Method of multiple scales in scalar field cosmology

V M Zhuravlev[1], **S V Chervon** [2,3,4]

[1] Inter-University Department of Space Research, Samara National Research University, 443086 Samara, Russia
[2] Bauman Moscow State Technical University, Moscow, Russia
[3] Ulyanovsk State Pedagogical University, Ulyanovsk, Russia
[4] Kazan Federal University, Kazan, Russia

E-mail: zhvictorm@gmail.com, chervon.sergey@gmail.com

Abstract. In this work, the method of multiple scales is applied to analysis of cosmological dynamics. The method is used to construct solutions to the dynamic equations of the Universe filled with a scalar field in the Friedman-Robertson-Walker metric. A general scheme is described for choosing small dimensionless parameters of the expansion of model functions and applying the method itself to the equations of cosmological dynamics. Solutions are given that are constructed for two different types of a small parameter - a small field value and a small slow roll parameter.

1. Introduction

At present time a scalar field cosmology is actively applied for studying early and later evolution of the Universe. Cosmological dynamic equation of self-interacting scalar field in Friedmann universe is described by the system of ordinary differential equations of the second order with respect to a scale factor or to a scalar field itself [1].

Many attempts were devoted to investigation of the Scalar Cosmology Equations (SCEs) on an inflationary stage by approximate methods (slow-roll and fast oscillation between them) in the first works on inflation by Starobinsky (1980)[2], Guth (1981)[3] Linde (1982)[4], Albrecht and Steinhardt (1982)[5]. Exact solution construction of SCEs is started in the works [6], [7], [8], [9]. More detail about exact solutions and methods of construction of them can be found in the monograph [1]. In the present article we propose new approach for searching approximate solutions of SCEs using method of multiple scales or multi-scale analysis.

Effective methods of asymptotic analysis of models of cosmological dynamics are usually built on the qualitative analysis of dynamical systems [10, 11, 12, 13]. Such methods provide useful information of a general nature and make it possible to understand the limiting states of the Universe evolution. Nevertheless, if it is necessary to obtain more accurate information about the nature of the solutions under certain initial conditions, the methods of asymptotic analysis turn out to be insufficiently effective. For solving problems of analyzing the current state of systems, methods of approximate analysis, built on the expansion of solutions in series in a small parameter, are much more useful. Among such methods, which is rarely used in investigation of cosmological dynamic, is the method of multiple scales or multiple scale analysis.

The method of multiple scales (MoMS) is one of the effective methods for the approximate analysis of the dynamics of various types of objects and processes. The initial field of application

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of MoMS, most likely, is the mechanics and theory of wave processes [14, 15, 16, 17]. The main idea of the MoMS is reduced to the representation of the dependence of the initial process on time & coordinates to the dependence on a wider range of variables, which are related to the initial variable scale factors and containing small parameters of the model in various degrees. The difference between the method of multiple scales and the standard method of a small parameter is that the introduction of new "slow" or "fast" variables makes it possible to improve the convergence of the series in this small parameter. To control the rate of convergence of series, the MoMS contains a specific tool, which is often called a procedure for eliminating resonances. The meaning of this procedure is that the introduction of additional independent variables ("slow" or "fast") generates a certain arbitrariness in the choice of the balance between the individual terms of the equations obtained as a result of expansions in a series. This arbitrariness allows the selection of dependence expansion elements in a series in such a way that the terms having a power-law form in independent variables (secular terms), which significantly worsen the rate of convergence of the series and reduce the radii of their convergence, are excluded from the series.

In this article, the MoMS is applied to analyzing dynamic of the Universe filled by the self-interacting scalar field in the Friedman-Robertson-Walker space-time. Although this model has been studied quite fully, it can be used as an example to show the features of the application of the MoMS to problems of cosmological dynamics, which can subsequently be usefully applied to more complex models.

The article is organized as follow. In Section 2 we derive cosmological dynamic equation in dimensionless form. Section 3 devoted to obtaining approximate equations for small scalar field by MoMS. In Section 4 we present solutions till the third order. Section 5 contain approximate solution for a scale factor till the second order. In Section 6 we consider approximation based on the first slow roll parameter and in Section 7 we derive equations and find solutions till the second order. In Section 8 we represent the general solution including that for the scale factor. In Section 9 we discuss obtained results.

2. Cosmological dynamic equations in dimensionless form

Let us study a cosmological dynamic of the self-interacting scalar field on the basis of MoMS. Cosmological dynamic equations [1] are:

