Exact solutions of Einstein and Einstein-Maxwell gravity minimally coupled to a scalar field

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Abstract. We present a general method for solving exactly the static field equations of Einstein and Einstein-Maxwell gravity minimally coupled to a scalar field. Through this method one can derive broad classes of static solutions with radial symmetry of the theory, which may play an important role in applications of the AdS/CFT correspondence to condensed matter and strongly coupled QFT's. Moreover, the method allows to prove a new no-hair theorem about the existence of hairy black brane and black hole solutions.

1. Introduction

In recent years there has been a renewed growing interest for the static black hole solutions of Einstein (and Einstein-Maxwell) gravity coupled to scalar fields. In particular, the recent advances in string theory and the AdS/CFT correspondence [1] have shifted the focus from the search of asymptotically flat to asymptotically anti-de Sitter (AdS) solutions. In fact, static black hole and black brane solutions with a non trivial profile of the scalar field (scalar 'hair') and AdS asymptotics can play a crucial role in applications of the AdS/CFT correspondence to holographic strongly coupled quantum field theories (QFTs). In these theories the nontrivial, coordinate-dependent scalar hair of the black hole solutions can be interpreted as a scalar condensate which gives rise to a rich phenomenology in the dual QFT, reminiscent of well-known condensed matter systems [2, 3, 4, 5].

However, despite the growing importance played by static black hole solutions with scalar hair and AdS asymptotics, very few exact analytical solutions of this kind are known [6, 7, 8].

Starting from these considerations, we propose a general method for solving the field equations of Einstein and Einstein-Maxwell gravity minimally coupled to a scalar field ϕ in the static, radially symmetric case. This method allows to find exact analytic solutions of Einstein and Einstein-Maxwell gravity with scalar hair in several situations: four or generic $d + 2$ spacetime dimensions, different topologies of the transverse space (planar, spherical, hyperbolic) and different asymptotics (anti-de Sitter, domain wall, conformal to Lifshitz spacetime). Moreover, the method can be used to prove a new 'no-hair' theorem about the general existence of black brane and black hole solutions of Einstein and Einstein-Maxwell gravity minimally coupled to a scalar field.

In this paper, mainly based on the work [9], we briefly present this solving method (Sect. 2), a panoramic of the exact solutions obtained (Sect.3, 4) and the formulation of the new no-hair theorem (Sect. 5). Finally, in Sect.6 we point out our concluding remarks.

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2. The solving method

We start from the action of Einstein-Maxwell gravity in $d+2$ dimensions, minimally coupled to a scalar field ϕ and with a generic self-interaction potential $V(\phi)$:

$$A = \int d^{d+2}x\sqrt{-g}\left(\mathcal{R} - 2(\partial\phi)^2 - F^2 - V(\phi)\right). \tag{1}$$

We will investigate static, radially symmetric and purely electric solutions. We adopt a Schwarzschild gauge to write the spacetime metric:

$$ds^2 = -U(r)dt^2 + U^{-1}(r)dr^2 + R^2(r)d\Omega^2_{(\varepsilon,d)}, \tag{2}$$

where $\varepsilon = 0, 1, -1$ denotes, respectively, the d-dimensional planar, spherical, or hyperbolic transverse space with metric $d\Omega^2_{(\varepsilon,d)}$. In these coordinates, the electric field reads: $F_{tr} = Q/R^d$, where Q is the electric charge. With the parametrization (2), the field equations take the form:

$$\frac{R''}{R} = -\frac{2}{d}(\phi')^2, \quad (UR^d\phi')' = \frac{1}{4}R^d\frac{dV}{d\phi}, \tag{3}$$

$$(UR^d)'' = \varepsilon d(d-1)R^{d-2} + 2\frac{d-2}{d}\frac{Q^2}{R^d} - \frac{d+2}{d}R^dV, \tag{4}$$

$$(UR^{d-1}R')' = \varepsilon(d-1)R^{d-2} - \frac{2}{d}\frac{Q^2}{R^d} - \frac{1}{d}R^dV. \tag{5}$$

Our method for solving the field equations (3)-(5) works as follows. Assuming that the r-dependence of the scalar field $\phi(r)$ is given, and introducing the new variables F, Y and u defined as:

$$F(r) = -\frac{2}{d}(\phi')^2, \quad R = e^{\int Y}, \quad u = UR^d, \tag{6}$$

the field equations (3)-(5) become:

$$Y' + Y^2 = F, \quad (u\phi')' = \frac{1}{4}e^{d\int Y}\frac{dV}{d\phi}, \tag{7}$$

$$u'' - (d+2)(uY)' = -2\varepsilon(d-1)e^{(d-2)\int Y} + 4Q^2e^{-d\int Y}, \tag{8}$$

$$u'' = \varepsilon d(d-1)e^{(d-2)\int Y} + 2\frac{d-2}{d}Q^2e^{-d\int Y} - \frac{d+2}{d}e^{d\int Y}V. \tag{9}$$

