Efimov Effect
in Pionless Effective Field Theory
and its Application to
Hadronic Molecules

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Preamble

In the past years various new hadronic states were discovered in the charm and bottom meson sector by experiments like Belle, BES and Babar. Many of these states fit into the conventional $c\bar{c}$ or $b\bar{b}$ meson spectrum, but a growing number of them does not. Especially, the charged ones which can definitely be no conventional hadrons motivate the term "exotic" hadrons to classify particles which are neither conventional baryons nor conventional mesons. Their substructure is not yet clear due to the often poor experimental knowledge and there are many different interpretations for these exotic particles. One very popular idea is the hadronic molecule interpretation which states that two ordinary hadrons can form a loosely bound pair similar to the deuteron made of a proton and a neutron. However, in the latter system it is known that after adding a third nucleon there appears a three particle bound state, the triton. The emergence of such a three-body state can be explained by the Efimov effect as a low-energy universality phenomenon. The question which will be answered in this work is if there exists also an Efimov effect in systems of hadronic molecules which scatter off a third meson or baryon. Since the number of candidate systems is large (and still growing due to ongoing experiments) it is useful to find a general expression that tells us if the Efimov effect is present. Moreover, this expression should only depend on the basic particle properties like mass, spin and isospin. Such a relation will be derived in this work for a generic three particle system that can be described in a pionless effective field theory. It will then be applied to a set of possible hadronic molecule systems in order to check if the Efimov effect occurs. The thesis is organized as follows: after an introduction of quantum chromodynamics and a review of the classification of hadrons we continue with a summary, firstly, of experimental observations concerning exotic hadrons, secondly, of the idea of effective field theories and thirdly, of basic scattering theory. In chapter 2 the three identical boson system will be discussed in order to explain the Efimov effect more quantitatively. With this knowledge it is possible to derive the mentioned general expression regarding the existence of Efimov physics in a generic three particle system. After the detailed derivation in chapter 3 (using the information given in the appendices) the results will be applied to scattering processes where known hadronic molecules as well as hypothetical ones are involved. This is done in order to either search for the Efimov effect or – for the hypothetical molecular states – to predict under which conditions an Efimov trimer is expected. For system without Efimov effect we will calculate in chapter 5 for a selected example S-wave observables of the in this case elastic scattering process. The thesis is completed by a summary of the results and an outlook on further work that can be done based on this work.

Chapter 1

Introduction and theoretical background

1.1 Quantum chromodynamics and conventional hadrons

Nature is described by four main forces: gravity, electromagnetic, weak and strong force. While gravity is up to now not yet – in a well established way – quantized, the latter three are combined within the standard model of particle physics. Namely, one uses the concept of quantum field theories with local gauge invariance (gauge theories) to describe electromagnetic, weak and strong interactions. On the one hand quantum electrodynamics (QED), i.e. the theory of the electromagnetic and – with some extensions – also of the weak force (electroweak unification), is perturbative. This is the case because the corresponding running coupling constant is much smaller than 1 for a wide energy range. On the other hand the strong force described by quantum chromodynamics (QCD) has a coupling constant whose running is much faster. Hence, for a wide energy range, especially in the region of hadron formation, it is of the order of 1 and thus, one cannot use perturbation theory. Apart from this subtlety (which is further discussed in section 1.3) the standard model of particle physics contains twelve fundamental fermions (as well as twelve anti-fermions) and twelve gauge bosons (the photon γ, $W^{\pm}$- and Z^0-boson as well as eight gluons) which are the carriers of the electroweak and strong force. One half of the fundamental fermions are leptons (electron (e^-), muon (μ^-), tauon (τ^-) and the three corresponding neutrinos ν_e, ν_μ, ν_τ) and the other half are quarks. As the latter are the constituents of hadrons we focus on them in more detail. The quarks are grouped into three (weak isospin) doublets

$$\begin{pmatrix} u \\ d \end{pmatrix}, \quad \begin{pmatrix} c \\ s \end{pmatrix}, \quad \begin{pmatrix} t \\ b \end{pmatrix}, \tag{1.1}$$