$$3H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{1}$$

$$2\dot{H} = -\dot{\phi}^2, \tag{2}$$

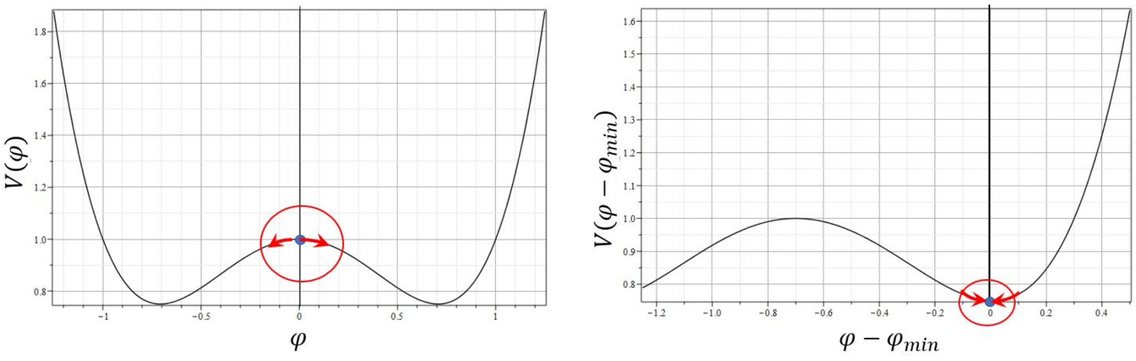
$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0. \tag{3}$$

Here $H = \dot{a}/a = \frac{da}{adt}$ is the Habble parameter, a is a scale factor, ϕ is a scalar field, $V(\phi)$ is the self-interacting potential of the scalar field, t is a cosmic time. Equations of gravitation dynamic (1) and (2) have the field equation (3) as a consequence [1]. Therefore we may exclude this equation from consideration.

To obtain dimensionless variables [11] we set $\xi = H/H_0,\ \eta = \phi/\Phi_0,\ \tau = t/T_0$. Here $H_0,\ \Phi_0$ and T_0 - some constants of the corresponding dimensions, which should be determined by the choice of the required asymptotic of the the initial system solutions. As a result of substitution in the initial system (1) and (2) of dimensionless variables, we arrive at a system of equations in the following form:

$$3H_0^2\xi^2 = \frac{1}{2}\Phi_0^2\dot{\eta}^2 + \mathcal{V}(\eta, \Phi_0), \tag{4}$$

$$2H_0T_0\dot{\xi} = -\Phi_0^2\dot{\eta}^2. \tag{5}$$



(a) (b)

Figure 1. The positions of the points of the expansion of the potential at the maximum (a) and minimum (b) for the Higgs potential (8).

Equivalently, in another form:

$$\frac{\alpha}{\beta}\dot{\xi} + 3\xi^2 = \frac{1}{H_0^2}\mathcal{V}(\eta, \Phi_0), \tag{6}$$

$$\dot{\xi} = -\frac{1}{2}\beta\dot{\eta}^2. \tag{7}$$

Here $V(\phi) = \mathcal{V}(\eta, \Phi_0)$ and:

$$\alpha = \frac{\Phi_0^2}{T_0^2 H_0^2}, \quad \beta = \frac{\Phi_0^2}{H_0 T_0}.$$

Dimensionless parameters α and β, under given form of the potential $V(\phi)$ and initial conditions, define dynamic of cosmological process. Because the parameter $\gamma = H_0 T_0$ is dimensionless by definition, then Φ_0 is dimensionless as well.

3. Expansion on a scale of the field $\Phi_0 = \sqrt{\varepsilon}$

Let us consider the first option for choosing a small parameter, setting $\gamma = H_0 T_0 = 1$ and a small parameter $\Phi_0 = \sqrt{\varepsilon}$. A small value of the field scale Φ_0 means a small deviation of the field from the point $\phi = 0$, which can be placed by the transformation: $\phi' = \phi - \phi_{min}$ to any other point. In this case, the position of the point $\phi = 0$ can be both at the minimum and maximum, as shown in Fig. 1, as well as at any other point on the ϕ axis. This fact is illustrated by the figures with the positions of the points $\phi = 0$ for the Higgs potential:

$$V = 1 + \phi^4 - \phi^2. \tag{8}$$

To select the field scale of ϕ as a small parameter we have $\alpha = \beta$. Using new variables, we can formally write:

$$\frac{1}{H_0^2}V(\phi) = \mathcal{V}(\eta, \sqrt{\varepsilon}).$$

The assumed smallness of the parameter ε allows, in accordance with the general ideology of multiple scales expansions, to seek solutions to the system (6)-(7) in the form of dependence on the usual τ and slow variables, which will have the following form:

$$\tau_1 = \sqrt{\varepsilon}\tau, \ \tau_2 = \varepsilon\tau, \ \ldots .$$

Hence, for any function we have $F(\tau) = \mathcal{F}(\tau, \tau_1, \tau_2, \dots)$:

$$\frac{d}{d\tau}F(t) = \left(\frac{\partial}{\partial\tau} + \sqrt{\varepsilon}\frac{\partial}{\partial\tau_1} + \varepsilon\frac{\partial}{\partial\tau_2} + \cdots\right)\mathcal{F}(\tau, \tau_1, \tau_2, \dots).$$

Further we represent the potential $V(\phi)$ as an expansion series in ϕ:

$$V(\phi) = V_0 + V_1\phi + V_2\phi^2 + \cdots.$$

From here:

$$\mathcal{V}(\eta, \sqrt{\varepsilon}) = v_0 + v_1\eta\sqrt{\varepsilon} + v_2\eta^2\varepsilon + \cdots,$$

where $v_i = H_0^{-2}V_i,\ i = 0, 1, \dots$ are dimensionless parameters of the potential.

