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RESEARCH ARTICLE

Some Soliton Hierarchies Associated with Lie Algebras $\mathfrak{sp}(4)$ and $\mathfrak{so}(5)$

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Abstract
Based on the symplectic Lie algebra $\mathfrak{sp}(4)$, we obtain two integrable hierarchies of $\mathfrak{sp}(4)$, and by using the trace identity, we give their Hamiltonian structures. Then, we use 2×2 Kronecker product, and construct integrable coupling systems of one soliton equation. Next, we consider two bases of Lie algebra $\mathfrak{so}(5)$, and we get the corresponding two integrable hierarchies. Finally, we discuss the relation between the integrable hierarchies of two different bases associated with Lie algebra $\mathfrak{so}(5)$.

Keywords Symplectic Lie algebra · Hamiltonian structures · Kronecker product · Integrable coupling

1 Introduction

The trace identity proposed by Tu is a simple and powerful tool for generating integrable hierarchies of soliton equations and their corresponding Hamiltonian structures [1]. So we first introduce the theory of Tu scheme. Usually, we start from a spectral problem

$$\begin{cases} \phi_x = U\phi, \quad U = U(u,\lambda), \; V = V(u,\lambda), \\ \phi_t = U\phi, \quad \phi = (\phi_1, \phi_2)^T, \quad \lambda_t = 0, \end{cases}$$

whose compatibility condition $\phi_{xt} = \phi_{tx}$ generates the zero curvature equation

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$$U_t - V_x + [U, V] = 0. \tag{1.1}$$

From (1.1), a hierarchy of soliton equations can be obtained

$$u_t = K(u) = J\frac{\delta H_n}{\delta u}, \tag{1.2}$$

where J is Hamiltonian operator.

In order to write Eq.(1.2) as a Hamiltonian form, the famous trace identity was established in [1] as follows:

$$\frac{\delta}{\delta u}\langle V, \frac{\partial U}{\partial \lambda}\rangle = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\langle V, \frac{\partial U}{\partial u}\rangle, \quad \gamma = \frac{\lambda}{2}\frac{d}{d\lambda}\ln \mathrm{tr(V\^{}2)},$$

where $\langle x, y\rangle = tr(xy)$. At the same time, based on Lie algebra A_1, Tu gave some integrable hierarchies of A_1 and constructed their Hamiltonian structures. For example, AKNS hierarchy, KN hierarchy and WKI hierarchy, etc.. [2] extended this method to discrete case and obtained some discrete integrable systems and their Hamiltonian structures. [3] extended to loop algebra $\tilde{A}_2$, and obtained some soliton hierarchies of $\tilde{A}_2$. [4] applied Tu scheme to Lie algebra B_2, and gave the generalized trace identity. [5] obtained integrable hierarchies of Lie algebra $\mathfrak{so}(3, \mathbb{R})$ and their Hamiltonian structures.

In order to obtain more integrable systems, [6] proposed the concept of integrable couplings, and obtained a large number of integrable coupling systems. Thereafter, a few ways to construct integrable couplings are presented. For example, by using perturbations [6], enlarging spectral problems and creating new loop algebras [7]. [8] gave Lax representations and zero curvature representations by Kronecker product, and obtained integrable coupling systems of soliton hierarchies. A large number of integrable couplings were constructed by this method [9, 10].

This paper is arranged as follows. In section 2, we extend trace identity to symplectic Lie algebra $\mathfrak{sp}(4)$, obtain two integrable hierarchies of $\mathfrak{sp}(4)$, and construct their Hamiltonian structures by using the trace identity. In section 3, we use 2×2 Kronecker product, and construct integrable coupling systems of $\mathfrak{sp}(4)$. In section 4, we consider two different bases of Lie algebra $\mathfrak{so}(5)$, and get the relation between their corresponding hierarchies.

2 Two Soliton Hierarchies Associated with $\mathfrak{sp}(4)$

2.1 The First Soliton Hierarchy Associated with $\mathfrak{sp}(4)$

The compact real form $\mathfrak{sp}(4)$ of complex symplectic Lie algebra $\mathfrak{sp}(4, \mathbb{C})$ is defined as [11, 12]

$$\mathfrak{sp}(4) = \{x \in \mathfrak{gl}(4, \mathbb{C}) | Hx + x^t H = 0\},$$

where $H = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$, and I_2 is the 2×2 identity matrix. We can obtain the bases of Lie algebra $\mathfrak{sp}(4)$

$$\begin{aligned} E_1 = e_{11} - e_{33},\ E_2 = e_{22} - e_{44},\ E_3 = e_{12} - e_{43},\ E_4 = e_{21} - e_{34}, \\ E_5 = e_{14} - e_{23},\ E_6 = e_{32} - e_{41},\ E_7 = e_{13},\ E_8 = e_{31},\ E_9 = e_{24},\ E_{10} = e_{42}, \end{aligned} \tag{2.3}$$

where e_{ij} is a 4×4 matrix with 1 in the (i, j)-th position and zero elsewhere.

Consider an isospectral problem

$$\begin{cases} \varphi_x = U_0 \varphi, \\ \varphi_t = V_0 \varphi, \quad \lambda_t = 0. \end{cases}$$

Set

$$\begin{aligned} U_0 = & E_1(1) + E_2(1) + u_1 E_5(0) + u_2 E_6(0) + u_3 E_7(0) + u_4 E_8(0) \\ & + u_5 E_9(0) + u_6 E_{10}(0), \end{aligned}$$

i.e.

$$U_0 = \begin{pmatrix} \lambda & 0 & u_3 & u_1 \\ 0 & \lambda & u_1 & u_5 \\ u_4 & u_2 & -\lambda & 0 \\ u_2 & u_6 & 0 & -\lambda \end{pmatrix}, \tag{2.4}$$

and

$$V_0 = \begin{pmatrix} a & c & g & e \\ d & b & e & y \\ h & f & -a & -d \\ f & z & -c & -b \end{pmatrix} = \sum_{i \geq 0} \begin{pmatrix} a_i & c_i & g_i & e_i \\ d_i & b_i & e_i & y_i \\ h_i & f_i & -a_i & -d_i \\ f_i & z_i & -c_i & -b_i \end{pmatrix} \lambda^{-i}. \tag{2.5}$$

The stationary zero curvature representation $V_{0,x} = [U_0, V_0]$ gives

$$\begin{cases} a_x = u_1 f - u_2 e + u_3 h - u_4 g, \\ b_x = u_1 f - u_2 e + u_5 z - u_6 y, \\ c_x = u_1 z - u_2 g + u_3 f - u_6 e, \\ d_x = u_1 h - u_2 y - u_4 e + u_5 f, \\ e_x = 2\lambda e - u_1 a - u_1 b - u_3 d - u_5 c, \\ f_x = -2\lambda f + u_2 a + u_2 b + u_4 c + u_6 d, \\ g_x = 2\lambda g - 2u_1 c - 2u_3 a, \\ h_x = -2\lambda h + 2u_2 d + 2u_4 a, \\ y_x = 2\lambda y - 2u_1 d - 2u_5 b, \\ z_x = -2\lambda z + 2u_2 c + 2u_6 b. \end{cases} \tag{2.6}$$

Take the initial values

$$a_0 = \alpha,\ b_0 = \beta,\ c_0 = d_0 = e_0 = f_0 = g_0 = h_0 = y_0 = z_0 = 0.$$

