Wormhole in higher-dimensional space-time

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Wormhole in higher-dimensional space-time

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Abstract. We introduce our recent studies on wormhole, especially its stability aspect in higher-dimensional space-time both in general relativity and in Gauss-Bonnet gravity. We derived the Ellis-type wormhole solution in n-dimensional general relativity, and found existence of an unstable mode in its linear perturbation analysis. We also evolved it numerically in dual-null coordinate system, and confirmed its instability. The wormhole throat will change into black hole horizons for the input of the (relatively) positive energy, while it will change into inflationary expansion for the (relatively) negative energy input. If we add Gauss-Bonnet terms (higher curvature correction terms in gravity), then wormhole tends to expand (or change to black hole) if the coupling constant α is positive (negative), and such bifurcation of the throat horizon is observed earlier in higher dimension.

1. Introduction

Wormhole is a popular tool for interstellar travel in science fiction. It connects two different space-time points directly, which does not contradict with the Einstein's field theory. Although such an astrophysical object is not yet observed, studying wormhole structure gives us many chances to understand unknown aspects of the gravitational field.

The most famous and the common starting model of wormhole is, we think, the one announced by Morris and Thorne as a "traversable wormhole" [1]. They considered "traversable conditions" for human travel through wormholes responding to Carl Sagan's idea for his novel *Contact*, and concluded that such a wormhole solution is available if we allow "exotic matter" (negative-energy matter).

The introduction of exotic matter sounds to be unusual for the first time, but such matter appears in quantum field theory and in alternative gravitational theories such as scalar-tensor theories. The Morris-Thorne solution is constructed with a massless Klein-Gordon field whose gravitational coupling takes the opposite sign to normal, which is found in Ellis's earlier work [2], so that we call it Ellis wormhole, hereafter. (See a review e.g. by Visser [3] for earlier works; See also e.g. Lobo [4] for recent works).

Ellis wormhole solution was studied in many contexts. Among them, we focus on its dynamical features in higher-dimensional space-time.

The first numerical simulation on Ellis wormhole in 4-dimensional space-time was reported by one of the authors [5]. They use a dual-null formulation for spherically symmetric space-time integration, and observed that the wormhole is unstable against Gaussian pulses in either exotic or normal massless Klein-Gordon fields. The wormhole throat suffers a bifurcation of horizons and either explodes to form an inflationary universe or collapses to a black hole, if the total input energy is negative or positive, respectively. These basic behaviors were repeatedly confirmed by other groups [6, 7].

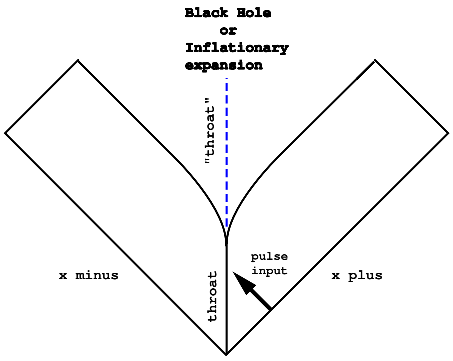

Figure 1. Partial Penrose diagram of the evolved space-time. Suppose we live in the right-side region and input a pulse to an Ellis-wormhole in the middle of the diagram. The wormhole throat suffers a bifurcation of horizons and either explodes to form an inflationary universe or collapses to a black hole, if the total input energy is negative or positive, respectively [5].

The changes of wormhole either to a black hole or an expanding throat support a unified understanding of black holes and traversable wormholes proposed by Hayward [8]. His proposal is that the two are dynamically interconvertible, and that traversable wormholes are understandable as black holes under the existence of negative energy density.

In this article, we introduce our recent studies on the stability in higher-dimensional space-time. The higher-dimensional theories such as string/M theories are applied for various unsolved problems in gravitational phenomena and cosmology, and gain new insights into them. We believe that wormholes will also give us new fundamental physical landscapes. We therefore demonstrate wormhole dynamics also in Gauss-Bonnet (GB) gravity, which is one of the modified gravity theory including higher-order corrections of curvatures, one of the string-motivated gravity theories.