Similarly, the solution for $\xi(t)$ and $\eta(t)$ will be sought in the following form:

$$\xi = \xi_0(\tau, \tau_1, \tau_2, \dots) + \sqrt{\varepsilon}\xi_1(\tau, \tau_1, \tau_2, \dots) + \varepsilon\xi_2(\tau, \tau_1, \tau_2, \dots) + \cdots, \quad (9)$$

$$\eta = \eta_0(\tau, \tau_1, \tau_2, \dots) + \sqrt{\varepsilon}\eta_1(\tau, \tau_1, \tau_2, \dots) + \varepsilon\eta_2(\tau, \tau_1, \tau_2, \dots) + \cdots. \quad (10)$$

Substituting (9) and (10) into (6), with respect to the first three expansion elements we have:

$$\begin{aligned}&\left(\frac{\partial}{\partial\tau} + \sqrt{\varepsilon}\frac{\partial}{\partial\tau_1} + \varepsilon\frac{\partial}{\partial\tau_2} + \cdots\right)\Big(\xi_0(\tau, \tau_1, \dots) + \sqrt{\varepsilon}\xi_1(\tau, \tau_1, \dots) + \varepsilon\xi_2(\tau, \tau_1, \dots) + \cdots\Big) + \\ &+3\Big(\xi_0(\tau, \tau_1, \dots) + \sqrt{\varepsilon}\xi_1(\tau, \tau_1, \dots) + \varepsilon\xi_2(\tau, \tau_1, \dots) + \cdots\Big)^2 = \\ &= v_0 + v_1\Big(\eta_0(\tau, \tau_1, \dots) + \sqrt{\varepsilon}\eta_1(\tau, \tau_1, \dots)\Big)\sqrt{\varepsilon} + v_2(\eta_0^2\varepsilon + 2\eta_0\eta_1\varepsilon^{3/2}) + v_3\eta_0^3\varepsilon^{3/2} + \cdots.\end{aligned} \quad (11)$$

The first four equations of the system (11) in the expansion in orders of $\varepsilon^{1/2}$ have the following form:

$$\varepsilon^0: \quad \frac{\partial\xi_0}{\partial\tau} + 3\xi_0^2 = v_0, \quad (12)$$

$$\varepsilon^{1/2}: \quad \frac{\partial\xi_1}{\partial\tau} + \frac{\partial\xi_0}{\partial\tau_1} + 6\xi_0\xi_1 = v_1\eta_0, \quad (13)$$

$$\varepsilon^1: \quad \frac{\partial\xi_2}{\partial\tau} + \frac{\partial\xi_1}{\partial\tau_1} + \frac{\partial\xi_0}{\partial\tau_2} + 6\xi_0\xi_2 + 3\xi_1^2 = v_1\eta_1 + v_2\eta_0^2, \quad (14)$$

$$\begin{aligned}\varepsilon^{3/2}: \quad &\frac{\partial\xi_3}{\partial\tau} + \frac{\partial\xi_2}{\partial\tau_1} + \frac{\partial\xi_1}{\partial\tau_2} + \frac{\partial\xi_0}{\partial\tau_3} + 6(\xi_0\xi_3 + \xi_1\xi_2) = \\ &= v_1\eta_2 + 2v_2\eta_0\eta_1 + v_3\eta_0^3 \\ &\cdots\end{aligned} \quad (15)$$

Similarly, we find the expansion of (7). After substitution of the series (9) and (10), we find:

$$\begin{aligned}&\left(\frac{\partial}{\partial\tau} + \sqrt{\varepsilon}\frac{\partial}{\partial\tau_1} + \varepsilon\frac{\partial}{\partial\tau_2} + \cdots\right)\Big(\xi_0(\tau, \tau_1, \dots) + \sqrt{\varepsilon}\xi_1(\tau, \tau_1, \dots) + \varepsilon\xi_2(\tau, \tau_1, \dots)\Big) = \\ &= -\frac{1}{2}\varepsilon\left(\frac{\partial\eta_0}{\partial\tau} + \sqrt{\varepsilon}\frac{\partial\eta_0}{\partial\tau_1} + \varepsilon\frac{\partial\eta_0}{\partial\tau_2} + \sqrt{\varepsilon}\frac{\partial\eta_1}{\partial\tau} + \varepsilon\frac{\partial\eta_0}{\partial\tau_1} + \varepsilon\frac{\partial\eta_2}{\partial\tau_2} + \dots\right)^2.\end{aligned} \quad (16)$$

The first four equations of (7) for the expansion (44) in orders $\varepsilon^{1/2}$ have the following form:

$$\varepsilon^0: \quad \frac{\partial\xi_0}{\partial\tau} = 0, \quad (17)$$

$$\varepsilon^{1/2}: \quad \frac{\partial \xi_1}{\partial \tau} + \frac{\partial \xi_0}{\partial \tau_1} = 0, \tag{18}$$

$$\varepsilon^{1}: \quad \frac{\partial \xi_2}{\partial \tau} + \frac{\partial \xi_1}{\partial \tau_1} + \frac{\partial \xi_0}{\partial \tau_2} = -\frac{1}{2}\left(\frac{\partial \eta_0}{\partial \tau}\right)^2, \tag{19}$$

$$\varepsilon^{3/2}: \quad \frac{\partial \xi_3}{\partial \tau} + \frac{\partial \xi_2}{\partial \tau_1} + \frac{\partial \xi_1}{\partial \tau_2} + \frac{\partial \xi_0}{\partial \tau_3} = -\frac{1}{2}\frac{\partial \eta_0}{\partial \tau}\left(\frac{\partial \eta_0}{\partial \tau_1} + \frac{\partial \eta_1}{\partial \tau}\right), \tag{20}$$

$$\ldots$$

Thus we can solve step by step the systems (12)-(15) and (17)-(20).

