Progress of Theoretical Physics, Vol. 128, No. 4, October 2012

Radiative Properties of Kinks in the $\sin^4(\varphi)$ System

Mohammad MOHAMMADI,[1] Nematollah RIAZI[1,2,*)] and Azizollah AZIZI[1]

[1] *Physics Department and Biruni Observatory, Shiraz University, Shiraz 71454, Iran*
[2] *Physics Department, Shahid Beheshti University, Evin, Tehran 19839, Iran*

(Received October 19, 2011; Revised August 18, 2012)

In this paper, we study the nonlinear $\sin^4(\varphi)$ system in 1+1 dimensions which exhibits interesting nonlinear properties. We have categorized the system as radiative, since the collision of a kink and an antikink with velocities less than a threshold velocity leads to the complete annihilation of the pair and production of two high-amplitude wave packets with zero topological charges. Our results show that the individual kinks and antikinks are stable even against strong (nonlinear) perturbations. Other radiative systems similar to the $\sin^4(\varphi)$ system are also studied. Finally, linear perturbations about the kink solution are examined in relation to the relaxation problem, by looking for the bound states of the resulting Schrödinger-like equation. Interestingly enough, the $\sin^4(\varphi)$ system has only one trivial bound state with the ω^2 eigenvalue residing exactly at the top of the potential well. The significance of this property on the relaxation of the kink in this system is examined and compared to other nonlinear systems.

Subject Index: 011, 034

§1. Introduction

Nonlinear Klein-Gordon type, field systems in one space and one time dimensions which possess soliton-like kink (antikink) solutions, have been studied for decades. The most well-known system in this field is the sine-Gordon (SG) system.[1)–4)] The integrable SG system has been under focus in recent investigations and has found various interesting applications in many branches of physics.[5)–11)] Another famous, albeit non-integrable system is the φ^4 system.[1),12)–16)] This system, too, has been used to model the behavior of physical systems in several branches of physics. Beside the SG and φ^4 systems, there are many other systems with kink solutions which are not so well known and well studied as the SG and φ^4 systems. These include different versions of the φ^6 system[17),19)] and double and multiple-sine-Gordon (DSG and MSG) systems.[20)–22)]

One way to compare different nonlinear systems which have kink solutions is to perturb the dynamical equation about the static kink solution and retain terms linear in the perturbation term. This leads to a Schrödinger-like equation with an attractive potential caused by the kink. Such a system has a finite number of bound states. For integrable systems like the SG equation, there exists only one bound state with zero frequency (the so-called trivial mode). Non-integrable systems like the ϕ^4 system show extra bound states. These bound states, sometimes called internal modes, refer to the eigen-solutions of the linearized perturbations around the static kink solution. It is interesting to note that having just one trivial bound state does

*) E-mail: riazi@physics.susc.ac.ir

not mean that the system is integrable and more restrictive properties like those considered in 17) are needed to make the system nearly integrable or rigorously integrable. If a system has extra bound states, the bound states provide internal channels for the absorption of energy from external triggers.[17),18)]

In spite of unlimited choices for kink-systems, it is only the SG system which is integrable. The solitary waves of integrable systems are genuine solitons, in other words, they reappear without any deformation after collisions with each other. Moreover, the SG kink solutions do not have any internal mode (except for a zero frequency, neutral mode) to absorb small perturbations and its associated Schrödinger kink-potential is completely reflection-less.[17),18)] Some of the non-integrable systems such as φ^4, DSG and φ^6, have extra internal modes which cause vibration of the kink or antikink after collision with other solitons or external perturbations.[16),17),19),20)] For the other non-integrable systems, which — like integrable ones — do not have extra modes, there are no vibrations in solitary waves solutions after collisions, but we can always detect some small radiative wave packets which are emitted after collisions.[17)] Neutral (zero topological charge), radiative wave packets move with a speed near to that of light and usually appear in all kink-systems after collisions except the SG system.

In this paper, we investigate — numerically — radiative kink-bearing systems. A typical example for such a system is the $\sin^4(\varphi)$ system which was introduced by Kulagin and Omel'yanov.[23),24)] They presented mathematical tools to study the fate of kinks and antikinks in collision using their asymptotic behavior. In the present paper, we base our study on a comparative investigation of the Schrödinger-like equations resulting from the perturbations of the kink solution to investigate radiative kink-bearing systems. We will observe that for radiative kink systems the kink and antikink with opposite topological charges annihilate each other, creating two radiative wave packets (with zero topological charges), a process resembling e^-e^+ annihilation. The created neutral, radiative wave packets have large amplitudes and move with (nearly) the speed of light. The kink solutions in these systems are very stable even under the large perturbations. The associated Schrödinger kink-potentials always have only one trivial bound state whose eigenvalue resides exactly at the top of their kink-potentials.

The organization of this paper is as follows. In the next section, we introduce the Lagrangian density, field equation, the kink and antikink solutions and other basic properties of the $\sin^4(\varphi)$ system. In §3, we will examine the kink-antikink collisions at various initial velocities, using a finite difference numerical algorithm. The Schrödinger kink-potential and the internal mode of the $\sin^4(\varphi)$ kinks are calculated in §4. In §5, we introduce similar radiative systems and will compare them to deduce some general properties of such systems. The last section contains a summary and conclusions.

