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Review

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Ordered Patterns of (3+1)-Dimensional Hadronic Gauged Solitons in the Low-Energy Limit of Quantum Chromodynamics at a Finite Baryon Density, Their Magnetic Fields and Novel BPS Bounds

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Abstract: In this paper, we will review two analytical approaches to the construction of non-homogeneous Baryonic condensates in the low-energy limit of QCD in (3+1) dimensions. In both cases, the minimal coupling with the Maxwell $U(1)$ gauge field can be taken explicitly into account. The first approach (which is related to the generalization of the usual spherical hedgehog ansatz to situations without spherical symmetry at a finite Baryon density) allows for the construction of ordered arrays of Baryonic tubes and layers. When the minimal coupling of the Pions to the $U(1)$ Maxwell gauge field is taken into account, one can show that the electromagnetic field generated by these inhomogeneous Baryonic condensates is of a force-free type (in which the electric and magnetic components have the same size). Thus, it is natural to wonder whether it is also possible to analytically describe magnetized hadronic condensates (namely, Hadronic distributions generating only a magnetic field). The idea of the second approach is to construct a novel BPS bound in the low-energy limit of QCD using the theory of the Hamilton–Jacobi equation. Such an approach allows us to derive a new topological bound which (unlike the usual one in the Skyrme model in terms of the Baryonic charge) can actually be saturated. The nicest example of this phenomenon is a BPS magnetized Baryonic layer. However, the topological charge appearing naturally in the BPS bound is a non-linear function of the Baryonic charge. Such an approach allows us to derive important physical quantities (which would be very difficult to compute with other methods), such as how much one should increase the magnetic flux in order to increase the Baryonic charge by one unit. The novel results of this work include an analysis of the extension of the Hamilton–Jacobi approach to the case in which Skyrme coupling is not negligible. We also discuss some relevant properties of the Dirac operator for quarks coupled to magnetized BPS layers.

Keywords: low-energy QCD; chiral models; non-homogeneous condensates; topological solitons; BPS bound

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1. Introduction

The theory of quantum chromodynamics (QCD), which describes strong interactions, has the fascinating feature that its coupling constant increases as the energy decreases (and vice versa). This results in a phase diagram divided into different regimes, where different tools or approaches are needed to study particular conditions. In the high-energy domain, the coupling constant has a low value, which allows the use of perturbative techniques. As far as the present theoretical understanding of QCD is concerned, the importance of perturbation theory (which is very effective at high energies due to asymptotic freedom) is widely recognized. Modern scattering amplitude techniques are well equipped to compute tree-level amplitudes of high complexity as well as higher-loop radiative corrections both at zero temperatures and at finite (but high) temperatures (see [1–3] and the references therein). Such techniques provide us with the main tools to analyze the perturbative phase of QCD and are extremely successful in this regime. However, as the coupling constant's order of magnitude increases at lower energies, a perturbative approach becomes unfeasible, and only a handful of refined numerical techniques and effective models have been shown to be useful. In fact, in this infra-red domain (IR henceforth), the physical properties of the phase diagram are largely determined by genuine non-perturbative configurations. Such configurations, called solitons, are smooth classical solutions of the field equations with a non-vanishing topological charge (which prevents them from decaying into the trivial vacuum).

At low energies and temperatures, far from the perturbative regime, the analysis of the interplay between strong interactions and electromagnetic fields still leads to challenging open questions. This is mainly because some of the available options for exploring QCD in the non-perturbative regime also become inefficient in extreme conditions of low temperatures, at finite Baryonic charge densities, or in the presence of electromagnetic fields. On the one hand, Lattice-QCD, which is the main numerical tool to simulate strongly interacting QCD systems, experiences some problems when dealing with both finite Baryon densities and background external fields (see [4–7]). On the other hand, analytical approaches based on AdS/CFT are also not very effective for non-homogeneous condensates. As it is clear from the classic references [8–10] (see also the references therein), the analysis of such condensates using holography heavily relies on numerical analysis anyway. Furthermore, when analytical solutions are available, new phenomena can be disclosed (such as the appearance of critical values of control parameters—like the external magnetic field—beyond which the behavior of the configurations changes considerably). Last but not least, in the cases in which the electromagnetic self-interactions of non-homogeneous Baryonic condensates are included, not only do the numerical methods to analyze the Baryonic condensates in [11–15] (and the references therein) become considerably more demanding, but the analytic and variational tools in [16–25] also partially lose their effectiveness. Indeed, it is not easy to generalize the results in these mentioned works to include the minimal coupling with a dynamical $U(1)$ gauge field in (3+1) dimensions. Thus, any novel analytic insight can be extremely useful. There is no doubt that a deeper understanding of the properties of the topological solitons of QCD is of fundamental importance in order to shed more light on the phase diagram (see [4,26–28] and the references therein). There are many physical situations where these issues are fundamental, and here we will name just a few of them. Firstly, several intriguing phenomena (which only appear at a finite Baryon density) deserve a much deeper theoretical analysis. In particular, the existence of non-homogeneous Baryonic condensates is a genuine result of the fact that a finite amount of topological charge is "forced to live" within a finite spatial region (otherwise, in free space, such condensates would not form). In (1+1)-dimensional models (where the "solitons" are kinks, such as in the Gross–Neveu model and its variants; see, for instance, [29–38]), it is clear by now that there is a finite region in the phase diagram (which appears at a finite density) where kink crystals (namely, ordered arrays of kinks) dominate. Similar results further supporting the appearance of non-homogeneous condensates have been obtained in higher-dimensional models (under the assumption that the main fields only depend on one spatial coordinate, see [16–20] and the references therein). Both types of results strongly suggest that similar phenomena should appear at the low-energy limit of QCD in (3+1) dimensions as well. The models analyzed in [29–38] are considered very good toy models of QCD at a finite Baryon density. In such models, the formation of ordered patterns of solitons is well understood theoretically (due to integrability). In (3+1) dimensions, there is strong numerical as well as phenomenological evidence (see [12–15,39–44] and the references therein) suggesting that non-homogeneous Baryonic condensates do appear at low energies and temperatures when a finite amount of Baryonic charge is forced into a finite spatial volume. The numerical simulations of these systems are very challenging;

hence, the development of novel theoretical tools would be very helpful. It is also worth noting that the elegant analytic approach introduced in [16–20] (and the references therein) cannot be applied directly to describe the inhomogeneous Baryonic distributions observed in [12–15,39–44] (and the references therein) since the energy and Baryon densities of such distributions depend in a non-trivial way on more than one spatial coordinate. Secondly, Lattice-QCD is not so effective at a finite Baryon density due to the infamous sign problem (see [4–6] and the references therein). Consequently, the study of the role of finite density effects in the analysis of non-perturbative configurations with a finite topological charge acquires a high priority. Thirdly, a detailed analysis of the effects of external electric and magnetic fields on the topological solitons of the low-energy limit of QCD could clarify the relevant physical properties of non-perturbative configurations in neutron stars and heavy-ion collisions.

Exploring this regimen of extreme conditions, where the above-mentioned methods seem to fail, is not only interesting because of the challenge itself, but because these conditions are of great intrinsic physical interest because they appear, for instance, in neutrons stars and heavy-ion collisions. Indeed, there is sound evidence (see [12–15,39–44] and the references therein) suggesting that non-homogeneous Baryonic distributions appear at low energies. However, the numerical simulations of these systems are very challenging, especially when coupling with the Maxwell gauge potential is included. Thus, the development of novel theoretical tools would be very helpful to complement the numerical analysis. Moreover, in situations where strong magnetic fields are present (see, for instance, [45–47] and the references therein), it is very natural to wonder whether or not the topological solitons of the theory (which play a prominent role in the phase diagram) are affected by strong external electromagnetic fields.

In this paper, we will describe three analytical approaches to describing non-homogeneous Baryonic condensates both without and with the inclusion of electromagnetic interactions, introduced in references [48–50]. These techniques allow us to explicitly compute interesting quantities of these Hadronic condensates of high physical interest (which would be difficult to compute otherwise). This paper is organized as follows: Firstly, in Section 2, we will present a short review on the chiral σ-model and the Skyrme model, emphasizing the existence of a conserved topological charge and a lower bound for the energy. We will also present two useful parametrization choices for the fields of these chiral models. Then, in Section 3, we comment on the hedgehog ansatz, and then introduce two types of ansätzes which represent inhomogeneous Baryonic condensates. In Section 4, we introduce the coupling between the Skyrme model and the electromagnetic field. In Section 5, we review recently obtained results where a coupling between the σ-model and Maxwell results in stable solutions in the shape of ordered magnetized Baryonic layers, as well as the method with which those solutions can be obtained and the implications of having a non-trivial lower bound of the energy. In Section 6, we discuss some future theoretical applications of the previous results of coupling Skyrme and Maxwell theories. Some comments, final remarks and projections are presented in Section 7.

2. Chiral Models

In this section, we introduce two non-linear models first presented in the 1950s and 1960s, which have the characteristic of being defined in a target manifold $SU(N_F)$, with $N_F = 2$ (for a generalization, see [51]). Due to their internal global symmetries, they are regarded as *chiral models*.

2.1. The σ-Model

The σ-model was first introduced by Gell-Mann and Lévy in 1960 in the context of nuclear physics in an attempt to construct a chirally invariant Lagrangian. Originally, this model consisted of a triplet ϕ^a of Pions, a scalar field ϕ_0 and a coupling with a fermionic iso-doublet ψ. We will not consider this fermionic coupling until later in the review. In modern notation, the Lagrangian of the model is:

$$\mathcal{L}_0 = \frac{F_\pi^2}{16}\,\mathrm{Tr}\Big\{\partial_\mu U \partial^\mu U^\dagger\Big\}, \tag{1}$$

where $U = U(t, \vec{x})$ is an $SU(2)$-valued scalar; it can be written explicitly in terms of the Pauli σ matrices as,

$$U = \phi_0 \mathbf{1} + \phi^a \cdot \mathbf{t}_a, \tag{2}$$

with $\mathbf{t}_a = i\sigma_a$. And, since $U \in SU(2)$, the fields (ϕ_0, ϕ^a) verify that

$$\phi_0^2 + \phi^a \phi_a = 1,$$

is restricted to the target manifold S^3 (the space manifold of the $SU(2)$) group.