From (2.6), we have

$$
\begin{aligned}
a_1 &= b_1 = 0,\ c_1 = \frac{1}{2}\partial^{-1}((u_2u_3 + u_1u_6)(\beta - \alpha)),\\
d_1 &= \frac{1}{2}\partial^{-1}((u_1u_4 + u_2u_5)(\alpha - \beta)),\\
e_1 &= \frac{1}{2}u_1(\alpha + \beta),\ f_1 = \frac{1}{2}u_2(\alpha + \beta),\ h_1 = u_4\alpha,\ g_1 = u_3\alpha,\ y_1 = u_5\beta,\\
z_1 &= u_6\beta,\\
e_2 &= \frac{1}{4}u_{1x}(\alpha + \beta) + \frac{1}{4}u_3\partial^{-1}(u_1u_4 + u_2u_5)(\alpha - \beta)\\
&\quad - \frac{1}{4}u_5\partial^{-1}(u_1u_6 + u_2u_3)(\alpha - \beta),\\
f_2 &= -\frac{1}{4}u_{2x}(\alpha + \beta) + \frac{1}{4}u_4\partial^{-1}(u_1u_6 + u_2u_3)(\beta - \alpha)\\
&\quad + \frac{1}{4}u_6\partial^{-1}(u_1u_4 + u_2u_5)(\alpha - \beta),\\
g_2 &= \frac{1}{2}u_{3x}\alpha + \frac{1}{2}u_1\partial^{-1}(u_1u_6 + u_2u_3)(\beta - \alpha),\\
h_2 &= -\frac{1}{2}u_{4x}\alpha - \frac{1}{2}u_2\partial^{-1}(u_1u_4 + u_2u_5)(\beta - \alpha),\\
y_2 &= \frac{1}{2}u_{5x}\beta - \frac{1}{2}u_1\partial^{-1}(u_1u_4 + u_2u_5)(\beta - \alpha),\\
z_2 &= -\frac{1}{2}u_{6x}\beta + \frac{1}{2}u_2\partial^{-1}(u_1u_6 + u_2u_3)(\beta - \alpha).
\end{aligned}
$$

Take $V_0^n = V_{0,+}^n$, then the zero curvature equation $U_{0,t} - V_{0,+x}^n + [U_0, V_{0,+}^n] = 0$ leads to the following Lax integrable hierarchy

$$
u_{t_n} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}_{t_n} = \begin{pmatrix} -2e_{n+1} \\ 2f_{n+1} \\ -2g_{n+1} \\ 2h_{n+1} \\ -2y_{n+1} \\ 2z_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2f_{n+1} \\ 2e_{n+1} \\ h_{n+1} \\ g_{n+1} \\ z_{n+1} \\ y_{n+1} \end{pmatrix} = J_1 P_{1,n+1}. \quad (2.7)
$$

From the recurrence relations (2.6), we have

$$
P_{1,n+1} = \begin{pmatrix} l_{11} & l_{12} & l_{13} & l_{14} & l_{15} & l_{16} \\ l_{21} & l_{22} & l_{23} & l_{24} & l_{25} & l_{26} \\ l_{31} & l_{32} & l_{33} & l_{34} & l_{35} & l_{36} \\ l_{41} & l_{42} & l_{43} & l_{44} & l_{45} & l_{46} \\ l_{51} & l_{52} & l_{53} & l_{54} & l_{55} & l_{56} \\ l_{61} & l_{62} & l_{63} & l_{64} & l_{65} & l_{66} \end{pmatrix} P_{1,n} = L_1 P_{1,n},
$$

where L_1 is a recurrence operator, and

$$l_{11}=-\frac{1}{2}\partial+u_2\partial^{-1}u_1+\frac{1}{2}(u_4\partial^{-1}u_3+u_6\partial^{-1}u_5),\ l_{12}=-u_2\partial^{-1}u_2-\frac{1}{2}(u_4\partial^{-1}u_6+u_6\partial^{-1}u_4),$$
$$l_{21}=u_1\partial^{-1}u_1+\frac{1}{2}(u_3\partial^{-1}u_5+u_5\partial^{-1}u_3),\ l_{22}=\frac{1}{2}\partial-u_1\partial^{-1}u_2-\frac{1}{2}(u_3\partial^{-1}u_4+u_5\partial^{-1}u_6),$$
$$l_{31}=\frac{1}{2}l_{15}=\frac{1}{2}(u_2\partial^{-1}u_5+u_4\partial^{-1}u_1),\ l_{32}=\frac{1}{2}l_{14}=-\frac{1}{2}(u_2\partial^{-1}u_4+u_4\partial^{-1}u_2),$$
$$l_{41}=\frac{1}{2}l_{23}=\frac{1}{2}(u_1\partial^{-1}u_3+u_3\partial^{-1}u_1),\ l_{42}=\frac{1}{2}l_{26}=-\frac{1}{2}(u_1\partial^{-1}u_6+u_3\partial^{-1}u_2),$$
$$l_{51}=\frac{1}{2}l_{13}=\frac{1}{2}(u_2\partial^{-1}u_3+u_6\partial^{-1}u_1),\ l_{52}=\frac{1}{2}l_{16}=-\frac{1}{2}(u_2\partial^{-1}u_6+u_6\partial^{-1}u_2),$$
$$l_{61}=\frac{1}{2}l_{25}=\frac{1}{2}(u_1\partial^{-1}u_5+u_5\partial^{-1}u_1),\ l_{62}=\frac{1}{2}l_{24}=-\frac{1}{2}(u_1\partial^{-1}u_4+u_5\partial^{-1}u_2),$$
$$l_{33}=-\frac{1}{2}\partial+u_2\partial^{-1}u_1+u_4\partial^{-1}u_3,\ l_{34}=-u_4\partial^{-1}u_4,\ l_{35}=0,\ l_{36}=-u_2\partial^{-1}u_2,$$
$$l_{43}=u_3\partial^{-1}u_3,\ l_{44}=\frac{1}{2}\partial-u_1\partial^{-1}u_2-u_3\partial^{-1}u_4,\ l_{45}=u_1\partial^{-1}u_1,\ l_{46}=0,$$
$$l_{53}=0,\ l_{54}=-u_2\partial^{-1}u_2,\ l_{55}=-\frac{1}{2}\partial+u_2\partial^{-1}u_1+u_6\partial^{-1}u_5,\ l_{56}=-u_6\partial^{-1}u_6,$$
$$l_{63}=u_1\partial^{-1}u_1,\ l_{64}=0,\ l_{65}=u_5\partial^{-1}u_5,\ l_{66}=\frac{1}{2}\partial-u_1\partial^{-1}u_2-u_5\partial^{-1}u_6.$$

To furnish Hamiltonian structures, we use the trace identity, and have

$$\langle V_0,\frac{\partial U_0}{\partial\lambda}\rangle=2a+2b,\ \langle V_0,\frac{\partial U_0}{\partial u_1}\rangle=2f,\ \langle V_0,\frac{\partial U_0}{\partial u_2}\rangle=2e,$$
$$\langle V_0,\frac{\partial U_0}{\partial u_3}\rangle=h,\ \langle V_0,\frac{\partial U_0}{\partial u_4}\rangle=g,\ \langle V_0,\frac{\partial U_0}{\partial u_5}\rangle=z,\ \langle V_0,\frac{\partial U_0}{\partial u_6}\rangle=y.$$

Substituting the above formulate into the trace identity yields

$$\frac{\delta}{\delta u}\int(2a+2b)dx=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{pmatrix}2f\\2e\\h\\g\\z\\y\end{pmatrix}.$$

Balancing coefficients of each power of λ in the above equality gives rise to

$$\frac{\delta}{\delta u}\int(2a_{n+1}+2b_{n+1})dx=(\gamma-n)\begin{pmatrix}2f_n\\2e_n\\h_n\\g_n\\z_n\\y_n\end{pmatrix}.$$

Taking $n=1$, gives $\gamma=0$. Thus, we see

$$u_t = J_1 P_{1,n+1} = J_1 \frac{\delta H_n^{(1)}}{\delta u}, \quad H_n^{(1)} = -2 \int (\frac{a_{n+2} + b_{n+2}}{n+1}) dx, \quad n \geq 0.$$

It is said that the hierarchy (2.7) have the Hamiltonian structure, and it is easy to verity that $J_1 L_1 = L_1^* J_1$. Therefore, the hierarchy (2.7) is Liouville integrable.