§2. Introducing the $\sin^4(\varphi)$ system

Real nonlinear Klein-Gordon-type field equations in 1+1 dimensions obey the following equation:

$$\Box\varphi = -\frac{dU}{d\phi}, \tag{2·1}$$

where natural units $\hbar = c = 1$ are used, and the the choice of metric is such that $x^\alpha = (t; x)$ and $x_\alpha = (t; -x)$. In this PDE, $U(\varphi)$ is the field potential. The general form of the lagrangian density which yields this PDE is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - U(\varphi). \tag{2·2}$$

Using the Noether theorem and the invariance of the action under the space-time translation $x^\mu \to x^\mu + a^\mu$, one easily obtains the following expression for the stress-energy tensor:

$$T^{\mu\nu} = \partial^\mu\varphi\partial^\nu\varphi - \eta^{\mu\nu}\mathcal{L}, \tag{2·3}$$

where $\eta^{\mu\nu}$ is the $(1+1)$ dimensional Minkowski metric. The T^{00} component of the energy-momentum tensor gives the energy density which in general reads:

$$\varepsilon(x,t) = T^{00} = \frac{1}{2}\varphi'^2 + \frac{1}{2}\dot{\varphi}^2 + U(\varphi). \tag{2·4}$$

Provided that the potential has at least two degenerate minima, such systems have at least two types of solitary wave solutions which are named kinks and anti-kinks, corresponding to every dual consecutive minimum points of $U(\varphi)$. We can define a topological current for the solutions of these systems according to

$$J^\mu = C\epsilon^{\mu\nu}\partial_\nu\varphi, \tag{2·5}$$

where $\epsilon^{\mu\nu}$ is the completely antisymmetric tensor and C is an arbitrary constant. This current is conserved locally for all dynamic solutions:

$$\partial_\mu J^\mu = C\epsilon^{\mu\nu}\partial_\mu\partial_\nu\varphi = 0. \tag{2·6}$$

The corresponding topological charge is

$$Q = \int J^0 dx, \tag{2·7}$$

which is a constant of motion. For kink solutions $Q > 0$, and for antikink solutions $Q < 0$. Moreover, it can be shown that if $\varphi_o(x)$ is the static kink solution, the moving kink solution will be

$$\varphi_v(x,t) = \varphi_o\left(\frac{x - vt}{\sqrt{1-v^2}}\right) = \varphi_o(\gamma(x - vt)), \tag{2·8}$$

in which $\gamma = (1-v^2)^{-1/2}$, and v is the kink or antikink velocity. The total energy is obtained by integrating the energy density over all space:

$$E = \int T^{00} dx. \tag{2·9}$$

Furthermore, it can be shown that a moving kink satisfies the famous relativistic relation

$$E^2(v) = E^2(0) + P^2, \tag{2·10}$$

where

$$P = \int T^{10} dx. \tag{2·11}$$

One of the most well-known kink-bearing systems in 1+1 dimensions is the SG system. The exceptional property of the SG system which distinguishes it from other kink-bearing systems is its integrability. Integrability makes its solitary wave solutions free from any chaotic behavior.[25)] Perhaps the most important influence of this property would show up in the kink-antikink collisions. For integrable systems like the SG system, solitons reappear without any deformation after collisions with each other. The self-interaction potential of the SG system is in following form

$$U(\varphi) = 1 - \cos(\varphi). \tag{2·12}$$

For the purposes of this paper, we use another equivalent form (within numerical factors) of the potential (2·12) in the form

$$U(\varphi) = \sin^2(\varphi). \tag{2·13}$$

Accordingly, we call this system as $\sin^2(\varphi)$ instead of SG. This potential can be easily generalized to $\sin^N(\varphi)$:

$$U(\varphi) = \sin^N(\varphi). \tag{2·14}$$

In this paper, we concentrate our study on the $\sin^4(\varphi)$ system, although some of the results can be easily extended to other even values of N. The Lagrangian density for the $\sin^4(\varphi)$ system becomes

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \sin^4(\varphi). \tag{2·15}$$

The Euler-Lagrange equation is easily obtained for this system (2·2):

$$\Box\varphi = -4\cos(\varphi)\sin^3(\varphi). \tag{2·16}$$

The potential of the system has infinite minima at $n\pi$ in which n is an integer number. These degenerate vacua lead to the appearance of stable kinks and antikinks. The kink and antikink solutions are easily obtained by directly solving Eq. (2·16) for the static field $\varphi(x)$ with appropriate boundary conditions:

$$\varphi_o = \pm\cot^{-1}(\sqrt{2}(x - x_0)), \tag{2·17}$$

with the $-$ $(+)$ signs corresponding to kink (antikink), respectively. The moving solutions are easily obtained by a relativistic boost (2·8):

$$\varphi_v(x,t) = \pm\cot^{-1}(\sqrt{2}\gamma(x - vt - x_0)). \tag{2·18}$$

We can define a topological current for solutions of this system according to

$$J^\mu = \frac{1}{\pi}\epsilon^{\mu\nu}\partial_\nu\varphi. \tag{2·19}$$

The corresponding topological charge (2·7) is $Q = \pm 1$ for the kink (antikink). The total energy for the static kink and antikink solutions turn out to be equal to $\frac{\sqrt{2}\pi}{2}$.

