

Chapter 2

Continuous and Discrete (Flavor) Symmetries

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Abstract

In this talk I give an overview over continuous and discrete groups and how these are used in the field of model building as flavor symmetries. The latter act on the space of the three generations of elementary particles. I mainly concentrate on discussing generic mathematical properties of these groups relevant for understanding their possible predictive power when applied to explain fermion mass and mixing patterns. I also put emphasis on the classification of discrete groups.

2.1. Introduction

The existence of three generations of elementary particles is well-established. They can be distinguished according to their mass as well as their mixing. The mass hierarchy among charged fermions is known to be strong, especially in the up quark sector

$$m_u : m_c : m_t \approx \lambda^8 : \lambda^4 : 1, \quad m_d : m_s : m_b \approx \lambda^4 : \lambda^2 : 1, \quad m_e : m_\mu : m_\tau \approx \lambda^{4\div 5} : \lambda^2 : 1 \quad \text{with} \quad \lambda \approx 0.22,$$

while it is much milder in the neutrino sector. Neutrino masses are experimentally constrained by the measurements of solar and atmospheric mass square differences (at the 2σ level) [1]

$$\delta m^2 \equiv \Delta m_{\text{sol}}^2 \equiv m_2^2 - m_1^2 = (7.58_{-0.42}^{+0.41}) \times 10^{-5} \text{ eV}^2, \quad |\Delta m^2| \equiv \left| m_3^2 - \frac{m_2^2 + m_1^2}{2} \right| = (2.35_{-0.18}^{+0.22}) \times 10^{-3} \text{ eV}^2.$$

The sign of Δm^2 is undetermined and the absolute neutrino mass scale m_0 is bounded, $m_0 \lesssim 0.3 \text{ eV}$, by cosmological measurements [2], searches for neutrinoless double beta [3] and Tritium beta decay [4]. The mixing among quarks is small

$$\theta_{12}^q \equiv \theta_C \approx 13^\circ, \quad \theta_{23}^q \approx 2.4^\circ \quad \text{and} \quad \theta_{13}^q \approx 0.21^\circ.$$

At the same time, lepton mixing, measured in neutrino oscillation experiments, involves two large mixing angles [1]

$$\sin^2 \theta_{12}^l = 0.306_{-0.031}^{+0.036}, \quad \sin^2 \theta_{23}^l = 0.42_{-0.06}^{+0.18} \quad (2\sigma \text{ level})$$

and a third small angle which is non-zero [1] - according to recent experimental indications [5]

$$\sin^2 \theta_{13}^l = 0.021_{-0.013}^{+0.015} \quad (2\sigma \text{ level}). \quad (2.1)$$

This value is obtained using the old estimates for reactor anti-neutrino fluxes. Neither the hierarchy among the charged fermion masses nor the fermion mixing patterns can be explained in the framework of the Standard Model (SM). The peculiar structure of the lepton mixing is compatible with special, theoretically motivated, patterns like

- $\mu\tau$ symmetric mixing [6]: $\sin^2 \theta_{23}^l = \frac{1}{2}$, $\sin^2 \theta_{13}^l = 0$ and $\theta_{12}^l \sim \mathcal{O}(1)$,
- tri-bimaximal (TB) mixing [7]: $\sin^2 \theta_{12}^l = \frac{1}{3}$, $\sin^2 \theta_{23}^l = \frac{1}{2}$, $\sin^2 \theta_{13}^l = 0$,
- golden ratio pattern [8]: $\sin^2 \theta_{12}^l = \frac{1}{\sqrt{5}\phi} \approx 0.276$, $\sin^2 \theta_{23}^l = \frac{1}{2}$, $\sin^2 \theta_{13}^l = 0$ with $\phi = (1 + \sqrt{5})/2$,
- bimaximal mixing [9]: $\sin^2 \theta_{12}^l = \frac{1}{2}$, $\sin^2 \theta_{23}^l = \frac{1}{2}$, $\sin^2 \theta_{13}^l = 0$,

up to small corrections. Taking into consideration the success of symmetries describing correctly the gauge interactions of the SM particles, it is tempting to assume a flavor symmetry G_f , i.e. a symmetry acting on the space of generations, to be responsible for the features of fermion masses and mixing.

In section 2.2 I present an overview over general properties of a flavor symmetry G_f . Section 2.3 is dedicated to a brief discussion of models with continuous G_f and section 2.4 contains the explanation of the connection between the non-trivial symmetry breaking of a discrete group G_f and the fermion mixing pattern. I show how TB mixing arises from the groups A_4 and S_4 and how predictions for elements of the mixing matrix are derived from (single-valued and double-valued) dihedral groups. In section 2.5 I sketch a supersymmetric D_{14} model explaining the Cabibbo angle θ_C and its extension to the lepton sector which leads to maximal atmospheric mixing, vanishing θ_{13}^l and generically large θ_{12}^l . I conclude in section 2.6.

