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REVIEW

Black hole spectroscopy: status report

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Abstract
A brief overview of the "Black hole spectroscopy program" status is presented. Albeit given from a personal angle, it constitutes an attempt to convey the impressive progress achieved within the field in the last few years. Modeling and observational aspects are touched upon, although both from an observationally-oriented perspective. Particular emphasis is given to recent advancements within general relativity and challenging open problems.

Keywords Black hole · Gravitational waves · Black hole spectroscopy · Searches for new physics

Contents

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1 Introduction

The idea that precise measurements of vibrational spectra can provide convincing evidence for the black hole (BH) nature of dark compact objects, and for the accuracy of a general relativistic (GR) description of large curvature dynamics is now decades old [1–3]. Interest in the mathematical description of black hole perturbations was initially fueled by the fundamental question of their stability, namely physical viability [4–6], and since then has evolved towards including BH perturbations in connection with astrophysical phenomena. For excellent reviews of early developments we refer to [7–11], while for pointers to the mathematical literature see [12–16].

More recently, focus of the physics community shifted towards a specific setting exciting BH perturbations: the coalescence of two compact objects, particularly in the form of a merger from a binary at the end of a long-lived bounded orbit. A robust understanding of this process fueled the opportunity of exploiting this type of signals to search for new physics ,[1] either in the form of non-BH dark compact objects [17], or modifications to the Einstein-Hilbert action [18, 19].

Consequently, the modeling of the resulting post-merger signal and the experimental extraction of characteristic BH vibrational frequencies has been a leading topic of gravitational wave (GW) physics. Significant milestones in these directions were achieved on the theoretical side through the first fully general relativistic simulation of a binary merger [20–22] and the subsequent extraction of the remnant BH vibrational properties [23, 24], while on the experimental one through the groundbreaking observations of GWs from a binary merger, GW150914 [25]. Beyond marking the first detection of a GW signal, the observation of GW150914 simultaneously happened to be the first time that a BH relaxation process has ever been observed [26]. The combination of these two advancements opened the era of modern *observational* BH spectroscopy.

Another intriguing possibility worth mentioning is the usage of this type of signals to verify if the BH background describing the process can be well-approximated by the Kerr solution, or if corrections due to dark-matter halos, accretion disks or other "environmental" effects might play a role [27–31]. The latter question is clearly of paramount importance before being able to draw any conclusions about "fundamental" gravity properties. Current investigations point to the fact that such environmental contaminations are expected to be negligible in the ringdown phase, for signals observed by current and planned observatory [32–34], albeit full numerical evolution are required to consolidate this picture [35].

Below, we concisely review past progress, current efforts and open problems of the physical description of BH spectra excitations in the aftermath of a binary merger, and the data analysis challenges and techniques required to extract information on such phenomena from interferometric data. For brevity, the analysis will focus on the (yet incomplete) understanding on the process within GR, albeit computation of QNM spectra relaxing the GR or Kerr assumptions have witnessed remarkable progress in recent times [36–58].

[1] These efforts are also commonly interpreted as *testing General Relativity* through the lenses of a framework inherited from previous experimental investigations [170].