When $n = 1$, the hierarchy (2.7) reduces to the first system

$$\begin{aligned} u_{1t} &= -\frac{1}{2}u_{1x}(\alpha+\beta) - \frac{1}{2}u_3\partial^{-1}(u_1u_4 + u_2u_5)(\alpha-\beta) \\ &\quad + \frac{1}{2}u_5\partial^{-1}(u_1u_6 + u_2u_3)(\alpha-\beta), \\ u_{2t} &= -\frac{1}{2}u_{2x}(\alpha+\beta) + \frac{1}{2}u_4\partial^{-1}(u_1u_6 + u_2u_3)(\beta-\alpha) \\ &\quad + \frac{1}{2}u_6\partial^{-1}(u_1u_4 + u_2u_5)(\alpha-\beta), \\ u_{3t} &= -u_{3x}\alpha - u_1\partial^{-1}(u_1u_6 + u_2u_3)(\beta-\alpha), \\ u_{4t} &= -u_{4x}\alpha - u_2\partial^{-1}(u_1u_4 + u_2u_5)(\beta-\alpha), \\ u_{5t} &= -u_{5x}\beta + u_1\partial^{-1}(u_1u_4 + u_2u_5)(\beta-\alpha), \\ u_{6t} &= -u_{6x}\beta + u_2\partial^{-1}(u_1u_6 + u_2u_3)(\beta-\alpha). \end{aligned} \tag{2.8}$$

2.2 The Second Soliton Hierarchy Associated with $\mathfrak{sp}(4)$

In this section, we consider another isospectral problem

$$\begin{cases} \varphi_x = U_1\varphi, \\ \varphi_t = V_0\varphi, \quad \lambda_t = 0. \end{cases}$$

Set

$$\begin{aligned} U_1 =& E_1(1) - E_2(1) + u_1'E_3(0) + u_2'E_4(0) + u_3'E_7(0) + u_4'E_8(0) \\ &+ u_5'E_9(0) + u_6'E_{10}(0), \end{aligned}$$

and V_0 is defined by (2.5),

$$U_1 = \begin{pmatrix} \lambda & u_1' & u_3' & 0 \\ u_2' & \lambda & 0 & u_5' \\ u_4' & 0 & -\lambda & -u_2' \\ 0 & u_6' & -u_1' & -\lambda \end{pmatrix}. \tag{2.9}$$

The stationary zero curvature representation $V_{0,x} = [U_1, V_0]$ gives

$$\begin{cases} a_x = u_1'd - u_2'c + u_3'h - u_4'g, \\ b_x = -u_1'd + u_2'c + u_5'z - u_6'y, \\ c_x = 2\lambda c + u_1'b - u_1'a + u_3'f - u_6'e, \\ d_x = -2\lambda d + u_2'a - u_2'b - u_4'e + u_5'f, \\ e_x = u_1'y + u_2'g - u_3'd - u_5'c, \\ f_x = -u_1'h - u_2'z + u_4'c + u_6'd, \\ g_x = 2\lambda g + 2u_1'e - 2u_3'a, \\ h_x = -2\lambda h - 2u_2'f + 2u_4'a, \\ y_x = -2\lambda y + 2u_2'e - 2u_5'b, \\ z_x = 2\lambda z - 2u_1'f + 2u_6'b. \end{cases} \tag{2.10}$$

Let

$$a = \sum_{i\geq 0} a_i\lambda^{-i},\ b = \sum_{i\geq 0} b_i\lambda^{-i},\ c = \sum_{i\geq 0} c_i\lambda^{-i},\ d = \sum_{i\geq 0} d_i\lambda^{-i},\ e = \sum_{i\geq 0} e_i\lambda^{-i},$$
$$f = \sum_{i\geq 0} f_i\lambda^{-i},\ g = \sum_{i\geq 0} g_i\lambda^{-i},\ h = \sum_{i\geq 0} h_i\lambda^{-i},\ y = \sum_{i\geq 0} y_i\lambda^{-i},\ z = \sum_{i\geq 0} z_i\lambda^{-i}.$$

Take the initial values

$$a_0 = \alpha,\ b_0 = \beta,\ c_0 = d_0 = e_0 = f_0 = g_0 = h_0 = y_0 = z_0 = 0.$$

From (2.10), we have

$$\begin{aligned}
a_1 &= b_1 = 0,\ c_1 = \frac{1}{2}u_1'(\alpha-\beta),\ d_1 = \frac{1}{2}u_2'(\alpha-\beta),\ g_1 = u_3'\alpha,\ h_1 = u_4'\alpha,\ y_1 = -u_5'\beta, \\
z_1 &= -u_6'\beta,\ e_1 = \partial^{-1}u_2'u_3'\alpha - \partial^{-1}u_1'u_5'\beta - \frac{1}{2}\partial^{-1}(u_1'u_5' + u_2'u_3')(\alpha-\beta), \\
f_1 &= \partial^{-1}u_2'u_6'\beta - \partial^{-1}u_1'u_4'\alpha + \frac{1}{2}\partial^{-1}(u_1'u_4' + u_2'u_6')(\alpha-\beta), \\
c_2 &= \frac{1}{4}u_{1x}'(\alpha-\beta) - \frac{1}{4}u_3'\partial^{-1}(u_1'u_4' + u_2'u_6')(\alpha-\beta) + \frac{1}{4}u_6'\partial^{-1}(u_1'u_5' + u_2'u_3')(\alpha-\beta) \\
&\quad - \frac{1}{2}u_3'\partial^{-1}u_2'u_6'\beta + \frac{1}{2}u_3'\partial^{-1}u_1'u_4'\alpha + \frac{1}{2}u_6'\partial^{-1}u_2'u_3'\alpha - \frac{1}{2}u_6'\partial^{-1}u_1'u_5'\beta, \\
d_2 &= -\frac{1}{4}u_{2x}'(\alpha-\beta) + \frac{1}{4}u_4'\partial^{-1}(u_1'u_5' + u_2'u_3')(\alpha-\beta) + \frac{1}{4}u_5'\partial^{-1}(u_1'u_4' + u_2'u_6')(\alpha-\beta) \\
&\quad - \frac{1}{2}u_4'\partial^{-1}u_2'u_3'\alpha + \frac{1}{2}u_4'\partial^{-1}u_1'u_5'\beta + \frac{1}{2}u_5'\partial^{-1}u_2'u_3'\beta - \frac{1}{2}u_5'\partial^{-1}u_1'u_4'\alpha, \\
g_2 &= \frac{1}{2}u_{3x}'\alpha + \frac{1}{2}u_1'\partial^{-1}(u_1'u_5' + u_2'u_3')(\alpha-\beta) - u_1'\partial^{-1}u_2'u_3'\alpha + u_1'\partial^{-1}u_1'u_5'\beta, \\
h_2 &= -\frac{1}{2}u_{4x}'\alpha - \frac{1}{2}u_2'\partial^{-1}(u_1'u_4' + u_2'u_6')(\alpha-\beta) - u_2'\partial^{-1}u_2'u_6'\beta + u_2'\partial^{-1}u_1'u_4'\alpha, \\
y_2 &= \frac{1}{2}u_{5x}'\beta - \frac{1}{2}u_2'\partial^{-1}(u_1'u_5' + u_2'u_3')(\alpha-\beta) + u_2'\partial^{-1}u_2'u_3'\alpha - u_2'\partial^{-1}u_1'u_5'\beta, \\
z_2 &= -\frac{1}{2}u_{6x}'\beta + \frac{1}{2}u_1'\partial^{-1}(u_1'u_4' + u_2'u_6')(\alpha-\beta) + u_1'\partial^{-1}u_2'u_6'\beta - u_1'\partial^{-1}u_1'u_4'\alpha.
\end{aligned}$$