2.2. General properties of G_f

Several properties of this new symmetry have to be fixed, namely

- whether it is abelian or non-abelian,
- whether it is continuous (like the gauge symmetries) or discrete (like parity),
- whether it is local (like the gauge symmetries) or global (like lepton number $U(1)_L$),
- whether it commutes with the other symmetries (e.g. gauge symmetries) of the theory or not,
- whether it is broken spontaneously or explicitly,
- whether it is broken at high energies (like at the seesaw or the GUT scale) or low energies (like the electroweak scale),
- whether it is broken in an arbitrary way or to one of its subgroups.

Apart from that the size of G_f depends on the gauge group of the model, e.g. in the framework of the SM without right-handed neutrinos the maximal symmetry of the kinetic terms of the fermions is $U(3)^5$, while it is reduced to $U(3)$ in the case of an $SO(10)$ model in which all fermions of one generation including the right-handed neutrino are unified into the 16-dimensional representation.

The presence of more than one generation can only be explained with a non-abelian symmetry which has two- and three-dimensional irreducible representations. An advantage of discrete over continuous symmetries is that the former in general contain several small representations suitable to host the three fermion generations. Furthermore, it is well-known that the spontaneous breaking of a continuous global (local) symmetry leads to the appearance of Goldstone (gauge) bosons, respectively. In order to avoid this I choose the symmetry to be discrete. However, also the breaking of the latter does in general lead to experimentally constrained effects, namely the generation of domain walls. This does not pose a problem as long as the scale at which the symmetry is broken is comparable with or larger than the scale of inflation. Thus high energy scales like the seesaw scale or the scale of grand unification seem to be preferred. In this case the flavor symmetry G_f is usually broken through non-vanishing vacuum expectation values (VEVs) of gauge singlets, called flavons. Models in which G_f is broken at a low energy scale contain instead several Higgs doublets which also transform non-trivially under G_f . Such models seem to be disfavored, because flavor changing neutral currents and lepton flavor violating processes, like $\mu \rightarrow e\gamma$, are induced. Another possibility is to explicitly break G_f , for example, at the boundaries of an extra spatial dimension.

A list of possible choices for G_f is

- continuous symmetries: $SU(2)$, $U(2)$, $SO(3)$, $SU(3)$ and $U(3)$.
- discrete symmetries:
 - permutation symmetries: symmetric groups S_N and alternating groups A_N with $N \in \mathbb{N}$,
 - dihedral symmetries: single-valued groups D_n and double-valued groups D'_n with $n \in \mathbb{N}$,
 - further double-valued groups: T' , O' , I' , ...
 - subgroups of $SU(3)$: $\Delta(3n^2)$ and $\Delta(6n^2)$ with $n \in \mathbb{N}$, as well as Σ groups,
 - subgroups of $U(3)$ such as $\Sigma(81)$ or of the listed groups, e.g. $T_7 \cong Z_7 \rtimes Z_3 \subset \Delta(147)$.

2.3. Overview over models with continuous G_f

In this section I give a brief, and incomplete, survey over models with continuous non-abelian flavor symmetries. In [10] supersymmetric models with a $U(2)$ group are discussed in the context of $SU(5)$ and $SO(10)$ grand unified theories (GUTs). Generic features of these models are: the three generations of fermions are assigned to $\mathbf{2} + \mathbf{1}$ which easily explains the heaviness of the third generation and allows to alleviate the so-called supersymmetric flavor problem; the breaking of the symmetry $U(2)$ proceeds in two steps, first to a $U(1)$ subgroup and then to nothing. These models predict nine relations among fermion masses and mixing, however, only θ_{23}^l is in general large in the lepton sector, while the two other mixing angles θ_{12}^l and θ_{13}^l are small. Models with flavor symmetries $SO(3)$ [11] and $SU(3)$ [12] are more promising, because they allow to unify all three generations of elementary particles. Furthermore, the largeness of two mixing angles, θ_{12}^l and θ_{23}^l , in the lepton sector indicates that all three generations are closely related instead of only two. Also in these models the breaking of the flavor symmetry proceeds usually in two steps, i.e. $SO(3) \rightarrow SO(2) \rightarrow \text{nothing}$ and $SU(3) \rightarrow SU(2) \rightarrow \text{nothing}$, respectively. The models are supersymmetric and the gauge group is the one of the SM, the Pati-Salam group or, in the case of an extra-dimensional model, $SO(10)$. The masses of the SM fermions arise usually only from non-renormalizable operators and additional symmetries, such as Z_n or $U(1)$, are imposed in order to forbid certain operators. Assuming different messenger mass scales allows to differentiate among the expansion parameters present in the up quark, down quark and charged lepton

