omega, \beta) = 0. \quad (55)$$

Assuming a separation of variables with the ansatz

$$\psi(\Omega, \beta) = \exp(\sqrt{3}\nu\beta)\chi(\Omega), \quad (56)$$

we observe that the operator  $P_\beta$  in the exponential in (55) will shift the wave function by a factor

$$\psi(\Omega, \beta - i\sqrt{3}\theta) = \exp(-3i\nu\theta)\psi(\Omega, \beta), \quad (57)$$

thus  $\chi(\Omega)$  must satisfy the equation

$$\left[ -\frac{d^2}{d\Omega^2} + 48\exp(-3i\nu\theta)\exp(-2\sqrt{3}\Omega) + 3\nu^2 \right] \chi(\Omega) = 0. \quad (58)$$

This equation can be solved, in the same manner as in the commutative case. Therefore the wave function will be given by

$$\psi_{\nu}^{\pm}(\Omega, \beta) = e^{\pm i\sqrt{3}\nu\beta} K_{i\nu} \left\{ 4\exp \left[ -\sqrt{3} \left( \Omega \mp \frac{\sqrt{3}}{2}\nu\theta \right) \right] \right\}. \quad (59)$$

These are the solutions to the Wheeler-DeWitt equation (55). Note that noncommutativity induces a difference of the arguments of the Bessel functions in these solutions. Moreover, from its form, we can expect that the noncommutativity effects are enhanced for  $\psi^+$ . Thus, for the particular model we have chosen, the solution (59) allows an exact analysis, without the need of a  $\theta$  expansion.

## 6. Conclusions and Outlook

We have presented different approaches and results in noncommutative gravity in particular in relation to topological gravity, self-dual gravity, and quantum cosmology. This results can be applied to other gravitational systems. These kind of ideas are being applied in connection to BF theories [19], topological invariants, and black hole thermodynamics [20], with the hope of better understanding the quantum behavior.

## 7. Acknowledgments

This work is supported in part by CONACyT Grants Nos. 37851E and 33951E, and by projects of Universidad de Guanajuato and PROMEP.

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