Then the zero curvature equation $U_{1,t} - V^n_{0,+x} + [U_1, V^n_{0,+}] = 0$ leads to the following Lax integrable hierarchy

$$u'_t = \begin{pmatrix} u'_1 \\ u'_2 \\ u'_3 \\ u'_4 \\ u'_5 \\ u'_6 \end{pmatrix}_t = \begin{pmatrix} -2c_{n+1} \\ 2d_{n+1} \\ -2g_{n+1} \\ 2h_{n+1} \\ 2y_{n+1} \\ -2z_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} 2d_{n+1} \\ 2c_{n+1} \\ h_{n+1} \\ g_{n+1} \\ z_{n+1} \\ y_{n+1} \end{pmatrix} = J_2 P_{2,n+1}. \quad (2.11)$$

From the recurrence relations (2.10), we have

$$P_{2,n+1} = \begin{pmatrix} l'_{11} & l'_{12} & l'_{13} & l'_{14} & l'_{15} & l'_{16} \\ l'_{21} & l'_{22} & l'_{23} & l'_{24} & l'_{25} & l'_{26} \\ l'_{31} & l'_{32} & l'_{33} & l'_{34} & l'_{35} & l'_{36} \\ l'_{41} & l'_{42} & l'_{43} & l'_{44} & l'_{45} & l'_{46} \\ l'_{51} & l'_{52} & l'_{53} & l'_{54} & l'_{55} & l'_{56} \\ l'_{61} & l'_{62} & l'_{63} & l'_{64} & l'_{65} & l'_{66} \end{pmatrix} P_{2,n} = L_2 P_{2,n},$$

where L_2 is a recurrence operator, and

$$\begin{aligned}
& l'_{11} = -\frac{1}{2}\partial + u'_2\partial^{-1}u'_1 + \frac{1}{2}(u'_4\partial^{-1}u'_3 + u'_5\partial^{-1}u'_6),\ l'_{12} = -u'_2\partial^{-1}u'_2 + \frac{1}{2}(u'_4\partial^{-1}u'_5 + u'_5\partial^{-1}u'_4),\\
& l'_{21} = u'_1\partial^{-1}u'_1 - \frac{1}{2}(u'_3\partial^{-1}u'_6 + u'_6\partial^{-1}u'_3),\ l'_{22} = \frac{1}{2}\partial - u'_1\partial^{-1}u'_2 - \frac{1}{2}(u'_3\partial^{-1}u'_4 + u'_6\partial^{-1}u'_5),\\
& l'_{31} = -\frac{1}{2}l'_{16} = -\frac{1}{2}(u'_2\partial^{-1}u'_6 + u'_4\partial^{-1}u'_1),\ \ l'_{32} = \frac{1}{2}l'_{14} = -\frac{1}{2}(u'_2\partial^{-1}u'_4 + u'_4\partial^{-1}u'_2),\\
& l'_{41} = \frac{1}{2}l'_{23} = \frac{1}{2}(u'_1\partial^{-1}u'_3 + u'_3\partial^{-1}u'_1),\ \ l'_{42} = -\frac{1}{2}l'_{25} = \frac{1}{2}(u'_1\partial^{-1}u'_5 - u'_3\partial^{-1}u'_2),\\
& l'_{51} = \frac{1}{2}l'_{26} = \frac{1}{2}(u'_1\partial^{-1}u'_6 + u'_6\partial^{-1}u'_1),\ \ l'_{52} = \frac{1}{2}l'_{24} = \frac{1}{2}(u'_1\partial^{-1}u'_4 - u'_6\partial^{-1}u'_2),\\
& l'_{61} = -\frac{1}{2}l'_{13} = \frac{1}{2}(u'_5\partial^{-1}u'_1 - u'_2\partial^{-1}u'_3),\ \ l'_{62} = \frac{1}{2}l'_{15} = -\frac{1}{2}(u'_2\partial^{-1}u'_5 + u'_5\partial^{-1}u'_2),\\
& l'_{33} = -\frac{1}{2}\partial + u'_2\partial^{-1}u'_1 + u'_4\partial^{-1}u'_3,\ \ l'_{34} = -u'_4\partial^{-1}u'_4,\ \ l'_{35} = u'_2\partial^{-1}u'_2,\ \ l'_{36} = 0,\\
& l'_{43} = u'_3\partial^{-1}u'_3,\ \ l'_{44} = \frac{1}{2}\partial - u'_1\partial^{-1}u'_2 - u'_3\partial^{-1}u'_4,\ \ l'_{45} = 0,\ \ l'_{46} = -u'_1\partial^{-1}u'_1,\\
& l'_{53} = -u'_1\partial^{-1}u'_1,\ \ l'_{54} = 0,\ \ l'_{55} = \frac{1}{2}\partial - u'_1\partial^{-1}u'_2 - u'_6\partial^{-1}u'_5,\ \ l'_{56} = u'_6\partial^{-1}u'_6,\\
& l'_{63} = 0,\ \ l'_{64} = u'_2\partial^{-1}u'_2,\ \ l'_{65} = -u'_5\partial^{-1}u'_5,\ \ l'_{66} = -\frac{1}{2}\partial + u'_2\partial^{-1}u'_1 + u'_5\partial^{-1}u'_6.
\end{aligned}$$

To furnish Hamiltonian structures, we use the trace identity, and have

$$\langle V_0, \frac{\partial U_1}{\partial \lambda}\rangle = 2a - 2b,\ \langle V_0, \frac{\partial U_1}{\partial u'_1}\rangle = 2d,\ \langle V_0, \frac{\partial U_1}{\partial u'_2}\rangle = 2c,$$
$$\langle V_0, \frac{\partial U_1}{\partial u'_3}\rangle = h,\ \langle V_0, \frac{\partial U_1}{\partial u'_4}\rangle = g,\ \langle V_0, \frac{\partial U_1}{\partial u'_5}\rangle = z,\ \langle V_0, \frac{\partial U_1}{\partial u'_6}\rangle = y.$$

Substituting the above formulate into the trace identity yields

$$\frac{\delta}{\delta u'}\int (2a-2b)dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{pmatrix} 2d \\ 2c \\ h \\ g \\ z \\ y \end{pmatrix}.$$

Balancing coefficients of each power of λ in the above equality gives rise to

$$\frac{\delta}{\delta u'}\int (2a_{n+1}-2b_{n+1})dx = (\gamma - n)\begin{pmatrix} 2d_n \\ 2c_n \\ h_n \\ g_n \\ z_n \\ y_n \end{pmatrix}.$$

Taking $n = 1$, gives $\gamma = 0$.Thus, we see

$$u'_t = J_2 P_{2,n+1} = J_2 \frac{\delta H_n^{(2)}}{\delta u'}, \quad H_n^{(2)} = -2\int (\frac{a_{n+2}-b_{n+2}}{n+1})dx, \quad n \geq 0.$$

It is said that the hierarchy (2.11) have the Hamiltonian structure, and it is easy to prove that $J_2 L_2 = L_2^* J_2$. Therefore, the hierarchy (2.11) is Liouville integrable.