The main four contents in this article are: (a) constructing Ellis solutions in higher-dimensional general relativity, (b) stability analysis using a linear perturbation method in n-dimensional general relativity (GR) [9], (c) stability analysis using a numerical evolution method in $5, 6, 7$-dimensional GR, and (d) the similar numerical analysis with the GB coupling.

2. Wormhole solution and its linear perturbation in higher-dimensional GR

2.1. Field equations

We start from the n-dimensional Einstein-Klein-Gordon system

$$S = \int_M d^n x \sqrt{-g} \left[\frac{1}{2\kappa_n^2} R - \frac{1}{2}\epsilon(\nabla\phi)^2 - V(\phi) \right], \tag{1}$$

where κ_n^2 is the n-dimensional gravitational constant. The scalar field ϕ is the *normal* (or *ghost*) field if $\epsilon = 1$ (-1).

The metric of the space-time is assumed to be

$$ds^2 = -f(t,r)e^{-2\delta(t,r)}dt^2 + f(t,r)^{-1}dr^2 + R(t,r)^2 h_{ij}dx^i dx^j, \tag{2}$$

where $h_{ij}dx^i dx^j$ represents the line element of a unit $(n-2)$-dimensional constant curvature space with curvature $k = \pm 1, 0$ and volume Σ_k. In order to construct a static wormhole solution, we restrict the metric function as $f = f(r)$, $R = R(r)$, $\phi = \phi(r)$, and $\delta = 0$.

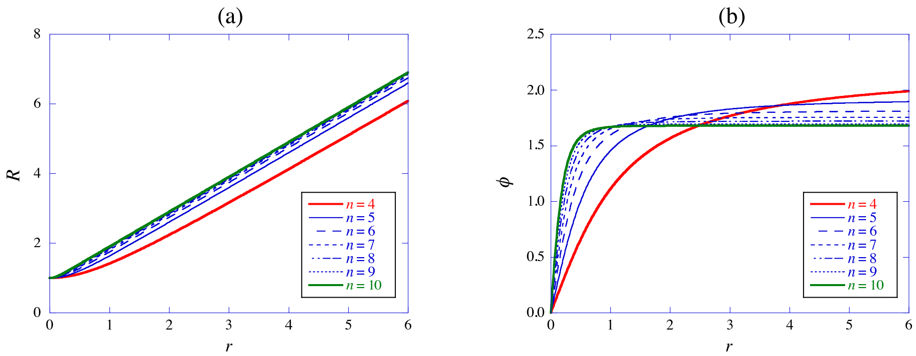

Figure 2. The n-dimensional wormhole solutions; (a) The circumference radius R and (b) the scalar field ϕ are plotted as a function of the radial coordinate r.

For massless ghost scalar field ($V(\phi) = 0$ and $\epsilon = -1$), the Klein-Gordon equation is integrated as $\phi' = C/(fR^{n-2})$, where C is an integration constant, and $' = d/dr$. By locating the throat of the wormhole at $r = 0$, and imposing the reflection symmetry at the throat, the solution of the field equations is obtained only for the spherically symmetric case $k = 1$, as

$$f \equiv 1, \qquad R' = \sqrt{1 - \left(\frac{a}{R}\right)^{2(n-3)}}, \qquad \phi = \frac{\sqrt{(n-2)(n-3)}}{\kappa_n} a^{n-3} \int \frac{1}{R(r)^{n-2}} dr, \tag{3}$$

where a is the radius of the throat, i.e. $R(0) = a$.

The throat of the wormhole has large curvature and the scalar field ϕ becomes steeper as n goes higher. We plotted these behaviors in Figure 2. For $n \to \infty$, the functions have the limiting solution, $R = r + a$ and $\phi = \pi/2$ ($r > 0$).

