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Stability in the higher derivative Abelian gauge field theories

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Abstract

We present an exact derivation of conserved tensors associated to the higher-order symmetries in the higher derivative Abelian gauge field theories. In our model, the wave operator of the derived theory is a n-th order polynomial expressed in terms of the usual Maxwell operator. Relying on this formalism and utilizing the extension of Noether's theorem, we acquire a series of conserved second-rank tensors which includes the standard canonical energy-momentum tensors. Moreover, with the aid of auxiliary fields, we succeed in obtaining the relations between the root decomposition of characteristic polynomial of the wave operator and the conserved energy-momentum tensors in the context of another equivalent lower-order representation. Under the certain conditions, although the canonical energy of the higher derivative dynamics is unbounded from below, the 00-component of the linear combination of these conserved quantities is bounded. By this reason, the original derived theory is considered stable. Finally, as an instructive example, we elaborate the third-order derived system and analyze the stabilities in different cases of root decomposition of the characteristic polynomial extensively.

1. Introduction

The research of the higher-order derivative systems dates back to the nineteenth century in Ostrogradsky's pioneering work [1] and has gathered a lot of attention along the years, mostly in the areas of effective low energy theories, cosmological behaviors and the modified gravities

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[2–9]. Historically, the first model of a higher-order derivative field theory was the Podolsky's generalized electrodynamics which is an extension of Maxwell's U(1) gauge theory [10,11]. The advantage of this modified form of the Maxwell's theory is that in this formulation, we are able to get rid of the infinities such as the electron self-energy and the vacuum polarization current associated to point particles [12]. Later on, Pais and Uhlenbeck proposed a class of classical higher derivative harmonic oscillators [13] which involve some novel physics related to the ultraviolet behavior and will result in a renormalizable quantum field theory [14]. In all these models, the Lagrangians yielding the higher-order equations of motion require more initial conditions than in usual dynamical systems. The standard analysis for dealing with these theories on Hamiltonian level is provided by Ostrogradski canonical approach [15–18]. Unfortunately, the Hamiltonian function obtained in such a way contains terms linear in momenta, which implies that the energy of the system can be lowered without any bound by increasing the momenta to large positive or negative values. Generally speaking, due to the presence of the higher derivatives, the existence of unbounded kinetic terms inevitably leads to runaway solutions if interactions are turned on. By these reasons, the higher-order dynamical systems may be unstable compared to the usual first-order Lagrangian [19–21]. However, one also can find out certain conditions that allow for stability, as it is the case for the Pais-Uhlenbeck oscillator which has been served as a toy model to understand several profound issues related to Ostrogradsky instability.

On the other hand, in contrast to the classical regime, quantizing the higher derivative dynamics imposes even more constraints. It is thought that the higher-order theories would possess propagators having poles with non-positive residues which give rise to the appearance of ghost states. At the quantum level, these ghost states have non-positive norms and due to this, they will violate the causality and spoil the unitarity evolution of the quantum theory which is unacceptable physically. In order to circumvent these problems in a physical allowed sector, various motivations and techniques have been put forward to avoid the Ostrogradsky ghosts in different higher derivative models [22–30]. For instance, in the context of Pais-Uhlenbeck's harmonic oscillator, it was argued by Raidal and Veermae that for the purpose of the energy spectrum of the theory be bounded, the ghost degrees of freedom should be necessarily complex [31]. In this sense, the resulting complex system can be consistently quantized by the rules of canonical quantization which yields a stable and unitary quantum correspondence and no Ostrogradsky instability. While in the usual Lee-Wick theories, by means of the polynomials with complex conjugate poles, it is possible to construct a unitary S-matrix of gravitational excitations to remove the negative effects of the ghosts [32,33]. Besides, by introducing form-factors with an analytic dependence on the propagating momenta [34], we are able to avoid the unphysical ghosts and this method preserves all fundamental properties of a quantum field theory including the basic positive definite Hamiltonian. Furthermore, in the non-Hermitian, $\mathcal{PT}$-symmetric model, i.e., symmetric under combined parity reflection and time reversal, it is essential to modify the dynamical inner product instead of using the standard Dirac inner product [35–38]. In this manner, one explicitly obtains the self-adjoint Hamiltonian and its ghost state is reinterpreted as an ordinary quantum state with positive $\mathcal{PT}$ norm which gives us the standard probabilistic interpretation.

Recently, a special class of linear higher derivative systems is worked out in [39] as an alternative approach to the problem of Ostrogradsky instability. For concreteness, the operators of the dynamic equations in these theories are supposed to be factorable in terms of a pair of distinct second-order operators satisfying some certain conditions. In this way, with the help of auxiliary fields absorbing the higher derivatives, it is direct to establish two equivalent systems that can be thought of as two different representations of the same theory. On the other hand, the famous Noether's theorem tells us that if the action functional is preserved under the spacetime translations, the system is equipped with canonical energy-momentum tensors and the 00-component is of particular importance since it has the sense of energy density of the theory. Especially, for the models of factorable type, we are capable of acquiring two families of integrals of motion which may be either bounded or unbounded depending on the specific values of parameters. Now as is explained in [39], the stability of the higher derivative system can be ensured if the 00-component in this family is positive definite, though the Noether's canonical energy usually is unbounded from below. So far, the efforts of this viewpoint have been focused mostly on various known classical models such as Pais-Uhlenbeck's harmonic oscillators, the higher derivative scalar fields and Podolsky's generalized electrodynamics.

After that, a more general and systematic method was carried out as a guide to investigate the stabilities in a wide class of higher derivative systems named derived type theories [40–44]. To explain that, these derived theories are based on simpler free primary models whose equations of motion only involve first- and second-order differential operators without the higher derivatives. In this setting, the wave operator which determines the dynamic equation of the higher derivative system is a polynomial in terms of the primary wave operator in the lower-order free theory. Then, every symmetry of the primary theory enables us to construct a n-parametric series of symmetries of the derived theory if the order of the characteristic polynomial of the wave operator is n. More importantly, these symmetries are connected to n independent conserved quantities from the perspective of more general correspondence between symmetries and conservation laws which is initially established by the Lagrange anchor [45,46]. Remarkably, when the primary wave operator commutes with the spacetime translation generators, the derived theory produces a n-parametric series of conserved second-rank tensors $(T_k)^{\mu}_{\nu}, k = 0, 1, ..., n-1$. In particular, the $k = 0$ term $(T_0)^{\mu}_{\nu}$ corresponds to the usual canonical energy density of the higher derivative system. Now although the canonical energy is unbounded due to the higher derivative nature, the linear combination of these tensors $(T_k)^{\mu}_{\nu}$ would give rise to bounded conserved charge. It is quite crucial to stress that this bounded quantity will stabilize the classical dynamics of derived model at free level, which also persists at quantum level. Moreover, as demonstrated at length in [41], when these conserved tensors are bounded in two different free theories, the inclusion of consistent interactions will not spoil the stability of the coupling system, at least at perturbative level.

The paper is organized as follows. In section 2, we simply illustrate the basic ingredients in the higher derivative Abelian gauge field theories involving the primary free model and the wave operator. Subsequently, we give a detailed derivation of the second-rank conserved tensors based on the higher-order symmetries and investigate the issue of the stability in this derived system. In section 3, according to the different root decompositions of the characteristic polynomials, we set up the formulae of the conserved tensors depending upon the real and complex roots by means of auxiliary fields. Then as an application, section 4 is devoted to the full analysis of the stabilities in the third-order derived system. The final section of this paper includes some concluding remarks and discussions.

2. Higher derivative Abelian gauge field theory

2.1. Conserved tensors

Let us start with the Lagrangian density of usual Maxwell electromagnetic theory which is described by the Abelian gauge fields A_μ in (1+3)-dimensional spacetime as follows

$$S = -\frac{1}{4}\int F^{\mu\nu}F_{\mu\nu}d^4x \tag{2.1}$$

here the metric $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is used to rise and lower the multi-indices. The dynamic equations of motion of (2.1) are given by

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = 0 \tag{2.2}$$

if we set

$$W_{\mu\nu} = \delta_{\mu\nu}\Box - \partial_\nu\partial_\mu \tag{2.3}$$

as the primary wave operator, then (2.3) defines the primary free field equations [41]

$$W_{\mu\nu}A^\nu = 0 \tag{2.4}$$

this primary model allows us to establish a more general Lagrangian of the higher-order extension of Maxwell's theory

$$S = \int d^4x A^\mu M_{\mu\nu} A^\nu \tag{2.5}$$

here M is termed as wave operator which is a polynomial in the formal variable W

$$M = a_n W^n + \ldots\ldots + a_2 W^2 + a_1 W + a_0 \tag{2.6}$$

and the equations of motion contain terms up to the $2n$-th time derivative

$$\sum_{l=0}^{n} a_l W^l_{\mu\nu} A^\nu = 0 \tag{2.7}$$

It is well known that the symmetry of a field theory plays a very important role in modern physics which has been regarded as one of the most powerful tools to analyze the behaviors of the gauge systems. Apparently, we should notice that (2.7) is invariant with respect to the local gauge transformation

$$\delta_\varepsilon A^\mu = R^\mu(\partial)\varepsilon, \qquad R^\mu = \partial^\mu \tag{2.8}$$

here $\varepsilon = \varepsilon(x)$ is a function of space-time coordinates and the gauge generators $R = (R^\mu)$ ought to be well understood as the right null-vectors of M in the sense of the matrix differential operators, namely we have

$$M(\partial)R(\partial) \equiv 0 \tag{2.9}$$

more generally, a symmetry of the linear theory (2.5) is a matrix differential operator L obeying the condition [40,41]

$$M(\partial)L(\partial) = Q(\partial)M(\partial) \tag{2.10}$$

where Q is some matrix differential operator. In this way, the appearance of the symmetry operator L leads to the following linear transformation of the fields which preserves the dynamical equations on-shell

$$\delta_\xi A = \xi L(\partial)A, \qquad \delta_\xi MA = \xi Q(\partial)MA \approx 0 \tag{2.11}$$

with constant ξ being the infinitesimal transformation parameter. A simple observation shows that the restriction (2.10) possesses a large number of trivial solutions in the form of

$$L(\partial) = U(\partial)M(\partial) + R(\partial)V(\partial) \tag{2.12}$$

here $U(\partial)$ and $V(\partial)$ are some matrix differential operators. As a matter of fact, these trivial symmetries exist in every free field theory and do not convey any significant information about the dynamics of the system. Hence we ignore them in the discussion below.