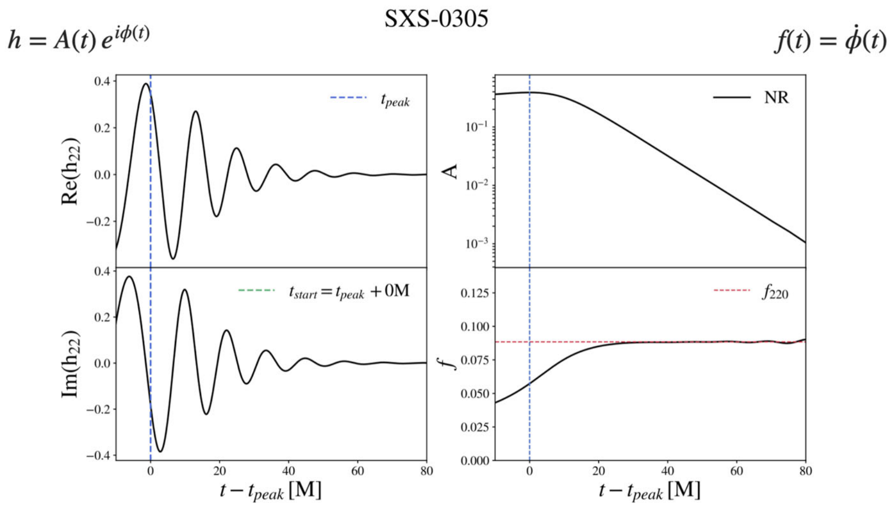

Fig. 1 Dominant quadrupolar mode (left: real and imaginary part; right: amplitude and frequency split, following the definitions given at the top of the figure) of the strain extracted from the `SXS:0305` highly-accurate simulation. The horizontal line marks the corresponding QNM frequency computed using the remnant mass and spin extracted at late times

2 The physical process

2.1 Introduction

The aftermath of a binary coalescence of two Kerr BHs in GR is an initially highly perturbed object, rapidly relaxing towards a stationary Kerr state. A large amount of analytical results and numerical experiments indicate that at intermediate times (after $\simeq 20M$ for a typical binary merger, where geometric units are used and M is the mass of the binary) the signal is well-approximated by a linear superposition of quasi-normal modes (QNMs), namely damped sinusoids carrying the characteristic frequencies of the background. This can be appreciated e.g. by looking at Fig. 1, displaying the dominant quadrupolar mode of the asymptotic gravitational wave strain from the highly accurate `SXS:0305` numerical simulation at highest resolution, representing a binary with parameters close to GW150914 [59, 60]. Around $\simeq 20M$, the frequency has approached a constant (equal to the corresponding longest-lived QNM frequency), and the amplitude scales exponentially according to the $A \cdot e^{-t/\tau}$,[2] with A a constant number, see top left of Fig. 3 in [61]. This is what will be referred to as the "stationary QNM regime", which has historically been the focus of spectroscopic analyses. For asymptotically flat spacetimes, the late-time $\sim 100M$ signal (not included in the above figure, since we won't be discussing it) is known to be dominated by power-law tails [62], which can be signifcantly more excited than previously expected when the binary eccentricity is sufficiently high [63–65].

[2] At first order only a single mode dominates. While mode-mixing effects and higher-order modes are present even at these intermediate times, they don't affect the conceptual picture discussed here.

During the stationary regime, the waveform observed far from the source will be of the form:

$$h^{\mathrm{Kerr}}_{\ell m}(t) = \sum_{\ell'=2}^{\infty} \sum_{n=0}^{\infty} \left[A^{+}_{\ell' mn}\, e^{i\left(\omega^{+}_{\ell' mn}(t-t_{\mathrm{ref}})+\phi^{+}_{\ell' mn}\right)} + A^{-}_{\ell' mn}\, e^{i\left(\omega^{-}_{\ell' mn}(t-t_{\mathrm{ref}})+\phi^{-}_{\ell' mn}\right)} \right] \quad (1)$$

with $\ell > 2$, $m \geq 0$, where $\omega^{\pm}_{\ell mn}$ correspond to Kerr complex QNM frequencies fixed by the final BH mass and angular momentum (spin). The $\pm$ labels refer to the two family of modes contributing to each harmonic, physically corresponding to perturbations that respectively co-rotate ("+") or counter-rotate ("-") with respect to the remnant BH spin. Albeit the complex QNMs frequencies can be predicted from known perturbative results in terms of the BH parameters, the amplitudes and phases are initial-data dependent. Chiefly, in this regime the amplitudes and phases are *constant* to a very good approximation.

Instead, the early stage of the process is (in principle) expected to be contaminated by all sorts of nonlinear effects, such as non-linear couplings, non-modal contributions, mass and spin evolution, mode-spreading etc. (see below for details). Beyond these, other features contributing to the early-time relaxation are instead connected to the dynamical nature of the QNMs excitation. Albeit less studied, this latter phenomenon is predicted even in linear theory [66], and is simply due to the fact that QNM excitation doesn't happen instantaneously. Instead, the modes will experience a transient "activation" regime, driven by the dynamical multipolar structure of the background. All these effects prevent an a-priori straightforward extension of the above description to the near-peak regime. Such extensions could possibly be achieved by including the time-dependence of the BH mass and spin, of the QNM amplitudes, together with other non-modal contributions [67, 68].