§3. Kink-antikink collision dynamics

We now let the kink and antikink collide at various initial velocities and examine what happens. In order to study the time evolution of the two soliton solutions of the $\sin^4(\varphi)$ equation we invoke numerical methods, since the non-trivial, time dependent solutions for this wave equation cannot be obtained analytically. In the finite difference method we use here, initial conditions are implemented in which a kink and an antikink start to move toward each other at the same initial speed. A code was generated in Matlab to compute the time evolution of the kink-antikink collision numerically. We used an essentially finite-deference method on a grid containing 800000 nodes. The bi-dimensional space-time is represented by a grid of spatial and temporal step-sizes h and k, respectively. The initial conditions at $t = 0$ was imposed according to

$$u(i,1) = \varphi(x_i, 0) = \varphi_o\left(\frac{x_i - a}{\sqrt{1-v_1^2}}\right) + \varphi_o\left(-\frac{x_i - b}{\sqrt{1-v_2^2}}\right), \quad b - a \gg 1, \tag{3·1}$$

in which $\varphi_o(x)$ is the static kink solution. Since this is basically a finite difference method we need also define the second time step ($t = k$) as initial condition. This can be approximately implemented as

$$u(i,2) = \varphi(x_i, k) = \varphi_o\left(\frac{x_i - v_1 k - a}{\sqrt{1-v_1^2}}\right) + \varphi_o\left(-\frac{x_i - v_2 k - b}{\sqrt{1-v_2^2}}\right), \quad b - a \gg 1. \tag{3·2}$$

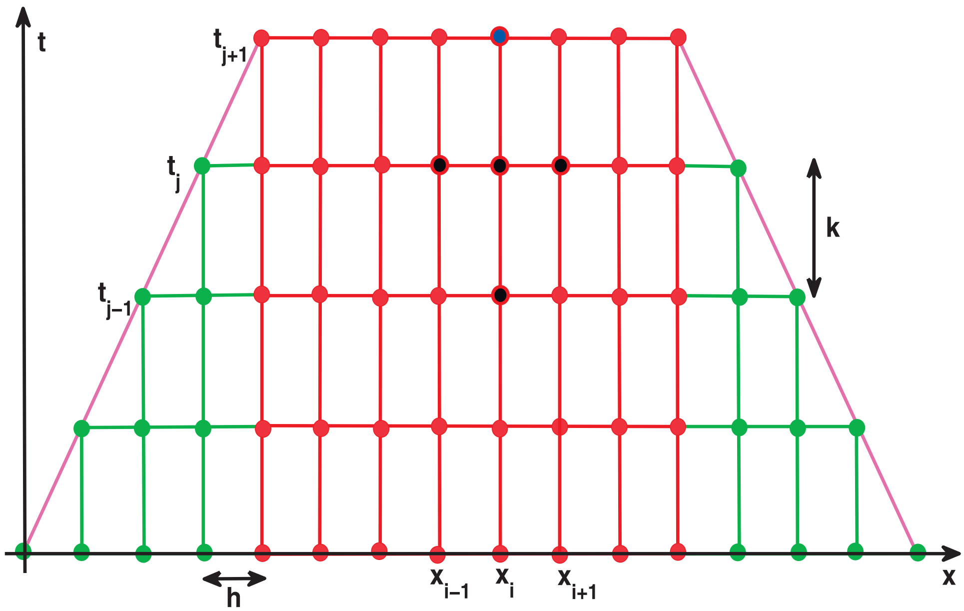

Fig. 1. A pyramid mesh is employed in our finite difference method in order to eliminate the effect of boundaries. To calculate field at each arbitrary mesh point, one needs the field values at four neighboring nodes in the grid. The field values at the first and second rows are approximated by the initial conditions (3·1) and (3·2).

To ensure the stability of the algorithm, the ratio $r = k/h$ must be smaller than 1. Moreover, to avoid the perturbing effect of rigid walls, we have adopted a pyramid grid (see Fig. 1). On this grid, we successively omit two end nodes on the x axis at each new time step. Finally, we only use results in the x-range of interest. The energy and other quantities of interest are easily calculated from the field values and finite-difference estimates for its temporal and spatial partial derivatives.

The results corresponding to different initial velocities, show the following situations with respect to the fate of the colliding kink and antikink:

1- For initial velocities well below the threshold v_{th} ($v \ll v_{th} = 0.714$), the kink and antikink are captured and quickly annihilate each other to produce two neutral (zero topological charge) radiative wave packets moving with nearly the speed of light, keeping their localized form (Fig. 2). It should be noted that pair annihilation has also been observed in solitary waves of a KdV-like equation.[26)]

2- For initial velocities near but less than the threshold ($v < v_{th}$), the kink and antikink are captured and quickly produce a pair of neutral radiative wave packets and a pair of kink and antikink which after a short time delay annihilate each other again and produce another pair of neutral radiative wave packets (Fig. 3).