2.2. Existence of unstable mode in linear perturbation analysis

We study linear stability of the solution obtained above. We focus on the "spherical" modes, where the $(n-2)$-dimensional constant curvature space is not perturbed. For metric variables f, δ, R and for scalar field ϕ, we assume the perturbed functions in the form of $x(t,r) = x_0(r) + x_1(r)e^{i\omega t}$, where x_0 denotes the static solution above. Following [7], we obtain the master equation for gauge-invariant variable. After some transformations of variable, we reach the single Schrödinger-type master equation, which potential does not diverge at the throat,

$$-\Psi_1'' + W(r)\Psi_1 = \omega^2\Psi_1, \quad \text{where} \quad W(r) = -\frac{1}{4R_0^2}\Big[\frac{3(n-2)^2}{R_0^{2(n-3)}} - (n-4)(n-6)\Big]. \tag{4}$$

We search eigenfunctions $\Psi_1(r)$ of eq. (4), and find that in any dimension n there exists one negative eigenvalue for ω^2, which implies that the solution is unstable. We find large negative ω^2 for higher n, which indicates the timescale of instability becomes shorter [9].

3. Numerical evolutions of wormhole solutions

3.1. Dual-null scheme

We implemented our evolution code [5] for higher-dimensional space-time, and with GB gravity terms. The system we consider is spherical symmetry, and expressed using dual-null coordinate.

The use of dual-null coordinate simplifies the treatment of horizon dynamics and radiation propagation clearly.

The Einstein-Gauss-Bonnet (EGB) action in n-dimensional space-time is described as

$$S = \int_{\mathcal{M}} d^n X \sqrt{-g}\left[\frac{1}{2\kappa^2}\left\{\alpha_{\rm GR}\left(\mathcal{R}-2\Lambda\right)+\alpha_{\rm GB}\mathcal{L}_{\rm GB}\right\}+\mathcal{L}_{\rm matter}\right], \tag{5}$$

$$\text{where}\quad \mathcal{L}_{\rm GB} = \mathcal{R}^2 - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}. \tag{6}$$

The additional term $\mathcal{L}_{\rm GB}$ forms ghost-free combinations [10] and does not give higher derivative equations but an ordinary set of equations with up to the second derivative in spite of the higher curvature combinations.

We adopt the line element

$$ds^2 = -2e^{f(x^+,x^-)}dx^+\,dx^- + r^2(x^+,x^-)h_{ij}dz^i dz^j, \tag{7}$$

where the coordinate (x^+, x^-) are along to null propagation directions.

For writing down the Einstein equations, we introduce the variables

$$\Omega = \frac{1}{r}, \qquad \vartheta_\pm = (n-2)\partial_\pm r, \qquad \nu_\pm = \partial_\pm f, \qquad p_\pm = \partial_\pm \phi, \tag{8}$$

which are conformal factor, expansions, in-affinities, and scalar momentum, respectively.

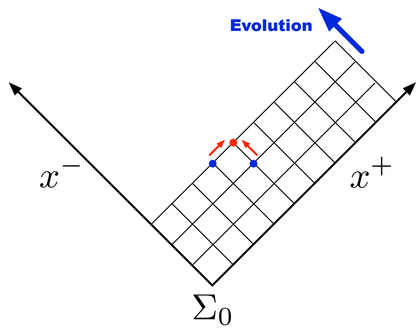

Figure 3. Numerical grid structure. Initial data are given on null hypersurfaces $\Sigma_\pm$ ($x^\mp = 0$, $x^\pm > 0$) and their intersection S.