Of particular relevance to the current study is the spacetime translation invariance of the primary wave operator, or in other words, the translation generators ∂_μ commute with the primary wave operator as follows

$$\left[\partial_\mu, W\right] = 0 \tag{2.13}$$

this fact signifies that the derived theory (2.5) enjoys the following higher-order symmetries

$$\delta_\varepsilon A^\mu = \varepsilon^\nu \partial_\nu (W^k A)^\mu \tag{2.14}$$

especially, $k = 0$ corresponds to the spacetime translation invariance of the action functional of the derived model. Now the extension of Noether's theorem implies each continuous symmetry of the action in (2.14) determines a conserved quantity

$$\partial_\mu (\Theta^k)^\mu_\nu = (\partial_\nu (W^k A)^\mu)(MA)_\mu \tag{2.15}$$

actually as pointed out in [40,41], these conserved tensors are linearly independent if the symmetry generators (2.14) are linearly independent modulo trivial symmetries. Thus in the present situation, from (2.6) we claim that the maximal number of independent conserved tensors equals to the order of the characteristic polynomial of the wave operator. More precisely, the $k = 0$ term in (2.15) corresponds to the Noether's canonical energy-momentum tensors, while the other $1 \le k \le n - 1$ terms are different integrals of motion linked to the higher-order symmetries of the gauge fields A_μ.

To find out the explicit expressions of $(\Theta^k)^\mu_\nu$, first of all, a simple calculation shows that

$$(W^k A)_\mu = \Box^{k-1} \partial^\rho F_{\rho\mu}, \qquad k = 1, 2, ..., n - 1 \tag{2.16}$$

for convenience, we define

$$\Box^{-1} \partial^\rho F_{\rho\mu} := A_\mu \tag{2.17}$$

in this way, the dynamical equations of motion (2.7) turn out to be

$$\sum_{l=0}^{n} a_l \Box^{l-1} \partial_\mu F^{\mu\nu} = 0 \tag{2.18}$$

subsequently, if $l = k$ in (2.15), it is clear to see that

$$(\partial_\nu (W^k A)^\mu)(W^k A)_\mu = \frac{1}{2} \partial_\nu (\Box^{k-1} \partial^\rho F_{\rho\lambda} \Box^{k-1} \partial_\tau F^{\tau\lambda}) \tag{2.19}$$

on the other hand, when $l \ge k + 1$, in view of

$$\partial_\nu \partial^\rho F_{\rho\mu} - \partial_\mu \partial^\rho F_{\rho\nu} = \Box F_{\nu\mu} \tag{2.20}$$

there is no difficulty in evaluating

$$\begin{aligned}
&(\partial_\nu (W^k A)^\mu)(W^l A)_\mu \\
=&(\partial_\nu \Box^{k-1} \partial^\rho F_{\rho\mu}) \Box^{l-1} \partial_\lambda F^{\lambda\mu} \\
=&\Box^k F_{\nu\mu} \Box^{l-1} \partial_\lambda F^{\lambda\mu} + (\partial_\mu \Box^{k-1} \partial^\rho F_{\rho\nu}) \Box^{l-1} \partial_\lambda F^{\lambda\mu} \\
=&\partial_\lambda (\Box^k F_{\nu\mu} \Box^{l-1} F^{\lambda\mu}) - (\Box^k \partial_\lambda F_{\nu\mu}) \Box^{l-1} F^{\lambda\mu} + \partial_\mu (\Box^{k-1} \partial^\rho F_{\rho\nu} \Box^{l-1} \partial_\lambda F^{\lambda\mu})
\end{aligned} \tag{2.21}$$

then making using of

$$\partial_\lambda F_{\nu\mu} + \partial_\nu F_{\mu\lambda} + \partial_\mu F_{\lambda\nu} = 0 \tag{2.22}$$

and taking into account of the symmetry among the indices λ, μ, one infers that

$$-(\Box^k \partial_\lambda F_{\nu\mu})\Box^{l-1} F^{\lambda\mu} = \frac{1}{2}(\Box^k \partial_\nu F_{\mu\lambda})\Box^{l-1} F^{\lambda\mu} \tag{2.23}$$

at this stage, if $l = k+1$ we easily get

$$\frac{1}{2}(\Box^k \partial_\nu F_{\mu\lambda})\Box^k F^{\lambda\mu} = \frac{1}{4}\partial_\nu(\Box^k F_{\mu\lambda}\Box^k F^{\lambda\mu}) \tag{2.24}$$

while $l \geq k+2$, applying the general procedure of integration by parts together with (2.22), we simply have

$$\begin{aligned}
&\frac{1}{2}(\Box^k \partial_\nu F_{\mu\lambda})\Box^{l-1} F^{\lambda\mu} \\
=&\frac{1}{2}\partial_\nu(\Box^k F_{\mu\lambda}\Box^{l-1} F^{\lambda\mu}) - \frac{1}{2}\Box^k F_{\mu\lambda}\Box^{l-1}\partial_\nu F^{\lambda\mu} \\
=&\frac{1}{2}\partial_\nu(\Box^k F_{\mu\lambda}\Box^{l-1} F^{\lambda\mu}) + \Box^k F^{\mu\lambda}\Box^{l-1}\partial_\mu F_{\nu\lambda} \\
=&\frac{1}{2}\partial_\nu(\Box^k F_{\mu\lambda}\Box^{l-1} F^{\lambda\mu}) + \partial_\mu(\Box^k F^{\mu\lambda}\Box^{l-1} F_{\nu\lambda}) - (\partial_\mu\Box^k F^{\mu\lambda})\Box^{l-1} F_{\nu\lambda}
\end{aligned} \tag{2.25}$$

as well as

$$\begin{aligned}
&-(\Box^k \partial_\mu F^{\mu\lambda})\Box^{l-1} F_{\nu\lambda} \\
=&-(\Box^k \partial_\mu F^{\mu\lambda})\Box^{l-2}(\partial_\nu\partial^\rho F_{\rho\lambda} - \partial_\lambda\partial^\rho F_{\rho\nu}) \\
=&-\partial_\nu(\Box^k \partial_\mu F^{\mu\lambda}\Box^{l-2}\partial^\rho F_{\rho\lambda}) + (\Box^k\partial_\nu\partial_\mu F^{\mu\lambda})\Box^{l-2}\partial^\rho F_{\rho\lambda} + \partial_\lambda(\Box^k\partial_\mu F^{\mu\lambda}\Box^{l-2}\partial^\rho F_{\rho\nu})
\end{aligned} \tag{2.26}$$

furthermore, utilizing (2.20) and after a straightforward computation, one certainly obtains

$$\begin{aligned}
&(\Box^k \partial_\nu\partial_\mu F^{\mu\lambda})\Box^{l-2}\partial^\rho F_{\rho\lambda} \\
=&(\Box^{k+1} F_{\nu\lambda})\Box^{l-2}\partial_\rho F^{\rho\lambda} + (\Box^k\partial_\lambda\partial^\mu F_{\mu\nu})\Box^{l-2}\partial_\rho F^{\rho\lambda} \\
=&\partial_\rho(\Box^{k+1} F_{\nu\lambda}\Box^{l-2} F^{\rho\lambda}) - (\Box^{k+1}\partial_\rho F_{\nu\lambda})\Box^{l-2} F^{\rho\lambda} + \partial_\lambda(\Box^k\partial^\mu F_{\mu\nu}\Box^{l-2}\partial_\rho F^{\rho\lambda}) \\
=&\partial_\rho(\Box^{k+1} F_{\nu\lambda}\Box^{l-2} F^{\rho\lambda}) + \frac{1}{2}\Box^{k+1}\partial_\nu F_{\lambda\rho}\Box^{l-2} F^{\rho\lambda} + \partial_\lambda(\Box^k\partial^\mu F_{\mu\nu}\Box^{l-2}\partial_\rho F^{\rho\lambda})
\end{aligned} \tag{2.27}$$

comparing (2.25), (2.26) to (2.27) and employing a recursive algorithm, we acquire the following equality

$$\begin{aligned}
&\frac{1}{2}(\Box^k \partial_\nu F_{\mu\lambda})\Box^{l-1} F^{\lambda\mu} \\
=&\sum_{i=0}^{l-k-2}\Bigg(\frac{1}{2}\partial_\nu(\Box^{k+i} F_{\mu\lambda}\Box^{l-1-i} F^{\lambda\mu}) + 2\partial_\mu(\Box^{k+i} F^{\mu\lambda}\Box^{l-1-i} F_{\nu\lambda}) \\
&-\partial_\nu(\Box^{k+i}\partial_\mu F^{\mu\lambda}\Box^{l-2-i}\partial^\rho F_{\rho\lambda}) + 2\partial_\mu(\Box^{k+i}\partial_\lambda F^{\lambda\mu}\Box^{l-2-i}\partial^\rho F_{\rho\nu})\Bigg) \\
&+\frac{1}{2}\Box^{l-1}\partial_\nu F^{\lambda\rho}\Box^k F_{\rho\lambda}
\end{aligned} \tag{2.28}$$