3- For initial velocities greater than threshold ($v > v_{th}$), the pair always reappear together with a pair of neutral radiative wave packets. Obviously, the outgoing kink antikink pair move with speed less than their initial speed (Fig. 4). In this case, if the initial velocities tend to the speed of light, the ratio of radiative wave energies to the total energy tends to zero and the collision process becomes very similar to the integrable SG system as pointed out in Ref. 24) (Fig. 5). In fact, if the initial speeds are nearly equal to the light speed, the kinetic kink and antikink energies are

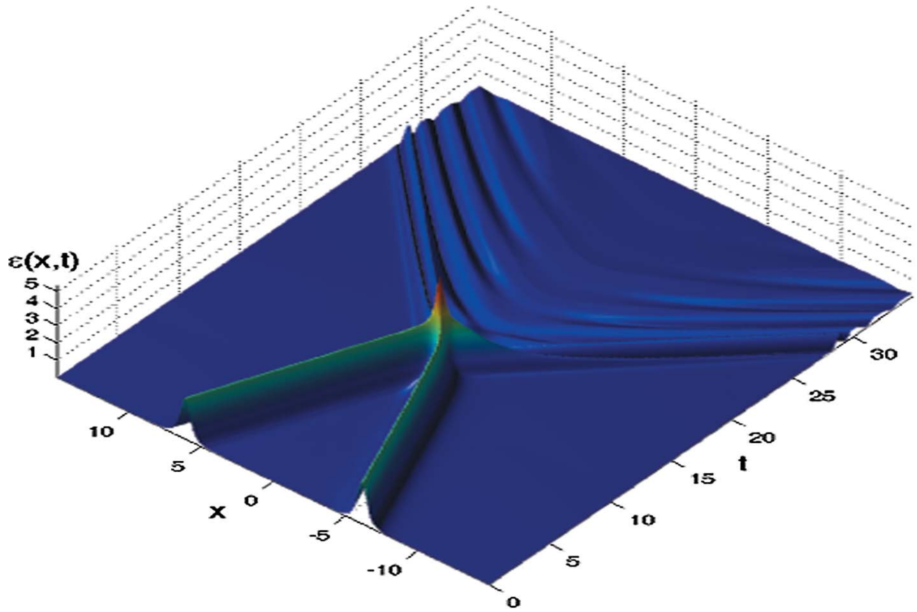

Fig. 2. Kink-antikink annihilation with the initial speed 0.40. The vertical axis in this figure and other similar figures represents energy density ($T^0_0 = \varepsilon(x,t)$).

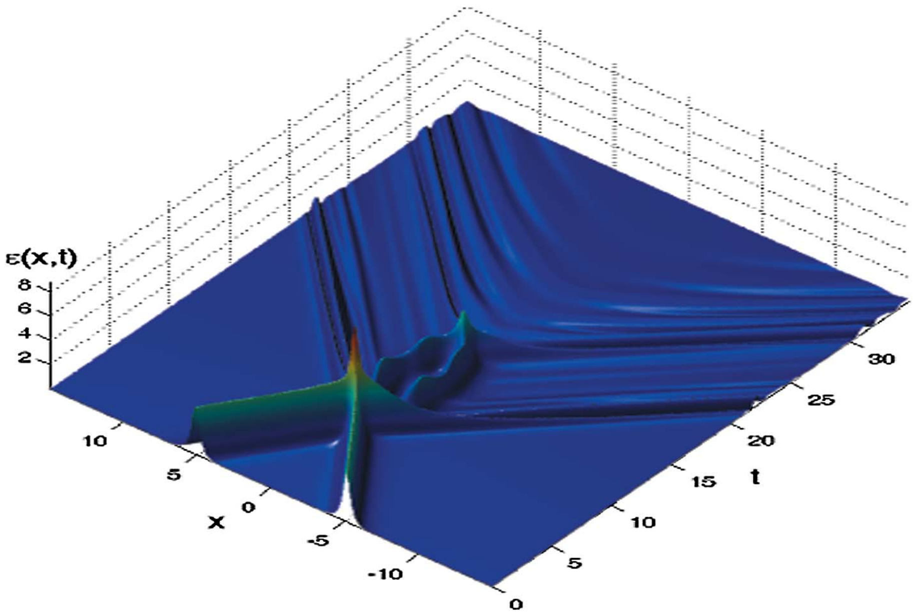

Fig. 3. Kink-antikink annihilation with initial kink speed 0.70.

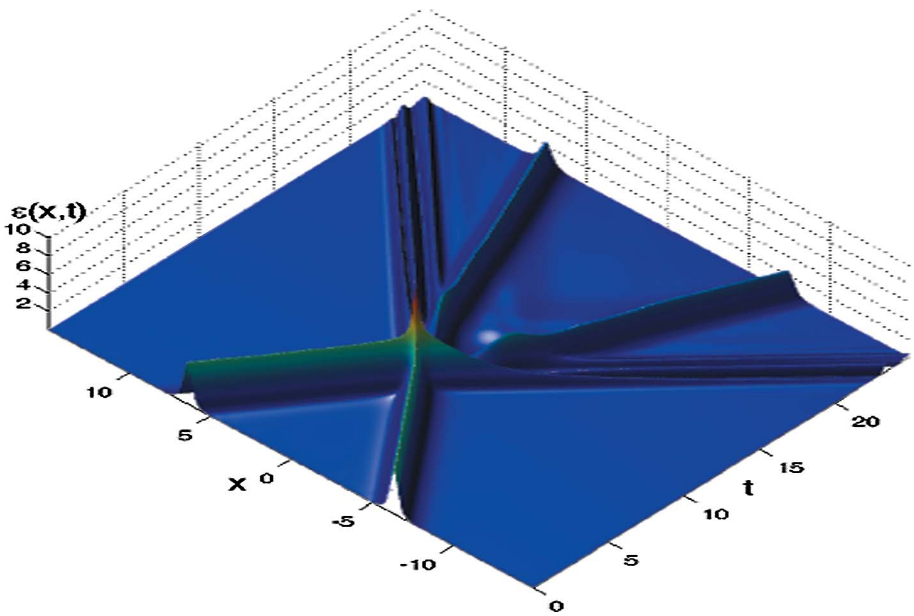

Fig. 4. Kink-antikink collision for the initial kink speed of 0.75.

so large that they do not change appreciably after the interaction in the collision process. This behavior seems to be true in general for similar kink bearing systems.