mass matrices. It has been shown that in such models TB neutrino mixing is predicted through constrained sequential dominance and a specific structure of the flavons VEVs. The latter can be achieved through F- as well as D-terms. The corrections arising from the charged lepton sector to the mixing angles are small. Subsequently, models in which the continuous symmetry has been replaced by a discrete one, like A_4 , $\Delta(27)$ and $\Delta(108)$, have been discussed [13]. The results of these models are very similar to those with continuous flavor groups. In general, the vacuum alignment is achieved in a simpler way; however, in the case of a discrete group D-terms associated with the flavor symmetry are obviously absent.

2.4. Non-trivial breaking of discrete G_f and fermion mixing

In the following I exemplify the idea of the non-trivial breaking of the flavor symmetry in different sectors of the theory, e.g. the charged lepton and the neutrino one, presenting several models: first, I briefly discuss how A_4 and S_4 give rise to TB mixing, then dihedral symmetries D_n in general as well as the group D_{14} in particular which can explain quark and lepton mixing patterns at the same time.

2.4.1. TB mixing and the groups A_4 and S_4

Probably, one of the most famous examples of a flavor symmetry broken in a non-trivial way is the one of the group A_4 which leads to the prediction of TB mixing. A simple and elegant model can be found in [14]. It is an extension of the minimal supersymmetric SM (MSSM) in which the group A_4 is spontaneously broken at high energies through flavon VEVs. The theory is formulated as an effective one with a cutoff Λ , assumed to be of the order of the GUT scale. An elaborate construction of the potential ensures the correct vacuum alignment. The supermultiplets containing left-handed leptons transform as irreducible triplet of A_4 and right-handed charged leptons are in three inequivalent one-dimensional representations. The two Higgs doublets $h_{u,d}$ do not transform under A_4 , while the flavons breaking A_4 correctly are two triplets, called φ_T and φ_S and two singlets $\xi, \tilde{\xi}$ (only one of them acquires a non-zero VEV). Apart from A_4 the model has a family independent Z_3 symmetry and a family dependent $U(1)_{FN}$ symmetry. The former allows to separate the two sets of flavons $\{\varphi_T\}$ and $\{\varphi_S, \xi, \tilde{\xi}\}$ and thus the charged lepton and neutrino sectors in the superpotential at the renormalizable level, while the latter which distinguishes among the right-handed charged leptons is used to explain the hierarchy among charged lepton masses. This $U(1)_{FN}$ symmetry is also assumed to be spontaneously broken, by a gauge singlet θ , carrying charge -1 .

The group A_4 is the symmetry group of the even permutations of four objects and is isomorphic to the symmetry group of a regular tetrahedron. It has 12 elements and four irreducible representations: three singlets $1, 1', 1''$ and a triplet 3 . The only non-trivial Kronecker product is $3 \times 3 = 1 + 1' + 1'' + 3 + 3$. The group can be defined in terms of two generators S and T which satisfy the relations (E is the neutral element of the group) [14]

$$S^2 = E, \quad T^3 = E, \quad (ST)^3 = E. \quad (2.2)$$

For the singlets S and T are simply 1 and the third roots of unity, $1, \omega^2, \omega$, respectively. For the triplet we use a basis in which T is represented through a diagonal matrix [14]

$$S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}. \quad (2.3)$$

With the vacuum alignment (for details see [14])

$$\langle \varphi_S \rangle = v_S (1, 1, 1), \quad \langle \xi \rangle = u, \quad \langle \tilde{\xi} \rangle = 0, \quad \langle \varphi_T \rangle = v_T (1, 0, 0), \quad (2.4)$$

the mass matrix $\mathcal{M}_l$ of the charged leptons is diagonal

$$\mathcal{M}_l = \frac{v_T}{\Lambda} \langle h_d \rangle \text{diag} \left(y_e \frac{\langle \theta \rangle^2}{\Lambda^2}, y_\mu \frac{\langle \theta \rangle}{\Lambda}, y_\tau \right), \quad (2.5)$$

while the one of the neutrinos M_ν arising from the Weinberg operator takes the form

$$M_\nu = \frac{\langle h_u \rangle^2}{\Lambda} \begin{pmatrix} a + 2b/3 & -b/3 & -b/3 \\ -b/3 & 2b/3 & a - b/3 \\ -b/3 & a - b/3 & 2b/3 \end{pmatrix}. \quad (2.6)$$