4. The solution of the system

From equation (12) and (17) one can find the following solution with respect to ξ_0:

$$\xi_0 = \pm\sqrt{\frac{v_0}{3}} = \pm\Lambda. \tag{21}$$

Hereinafter, the notation is introduced: $\Lambda = \sqrt{v_0/3}$. The solution (39) makes sense if $v_0 \geq 0$. The last inequality is known the consequence of the energy dominance condition. Since $v_0 = \text{const}$, we have: $\xi_0 = \text{const}$. As a result, the solution of (13) and (19) in the first order looks like this:

$$\xi_1 = \pm\frac{v_1}{6\Lambda}\eta_0(\tau_1, \tau_2, \ldots). \tag{22}$$

Moreover, the functions ξ_1 and η_0 depend only on slow variables: $\xi_1 = \xi_1(\tau_1, \tau_2, \ldots)$, $\eta_0 = \eta_0(\tau_1, \tau_2, \ldots)$.

The solution of (14) and (19) in the second order can now be written like this:

$$\xi_2 = \pm\frac{1}{8\sqrt{3v_0}}\Big(4v_0v_1\eta_1(\tau, \tau_1, \ldots) + (4v_0v_2 - v_1^2)\eta_0^2(\tau_1, \tau_2, \ldots)\Big). \tag{23}$$

In this case, from (20) for the function $\eta_1(\tau, \tau_1, \ldots)$ we obtain the equation:

$$\frac{\partial \eta_1}{\partial \tau} + \frac{\partial \eta_0}{\partial \tau_1} = 0.$$

From here we get:

$$\eta_1 = -\tau\frac{\partial \eta_0}{\partial \tau_1} + C_1(\tau_1, \tau_2, \ldots),$$

where $C_1(\tau_1, \tau_2, \ldots)$ is an integration constant wrt τ. Note, the dependence $\eta_0 = \eta_0(\tau_1, \tau_2, \ldots)$ on τ_1 does not defined yet. This can be used to eliminate in the solution for η_1 a term that grows linearly together with τ. Such terms are often called resonant or secular. The presence of such terms leads to a rapid divergence of the series in ε with increasing τ. By setting $\eta_0 = \eta_0(\tau_2, \ldots)$, we eliminate the dependence of η_1 on τ. This procedure is called the resonance elimination procedure. In this case, ξ_2 will also not depend on τ, but will depend on slow variables.

The third order scale factor equation can now be written in the following form:

$$\xi_3 = \frac{1}{6}\sqrt{\frac{3}{v_0}}\left(v_1\eta_2 + 2v_2\eta_0\eta_1 + v_3\eta_0^3 - 6\xi_1\xi_2)\right). \tag{24}$$

In this case, the following relations are also fulfilled:

$$\xi_2(\tau_1, \tau_2, \ldots) = -\tau_1\frac{\partial \xi_1}{\partial \tau_2} + \xi_1^{(0)}(\tau_2, \ldots).$$

The last relations define dependence ξ_2 on slow variable τ_1.

From the formal point of view, the linear dependence of $\xi_2(\tau_1, \tau_2, \ldots)$ on τ_1 can also be considered as a 'resonant' term leading to an accelerated divergence of the series. Therefore, it can also be eliminated by setting $\xi_1 = \xi_1(\tau_2, \ldots)$. In the next orders, one can continue to apply further a similar procedure.

5. Solution for the scale factor

Since ξ_0, ξ_1 and ξ_2 do not depend on the variable τ, it is possible to find out the character of the cosmological expansion. From definition of ξ up to $\varepsilon^{3/2}$ we have:

$$H = H_0\Big(\xi_0 + \sqrt{\varepsilon}\xi_1 + \varepsilon\xi_2 + \varepsilon^{3/2}\xi_3 + \ldots\Big) = H_0\frac{\dot{a}}{a} = \frac{H_0}{a_0 + \sqrt{\varepsilon}a_1 + \varepsilon a_2 + \varepsilon^{3/2}a_3 + \ldots}\left(\frac{\partial a_0}{\partial \tau} + \right.$$
$$\left. + \sqrt{\varepsilon}\left(\frac{\partial a_0}{\partial \tau_1} + \frac{\partial a_1}{\partial \tau}\right) + \varepsilon\left(\frac{\partial a_0}{\partial \tau_1} + \frac{\partial a_1}{\partial \tau_1} + \frac{\partial a_2}{\partial \tau} + \ldots\right) + \ldots\right).$$

