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A FASTER ALGORITHM TO COMPUTE LOWEST ORDER LONGITUDINAL AND TRANSVERSE RESISTIVE WALL WAKE FOR NON-ULTRARELATIVISTIC CASE

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Abstract

With the development of the steady state micro bunching (SSMB) storage ring, its parameters reveal that the ultra relativistic assumption which is wildly used is not valid for the electron beam bunch train, which has length in the 100 nm range, spacing of $1\mu m$ and energy in hundreds MeV range. The strength of the interaction between such bunches and the potential instability may need careful evaluation. At the same time, the effect of the space charge inside a single bunch due to space charge effect also needs to be considered. In this article, we reorganized the lowest-order longitudinal wakefield under non-ultra relativistic conditions, and the lowest-order transverse wakefield. We present the modified theoretical results and analysis. Then based on the result we have derived, we give a algorithm which is thousands time faster than direct calculation. It lays foundation in future research.

INTRODUCTION

Resistive wall wakefield has been studied by [1] [2] [3] and [4].

We have developed a faster numerical calculation algorithm to calculate the inverse fourier transformation of the frequency domain result.

Firstly, we will show the reorganized result of the ring model of the ultrarelativistic resistive wall case. Then, we will give the accelerate algorithm, which devides the inverse fourier transformation into two parts — space charge and resident resistive wall. We will find that the space charge part can be calculated directly by analytical solution. In the mean time, we can calculate the resident resistive wall part quickly by numerical integration.

M=0 RING MODEL

Here we give the ring model of a monopole

$$\rho_0 = \frac{I_0}{2\pi a}\delta\left(s-vt\right)\delta\left(r-a\right) = -\frac{Q_0}{a}\delta\left(s-vt\right)\delta\left(r-a\right) \quad (1)$$

$$j_0 = v\rho_0\hat{s} \quad (2)$$

Solving Maxwell equation in different area which is $r<a$, $a<r<b$ and $r>b$. We express the general solution of the vector potential and potential in three region.

$r<a$

$$\begin{aligned} A_r &= 0 \\ A_\varphi &= 0 \\ A_s &= p_s^c I_0\left(k_r r\right)\cos\varphi\ e^{ikz} \\ \phi &= p_0^c I_0\left(k_r r\right)\cos\varphi\ e^{ikz} \end{aligned} \quad (3)$$

where Lorentz Gauge requires that

$$p_0^c = \frac{c^2 k}{\omega}p_s^c \quad (4)$$

then

$a<r<b$

$$\begin{aligned} A_r &= 0 \\ A_\varphi &= 0 \\ A_s &= \left(p_s I_0\left(k_r r\right) + q_s K_0\left(k_r r\right)\right)\cos\varphi\ e^{ikz} \\ \phi &= \left(p_0 I_0\left(k_r r\right) + q_0 K_0\left(k_r r\right)\right)\cos\varphi\ e^{ikz} \end{aligned} \quad (5)$$

where Lorentz Gauge requires that

$$p_0 = \frac{c^2 k}{\omega}p_s,\ \ q_0 = \frac{c^2 k}{\omega}q_s \quad (6)$$

and

$r>b$

$$\begin{aligned} A_r &= 0 \\ A_\varphi &= 0 \\ A_s &= q_s^w K_1\left(\lambda r\right)\cos\varphi\ e^{ikz} \\ \phi &= q_0^w K_1\left(\lambda r\right)\cos\varphi\ e^{ikz} \end{aligned} \quad (7)$$

where Lorentz Gauge requires that

$$q_0^w = \frac{1}{\frac{\omega}{kc^2} + i\frac{\mu_0\sigma}{k}}q_s^w \quad (8)$$