The σ-model does not posses static solitonic solutions due to Derrick's scaling argument [52].

2.2. Skyrme Model

In 1961, the physicist T. Skyrme proposed a new non-linear theory for Pions [53], which features a modification to the σ-model, introducing a new quartic term in the Lagrangian that avoids Derrick's scaling argument, hence allowing the existence of stable, finite energy solitonic solutions called Skyrmions. Although in its beginnings, the relevance of this model was not widely recognized, today it definitely is. Indeed, it has been shown that the low-energy limit of QCD (minimally coupled with Maxwell $U(1)$ gauge theory) at leading order in the 't Hooft expansion [54–58] is the gauged Skyrme model [59–61] (we will study the $U(1)$ gauged version of the model later on). Skyrme theory possesses both (topologically trivial) small excitations describing mesons (Pions in the $SU(2)$ case) and topological solitons describing Baryons [26,60–66], with the Baryonic charge being a topological invariant. Hence, Skyrmions are the topological solitons of the low-energy limit of QCD.

The Lagrangian is:

$$\mathcal{L} = \frac{f_\pi^2}{16}\,\mathrm{Tr}\Big\{\Big(U^\dagger \partial_\mu U\Big)\Big(U^\dagger \partial^\mu U\Big)\Big\} + \frac{1}{32e^2}\,\mathrm{Tr}\Big\{\Big[\partial_\mu U U^\dagger,\, \partial_\nu U U^\dagger\Big]^2\Big\}, \tag{3}$$

where the first term is the Lagrangian of the σ-model, and the second term is known as the Skyrme term. Note the first term of the Lagrangian has been rewritten in a more convenient form, $\mathrm{tr}\{\partial_\mu U \partial_\mu U^{-1}\} = \mathrm{tr}\{(U^{-1}\partial_\mu U)^2\}$, which will be explained later. The parameters f_π, the Pion decay constant, and e, an adimensional constant which mediates the self-interaction that stabilizes the Skyrmions, are positive constants fixed experimentally (see [63,64]). The third term which gives the Pions a tree-level mass is deliberately neglected since the Pion mass is negligible with respect to the mass of the non-homogeneous condensates (which, as we will see, grows with the Baryonic charge).

Due to restrictions from QCD, we expect that the Lagrangian (3) is invariant under the action of the $SU(2)_L \times SU(2)_R$ chiral group (actually, a generalization of the Skyrme Lagrangian can be derived from this kind of consideration, see, e.g., [51]). Taking this into account, a chirally invariant action is constructed using left/right-invariant Maurer–Cartan forms.

The left-invariant L_μ and the right-invariant R_μ forms are:

$$\begin{aligned} L_\mu &= U^\dagger \partial_\mu U \\ R_\mu &= \partial_\mu U U^\dagger, \end{aligned} \tag{4}$$

which both transform under $(\hat{T}_L, \hat{T}_R) \in SU(2)_L \times SU(2)_R$ to:

$$\begin{aligned} L_\mu &\to \hat{T}_R L_\mu \hat{T}_R^\dagger, \\ R_\mu &\to \hat{T}_L R_\mu \hat{T}_L. \end{aligned} \tag{5}$$

In the following, we will stick to using the left-invariant form. Thus, we can re-write the Lagrangian in (3) to obtain an action which depends only on one of the chiral fields:

$$I_S = \frac{K}{2} \int d^4x \sqrt{-g}\, \mathrm{Tr}\left\{ L_\mu L^\mu + \frac{\lambda}{8} G_{\mu\nu} G^{\mu\nu} \right\}, \tag{6}$$

where $G_{\mu\nu} = [L_\mu, L_\nu]$ and $K = \frac{f_\pi^2}{4}$, $\lambda = \frac{4}{e^2 f_\pi^2}$, with g as the metric determinant. The field equations derived by varying the action (6) with respect to U are

$$\partial_\mu \left(L^\mu + \frac{\lambda}{4} [L_\nu, G^{\mu\nu}] \right) = 0, \tag{7}$$

and the energy-momentum tensor of the theory is

$$T_{\mu\nu} = -\frac{K}{2} \mathrm{Tr}\left\{ L_\mu L_\nu - \frac{1}{2} g_{\mu\nu} L^\alpha L_\alpha + \frac{\lambda}{4} \left(g^{\alpha\beta} G_{\mu\alpha} G_{\nu\beta} - \frac{1}{4} G_{\sigma\rho} G_{\sigma\rho} \right) \right\}. \tag{8}$$

In the case in which the theory is defined in $\mathbb{R}^3$, the requirement of having a finite energy is that $U \to \mathbf{1}$ at spatial infinity, $\vec{r} \to \infty$. As a result, $U(t, x)$ is a map from a compactified coordinate space S^3 to the $SU(2)$ group-manifold space, S^3. This map can now be labeled by a topological invariant $Q = \pi_3(S^3)$. This is the topological charge (or winding number) of the configuration, and it can be shown that this quantity is conserved (see [51], chapter 13). Moreover, in the classic references [53,60,61,67], it has been shown that Q represents the Baryonic charge of the configuration. This topological charge can be derived from the conserved topological current:

$$J_\mu = \epsilon_{\mu\nu\alpha\beta} \mathrm{Tr}\{ L_\nu L_\alpha L_\beta \}, \tag{9}$$

and hence

$$Q = -\frac{1}{24\pi^2} \int d^3x \epsilon^{ijk} \mathrm{Tr}\{ L_i L_j L_k \}. \tag{10}$$

A feature of great interest in the theory of topological solitons is the presence of a lower bound for the energy in terms of the topological charge that can be derived by manipulating the energy density (see [26,51,68,69]). This lower bound is called a BPS bound (named after E. Bogomolny, M.K. Prasad and C. Sommerfield, see [70,71]) and is topological in nature, since it only depends on the boundary conditions of the space in which the configuration lives. Hence,

$$E = k|Q|, \tag{11}$$

where k is a constant (that carries energy units) and $Q \in \mathbb{Q}$ is a topological index.

An important feature of the Skyrme theory, which is very different from the case of Yang–Mills Higgs Theory (with the Higgs field in the adjoint) is that, although a BPS lower bound on the energy of static solitonic solutions in terms of the Baryonic charge can be derived, $E \geq k|B|$, such a bound cannot be saturated on flat spacetimes. We will come back to this issue in the following sections.

2.3. Parametrization of the U Fields

As previously mentioned, the U fields in the σ-model and in the Skyrme model are $SU(2)$-valued fields. We can easily parameterize an expression of the form of Equation (2) by:

$$U(t, \vec{x}) = \cos(\alpha) \mathbf{1} + \sin(\alpha)\ \mathbf{t}_a \hat{r}^a \tag{12}$$

where

$$\hat{r} = \begin{pmatrix} \cos\Phi\sin\Theta \\ \sin\Phi\sin\Theta \\ \cos\Theta \end{pmatrix}. \tag{13}$$

The fields α, Φ and Θ are the scalar degrees of freedom, or profiles, which parameterize the U field. This parametrization is usually known as the *exponential parametrization*.

Firstly, let us show that the chiral current L_μ can be written in terms of this parametrization as:

$$\begin{aligned} L_\mu = {} & (\mathbf{t}\cdot\hat{r})\,\partial_\mu\alpha + \frac{\sin(2\alpha)}{2}\left[(\mathbf{t}\cdot\hat{\Theta})\partial_\mu\Theta + (\mathbf{t}\cdot\hat{\Phi})\partial_\mu\Phi\right] \\ & - \sin^2(\alpha)\left[(\mathbf{t}\cdot\hat{r})(\mathbf{t}\cdot\hat{\Theta})\partial_\mu\Theta + (\mathbf{t}\cdot\hat{r})(\mathbf{t}\cdot\hat{\phi})\partial_\mu\Phi\right], \end{aligned} \tag{14}$$

where

$$\hat{\Theta} = \begin{pmatrix} \cos\Phi\cos\Theta \\ \sin\Phi\cos\Theta \\ -\sin\Theta \end{pmatrix}, \quad \hat{\Phi} = \begin{pmatrix} -\sin\Phi\sin\Theta \\ \cos\Phi\sin\Theta \\ 0 \end{pmatrix}. \tag{15}$$

Now, we can rewrite Skyrme's action in (6) in terms of these parameters. Firstly, let us consider the general element $U = \phi^0 + \mathbf{t}_a\phi^a$ and let us introduce the following tensor [48]:

$$Y_{\mu\nu} = G_{ij}\partial_\mu\phi^i\partial_\nu\phi^j, \tag{16}$$

with

$$G_{ij} = \delta_{ij} + \frac{\phi^i\phi^j}{1-\phi^k\phi_k} \tag{17}$$

acting as the metric of an intern space of functions in $SU(2)$. With this, the action can be rewritten as:

$$I = \int d^4x\sqrt{-g}\left[\frac{1}{2}Y^\mu{}_\mu + \frac{\lambda}{4}\left((Y^\mu{}_\mu)^2 - Y^{\mu\nu}Y_{\mu\nu}\right)\right]. \tag{18}$$