The lack of understanding of the early ringdown regime, and possibility of contamination to the QNM extraction were circumvented by initial BH spectroscopy strategies by assuming to start the analysis during the intermediate stationary regime. This comes at the price of renouncing to the largest portion of the signal-to-noise ratio (SNR), but ensures a clean interpretation of the measurement, and an unbiased extraction of the physical QNM BH spectra. Indeed, these investigations were mainly targeted at high-SNR signals observed by future detectors [3, 69].

2.2 Past work

We now briefly review the current status of QNM waveform modeling, with sparse pointers to key early developments.

2.2.1 Early models and linear attempts

While early studies had to rely on "educated guesses" [61] concerning modes excitations and validity of perturbative results [70], as soon as the first numerical simulations of binary mergers became available, the question of the domain of validity of linear perturbation theory was investigated, which sparked significant interest also in the development of second-order perturbation theory [71–80]. Later, when simulations of mergers of quasi-circular binaries were achieved [20–22], studies along these lines were conducted in [23, 24]. An immediate attempt underwent to extend the QNM description up to the waveform peak by including multiple overtones in a fit. Indeed, early effective-one-body (EOB) models calibrated to numerical relativity featured a post-peak signal completion in terms of a superposition of up to $n = 7$ overtones. Additionally, they included "pseudo-QNM" contributions, found to be necessary to obtain a more stable fit that could "bridge" the post-merger frequency to the pre-merger orbital one [81]. However, these attempts were explicitly recognised as phenomenological (see e.g. [24]), since they were not accounting for the wealth of beyond-linear effects previously mentioned and described in more detail below. It is interesting to note that similar considerations were already presented in [82], as recounted in [83]. Later on, pure QNM superpositions were abandoned in the construction of phenomenological post-peak models employed in inspiral-merger-ringdown (IMR) waveforms, since the (resummation-inspired) strategy of [84] proved to be more accurate and stable. See also [85] for earlier proposals. Such models include a phenomenological time-dependence of QNM complex amplitudes, chosen to mimick the waveform morphology observed in Fig. 1.

More recently, considerable interest was again devoted to the problem following the attempts of [86] to propose a picture based once more on a superposition of a large number of overtones, again up to $n = 7$, carrying 16 free parameters when constructing a fit. This proposal suffered from the same conceptual and practical shortcomings of earlier attempts discussed above, and a critical assessment of this newer study was investigated in the by [61, 87], see also [88]. A fruitful lesson from [61] (surprisingly ignored in the many subsequent studies inspired by [86], which were thus following similar assumptions), is that models based on linear superpositions of constant-amplitude QNMs should be tested for self-consistency, especially before physically interpreting any improvement based on fits with large numbers of parameters. The simplest of such self-consistency tests is the complex amplitude *independence* on the fit starting time (within the stationary model validity regime), as already internally applied in e.g. [89, 90]. It is of fundamental importance to stress that, under the assumptions through which the model is constructed, such consistency requirement is implied for *all* modes, not just a subset of them [91, 92]. In the opposite scenario, it may be the case that "inconsistent modes" fit away non-QNM or non-linear features, preventing a clean interpretation of the remaining modes. The second requirement is a smooth variation of the extracted modes amplitudes upon changes of the initial conditions (e.g. binary mass ratio). Again this was well-known in earlier works [81, 89, 90], but surprisingly ignored by a large body of literature in more recent overtones investigations. Simple superpositions of a large number of constant-amplitude overtones extended up to the signal peak do not respect neither of these consistency criteria, as shown in [61, 88].

Before moving on, a word of caution is needed regarding the interpretation of these latter studies. A superficial reading could lead to think that such latter studies argue for the complete absence of overtones in the post-merger binary signal. Such conclusion would be obviously flawed, as some of the same authors have analytically computed the physical overtones excitations up to $N = 3$ in simplified post-merger perturbative settings [93]. It might well be the case that from a certain point onwards the post-peak signal is indeed well-described by a (time-dependent) linear description, for example because most of the non-linear features are cloacked behind the newly-formed horizon [94]. The key message of [61] is that in such a dynamical and complex evolution many effects are at play, and before being able to confidently identify short-lived overtones components, these effects must be investigated in detail before being possibly excluded. Only after achieving a robust modeling of beyond-linear effects and QNM time dependence can a physical interpretation of early-time modes superpositions be assessed.