As one immediately sees the latter leads to TB mixing and the neutrino masses are $|a + b|$, $|a|$, $|-a + b|$ in units of $\langle h_u \rangle^2/\Lambda$ with $a = x_a u/\Lambda$, $b = x_b v_S/\Lambda$. Apart from its simplicity this model is very appealing, because, as has been observed, the group A_4 is broken in a specific way. The flavons φ_S (ξ) break A_4 to a Z_2 subgroup in the neutrino sector, since the VEVs of these flavons are eigenvectors to the eigenvalue +1 of the element S of A_4 . From eq.(2.2) it is obvious that S generates a Z_2 group. The VEV of φ_T breaks A_4 to a Z_3 subgroup in the charged lepton sector, because it is an eigenvector to the eigenvalue +1 of the matrix representing the element T of A_4 which has order three, see eq.(2.2). Note that in this way A_4 is broken completely in the whole theory. The mismatch of the two different subgroups of A_4 , preserved in neutrino and charged lepton sectors, reflects the mismatch between neutrinos and charged leptons in the flavor space and thus allows for a neat interpretation of lepton mixing. All this is independent of the particular value of the fermion masses. Similarly, small mixing in the quark sector might be understood as sign that the subgroups preserved in the up and down quark sectors are the same (for a model realizing this idea with the help of the group T' , the double covering of A_4 , see [15]).

From the mathematical point of view two things are interesting to notice: first of all, not only non-zero VEVs of fields forming a 1 or a 3 can preserve a Z_2 group, but also those of fields transforming as $1'$ or $1''$, and second, apart from Z_2 and Z_3 D_2 is a subgroup of A_4 (whose preservation is however only compatible with non-zero VEVs of flavons being singlets of A_4). The first aspect can be seen in a different way: the matrix in eq.(2.6) is not only invariant under the element S of A_4 , but also under the matrix P_{23} which is the matrix representing the permutation of second and third rows and columns. The latter generates also a Z_2 group and commutes with the element S . Thus, the neutrino mass matrix M_ν is invariant under $Z_2 \times Z_2$, with one of the two being a subgroup of A_4 , while the other one is an accidental symmetry which arises through the particular choice of the flavor symmetry breaking fields. This fact has been used as argument against A_4 and it has been shown [16] that indeed from the mathematical view point the group S_4 is more appropriate for predicting TB mixing, because it contains the element S as well as P_{23} . The simple idea is then to use S_4 as flavor group and break it to $Z_2 \times Z_2$ in the neutrino sector, while the breaking in the charged lepton sector remains the same, i.e. $S_4 \rightarrow Z_3$. Again, the flavor group is broken completely in the whole theory. However, one problem which might arise in S_4 models is related to the question of how to achieve the mass hierarchy among charged leptons naturally, because S_4 only contains two inequivalent singlets in contrast to A_4 . Thus, it might not be simple to distinguish the three (right-handed) charged leptons. This problem can however be easily solved [17] by extending the flavor group to $S_4 \times Z_3$ and breaking the latter in the charged lepton sector to $Z_3^{(D)}$ which is the diagonal subgroup of the external Z_3 factor and the Z_3 contained in S_4 and generated by T . In order to distinguish the three right-handed charged leptons which transform trivially under S_4 , one assigns them to 1, ω^2 and ω under the additional Z_3 group.

Before discussing other flavor groups let me briefly comment on the question whether or not models predicting TB mixing are still favored in the light of the recent experimental results [5] which indicate a non-zero value for θ_{13}^l . Indeed, in all models the leading order result, in this case TB mixing, receives corrections from various sources, e.g. higher-dimensional operators. Thus, in all models a non-zero value of θ_{13}^l is aspected. Concerning its size one can roughly say: let us assume the size of the corrections to be δ and that the latter contribute in the same manner to all three mixing angles, then the request to not perturb too much the result for the solar mixing angle implies that $\delta \lesssim 0.05$. Thus, one might expect $\sin \theta_{13}^l \sim \delta \lesssim 0.05$ which is too small to explain the recent experimental indication. Since in many models several operators give rise to a correction to θ_{13}^l , we get in general $\sin \theta_{13}^l \approx |c|\delta$ with c complex. If c is largish, i.e. the corrections add up, larger values of θ_{13}^l can be explained. Alternatively, one can consider models in which the corrections to the

reactor mixing angle are generically larger than to the two other angles. This happens for example if there are two sets of flavor symmetry breaking fields whose VEVs are different in size [18]. Yet, another possibility is to assume that the mixing pattern at leading order is bimaximal mixing. Then, the reactor as well as the solar mixing angle have to receive corrections of order $\delta \sim \lambda \sim 0.2$ [19]. The crucial issue is then to protect the atmospheric mixing angle θ_{23}^l from too large corrections.