and named *up* (u), *down* (d), *strange* (s), *charm* (c), *bottom* (b) and *top* (t), also known as quark *flavors*. Besides the properties summarized in Tab. 1.1 quarks also carry a *color* charge with three degrees of freedom (sometimes referred to as "red", "blue" and "green"). The color interactions are described in terms of quantum chromodynamics whose underlying group is the $SU(3)_c$ color gauge group. There are eight gauge bosons in the theory called gluons (g) which themselves carry color and anti-color. As already mentioned a bound state made of quarks is

	d	u	s	c	b	t
Q – electric charge	$-\frac{1}{3}$	$+\frac{2}{3}$	$-\frac{1}{3}$	$+\frac{2}{3}$	$-\frac{1}{3}$	$+\frac{2}{3}$
I – isospin	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0
I_3 – isospin z-component	$-\frac{1}{2}$	$+\frac{1}{2}$	0	0	0	0
S – strangeness	0	0	-1	0	0	0
C – charm	0	0	0	1	0	0
B – bottomness	0	0	0	0	-1	0
T – topness	0	0	0	0	0	1

Table 1.1: *Additive quantum numbers of the quarks, from Ref. [2].*

called a hadron. Furthermore, we know that due to *confinement* (i.e. due to the fact that there is no experimental evidence for free colored particles (e.g. free quarks), see e.g. Ref. [2]) such a bound state must be color-neutral, i.e. it must be a color singlet which is invariant under $SU(3)_c$ transformations. Following Ref. [1] a quark q_i with color index i transforms under the fundamental representation $\mathbf{3}_c$ of $SU(3)_c$ and an anti-quark $\bar{q}_i$ under the $\bar{\mathbf{3}}_c$. Therefore one can form a $SU(3)_c$ invariant object in two ways: firstly, one can combine a quark and an anti-quark via $q_i \delta_{ij} \bar{q}_j$ so that it is in the $\mathbf{1}_c$ (the singlet) of $\mathbf{3}_c \otimes \bar{\mathbf{3}}_c = \mathbf{1}_c \oplus \mathbf{8}_c$. These $q_i \bar{q}_i$ objects are called *mesons*. Secondly, one can use the total antisymmetric tensor ε_{ijk} to form a three quark color-neutral object $\varepsilon_{ijk}\, q_i q_j q_k$ known as *baryon* or a color singlet three anti-quark state $\varepsilon_{ijk}\, \bar{q}_i \bar{q}_j \bar{q}_k$, the *anti-baryon*. At first glance, it seems that conventional mesons, baryons and anti-baryons are the only possible hadrons in the standard model. However, one can combine these already color-neutral objects to color-singlets with more than three quarks. These states are referred to as *exotic* hadrons and will be discussed in more detail in section 1.2.

It is known that the masses of the three lightest quarks u, d and s (obtained in the $\overline{\mathrm{MS}}$ scheme at a scale of 2 GeV) are [2]

$$m_u = 2.3^{+0.7}_{-0.5}\,\mathrm{MeV}\,,\quad m_d = 4.8^{+0.5}_{-0.3}\,\mathrm{MeV}\,,\quad m_s = 95 \pm 5\,\mathrm{MeV}\,, \tag{1.2}$$

and thus relatively close to each other especially compared to the much heavier charm ($m_c \sim 1.3\,\mathrm{GeV}$), bottom ($m_b \sim 4.2\,\mathrm{GeV}$) and top ($m_t \sim 173\,\mathrm{GeV}$) quarks (for a detailed discussion of the methods used to deduce the quark masses see Ref. [2]). Therefore one assumes an approximate $SU(3)_f$ flavor symmetry for the three lightest quark flavors. Hence, quantum mechanics induce that the bound states of the three lightest quarks will mix. For mesons this motivates their representation in a multiplet with $\mathbf{3}_f \otimes \bar{\mathbf{3}}_f = \mathbf{1}_f \oplus \mathbf{8}_f$ ground states with angular momentum $L = 0$ classified by charge Q, isospin I and strangeness S. Since the quarks a fermions with spin $1/2$ the total spin of the corresponding mesons can be either 0 or 1. Thus, there are two nonets (i.e. a singlet plus an octet), one for pseudoscalar ($J^P = 0^-$) mesons shown in Fig. 1.1(a) and one for vector mesons ($J^P = 1^-$), see Fig. 1.1(b). The former pseudoscalar multiplet contains an isospin doublet with strangeness $S = +1$ identified with the two pseudoscalar kaons, an isospin anti-doublet with strangeness $S = -1$ (the pseudoscalar anti-kaons), an isospin triplet of π^+, π^0,