So we have four unknown parameters to determine, which is

$$p_s^c,\quad p_s,\quad q_s,\quad q_s^w \quad (9)$$

we can decide the four parameters via boundary conditions. Then the four parameters could be solve as

$$p_s^c = p_s - \frac{\mu_0 Q_m \omega K_0\left(ak_r\right)}{k} \quad (10)$$

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$$q_s = -\frac{\mu_0 Q_m \omega I_0(ak_r)}{k} \tag{11}$$

$$q_s^w = \frac{-k_r p_s I_1(bk_r) + k_r q_s K_1(bk_r)}{\lambda K_1(b\lambda)} \tag{12}$$

$$\begin{aligned} p_s &= q_s \frac{N_{p_s}}{D_{p_s}} \\ N_{p_s} &= \omega^2 \lambda K_0(\lambda b) K_1(bk_r) + \\ &\quad k_r c^2 \left(\lambda^2 - k^2\right) K_1(b\lambda) K_0(bk_r) \\ D_{p_s} &= \omega^2 \lambda K_0(\lambda b) I_1(bk_r) - \\ &\quad k_r c^2 \left(\lambda^2 - k^2\right) K_1(\lambda b) I_0(bk_r) \end{aligned} \tag{13}$$

It should be noticed that when a approaches to zero, the ring model becomes the point model. By the way, Eq(13) has the same structure as point model did.

M=1 RING MODEL

We can give a *dipole ring model* and solve the electromagnetic field surround it. We explicitly give the source term of the dipole ring, which is Eq(14)

$$\begin{aligned} \rho_1 &= \frac{I_1}{\pi a^2} \delta(s - vt)\, \delta(r - a) \cos\theta \\ &= -\frac{Q_1}{a} \delta(s - vt)\, \delta(r - a) \cos\theta \\ j_1 &= c \rho_1 \hat{s} \end{aligned} \tag{14}$$

where I_1 is the dipole moment, a is the radius of the ring. Solving the equation, we give the general solution of vector potential A_r, A_φ, A_s and potential ϕ. In different area, we have

$$\begin{aligned} & r < a \\ A_r &= \frac{1}{2}\left(p_+^c I_2(k_r r) + p_-^c I_0(k_r r)\right) \cos\varphi\, e^{ikz} \\ A_\varphi &= \frac{1}{2}\left(p_+^c I_2(k_r r) - p_-^c I_0(k_r r)\right) \sin\varphi\, e^{ikz} \\ A_s &= p_s^c I_1(k_r r) \cos\varphi\, e^{ikz} \\ \phi &= p_0^c I_1(k_r r) \cos\varphi\, e^{ikz} \end{aligned} \tag{15}$$

where Lorentz Gauge requires that

$$p_+^c = -p_-^c, \quad p_0^c = \frac{c^2 k}{\omega} p_s^c \tag{16}$$

then

$$\begin{aligned} & a < r < b \\ A_r &= \frac{1}{2}\left(p_+ I_2(k_r r) + q_+ K_2(k_r r) + p_- I_0(k_r r)\right. \\ &\quad \left. + q_- K_0(k_r r)\right) \cos\varphi\, e^{ikz} \\ A_\varphi &= \frac{1}{2}\left(p_+ I_2(k_r r) + q_+ K_2(k_r r) - p_- I_0(k_r r)\right. \\ &\quad \left. - q_- K_0(k_r r)\right) \sin\varphi\, e^{ikz} \\ A_s &= \left(p_s I_1(k_r r) + q_s K_1(k_r r)\right) \cos\varphi\, e^{ikz} \\ \phi &= \left(p_0 I_1(k_r r) + q_0 K_1(k_r r)\right) \cos\varphi\, e^{ikz} \end{aligned} \tag{17}$$

where Lorentz Gauge requires that

$$p_- = -p_+, \quad p_0 = \frac{c^2 k}{\omega} p_s, \quad q_- = -q_+, \quad q_0 = \frac{c^2 k}{\omega} q_s \tag{18}$$