Furthermore, it was observed that the collision between solitary waves with the same topological charge is elastic and leads to the reappearance of the same solitary waves as before the collision. Collision between a kink and a radiative wave packet,

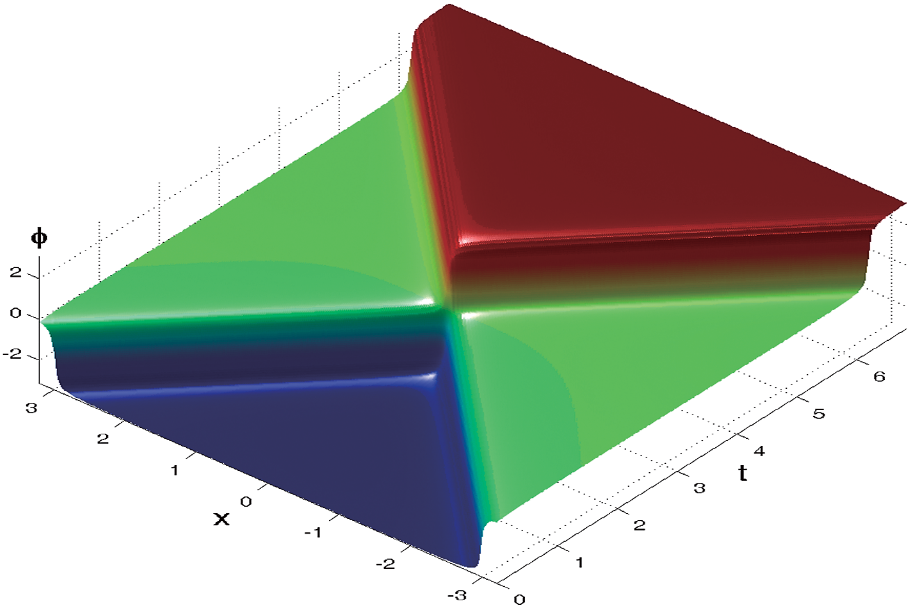

Fig. 5. Kink-antikink collision for an initial kink speed of 0.999. Since the initial speed is very near to the speed of light, the outgoing kink-antikink pair reappears almost intact as for genuine solitons. In this figure, we have used field representation instead of energy density representation in order to show the fate of the kink and antikink in a more transparent way.

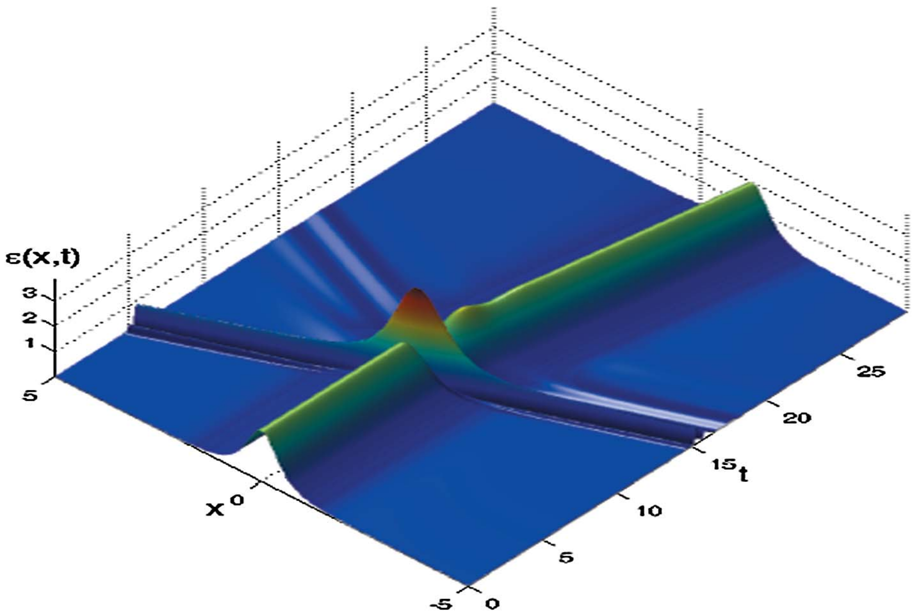

Fig. 6. Kink-radiative wave packet collision in the $\sin^4(\varphi)$ system. The radiative wave packet has zero topological charge. Note that part of the wave packet is reflected and part of it is transmitted. The kink recoils accordingly.