The variation in the action with respect to the α field leads us to:

$$\begin{aligned} & (-\partial_\mu\partial^\mu\alpha + \sin\alpha\cos\alpha)(\partial_\mu\Theta\partial^\mu\Theta + \sin^2\Theta\partial_\mu\Phi\partial^\mu\Phi) \\ & +\lambda\Big[\sin\alpha\cos\alpha((\partial_\mu\alpha\partial^\mu\alpha)(\partial_\nu\Theta\partial^\nu\Theta) - (\partial_\mu\alpha\partial^\mu\Theta)^2) \\ & + \sin\alpha\cos\alpha\sin^2\Theta((\partial_\mu\alpha\partial^\mu\alpha)(\partial_\nu\Phi\partial^\nu\Phi) - (\partial_\mu\alpha\partial^\mu\Phi)^2) \\ & +2\sin^3\alpha\cos\alpha\sin^2\Theta((\partial_\mu\Theta\partial^\mu\Theta)(\partial_\nu\Phi\partial^\nu\Phi) - (\partial_\nu\Theta\partial^\nu\Phi)^2) \\ & -\partial_\nu(\sin^2\alpha(\partial_\mu\Theta\partial^\mu\Theta)\partial^\nu\alpha) + \partial_\mu(\sin^2\alpha(\partial_\nu\alpha\partial^\nu\Theta)\partial^\mu\Theta) \\ & -\partial_\mu(\sin^2\alpha\sin^2\Theta(\partial_\nu\Phi\partial^\nu\Phi)\partial^\mu\alpha) + \partial_\mu(\sin^2\alpha\sin^2\Theta(\partial_\nu\alpha\partial^\nu\Phi)\partial^\mu\Phi)\Big] = 0, \end{aligned} \tag{19}$$

with the Θ field, we obtain:

$$\begin{aligned} & (-\sin^2\alpha\partial_\mu\partial^\mu\Theta - 2\sin\alpha\cos\alpha\partial_\mu\alpha\partial^\mu\Theta + \sin^2\alpha\sin\Theta\cos\Theta\partial_\mu\Phi\partial^\mu\Phi) \\ & +\lambda\Big[\sin^2\alpha\sin\Theta\cos\Theta((\partial_\mu\alpha\partial^\mu\alpha)(\partial_\nu\Phi\partial^\nu\Phi) - (\partial_\mu\alpha\partial^\mu\Phi)^2) \\ & + \sin^4\alpha\sin\Theta\cos\Theta((\partial_\mu\Theta\partial^\mu\Theta)(\partial_\nu\Phi\partial^\nu\Phi) - (\partial_\mu\Theta\partial^\mu\Phi)^2) \\ & -\partial_\mu(\sin^2\alpha(\partial_\nu\alpha\partial^\nu\alpha)\partial^\mu\Theta) + \partial_\mu(\sin^2\alpha(\partial_\nu\alpha\partial^\nu\Theta)\partial^\mu\alpha) \\ & -\partial_\mu(\sin^4\alpha\sin^2\Theta(\partial_\nu\Phi\partial^\nu\Phi)\partial^\mu\Theta) + \partial_\mu(\sin^4\alpha\sin^2\Theta(\partial_\nu\Theta\partial^\nu\Phi)\partial^\mu\Phi)\Big] = 0 \end{aligned} \tag{20}$$

and finally, varying with respect to Φ gives us:

$$\begin{aligned}&(-\sin^2\alpha\sin^2\Theta\;\partial_\mu\partial^\mu\Phi-2\sin\alpha\cos\alpha\sin^2\Theta\partial_\mu\alpha\partial^\mu\Phi-\sin^2\alpha\sin\Theta\cos\Theta\partial_\mu\Theta\partial^\mu\Phi)\\&+\lambda\Big[\partial_\mu(\sin^2\alpha\sin^2\Theta(\partial_\nu\alpha\partial^\nu\Phi)\partial^\mu\alpha)-\partial_\mu(\sin^2\alpha\sin^2\Theta(\partial_\nu\alpha\partial^\nu\alpha)\partial^\mu\Phi)\\&+\partial_\mu(\sin^4\alpha\sin^2\Theta(\partial_\nu\Theta\partial^\nu\Phi)\partial^\mu\Theta)-\partial_\mu(\sin^4\alpha\sin^2\Theta(\partial_\nu\Theta\partial^\nu\Theta)\partial^\mu\Phi)\Big]=0\end{aligned}\tag{21}$$

We must insist that Equations (19)–(21) are just the expansion of the previously shown field equations, but in terms of this particular parametrization.

Another useful parametrization is achieved by referring to the Euler angles of the sphere, then:

$$U(t,\vec{x})=e^{\mathbf{t}_3F}e^{\mathbf{t}_2H}e^{\mathbf{t}_3G}\tag{22}$$

where F,H,G are the three profiles which determine the U field in this parametrization. A way of connecting both the Euler angles and the exponential parametrization is shown in Appendix A.

In this parametrization, the chiral current L_μ is of the form:

$$L_\mu=\mathbf{t}_3\partial_\mu G+(\hat{O}_3+\mathbf{t}_3)\partial_\mu F+\left[(\hat{O}_2+\mathbf{t}_2)\cos(2F)+(\hat{O}_1+\mathbf{t}_1)\sin(2F)\right]\partial_\mu H,\tag{23}$$

where

$$\hat{O}_i=U^{-1}[\mathbf{t}_i,U]=U^{-1}\mathbf{t}_iU-\mathbf{t}_i.\tag{24}$$

The respective $\hat{O}_i$ matrices are written explicitly in Appendix B, together with some useful commutation relations. Using these matrices, we can rewrite the chiral current in a component-wise manner as:

$$\begin{aligned}L_\mu=&\{\sin(2H)\cos(2G)\partial_\mu F-\sin(2G)\partial_\mu H\}\mathbf{t}_1\\&+\{\sin(2H)\sin(2G)\partial_\mu F+\cos(2G)\partial_\mu H\}\mathbf{t}_2\\&+\{\partial_\mu G+\cos(2H)\partial_\mu F\}\mathbf{t}_3.\end{aligned}\tag{25}$$

With this, the Skyrme action can be written explicitly in terms of the three profiles as follows:

$$\begin{aligned}I(F,H,G)=&-\frac{K}{2}\int d^4x\sqrt{-g}[\partial_\mu H\partial^\mu H+\partial_\mu F\partial^\mu F+\partial_\mu G\partial^\mu G+2\cos(2H)(\partial_\mu F\partial^\mu G)\\&-\lambda(2\cos(2H)(\partial_\mu H\partial^\mu F)(\partial_\mu H\partial^\mu G)-(\partial_\mu H\partial^\mu H)(\partial_\mu F\partial^\mu G))\\&+4\sin^2(H)\cos^2(H)((\partial_\mu F\partial^\mu G)^2-(\partial_\mu F\partial^\mu F)(\partial_\mu G\partial^\mu G))\\&+(\partial_\mu H\partial^\mu F)^2+(\partial_\mu H\partial^\mu G)^2-(\partial_\mu H\partial^\mu H)(\partial_\mu F\partial^\mu F)-(\partial_\mu H\partial^\mu H)(\partial_\mu G\partial^\mu G)].\end{aligned}\tag{26}$$

By varying the action with respect to the profiles F, H and G, field equations are obtained:

$$\begin{aligned}0=&\partial_\mu[\cos(2G)\sin(2H)\partial^\mu F-\sin(2G)\partial^\mu H\\&-\lambda\sin(2G)((\partial_\nu F\partial^\nu F)\partial^\mu H-(\partial_\nu F\partial^\nu H)\partial^\mu F+(\partial_\nu G\partial^\nu G)\partial^\mu H-(\partial_\nu G\partial^\nu H)\partial^\mu G\\&+\cos(2H)(2(\partial_\nu F\partial^\nu G)\partial^\mu H-(\partial_\nu F\partial^\nu H)\partial^\mu G)-(\partial_\nu H\partial^\nu G)\partial^\mu F)\\&-\lambda\cos(2G)\sin(2H)((\partial_\nu F\partial^\nu G)\partial^\mu G+(\partial_\nu F\partial^\nu H)\partial^\mu H-(\partial_\nu G\partial^\nu G)\partial^\mu F\\&-(\partial_\nu H\partial^\nu H)\partial^\mu F-\cos(2H)((\partial_\nu F\partial^\nu G)\partial^\mu F-(\partial_\nu F\partial^\nu F)\partial^\mu G))],\end{aligned}\tag{27}$$

$$\begin{aligned}0=&\partial_\mu[\sin(2G)\sin(2H)\partial^\mu F-\cos(2G)\partial^\mu H\\&+\lambda\cos(2G)((\partial_\nu G\partial^\nu G)\partial^\mu H-(\partial_\nu G\partial^\nu H)\partial^\mu G+(\partial_\nu F\partial^\nu F)\partial^\mu H-(\partial_\nu F\partial^\nu H)\partial^\mu F\\&+\cos(2H)(2(\partial_\nu F\partial^\nu G)\partial^\mu H-(\partial_\nu F\partial^\nu H)\partial^\mu G)-(\partial_\nu G\partial^\nu H)\partial^\mu F)\\&+\lambda\sin(2G)\sin(2H)((\partial_\nu H\partial^\nu H)\partial^\mu F-(\partial_\nu H\partial^\nu F)\partial^\mu H+(\partial_\nu G\partial^\nu G)\partial^\mu F\\&-(\partial_\nu G\partial^\nu F)\partial^\mu G-\cos(2H)((\partial_\nu F\partial^\nu G)\partial^\mu F-(\partial_\nu F\partial^\nu F)\partial^\mu G))],\end{aligned}\tag{28}$$

$$
0 = \partial_\mu[\cos(2H)\partial^\mu F + \partial^\mu G - \lambda \sin^2(2H)((\partial_\nu F\partial^\nu G)\partial^\mu F - (\partial_\nu F\partial^\nu F)\partial^\mu G) \\ + \lambda\cos(2H)((\partial_\nu H\partial^\nu H)\partial^\mu F - (\partial_\nu H\partial^\nu F)\partial^\mu H) + \lambda((\partial_\nu H\partial^\nu H)\partial^\mu G - (\partial_\nu H\partial^\nu G)\partial^\mu H)]. \tag{29}
$$

3. The First Approach: The Generalized Hedgehog Ansatz, Baryonic Tubes and Layers

3.1. Hedgehog Ansatz

As previously mentioned, the main advantage of the Skyrme model is that it can support stable solitonic solutions. However, finding such solutions is a non-trivial task. The first ansatz to simplify the Skyrme field equations was proposed by Skyrme himself [53]. He constructed the first example of a spherical "hedgehog configuration", characterized by a spherically symmetric energy-momentum tensor and at the same time a non-vanishing topological charge. Skyrme required:

$$
-i(x \times \nabla)_i U(\vec{x}) + \left[\frac{\tau_i}{2}, U(\vec{x})\right] = 0, \tag{30}
$$

that is, the requirement that even if the chiral field itself is not spherically symmetric, the lack of spherical symmetry can be compensated for by an isospin transformation. The most general solution for Equation (30) is:

$$
U(r) = \cos\alpha(r)\mathbf{1} + \hat{r}^i \cdot \tau_i \sin\alpha(r), \tag{31}
$$

also known as the spherical *hedgehog ansatz* (HA) (the name "hedgehog" is due the form of the Pionic r^a vector field, which is also spherically symmetric). With this ansatz, the complete set of Skyrme equations (7) reduces to a single O.D.E. for $\alpha(r)$, which unfortunately cannot be solved analytically [53].