particularly, in the above derivation of (2.28), we have used the identities

$$
\begin{aligned}
&\sum_{i=0}^{l-k-2} \partial_\rho(\Box^{k+1+i} F_{\nu\lambda}\Box^{l-2-i} F^{\rho\lambda}) = \sum_{j=0}^{l-k-2} \partial_\rho(\Box^{k+j} F^{\rho\lambda}\Box^{l-1-j} F_{\nu\lambda}),\\
&\sum_{i=0}^{l-k-2} \partial_\mu(\Box^{k+i}\partial^\lambda F_{\lambda\nu}\Box^{l-2-i}\partial_\rho F^{\rho\mu}) = \sum_{j=0}^{l-k-2} \partial_\mu(\Box^{k+j}\partial_\lambda F^{\lambda\mu}\Box^{l-2-j}\partial^\rho F_{\rho\nu})
\end{aligned}
\tag{2.29}
$$

this can be done easily by performing $j = l - k - 2 - i$. Then, we proceed by putting all of these results together and formulate the higher-order conserved tensors in the form of

$$
(\Theta^k)^\mu_\nu = \sum_{l=k+1}^{n} a_l (t_1^{k,l})^\mu_\nu + \frac{1}{2}\delta^\mu_\nu a_k(\Box^{k-1}\partial^\rho F_{\rho\lambda}\Box^{k-1}\partial_\tau F^{\tau\lambda}) + \sum_{l=0}^{k-1} a_l (t_2^{k,l})^\mu_\nu \tag{2.30}
$$

here we set $a_{-1} = a_{n+1} = 0$ and

$$
\begin{aligned}
l = k+1 : (t_1^{k,l})^\mu_\nu =& \Box^k F_{\nu\lambda}\Box^k F^{\mu\lambda} - \frac{1}{4}\delta^\mu_\nu(\Box^k F_{\rho\lambda})\Box^k F^{\rho\lambda} + \Box^{k-1}\partial^\rho F_{\rho\nu}\Box^k\partial_\tau F^{\tau\mu},\\
l \geq k+2 : (t_1^{k,l})^\mu_\nu =& \Box^k F_{\nu\lambda}\Box^{l-1} F^{\mu\lambda} - \frac{1}{4}\delta^\mu_\nu(\Box^k F_{\rho\lambda})\Box^{l-1} F^{\rho\lambda}\\
&+ \sum_{i=0}^{l-k-2}(\Box^{k+i} F^{\mu\lambda}\Box^{l-1-i} F_{\nu\lambda} - \frac{1}{4}\delta^\mu_\nu\Box^{k+i} F_{\rho\lambda}\Box^{l-1-i} F^{\rho\lambda}\\
&+ \Box^{k+i}\partial_\lambda F^{\lambda\mu}\Box^{l-2-i}\partial^\rho F_{\rho\nu} - \frac{1}{2}\delta^\mu_\nu\Box^{k+i}\partial_\tau F^{\tau\lambda}\Box^{l-2-i}\partial^\rho F_{\rho\lambda})\\
&+ \Box^{k-1}\partial^\rho F_{\rho\nu}\Box^{l-1}\partial_\tau F^{\tau\mu}
\end{aligned}
\tag{2.31}
$$

this is an important step for our subsequent analysis.

While in the case of $l \leq k-1$, after integration by parts, one simply gets

$$
(\partial_\nu(W^k A)^\mu)(W^l A)_\mu = \partial_\nu((W^k A)^\mu(W^l A)_\mu) - (W^k A)^\mu\partial_\nu(W^l A)_\mu \tag{2.32}
$$

in this way, taking a similar tactic in the case of $l > k$, it is not difficult to re-express the $(W^k A)^\mu\partial_\nu(W^l A)_\mu$ as total derivative terms and analogously, the exact expressions of $(t_2^{k,l})^\mu_\nu$ are given by

$$
\begin{aligned}
l = k-1 : (t_2^{k,l})^\mu_\nu =& \delta^\mu_\nu\Box^{k-1}\partial^\rho F_{\rho\lambda}\Box^{k-2}\partial_\tau F^{\tau\lambda} - \Box^{k-1} F_{\nu\lambda}\Box^{k-1} F^{\mu\lambda}\\
&+ \frac{1}{4}\delta^\mu_\nu(\Box^{k-1} F_{\rho\lambda})\Box^{k-1} F^{\rho\lambda} - \Box^{k-2}\partial^\rho F_{\rho\nu}\Box^{k-1}\partial_\tau F^{\tau\mu},\\
l \leq k-2 : (t_2^{k,l})^\mu_\nu =& \delta^\mu_\nu\Box^{k-1}\partial^\rho F_{\rho\lambda}\Box^{l-1}\partial_\tau F^{\tau\lambda} - \Box^l F_{\nu\lambda}\Box^{k-1} F^{\mu\lambda}\\
&+ \frac{1}{4}\delta^\mu_\nu(\Box^l F_{\rho\lambda})\Box^{k-1} F^{\rho\lambda}\\
&- \sum_{i=0}^{k-l-2}(\Box^{l+i} F^{\mu\lambda}\Box^{k-1-i} F_{\nu\lambda} - \frac{1}{4}\delta^\mu_\nu\Box^{l+i} F_{\rho\lambda}\Box^{k-1-i} F^{\rho\lambda}\\
&+ \Box^{l+i}\partial_\lambda F^{\lambda\mu}\Box^{k-2-i}\partial^\rho F_{\rho\nu} - \frac{1}{2}\delta^\mu_\nu\Box^{l+i}\partial_\tau F^{\tau\lambda}\Box^{k-2-i}\partial^\rho F_{\rho\lambda})\\
&- \Box^{l-1}\partial^\rho F_{\rho\nu}\Box^{k-1}\partial_\tau F^{\tau\mu}
\end{aligned}
\tag{2.33}
$$

2.2. Stability

Following the above approach, once obtaining the explicit expressions of $(\Theta^k)^\mu_\nu$, let us investigate the problem of stability in (2.5) by introducing n independent parameters

$$\beta_0, \quad \beta_1, \quad ......, \quad \beta_{n-1} \tag{2.34}$$

and the total series of second-rank energy-momentum tensors of the derived theory under study reads as [40,41]

$$\Theta^\mu_\nu(a,\beta,A)=\sum_{k=0}^{n-1}\beta_k(\Theta^k)^\mu_\nu \tag{2.35}$$

this family of conserved tensors consists of the canonical energy-momentum $(\Theta^0)^\mu_\nu$ of the derived model (2.5) when $\beta_0=1, \beta_1=......=\beta_{n-1}=0$, though it is always unbounded below. In the light of this, the 00-component of $\Theta^\mu_\nu(a,\beta,A)$ captures the meanings of the energy density of the higher derivative system and hence the total energy of the derived theory is provided by the integral

$$E=\int d^3x\Theta^0_0 \tag{2.36}$$

as far as the issue of stability is concerned, our strategy here is to guarantee the positive definite of the total energy which is admissible just by the requirement $\Theta^0_0\geq 0$.

Currently, based on the present results, choosing $\mu=\nu=0$ in (2.30) and with the aid of metric $g_{\mu\nu}=\mathrm{diag}(1,-1,-1,-1)$, it is appropriate to cast the 00-component of $(\Theta^k)^\mu_\nu$ in the form of

$$\begin{aligned}(\Theta^k)^0_0=&-\frac{1}{4}\sum_{l=k+1}^{n}a_l\Box^k F_{\rho\lambda}\Box^{l-1}F_{\rho\lambda}+\frac{1}{4}\sum_{l=0}^{k-1}a_l\Box^l F_{\rho\lambda}\Box^{k-1}F_{\rho\lambda}\\&-\frac{1}{2}\sum_{l=k+2}^{n}\sum_{i=0}^{l-k-2}a_l(\frac{1}{2}\Box^{k+i}F_{\rho\lambda}\Box^{l-1-i}F_{\rho\lambda}-\Box^{k+i}\partial^\mu F_{\mu\lambda}\Box^{l-2-i}\partial^\rho F_{\rho\lambda})\\&+\frac{1}{2}\sum_{l=0}^{k-2}\sum_{i=0}^{k-l-2}a_l(\frac{1}{2}\Box^{l+i}F_{\rho\lambda}\Box^{k-1-i}F_{\rho\lambda}-\Box^{l+i}\partial^\mu F_{\mu\lambda}\Box^{k-2-i}\partial^\rho F_{\rho\lambda})\\&+\sum_{l=k+1}^{n}a_l\Box^{k-1}\partial^\rho F_{\rho 0}\Box^{l-1}\partial_\tau F^{\tau 0}+\sum_{l=0}^{k-1}a_l\Box^{k-1}\partial^\rho F_{\rho\lambda}\Box^{l-1}\partial_\tau F^{\tau\lambda}\\&-\sum_{l=0}^{k-1}a_l\Box^{l-1}\partial^\rho F_{\rho 0}\Box^{k-1}\partial_\tau F^{\tau 0}+\frac{1}{2}a_k(\Box^{k-1}\partial^\rho F_{\rho\lambda}\Box^{k-1}\partial_\tau F^{\tau\lambda})\end{aligned} \tag{2.37}$$

at first glance, the expression of $(\Theta^k)^0_0$ is a little more complicated, however, this can be simplified by noting that if we set $\nu=0$ in (2.18), the dynamic equation of motion becomes

$$\sum_{l=k+1}^{n}a_l\Box^{l-1}\partial_\tau F^{\tau 0}=-a_k\Box^{k-1}\partial_\tau F^{\tau 0}-\sum_{l=0}^{k-1}a_l\Box^{l-1}\partial_\tau F^{\tau 0} \tag{2.38}$$

inserting this relation back into $(\Theta^k)^0_0$, one discovers that the last four terms in (2.37) could be rewritten as