and

$$\begin{aligned} & r > b \\ A_r &= \frac{1}{2}\left(q_+^w K_2(\lambda r) + q_-^w K_0(\lambda r)\right) \cos\varphi\, e^{ikz} \\ A_\varphi &= \frac{1}{2}\left(q_+^w \left(K_2(\lambda r) - q_-^w K_0(\lambda r)\right)\right) \sin\varphi\, e^{ikz} \\ A_s &= q_s^w K_1(\lambda r) \cos\varphi\, e^{ikz} \\ \phi &= q_0^w K_1(\lambda r) \cos\varphi\, e^{ikz} \end{aligned} \tag{19}$$

where Lorentz Gauge requires that

$$q_-^w = -q_+^w, \quad q_0^w = \frac{1}{\frac{\omega}{kc^2} + i\frac{\mu_0 \sigma}{k}} q_s^w \tag{20}$$

So we have eight unknown parameters to determine, which is

$$p_+^c, \quad p_s^c, \quad p_+, \quad q_+, \quad p_s, \quad q_s, \quad q_+^w, \quad q_s^w \tag{21}$$

By linear algebra manipulation we solve eight coefficients in the end, and we will not list it because it is too long.

SPACE CHARGE CALCULATION

We notice that the resident resistive wall impedance is constraint in an relatively low frequency domain no matter how small the radius we are thinking about. This means the numerical integration will have a relatively low cutoff frequency. This will significantly save the time consuming.

After we seperate out the resident resistive wall part, we need to analytically solve the ring dipole space charge in the free space. First we consider the ring in the rest frame moving together with the ring. In the rest frame, there will only exists Electric field. Then, we use Lorentz Transformation to get the space charge in the laboratory frame. The settings of relevant parameters are shown in the Fig.1 Take dipole

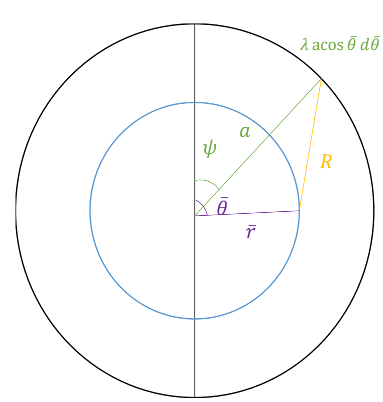

Figure 1: Dipole Ring Model.

model Fig.1 for example, the derived electric field from the

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potential is given by the following expression

$$\bar{E}_r\left(\bar{r},\bar{\theta},\bar{z}\right) = \int_0^{2\pi}\frac{1}{4\pi\varepsilon_0}\left(-\frac{Q_m\cos\psi\ \left(2\bar{r}-2a\cos\left(\bar{\theta}-\psi\right)\right)}{2\left(\bar{r}^2+a^2-2\bar{r}a\cos\left(\bar{\theta}-\psi\right)\ +\bar{z}^2\right)^{\frac{3}{2}}}\right)d\psi$$
$$\bar{E}_\theta\left(\bar{r},\bar{\theta},\bar{z}\right) = \int_0^{2\pi}\frac{1}{4\pi\varepsilon_0}\left(-\frac{aQ_m\cos\psi\sin\left(\bar{\theta}-\psi\right)}{\left(\bar{r}^2+a^2-2\bar{r}a\cos\left(\bar{\theta}-\psi\right)+\bar{z}^2\right)^{\frac{3}{2}}}\right)d\psi$$
$$\bar{E}_z\left(\bar{r},\bar{\theta},\bar{z}\right) = \int_0^{2\pi}\frac{1}{4\pi\varepsilon_0}\left(-\frac{\bar{z}Q_m\cos\psi}{\left(\bar{r}^2+a^2-2\bar{r}a\cos\left(\bar{\theta}-\psi\right)+\bar{z}^2\right)^{\frac{3}{2}}}\right)d\psi \tag{22}$$

we explicitly give the Lorentz Transformation in the Cylindrical coordinates of our case

$$E_r=\gamma\bar{E}_r,\quad E_\theta=\gamma\bar{E}_\theta,\quad E_z=\bar{E}_z$$
$$B_r=-\frac{1}{c}v\gamma\bar{E}_\theta,\quad ,B_\theta=\frac{1}{c^2}v\gamma\bar{E}_r,\quad B_z=0 \tag{23}$$