In the case of these spherical hedgehogs, one can achieve a higher Baryonic B charge by increasing $\alpha(r \to \infty) - \alpha(r = 0)$. But for $SU(2)$, only $|B| = 1$ is topologically stable, since any attempts to construct Skyrmions with $|B| \geq 2$ using the spherical HA will result in unstable configurations (see [51,69] and the references therein). On the other hand, for $SU(N_F)$ (with $N_F \geq 3$), spherically symmetric HAs with higher Baryonic charges which are stable have been constructed (see [72–76]).

3.2. Finite-Density Condensates

Now, we will introduce two ansätzes presented in [48,77], where the main aim is to find analytical solutions for the Skyrme field equations with an arbitrary Baryonic charge, defined in a compact spacetime and without spherical symmetry. For this purpose, we start by introducing the following flat metric:

$$
ds^2 = -dt^2 + L^2(dx^2 + dy^2) + L_z^2 dz, \tag{32}
$$

where x, y, z are Cartesian adimensional coordinates.

$$
0 \leq x \leq 2\pi, \quad 0 \leq y \leq \pi \quad 0 \leq z \leq 2\pi. \tag{33}
$$

Hence, this metric describes a box with volume $V = 4\pi^3 L_z L^2$.

The aim of considering a compact spacetime is to analyze the peculiar effects to force a finite amount of Baryonic charge to live in a finite volume.

3.2.1. Baryonic Tubes

The following ansatz was introduced in [48]:

$$
\alpha = \alpha(x), \quad \Theta = \Theta(y) := qy, \quad \Phi = \Phi(t, z) := p(\tfrac{t}{L} - z), \tag{34}
$$

where $q = 2v + 1$ with $p, vs. \in \mathbb{N}$ and $p \neq 0$.

Notice that the ansatz was chosen in such a way that $\Phi(t,z)$ actually depends on the light-like coordinate $t/L - z$, while the profile is linear in x. This ansatz reduces the Skyrme field equations to the following quadrature for $\alpha(x)$:

$$\partial_x\left[\frac{1}{2}(L^2+q^2\lambda\sin^2\alpha)(\partial_x\alpha)^2-\frac{q^2L^2}{2}sin^2(\alpha)-E_0\right]=0, \tag{35}$$

with E_0 as an integration constant. Hence, when (35) is satisfied, it is guaranteed that the three Skyrme field equations are also satisfied.

A quite important feature of this ansatz is that even if the field equations are greatly simplified, the Baryonic charge of the configuration is non-vanishing:

$$\rho_B=\Big[12pq\sin(qy)\sin^2(\alpha)\Big]\frac{\partial\alpha}{\partial x}\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z. \tag{36}$$

The properties of these Baryonic tubes have been analyzed in the following references: [78,79].

3.2.2. Baryonic Layers

In this section, we will work with the second parametrization (22) which uses the Euler angles. The following choice of ansatz for the profiles, first presented in [49], allows us to simplify the Skyrme field equations:

$$H(x^\mu)=H(t,x),\quad F(x^\mu)=\frac{q}{2}y,\quad G(x^\mu)=\frac{p}{2}z. \tag{37}$$

The Skyrme action is reduced to the following effective sine-Gordon theory:

$$I_{SG}=\int\left(-\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi+M_0(\cos(\beta\varphi)-1)\right)dtdx, \tag{38}$$

where

$$\varphi=\frac{4}{\beta}H,\quad M_0=\frac{\pi^2K}{8L^2}B^2\lambda,\quad \beta=\frac{2}{\pi[K(2L^2+B\lambda)]^{1/2}}, \tag{39}$$

Here, we consider $p=q$ and $B\equiv p^2>0$ to simplify the formulas.

The field equation for the field φ is given by:

$$-\partial_t^2\varphi+\partial_x^2\varphi-M_0\beta\sin(\beta\varphi)=0. \tag{40}$$

4. Minimal Coupling with Maxwell: Force-Free Plasma and Inhomogeneous Baryonic Condensates

The Skyrme–Maxwell Model

We can extend theory (6) to allow for invariance under gauge $U(1)$ transformations. We consider that a Skyrme field transforms due to the elements $e^{i\chi(x)}\in U(1)$ to:

$$U\to U'=U+i\chi(x)[t_3,U]$$

with the covariant derivative

$$D_\mu U=\partial_\mu U+A_\mu[t_3,U], \tag{41}$$

where the gauge invariance is given by introducing the gauge field A_μ, which transforms to:

$$A_\mu\to A_\mu=A_\mu-\partial_\mu\chi(x).$$

Therefore, the Skyrme–Maxwell action is:

$$I=\int\mathrm{d}^4x\sqrt{-g}\left[\frac{K}{2}\mathrm{Tr}\left\{\frac{1}{2}\Sigma^\mu\Sigma_\mu+\frac{\lambda}{16}G_{\mu\nu}G^{\mu\nu}\right\}-\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu}\right], \tag{42}$$

where Σ_μ is the gauged left-invariant chiral current:

$$\Sigma_\mu = U^{-1} D_\mu U = L_\mu + A_\mu \hat{O}_3, \tag{43}$$

with $G_{\mu\nu} = [\Sigma_\mu, \Sigma_\nu]$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. By varying the action, one can find that the field equations are:

$$\begin{aligned} D^\mu\left(\Sigma^\mu + \frac{\lambda}{4}[\Sigma_\nu, G^{\mu\nu}]\right) &= 0, \\ \partial_\mu F^{\mu\nu} &= J^\nu, \end{aligned} \tag{44}$$

where J^ν is the conserved current depending on the field configuration and the field:

$$J^\mu = \frac{K}{2}\,\mathrm{Tr}\left\{\hat{O}_3\Sigma^\mu + \frac{\lambda}{4}\hat{O}_3[\Sigma_\nu, G^{\mu\nu}]\right\}, \tag{45}$$

where we have already defined $\hat{O}_3 = U^{-1}t_3 U - t_3$.

On the other hand, the energy-momentum tensor in the Skyrme–Maxwell theory is

$$T_{\mu\nu} = -\frac{K}{2}\,\mathrm{Tr}\left\{\Sigma_\mu\Sigma_\nu - \frac{1}{2}g_{\mu\nu}\Sigma^\alpha\Sigma_\alpha + \frac{\lambda}{4}\left(g^{\alpha\beta}G_{\mu\alpha}G_{\nu\beta} - \frac{1}{4}g_{\mu\nu}G_{\sigma\rho}G_{\sigma\rho}\right)\right\} + \bar{T}_{\mu\nu}, \tag{46}$$

where $\bar{T}_{\mu\nu}$ is an extra term associated with the electromagnetic field:

$$\bar{T}_{\mu\nu} = F_{\mu\alpha}F^\alpha_\nu - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g_{\mu\nu}. \tag{47}$$

A consequence of this coupling is that the topological charge (10) is not gauge invariant. A new topological invariant charge is found in [60],

$$Q_B = \frac{1}{24\pi^2}\int(\rho_{B_1} + \rho_{B_2}) \tag{48}$$

where the density ρ_B is constructed by the contributions $\rho_B = \rho_{B1} + \rho_{B2}$, defined as:

$$\rho_{B1} = \epsilon^{ijk}\,\mathrm{Tr}\{L_i L_j L_k\}, \tag{49}$$

$$\rho_{B2} = -\epsilon^{ijk}\,\mathrm{Tr}\{\partial_i[3A_j t_3(R_k + L_k)]\}, \tag{50}$$

where the second term is responsible for the Callan–Witten effect. Note that ρ_{B_1}, ρ_{B_2} both depend on the Maurer–Cartan invariant forms L_μ, R_μ, which are different from the gauged chiral invariant form Σ_μ.

5. The Second Approach: Hamilton–Jacobi Equation and a Novel BPS Bound

5.1. Magnetized BPS Solitons

As we mentioned earlier, there is a strong argument that prevents one from finding any BPS configurations for the σ-model: Derrick's scaling argument [52]. Skyrme avoided this argument by introducing his famous term. However, the simplest possibility is the minimal coupling with the $U(1)$ gauge field (in analogy with the Abelian Higgs model, where the presence of the gauge field supports the existence of topologically stable vortices which cannot exist in the absence of gauge coupling).

Let us consider the gauged non-linear sigma-model (G-NLSM) action:

$$I[U] = -\frac{1}{4}\int \mathrm{d}^4x\{K\,\mathrm{Tr}\{\Sigma^\mu\Sigma_\mu\} - F_{\mu\nu}F^{\mu\nu}\}. \tag{51}$$

As in the Skryme–Maxwell model, the field equations are

$$\begin{aligned} D_\mu \Sigma^\mu &= 0 \\ \partial_\mu F^{\mu\nu} &= J^\nu, \end{aligned} \tag{52}$$

with the conserved current defined in (45) and the energy-momentum tensor from (46).