Using this expansion we arrive at the system of equations for calculating the evolution of the scale factor $a = a_0 + \sqrt{\varepsilon}a_1 + \varepsilon a_2 + \ldots$. Thus, we have:

$$\begin{aligned}
\xi_0 &= \frac{\partial \ln a_0}{\partial \tau}, \\
\xi_1 &= \frac{1}{a_0}\left(\frac{\partial a_0}{\partial \tau_1} - \frac{a_1}{a_0}\frac{\partial a_0}{\partial \tau} + \frac{\partial a_1}{\partial \tau}\right), \\
\xi_2 &= -\frac{\partial a_0}{\partial \tau}\left(\frac{a_2}{a_0^2} - \frac{a_1^2}{a_0^3}\right) - \left(\frac{\partial a_0}{\partial \tau_1} + \frac{\partial a_1}{\partial \tau}\right)\frac{a_1}{a_0^2} + \left(\frac{\partial a_0}{\partial \tau_2} + \frac{\partial a_1}{\partial \tau_1} + \frac{\partial a_2}{\partial \tau}\right)\frac{1}{a_0}.
\end{aligned}$$

Using the solution for ξ_0 (39) we find:

$$a_0 = A_0(\tau_1, \tau_2, \ldots)e^{\tau\sqrt{v_0/3}}.$$

Thus, in the zero order, evolution is an inflationary expansion with the Hubble parameter $H = H_0\Lambda$ at the sign of $+$ in the exponent. It is this option that is discussed below.

In the first order, the equation for a_1 is reduced to the form:

$$\frac{\partial a_1}{\partial \tau} - \sqrt{\Lambda}a_1 = A_0 e^{\Lambda\tau}\left(\frac{v_1}{6\Lambda}\eta_0 - \frac{\partial \ln A_0}{\partial \tau_1}\right).$$

The condition for the absence of resonance in this equation is the vanishing of the right side of this equation, which leads to the equation:

$$A_0\frac{v_1}{6\Lambda}\eta_0 - \frac{\partial A_0}{\partial \tau_1} = 0.$$

Hence, under the condition $\xi_1 = \text{const}$ (from (22) this means that $\eta_0 = \text{const}$ too) it follows:

$$A_0 = A_{00}\exp\Big(\frac{v_1}{6\Lambda}\eta_0\tau_1\Big). \tag{25}$$

Integration constant A_{00} may depend on $\tau_2, \ldots$. Note that the sign of the exponent in (25) is defined both the sign of v_1 (i.e., by the sign of the coefficient of the Taylor series of the expansion of the potential at zero) and the sign of the field itself in the zero order. Consequently, with the opposite signs of η_0 and v_1, the inflation rate will decrease in comparison with the zero order.

The solutions for the scale factor obtained in the first two orders can be represented as follows:

$$a = e^{\Lambda\tau}\left(A_{00}\exp\left(\frac{v_1}{6\Lambda}\eta_0\tau\sqrt{\varepsilon}\right) + \sqrt{\varepsilon}A_{10}\right) + O(\varepsilon), \tag{26}$$

where A_{10} is the constant of integration.

Considering η_0 as a constant in (26) and expanding the exponent $\exp\left(\frac{v_1}{6\Lambda}\eta_0\tau\sqrt{\varepsilon}\right)$ in Taylor series in $\sqrt{\varepsilon}$ one can obtain the next form of the solution

$$a = e^{\Lambda\tau}\left(A_{00} + \sqrt{\varepsilon}\left(\tilde{\eta}\tau + A_{10}\right)\right) + O(\varepsilon), \tag{27}$$

where $\tilde{\eta} = \frac{v_1}{6\Lambda}\eta_0$. The solution in the form (27) corresponds to exponent power-law inflation, considered, for example, in the work [18] (p.4, formula (14)).

6. Expansion on the slow roll parameter

Let us now consider the results of the analysis of equations in the case of choosing the scales in accordance with the relations: $\Phi_0 = 1,\ \ H_0T_0 = \varepsilon^{-1}$, where $\varepsilon \ll 1$ is a new small parameter. In dimensionless form, the slow roll parameter

$$p = -\frac{\dot{H}}{H^2},$$

will be like follows:

$$p = -\frac{1}{H_0T_0}\frac{\dot{\xi}}{\xi^2} = -\varepsilon\frac{\dot{\xi}}{\xi^2},$$

that is, it is a quantity of the first order of smallness in ε. This means that $\varepsilon = (H_0T_0)^{-1}$ can be interpreted as a slow roll scale.

Passing to the dimensionless equations (6), we find in this case:

$$\alpha = \varepsilon^2, \quad \beta = \varepsilon.$$

With this choice of parameters, the equations (6) take the following form:

$$\varepsilon\dot{\xi} + 3\xi^2 = \frac{1}{H_0^2}\mathcal{V}(\eta), \tag{28}$$

$$\dot{\xi} = -\frac{1}{2}\varepsilon\dot{\eta}^2. \tag{29}$$

This system is singular, since in two equations the small parameter stands at the highest derivative. To eliminate this drawback, we introduce a new time variable, setting $\theta = \tau\varepsilon^{-1} = H_0t$. For the new variable, the equations will look like this:

$$\dot{\xi} + 3\xi^2 = \frac{1}{H_0^2}\mathcal{V}(\eta), \tag{30}$$

$$\dot{\xi} = -\frac{1}{2}\dot{\eta}^2. \tag{31}$$

Hereinafter dot over the function means the derivative wrt new variable θ.