$$-\sum_{l=0}^{k-1} a_l \Box^{k-1}\partial^\rho F_{\rho\lambda}\Box^{l-1}\partial^\tau F_{\tau\lambda} - \frac{1}{2}a_k(\Box^{k-1}\partial^\rho F_{\rho\lambda}\Box^{k-1}\partial^\tau F_{\tau\lambda}) \tag{2.39}$$

which permits us to express the total energy density in a more concise and compact way

$$\Theta^0_0 = \sum_{k=0}^{n-1}\beta_k(\Theta^k)^0_0 = \sum_{i,j=0}^{n-1}\left(A_{ij}(a,\beta)\Box^i F_{\rho\lambda}\Box^j F_{\rho\lambda} + B_{ij}(a,\beta)\Box^i\partial^\rho F_{\rho\lambda}\Box^j\partial^\tau F_{\tau\lambda}\right) \tag{2.40}$$

here $A_{ij}(a,\beta), B_{ij}(a,\beta)$ are polynomial functions of the variables a_l, β_k and without loss of generality, these elements are assumed to be symmetric with respect to the indices i, j, namely

$$A_{ij}(a,\beta) = A_{ji}(a,\beta), \qquad B_{ij}(a,\beta) = B_{ji}(a,\beta) \tag{2.41}$$

inspecting the formula (2.40), we are aware of the fact that the 00-component of energy-momentum density is a pure quadratic form of the formal variables $\Box^i F_{\rho\lambda}, \Box^i\partial^\rho F_{\rho\lambda}$, therefore Θ^0_0 is positive if

$$A_{ij}(a,\beta), \quad B_{ij}(a,\beta) \quad \text{are all positive definite matrices} \tag{2.42}$$

in a conclusion, as long as the coefficients a_l and parameters β_k satisfy these positive definite conditions, the original free Abelian derived theory (2.5) admits bounded conserved quantities. Thus the higher derivative dynamics is considered stable, though its canonical energy is unbounded from below.

Our goal now is to deduce the explicit expressions of $A_{ij}(a,\beta)$ and $B_{ij}(a,\beta)$, initially noting that

$$\sum_{k=0}^{n-1}\sum_{l=k+2}^{n}\sum_{i=0}^{l-k-2}\beta_k a_l \Box^{k+i}F_{\rho\lambda}\Box^{l-1-i}F_{\rho\lambda} = \sum_{k=0}^{n-1}\sum_{l=k+2}^{n}\sum_{j=k}^{l-2}\beta_k a_l \Box^j F_{\rho\lambda}\Box^{l+k-1-j}F_{\rho\lambda} \tag{2.43}$$

then for specific k, it is reasonable to list the summations in (2.43) for every $k+2 \le l \le n$, these are

$$\begin{aligned}
l = k+2: &\quad \beta_k a_{k+2}(\Box^k F_{\rho\lambda}\Box^{k+1}F_{\rho\lambda}),\\
l = k+3: &\quad \beta_k a_{k+3}(\Box^k F_{\rho\lambda}\Box^{k+2}F_{\rho\lambda} + \Box^{k+1}F_{\rho\lambda}\Box^{k+1}F_{\rho\lambda}),\\
l = k+4: &\quad \beta_k a_{k+4}(\Box^k F_{\rho\lambda}\Box^{k+3}F_{\rho\lambda} + \Box^{k+1}F_{\rho\lambda}\Box^{k+2}F_{\rho\lambda} + \Box^{k+2}F_{\rho\lambda}\Box^{k+1}F_{\rho\lambda}),\\
l = k+5: &\quad \beta_k a_{k+5}(\Box^k F_{\rho\lambda}\Box^{k+4}F_{\rho\lambda} + \Box^{k+1}F_{\rho\lambda}\Box^{k+3}F_{\rho\lambda} + \Box^{k+2}F_{\rho\lambda}\Box^{k+2}F_{\rho\lambda}\\
&\qquad + \Box^{k+3}F_{\rho\lambda}\Box^{k+1}F_{\rho\lambda}),\\
&\ldots\ldots
\end{aligned} \tag{2.44}$$

following this idea, the coefficients of $\Box^k F_{\rho\lambda}\Box^{k+j}F_{\rho\lambda}$, $j \ge 0$ in (2.43) are given by

$$\begin{aligned}
&\Box^k F_{\rho\lambda}\Box^k F_{\rho\lambda}: \quad \beta_{k-1}a_{k+2}+\beta_{k-2}a_{k+3}+\beta_{k-3}a_{k+4}+\ldots\ldots=\sum_{i=1}^{k}\beta_{k-i}a_{k+i+1},\\
&\Box^k F_{\rho\lambda}\Box^{k+1} F_{\rho\lambda}: \quad \beta_{k}a_{k+2}+2\beta_{k-1}a_{k+3}+2\beta_{k-2}a_{k+4}+\ldots\ldots\\
&\qquad\qquad =\beta_k a_{k+2}+2\sum_{i=1}^{k}\beta_{k-i}a_{k+i+2},\\
&\Box^k F_{\rho\lambda}\Box^{k+2} F_{\rho\lambda}: \quad \beta_{k}a_{k+3}+2\beta_{k-1}a_{k+4}+2\beta_{k-2}a_{k+5}+\ldots\ldots\\
&\qquad\qquad =\beta_k a_{k+3}+2\sum_{i=1}^{k}\beta_{k-i}a_{k+i+3},\\
&\ldots\ldots
\end{aligned} \tag{2.45}$$

or more explicitly, we find

$$\begin{aligned}
&\Box^k F_{\rho\lambda}\Box^k F_{\rho\lambda}: \quad \sum_{i=1}^{k}\beta_{k-i}a_{k+i+1},\\
&\Box^k F_{\rho\lambda}\Box^{k+j} F_{\rho\lambda}: \quad \beta_k a_{k+j+1}+2\sum_{i=1}^{k}\beta_{k-i}a_{k+i+j+1}, \quad j>0
\end{aligned} \tag{2.46}$$

in a similar way, from

$$\sum_{k=0}^{n-1}\sum_{l=0}^{k-2}\sum_{i=0}^{k-l-2}\beta_k a_l \Box^{l+i}F_{\rho\lambda}\Box^{k-1-i}F_{\rho\lambda}=\sum_{k=0}^{n-1}\sum_{l=0}^{k-2}\sum_{j=l}^{k-2}\beta_k a_l \Box^{j}F_{\rho\lambda}\Box^{l+k-1-j}F_{\rho\lambda} \tag{2.47}$$

we carefully figure out the summations in (2.47)

$$\begin{aligned}
&l=k-2: \quad \beta_k a_{k-2}(\Box^{k-2}F_{\rho\lambda}\Box^{k-1}F_{\rho\lambda}),\\
&l=k-3: \quad \beta_k a_{k-3}(\Box^{k-3}F_{\rho\lambda}\Box^{k-1}F_{\rho\lambda}+\Box^{k-2}F_{\rho\lambda}\Box^{k-2}F_{\rho\lambda}),\\
&l=k-4: \quad \beta_k a_{k-4}(\Box^{k-4}F_{\rho\lambda}\Box^{k-1}F_{\rho\lambda}+\Box^{k-3}F_{\rho\lambda}\Box^{k-2}F_{\rho\lambda}+\Box^{k-2}F_{\rho\lambda}\Box^{k-3}F_{\rho\lambda}),\\
&l=k-5: \quad \beta_k a_{k-5}(\Box^{k-5}F_{\rho\lambda}\Box^{k-1}F_{\rho\lambda}+\Box^{k-4}F_{\rho\lambda}\Box^{k-2}F_{\rho\lambda}+\Box^{k-3}F_{\rho\lambda}\Box^{k-3}F_{\rho\lambda}\\
&\qquad\qquad +\Box^{k-2}F_{\rho\lambda}\Box^{k-4}F_{\rho\lambda}),\\
&\ldots\ldots
\end{aligned} \tag{2.48}$$

which result in the coefficients of $\Box^k F_{\rho\lambda}\Box^{k+j}F_{\rho\lambda}$, $j\geq 0$

$$\begin{aligned}
&\Box^k F_{\rho\lambda}\Box^k F_{\rho\lambda}: \quad \beta_{k+2}a_{k-1}+\beta_{k+3}a_{k-2}+\beta_{k+4}a_{k-3}+\ldots\ldots=\sum_{i=1}^{k}\beta_{k+i+1}a_{k-i},\\
&\Box^k F_{\rho\lambda}\Box^{k+1} F_{\rho\lambda}: \quad \beta_{k+2}a_{k}+2\beta_{k+3}a_{k-1}+2\beta_{k+4}a_{k-2}+\ldots\ldots\\
&\qquad\qquad =\beta_{k+2}a_k+2\sum_{i=1}^{k}\beta_{k+i+2}a_{k-i},
\end{aligned} \tag{2.49}$$

$$\Box^k F_{\rho\lambda}\Box^{k+2}F_{\rho\lambda}: \quad \beta_{k+3}a_k + 2\beta_{k+4}a_{k-1} + 2\beta_{k+5}a_{k-2} + \ldots\ldots$$
$$= \beta_{k+3}a_k + 2\sum_{i=1}^{k}\beta_{k+i+3}a_{k-i},$$

......

and we read off

$$\begin{aligned}&\Box^k F_{\rho\lambda}\Box^{k}F_{\rho\lambda}: \quad \sum_{i=1}^{k}\beta_{k+i+1}a_{k-i},\\ &\Box^k F_{\rho\lambda}\Box^{k+j}F_{\rho\lambda}: \quad \beta_{k+j+1}a_k + 2\sum_{i=1}^{k}\beta_{k+i+j+1}a_{k-i}, \quad j>0\end{aligned} \tag{2.50}$$

now combining (2.46) with (2.50), in the consideration of the coefficients in (2.37) and by virtue of (2.41), we are thus led to the exact formulae of $A_{k,k+j}(a,\beta)$ for $j \geq 0$

$$A_{k,k+j}(a,\beta) = A_{k+j,k}(a,\beta) = \frac{1}{4}\sum_{i=0}^{k}(\beta_{k+i+j+1}a_{k-i} - \beta_{k-i}a_{k+i+j+1}) \tag{2.51}$$

analogously, it is not hard to derive

$$\begin{aligned}B_{k,k+j}(a,\beta) = B_{k+j,k}(a,\beta) &= \frac{1}{2}\sum_{i=1}^{k}(\beta_{k-i+1}a_{k+i+j+1} - \beta_{k+i+j+1}a_{k-i+1})\\ &- \frac{1}{4}(\beta_{k+1}a_{k+j+1} + \beta_{k+j+1}a_{k+1})\end{aligned} \tag{2.52}$$

as a final comment, we have set $a_k = \beta_k = 0$ for $k < 0$ and $k \geq n$ in the above expressions (2.51), (2.52).