2.2.2 Beyond the stationary linear picture

Indeed, recent years have witnessed an impressive effort by the community to model non-linear and time-dependent effects, leading to a wealth of new results, with rapid progress still undergoing. Towards this goal, novel results in the definition of a "QNM scalar-product" [95, 96] hold great promise in advancing the understanding of such process.

Key effects that have been targeted include:

- *Dynamical background* from the moment a common horizon forms, to the one a stationary state is reached, the BH background parameters (mass and spin, for an astrophysical Kerr black hole) will be changing with time [97], see also [23, 24, 98–100].
  This will be driving a complex frequency drift, but also induce "mode-spreading" [97, 101].
  Namely, a given QNM generated around the initial background will experience an amplitude variation due to the background dynamical evolution, while also giving rise to a family of additional modes with respect to the final background.
  Albeit this effect will certainly be present, it seems unlikely that the entire frequency increase in the waveform might be captured by it, given the percentage-level variation of mass and spin, which should account for the order-unity variation displayed by the waveform frequency;
- *Modes coupling* higher-order perturbative contributions imply that two "parent" modes can interact, giving rise to "child" modes with frequencies composed of linear combinations of parent frequencies, and an angular structure dictated by Clebsch-Gordan rules [66].
  The most-studied among these couplings has been the $(2, 2, 0) \times (2, 2, 0) \rightarrow (4, 4, 0)$ one, given that the $(\ell, m, n) = (2, 2, 0)$ mode is typically the dominant one for such systems.
  After the first extraction of such coupling from nonlinear simulations [102, 103] (see earlier work in [89]), a wealth of perturbative studies investigated the quadratic amplitude generation dependence on initial conditions and BH spin, both in numerical [61, 104–106] and analytical [107–111] settings.
  While this effect has a significant impact on non-quadrupolar harmonics, and its inclusion will certainly be required to achieve high-fidelity templates for e.g. $(\ell, m, n) = (4, 4, 0)$, nonlinear couplings relevance to model the early ringdown regime seems to be limited.

In fact, angular selection rules imply that in order for nonlinear couplings involving the highly excited $(\ell, m, n) = (2, 2, 0)$ to enter back the dominant $(\ell, m) = (2, 2)$ spherical mode, interactions with typical sub-dominant modes such as the $(\ell, m) = (2, 0)$ or counter-rotating modes are required, which are expected to deliver an overall small contribution.

These arguments apply as well to third-order couplings, which are expected to deliver an even larger suppression.

2.2.3 Numerical extraction of QNM excitation amplitudes

Another aspect actively pursued in recent years is the numerical extraction of QNM constant amplitude values from the stationary portion of the ringdown signal for comparable-mass binaries, and their modeling in terms of the merger initial conditions (e.g. the binary masses and spins). The quasi-circular case with binary spins aligned to the orbital angular momentum has now been tackled, reaching very high accuracy, with a variety of methodologies [89, 90, 104, 112–114].

The mis-aligned spins case (determing spin-induced precession, hence dubbed "precessing case" below) has witnessed less attention from the QNM amplitudes modeling side ,[3] albeit significant progress has been recently achieved [115, 116]. In particular, the methodology leveraged in [113, 114] holds great promise in obtaining a robust extraction of these quantities. Extensions of the phenomenological models akin to [84] to spin-precessing configurations were instead included in e.g. [117, 118], with [119] including for the first time mode-mixing contributions in these type of templates.

Due to the fact that gravitational radiation is highly efficient in circularising binaries evolving in bounded orbits, the non-circular case has also received little attention in the past. Recently, interest was revived by the possibility that a non-negligible fraction of binaries observed by the LIGO-Virgo-Kagra (LVK) detectors might have formed dynamically or that BH captures could be observed directly in the detectors band [120–124]. In Ref. [125], the extraction of QNM amplitudes beyond the quasi-circular case was achieved in the case of a non-spinning progenitors binary. The methodology employed stems from [63, 126], which have introduced a gauge-invariant parameterisation of non-circular orbits based on combinations of energy and angular momentum parameters, applicable during the entire coalescence process.