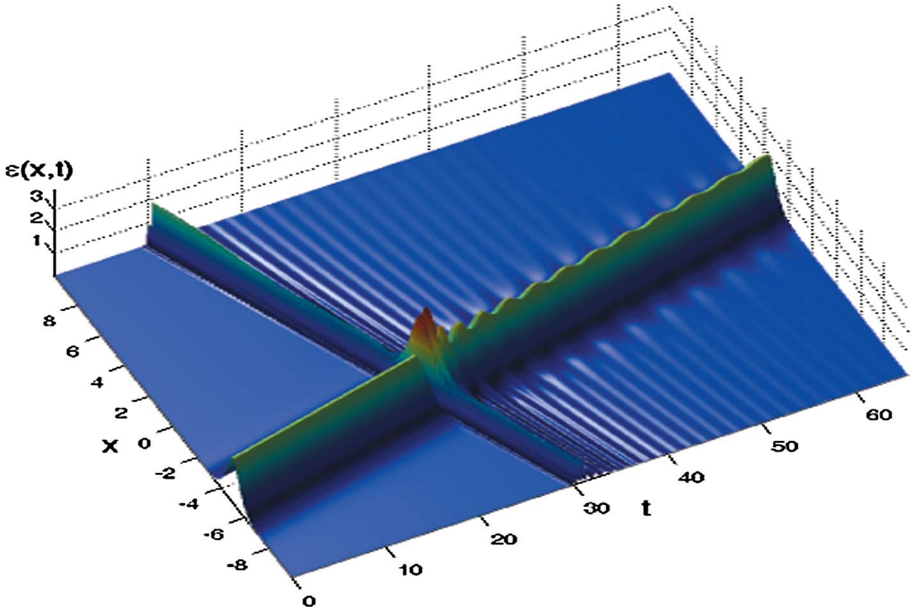

Fig. 7. Kink-radiative wave packet collision in the SG system. The radiative wave packet has zero topological charge.

leads to the reappearance of the kink with minor vibration after collision (Fig. 6).

We conclude that the solitary waves in the $\sin^4(\varphi)$ system are very stable even against large amplitude perturbations. This is partly due to the fact that the $\sin^4(\varphi)$ kinks or antikinks do not have internal modes for the absorbtion of energy via collisions with external perturbation. In Fig. 6 the ratio of the incident radiative wave energy to energy of the kink before collision is 0.36, which is relatively considerable and well beyond a linear perturbation. Note that this situation does not hold for many well-known systems such as the φ^4 system. Even for the integrable SG system, despite the nonexistence of any internal modes, we have observed long-lasting oscillations of the kink after large amplitude external perturbations (Fig. 7). In order to investigate the role of internal modes, we consider them separately in the next section.

§4. Internal modes of the $\sin^4(\varphi)$ kink

In order to obtain the internal modes of the $\sin^4(\varphi)$ kink, we add a small time dependent perturbation term to the static solution

$$\phi(x,t) = \varphi_s(x) + \psi e^{-i\omega t} \tag{4·1}$$

in which $\varphi_s(x)$ is the static solitary wave solution and ψ is a small perturbation amplitude. After substituting this ansatz in the wave equation

$$\Box\phi = -\frac{\partial U(\phi)}{\partial \phi}, \tag{4·2}$$

we expand the potential and keep only the linear terms in ψ. Finally, we obtain an eigenvalue equation for ψ, which looks like the Schrödinger equation:

$$-\frac{d^2\psi}{dx^2}+V(x)\psi=E\psi, \tag{4·3}$$

where

$$E=\omega^2 \quad \text{and} \quad V(x)=\frac{d^2U(\phi_s)}{d\phi_s^2}, \tag{4·4}$$

and $\phi_s(x)$ is the static kink solution. One can prove easily that this differential equation in general has a trivial solution which corresponds to $\omega=0$,

$$\psi_1(x)=\frac{d\phi_s}{dx}. \tag{4·5}$$

But, this trivial solution is in fact associated with an infinitesimal translation of the static kink:

$$\phi(x,t)=\phi_s(x)+\varepsilon\frac{d\phi_o}{dx}=\phi_s(x+\varepsilon), \quad \varepsilon\to 0. \tag{4·6}$$

If we apply this procedure to the integrable SG system, no modes except the trivial mode with zero frequency would be found. For some non-integrable systems such as φ^4 or DSG (double sine-Gordon) systems, there appear extra internal modes. One can argue that if a solitary wave solution has such extra modes, extra channels for energy absorption exist. Thus, after a collision, these extra modes can be excited and the shape of kink (or anti-kink) is deformed accordingly, causing the kink to vibrate. The necessary energy for these vibrations is taken from the incoming translational kinetic energy.

If we apply this procedure to the $\sin^4(\varphi)$ system, we obtain the following Schrödinger kink-potential:

$$V(x)=\frac{24x^2-4}{(1+2x^2)^2} \tag{4·7}$$

which is a symmetric potential with a minimum at $x=0$ ($V_{min}=V(0)=-4$). It must be noticed that the square of bound state frequencies ($E=\omega^2$) could lay anywhere between V_{min} and the asymptotic potential $V_a=V(\pm\infty)$. For the $\sin^4$ system, asymptotic kink-potential is exactly zero $V_a=0$. Moreover, it is possible to show numerically that this potential has only one trivial bound state exactly like the SG system. Therefore, we have found a system which has a single bound state eigenvalue lying exactly on the top of the kink-potential ($E=\omega^2=V_a$). Let us compare this bound state with the bound states of three famous nonlinear systems (see Fig. 8). Note that the $\omega^2=0$ trivial mode is present in all cases. For the $\sin^4(\varphi)$ and SG systems, there is only one trivial mode. For the $\sin^4(\varphi)$ system, however, the trivial $\omega^2=0$ eigenvalue coincides with the asymptotic value of the potential. As expected, φ^4 and DSG have extra non-trivial modes.