2.4.2. Dihedral symmetries D_n as G_f

In order to show the interesting and amusing properties, a non-trivial breaking of a discrete symmetry can have, I discuss the case of a dihedral flavor group D_n which is broken to two distinct Z_2 subgroups. If the former is generated by A and B, the generating elements of the latter are BA^{k_1} and BA^{k_2} (k_i integers), respectively. If two of the left-handed fermion (quark or lepton) generations transform as irreducible two-dimensional representation $\underline{2}_j$, one of the elements of the mixing matrix V is of the form

$$|V_{\alpha\beta}| = \left| \cos \left(\frac{\pi (k_1 - k_2) j}{n} \right) \right|. \quad (2.7)$$

As one can nicely see, $|V_{\alpha\beta}|$ only depends on group-theoretical quantities, i.e. the index n of the dihedral group D_n , the indices characterizing the two Z_2 subgroups $k_{1,2}$ and the index j of the two-dimensional representation. Note that for $k_1 = k_2$ the mixing is trivial, as expected. The way to reach the result in eq.(2.7) can be understood performing the following seven steps: first, one has to choose the group D_n and select the indices $k_{1,2}$ of the two Z_2 subgroups which are associated with the two different fermion sectors 1 and 2. Then, one assigns left-handed fields to $\underline{1}_s + \underline{2}_j$, while the three generations of right-handed fermions can either transform as singlets $\underline{1}_{ip}$ or also as singlet and doublet $\underline{1}_l + \underline{2}_m$. We then consider a generic model in which two sets of scalars exist $\{\Phi_1\}$ and $\{\Phi_2\}$. These sets contain fields in all representations μ of D_n and we assume their VEVs to be such that they leave invariant a Z_2 group generated by $BA^{k_{1,2}}$, respectively. As one can check, the most general matrices $\mathcal{M}_i$ with $i = 1, 2$ for fermions of sector 1 and 2 are of the form (always given in the left-right basis)

$$\mathcal{M}_i = \begin{pmatrix} 0 & A_i & B_i \\ C_i & D_i & E_i \\ -C_i e^{-i\varphi_i j} & D_i e^{-i\varphi_i j} & E_i e^{-i\varphi_i j} \end{pmatrix}, \quad (2.8)$$

if right-handed fermions are singlets under the group D_n , or take the form

$$\mathcal{M}_i = \begin{pmatrix} A_i & C_i & C_i e^{-i\varphi_i m} \\ B_i & D_i & E_i \\ B_i e^{-i\varphi_i j} & E_i e^{-i\varphi_i (j-m)} & D_i e^{-i\varphi_i (j+m)} \end{pmatrix}, \quad (2.9)$$

if right-handed fermions transform as $\underline{1}_l + \underline{2}_m$. The parameters $A_i, \dots, E_i$ are in general complex and not further constrained by the flavor symmetry D_n . The phase φ_i which appears in both cases only depends on group-theoretical quantities

$$\varphi_i = \frac{2\pi k_i}{n}. \quad (2.10)$$

For both matrices the combination $\mathcal{M}_i \mathcal{M}_i^\dagger$ can be written as

$$\mathcal{M}_i \mathcal{M}_i^\dagger = \begin{pmatrix} a_i & b_i e^{i\zeta_i} & b_i e^{i(\zeta_i + \varphi_i j)} \\ b_i e^{-i\zeta_i} & c_i & d_i e^{i\varphi_i j} \\ b_i e^{-i(\zeta_i + \varphi_i j)} & d_i e^{-i\varphi_i j} & c_i \end{pmatrix} \quad (2.11)$$

with $a_i, \dots, d_i$ and ζ_i depending on the parameters present in the matrices in eqs.(2.8)-(2.9). The matrix U_i diagonalizing the combination $\mathcal{M}_i \mathcal{M}_i^\dagger$ is

$$U_i = \begin{pmatrix} 0 & \cos \theta_i e^{i\zeta_i} & \sin \theta_i e^{i\zeta_i} \\ -\frac{1}{\sqrt{2}} e^{i\varphi_i j} & -\frac{\sin \theta_i}{\sqrt{2}} & \frac{\cos \theta_i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{\sin \theta_i}{\sqrt{2}} e^{-i\varphi_i j} & \frac{\cos \theta_i}{\sqrt{2}} e^{-i\varphi_i j} \end{pmatrix} \quad (2.12)$$