An unclear element of the last system of equations is the term on the right-hand side of (30) associated with the potential. The crux of the problem is how the values of the $\mathcal{V}(\eta)$ function relate to H_0^2. There are two possibilities. The first is that the values of $H_0^{-2}\mathcal{V}(\eta)$ are of order 1. The second is that $T_0^2\mathcal{V}(\eta)$ is the order 1. In the first case, the equations do not contain the small parameter ε, and in the second, the term with the potential in the right-hand side of (30) is of order ε^2. In the general case, it can be assumed that the functional dependence of the potential on η can be split into two parts. The first part is of order 1, according to the case of $H_0^{-2}\mathcal{V}$. The second part, corresponding to the case when $T_0^2\mathcal{V}$, is of order 1. Thus, we assume:

$$H_0^{-2}\mathcal{V} = V_0(\eta) + \varepsilon^2 U(\eta),$$

where $\mathcal{V}_0(\eta)$ and $U(\eta)$ are functions of order 1.

7. Model's equations and their solutions

Let us study the case $V_0 = v_0 = \text{const}$. Under this condition v_0 can be considered as cosmological constant. We will construct an expansion in ε, assuming:

$$\xi = \xi_0 + \varepsilon\xi_1 + \ldots, \quad \eta = \eta_0 + \varepsilon\eta_0 + \ldots.$$

The dependence of the expansion coefficients on slow variables will be determined by the variables $\theta_i = \varepsilon^i\theta,\ i = 0, 1, 2, \ldots$. For this expansion, we restrict ourselves to only the first two orders. As a result, we have the following system of equations in the orders of the equation (30):

$$\dot{\xi}_0 + 3\xi_0^2 = v_0, \tag{32}$$

$$\dot{\xi}_1 + \frac{\partial\xi_0}{\partial\theta_1} + 6\xi_0\xi_1 = 0, \tag{33}$$

$$\dot{\xi}_2 + \frac{\partial\xi_1}{\partial\theta_1} + \frac{\partial\xi_0}{\partial\theta_2} + 6\xi_0\xi_2 + 3\xi_1^2 = U(\eta_0). \tag{34}$$

Similarly, for the equation (31) we have:

$$\dot{\xi}_0 = -\frac{1}{2}\dot{\eta_0}^2, \tag{35}$$

$$\dot{\xi}_1 + \frac{\partial\xi_0}{\partial\theta_1} = -\dot{\eta_0}\left(\dot{\eta_1} + \frac{\partial\eta_0}{\partial\theta_1}\right), \tag{36}$$

$$\dot{\xi}_2 + \frac{\partial\xi_1}{\partial\theta_1} + \frac{\partial\xi_0}{\partial\theta_2} = -\dot{\eta_0}\left(\dot{\eta_2} + \frac{\partial\eta_1}{\partial\theta_1} + \frac{\partial\eta_0}{\partial\theta_2}\right) - \frac{1}{2}\left(\dot{\eta_1}\frac{\partial\eta_0}{\partial\theta_1}\right)^2. \tag{37}$$

7.1. Zero order

The zero-order solution for ξ_0 is constructed using the formal replacement:

$$\xi_0 = \frac{1}{3}\frac{d\ln u}{d\theta}. \tag{38}$$

For new function $u(\theta)$ we have the equation:

$$\ddot{u} = 3v_0 u.$$

General solution for u has the following form:

$$u = C_1 e^{\mu\theta} + C_2 e^{-\mu\theta}.$$

Here $\mu^2 = 3v_0$. Solution for ξ_0 is:

$$\xi_0 = \frac{\mu}{3}\left[\frac{1 - Ce^{-2\mu\theta}}{1 + Ce^{-2\mu\theta}}\right].$$

Here $C = C_2/C_1$. Since $\dot{\xi}_0 < 0$, then there are restrictions on the choice of C. For the function ξ_0 to be monotonically decreasing, it is necessary that $C < 0$. The value of C determines in this case the position of the cosmological singularity. If we assume that the instant of time of the cosmological singularity is chosen equal to $t = 0$, then $C = -1$. As a result, the solution for ξ_0 can be written as follows:

$$\xi_0 = \frac{\mu}{3\tanh(\mu\theta)}. \tag{39}$$

From here:

$$\dot{\xi}_0 = -\frac{\mu^2}{3\sinh^2(\mu\theta)} < 0, \tag{40}$$

$$a_0 = A_0 \sinh^{1/3}(\mu\theta). \tag{41}$$

In this formula $a_0(\theta)$ is a scale factor in zero-order, A_0 - the constant of integration. Note, that such type of solution for arbitrary exponent α instead of 1/3 was presented in [1] (p.22, formula (2.28).