3. Root decompositions

As is well known, every polynomial has a basic feature that, in principle, it can be formulated in terms of its real and complex roots. Now adopting a more extensive viewpoint, we wish to explore some aspects of the conserved tensors in connection with the structure of roots of characteristic polynomial of the wave operator. The main reason to address the problem in this way is that it seems more accessible to obtain the bounded 00-component of the conserved quantities which may not been seen directly from the general expression (2.6). Motivated by this, we suppose the wave operator of the higher derivative derived theory has the following factorable structure

$$M = \sum_{l=0}^{n}a_l W^l = \prod_{i=1}^{p}(W-\lambda_i)^{p_i}\prod_{j=1}^{q}(W^2 - (\omega_j + \bar{\omega}_j)W + \omega_j\bar{\omega}_j)^{q_j} \tag{3.1}$$

without loss of generality, here we impose $a_n = 1$ and the numbers $\lambda_i, \omega_j, \bar{\omega}_j$ label different real and complex roots respectively. In addition, the numbers p_i, q_j are the corresponding multiplicities of the roots and the indices p, q surely fulfill the restriction

$$\sum_{i=1}^{p}p_i + 2\sum_{j=1}^{q}q_j = n \tag{3.2}$$

following the usual steps, for every root in (3.1), it is useful to define the new dynamic fields to absorb the higher derivatives of the original fields

$$\begin{aligned}\xi_k &= \prod_{\substack{i=1,\\ i\neq k}}^{p} (W-\lambda_i)^{p_i} \prod_{j=1}^{q} (W^2-(\omega_j+\bar{\omega}_j)W+\omega_j\bar{\omega}_j)^{q_j} A,\\ \eta_k &= \prod_{i=1}^{p} (W-\lambda_i)^{p_i} \prod_{\substack{j=1,\\ j\neq k}}^{q} (W^2-(\omega_j+\bar{\omega}_j)W+\omega_j\bar{\omega}_j)^{q_j} A\end{aligned} \tag{3.3}$$

by this construction, when the original fields A_μ are subject to the higher derivative field equations (2.7), these new component fields of course satisfy the lower-order derived equations

$$(W-\lambda_i)^{p_i}\xi_i = 0, \qquad (W^2-(\omega_j+\bar{\omega}_j)W+\omega_j\bar{\omega}_j)^{q_j}\eta_j = 0 \tag{3.4}$$

for $i = 1, 2, ..., p$ and $j = 1, 2, ..., q$. Meanwhile, we recognize that these dynamic equations also come from the following action functional

$$S = \int d^4x \left[\sum_{i=1}^{p} \xi_i (W-\lambda_i)^{p_i}\xi_i + \sum_{j=1}^{q} \eta_j (W^2-(\omega_j+\bar{\omega}_j)W+\omega_j\bar{\omega}_j)^{q_j}\eta_j \right] \tag{3.5}$$

at this point, obviously the relations in (3.3) enable us to establish one-to-one correspondence between the solutions to the higher derivative Abelian gauge theory (2.5) and the lower-order dynamical system (3.5). In other words, these two systems are equivalent and can be viewed as two different representations of the same theory which are usually called A- and $\xi_i\eta_i$-representations.

3.1. Real roots case

To proceed further, realizing the fact that all the fields ξ_i, η_j in (3.5) are independent degrees of freedom, hence we can treat them on the same footing. Firstly we consider the following variational symmetries of the action functional

$$\delta_\varepsilon \xi_i = \varepsilon^\mu \partial_\mu (W-\lambda_i)^k \xi_i, \qquad k = 0, 1, ..., p_i - 1 \tag{3.6}$$

for ξ_i. Along this line, it immediately follows that the spacetime translation invariance of the primary operator W gives us a series of conserved tensors

$$\partial_\mu (T_i^k)^\mu_\nu = \partial_\nu((W-\lambda_i)^k\xi_i)(W-\lambda_i)^{p_i}\xi_i, \qquad i = 1, 2, ..., p \tag{3.7}$$

then a direct calculation leads to

$$\partial_\nu((W-\lambda_i)^k\xi_i)(W-\lambda_i)^{p_i}\xi_i = \sum_{j=0}^{k}\sum_{l=0}^{p_i} C_k^j(-\lambda_i)^{k-j}\partial_\nu W^j\xi_i C_{p_i}^l(-\lambda_i)^{p_i-l}W^l\xi_i \tag{3.8}$$

this resembles the formula studied in (2.15) and recalling (2.30), we are able to cast the second-rank conserved tensors $(T_i^k)^\mu_\nu$ in the form of

$$(T_i^k)^\mu_\nu(\xi_i) = \sum_{l=0}^{p_i}\left(\sum_{j=0}^{l-1} C_k^j C_{p_i}^l (-\lambda_i)^{k+p_i-j-l}(\tilde{t}_1^{j,l})^\mu_\nu + \sum_{j=l+1}^{k} C_k^j C_{p_i}^l (-\lambda_i)^{k+p_i-j-l}(\tilde{t}_2^{j,l})^\mu_\nu\right) + \frac{1}{2}\sum_{l=0}^{p_i} C_k^l C_{p_i}^l (-\lambda_i)^{k+p_i-2l}\delta^\mu_\nu(\Box^{l-1}\partial^\rho \tilde{F}^i_{\rho\lambda}\Box^{l-1}\partial_\tau \tilde{F}_i^{\tau\lambda}) \tag{3.9}$$

for convenience, here we use the notations

$$(\tilde{t}_1^{j,l})^\mu_\nu = (t_1^{j,l})^\mu_\nu(\xi_i), \quad (\tilde{t}_2^{j,l})^\mu_\nu = (t_2^{j,l})^\mu_\nu(\xi_i), \quad \tilde{F}^i_{\rho\lambda} = \partial_\rho \xi_{i\lambda} - \partial_\lambda \xi_{i\rho} \tag{3.10}$$

At this stage, let us pay attentions that upon substitution of (3.3) into (3.9), these $(T_i^k)^\mu_\nu$ are just the linear combinations of $(\Theta^k)^\mu_\nu$ in (2.30). By this reason, perhaps it is more convenient to adopt this description to deal with the issue of stability in original higher-order derived theory. Furthermore, we would like to remark here that the action functional (3.5) is also equipped with the symmetries

$$\delta_\varepsilon \xi_i = \varepsilon^\mu \partial_\mu (W^k \xi_i) \tag{3.11}$$

which allows us to modify the conserved tensors as

$$(T_i^k)^\mu_\nu(\xi_i) = \sum_{l=k+1}^{p_i} C_{p_i}^l (-\lambda_i)^{p_i-l}(\tilde{t}_1^{k,l})^\mu_\nu + \sum_{l=0}^{k-1} C_{p_i}^l (-\lambda_i)^{p_i-l}(\tilde{t}_2^{k,l})^\mu_\nu + \frac{1}{2}C_{p_i}^k(-\lambda_i)^{p_i-k}\delta^\mu_\nu(\Box^{k-1}\partial^\rho \tilde{F}^i_{\rho\lambda}\Box^{k-1}\partial_\tau \tilde{F}_i^{\tau\lambda}) \tag{3.12}$$

3.2. *Complex roots case*

In a similar way, for the fields η_j, we have the following independent higher-order symmetry transformations of the action functional S

$$\delta_\varepsilon \eta_j = \varepsilon^\mu \partial_\mu (W^k \eta_j) \tag{3.13}$$

which are parameterized by the indices $k = 0, 1, ..., 2q_j - 1$. Very much in the spirit of above analysis, it is evident to see that the corresponding conserved quantities obey

$$\begin{aligned}\partial_\mu (U_j^k)^\mu_\nu =& \partial_\nu (W^k \eta_j)(W^2 - (\omega_j + \bar{\omega}_j)W + \omega_j\bar{\omega}_j)^{q_j}\eta_j \\ =& \sum_{r,s=0}^{q_j} C_{q_j}^{r,s}\partial_\nu(W^k\eta_j)W^{2r}(-\omega_j - \bar{\omega}_j)^s W^s(\omega_j\bar{\omega}_j)^{q_j-r-s}\eta_j\end{aligned} \tag{3.14}$$

here

$$C_{q_j}^{r,s} = \frac{q_j!}{r!s!(q_j-r-s)!} \tag{3.15}$$

then by means of (2.30), the explicit expressions of the second-rank conserved tensors associated with complex roots take the form of

$$
\begin{aligned}
(U_j^k)^{\mu}_{\nu}(\eta_j) = & \sum_{2r+s>k} C_{q_j}^{r,s}(-\omega_j - \bar{\omega}_j)^s(\omega_j\bar{\omega}_j)^{q_j-r-s}(t_1^{k,2r+s})^{\mu}_{\nu}(\eta_j) \\
& + \sum_{2r+s<k} C_{q_j}^{r,s}(-\omega_j - \bar{\omega}_j)^s(\omega_j\bar{\omega}_j)^{q_j-r-s}(t_2^{k,2r+s})^{\mu}_{\nu}(\eta_j) \\
& + \frac{1}{2}\sum_{2r+s=k} C_{q_j}^{r,s}(-\omega_j - \bar{\omega}_j)^s(\omega_j\bar{\omega}_j)^{q_j-r-s}\delta^{\mu}_{\nu}(\Box^{k-1}\partial^{\rho}\bar{F}^j_{\rho\lambda}\Box^{k-1}\partial_{\tau}\bar{F}_j^{\tau\lambda})
\end{aligned} \tag{3.16}
$$

here we use $\bar{F}^j_{\rho\lambda} = \partial_\rho \eta_{j\lambda} - \partial_\lambda \eta_{j\rho}$.