We already know that systems which have extra non-trivial modes show vibrational motion after kink-antikink collisions. However, this general statement is based on the study of small internal perturbations.[17),18)] Fortunately, if we use this

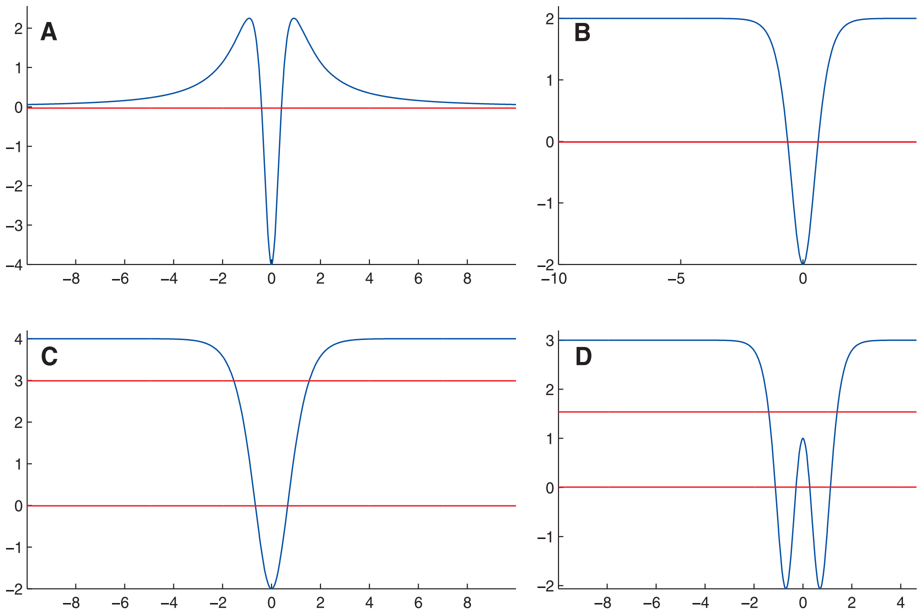

Fig. 8. A, B, C and D represent the kink-potential of the $\sin^4(\varphi)$, SG, φ^4 and DSG systems, respectively. The horizontal lines correspond to the value of the ω^2 eigenvalues.

method for systems which have potentials similar to that of the $\sin^4(\varphi)$ system, we find many similarities between the behavior of such systems during and after the collisions. This numerical study has led us to introduce a new group of kink bearing systems which we call radiative systems. Such interesting systems are studied in the next section.

§5. Radiative systems

In the previous sections we examined the $\sin^4(\varphi)$ system as a radiative system. In this section, we examine such a radiative system in a more general context. The main aspect of a radiative system is that it exhibits a threshold velocity (v_{th}) in the kink-antikink collisions. If the velocity of kink (antikink) before collision is less than the threshold velocity v_{th}, the pair completely annihilate each other and produce two radiative wave packets with zero topological charge. If the initial velocities are larger than v_{th}, a new kink-antikink pair is produced, together with a pair of radiative wave packets which appear just after the collision. Such radiative systems do not show any windows for velocities less than v_{th}.

There are many radiative systems which behave like the $\sin^4(\varphi)$ system in collisions. Let us turn to the following version of the φ^6 system which has the potential:

$$U(\varphi) = \frac{1}{2}|1 - \varphi^2|^3, \tag{5·1}$$

where the associated kink solution is

$$\varphi = \frac{x}{\sqrt{1 + x^2}}. \tag{5·2}$$

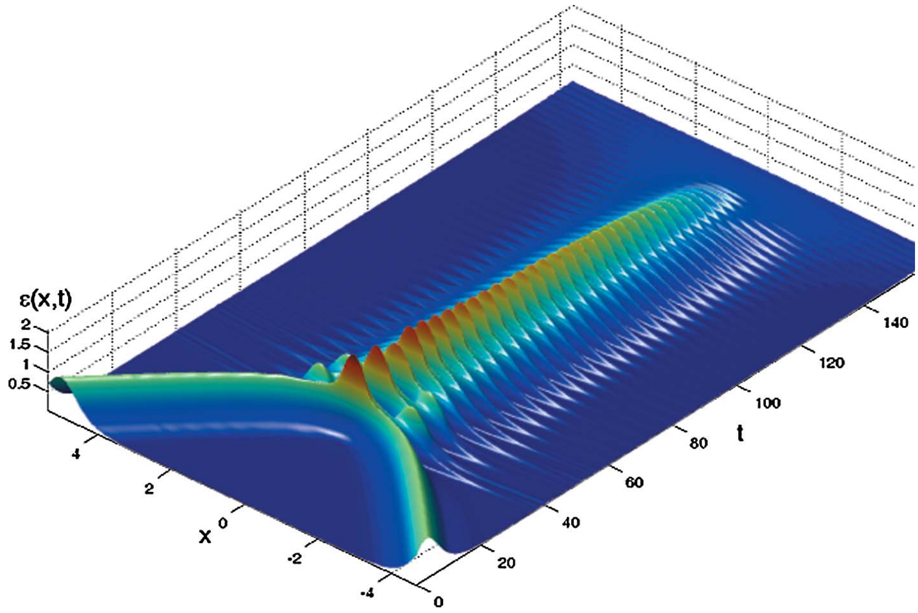

Fig. 9. Kink-antikink collision in the φ^6 system with the initial speed 0.10. For $v < v_{th}$, one observes that the kink and antikink finally annihilate each other after a long series of oscillations. The outcome is a series of small amplitude radiative waves.