We prepare our numerical integration range as drawn in Figure 3. The grid will cover both universes connected by the wormhole throat $x^+ = x^-$. We give initial data on a surface S and the two null hypersurfaces $\Sigma_\pm$ generated from it. Generally the initial data have to be given as

$$(\Omega, f, \vartheta_\pm, \phi) \quad \text{on } S\text{: } x^+ = x^- = 0 \tag{9}$$

$$(\nu_\pm, p_\pm) \quad \text{on } \Sigma_\pm\text{: } x^\mp = 0,\ x^\pm > 0. \tag{10}$$

We then evolve the data $u = (\Omega, \vartheta_\pm, f, \nu_\pm, \phi, p_\pm)$ on a constant-x^- slice to the next.

As a virtue of the dual-null scheme, we can follow the wormhole throat or black hole horizons easily. They are both trapping horizons, hypersurfaces where $\vartheta_+ = 0$ or $\vartheta_- = 0$. In order to evaluate the energy, we apply the Misner-Sharp mass in n-dimensional EGB gravity [11].

3.2. Evolutions of 4, 5, and 6-dimensional wormhole solutions in GR

We checked our numerical code whether it reproduces the static wormhole solution obtained in §2. We next add a perturbation to the scalar momentum on the static wormhole solution in the form of Gaussian pulses from the right-hand universe, $\delta p_+ = c_1 \exp(-c_2(l-c_3)^2)$,where

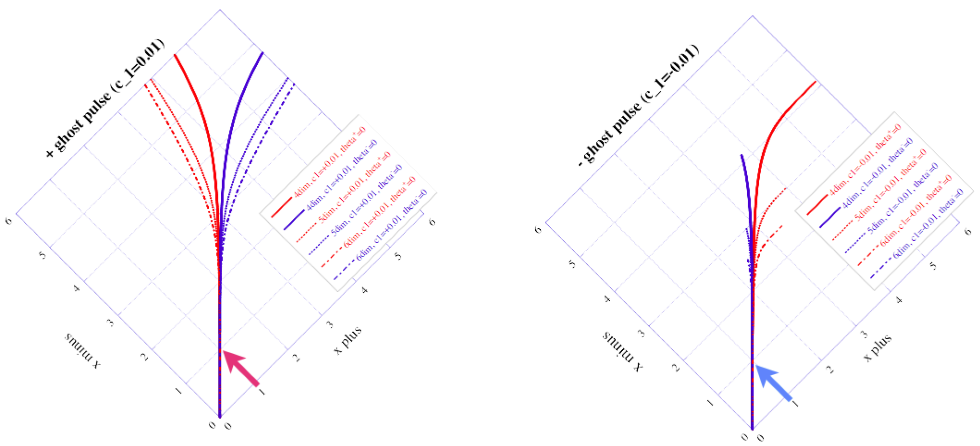

Figure 4. Location of the expansion ϑ_+ (red lines) and ϑ_- (blue lines) for evolutions of a solution in 4, 5, and 6-dimensional General Relativity ($\alpha_{\rm GB} = 0$) as a function of (x_+, x_-). The throat begins expanding if we input negative energy scalar flux (left panel), while the throat turns to be a black hole if we input positive energy scalar flux (right panel).

c_1, c_2, c_3 are parameters. In Figure 4, we plotted the cases with small amplitude $c_1 = \pm 0.01$ and width $c_2 = 3$, and the initial location $c_3 = 1$. That is, the pulse will hit the wormhole throat at $x^+ = x^- = 1$. Positive (or negative) c_1 corresponds to enhancing (or reducing) the supporting ghost field.

Figure 4 shows the results of $n = 4, 5$, and 6 dimensional wormhole solution with above perturbation. The plot shows locations of vanishing expansions, $\vartheta_\pm = 0$, in (x^+, x^-) plane. We see first the wormhole throat locates where $\vartheta_\pm = 0$, but after a small pulse hit it, then the throat (or horizon) split into two ($\vartheta_+ = 0$ and $\vartheta_- = 0$), depending on the signature of the energy of pulse.