To this end, once the conserved tensors for the real and complex roots are known, we need two collections of parameters

$$
\beta_i^r, \quad r = 0, ..., p_i - 1, \quad i = 1, ..., p, \quad \gamma_j^s, \quad s = 0, ..., q_j, \quad j = 1, ..., q \tag{3.17}
$$

to encode all of these consequences into a single expression, thus the total second-rank conserved tensors of the lower-order action functional (3.5) is displayed as

$$
\Theta^{\mu}_{\nu} = \sum_{i=1}^{p}\sum_{r=0}^{p_i-1}\beta_i^r(T_i^r)^{\mu}_{\nu}(\xi_i) + \sum_{j=1}^{q}\sum_{s=0}^{2q_j-1}\gamma_j^s(U_j^s)^{\mu}_{\nu}(\eta_j) \tag{3.18}
$$

with this result in hand, there is no difficulty in obtaining the conserved tensors of the original derived theory by plugging (3.12) and (3.16) into (3.18). When it comes to the stability of the higher derivative system, as mentioned before, the positive 00-component is relevant. Specifically due to the independence of the fields ξ_i and η_j, the positive condition can be met only if the parameters β_i^r, γ_j^s satisfy

$$
\sum_{r=0}^{p_i-1}\beta_i^r(T_i^r)^0_0(\xi_i) \geq 0, \qquad \sum_{s=0}^{2q_j-1}\gamma_j^s(U_j^s)^0_0(\eta_j) \geq 0 \tag{3.19}
$$

completely for any solutions of ξ_i and η_j. In short, we demonstrate that the derived theory is stable if and only if all the component fields are stable.

4. Third-order derived theory

For simplicity, an instructive example to illustrate the spirit in previous section is the third-order derived theory which naturally possesses a three-parameter family of conserved tensors. The behavior of 00-component of the conserved tensors in such model strongly relies on the structure of the roots of the characteristic polynomial

$$
Z^3 + a_2 Z^2 + a_1 Z + a_0 = 0 \tag{4.1}
$$

in what follows, we intend to make a detailed investigation for different situations of the root decomposition of the third-order equation (4.1).

To begin with, let us concentrate on the case when the polynomial (4.1) owns three different real roots λ_i, or in other words, the wave operator (2.6) is decomposed as follows

$$
M = (W - \lambda_1)(W - \lambda_2)(W - \lambda_3) \tag{4.2}
$$

obviously, such situation corresponds to

$$p=3, \qquad q=0, \qquad p_i=1 \tag{4.3}$$

in (3.1). Now from the practice in above discussion, it is convenient to define the auxiliary fields ξ_i

$$\xi_1=(W-\lambda_2)(W-\lambda_3)A, \quad \xi_2=(W-\lambda_1)(W-\lambda_3)A, \quad \xi_3=(W-\lambda_1)(W-\lambda_2)A \tag{4.4}$$

which provide the relations

$$(W-\lambda_1)\xi_1=0, \quad (W-\lambda_2)\xi_2=0, \quad (W-\lambda_3)\xi_3=0 \tag{4.5}$$

then according to the results (3.12), we acquire the following second-rank conserved tensors

$$(T_i)^\mu_\nu(\xi_i)=(t_1^{0,1})^\mu_\nu(\xi_i)-\frac{1}{2}\lambda_i\delta^\mu_\nu\xi_{i\rho}\xi_i^\rho \tag{4.6}$$

and taking advantage of (2.31), it is not hard to write down

$$(t_1^{0,1})^\mu_\nu(\xi_i)=\tilde{F}^i_{\nu\lambda}\tilde{F}^{\mu\lambda}_i-\frac{1}{4}\delta^\mu_\nu\tilde{F}^i_{\rho\lambda}\tilde{F}^{\rho\lambda}_i+\xi_{i\nu}\partial_\tau\tilde{F}^{\tau\mu}_i \tag{4.7}$$

due to the relations in (4.5), the equations of motion for ξ_i turn out to be

$$\partial_\rho\tilde{F}^{\rho 0}_i-\lambda_i\xi^0_i=0 \tag{4.8}$$

with the aid of these equalities and in view of the metric $g_{\mu\nu}=\mathrm{diag}(1,-1,-1,-1)$, it would be greatly helpful for us to simplify the $(T_i)^0_0$ in the form of

$$(T_i)^0_0(\xi_i)=-\frac{1}{4}\tilde{F}^i_{\mu\nu}\tilde{F}^i_{\mu\nu}+\frac{1}{2}\lambda_i\xi_{i\rho}\xi_{i\rho} \tag{4.9}$$

in this manner, the 00-component of the linear combination of $(T_i)^0_0$ is given by

$$T^0_0=\sum_{i=1}^{3}\beta_i(-\frac{1}{4}\tilde{F}^i_{\mu\nu}\tilde{F}^i_{\mu\nu}+\frac{1}{2}\lambda_i\xi_{i\rho}\xi_{i\rho}) \tag{4.10}$$

typically, by imposing

$$\beta_i<0, \qquad \lambda_i<0, \qquad i=1,2,3 \tag{4.11}$$

the contributions of all the component fields are positive which bring us to the stability of the higher derivative gauge system defined by the third-order wave operator.

We can take this result one step further by a second application of the third-order equation possessing a simple real root λ_1 and a real root λ_2 of multiplicity of 2

$$M=(W-\lambda_1)(W-\lambda_2)^2 \tag{4.12}$$

which correctly corresponds to the case of

$$p=3, \qquad p_1=1, \qquad p_2=2, \qquad q=0 \tag{4.13}$$

in (3.1). Following the prescription of the method, we define the auxiliary fields

$$\xi_1=(W-\lambda_2)^2A, \qquad \xi_2=(W-\lambda_1)A \tag{4.14}$$

which yield the relations

$$(W-\lambda_1)\xi_1=0, \qquad (W-\lambda_2)^2\xi_2=0 \tag{4.15}$$

then by virtue of formula (3.12), the second-rank conserved tensors in current situation have the form

$$\begin{aligned}
(T_1)^\mu_\nu(\xi_1) =&(t_1^{0,1})^\mu_\nu(\xi_1)-\frac{1}{2}\lambda_1\delta^\mu_\nu\xi_{1\rho}\xi_1^\rho,\\
(T_2^0)^\mu_\nu(\xi_2) =&-2\lambda_2(t_1^{0,1})^\mu_\nu(\xi_2)+(t_1^{0,2})^\mu_\nu(\xi_2)+\frac{1}{2}\lambda_2^2\delta^\mu_\nu\xi_{2\rho}\xi_2^\rho,\\
(T_2^1)^\mu_\nu(\xi_2) =&(t_1^{1,2})^\mu_\nu(\xi_2)+\lambda_2^2(t_2^{1,0})^\mu_\nu(\xi_2)-\lambda_2\delta^\mu_\nu\partial^\rho\tilde F_{\rho\lambda}\partial_\tau\tilde F^{\tau\lambda}
\end{aligned}\tag{4.16}$$

in these expressions, the notations

$$F_{\rho\lambda}=\partial_\rho\xi_{1\lambda}-\partial_\lambda\xi_{1\rho}, \qquad \tilde F_{\rho\lambda}=\partial_\rho\xi_{2\lambda}-\partial_\lambda\xi_{2\rho} \tag{4.17}$$

are adopted. After a straightforward calculation of $(t_i^{k,l})^\mu_\nu$ in (2.31) and (2.33), we successfully get

$$\begin{aligned}
(t_1^{0,2})^\mu_\nu(\xi_2) =&\tilde F_{\nu\lambda}\Box\tilde F^{\mu\lambda}-\frac{1}{4}\delta^\mu_\nu\tilde F_{\rho\lambda}\Box\tilde F^{\rho\lambda}+\tilde F^{\mu\lambda}\Box\tilde F_{\nu\lambda}-\frac{1}{4}\delta^\mu_\nu\tilde F_{\rho\lambda}\Box\tilde F^{\rho\lambda}\\
&+\partial_\lambda\tilde F^{\lambda\mu}\partial^\rho\tilde F_{\rho\nu}-\frac{1}{2}\delta^\mu_\nu\partial_\tau\tilde F^{\tau\lambda}\partial^\rho\tilde F_{\rho\lambda}+\xi_{2\nu}\Box\partial_\tau\tilde F^{\tau\mu},\\
(t_1^{1,2})^\mu_\nu(\xi_2) =&\Box\tilde F_{\nu\lambda}\Box\tilde F^{\mu\lambda}-\frac{1}{4}\delta^\mu_\nu\Box\tilde F_{\rho\lambda}\Box\tilde F^{\rho\lambda}+\partial^\rho\tilde F_{\rho\nu}\Box\partial_\tau\tilde F^{\tau\mu},\\
(t_2^{1,0})^\mu_\nu(\xi_2) =&\delta^\mu_\nu\partial^\rho\tilde F_{\rho\lambda}\xi_2^\lambda-\tilde F_{\nu\lambda}\tilde F^{\mu\lambda}+\frac{1}{4}\delta^\mu_\nu\tilde F_{\rho\lambda}\tilde F^{\rho\lambda}-\xi_{2\nu}\partial_\tau\tilde F^{\tau\mu}
\end{aligned}\tag{4.18}$$