with θ_i given in terms of the parameters appearing in eq.(2.11). As one clearly sees, one of the eigenvectors only contains the phase φ_i and thus only depends on group-theoretical quantities. The physical mixing arises from the misalignment of the matrices U_1 and U_2 :

$$V = U_1^\dagger U_2. \quad (2.13)$$

Then one of the elements $|V_{\alpha\beta}|$ is of the form

$$|V_{\alpha\beta}| = \frac{1}{2} \left| 1 + e^{i(\varphi_1 - \varphi_2)j} \right| = \left| \cos \left((\varphi_1 - \varphi_2) \frac{j}{2} \right) \right| = \left| \cos \left(\frac{\pi (k_1 - k_2)j}{n} \right) \right|, \quad (2.14)$$

while the rest depends also on the angles $\theta_{1,2}$ and the phases $\zeta_{1,2}$.

It has been shown [20] that $\mu\tau$ symmetric lepton mixing can be realized in this way with the group D_4 [21] and that D_{14} can be used to predict the correct size of the Cabibbio angle [22]. In the subsequent section a model is outlined in which D_{14} plays the role of the flavor symmetry and which explains the Cabibbio angle as well as nearly $\mu\tau$ symmetric lepton mixing [23].

2.5. D_{14} - a symmetry for quarks and leptons

The group D_{14} belongs to the dihedral symmetries and has 28 elements. As all D_n groups, it only contains one- and two-dimensional representations, called $\underline{1}_i$, $i = 1, \dots, 4$ and $\underline{2}_j$, $j = 1, \dots, 6$. It can be described in terms of two generators A and B which satisfy the relations

$$A^{14} = E, \quad B^2 = E, \quad A B A = B. \quad (2.15)$$

The generators A and B for the singlets are just ± 1 and the two-by-two matrices for $\underline{2}_j$ are in a convenient basis

$$A = \begin{pmatrix} e^{(\frac{\pi i}{7})j} & 0 \\ 0 & e^{-(\frac{\pi i}{7})j} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (2.16)$$

The main properties of the original D_{14} model [22] which only contains quarks are the following: it is an extension of the MSSM formulated as effective theory with a cutoff Λ , the flavor symmetry G_f is the product $D_{14} \times U(1)_{FN} \times Z_3$, D_{14} and $U(1)_{FN}$ together explain the quark mass hierarchies and Z_3 is used for separating appropriately up and down quark sectors. All flavor symmetries are broken spontaneously at high energies through flavon VEVs. The two sets of flavons breaking the group $D_{14} \times Z_3$ are $\{\psi_{1,2}^u, \chi_{1,2}^u, \xi_{1,2}^u, \eta^u\}$ and $\{\psi_{1,2}^d, \chi_{1,2}^d, \xi_{1,2}^d, \eta^d, \sigma\}$, while θ only carries non-trivial $U(1)_{FN}$ charge. The left-handed quark doublets $Q_D = (Q_1, Q_2)$ and Q_3 are in the representations $\underline{2}_1$ and $\underline{1}_1$, while the right-handed quarks u^c, c^c, t^c and d^c, s^c, b^c are singlets under D_{14} and $h_{u,d}$ are trivial singlets of D_{14} .

The leading operators contributing to up quark masses are (order one coefficients are omitted in the follow-

ing and $(\dots)$ denotes the contraction to a D_{14} -invariant)

$$Q_3 t^c h_u + \frac{1}{\Lambda} (Q_D \psi^u) t^c h_u + \frac{1}{\Lambda} Q_3 (c^c \eta^u) h_u \quad (2.17)$$

$$+ \frac{\theta^2}{\Lambda^4} (Q_D u^c \chi^u \xi^u) h_u + \frac{\theta^2}{\Lambda^4} (Q_D u^c (\xi^u)^2) h_u + \frac{\theta^2}{\Lambda^4} (Q_D \psi^u \eta^u u^c) h_u \quad (2.18)$$

$$+ \frac{1}{\Lambda^2} (Q_D c^c \chi^u \xi^u) h_u + \frac{1}{\Lambda^2} (Q_D c^c (\xi^u)^2) h_u + \frac{1}{\Lambda^2} (Q_D \psi^u) (\eta^u c^c) h_u, \quad (2.19)$$

while the ones contributing to the masses of the down quarks read

$$\frac{1}{\Lambda} Q_3 (b^c \eta^d) h_d + \frac{\theta}{\Lambda^2} Q_3 s^c \sigma h_d + \frac{\theta}{\Lambda^2} (Q_D \psi^d) s^c h_d. \quad (2.20)$$