If the location of ϑ_+ is outer (in x^+-direction) than that of ϑ_-, then the region $\vartheta_- < x < \vartheta_+$ is judged as a black hole. Otherwise the region $\vartheta_+ < x < \vartheta_-$ can be judged as an expanding throat. The throat begins expanding if we input negative energy scalar flux (left panel in Figure 4), while the throat turns to be a black hole if we input positive energy scalar flux (right panel).

3.3. Evolutions of wormhole solution in EGB gravity
We also construct the initial data with GB terms $\alpha_{\rm GB} \neq 0$, and study their effects to the evolutions. We confirm 4-dimensional EGB equation reproduces the equivalent results with those of GR.

Figure 5 shows the case of 5 and 6-dimensional EGB gravity. The initial data of wormhole on $\Sigma_\pm = 0$ are obtained numerically, solving the set of equations. The evolutions with $\alpha_{\rm GB} \neq 0$ are quite unstable, and we are hard to keep the static configurations long enough. We see if $\alpha_{\rm GB} < 0$ the throat turns to be a black hole (left panel in Figure 5). On the contrary, if $\alpha_{\rm GB} > 0$, then the throat begins to expand (right two panels).

4. Conclusions and Discussions
We studied the simplest wormhole solutions and their stability in higher-dimensional space-time both in GR and EGB gravity. The space-time is assumed to be static and spherically symmetric, has ghost scalar field, and has reflection symmetry at the throat.

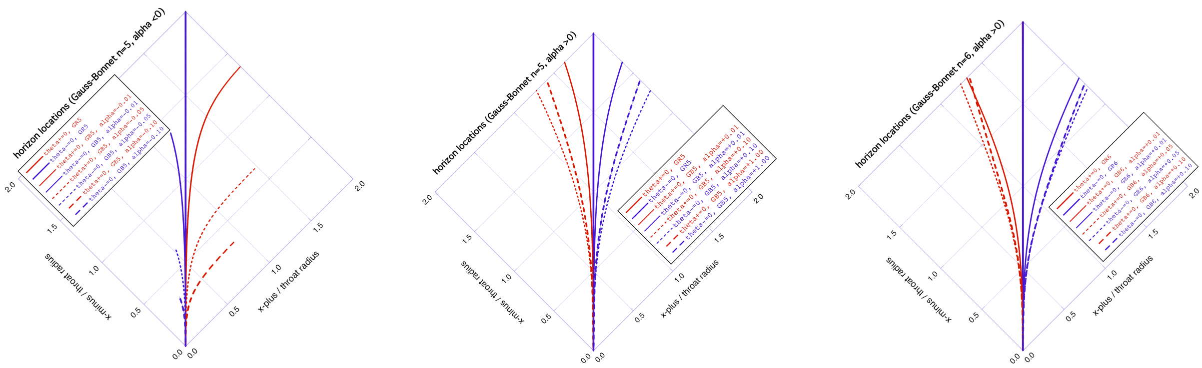

Figure 5. The same with Figure 4, but for Einstein-Gauss-Bonnet gravity. When $\alpha_{\rm GB} < 0$ (left panel), the throat turns to be a black hole, while for $\alpha_{\rm GB} > 0$ the throat begins expanding (middle/left panel).

Using the linear perturbation technique, we showed that the solutions have at least one negative mode, which concludes that all wormholes are linearly unstable. The time scale of instability becomes shorter as n becomes large. This prediction is confirmed using numerical evolutions.

Numerical evolutions show that if we put a scalar pulse with small amplitude to the throat, then the wormhole throat bifurcates, its horizon structure changes into a black hole or an expanding throat depending the pulse energy is positive or negative, respectively. We also investigated the effect of the higher-order curvature corrections using Gauss-Bonnet terms, and found that such corrections do not work for stabilization of wormholes.

All the behavior of wormholes may be explained simply with energy balance. In [5], for small perturbations, an existence of critical solution is suggested. The similar behaviors are also observed in our investigations, which will be reported elsewhere.

We guess a wormhole with exotic matter is a disguise to avoid public notice, and does prefer to appear as a black hole or an expanding universe.

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