for ξ_2. Similarly making use of the equations of motion from (4.15)

$$\Box\partial_\rho\tilde F^{\rho0}-2\lambda_2\partial_\rho\tilde F^{\rho0}+\lambda_2^2\xi_2^0=0 \tag{4.19}$$

we are capable of formulating the 00-components of $(T_2^i)^0_0(\xi_2)$ in a more compact form

$$\begin{aligned}
(T_1)^0_0(\xi_1) =&-\frac{1}{4}F_{\mu\nu}F_{\mu\nu}+\frac{1}{2}\lambda_i\xi_{1\rho}\xi_{1\rho},\\
(T_2^0)^0_0(\xi_2) =&-\frac{1}{2}\tilde F_{\mu\nu}\Box\tilde F_{\mu\nu}+\frac{1}{2}\partial^\mu\tilde F_{\mu\rho}\partial^\nu\tilde F_{\nu\rho}+\frac{1}{2}\lambda_2\tilde F_{\mu\nu}\tilde F_{\mu\nu}-\frac{1}{2}\lambda_2^2\xi_{2\rho}\xi_{2\rho},\\
(T_2^1)^0_0(\xi_2) =&-\frac{1}{4}\Box\tilde F_{\mu\nu}\Box\tilde F_{\mu\nu}+\lambda_2\partial^\mu\tilde F_{\mu\rho}\partial^\nu\tilde F_{\nu\rho}-\lambda_2^2\xi_{2\mu}\partial^\rho\tilde F_{\rho\mu}+\frac{1}{4}\lambda_2^2\tilde F_{\mu\nu}\tilde F_{\mu\nu}
\end{aligned}\tag{4.20}$$

subsequently, by introducing a series of parameters β, β_0 and β_1, the total energy density of the system is expressed as

$$\begin{aligned}
T^0_0 =&\beta(T_1)^0_0+\beta_0(T_2^0)^0_0+\beta_1(T_2^1)^0_0\\
=&\beta(-\frac{1}{4}F_{\mu\nu}F_{\mu\nu}+\frac{1}{2}\lambda_1\xi_{1\rho}\xi_{1\rho})-\frac{1}{4}\beta_1\Box\tilde F_{\mu\nu}\Box\tilde F_{\mu\nu}-\frac{1}{2}\beta_0\tilde F_{\mu\nu}\Box\tilde F_{\mu\nu}\\
&+(\frac{1}{2}\lambda_2\beta_0+\frac{1}{4}\lambda_2^2\beta_1)\tilde F_{\mu\nu}\tilde F_{\mu\nu}+(\frac{1}{2}\beta_0+\beta_1\lambda_2)\partial^\mu\tilde F_{\mu\rho}\partial^\nu\tilde F_{\nu\rho}\\
&-\beta_1\lambda_2^2\xi_{2\mu}\partial^\rho\tilde F_{\rho\mu}-\frac{1}{2}\beta_0\lambda_2^2\xi_{2\rho}\xi_{2\rho}
\end{aligned}\tag{4.21}$$

once again, the condition

$$\beta < 0, \qquad \lambda_1 < 0 \tag{4.22}$$

is sufficient to guarantee the stability of dynamics of ξ_1. While for the field ξ_2, to ensure the positive definite of the quadratic form, it is better to choose

$$\beta_1 < 0, \qquad \beta_0 = -\beta_1\lambda_2, \qquad \lambda_2 < 0 \tag{4.23}$$

indeed, one can verify this assertion through the evaluation of the discriminant in the quadratic form directly. Analogously, in the case of a pair of complex conjugate roots, the linear combination of $(U_2^0)_0^0$ and $(U_2^1)_0^0$ will not give us a positive conserved tensor unless the imaginary part of complex root is set to zero. This turns out to be the case we just discussed.

Finally, we restrict our attention to the situation where the third-order equation merely has real root λ of multiplicity 3

$$M = (W-\lambda)^3 \tag{4.24}$$

which belongs to the case of

$$p = 3, \qquad q = 0, \qquad p_1 = 3 \tag{4.25}$$

in (3.1). Then from (3.12) and after some algebraic manipulations, it is not difficult to derive the explicit formulae of the conserved tensors

$$\begin{aligned}
(T^0)^\mu_\nu =&3\lambda^2(t_1^{0,1})^\mu_\nu - 3\lambda(t_1^{0,2})^\mu_\nu + (t_1^{0,3})^\mu_\nu - \frac{1}{2}\lambda^3\delta^\mu_\nu A_\rho A^\rho,\\
(T^1)^\mu_\nu =&-3\lambda(t_1^{1,2})^\mu_\nu + (t_1^{1,3})^\mu_\nu - \lambda^3(t_2^{1,0})^\mu_\nu + \frac{3}{2}\lambda^2\delta^\mu_\nu\partial^\rho F_{\rho\lambda}\partial_\tau F^{\tau\lambda},\\
(T^2)^\mu_\nu =&(t_1^{2,3})^\mu_\nu - \lambda^3(t_2^{2,0})^\mu_\nu + 3\lambda^2(t_2^{2,1})^\mu_\nu - \frac{3}{2}\lambda\delta^\mu_\nu\Box\partial^\rho F_{\rho\lambda}\Box\partial_\tau F^{\tau\lambda}
\end{aligned} \tag{4.26}$$

taking into account of (2.31) and (2.33), our task here is to work out the $(t_i^{k,l})^\mu_\nu$ in the form of

$$\begin{aligned}
(t_1^{0,3})^\mu_\nu =&F_{\nu\lambda}\Box^2F^{\mu\lambda} - \frac{1}{4}\delta^\mu_\nu F_{\rho\lambda}\Box^2F^{\rho\lambda} + F^{\mu\lambda}\Box^2F_{\nu\lambda} - \frac{1}{4}\delta^\mu_\nu F_{\rho\lambda}\Box^2F^{\rho\lambda}\\
&+\partial_\lambda F^{\lambda\mu}\Box\partial^\rho F_{\rho\nu} - \frac{1}{2}\delta^\mu_\nu\partial_\tau F^{\tau\lambda}\Box\partial^\rho F_{\rho\lambda} + \Box F^{\mu\lambda}\Box F_{\nu\lambda}\\
&-\frac{1}{4}\delta^\mu_\nu\Box F_{\rho\lambda}\Box F^{\rho\lambda} + \Box\partial_\lambda F^{\lambda\mu}\partial^\rho F_{\rho\nu} - \frac{1}{2}\delta^\mu_\nu\Box\partial_\tau F^{\tau\lambda}\partial^\rho F_{\rho\lambda}\\
&+A_\nu\Box^2\partial_\tau F^{\tau\mu},\\
(t_1^{1,3})^\mu_\nu =&\Box F_{\nu\lambda}\Box^2F^{\mu\lambda} - \frac{1}{4}\delta^\mu_\nu\Box F_{\rho\lambda}\Box^2F^{\rho\lambda} + \Box F^{\mu\lambda}\Box^2F_{\nu\lambda} - \frac{1}{4}\delta^\mu_\nu\Box F_{\rho\lambda}\Box^2F^{\rho\lambda}\\
&+\Box\partial_\lambda F^{\lambda\mu}\Box\partial^\rho F_{\rho\nu} - \frac{1}{2}\delta^\mu_\nu\Box\partial_\tau F^{\tau\lambda}\Box\partial^\rho F_{\rho\lambda} + \partial^\rho F_{\rho\nu}\Box^2\partial_\tau F^{\tau\mu},\\
(t_1^{2,3})^\mu_\nu =&\Box^2F_{\nu\lambda}\Box^2F^{\mu\lambda} - \frac{1}{4}\delta^\mu_\nu\Box^2F_{\rho\lambda}\Box^2F^{\rho\lambda} + \Box\partial^\rho F_{\rho\nu}\Box^2\partial_\tau F^{\tau\mu},\\
(t_2^{2,0})^\mu_\nu =&\delta^\mu_\nu\Box\partial^\rho F_{\rho\lambda}A^\lambda - F_{\nu\lambda}\Box F^{\mu\lambda} + \frac{1}{4}\delta^\mu_\nu F_{\rho\lambda}\Box F^{\rho\lambda} - (F^{\mu\lambda}\Box F_{\nu\lambda}\\
&-\frac{1}{4}\delta^\mu_\nu F_{\rho\lambda}\Box F^{\rho\lambda} + \partial_\lambda F^{\lambda\mu}\partial^\rho F_{\rho\nu} - \frac{1}{2}\delta^\mu_\nu\partial_\tau F^{\tau\lambda}\partial^\rho F_{\rho\lambda}) - A_\nu\Box\partial_\tau F^{\tau\mu},\\
(t_2^{2,1})^\mu_\nu =&\delta^\mu_\nu\Box\partial^\rho F_{\rho\lambda}\partial_\tau F^{\tau\lambda} - \Box F_{\nu\lambda}\Box F^{\mu\lambda} + \frac{1}{4}\delta^\mu_\nu\Box F_{\rho\lambda}\Box F^{\rho\lambda} - \partial^\rho F_{\rho\nu}\Box\partial_\tau F^{\tau\mu}
\end{aligned} \tag{4.27}$$

when expanding the operator (4.24), we obtain the equation of motion for the gauge fields

$$\Box^2\partial_\rho F^{\rho 0} - 3\lambda\Box\partial_\rho F^{\rho 0} + 3\lambda^2\partial_\rho F^{\rho 0} - \lambda^3 A^0 = 0 \tag{4.28}$$

under these constraints, it would be tempting to look for the expressions of 00-components of the conserved tensors