A natural question arises: is there any way of obtaining solitonic solutions for this theory, given a convenient choice of Maxwell field, which are topologically stable, as is the case for vortices in critical superconductors? We follow the technique introduced in [50]. Using the metric in (32), the ansatz for the F, H, G profiles reads:

$$F(x^\mu) = py, \quad H(x^\mu) = H(z), \quad G(x^\mu) = px, \tag{53}$$

and the gauge field components are:

$$A_0 = A_1 = 0, \quad A_1 = \tfrac{p}{2} - u(z), \quad A_3 = -\tfrac{p}{2} + u(z). \tag{54}$$

with p being n integer. As a result for choosing this ansatz, the field equations reduce to:

$$\begin{cases} u'' - 4KL_z^2\, u\, \sin^2(H) &= 0 \\ H'' + 4\left(\frac{L_z}{L}\right)^2 \sin(2H)\left(\frac{p^2}{4} - u^2\right) &= 0 \end{cases} \tag{55}$$

with the resulting energy density:

$$T_{00} = \frac{K}{L^2}\left[p^2\cos^2(H) + 4\sin^2(H)u^2\right] + \frac{K}{2L_z^2}(H')^2 + \frac{1}{(L_zL)^2}(u')^2. \tag{56}$$

The Bayronic density, which has contributions both from the U field and from the gauge field, is:

$$\rho_B = -12p\,\frac{\mathrm{d}}{\mathrm{d}r}[u(1+\cos 2H)].$$

5.2. Hamilton–Jacobi Method

Now, we will review a novel strategy for finding BPS bounds, presented in [50], which depends on the possibility of constructing a Hamilton–Jacobi equation from a non-negative energy density, T_{00}, which depends on two (or more) scalar degrees of freedom A, B. Suppose we have an energy density of the form:

$$T_{00} = \frac{1}{2}\left(\frac{\partial A}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial B}{\partial x}\right)^2 + V(A,B), \tag{57}$$

where x is the relevant space-like coordinate and $V(A,B)$ is the given interaction potential between the two degrees of freedom. We would like to write this energy density in a "BPS-style" of the form:

$$T_{00} \sim (\partial A \pm \Gamma_A)^2 + (\partial B \pm \Gamma_B)^2 \pm \frac{\mathrm{d}W}{\mathrm{d}x}, \tag{58}$$

where Γ_A and Γ_B are two auxiliary quantities, both functions of the profiles A and B, which allow us to construct the "BPS-style" expression. From the above considerations, it is clear that we require:

$$\frac{(\partial A \pm \Gamma_A)^2}{2} + \frac{(\partial B \pm \Gamma_B)^2}{2} - \frac{\mathrm{d}W}{\mathrm{d}x} = T_{00}, \tag{59}$$

which is equivalent to

$$\frac{\Gamma_A^2}{2} + \frac{\Gamma_B^2}{2} = V(A, B), \tag{60}$$

$$\Gamma_A \frac{\partial A}{\partial x} + \Gamma_B \frac{\partial B}{\partial x} = \frac{\mathrm{d}W}{\mathrm{d}x}. \tag{61}$$

A natural way of fulfilling (61) is:

$$\Gamma_A = \frac{\partial W}{\partial A}, \qquad \Gamma_B = \frac{\partial W}{\partial B}. \tag{62}$$

Replacing condition (62) in (60), we arrive arrive at:

$$\left(\frac{\partial W}{\partial A}\right)^2 + \left(\frac{\partial W}{\partial B}\right)^2 = 2V(A, B), \tag{63}$$

When the BPS equations are satisfied, the only contribution to the energy is the lower BPS bound:

$$E \geq \left| \int_S \mathrm{d}x \, \frac{\mathrm{d}W}{\mathrm{d}x} \right|. \tag{64}$$

A Novel BPS Bound

Using the novel technique describe above, the Hamilton–Jacobi for the energy density in Equation (56), is given by:

$$\frac{L_z^2}{2K} \left(\frac{\partial W}{\partial H}\right)^2 + \left(\frac{L_z L}{2}\right)^2 \left(\frac{\partial W}{\partial u}\right)^2 = \frac{K}{L^2} \left[p^2 \cos^2(H) + 4 \sin^2(H) u^2\right]. \tag{65}$$

The solution of the above Hamilton–Jacobi equation is:

$$W = \frac{4K^{3/2}}{pL_z} u \cos H, \tag{66}$$

provided the following relations hold:

$$L = \frac{p}{\sqrt{2K}} \Leftrightarrow A = \pi^2 \frac{p^2}{K} \Leftrightarrow A = \left(\frac{2\pi p}{e f_\pi}\right)^2, \tag{67}$$

Thus, the energy density T_{00} can be rewritten in "BPS-style" as follows:

$$T_{00} = \frac{K}{2(pL_z)}^2 \left[(pH' \pm 4\sqrt{K} L_z u \sin H)^2 + 4(u' \mp p\sqrt{K} L_z \cos H)^2\right] \pm \frac{\mathrm{d}W}{\mathrm{d}r}. \tag{68}$$

The first-order BPS equations are:

$$pH' \pm 4\sqrt{K} L_z u \sin(H) = 0, \tag{69}$$

$$u' \mp p\sqrt{K} L_z \cos(H) = 0. \tag{70}$$

It can easily be shown that the second-order BPS equations imply the field Equation (55).

The lower bound for the energy is:

$$E = \int \sqrt{-g} \, d^3x T_{00} = AL_z \int_0^{2\pi} T_{00} \, dx \geq |Q|,$$

with

$$Q = AL_z |W(2\pi) - W(0)|.$$

Two important observations arise from this analysis: first, we have found that for the G-NLSM, one can define a lower bound that can be saturated, unlike the BPS bound for the Skyrme model; second, the corresponding topological charge differs from the Baryonic charge of the configuration.

6. Applications

6.1. Fermions Coupled to Skyrmions

The analysis of the coupling between the Dirac equation for quarks and the analytic non-homogeneous Baryonic condensates described previously (and in [50]) is of great interest since it will disclose how the non-homogeneous Baryonic condensates affect the spectrum of the Dirac operator, a task whose relevance is comparable with the importance of the analysis of the Dirac spectrum in an Instanton background.

The original sigma model presented by Gell-Mann in [80] (also see [81]) represents a triplet of Pions and a scalar field (the U field) coupled with fermions. This kind of coupling is of great interest in some theoretical studies related to atomic and nuclear physics, and was extensively studied in the following decade (see [82–84] and the references therein). Unfortunately, there are two obstacles that historically made this kind of problem very difficult to solve. First, due to the very nature of the equations that we are dealing with, until very recently, there was no explicit analytic solution for the chiral field U (and only a few numerical solutions for the chiral solitons were known). Due to the difficulty in analyzing the Dirac equation for quarks, coupled with the chiral field U, whose profile is known only numerically, Hiller [85] and Zhao [86] introduced an analytic profile for the chiral field to derive relevant physical properties. However, such a profile was not a solution of the field equations.

The fermion–chiral soliton coupling is also known as a "chiral coupling" (see [87]) and has the form

$$\mathcal{L}_{int} = \bar{\psi}\left[i\gamma^{\mu}\partial_{\mu} + gU^{\gamma_5}\right]\psi, \tag{71}$$

where

$$U^{\gamma_5} = g[\phi_0 + i\gamma_5[\phi^a \cdot \tau_a]]. \tag{72}$$

In our analysis, we will consider that this coupling is mediated by the coupling constant g (see [85–87]). We will assume the backreaction effects of the quarks on the Hadronic condensates to be negligible (see [85,86,88–91]). The usual method is to study the coupling by analyzing the Hamiltonian:

$$\hat{H}_0 = \boldsymbol{\alpha} \cdot \vec{p} + g\hat{\beta}U^{\gamma_5} \tag{73}$$

where $\boldsymbol{\alpha} = (\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)$ and $\hat{\beta}$ are the Dirac matrices.

In the exponential representation, $U = \cos(\alpha) + \mathbf{t}_a\hat{r}^a(\Theta, \Phi)\sin(\alpha)$. Hence, because of the properties of the γ_5 matrix, it is directly obtained that:

$$U^{\gamma_5} = \cos(\alpha) + \gamma_5\mathbf{t}_a\hat{r}^a(\Theta, \Phi)\sin(\alpha) \tag{74}$$

which can be rewritten in the useful form:

$$U^{\gamma_5} = \frac{\mathbf{1} + \gamma_5}{2}U + \frac{\mathbf{1} - \gamma_5}{2}U^{-1} \tag{75}$$

Hence,

$$\hat{H}_0 = \boldsymbol{\alpha} \cdot \vec{p} + g\hat{\beta}\left(\frac{\mathbf{1} + \gamma_5}{2}U + \frac{\mathbf{1} - \gamma_5}{2}U^{-1}\right). \tag{76}$$

However, we are interested in studying the case of the the G-NLSM, so the minimal coupling to the $U(1)$ gauge field must be taken into account:

$$\hat{H}_1 = \boldsymbol{\alpha} \cdot \vec{\pi} + g\hat{\beta}\left(\frac{\mathbf{1} + \gamma_5}{2}U + \frac{\mathbf{1} - \gamma_5}{2}U^{-1}\right), \tag{77}$$

with $\vec{\pi} = \vec{p} - q\vec{A}$.

Following the ansatz (53), an important novel technical result is that one can construct two Hermitian operators that commute with the Hamiltonian and between themselves:

$$\hat{P}_x := \hat{p}_x - ip\frac{1+\gamma_5}{2}\tau_3, \tag{78}$$

$$\hat{P}_y := \hat{p}_y + ip\frac{1-\gamma_5}{2}\tau_3, \tag{79}$$

where the integer p is defined as in the ansatz (53).

The above commuting operators can be interpreted as the generalized momenta (with extra isospin terms). The isospin corrections in these generalized momenta are related to the fact that although the configuration in Equations (69) and (70) is not translationally invariant along the (x, y) directions, the lack of invariance can be compensated for by isospin transformations.

These operators will be useful to analyze how the spectrum of the Dirac operators for quarks is affected by the presence of the magnetized BPS Hadronic layers (see [85–87] and the references therein).