This system shows a threshold velocity $v_{th} = 0.28$ for kink-antikink collisions. The kink and antikink reappear after the collision without any vibration with two radiative wave packets for initial velocities larger than $v_{th} = 0.28$ similar to the $\sin^4(\varphi)$ system. But, there is an important difference between $\sin^4(\varphi)$ and φ^6 systems for initial velocities less than $v_{th} = 0.28$. In this range of initial velocities, the full kink-antikink annihilation is completed very smoothly after some vibrations (see Fig. 9). The kink solutions in the φ^6 system are also stable under large perturbations as one expects. The associated Schrödinger kink-potential has only one trivial bound state exactly equal to asymptotic kink-potential ($\omega^2 = V_a = 0$) like the $\sin^4(\varphi)$ system.

Besides the $\sin^4(\varphi)$ and φ^6 systems, we can introduce more systems which are radiative. We have performed numerical simulations of kink-antikink annihilation for $\sin^N(\varphi)$ systems up to $N = 10$. All $\sin^N(\varphi)$ systems ($N = 2, 4, 6, 8, 10$) show similar behavior. All of these systems have zero asymptotic Schrödinger kink-potentials with a single trivial bound state which rests on the top of the kink potential ($\omega^2 = V_a = 0$). Furthermore, all of them show a threshold velocity and behave closely like the $\sin^4(\varphi)$ system in collisions, as far as we could check numerically for various values of N. We therefore propose that systems which have only one trivial bound state residing at the top of the kink potential are radiative. This conjecture is supported by a comparative (numerical) investigation of internal kink (antikink) modes and the system behavior during collisions for various kink-bearing potentials.

§6. Summary and conclusions

We studied the nonlinear $\sin^4(\varphi)$ system in 1+1 dimensions. The kink-antikink collisions were examined and it was shown that this and similar systems (which we call radiative systems) exhibit a threshold velocity, beyond which the pair scatter, leading to the formation of a pair of zero topological charge wave packets. Below the threshold velocity, the pair annihilate each other, producing two neutral wave packets. The kink solutions of the proposed system exhibit interesting behavior when they are fired with large amplitude radiative wave packets. Our simulations confirmed that kink (antikink) solutions in this system are very stable even against large amplitude, nonlinear impacts. Even the kink solution of the integrable SG system was observed to undergo large amplitude oscillations after such a large impact. The Schrödinger kink-potential of the $\sin^4(\varphi)$ system was found to have only one trivial bound state exactly like SG system. However, the associated eigenvalue resides exactly on top of the kink-potential.

Finally, we introduced some other systems which share these interesting properties with the $\sin^4(\varphi)$ system.

References

1) R. Rajaraman, *Solitons and Instantons* (North Holland, Elsevier, Amsterdam, 1982).
2) A. Das, *Integrable Models* (World Scientific, 1989).
3) G. L. Lamb, Jr., *Elements of Soliton Theory* (John Wiley and Sons, USA, 1980).
4) P. G. Drazin and R. S. Johnson, *Solitons: an Introduction* (Cambridge University Press, 1989).
5) R. Khomeriki and J. Leon, Phys. Rev. E **71** (2005), 056620.
6) G. Fiore, math-ph/0512002.
7) N. Riazi, A. Azizi and S. M. Zebarjad, Phys. Rev. D **66** (2002), 065003.
8) L. V. Yakushevich, *Nonlinear Physics of DNA* (Wiley, 2004).
9) L. V. Yakushevich, A. V. Savin and L. I. Manevitch, Phys. Rev. E **66** (2002), 016614.
10) S. Cuenda, A. Sanchez and N. R. Quintero, Physica D **223** (2006), 214.
11) J. Timonen, M. Stirland, D. J. Pilling, Y. Cheng and R. K. Bullough, Phys. Rev. Lett. **56** (1986), 2233.
12) D. K. Campbell and M. Peyrard, Physica D **19** (1986), 165.
13) D. K. Campbell and M. Peyrard, Physica D **18** (1986), 47.
14) D. K. Campbell, J. S. Schonfeld and C. A. Wingate, Physica D **9** (1983), 1.
15) M. Peyrard and D. K. Campbell, Physica D **9** (1983), 33.
16) R. H. Goodman and R. Haberman, SIAM J. Appl. Dyn. Sys. **4** (2005), 1195.
17) M. Mohammadi and N. Riazi, Prog. Theor. Phys. **126** (2011), 237.
18) A. R. Gharaati, N. Riazi and F. Mohebbi, Int. J. Theor. Phys. **45** (2006), 57.
19) S. Hoseinmardi and N. Riazi, Int. J. Mod. Phys. A **25** (2010), 3261.
20) V. A. Gani and A. E. Kudryavtsev, Phys. Rev. E **60** (1999), 3305.
21) C. A. Popov, Wave Motion **42** (2006), 309.
22) M. Peyravi, A. Montakhab, N. Riazi and A. Gharaati, Eur. Phys. J. B **72** (2009), 269.
23) D. A. Kulagin and G. A. Omel'yanov, Math. Notes **75** (2004), 563.
24) G. A. Omel'yanov, Electron. J. Diff. Equ. **2010** (2010), No. 150.
25) M. V. Berry, AIP Conf. Proc. **46** (1978), 16.
26) A. R. Santarelli, Nuovo Cim. B **46** (1978), 179.