When $n = 1$, the hierarchy (2.11) reduces to the first system

$$\begin{aligned}
u'_{1t} =& -\frac{1}{2}u'_{1x}(\alpha-\beta) + \frac{1}{2}u'_3\partial^{-1}(u'_1u'_4 + u'_2u'_6)(\alpha-\beta) - \frac{1}{2}u'_6\partial^{-1}(u'_1u'_5 + u'_2u'_3)(\alpha-\beta) \\
&+ u'_3\partial^{-1}u'_2u'_6\beta - u'_3\partial^{-1}u'_1u'_4\alpha - u'_6\partial^{-1}u'_2u'_3\alpha + u'_6\partial^{-1}u'_1u'_5\beta, \\
u'_{2t} =& -\frac{1}{2}u'_{2x}(\alpha-\beta) + \frac{1}{2}u'_4\partial^{-1}(u'_1u'_5 + u'_2u'_3)(\alpha-\beta) + \frac{1}{2}u'_5\partial^{-1}(u'_1u'_4 + u'_2u'_6)(\alpha-\beta) \\
&- u'_4\partial^{-1}u'_2u'_3\alpha + u'_4\partial^{-1}u'_1u'_5\beta + u'_5\partial^{-1}u'_2u'_3\beta - u'_5\partial^{-1}u'_1u'_4\alpha, \\
u'_{3t} =& -u'_{3x}\alpha - u'_1\partial^{-1}(u'_1u'_5 + u'_2u'_3)(\alpha-\beta) + 2u'_1\partial^{-1}u'_2u'_3\alpha - 2u'_1\partial^{-1}u'_1u'_5\beta, \\
u'_{4t} =& -u'_{4x}\alpha - u'_2\partial^{-1}(u'_1u'_4 + u'_2u'_6)(\alpha-\beta) - 2u'_2\partial^{-1}u'_2u'_6\beta + 2u'_2\partial^{-1}u'_1u'_4\alpha, \\
u'_{6t} =& u'_{6x}\beta - u'_1\partial^{-1}(u'_1u'_4 + u'_2u'_6)(\alpha-\beta) - 2u'_1\partial^{-1}u'_2u'_6\beta + 2u'_1\partial^{-1}u'_1u'_4\alpha.
\end{aligned}$$

3 Integrable Coupling Systems for the Hierarchy of $\mathfrak{sp}(4)$

In this section, we will use 2×2 Kronecker product [10] to construct integrable coupling systems of Lie algebra $\mathfrak{sp}(4)$.

Let U_2 and V_2 have the forms [10]

$$U_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes U_0 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes U_1,$$
$$V_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes V_0 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes V_1,$$

then that can get a new pair of U_2 and V_2,

$$U_2 = \begin{pmatrix} U_0 & U_1 \\ 0 & U_0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} V_0 & V_1 \\ 0 & V_0 \end{pmatrix}.$$

Therefore, the corresponding enlarged zero curvature equation $U_{2,t} - V_{2,x} + [U_2, V_2] = 0$ is equivalent to

$$U_{0,t} - V_{0,x} + [U_0, V_0] = 0,$$
$$U_{1,t} - V_{1,x} + [U_0, V_1] + [U_1, V_0] = 0,$$

where

$$U_0 = \begin{pmatrix} \lambda & 0 & u_3 & u_1 \\ 0 & \lambda & u_1 & u_5 \\ u_4 & u_2 & -\lambda & 0 \\ u_2 & u_6 & 0 & -\lambda \end{pmatrix}, \quad U_1 = \begin{pmatrix} 0 & 0 & u_3'' & u_1'' \\ 0 & 0 & u_1'' & u_5'' \\ u_4'' & u_2'' & 0 & 0 \\ u_2'' & u_6'' & 0 & 0 \end{pmatrix},$$
$$V_0 = \begin{pmatrix} a & c & g & e \\ d & b & e & y \\ h & f & -a & -d \\ f & z & -c & -b \end{pmatrix}, \quad V_1 = \begin{pmatrix} a'' & c'' & g'' & e'' \\ d'' & b'' & e'' & y'' \\ h'' & f'' & -a'' & -d'' \\ f'' & z'' & -c'' & -b'' \end{pmatrix}.$$

From the stationary zero curvature equation $V_{2,x} = [U_2, V_2]$, we obtain (2.6) and

$$\begin{cases} a''_x = u_1 f'' - u_2 e'' + u_3 h'' - u_4 g'' + u''_1 f - u''_2 e + u''_3 h - u''_4 g, \\ b''_x = u_1 f'' - u_2 e'' + u_5 z'' - u_6 y'' + u''_1 f - u''_2 e + u''_5 z - u''_6 y, \\ c''_x = u_1 z'' - u_2 g'' + u_3 f'' - u_6 e'' + u''_1 z - u''_2 g + u''_3 f - u''_6 e, \\ d''_x = u_1 h'' - u_2 y'' - u_4 e'' + u_5 f'' + u''_1 h - u''_2 y - u''_4 e + u''_5 f, \\ e''_x = 2\lambda e'' - u_1 a'' - u_1 b'' - u_3 d'' - u_5 c'' - u''_1 a - u''_1 b - u''_3 d - u''_5 c, \\ f''_x = -2\lambda f'' + u_2 a'' + u_2 b'' + u_4 c'' + u_6 d'' + u''_2 a + u''_2 b + u''_4 c + u''_6 d, \\ g''_x = 2\lambda g'' - 2u_1 c'' - 2u_3 a'' - 2u''_1 c - 2u''_3 a, \\ h''_x = -2\lambda h'' + 2u_2 d'' + 2u_4 a'' + 2u''_2 d + 2u''_4 a, \\ y''_x = 2\lambda y'' - 2u_1 d'' - 2u_5 b'' - 2u''_1 d - 2u''_5 b, \\ z''_x = -2\lambda z'' + 2u_2 c'' + 2u_6 b'' + 2u''_2 c + 2u''_6 b. \end{cases} \tag{3.12}$$

Let $a, b, c, d, e, f, g, h, y, z$ be defined as (2.5), and

$$\begin{aligned} & a'' = \sum_{i\geq 0} a''_i \lambda^{-i},\ b'' = \sum_{i\geq 0} b''_i \lambda^{-i},\ c'' = \sum_{i\geq 0} c''_i \lambda^{-i}, \\ & d'' = \sum_{i\geq 0} d''_i \lambda^{-i},\ e'' = \sum_{i\geq 0} e''_i \lambda^{-i}, \\ & f'' = \sum_{i\geq 0} f''_i \lambda^{-i},\ g'' = \sum_{i\geq 0} g''_i \lambda^{-i},\ h'' = \sum_{i\geq 0} h''_i \lambda^{-i},\ y'' = \sum_{i\geq 0} y''_i \lambda^{-i}, \\ & z'' = \sum_{i\geq 0} z''_i \lambda^{-i}. \end{aligned}$$

Similarly, take the initial values

$$a''_0 = \alpha',\ b''_0 = \beta',\ c''_0 = d''_0 = e''_0 = f''_0 = g''_0 = h''_0 = y''_0 = z''_0 = 0.$$