6.2. Hamilton–Jacobi Equation with the Skyrme Term

It is natural, after studying the Hamilton–Jacobi equation for the G-NLSM, to generalize these results for the full Skyrme model and analyze the relevance of the Skyrme term for the physical properties of these magnetized BPS layers. In this case, substituting the ansatz shown previously (53) into the field Equation (44) (which included the Skyrme term), we obtain

$$\begin{aligned} 0 = H''\Big[L^2 + 2\lambda p^2 - 2\lambda \sin^2(H)\Big(p^2 - 4u^2\Big)\Big] + 16\lambda \sin^2(H) u H' u' \\ + \sin(2H)\left[\Big(p^2 - 4u^2\Big)\Big(L_z^2 - \lambda H'^2\Big) - \frac{8\lambda L_z^2 p^2 u^2 \cos(2H)}{L^2}\right], \end{aligned} \tag{80}$$

$$0 = u'' + \frac{4Ku \sin^2(H)\big(\lambda L^2 H'^2 + L_z^2 (2\lambda p^2 \cos^2(H) + L^2)\big)}{L^2}. \tag{81}$$

and the energy- density T_{00} reduces to

$$\begin{aligned} T_{00} = \frac{KH'^2 (2\lambda p^2 \cos^2(H) + 8\lambda u^2 \sin^2(H) + L^2)}{2L^2 L_z^2} + \frac{u'^2}{L^2 L_z^2} \\ + \frac{K\big(4u^2 \sin^2(H)(2\lambda p^2 \cos^2(H) + L^2) + L^2 p^2 \cos^2(H)\big)}{L^4}. \end{aligned} \tag{82}$$

Using the strategy shown previously, we can derive the Hamilton–Jacobi equation for the full Skyrme model:

$$a\left(\frac{\partial W}{\partial H}\right)^2 + b\left(\frac{\partial W}{\partial u}\right)^2 = V(u, H) \tag{83}$$

with

$$a = \frac{L_z^2 L^2}{2K(2\lambda p^2 \cos^2(H) + 8\lambda u^2 \sin^2(H) + L^2)} \tag{84}$$

$$b = \left(\frac{L_z L}{2}\right)^2 \tag{85}$$

$$V(u, H) = \frac{K\big(4u^2 \sin^2(H)(2\lambda p^2 \cos^2(H) + L^2) + L^2 p^2 \cos^2(H)\big)}{L^4}. \tag{86}$$

The above novel technical result is very relevant, since it allows us to reduce the analysis of the physical effects of the Skyrme term on the magnetized BPS Baryonic layers to the analysis of a suitable Hamilton–Jacobi equation in two dimensions. Due to the huge amount of available results on the Hamilton–Jacobi equation, this approach can be extremely useful in the context of strongly interacting Baryonic matter. We will come back to the physical implications in a future publication.

7. Conclusions and Perspectives

In the present paper, we have reviewed two strategies for the analysis of inhomogeneous condensates in the low-energy limit of QCD minimally coupled with Maxwell theory.

The first strategy is based on the generalization of the hedgehog ansatz to situations without spherical symmetry (in particular when a finite amount of Baryonic charge is forced to live within a finite spatial volume). This approach allows us to analytically derive many properties of inhomogeneous condensates, such as the Hadronic and energy-density profiles. This technique is especially effective when describing Baryonic layers and tubes (which appear in the numerical simulations of multi-Baryonic systems at a finite density). On the other hand, the configurations which have been constructed in this way generate electric and magnetic fields of the same size. Thus, the natural question is whether or not one can find a new analytic tool able to generate magnetized Baryonic configurations (as in many situations of physical interest, such as within neutron stars, where the magnetic field is expected to be much more intense than the electric field). The second strategy is the first step in this direction. In the case of the G-NLSM (using the theory of Hamilton–Jacobi equation), it is possible to derive a novel BPS bound for the energy of Baryonic magnetized layers, which (unlike the usual case in the Skyrme model) can actually be saturated. The surprising feature of this result is that the right-hand side of the topological bound is not the Baryonic charge but, rather, a non-linear function of it. The novel results of the present paper are the following. First of all, we have generalized the Hamilton–Jacobi approach to the case in which the Skyrme coupling is non-vanishing. Secondly, we have discussed the Dirac equation coupled to such BPS Baryonic layers. We have constructed two Hermitian operators which commute between themselves and with the Dirac Hamiltonian. These results will allow for, in the near future, a detailed analysis of the physical effects of the Skyrme term on the BPS Baryonic layers as well as a direct study of how the spectrum of the Dirac operator is affected by the Baryonic condensates themselves.

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Abbreviations

The following abbreviations are used in this manuscript:

QCD	Quantum Chromodynamics
NLSM	Non-Linear σ-model
G-NLSM	Gauged Non-Linear σ-model
HA	Hedgehog Ansatz
HJ	Hamilton–Jacobi (equation)

Appendix A. Euler Representation and Exponential Representation of the $SU(2)$ Skyrme Fields

Given a $SU(2)$ scalar written as in (22), we can rewrite it in an exponential representation. Firstly, consider that:

$$U = e^{\mathbf{t}_3 F} e^{\mathbf{t}_2 H} e^{\mathbf{t}_3 G} = \cos(H)\cos(F+G)\mathbf{1} + \sin(H)\sin(F-G)\mathbf{t}_1 + \sin(H)\cos(F-G)\mathbf{t}_2 + \cos(H)\sin(F+G)\mathbf{t}_3, \tag{A1}$$

Now, define

$$\cos(\alpha) = \cos(H)\cos(F+G), \tag{A2}$$

Hence,

$$\sin(\alpha) = \sqrt{\sin^2(H) + \cos^2(H)\sin^2(G+F)}, \tag{A3}$$

and then, by factorization, we can easily achieve the exponential representation:

$$U = \cos(\alpha)\mathbf{1} + \mathbf{t}_i \hat{r}^i \sin\alpha, \tag{A4}$$

where

$$\hat{r} = \frac{1}{\sin(\alpha)} \begin{pmatrix} \sin(H)\sin(F-G) \\ \sin(H)\cos(F-G) \\ \cos(H)\sin(F+G) \end{pmatrix}. \tag{A5}$$

Then, the remaining coordinates Θ, Φ in the exponential representation are defined by:

$$\cos(\Theta) = \frac{\cos(H)\sin(F+G)}{\sqrt{\sin^2(H) + \cos^2(H)\sin^2(F+G)}}, \tag{A6}$$

$$\sin(\Theta) = \frac{\sin(H)}{\sqrt{\sin^2(H) + \cos^2(H)\sin^2(G+F)}}. \tag{A7}$$

Appendix B. $\hat{O}_i$ Matrices and Some Commutation Relations

Consider the $SU(2)$-valued scalar field U in Euler angles representation:

$$U = e^{\mathbf{t}_3 F} e^{\mathbf{t}_2 H} e^{\mathbf{t}_3 G}. \tag{A8}$$

In this section, we will show the $\hat{O}_i = U^{-1}[\mathbf{t}_i, U]$ matrices, which are needed in order to compute the chiral current (23) and other quantities related to the equations of motion in the Skyrme model in terms of the Euler parametrization.

$$\hat{O}_1 = -2\Big(\cos^2(H)\sin^2(F+G) + \sin^2(H)\cos^2(F-G)\Big)\mathbf{t}_1 + \Big(\cos^2(H)\sin(2(F+G)) + \sin^2(H)\sin(2(F-G))\Big)\mathbf{t}_2 - \sin(2H)\cos(2H)\mathbf{t}_3, \tag{A9}$$

$$\hat{O}_2 = \Big(\sin^2(H)\sin(2(F-G)) - \cos^2(H)\sin(2(F+G))\Big)\mathbf{t}_1 - 2\Big(\sin^2(H)\sin^2(F-G) + \cos^2(H)\sin^2(F+G)\Big)\mathbf{t}_2 + \sin(2H)\sin(2F)\mathbf{t}_3, \tag{A10}$$

$$\hat{O}_3 = \sin(2H)\cos(2G)\mathbf{t}_1 + \sin(2H)\sin(2G)\mathbf{t}_2 - 2\sin^2(H)\mathbf{t}_3 \tag{A11}$$

To compute these results, it is necessary to iterate some commutation relations. These are of the form $[\mathbf{t}_i, U]$:

$$\begin{aligned}[\mathbf{t}_1, U] &= -2\sin(G)\ e^{\mathbf{t}_3 F} e^{\mathbf{t}_2 H}\mathbf{t}_2 - 2\sin(H)\ e^{\mathbf{t}_3 F} e^{\mathbf{t}_3 G}\mathbf{t}_3 + 2\sin(F)\ \mathbf{t}_2 e^{\mathbf{t}_2 H} e^{\mathbf{t}_3 G} \\ &= 2\cos(F+G)\cos(H)\mathbf{t}_2 - 2\cos(F-G)\sin(H)\mathbf{t}_3 \\ &= \begin{pmatrix} -2i\cos(F-G)\sin H & 2\sin(F+G)\cos H \\ -2\sin(F+G)\cos H & 2i\cos(F-G)\sin H \end{pmatrix},\end{aligned} \tag{A12}$$

$$\begin{aligned}[\mathbf{t}_2, U] &= -2\sin(F)\ \mathbf{t}_1 e^{\mathbf{t}_2 H} e^{\mathbf{t}_3 G} - 2\sin(G)\ e^{\mathbf{t}_3 F} e^{\mathbf{t}_2 H}\mathbf{t}_1 \\ &= 2\sin(F-G)\sin(H)\mathbf{t}_3 - 2\sin(F+G)\cos(H)\mathbf{t}_1 \\ &= \begin{pmatrix} 2i\sin(F-G)\sin(H) & -2i\sin(F+G)\cos(H) \\ -2i\sin(F+G)\cos(H) & -2i\sin(F-G)\sin(H) \end{pmatrix},\end{aligned} \tag{A13}$$

$$\begin{aligned}[\mathbf{t}_3, U] &= 2\sin(H)\ e^{\mathbf{t}_3 F}\mathbf{t}_1 e^{\mathbf{t}_3 G} \\ &= 2\cos(F-G)\sin H\mathbf{t}_1 - 2\sin(F-G)\sin(H)\mathbf{t}_2 \\ &= \begin{pmatrix} 0 & 2ie^{i(F-G)}\sin(H) \\ 2ie^{-i(F-G)}\sin H & 0 \end{pmatrix}.\end{aligned} \tag{A14}$$

which are also computed by iterating some of the following commutators:

$$\left[\mathbf{t}_1, e^{\pm \mathbf{t}_3 F}\right] = \pm 2\sin(F)\mathbf{t}_2, \tag{A15}$$

$$\left[\mathbf{t}_2, e^{\pm \mathbf{t}_3 F}\right] = \mp 2\sin(F)\mathbf{t}_1, \tag{A16}$$

$$\left[\mathbf{t}_1, e^{\pm \mathbf{t}_2 H}\right] = \mp 2\sin(H)\mathbf{t}_3, \tag{A17}$$

$$\left[\mathbf{t}_3, e^{\pm \mathbf{t}_2 H}\right] = \pm 2\sin(H)\mathbf{t}_1. \tag{A18}$$