Recent advancements have shown that although the initial main motivation for QNM studies was to search for new physics, modeling efforts arising within the BH spectroscopy program can also have direct astrophysical implications, informing IMR models used in LVK searches and binary parameters extraction. Even further, physically-motivated models of QNM excitations and prompt ringdown, required to advance the BH spectroscopy program, could deliver a first-principles model with much higher interpretability properties. This feature would be key to allow for easier extensions to include environmental effects or beyond-GR/Kerr signatures, saving the need for costly numerical simulations that become hardly feasible when considering such host of additional effects.

[3] Phenomenological descriptions entering inspiral-merger-ringdown models that do not focus on the modeling of QNM components are not discussed here, see references in [116].

2.3 Future developments and open problems

Future milestones and open problems include:

- *QNM amplitudes in generic orbits* the extension of closed-form QNM amplitudes models to the quasi-circular precessing case, the spin-aligned noncircular case, and finally to the generic noncircular-precessing one.
  Noncircular extensions of post-merger phenomenological models similar to [63, 84], together with investigations on whether standard twisting-up techniques [117, 127] are sufficient to accurately extend these noncircular templates to the spin-precessing case are also future key milestones.
- *Amplitude growth* as detailed in e.g. [66], transient effects induced by initial data impart a time-dependent amplitude growth even within the context of linear perturbation theory. Extensions of these models to the realistic case of initial data induced by the inspiral of comparable mass binaries appears challenging, but would be necessary to achieve physically motivated templates of the prompt ringdown emission.
- *Orbital imprint* already from the first investigation concerning QNM amplitudes models as a function of binary parameters [112], it was understood that for many QNM modes such dependence closely resembles the leading-order post-Newtonian scaling governing inspiral amplitudes.
  This observation was key in the construction of subsequent high-accuracy closed-form models, and was investigated numerically in a systematic way in [128].
  Analytical models predicting this dependence are currently missing, but as argued in [90], the simplicity of these expressions gives hope that such computation may be achieved in the near future.

3 The data analysis problem

3.1 Introduction and past work

3.1.1 Frequency-domain vs. time-domain formulations

The extraction of BH vibrational spectra from GWs data has been proposed long ago with the key motivation of investigating the Kerr nature of the emitting remnant compact object. Initial analyses were working in an idealised frequency-domain framework [3, 69, 129], where the simulated data consisted of a Fourier transformed QNM-only template with constant amplitudes. This approach ignored altogether contaminations from the pre-stationary ringdown signal and the pre-merger signal. Such setup is sufficient to obtain a reasonable estimate of measurable ringdown properties. However, for real observational scenarios in which a pre-stationary (or even pre-ringdown) signal is always present, these portions will pollute the QNM analysis if not properly taken into account.

A way to get around this problem when analysing complete signals is to multiply the data and the template by a "windowing" function, manually setting to zero previous contributions, as done in [130]. This method is valid but cumbersome, since the steepness of the window needs to be optimised to strike a balance between signal loss (decreased by a steeply-rising window) and spurious Gibbs-phenomena (increased by a steeply-rising window). Formulating the test directly in the time domain can overcome this problem, and this methodology has been applied in the white-noise case (process with equal power over all frequencies, thus uncorrelated in time) in [131], then generalised to the coloured noise (correlated over time) weakly-sense stationary case in [132], when assuming small correlation between ringdown and inspiral components. The full solution for a fixed analysis start time was given in [133] through a truncated likelihood formulation, implemented in the `ringdown` package [134]. An alternative, but mathematically equivalent formulation in the frequency domain was presented in [135], building on [136]. For a comparison of the latter with time-domain methods see [134].