From (3.12), we have

$$
\begin{aligned}
a_1'' =& b_1'' = 0,\ e_1'' = \frac{1}{2}u_1(\alpha' + \beta') + \frac{1}{2}u_1''(\alpha + \beta),\ f_1'' = \frac{1}{2}u_2(\alpha' + \beta') + \frac{1}{2}u_2''(\alpha + \beta),\\
g_1'' =& u_3\alpha' + u_3''\alpha,\ h_1'' = u_4\alpha' + u_4''\alpha,\ y_1'' = u_5\beta' + u_5''\beta,\ z_1'' = u_6\beta' + u_6''\beta,\\
c_1'' =& \frac{1}{2}\partial^{-1}(u_2u_3 - u_1u_6)(\alpha' + \beta') + \frac{1}{2}\partial^{-1}(u_2u_3'' + u_2''u_3 - u_1u_6'' - u_1''u_6)(\alpha + \beta)\\
&+ \partial^{-1}u_1u_6\beta' - \partial^{-1}u_2u_3\alpha' + \partial^{-1}(u_1u_6'' + u_1''u_6)\beta - \partial^{-1}(u_2u_3'' + u_2''u_3)\alpha,\\
d_1'' =& \frac{1}{2}\partial^{-1}(u_2u_5 - u_1u_4)(\alpha' + \beta') + \frac{1}{2}\partial^{-1}(u_2u_5'' + u_2''u_5 - u_1u_4'' - u_1''u_4)(\alpha + \beta)\\
&+ \partial^{-1}u_1u_4\alpha' - \partial^{-1}u_2u_5\beta' + \partial^{-1}(u_1u_4'' + u_1''u_4)\alpha - \partial^{-1}(u_2u_5'' + u_2''u_5)\beta,\\
e_2'' =& \frac{1}{4}u_{1x}(\alpha' + \beta') + \frac{1}{4}u_{1x}''(\alpha + \beta) + \frac{1}{4}u_3\partial^{-1}(u_1u_4 + u_2u_5)(\alpha' - \beta')\\
&+ \frac{1}{4}u_5''\partial^{-1}(u_1u_6 + u_2u_3)(\beta - \alpha) + \frac{1}{4}u_3''\partial^{-1}(u_1u_4 + u_2u_5)(\alpha - \beta)\\
&+ \frac{1}{4}u_3\partial^{-1}(u_1u_4'' + u_1''u_4 + u_2u_5'' + u_2''u_5)(\alpha - \beta) + \frac{1}{4}u_5\partial^{-1}(u_1u_6 + u_2u_3)(\beta' - \alpha')\\
&+ \frac{1}{4}u_5\partial^{-1}(u_1u_6'' + u_1''u_6 + u_2u_3'' + u_2''u_3)(\beta - \alpha)\\
f_2'' =& -\frac{1}{4}u_{2x}(\alpha' + \beta') - \frac{1}{4}u_{2x}''(\alpha + \beta) + \frac{1}{4}u_4\partial^{-1}(u_1u_6 + u_2u_3)(\beta' - \alpha')\\
&+ \frac{1}{4}u_6''\partial^{-1}(u_1u_4 + u_2u_5)(\alpha - \beta) + \frac{1}{4}u_4''\partial^{-1}(u_1u_6 + u_2u_3)(\beta - \alpha)\\
&+ \frac{1}{4}u_4\partial^{-1}(u_1u_6'' + u_1''u_6 + u_2u_3'' + u_2''u_3)(\beta - \alpha) + \frac{1}{4}u_6\partial^{-1}(u_1u_4 + u_2u_5)(\alpha' - \beta')\\
&+ \frac{1}{4}u_6\partial^{-1}(u_1u_4'' + u_1''u_4 + u_2u_5'' + u_2''u_5)(\alpha - \beta)\\
g_2'' =& \frac{1}{2}u_{3x}\alpha' + \frac{1}{2}u_{3x}''\alpha + \frac{1}{2}u_1\partial^{-1}(u_1u_6 + u_2u_3)(\beta' - \alpha') + \frac{1}{2}u_1''\partial^{-1}(u_1u_6 + u_2u_3)(\beta - \alpha)\\
&+ \frac{1}{2}u_1\partial^{-1}(u_1u_6'' + u_1''u_6 + u_2u_3'' + u_2''u_3)(\beta - \alpha),\\
h_2'' =& -\frac{1}{2}u_{4x}\alpha' - \frac{1}{2}u_{4x}''\alpha + \frac{1}{2}u_2\partial^{-1}(u_1u_4 + u_2u_5)(\alpha' - \beta')\\
&+ \frac{1}{2}u_2''\partial^{-1}(u_1u_4 + u_2u_5)(\alpha - \beta) + \frac{1}{2}u_2\partial^{-1}(u_1u_4'' + u_1''u_4 + u_2u_5'' + u_2''u_5)(\alpha - \beta),\\
y_2'' =& \frac{1}{2}u_{5x}\beta' + \frac{1}{2}u_{5x}''\beta + \frac{1}{2}u_1\partial^{-1}(u_1u_4 + u_2u_5)(\alpha' - \beta')\\
&+ \frac{1}{2}u_1''\partial^{-1}(u_1u_4 + u_2u_5)(\alpha - \beta) + \frac{1}{2}u_1\partial^{-1}(u_1u_4'' + u_1''u_4 + u_2u_5'' + u_2''u_5)(\alpha - \beta),\\
z_2'' =& -\frac{1}{2}u_{6x}\beta' - \frac{1}{2}u_{6x}''\beta + \frac{1}{2}u_2\partial^{-1}(u_1u_6 + u_2u_3)(\beta' - \alpha')\\
&+ \frac{1}{2}u_2''\partial^{-1}(u_1u_6 + u_2u_3)(\beta - \alpha) + \frac{1}{2}u_2\partial^{-1}(u_1u_6'' + u_1''u_6 + u_2u_3'' + u_2''u_3)(\beta - \alpha).
\end{aligned}
$$

From the zero curvature equation $U_{2,t} - V^n_{2,+x} + [U_2, V^n_{2,+}] = 0$, we can obtain the following Lax integrable hierarchy

$$\bar{u}_t = \begin{pmatrix} u \\ u'' \end{pmatrix}_t = \begin{pmatrix} J_1 & 0 \\ 0 & J_1 \end{pmatrix} \begin{pmatrix} P_{1,n+1} \\ P''_{1,n+1} \end{pmatrix} = JP_{n+1}, \tag{3.13}$$

where J_1 is defined by (2.7), and

$$u_t = \begin{pmatrix} -2e_{n+1} \\ 2f_{n+1} \\ -2g_{n+1} \\ 2h_{n+1} \\ 2y_{n+1} \\ -2z_{n+1} \end{pmatrix}, \quad u''_t = \begin{pmatrix} -2e''_{n+1} \\ 2f''_{n+1} \\ -2g''_{n+1} \\ 2h''_{n+1} \\ 2y''_{n+1} \\ -2z''_{n+1} \end{pmatrix}, \quad P_{1,n+1} = \begin{pmatrix} 2f_{n+1} \\ 2e_{n+1} \\ h_{n+1} \\ g_{n+1} \\ z_{n+1} \\ y_{n+1} \end{pmatrix}, \quad P''_{1,n+1} = \begin{pmatrix} 2f''_{n+1} \\ 2e''_{n+1} \\ h''_{n+1} \\ g''_{n+1} \\ z''_{n+1} \\ y''_{n+1} \end{pmatrix}.$$

We obtain the integrable coupling system of the soliton hierarchy (2.7) by using the Kronecker product. So this method is an efficient and new method for constructing the integrable coupling systems of soliton hierarchies.

When $n = 1$, the hierarchy (3.13) reduces to the first system (2.8) and

$$\begin{aligned}
u''_{1t} =& -\frac{1}{2}u_{1x}(\alpha' + \beta') - \frac{1}{2}u''_{1x}(\alpha + \beta) - \frac{1}{2}u_3\partial^{-1}(u_1u_4 + u_2u_5)(\alpha' - \beta') \\
& - \frac{1}{2}u''_5\partial^{-1}(u_1u_6 + u_2u_3)(\beta - \alpha) - \frac{1}{2}u''_3\partial^{-1}(u_1u_4 + u_2u_5)(\alpha - \beta) \\
& - \frac{1}{2}u_3\partial^{-1}(u_1u''_4 + u''_1u_4 + u_2u''_5 + u''_2u_5)(\alpha - \beta) - \frac{1}{2}u_5\partial^{-1}(u_1u_6 + u_2u_3)(\beta' - \alpha') \\
& - \frac{1}{2}u_5\partial^{-1}(u_1u''_6 + u''_1u_6 + u_2u''_3 + u''_2u_3)(\beta - \alpha), \\
u''_{2t} =& -\frac{1}{2}u_{2x}(\alpha' + \beta') - \frac{1}{2}u''_{2x}(\alpha + \beta) + \frac{1}{2}u_4\partial^{-1}(u_1u_6 + u_2u_3)(\beta' - \alpha') \\
& + \frac{1}{2}u''_6\partial^{-1}(u_1u_4 + u_2u_5)(\alpha - \beta) + \frac{1}{2}u''_4\partial^{-1}(u_1u_6 + u_2u_3)(\beta - \alpha) \\
& + \frac{1}{2}u_4\partial^{-1}(u_1u''_6 + u''_1u_6 + u_2u''_3 + u''_2u_3)(\beta - \alpha) \\
& + \frac{1}{2}u_6\partial^{-1}(u_1u_4 + u_2u_5)(\alpha' - \beta') + \frac{1}{2}u_6\partial^{-1}(u_1u''_4 + u''_1u_4 + u_2u''_5 + u''_2u_5)(\alpha - \beta), \\
u''_{3t} =& -u_{3x}\alpha' - u''_{3x}\alpha - u_1\partial^{-1}(u_1u_6 + u_2u_3)(\beta' - \alpha') \\
& - u''_1\partial^{-1}(u_1u_6 + u_2u_3)(\beta - \alpha) - u_1\partial^{-1}(u_1u''_6 + u''_1u_6 + u_2u''_3 + u''_2u_3)(\beta - \alpha), \\
u''_{4t} =& -u_{4x}\alpha' - u''_{4x}\alpha + u_2\partial^{-1}(u_1u_4 + u_2u_5)(\alpha' - \beta') \\
& + u''_2\partial^{-1}(u_1u_4 + u_2u_5)(\alpha - \beta) \\
& + u_2\partial^{-1}(u_1u''_4 + u''_1u_4 + u_2u''_5 + u''_2u_5)(\alpha - \beta), \\
u''_{5t} =& u_{5x}\beta' + u''_{5x}\beta + u_1\partial^{-1}(u_1u_4 + u_2u_5)(\alpha' - \beta') \\
& + u''_1\partial^{-1}(u_1u_4 + u_2u_5)(\alpha - \beta) + u_1\partial^{-1}(u_1u''_4 + u''_1u_4 + u_2u''_5 + u''_2u_5)(\alpha - \beta), \\
u''_{6t} =& u_{6x}\beta' + u''_{6x}\beta - u_2\partial^{-1}(u_1u_6 + u_2u_3)(\beta' - \alpha') \\
& - u''_2\partial^{-1}(u_1u_6 + u_2u_3)(\beta - \alpha) - u_2\partial^{-1}(u_1u''_6 + u''_1u_6 + u_2u''_3 + u''_2u_3)(\beta - \alpha).
\end{aligned}$$