because the coordinates transformation,

$$\bar{r}=r,\quad \bar{\theta}=\theta,\quad \bar{z}=\gamma z$$

the electromagnetic field in the laboratory frame can be expressed as

$$B_r=-\frac{1}{c^2}v\frac{\gamma}{4\pi\varepsilon_0}\int_0^{2\pi}\left(-\frac{aQ_m\cos\psi\ sin\left(\theta-\psi\right)}{\left(r^2+a^2-2ra\cos\left(\theta-\psi\right)+\left(\gamma z\right)^2\right)^{\frac{3}{2}}}\right)d\psi \tag{24}$$

$$B_\theta=\frac{1}{c^2}v\frac{\gamma}{4\pi\varepsilon_0}\int_0^{2\pi}\left(-\frac{Q_m\cos\psi\ \left(2r-2a\cos\left(\theta-\psi\right)\right)}{2\left(r^2+a^2-2ra\cos\left(\theta-\psi\right)+\left(\gamma z\right)^2\right)^{\frac{3}{2}}}\right)d\psi \tag{25}$$

$$B_z=0 \tag{26}$$

$$E_r=\frac{\gamma}{4\pi\varepsilon_0}\int_0^{2\pi}\left(-\frac{Q_m\cos\psi\ \left(2r-2a\cos\left(\theta-\psi\right)\right)}{2\left(r^2+a^2-2ra\cos\left(\theta-\psi\right)\ +\left(\gamma z\right)^2\right)^{\frac{3}{2}}}\right)d\psi \tag{27}$$

$$E_\theta=\frac{\gamma}{4\pi\varepsilon_0}\int_0^{2\pi}\left(-\frac{aQ_m\cos\psi\sin\left(\theta-\psi\right)}{\left(r^2+a^2-2ra\cos\left(\theta-\psi\right)+\left(\gamma z\right)^2\right)^{\frac{3}{2}}}\right)d\psi \tag{28}$$

$$E_z=\frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\left(-\frac{\gamma zQ_m\cos\psi}{\left(r^2+a^2-2ra\cos\left(\theta-\psi\right)+\left(\gamma z\right)^2\right)^{\frac{3}{2}}}\right)d\psi \tag{29}$$

Thus, we can derive a faster numerical algorithm to calculate the Lorentz Force in the time domain, the steps of the algorithm can be explained as follows. On the one hand, we calculate the Space Charge of the ring model, and give the Lorentz Force of the Space Charge in the time domain. On the other hand, we calculate the numerical Inverse Fourier Transformation of the resident Resistive Wall wake in the frequency domain. Finally, we combine the result to get the total effect of the nonultra-relativistic Resistive Wall.

CONCLUSION

We have organized and computed the impedance wall electromagnetic field models for monopoles and dipoles, and have accelerated the numerical calculation process involved in the inverse Fourier transform. After this work, we can move forward to dynamic analysis.

REFERENCES

[1] A. W. Chao, *Lectures on Accelerator Physics*. World Scientific, 2020. `doi:10.1142/12004`

[2] K. L. F. Bane and M. Sands, "The Short-Range Resistive Wall Wakefields," in *Micro bunches workshop*, *AIP Conference Proceedings*, vol. 367, no. 1, pp. 131–149, 04 1996. `doi:10.1063/1.50300`

[3] G. Stupakov, "Resistive-wall wake for nonrelativistic beams revisited,"*Phys. Rev. Accel. Beams*, vol. 23, no. 9, p. 094401, Sep. 2020. `doi:10.1103/physrevaccelbeams.23.094401`

[4] F. Zimmermann and K. Oide, "Resistive-wall wake and impedance for nonultrarelativistic beams,"*Phys. Rev. Spec. Top. Accel Beams*, vol. 7, no. 4, p. 044201, Apr. 2004. `doi:10.1103/physrevstab.7.044201`