3.1.2 The LVK search in a nutshell

These works led to the time-domain ringdown analysis pipeline built around the `pyRing` package [132, 137], employed by the LVK collaboration to isolate and analyse ringdown signals observed throughout the first, second and third observing runs [138–141]. The pipeline explored a wide range of starting times, and employed models of increasing complexity, ranging from agnostic superpositions of damped sinusoids to templates calibrated against binary BH numerical simulations [90]. All these analyses are complementary to each other. The most agnostic ones are supposed to be able to capture very generic beyond-Kerr/GR signatures, such as additional non-tensorial modes [142], but also to be less sensitive to small deviations from Kerr GR templates (i.e. will require a much larger SNR to flag possible deviations). Conversely, highly-informed templates such as [90, 143] are expected to have a harder time faithfully characterising features that strongly deviate from the GR predicted behaviour, but are expected to be more sensitive to minute deviations (also because less affected by Occam's penalty). Beyond the most-informed analyses performed through `pyRing`, an even more informed analysis is the `pSEOB` one [144–146], where now the pre-merger signal is assumed to correspond to the GR one (unlike time-domain spectroscopic analyses that do not make such assumption, and remove contributions from the pre-merger). The analysis proceeds by retaining the entire NR calibration of a standard full-signal waveform, and adding deviations to QNM parameters. Thus, it will be more sensitive than spectroscopic analyses to small GR deviations, although the advantages of interpretability, and flexibility of damped-sinusoids superpositions in capturing more exotic signals are lost. Details of the actual sensitivity to new physics signatures will likely depend on the details of the modification searched for. This extensive LVK search, together with the subsequent extension [143], did not find robust evidence for the presence of subdominant modes, within the physical regime of applicability of respective models, nor evidence for new physics signatures.

3.1.3 Multiple overtones claims

Evidence for multiple modes was instead reported in [133] regarding an additional overtone ($n = 1$) in GW150914. The overtone detection was challenged in [147], with subsequent responses in [148–150] and analyses from independent groups in [151–154]. Leaving aside the (previously described) fundamental issues with the underlying physical model, through which detection was claimed, many of these analyses initially seemed to return different results regarding the statistical significance of an overtone component starting at the peak of the waveform.

Part of this descrepancy was due to some analyses neglecting certain sources of uncertainty or using an incorrect labeling of the time axis. For example, in [147] a mislabeling of the time stamps induced a systematic error around $0.06\,ms$, while the initial 3.6σ detection claim was ignoring the statistical uncertainty on the determination of t_{start}, amounting to neglect a statistical uncertainty of around $2.5\,ms$. For reference, the damping time of the putative $n = 1$ overtone is $\tau_{221} \simeq 1.4\,ms$, while for the fundamental mode $\tau_{220} \simeq 4.2\,ms$. Once all these elements had been fixed and start time uncertainty was kept into account, analyses converged towards a lower overtone significance, slightly smaller than 2σ with a negative Bayes Factor for the overtone presence, for most of the independent analyses conducted by groups not part of the initial debate [151, 152]. Initially the authors of [153] found higher significance, but later some of the same authors obtained low evidence for an overtone presence with similar methods, but now marginalising over the time uncertainty [154].

Albeit different techniques have been applied by different groups, hence it is not always immediate to compare all these results, some of these analyses did yield different results even when using mathematically equivalent methods. The reason behind this apparent inconsistency is that for the type of signal analysed (very short overtone component, low signal-to-noise ratio) even small differences in data conditioning (sampling rate, bandpassing) and noise estimation methods (namely the computation of the autocorrelation function setting the variance of the gaussian likelihood employed) can lead to non-negligible differences. Indeed these inputs were not always uniform across analyses, as recollected e.g. in [150], which motivates these discrepancies. Instead, in [155] is it shown how similar analyses do obtain compatible outputs when employing similar inputs.

A clear lesson from the ensuing debate is that more attention needs to be paid to develop robust and commonly agreed procedures to include all statistical and systematic uncertainties, together with robust detection thresholds (see below). The recounted debate has certainly been extremely useful in investigating the details of data-analysis methods in spectroscopic settings, hence useful in the context of future detections. However, the view of the author is that for what concerns the physical interpretation of the analysis result, the outcome is moot, independently of the claimed significance. This is because, as extensively motivated in [61] and summarised above, the underlying model ignores a vast array of physical effects that are known to come into play when analysing data close to the peak (time-varying background, nonlinearities, dynamical QNM excitation,...), hence simply cannot be interpreted as a spectroscopic measurement.