4 Integrable Hierarchies of Lie Algebra $\mathfrak{so}(5)$

Compact real form $\mathfrak{so}(5)$ is defined as [13]

$$\mathfrak{so}(5) = \mathfrak{o}(5, R) = \{x \in \mathfrak{gl}(5, R) \mid x^t H + Hx = 0, H^t = H\}.$$

which means that Lie algebra $\mathfrak{so}(5)$ is constituted by 5×5 real antisymmetric matrix.

If set [13]

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_2 \\ 0 & I_2 & 0 \end{pmatrix}, \tag{4.14}$$

we can obtain the bases of Lie algebra $\mathfrak{so}(5)$

$$\begin{aligned} &E_1' = e_{22} - e_{44},\ E_2' = e_{33} - e_{55},\ E_3' = e_{32} - e_{45},\ E_4' = e_{23} - e_{54},\ E_5' = e_{52} - e_{43}, \\ &E_6' = e_{25} - e_{34},\ E_7' = e_{13} - e_{51},\ E_8' = e_{31} - e_{15},\ E_9' = e_{12} - e_{41},\ E_{10}' = e_{21} - e_{14}, \end{aligned} \tag{4.15}$$

where e_{ij} is a 5×5 matrix with 1 in the (i, j)-th position and zero elsewhere.

Consider an isospectral matrices

$$\begin{aligned} U_3 = &E_1'(1) + E_2'(1) + u_1 E_5'(0) + u_2 E_6'(0) + u_3 E_7'(0) + u_4 E_8'(0) \\ &+ u_5 E_9'(0) + u_6 E_{10}'(0), \end{aligned}$$

where

$$U_3 = \begin{pmatrix} 0 & u_5 & u_3 & -u_6 & -u_4 \\ u_6 & \lambda & 0 & 0 & u_2 \\ u_4 & 0 & \lambda & -u_2 & 0 \\ -u_5 & 0 & -u_1 & -\lambda & 0 \\ -u_3 & u_1 & 0 & 0 & -\lambda \end{pmatrix},$$

and

$$V_3 = aE_1' + bE_2' + cE_3' + dE_4' + eE_5' + fE_6' + gE_7' + hE_8' + yE_9' + zE_{10}',$$

where

$$V_3 = \begin{pmatrix} 0 & y & g & -z & -h \\ z & a & d & 0 & f \\ h & c & b & -f & 0 \\ -y & 0 & -e & -a & -c \\ -g & e & 0 & -d & -b \end{pmatrix}.$$

The stationary zero curvature representation $V_{3,x} = [U_3, V_3]$ gives

$$\begin{cases} a_x = u_6 y + u_2 e - u_5 z - u_1 f, \\ b_x = u_4 g + u_2 e - u_3 h - u_1 f, \\ c_x = u_4 y - u_5 h, \\ d_x = u_6 g - u_3 z, \\ e_x = -2\lambda e - u_3 y + u_1 a + u_5 g + u_1 b, \\ f_x = 2\lambda f - u_6 h - u_2 b + u_4 z - u_2 a, \\ g_x = -\lambda g + u_5 d + u_3 b + u_6 e - u_1 z, \\ h_x = \lambda h + u_2 y - u_6 c - u_4 b - u_5 f, \\ y_x = -\lambda y + u_5 a + u_3 c - u_4 e + u_1 h, \\ z_x = \lambda z - u_2 g - u_6 a - u_4 d + u_3 f. \end{cases} \tag{4.16}$$

Let

$$a = \sum_{i\geq 0} a_i \lambda^{-i},\ b = \sum_{i\geq 0} b_i \lambda^{-i},\ c = \sum_{i\geq 0} c_i \lambda^{-i},\ d = \sum_{i\geq 0} d_i \lambda^{-i},\ e = \sum_{i\geq 0} e_i \lambda^{-i},$$
$$f = \sum_{i\geq 0} f_i \lambda^{-i},\ g = \sum_{i\geq 0} g_i \lambda^{-i},\ h = \sum_{i\geq 0} h_i \lambda^{-i},\ y = \sum_{i\geq 0} y_i \lambda^{-i},\ z = \sum_{i\geq 0} z_i \lambda^{-i}.$$

Similarly, take the initial values

$$a_0 = \alpha,\ b_0 = \beta,\ c_0 = d_0 = e_0 = f_0 = g_0 = h_0 = y_0 = z_0 = 0.$$

From (4.16), we have

$$a_1 = b_1 = 0,\ c_1 = \partial^{-1} u_4 u_5 (\alpha - \beta),\ d_1 = \partial^{-1} u_3 u_6 (\beta - \alpha),\ e_1 = \frac{1}{2} u_1 (\alpha + \beta),$$
$$f_1 = \frac{1}{2} u_2 (\alpha + \beta),\ g_1 = u_3 \beta,\ h_1 = u_4 \beta,\ y_1 = u_5 \alpha,\ z_1 = u_6 \alpha,$$
$$e_2 = -\frac{1}{4} u_{1x} (\alpha + \beta) + \frac{1}{2} u_3 u_5 (\beta - \alpha),\ f_2 = \frac{1}{4} u_{2x} (\alpha + \beta) + \frac{1}{2} u_4 u_6 (\beta - \alpha),$$
$$g_2 = -u_{3x} \beta + u_5 \partial^{-1} u_3 u_6 (\beta - \alpha) + \frac{1}{2} u_1 u_6 (\beta - \alpha),$$
$$h_2 = u_{4x} \alpha + u_6 \partial^{-1} u_4 u_5 (\alpha - \beta) + \frac{1}{2} u_2 u_5 (\beta - \alpha),$$
$$y_2 = -u_{5x} \alpha + u_3 \partial^{-1} u_4 u_5 (\alpha - \beta) + \frac{1}{2} u_1 u_4 (\beta - \alpha),$$
$$z_2 = u_{6x} \alpha + u_4 \partial^{-1} u_3 u_6 (\beta - \alpha) + \frac{1}{2} u_2 u_3 (\beta - \alpha).$$