Finally, it is also worth mentioning that

$$\left[e^{\mathbf{t}_3 F}, e^{\mathbf{t}_2 H}\right] = 2\sin(F)\sin(H)\mathbf{t}_1. \tag{A19}$$

References

1. Maniatis, M. The tree world. *arXiv* **2021**, arXiv:2108.07266.
2. Laine, M.; Vuorinen, A. Basics of thermal field theory. *Lect. Notes Phys.* **2016**, *925*, 1701–01554.
3. Schroder, Y. Weak-coupling expansion of the hot QCD pressure. *arXiv* **2006**, arXiv:hep-ph/0605057.
4. Kogut, J.B.; Stephanov, M.A. *The Phases of Quantum Chromodynamics: From Confinement to Extreme Environments*; Cambridge University Press: Cambridge, UK , 2003; Volume 21.
5. Bzdak, A.; Esumi, S.; Koch, V.; Liao, J.; Stephanov, M.; Xu, N. Mapping the phases of quantum chromodynamics with beam energy scan. *Phys. Rep.* **2020**, *853*, 1–87. [CrossRef]
6. Nagata, K. Finite-density lattice QCD and sign problem: Current status and open problems. *Prog. Part. Nucl. Phys.* **2022**, *127*, 103991. [CrossRef]
7. Astrakhantsev, N.Y.; Braguta, V.; Kolomoyets, N.; Kotov, A.Y.; Kuznedelev, D.; Nikolaev, A.; Roenko, A. Lattice Study of QCD Properties under Extreme Conditions: Temperature, Density, Rotation, and Magnetic Field. *Phys. Part. Nucl.* **2021**, *52*, 536–541. [CrossRef]
8. Sakai, T.; Sugimoto, S. More on a holographic dual of QCD. *Prog. Theor. Phys.* **2005**, *114*, 1083–1118. [CrossRef]
9. Hata, H.; Sakai, T.; Sugimoto, S.; Yamato, S. Baryons from instantons in holographic QCD. *Prog. Theor. Phys.* **2007**, *117*, 1157–1180. [CrossRef]
10. Sutcliffe, P. Holographic skyrmions. *Mod. Phys. Lett. B* **2015**, *29*, 1540051. [CrossRef]
11. López, J.A.; Dorso, C.O.; Frank, G. Properties of nuclear pastas. *Front. Phys.* **2021**, *16*, 24301. [CrossRef]
12. Dorso, C.O.; Frank, G.A.; López, J.A. Phase transitions and symmetry energy in nuclear pasta. *Nucl. Phys. A* **2018**, *978*, 35–64. [CrossRef]

13. Pelicer, M.; Antonelli, M.; Menezes, D.; Gulminelli, F. Anisotropic electron transport in the nuclear pasta phase. *Mon. Not. R. Astron. Soc.* **2023**, *521*, 743–759. [CrossRef]
14. Nandi, R.; Schramm, S. Transport properties of the nuclear pasta phase with quantum molecular dynamics. *Astrophys. J.* **2018**, *852*, 135. [CrossRef]
15. Yakovlev, D. Electron transport through nuclear pasta in magnetized neutron stars. *Mon. Not. R. Astron. Soc.* **2015**, *453*, 581–590. [CrossRef]
16. Brauner, T.; Yamamoto, N. Chiral soliton lattice and charged pion condensation in strong magnetic fields. *J. High Energy Phys.* **2017**, *2017*, 132. [CrossRef]
17. Kawaguchi, M.; Ma, Y.L.; Matsuzaki, S. Chiral soliton lattice effect on baryonic matter from a skyrmion crystal model. *Phys. Rev. C* **2019**, *100*, 025207. [CrossRef]
18. Brauner, T.; Filios, G.; Kolešová, H. Chiral soliton lattice in QCD-like theories. *J. High Energy Phys.* **2019**, *2019*, 29. [CrossRef]
19. Higaki, T.; Kamada, K.; Nishimura, K. Formation of a chiral soliton lattice. *Phys. Rev. D* **2022**, *106*, 096022. [CrossRef]
20. Casalbuoni, R.; Nardulli, G. Inhomogeneous superconductivity in condensed matter and QCD. *Rev. Mod. Phys.* **2004**, *76*, 263. [CrossRef]
21. Klebanov, I. Nuclear matter in the Skyrme model. *Nucl. Phys. B* **1985**, *262*, 133–143. [CrossRef]
22. Kugler, M.; Shtrikman, S. Skyrmion crystals and their symmetries. *Phys. Rev. D* **1989**, *40*, 3421. [CrossRef]
23. Castillejo, L.; Jones, P.; Jackson, A.D.; Verbaarschot, J.J.; Jackson, A. Dense skyrmion systems. *Nucl. Phys. A* **1989**, *501*, 801–812. [CrossRef]
24. Adam, C.; Martín-Caro, A.G.; Huidobro, M.; Vázquez, R.; Wereszczynski, A. Dense matter equation of state and phase transitions from a generalized Skyrme model. *Phys. Rev. D* **2022**, *105*, 074019. [CrossRef]
25. Adam, C.; Castelo, J.; Martín-Caro, A.G.; Huidobro, M.; Wereszczynski, A. Effective no-hair relations for spinning Boson Stars. *arXiv* **2023**, arXiv:2305.06181.
26. Manton, N.; Sutcliffe, P. *Topological Solitons*; Cambridge Monographs on Mathematical Physics; Cambridge University Press: Cambridge, UK, 2004. [CrossRef]
27. Shuryak, E. *Nonperturbative Topological Phenomena in QCD and Related Theories*; Springer: Berlin/Heidelberg, Germany, 2021; Volume 977.
28. Shifman, M. *Advanced Topics in Quantum Field Theory: A Lecture Course*; Cambridge University Press: Cambridge, UK, 2022.
29. Dunne, G.V.; Thies, M. Time-Dependent Hartree-Fock Solution of Gross-Neveu Models: Twisted-Kink Constituents of Baryons and Breathers. *Phys. Rev. Lett.* **2013**, *111*, 121602. [CrossRef] [PubMed]
30. Başar, G.; Dunne, G.V.; Thies, M. Inhomogeneous condensates in the thermodynamics of the chiral NJL 2 model. *Phys. Rev. D* **2009**, *79*, 105012. [CrossRef]
31. Schnetz, O.; Thies, M.; Urlichs, K. Full phase diagram of the massive Gross–Neveu model. *Ann. Phys.* **2006**, *321*, 2604–2637. [CrossRef]
32. Thies, M. Chiral spiral in the presence of chiral imbalance. *Phys. Rev. D* **2018**, *98*, 096019. [CrossRef]
33. Thies, M. Duality study of the chiral Heisenberg-Gross-Neveu model in 1+1 dimensions. *Phys. Rev. D* **2020**, *102*, 096006. [CrossRef]
34. Thies, M. First-order phase boundaries of the massive (1+1)-dimensional Nambu–Jona-Lasinio model with isospin. *Phys. Rev. D* **2020**, *101*, 074013. [CrossRef]
35. Thies, M. Phase structure of the (1+1)-dimensional Nambu–Jona-Lasinio model with isospin. *Phys. Rev. D* **2020**, *101*, 014010. [CrossRef]
36. Thies, M. Gross-Neveu model with O (2) L× O (2) R chiral symmetry: Duality with Zakharov-Mikhailov model and large N solution. *Phys. Rev. D* **2023**, *107*, 076024. [CrossRef]
37. Adhikari, P.; Andersen, J.O.; Kneschke, P. Inhomogeneous chiral condensate in the quark-meson model. *Phys. Rev. D* **2017**, *96*, 016013. [CrossRef]
38. Adhikari, P.; Andersen, J.O.; Kneschke, P. Erratum: Inhomogeneous chiral condensate in the quark-meson model [Phys. Rev. D 96, 016013 (2017)]. *Phys. Rev. D* **2018**, *98*, 099902. [CrossRef]
39. Ravenhall, D.; Pethick, C.; Wilson, J. Structure of matter below nuclear saturation density. *Phys. Rev. Lett.* **1983**, *50*, 2066. [CrossRef]
40. Hashimoto, M.a.; Seki, H.; Yamada, M. Shape of nuclei in the crust of neutron star. *Prog. Theor. Phys.* **1984**, *71*, 320–326. [CrossRef]
41. Horowitz, C.; Berry, D.; Briggs, C.; Caplan, M.; Cumming, A.; Schneider, A. Disordered nuclear pasta, magnetic field decay, and crust cooling in neutron stars. *Phys. Rev. Lett.* **2015**, *114*, 031102. [CrossRef] [PubMed]
42. Schneider, A.S.; Caplan, M.E.; Berry, D.K.; Horowitz, C.J. Domains and defects in nuclear pasta. *Phys. Rev. C* **2018**, *98*, 055801. [CrossRef]
43. Caplan, M.E.; Schneider, A.; Horowitz, C.J. Elasticity of nuclear pasta. *Phys. Rev. Lett.* **2018**, *121*, 132701. [CrossRef]
44. Dorso, C.O.; Strachan, A.; Frank, G.A. The nucleonic thermal conductivity of "pastas" in neutron star matter. *Nucl. Phys. A* **2020**, *1002*, 122004. [CrossRef]
45. Kaspi, V.M.; Beloborodov, A.M. Magnetars. *Annu. Rev. Astron. Astrophys.* **2017**, *55*, 261–301. [CrossRef]
46. Kharzeev, D.E.; McLerran, L.D.; Warringa, H.J. The effects of topological charge change in heavy ion collisions: "Event by event P and CP violation". *Nucl. Phys. A* **2008**, *803*, 227–253. [CrossRef]