3.1.4 Multiple fundamental modes claims

For what concerns detection claims of fundamental modes corresponding to harmonics different than $(\ell, m) = (2, 2)$ (dominant mode), the observability of the sub-dominant

$(\ell, m) = (3, 3)$ was claimed in GW190521 data in [135]. Hints of higher harmonics were uncovered in [155] for the $(\ell, m) = (2, 1)$ mode in GW190521, and in several events for both $(\ell, m) = [(3, 3), (2, 1)]$ modes by [143]. Both these latter analyses did not make any strong detection claim.

Focusing on GW190521, most of the aforementioned problems affecting overtones detection have a significantly smaller impact, given the longer duration of the $n = 0$ components with respect to an overtone. However, many more complications arise in this case due to the morphology of the investigated signal. In fact, GW190521 was originated by a highly massive system (remnant mass $M_f \simeq 250 M_\odot$ in the detector frame), thus spending a very short duration in the detector band. For this reason, it is not possible to discriminate whether this event had measurable spin-induced precession or orbital eccentricity, or a combination of both [4] [124, 156]. However, all inspiral-merger-ringdown analyses conducted by the LVK and external groups [138, 139, 157] agree on the presence of at least one of the two effects, and on the incompatibility of such signal with a spin-aligned quasi-circular binary.

The spectroscopic analysis of [135], claiming detection of the $(\ell, m, n) = (3, 3, 0)$ mode, suffered from two major issues related to the aforementioned signal morphology:

- The amplitude model, employed to reported Bayes Factor for multiple Kerr harmonics, assumed an upper cut on the amplitudes ratio valid for spin-aligned quasi-circular systems only. Additionally, almost all results are quoted assuming reflection symmetry around the orbital plane, broken in generic spin-precessing binaries;
- The chosen t_{start} was based on a frequency-domain approximant family [158] known to provide a sub-optimal signal description than the `NRSur7dq4` model [159].
  The latter model was employed by the LVK collaboration when quoting main results [138] precisely because of its higher accuracy, especially in the region of parameter space probed by this signal.
  When translating the t_{start} at which detection is claimed to the peaktime t_{peak} of $h_+^2 + h_\times^2$ as reconstructed using `NRSur7dq4`, this corresponds to $t_{\mathrm{start}} \simeq t_{\mathrm{peak}} + [2-5] M_f$, where a stationary QNM model based on fundamental harmonics with constant amplitudes is not valid.

Albeit the investigations presented in [135] are certainly tantalising and indicative of a non-trivial signal structure, both these factors prevent a rigorous interpretation of the measurement as a multi-modal spectroscopic detection. Indeed, when keeping these two elements into account, the analysis of [155] finds preference not for the $(\ell, m, n) = (3, 3, 0)$ mode, but for the $(\ell, m, n) = (2, 1, 0)$ mode, but at a much lower significance for which no strong detection claim was made.

In summary, the early ringdown regime bears a significant prize in terms of signal-to-noise ratio (hence resolving power) gained, and it is only natural that analyses will attempt at leveraging this increase. However, as extensively argued in the previous

[4] Further, the gaussian-noise simulation study conducted in [121] indicates that some amount of non-circular behaviour was present, although the same exercise should be repeated using real interferometric noise to validate this conclusion.

section, the risk is that the outcome of such analyses becomes similar to phenomenological fits already performed at the time of the first detection [26], constituting a valid consistency test of GR, but not a spectroscopic measurement.

3.1.5 Start time marginalisation

In [132] the ringdown analysis starting time $t_{\rm start}$ was recognised as the parameter controlling the largest systematic uncertainty of these type of analyses, and its marginalisation attempted, producing a posterior on this quantity. Unfortunately, when adopting the unbiased truncated formulation, this marginalisation does not appear to be possible in a straightforward way. The intuitive reason behind this is that $t_{\rm start}$ does not only control when a QNM-superposition template starts to be non-zero, but also the amount of data included in the likelihood calculation. Hence, the analysis output will tend to push $t_{\rm start}$ as early as possible, to include as much signal as possible, increasing the likelihood function.

Note that this behaviour is dependent on the signal/template features. In fact, it is partly due to the flexibility of damped-sinusoids templates being used, and on the smooth morphology of the binary BH waveform signals (similar to what predicted by effective-one-body arguments [70]), for which no abrupt changes between the merger and ringdown portions can be easily discerned using feature-detection algorithms. Such complication would have instead been avoided, if merger waveforms would have given rise to high-frequency complicated emission patterns, as sometimes foreseen before numerical simulations of binary mergers were achieved [160]. Also note that depending on the problem formulation, the opposite behaviour might arise, with $t_{\rm start}$ being pushed as late as possible to have a close-to-zero signal, consistent with gaussian noise [154].