Note that all these operators respect the separation among the up and down quark sectors, as up-type flavons only contribute to up quark masses and down-type ones only to down quark masses. With the appropriate vacuum configuration of the flavons, see [22] for details of the superpotential, two distinct Z_2 groups are preserved. One is characterized with an even index, see above, which can be set without loss of generality to zero, while the second one has an odd index k which remains undetermined. At the Z_2 symmetry preserving level, the up and down quark mass matrices are of the form

$$\mathcal{M}_u = \begin{pmatrix} -\alpha_1^u t^2 \epsilon^2 & \alpha_2^u \epsilon^2 & \alpha_3^u \epsilon \\ \alpha_1^u t^2 \epsilon^2 & \alpha_2^u \epsilon^2 & \alpha_3^u \epsilon \\ 0 & \alpha_4^u \epsilon & y_t \end{pmatrix} \langle h_u \rangle \quad \text{and} \quad \mathcal{M}_d = \begin{pmatrix} 0 & \alpha_1^d t \epsilon & 0 \\ 0 & \alpha_1^d e^{-\pi i k / 7} t \epsilon & 0 \\ 0 & \alpha_2^d t \epsilon & y_b \epsilon \end{pmatrix} \langle h_d \rangle \quad (2.21)$$

with $\langle \Phi^u \rangle / \Lambda \approx \epsilon$, $\langle \Phi^d \rangle / \Lambda \approx \epsilon$, $t = \langle \theta \rangle / \Lambda \approx \epsilon \approx \lambda^2 \approx 0.04$. As one easily computes, the quark masses fulfill the relations

$$m_u^2 : m_c^2 : m_t^2 \sim \epsilon^8 : \epsilon^4 : 1, \quad m_d^2 : m_s^2 : m_b^2 \sim 0 : \epsilon^2 : 1, \quad (2.22)$$

$$m_b^2 : m_t^2 \sim \epsilon^2 : 1 \quad \text{for small } \tan \beta = \langle h_u \rangle / \langle h_d \rangle$$

and the CKM mixing matrix takes the form

$$|V_{CKM}| = \begin{pmatrix} |\cos(\frac{k\pi}{14})| & |\sin(\frac{k\pi}{14})| & 0 \\ |\sin(\frac{k\pi}{14})| & |\cos(\frac{k\pi}{14})| & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \mathcal{O}(\epsilon^4) & \mathcal{O}(\epsilon^2) \\ \mathcal{O}(\epsilon^2) & \mathcal{O}(\epsilon^2) & \mathcal{O}(\epsilon) \\ \mathcal{O}(\epsilon) & \mathcal{O}(\epsilon) & \mathcal{O}(\epsilon^2) \end{pmatrix}. \quad (2.23)$$

As one can see, for the choices $k = 1$ and $k = 13$ the matrix elements involving the first two generations are very close to their experimental best fit values, e.g. $|V_{ud}| \approx 0.97493$ should be compared with $|V_{ud}|_{\text{exp}} = 0.97419_{-0.00022}^{+0.00022}$. Subleading corrections lead to the down quark mass of the correct size as well as allow all elements of V_{CKM} and the Jarlskog invariant J_{CP} to be accommodated.

In a second step I would like to outline how to extend this model to the lepton sector in a minimalistic way, i.e. to add only a small number of new fields, to not perturb the results achieved for quarks too much and at the same time to predict $\mu\tau$ symmetric lepton mixing. The setup which fulfills these requirements is the following [23]: the Z_3 factor of the flavor symmetry is replaced by Z_7 and still has the role to segregate the different symmetry breaking sectors, the left-handed leptons L_1 and $L_D = (L_2, L_3)$ transform as $\underline{1}_3$ and $\underline{2}_2$, also the right-handed neutrinos ν_1^c and $\nu_D^c = (\nu_2^c, \nu_3^c)$ transform as $\underline{1}_2 + \underline{2}_3$, while all right-handed charged leptons e^c, μ^c, τ^c are trivial singlets of D_{14} . Furthermore, one flavon $\chi_{1,2}^e$ is added which is a doublet $\underline{2}_2$ of D_{14} .