The first equation in (7) is a first-order nonlinear equation for Y, known as the Riccati equation, which can be solved in a number of cases. Once the solution for Y has been found, we can integrate Eq. (8), which is linear in u, to obtain:

$$u = R^{d+2}\left[\int\left(4Q^2\int\frac{1}{R^d} - 2\varepsilon(d-1)\int R^{d-2} - C_1\right)\frac{1}{R^{d+2}} + C_2\right], \tag{10}$$

where C_1 and C_2 are integration constants. Finally, we can determine the potential $V(\phi)$ by using Eq. (9):

$$V = \frac{d^2(d-1)}{d+2}\frac{\varepsilon}{R^2} + 2\frac{d-2}{d+2}\frac{Q^2}{R^{2d}} - \frac{d}{d+2}\frac{u''}{R^d}, \tag{11}$$

while the metric functions read (cfr. 6): $R = \Lambda e^{\int Y}$, $U = u/R^d$, where the integration constant Λ coming from the integral of Y.

The key feature of this method is to assume a given profile $\phi(r)$ for the scalar field and to solve the system for the metric functions and the potential $V(\phi)$. This unusual approach is actually very useful because, focusing on solutions with AdS asymptotics, in particular for what concerns application to the AdS/CFT correspondence, the actual exact form of the potential $V(\phi)$ is not particularly relevant, while is often more important the behavior of the scalar field $\phi(r)$, in particular its fall-off behavior at $r = \infty$.

3. Uncharged solutions

In this section we will consider the case of (3 + 1)-dimensional spacetime, i.e. $d = 2$, and uncharged ($Q = 0$) black brane ($\varepsilon = 0$) solutions. In particular we are interested in asymptotically domain wall and asymptotically anti de-Sitter solutions.

3.1. Domain wall solutions

Our method for solving the field equations (3)-(5) requires an ansatz for the scalar field. Usually, domain wall solutions appear when the scalar behaves as $\log r$ [10, 11] . The most natural ansatz is therefore:

$$\gamma\phi = \log \frac{r}{r_-}, \tag{12}$$

where γ is a constant and r_- sets a length-scale.

With the ansatz (12), the Riccati equation in (7) is solved by:

$$Y = \frac{\alpha}{r}, \quad \alpha(\alpha - 1) = -\frac{1}{\gamma^2}. \tag{13}$$

Parametrizing α and γ as: $\gamma^{-1} = h\alpha = h/(h^2 + 1)$, the solution takes the form:

$$U = \left(\frac{r}{r_-}\right)^{\frac{2}{1+h^2}} - C_1 \left(\frac{r}{r_-}\right)^{-\frac{1-h^2}{1+h^2}}, \qquad R = \left(\frac{r}{r_-}\right)^{\frac{1}{1+h^2}}, \tag{14}$$

while the potential has a simple exponential form:

$$V = -\frac{2(3 - h^2)}{(1 + h^2)^2 r_-^2}\, e^{-2h\phi}. \tag{15}$$

This solution has the following main features:

- In the 'extremal' zero-temperature case, $C_1 = 0$, the solution (14) has the typical form of a single-scalar domain wall solution: $ds^2 = (Ar)^\delta(\eta_{\mu\nu}dx^\mu dx^\nu) + (Ar)^{-\delta}dr^2$, which preserves only the Poincaré isometry of the 3D transverse space [11];
- For $h^2 \leq 3$ and $C_1 > 0$, our solution (14) represents a black brane with domain wall asymptotics, a singularity at $r = 0$ and a regular horizon at $r_h = C_1^{(1+h^2)/(3-h^2)} r_-$.

3.2. Asymptotically AdS solutions

Inspired by known solutions in flat spacetime and in gauged supergravity [6], to search for asymptotically AdS solutions we use an ansatz in which ϕ is expressed in terms of an harmonic function X:

$$\gamma\phi = \log X, \qquad X = 1 - \frac{r_-}{r}, \tag{16}$$

where γ and r_- are constants.

With the ansatz (16), the Riccati equation (7) can be solved in terms of the harmonic function X to give:

$$R = \Lambda r X^{\beta + \frac{1}{2}}, \qquad \beta^2 - \frac{1}{4} = -\frac{1}{\gamma^2}. \tag{17}$$

We consider the $C_1 = 0$ extremal solution. The constant C_2 essentially determines the normalization of the potential and it can be fixed by choosing $C_2 = 1/L^2$. With these assumptions, Eq. (10) and (11) give, respectively, the solution for the metric and the scalar potential:

$$U = R^2 = \frac{r^2}{L^2}\left(1 - \frac{r_-}{r}\right)^{2\beta+1}, \tag{18}$$

$$V(\gamma,\phi) = -\frac{2}{L^2} e^{2\gamma\beta\phi}[2 - 8\beta^2 + (1 + 8\beta^2)\cosh(\gamma\phi) - 6\beta\sinh(\gamma\phi)]. \tag{19}$$

One can easily check that the hairy extremal solution (18) interpolates between an AdS vacuum: $ds^2 = (r/L)^2(\eta_{\mu\nu}dx^\mu dx^\nu) + (r/L)^{-2}dr^2$ at $r = \infty$ and a domain wall solution like the (14), with $C_1 = 0$, near $r = r_-$.