Current strategies that do not assume pre-merger information simply involve repeating the analysis at multiple times, veryfing the robustness of a given result within the a certain confidence region (typically 90%) of $t_{\rm start}$ as inferred from IMR analyses. This effectively "discretises" the $t_{\rm start}$ support, providing some sort of "marginalisation" of this parameter. However, combining the result of these discrete inference steps is not immediate. This is because each inference with a given $t_{\rm start}$ removes all data before $t_{\rm start}$, implying that different runs are performed on different subsets of the same *data*. Hence, computing a meaningful Bayes Factor between runs at different times (and producing averaged posteriors), is not straightforward. Note, however, that Bayes Factors between different models at a fixed start time retain a standard Bayesian interpretation. A conservative way to avoid this problem when interested in e.g. the detection of a mode (or a posterior of a certain parameter attaining a given value), is to require that such Bayes Factor is larger than a given threshold throughout the uncertainty band of $t_{\rm start}$, as done e.g. in [143]. This is clearly sub-optimal, but valid.

Another way to resolve the issue is to avoid ignoring the pre-ringdown data, making an assumption on the pre-merger signal, in which case $t_{\rm start}$ becomes again a "standard parameter" (i.e. it does not control the amount of data included). This was the strategy employed in [154] by assuming a GR template. This strategy is valid, but loses some of the agnostic character of the search, going closer to `pSEOB`-like searches. In [152] an agnostic superposition of wavelets for the pre-ringdown data was considered. However, without imposing any GR prediction on when the QNM-driven signal starts, the authors had to resort to a series of narrow Gaussian priors over $t_{\rm start}$, which is more computationally efficient than the discretisation strategy employed in time-domain analyses, but ultimately equivalent.

3.2 Open issues

From the above discussion, open issues and challenges that should be addressed in the future to advance spectroscopic analyses are:

- Agreement on a mode detection threshold, dependent on the binary parameter space and instrumental background;
- Marginalisation of $t_{\rm start}$ and sky-position parameters with as little pre-ringdown information as possible;
- Inclusion of systematics in the determination of $t_{\rm start}$, induced by systematic uncertainties in IMR models;
- Identification of the (SNR-dependent) validity regime of the different classes of stationary QNM-like models, as a function of the binary parameter space;
- Autocorrelation methods comparisons, and inclusion of systematic uncertainties induced by different methods or off-source computations [161, 162];

Albeit all these items need to be thoroughly addressed, the prospects of a multi-modal spectroscopic detection in the stationary QNM regime, applied to loud "golden" signals sourced by well-understood binary mergers remain very optimistic [163, 164]. A typical example is the determination of the $(\ell, m, n) = (3, 3, 0)$ mode from a slowly-spinning binary with $m_1/m_2 = [2, 3]$. Past literature detection debates were instead mostly sparkled by attempts to apply spectroscopic analyses to the least understood signals available (GW190521), for which many uncertainties remain even on the IMR side, or to extend QNM templates to the pre-stationary regime, for which robust models are currently lacking. All these complications are instead expected to fade away for aforementioned golden signals, albeit past debates have clearly shown that robust spectroscopic analyses require more work than initially thought. A study rigorously re-evaluating spectroscopic detections on golden binaries, now applying all the analyses developments stemmed from recent debates, appears timely and necessary.

Concerning the analysis of high-SNR signals that are going to be observed by planned detectors [165, 166], many complications that normally affect future detectors measurements (noise non-stationarities, signals overlap,...) are typically greatly reduced by the short ringdown signal duration. However, the problem remains complicated, and a discussion of all the complications of high-accuracy measurements achieved next-generation detectors, such as waveform systematics (including remnant recoil [167, 168] and hereditary effects [65, 169]) and the simultaneous modelling of many QNM contributions, is beyond the scope of this overview. Despite presenting a clear challenge, on a final optimistic note, such analysis does not seem to present any fundamental obstacle, assuming accurate enough theoretical models will be developed.

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