The zero curvature equation $U_{3,t} - V^n_{3,+x} + [U_3, V^n_{3,+}] = 0$ leads to the integrable hierarchies

$$u_t = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}_t = \begin{pmatrix} 2e_{n+1} \\ -2f_{n+1} \\ g_{n+1} \\ -h_{n+1} \\ y_{n+1} \\ -z_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 2f_{n+1} \\ 2e_{n+1} \\ 2h_{n+1} \\ 2g_{n+1} \\ 2z_{n+1} \\ 2y_{n+1} \end{pmatrix} = J_3 P_{3,n+1}. \quad (4.17)$$

From the recurrence relations (4.16), we have

$$P_{3,n+1} = \begin{pmatrix} l''_{11} & l''_{12} & l''_{13} & l''_{14} & l''_{15} & l''_{16} \\ l''_{21} & l''_{22} & l''_{23} & l''_{24} & l''_{25} & l''_{26} \\ l''_{31} & l''_{32} & l''_{33} & l''_{34} & l''_{35} & l''_{36} \\ l''_{41} & l''_{42} & l''_{43} & l''_{44} & l''_{45} & l''_{46} \\ l''_{51} & l''_{52} & l''_{53} & l''_{54} & l''_{55} & l''_{56} \\ l''_{61} & l''_{62} & l''_{63} & l''_{64} & l''_{65} & l''_{66} \end{pmatrix} P_{3,n} = L_3 P_{3,n},$$

where L_3 is a recurrence operator, and

$$\begin{aligned}
&l''_{11} = \frac{1}{2}\partial - u_2\partial^{-1}u_1, \;\; l''_{12} = u_2\partial^{-1}u_2, \;\; l''_{13} = \frac{1}{2}(u_6 - u_2\partial^{-1}u_3), \\
&l''_{14} = \frac{1}{2}u_2\partial^{-1}u_4, \\
&l''_{15} = \frac{1}{2}(u_4 - u_2\partial^{-1}u_5), \;\; l''_{16} = \frac{1}{2}u_2\partial^{-1}u_6, \;\; l''_{21} = -u_1\partial^{-1}u_1, \\
&l''_{22} = -\frac{1}{2}\partial^{-1} + u_1\partial^{-1}u_2), \\
&l''_{23} = -\frac{1}{2}u_1\partial^{-1}u_3, \;\; l''_{24} = \frac{1}{2}(u_5 + u_1\partial^{-1}u_4), \; l''_{25} = -\frac{1}{2}u_1\partial^{-1}u_5, \\
&l''_{26} = \frac{1}{2}(-u_3 + u_1\partial^{-1}u_6), \\
&l''_{31} = -u_4\partial^{-1}u_1 + u_5, \;\; l''_{32} = u_4\partial^{-1}u_2, \;\; l''_{33} = \partial - u_6\partial^{-1}u_5 - u_4\partial^{-1}u_3, \\
&l''_{34} = u_4\partial^{-1}u_4, \\
&l''_{35} = 0, \;\; l''_{36} = -u_2 + u_6\partial^{-1}u_4, \;\; l''_{41} = -u_3\partial^{-1}u_1, \;\; l''_{42} = u_3\partial^{-1}u_2 + u_6, \\
&l''_{43} = -u_3\partial^{-1}u_3, \\
&l''_{44} = -\partial + u_5\partial^{-1}u_6 + u_3\partial^{-1}u_4, \;\; l''_{45} = -u_5\partial^{-1}u_3 - u_1, \;\; l''_{46} = 0, \\
&l''_{51} = -u_6\partial^{-1}u_1 - u_3, \\
&l''_{52} = u_6\partial^{-1}u_2, \;\; l''_{53} = 0, \;\; l''_{54} = u_2 + u_4\partial^{-1}u_6, \;\; l''_{55} = \partial - u_6\partial^{-1}u_5 - u_4\partial^{-1}u_3, \\
&l''_{56} = u_6\partial^{-1}u_6, \;\; l''_{61} = -u_5\partial^{-1}u_1, \;\; l''_{62} = u_5\partial^{-1}u_2 - u_4, \;\; l''_{63} = -u_3\partial^{-1}u_5 + u_1, \\
&l''_{64} = 0, \;\; l''_{65} = -u_5\partial^{-1}u_5, \;\; l''_{66} = -\partial + u_5\partial^{-1}u_6 + u_3\partial^{-1}u_4.
\end{aligned}$$

When $n = 1$, the hierarchy (4.17) reduces to the first system

$$
\begin{aligned}
u_{1t} &= -\frac{1}{2}u_{1x}(\alpha+\beta) + u_3u_5(\beta-\alpha),\\
u_{2t} &= -\frac{1}{2}u_{2x}(\alpha+\beta) - u_4u_6(\beta-\alpha),\\
u_{3t} &= -u_{3x}\beta + u_5\partial^{-1}u_3u_6(\beta-\alpha) + \frac{1}{2}u_1u_6(\beta-\alpha),\\
u_{4t} &= -u_{4x}\alpha - u_6\partial^{-1}u_4u_5(\alpha-\beta) - \frac{1}{2}u_2u_5(\beta-\alpha),\\
u_{5t} &= -u_{5x}\alpha + u_3\partial^{-1}u_4u_5(\alpha-\beta) + \frac{1}{2}u_1u_4(\beta-\alpha),\\
u_{6t} &= -u_{6x}\alpha - u_4\partial^{-1}u_3u_6(\beta-\alpha) - \frac{1}{2}u_2u_3(\beta-\alpha).
\end{aligned}
$$

At the same time, we can easy to get another bases of Lie algebra $\mathfrak{so}(5)$

$$
\begin{aligned}
E_1'' &= e_{11} - e_{33},\ E_2'' = e_{22} - e_{44},\ E_3'' = e_{21} - e_{34},\ E_4'' = e_{12} - e_{43},\ E_5'' = e_{41} - e_{32},\\
E_6'' &= e_{14} - e_{23},\ E_7'' = e_{52} - e_{45},\ E_8'' = e_{25} - e_{54},\ E_9'' = e_{51} - e_{35},\ E_{10}'' = e_{15} - e_{53}.
\end{aligned} \tag{4.18}
$$

We consider time spectral matrix

$$
V_3' = aE_1'' + bE_2'' + cE_3'' + dE_4'' + eE_5'' + fE_6'' + gE_7'' + hE_8'' + yE_9'' + zE_{10}'',
$$

where

$$
V_3' = \begin{pmatrix} a & d & 0 & f & z\\ c & b & -f & 0 & h\\ 0 & -e & -a & -c & -y\\ e & 0 & -d & -b & -g\\ y & g & -z & -h & 0 \end{pmatrix}.
$$

We can easy to obtain the relation between V_3 and V_3' satisfy

$$
\varphi : V_3 \to V_3',\ \ \varphi(MV_3M^t) = V_3',
$$

where $M = \begin{pmatrix} 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 1&0&0&0&0 \end{pmatrix}$.

So we have the following spectral matrix

$$
\begin{aligned}
U_3' =& E_1''(1) + E_2''(1) + u_1E_5''(0) + u_2E_6''(0) + u_3E_7''(0) + u_4E_8''(0)\\
&+ u_5E_9''(0) + u_6E_{10}''(0),
\end{aligned}
$$

where

$$U'_3 = \begin{pmatrix} \lambda & 0 & 0 & u_2 & u_6 \\ 0 & \lambda & -u_2 & 0 & u_4 \\ 0 & -u_1 & -\lambda & 0 & -u_5 \\ u_1 & 0 & 0 & -\lambda & -u_3 \\ u_5 & u_3 & -u_6 & -u_4 & 0 \end{pmatrix}.$$

From the stationary zero curvature representation $V'_{3,x} = [U'_3, V'_3]$, we can get the same recurrence relations as the equations (4.16). So we can obtain the same integrable hierarchies, Hamiltonian operator and the recurrence operator. So there are the same integrable hierarchies of two different bases associated with Lie algebra $\mathfrak{so}(5)$.

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