In the neutrino sector, the leading contributions to the right-handed neutrino mass matrix $\mathcal{M}_R$ are

$$\nu_1^c \nu_1^c \sigma + (\nu_D^c \nu_D^c) \sigma \quad \text{giving rise to} \quad \mathcal{M}_R = \begin{pmatrix} \alpha_1^M & 0 & 0 \\ 0 & 0 & \alpha_2^M \\ 0 & \alpha_2^M & 0 \end{pmatrix} \epsilon \Lambda, \quad (2.24)$$

while the Dirac neutrino mass matrix $\mathcal{M}_\nu^D$ takes the form

$$\mathcal{M}_\nu^D = \begin{pmatrix} 0 & \alpha_1^D & \alpha_1^D \\ \alpha_2^D & 0 & \alpha_3^D \\ -\alpha_2^D & \alpha_3^D & 0 \end{pmatrix} \epsilon \langle h_u \rangle \quad (2.25)$$

and receives at leading level contributions from the operators

$$\frac{1}{\Lambda} (L_1 \nu_D^c \xi^u) h_u + \frac{1}{\Lambda} (L_D \nu_1^c \chi^u) h_u + \frac{1}{\Lambda} (L_D \nu_D^c \psi^u) h_u. \quad (2.26)$$

Consequently, the light neutrino mass matrix $\mathcal{M}_\nu$ can be cast into the form

$$\mathcal{M}_\nu = \begin{pmatrix} 2x^2/v & x & x \\ x & z & v-z \\ x & v-z & z \end{pmatrix} \epsilon \langle h_u \rangle^2 / \Lambda. \quad (2.27)$$

The predictions are then: $\mu\tau$ symmetric neutrino mixing, $\tan(\theta_{12}^\nu) = \sqrt{2} \left| \frac{x}{v} \right|$ and normal ordering with $m_1 = 0$. In addition, one finds $|m_{ee}| = m_2 \sin^2(\theta_{12}^\nu) = \sqrt{\delta m^2} \sin^2(\theta_{12}^\nu)$. For the charged lepton sector we use the additional flavon $\chi_{1,2}^e$ which acquires a VEV of the form $v^e (1, 0)$. Such an alignment is easily achieved with a driving field being a trivial singlet under D_{14} and carrying an appropriate Z_7 charge. Interestingly enough, also this alignment preserves a Z_2 subgroup of D_{14} which, however, does not coincide with one of the Z_2 symmetries present in the up and down quark sectors. The main contributions to the charged lepton mass matrix $\mathcal{M}_l$ originate from (α stands for e, μ and τ)

$$\frac{1}{\Lambda} (L_D \chi^e) \alpha^c h_d + \frac{1}{\Lambda^2} (L_D \chi^e \xi^u) \alpha^c h_d + \frac{1}{\Lambda^3} (L_1 \chi^e \psi^u \xi^u) \alpha^c h_d + \frac{1}{\Lambda^3} (L_1 \eta^u) (\chi^e \chi^u) \alpha^c h_d \quad (2.28)$$

which give rise to

$$\mathcal{M}_l = \begin{pmatrix} \alpha_1^e \epsilon^3 & \alpha_2^e \epsilon^3 & \alpha_3^e \epsilon^3 \\ \alpha_4^e \epsilon^2 & \alpha_5^e \epsilon^2 & \alpha_6^e \epsilon^2 \\ \alpha_7^e \epsilon & \alpha_8^e \epsilon & \alpha_9^e \epsilon \end{pmatrix} \langle h_d \rangle \quad \text{for} \quad v^e / \Lambda \approx \epsilon \approx \lambda^2. \quad (2.29)$$

As one can see, the charged lepton mass hierarchy is correctly predicted, $m_e : m_\mu : m_\tau \sim \epsilon^2 : \epsilon : 1$, and the three mixing angles in the charged lepton sector are small

$$\theta_{12}^e \sim \epsilon, \quad \theta_{13}^e \sim \epsilon^2, \quad \theta_{23}^e \sim \epsilon. \quad (2.30)$$

Thus, the lepton mixing is nearly $\mu\tau$ symmetric, i.e.

$$\sin^2 \theta_{23}^l = \frac{1}{2} + \mathcal{O}(\epsilon), \quad \sin \theta_{13}^l = \mathcal{O}(\epsilon), \quad \sin^2 \theta_{12}^l = \mathcal{O}(1). \quad (2.31)$$

As regards the recent indication for $\theta_{13}^l \neq 0$ [5], we note that the model naturally leads only to rather small θ_{13}^l which might seem to be disfavored at the moment.

2.6. Conclusions

I have presented an overview over flavor symmetries in general and their properties. I commented briefly on the case of a continuous G_f and then focussed on discrete symmetries playing the role of G_f with special emphasis on those cases in which they are broken in a non-trivial way. I showed a variety of examples: A_4 and S_4 and their relation to TB mixing, predictions from dihedral groups D_n in general and from D_{14} in particular. Lastly, I have sketched a D_{14} model in which the Cabibbo angle as well as $\mu\tau$ symmetric lepton mixing are closely related to how D_{14} is broken and at the same time all fermion mass hierarchies are naturally accommodated.

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