Using (40), the zero-order field equation (35) can be reduced to the following form:

$$\dot{\eta}_0 = s\frac{\sqrt{2}\mu}{\sqrt{3}\sinh(\mu\theta)}, \quad s = \pm 1.$$

From here we find:

$$\eta_0 = f_0 + \int \frac{\mu d\theta}{\sinh(\mu\theta)} = f_0 + \frac{s\sqrt{2}}{\sqrt{3}} \int \frac{dz}{z^2 - 1} = f_0 + \frac{s}{\sqrt{6}} \ln\left(\frac{\cosh(\mu\theta) - 1}{\cosh(\mu\theta) + 1}\right). \tag{42}$$

7.2. First order

In order zero, the solution for ξ_0 does not contain arbitrary constants that might depend on slow variables. Therefore, from (33) in the first order for ξ_1 we have the following equation:

$$\dot{\xi}_1 = -\frac{2\mu}{\tanh(\mu t)}\xi_1.$$

From here we find:

$$\xi_1 = \frac{q_1(\tau_1, \ldots)}{\sinh^2(\mu\theta)}, \tag{43}$$

where $q_1 = q_1(\theta_1, \ldots)$ is the integration constant depending on slow variables.

Unlike ξ_0, the solution for η_0 contains a constant of integration, which may depend on slow variables. As a result, for η_1 we have from (36) the following equation:

$$\dot{\eta}_1 = -\frac{1}{\dot{\eta}_0}\left(\dot{\xi}_1 + \dot{\eta}_0 \frac{\partial f_0}{\partial \theta_1}\right) = \frac{2sq_1}{\sqrt{3}} \frac{\cosh(\mu\theta)}{\sinh^2(\mu\theta)} - \frac{\partial f_0}{\partial \theta_1}. \tag{44}$$

From here:

$$\eta_1 = -\frac{2sq_1}{\sqrt{3}\mu} \frac{1}{\sinh(\mu\theta)} - \frac{\partial f_0}{\partial \theta_1}\theta + f_1(\theta_1, \ldots). \tag{45}$$

The second term in (45) is linear in θ and it is 'resonant'. To exclude it, one need to put

$$\frac{\partial f_0}{\partial \theta_1} = 0, \quad f_0 = f_0(\theta_2, \ldots).$$

Thus, the solution for η_1 is:

$$\eta_1 = -\frac{2sq_1}{\sqrt{3}\mu} \frac{1}{\sinh(\mu\theta)} + f_1(\theta_1, \ldots). \tag{46}$$

7.3. Second order

The equation (34) for the scale factor in the second order, taking into account the first two: (39) and (43), can be written as follows:

$$\dot{\xi}_2 + \frac{\partial q_1}{\partial \theta_1}\frac{1}{\sinh^2(\mu\theta)} + \frac{2\mu}{\tanh(\mu t)}\xi_2 + 3\frac{q_1^2}{\sinh^4(\mu\theta)} = U(\eta_0). \tag{47}$$

The solution of (47) depends on the form of the function $U(\eta_0)$. Therefore, to obtain its explicit form, it is necessary to select a certain type of this dependence. However, to clarify the presence of resonance terms in this equation, it is sufficient to consider only some components of the Taylor series of the function $U(\eta_0)$ at zero. The eigenfunction of the equation (47), up to a constant factor, coincides with $\sinh^{-2}(\mu\theta)$. Consequently, only the Taylor series component quadratic in η_0 is of interest to eliminate resonances, since $\eta_0 \sim \sinh^{-1}(\mu\theta)$. Denoting this term by $u_2\eta_0^2$, we find an equation for $q_0(\tau_1)$, which is equivalent to the condition for the absence of resonances in the following form:

$$\frac{\partial q_1}{\partial \theta_1} = u_2. \tag{48}$$

From here:

$$q_1 = u_2\theta_1 + q_{10}.$$

As a result, the final solution for ξ_2 can be represented as follows:

$$\xi_2 = \frac{1}{\sinh^2(\mu\theta)}\left(q_2 + 3\frac{(u_2\theta_1 + q_{10})^2}{\sinh(\mu\theta)} + \int\limits_0^\theta \Big(U(\eta_0(z)) - u_2\eta_0^2(z)\Big)\sinh^2(\mu z)dz\right). \tag{49}$$

Here q_2 is the constant of integration, depending on slow variables.

The field equation in the second order (37) now has the following form:

$$\frac{1}{\dot{\eta}_0}\left(\dot{\xi}_2 + \frac{u_2}{\sinh^2(\mu\theta)}\right) - \frac{2su_2}{\sqrt{3}\mu}\frac{1}{\sinh(\mu\theta)} + \frac{\partial f_1}{\partial \theta_1} + \frac{\partial f_0}{\partial \theta_2} = -\dot{\eta}_2.$$

This equation is easy to integrate, but like the equation for ξ_2, it depends on the form of the function $U(\eta)$. Therefore, to construct solutions in the second and subsequent orders, it is necessary to clarify the general ideas about the form of the self-interacting potential. In the first two orders, the form of the function does not matter, and the entire dynamics is determined by the initial conditions, as well as the value of the constant v_0, which plays the role of the cosmological constant.