47. d'Enterria, D.; Drewes, M.; Giammanco, A.; Hajer, J.; Bratkovskaya, E.; Bruce, R.; Burmasov, N.; Dyndal, M.; Gould, O.; Grabowska-Bold, I.; et al. Opportunities for new physics searches with heavy ions at colliders. *J. Phys. G Nucl. Part. Phys.* **2023**, *50*, 050501. [CrossRef]
48. Canfora, F. Ordered Arrays of Baryonic Tubes in the Skyrme Model in (3+1) Dimensions at Finite Density. *Eur. Phys. J. C* **2018**, *78*, 929. [CrossRef]
49. Canfora, F.; Lagos, M.; Pais, P.; Vera, A. Exact mapping from the (3+1)-dimensional Skyrme model to the (1+1)-dimensional sine-Gordon theory and some applications. *Phys. Rev. D* **2023**, *108*, 114027. [CrossRef]
50. Canfora, F. Magnetized Baryonic Layer and a Novel BPS Bound in the Gauged-Non-Linear-Sigma-Model-Maxwell Theory in (3+1)-Dimensions through Hamilton–Jacobi Equation. *J. High Energy Phys.* **2023**, *2023*, 7. [CrossRef]
51. Balachandran, A.P.; Marmo, G.; Skagerstam, B.S.; Stern, A. *Classical Topology and Quantum States*; World Scientific: Singapore, 1991. [CrossRef]
52. Derrick, G. Comments on nonlinear wave equations as models for elementary particles. *J. Math. Phys.* **1964**, *5*, 1252–1254. [CrossRef]
53. Skyrme, T.H.R. A unified field theory of mesons and baryons. *Nucl. Phys.* **1962**, *31*, 556–569. [CrossRef]
54. 't Hooft, G. A Planar Diagram Theory for Strong Interactions. *Nucl. Phys. B* **1974**, *72*, 461. [CrossRef]
55. 't Hooft, G. A two-dimensional model for mesons. *Nucl. Phys. B* **1974**, *75*, 461–470. [CrossRef]
56. 't Hooft, G. A planar diagram theory for strong interactions. In *The Large N Expansion in Quantum Field Theory and Statistical Physics: From Spin Systems to 2-Dimensional Gravity*; World Scientific: Singapore, 1993; pp. 80–92.
57. Veneziano, G. Some aspects of a unified approach to gauge, dual and Gribov theories. *Nucl. Phys. B* **1976**, *117*, 519–545. [CrossRef]
58. Witten, E. Baryons in the 1N expansion. *Nucl. Phys. B* **1979**, *160*, 57–115. [CrossRef]
59. Skyrme, T.H.R. A non-linear field theory. *Proc. R. Soc. Lond. Ser. A Math. Phys. Sci.* **1961**, *260*, 127–138.
60. Callan, C.G.; Witten, E. Monopole Catalysis of Skyrmion Decay. *Nucl. Phys. B* **1984**, *239*, 161–176. [CrossRef]
61. Witten, E. Current algebra, baryons, and quark confinement. *Nucl. Phys. B* **1983**, *223*, 433–444. [CrossRef]
62. Balachandran, A.; Barducci, A.; Lizzi, F.; Rodgers, V.; Stern, A. Doubly strange dibaryon in the chiral model. *Phys. Rev. Lett.* **1984**, *52*, 887. [CrossRef]
63. Adkins, G.S.; Nappi, C.R.; Witten, E. Static properties of nucleons in the Skyrme model. *Nucl. Phys. B* **1983**, *228*, 552–566. [CrossRef]
64. Guadagnini, E. Baryons as solitons and mass formulae. *Nucl. Phys. B* **1984**, *236*, 35–47. [CrossRef]
65. Manton, N. Is the B = 2 Skyrmion axially symmetric? *Phys. Lett. B* **1987**, *192*, 177–179. [CrossRef]
66. Goldhaber, A.S.; Manton, N. Maximal symmetry of the Skyrme crystal. *Phys. Lett. B* **1987**, *198*, 231–234. [CrossRef]
67. Balachandran, A.; Nair, V.P.; Rajeev, S.; Stern, A. Soliton states in the quantum-chromodynamic effective Lagrangian. *Phys. Rev. D* **1983**, *27*, 1153. [CrossRef]
68. Shnir, Y.M. *Magnetic Monopoles*; Springer: Berlin/Heidelberg, Germany, 2006.
69. Shnir, Y.M. *Topological and Non-Topological Solitons in Scalar Field Theories*; Cambridge University Press: Cambridge, UK, 2018.
70. Prasad, M.; Sommerfield, C.M. Exact classical solution for the't Hooft monopole and the Julia-Zee dyon. *Phys. Rev. Lett.* **1975**, *35*, 760. [CrossRef]
71. Bogomolny, E. The stability of classical solutions. *Sov. J. Nucl. Phys.* **1976**, *24*, 861–870.
72. Balachandran, A.; Lizzi, F.; Rodgers, V.; Stern, A. Dibaryons as chiral solitons. *Nucl. Phys. B* **1985**, *256*, 525–556. [CrossRef]
73. Kopeliovich, V.; Stern, B.; Schwesinger, B. SU (3)-Skyrmion with B = 2 and Large strangeness content. *Pis' Ma V Zhurnal Ehksperimental'noj I Teor. Fiz.* **1995**, *62*, 177–181.
74. Din, A.M.; Zakrzewski, W.J. General classical solutions in the CPn-1 model. *Nucl. Phys. B* **1980**, *174*, 397–406. [CrossRef]
75. Ioannidou, T.; Piette, B.; Zakrzewski, W. Spherically symmetric solutions of the SU (N) Skyrme models. *J. Math. Phys.* **1999**, *40*, 6223–6233. [CrossRef]
76. Brihaye, Y.; Hartmann, B.; Ioannidou, T.; Zakrzewski, W. Harmonic map analysis of SU (N) gravitating Skyrmions. *Phys. Rev. D* **2004**, *69*, 124035. [CrossRef]
77. Canfora, F.; Carignano, S.; Lagos, M.; Mannarelli, M.; Vera, A. Pion Crystals Hosting Topologically Stable Baryons. *Phys. Rev. D* **2021**, *103*, 076003. [CrossRef]
78. Canfora, F.; Lagos, M.; Oh, S.H.; Oliva, J.; Vera, A. Analytic (3+1)-Dimensional Gauged Skyrmions, Heun, and Whittaker-Hill Equations and Resurgence. *Phys. Rev. D* **2018**, *98*, 085003. [CrossRef]
79. Mannarelli, M.; Canfora, F.; Carignano, S.; Lagos, M.; Vera, A. Inhomogeneous pion condensed phase hosting topologically stable baryons. In Proceedings of the EPJ Web of Conferences, Daejeon, Republic of Korea, 22–26 April 2022; EDP Sciences: Les Ulis, France, 2022; Volume 258, p. 07003.
80. Gell-Mann, M.; Lévy, M.M. The Axial Vector Current in Beta Decay. *Il Nuovo C.* **1960**, *16*, 705–726. [CrossRef]
81. Itzykson, C.; Zuber, J.B. *Quantum Field Theory*; Courier Corporation: Chelmsford, MA, USA, 2006; Chapter 11-4 .
82. Goldstone, J.; Jaffe, R.L. Baryon number in chiral bag models. *Phys. Rev. Lett.* **1983**, *51*, 1518. [CrossRef]
83. Niemi, A.J.; Semenoff, G.W. Fermion number fractionization in quantum field theory. *Phys. Rep.* **1986**, *135*, 99–193. [CrossRef]
84. Hosaka, A.; Toki, H. Chiral bag model for the nucleon. *Phys. Rep.* **1996**, *277*, 65–188. [CrossRef]
85. Hiller, J.R.; Jordan, T.F. Solutions of the Dirac Equation for Fermions in Skyrme Fields. *Phys. Rev. D* **1986**, *34*, 1176–1183. [CrossRef] [PubMed]

86. Zhao, M.; Hiller, J.R. Eigenenergies of Fermions Bound in Skyrme Fields. *Phys. Rev. D* **1989**, *40*, 1329–1335. [CrossRef] [PubMed]
87. Shnir, Ya.M. Collective Coordinates of the Skyrme Model Coupled with Fermions. *Phys. Scr.* **2003**, *67*, 361–365. [CrossRef]
88. Kahana, S.; Ripka, G. Baryon Density of Quarks Coupled to a Chiral Field. *Nucl. Phys. A* **1984**, *429*, 462–476. [CrossRef]
89. Banerjee, B.; Glendenning, N.K.; Soni, V. Soliton Matter and the Onset of Color Conductivity. *Phys. Lett. B* **1985**, *155*, 213–216. [CrossRef]
90. Glendenning, N.K.; Banerjee, B. Soliton Matter as a Model of Dense Nuclear Matter. *Phys. Rev. C* **1986**, *34*, 1072–1080. [CrossRef]
91. Rho, M.; Goldhaber, A.S.; Brown, G.E. Topological Soliton Bag Model for Baryons. *Phys. Rev. Lett.* **1983**, *51*, 747–750. [CrossRef]