$$\begin{aligned}
(T^0)^0_0 =& -\frac{3}{4}\lambda^2 F_{\mu\nu}F_{\mu\nu} + \frac{3}{2}\lambda F_{\mu\nu}\Box F_{\mu\nu} - \frac{1}{2}F_{\mu\nu}\Box^2 F_{\mu\nu} - \frac{1}{4}\Box F_{\mu\nu}\Box F_{\mu\nu}\\
&-\frac{3}{2}\lambda\partial^\rho F_{\rho\mu}\partial^\tau F_{\tau\mu} + \partial^\rho F_{\rho\mu}\Box\partial^\tau F_{\tau\mu} + \frac{1}{2}\lambda^3 A_\rho A_\rho,\\
(T^1)^0_0 =& -\frac{1}{4}\lambda^3 F_{\mu\nu}F_{\mu\nu} + \frac{3}{4}\lambda\Box F_{\mu\nu}\Box F_{\mu\nu} - \frac{1}{2}\Box F_{\mu\nu}\Box^2 F_{\mu\nu} + \frac{1}{2}\Box\partial^\rho F_{\rho\mu}\Box\partial^\tau F_{\tau\mu}\\
&-\frac{3}{2}\lambda^2\partial^\rho F_{\rho\mu}\partial^\tau F_{\tau\mu} + \lambda^3 A_\rho\partial^\tau F_{\tau\rho},\\
(T^2)^0_0 =& -\frac{1}{4}\Box^2 F_{\mu\nu}\Box^2 F_{\mu\nu} + \frac{3}{4}\lambda^2\Box F_{\mu\nu}\Box F_{\mu\nu} - \frac{1}{2}\lambda^3 F_{\mu\nu}\Box F_{\mu\nu} + \frac{1}{2}\lambda^3\partial^\rho F_{\rho\mu}\partial^\tau F_{\tau\mu}\\
&+\frac{3}{2}\lambda\Box\partial^\rho F_{\rho\mu}\Box\partial^\tau F_{\tau\mu} - 3\lambda^2\partial^\rho F_{\rho\mu}\Box\partial^\tau F_{\tau\mu} + \lambda^3 A_\mu\Box\partial^\tau F_{\tau\mu}
\end{aligned} \tag{4.29}$$

then to fix the instability of the higher derivative system, we require parameters β_i to enter the total energy density which can be presented as follows

$$\begin{aligned}
T^0_0 =& \sum_{i=0}^{2}\beta_i (T^i)^0_0\\
=& -\frac{1}{4}\beta_2\Box^2 F_{\mu\nu}\Box^2 F_{\mu\nu} + (\frac{3}{4}\lambda^2\beta_2 + \frac{3}{4}\lambda\beta_1 - \frac{1}{4}\beta_0)\Box F_{\mu\nu}\Box F_{\mu\nu}\\
&-(\frac{3}{4}\lambda^2\beta_0 + \frac{1}{4}\lambda^3\beta_1)F_{\mu\nu}F_{\mu\nu}\\
&-\frac{1}{2}\beta_0 F_{\mu\nu}\Box^2 F_{\mu\nu} + (\frac{3}{2}\lambda\beta_0 - \frac{1}{2}\lambda^3\beta_2)F_{\mu\nu}\Box F_{\mu\nu} - \frac{1}{2}\beta_1\Box F_{\mu\nu}\Box^2 F_{\mu\nu}\\
&+(\frac{1}{2}\beta_1 + \frac{3}{2}\lambda\beta_2)\Box\partial^\rho F_{\rho\mu}\Box\partial^\tau F_{\tau\mu} - (\frac{3}{2}\lambda\beta_0 + \frac{3}{2}\lambda^2\beta_1 - \frac{1}{2}\lambda^3\beta_2)\ \partial^\rho F_{\rho\mu}\partial^\tau F_{\tau\mu}\\
&+\frac{1}{2}\lambda^3\beta_0 A_\rho A_\rho + (\beta_0 - 3\lambda^2\beta_2)\partial^\rho F_{\rho\mu}\Box\partial^\tau F_{\tau\mu} + \lambda^3\beta_2 A_\mu\Box\partial^\tau F_{\tau\mu} + \lambda^3\beta_2 A_\rho\partial^\tau F_{\tau\rho}
\end{aligned} \tag{4.30}$$

in this formalism, as a final result, T^0_0 is positive and bounded only if the matrices

$$\begin{pmatrix}
-\frac{1}{4}\beta_2 & -\frac{1}{4}\beta_1 & -\frac{1}{4}\beta_0\\
-\frac{1}{4}\beta_1 & \frac{3}{4}\lambda^2\beta_2 + \frac{3}{4}\lambda\beta_1 - \frac{1}{4}\beta_0 & \frac{3}{4}\lambda\beta_0 - \frac{1}{4}\lambda^3\beta_2\\
-\frac{1}{4}\beta_0 & \frac{3}{4}\lambda\beta_0 - \frac{1}{4}\lambda^3\beta_2 & -(\frac{3}{4}\lambda^2\beta_0 + \frac{1}{4}\lambda^3\beta_1)
\end{pmatrix}$$

and

$$\begin{pmatrix}
\frac{1}{2}\beta_1 + \frac{3}{2}\lambda\beta_2 & \frac{1}{2}(\beta_0 - 3\lambda^2\beta_2) & \frac{1}{2}\lambda^3\beta_2\\
\frac{1}{2}(\beta_0 - 3\lambda^2\beta_2) & -(\frac{3}{2}\lambda\beta_0 + \frac{3}{2}\lambda^2\beta_1 - \frac{1}{2}\lambda^3\beta_2) & \frac{1}{2}\lambda^3\beta_2\\
\frac{1}{2}\lambda^3\beta_2 & \frac{1}{2}\lambda^3\beta_2 & \frac{1}{2}\lambda^3\beta_0
\end{pmatrix}$$

are all positive definite matrices.

5. Conclusion and discussions

In this paper, we investigate the issue of stability of the Abelian higher derivative gauge field theory whose wave operator is a polynomial of arbitrary finite order n in terms of the usual lower-order Maxwell operator. In this setup, the Lagrangian is immediately seen to admit a n-parameter series of conserved quantities by using the extension of Noether's theorem if there exists some linear operators commuting with the primary wave operator. Essentially, we obtain n independent second-rank conserved tensors which are connected with the spacetime translation invariance of the action functional. To be more precise, one of the entries of the series is the Noether's canonical energy-momentum tensors, and the others are different integrals of motion corresponding to the higher-order symmetries of the gauge fields. Owing to this, the linear combination of these conserved tensors contains the standard canonical energy-momentum tensors, though it is always unbounded. This suggests us that once the positive conserved quantity is included into the series, the free derived theory will be stable in the classical regime. Moreover, such stability can be promoted to quantum level which implies the bounded spectrum of energy in the corresponding quantum theory. Next, for the generic derived system of order n, we propose a procedure of constructing n-parametric family of conserved tensors whose structure depends on the roots of the characteristic polynomial of the wave operator. In this scheme, one may expect a deeper understanding of stability of the higher derivative system. As a concrete example, for all considered cases, we explicitly deduce the conserved tensors as well as the 00-components for the derived theory involving terms up to third order. Under certain assumptions about the parameters and the roots, we make a claim that the third-order free derived model admits bounded conserved quantities and it is thus considered stable.

As a matter of fact, the existence of these additional conserved quantities can be viewed as a consequence of the so-called Lagrange anchor which may be traced back to the quantization of not necessarily Lagrangian dynamics [47]. In the context of general Lagrangian system or not, the Lagrange anchor maps the conserved quantities to symmetries for the field equations [48] and offers us a new insight into the characterization of the higher derivative dynamic equations for the non-Lagrangian systems. As a general result, every Lagrange anchor will bring about Poisson brackets and Hamiltonian in the first-order formalism which are responsible for the quantization of the classical stable Abelian theory without loss of stability. Roughly speaking, the Lagrange anchor is not unique in the higher derivative system and if the dynamic equations are equipped with multiple Lagrange anchors, the same symmetry can be related to different conserved quantities. Indeed, when the field equations possess various Lagrange anchors, the inequivalent ones will result in the canonically inequivalent Poisson brackets, thus the theory turns out to be multi-Hamiltonian in the first-order formulation [49]. Especially in this class of derived theories, a suitable choice of parameters brings the corresponding Hamiltonian bounded from below.

Like the general couplings among electromagnetic and electron fields, the Lagrange anchor allows us to systematically add consistent and stable interactions between two different derived field theories, namely the Abelian higher derivative gauge model and the higher derivative matter systems including the complex scalar fields and the Dirac spinor fields. More concretely, once two primary theories admit consistent interactions, it would be nice to gain a $n+N$-parametric series of consistent interactions among the two derived models, where n and N are the orders of characteristic polynomials of the wave operators respectively. In this sense, the conserved tensors in the resulting coupling system can be regarded as the deformations of the conserved quantities of the primary theories at the free level [50]. It was argued that as long as the free derived theory admits bounded conserved quantity, the system remains stable upon inclusion of consistent interactions. Generally speaking, in derived theories, the vertices of stable interactions are always non-Lagrangian in the higher derivative equations, but the system can still admit the Hamiltonian formulation. Being bounded from below, the Hamiltonian arising from the positive conserved quantity of the interacting theory is not canonically equivalent to any one in the Ostrogradski formalism. Furthermore, it is worth noting that the two-form gauge field system is another generalization of Maxwell theory which is not only interesting in its own right, but plays a central role in string theory and various supergravity models. Analogously, we may add higher derivative terms into the usual two-form or more general p-form gauge theories and the analysis of stabilities of these systems can be worked out in a similar way. Then it is also significant to try to build consistent and stable interactions among the two-form gauge and matter fields in the standard method. All of these would be interesting to exploit in future.

CRediT authorship contribution statement

All of the contributions of this paper are completed by the author Jialiang Dai.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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