Unfortunately, our method does not allow to find a full $C_1 \neq 0$ one-parameter family of black brane solutions. If they exist, have to be found numerically.

4. Other solutions

We now present a brief overview on charged, spherical and $d+2$-dimensional solutions obtained through the solving method. For more details see [9].

4.1. Solutions conformal to Lifshitz spacetime

Starting again from the ansatz (12), we can obtain an exact solution of the field equations also when $Q \neq 0$. In this case, the potential has also a simple exponential form, while in the extremal $C_1 = 0$ case the solution is conformal to the Lifshitz spacetime: $ds^2 = l^2\left(-r^{2z}dt^2 + \frac{dr^2}{r^2} + r^2 dx^i dx^i\right)$, where z is the critical exponent [12].

For $C_1 > 0$ and in a specific range of the parameters, the solution represents a black brane with asymptotics conformal to Lifshitz, a singularity at $r = 0$ and a regular horizon.

4.2. Charged asymptotically AdS solutions

Solving the field equations with the ansatz (16), $Q \neq 0$ and $C_1 = 0$, we obtain a solution interpolating between an asymptotic AdS vacuum at $r = \infty$ and a conformal Lifshitz solution in the near-horizon limit $r \sim r_-$. This solution describes a one-parameter (the charge Q) family of black brane solutions, while also in this case the method does not allow to find a full $C_1 \neq 0$ two-parameter (mass and charge) family of solutions.

4.3. Other solutions

Several of the $\varepsilon = 0$ and $d = 2$ solutions found can be easily generalized to the case of spherical or hyperbolic topology $\varepsilon = \pm 1$ (black holes) and to $d + 2$ dimensions. In particular one can derive:

- Uncharged and charged black holes with Lifshitz and AdS asymptotics;
- Spherical solutions generated from the planar ones;
- $(d+2)$-dimensional domain wall, Lifshitz-like and uncharged asymptotically AdS solutions;

5. No-hair theorem

An important issue in this context is the question about the existence of regular, static black brane (hole) solutions of Einstein-scalar gravity with AdS asymptotics beyond the Schwarzschild and Reissner-Nordstrom solutions, i.e. of solutions endowed with non trivial scalar hair.

This problem has been already discussed in the literature. In particular, it has been argued that a necessary condition for the existence of such black hole solutions is the violation of the Positive Energy Theorem (PET) [13]. This result rules out black hole solutions with scalar hair and positive squared-mass m^2, but allows for the existence of these solutions when m^2 is negative but above the BF bound [14].

Through the solving method discussed it is possible to prove a new no-hair theorem about the existence of regular hairy black brane solutions of Einstein-scalar gravity, in which a key ingredient is the existence of an extremal $T = 0$ hairy black brane solution. The enunciate of our theorem is the following (for a detailed demonstration see [9]):

1) *One-parameter families of asymptotically AdS black brane solutions with nontrivial scalar hair exist only if the field equations (3)-(5) admit an extremal* $T = 0$, $U = R^2$ *solution.*
2) *Black-brane solutions that asymptotically approach the domain wall solution (14) exist in some range of the parameters for the case of an exponential potential* $V(\phi)$.
3) *The allowed asymptotically AdS hairy black brane solutions necessarily have a scalar hair that depends on the black brane temperature* T. *Solutions with temperature-independent scalar hair exist only for the case of domain wall spacetimes (14).*

It is important to stress that this no-hair theorem can be easily extended to the charged case $Q \neq 0$ and to the spherical (hyperbolic) case $\varepsilon = \pm 1$ (black holes).

6. Conclusions

We have presented a general method for finding exact, static and radially symmetric analytic solutions of Einstein and Einstein-Maxwell gravity minimally coupled to a scalar field. Rather than assuming a particular form of the scalar self-interaction potential, our method starts from an ansatz for the scalar field profile and determines, together with the metric functions, the corresponding form of the potential. For this reason it is particularly suitable for applications to the AdS/CFT correspondence. Through this method one can derive broad classes of exact analytic hairy solutions of Einstein and Einstein-Maxwell gravity minimally coupled to a scalar, and a new no-hair theorem about the existence of black brane (hole) solutions of Einstein gravity with scalar hair.

Our approach has a main drawback: in some situations it does not allow to find a full one-parameter family of black holes, i.e. the full spectrum of solutions for different temperatures, but only 'extremal' $T = 0$ solutions, and it is not even clear whether or not such solutions exist. A final answer to this question involves numerical computation, currently in progress.

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