8. General solution

Combining the obtained solutions in the first two orders, we find:

$$\xi = \frac{\mu}{3\tanh(\mu\theta)} + \varepsilon\frac{u_2\theta_1 + q_{10}}{\sinh^2(\mu\theta)} + O(\varepsilon^2), \tag{50}$$

$$\eta = f_0 + \frac{s}{\sqrt{6}}\ln\left(\frac{\cosh(\mu\theta)-1}{\cosh(\mu\theta)+1}\right) - \varepsilon\left(\frac{2s(u_2\theta_1+q_{10})}{\sqrt{3}\mu}\frac{1}{\sinh(\mu\theta)} - f_1(\tau_1,\ldots)\right) + O(\varepsilon^2) \tag{51}$$

Accordingly, the solution for the scale factor is:

$$a = \left(A_0\mu + \varepsilon\Big(A_1 - A_0(u_2\theta_1 + q_{10})\frac{1}{\tanh(\mu\theta)}\Big)\right)\sinh^{1/3}(\mu\theta) + O(\varepsilon^2). \tag{52}$$

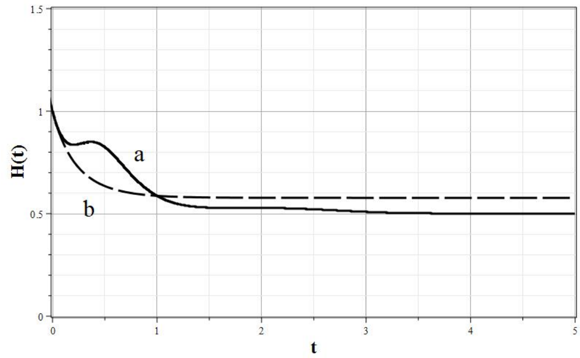


Figure 2. Comparison of the Hubble parameter variation curve obtained numerically (a) and using the expansion in a series in ε (b) for the Higgs potential in zeroth order (8).

Here A_0 and A_1 are integration constants.

To illustrate the nature of the obtained approximation in the zeroth order, let us compare the numerical solution of the problem with the Higgs potential (8) and the solution obtained using the MoMS. The corresponding curves are shown in Fig. 2. It can be seen that the zero order in the considered model for the Hubble parameter is universal, i.e. depends only on the values of the self-interacting potential at zero and does not depend on its specific form. Higher-order corrections require certain calculations related to the satisfaction of the initial conditions, which is beyond the scope of this work.

9. Conclusions

The calculations presented in this work demonstrate the general principles of using the method of multiple scales in scalar field cosmology. As has been shown, dynamical systems used to describe cosmological dynamics contain, as a rule, implicitly several dimensionless constants that determine the type of solutions. In the considered model, such dimensionless constants are the field scale Φ_0 and the slow roll parameter: $p = (H_0T_0)^{-1}$. The use of MoMS allows one to obtain approximate solutions for physical conditions corresponding to the smallness of these parameters, with a correction of the convergence rate. In this case, the information about the solution itself obtained with the help of MoMS supplements the general information that can be obtained by numerical methods and methods of qualitative analysis of dynamical systems. The results obtained indicate the possibility of using this approach in cosmology for more complex models than that considered in this work.

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References

[1] Chervon S, Fomin I , Yurov V, Yurov A 2019 *World Scientific Publisher* **13**
[2] Starobinsky A A 1980 *Phys. Lett. B* **91** 99
[3] Guth A H 1981 *Phys. Rev. D* **23** 347
[4] Linde A D 1982 *Phys. Lett. B* **108** 389

[5] Albrecht A and Steinhardt P J 1982 *Phys. Rev. Lett.* **48** 1220
[6] Ivanov G G 1981 *Gravitation and Theory of Relativity* (Kazan university publishing house) **18** 54
[7] Lucchin F, Matarrese S 1985 *Phys. Rev. D* **32** 1316
[8] Muslimov A G 1990 *Class. Quant. Grav.* **7** 231
[9] Barrow J D 1990 *Phys. Lett. B* **235** 40
[10] Zhuravlev V M 2016 *Space, Time and Fundamental Interactions* **4** 39
[11] Zhuravlev V M, Chervon A V 2020 *Universe* **6** 195
[12] Lowski M S, Hrycyna O, Stachowski A 2014 *Int. J. Geom. Methods Mod. Phys.* **11** 1460012
[13] Paliathanasis A, Tsamparlis M, Basilakos S, Barrow J D 2015 *Phys. Rev. D* **91** 123535
[14] Bhatnagar P L 1979 *Nonlinear waves in one-dimensional dispersive systems* (Oxford)
[15] Nayfen A H 1981 *Introduction to perturbation techniques* (John Wiley & Sons)
[16] Dodd R K , Eilbeck C J, Gibbon J D, Morris H C 1982 *Soliton and nonlinear wave equations* (Academic Press Inc. [Harcourt Brace Jovanovich Publishers])
[17] Eliseev V V 1999 *Mechanics of elastic bodies* (SPb: Publishing house of SPbSTU) p. 341
[18] Fomin I V, Chervon S V 2020 *J.Phys.Conf.Ser.* **1557** 012020