π^- without strangeness and in addition the two particles η and η' with $Q = I = S = 0$ which are superpositions of the $SU(3)_f$ flavor octet and singlet states η_8 and η_1 (see for example Ref. [2,3]). The same pattern is also found in the vector meson nonet where the pseudoscalar kaons are replaced by their vector counterparts (indicated by a $*$ superscript). The vector iso-triplet gets the symbol ρ and the two $SU(3)_f$ octet / singlet superpositional states with $Q = I = S = 0$ are named ω and φ (note, that their mixing is different from that of η and η' [2,3]).

For the baryons one has to keep in mind that they are fermions, consequently, their wave function

$$|\text{baryon}\rangle = |\text{color}\rangle \times |\text{flavor}\rangle \times |\text{spin}\rangle \times |\text{space}\rangle \ , \tag{1.3}$$

must be antisymmetric. Following Ref. [2] it is known that the color wave function is antisymmetric and the spatial part is symmetric for angular momentum $L = 0$ (ground state). Thus, $|\text{flavor, spin}\rangle$ must be symmetric, too. Combining both the $SU(3)_f$ flavor group of the light quarks and the $SU(2)$ spin group, one can treat a quark q as an element of a $SU(6)$. Group theory states (see e.g. Ref. [2]) that the direct product of three $\mathbf{6}$'s can be decomposed into four multiplets which are either symmetric, antisymmetric or of mixed symmetry:

$$\mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6} = \mathbf{56}_{\text{sym}} \oplus \mathbf{70}_{\text{mixed}} \oplus \mathbf{70}_{\text{mixed}} \oplus \mathbf{20}_{\text{antisym}} \ . \tag{1.4}$$

As mentioned above we need a symmetric representation for flavor and spin. Hence, the $\mathbf{56}_{\text{sym}}$ is the right choice. Since it is more common to treat quark flavor and spin separately we decompose the symmetric 56-plet again into two ground state $SU(3)_f$ flavor multiplets where each belongs to a fixed spin quantum number [2]:

$$\mathbf{56}_{\text{sym}} = {}^4\mathbf{10}_f \oplus {}^2\mathbf{8}_f \ , \tag{1.5}$$

with a superscript $(2J + 1)$ which defines the total spin J of the three quark state. Namely, one finds a spin $3/2$ flavor decuplet and a spin $1/2$ flavor octet, where the elements in both are classified by their electric charge Q, strangeness S and isospin I. In the octet (Fig. 1.1(c)) one has the nucleon isospin doublet of the proton and neutron as well as a strangeness $S = -1$ iso-triplet (Σ^+, Σ^0, Σ^-) and iso-singlet (Λ) and finally, a double-strange iso-doublet containing the two Ξ baryons with zero and negative charge. In the decuplet (Fig. 1.1(d)) one finds the Δ resonances and the spin $3/2$ states of Σ and Ξ and furthermore the Ω^- with strangeness $S = -3$.

Up to know we have only considered the three lightest quarks, but of course there are also hadrons made of charm and bottom quarks (note, that due to its very short life time the top quark is not relevant in hadron formation [2]). In order to take such states into account one can extend the $SU(3)_f$ flavor group to a $SU(4)_f$ or even a $SU(5)_f$ flavor symmetry by adding c and b quarks. However, since charm and bottom quarks are much heavier than u, d and s these symmetries are badly broken. Hence, the quantum mechanical mixture between a state with e.g. an u quark replaced by a c is negligible small. The three dimensional multiplets for the $SU(4)_f$ containing the charm quark can be found for example in Ref. [2].

4
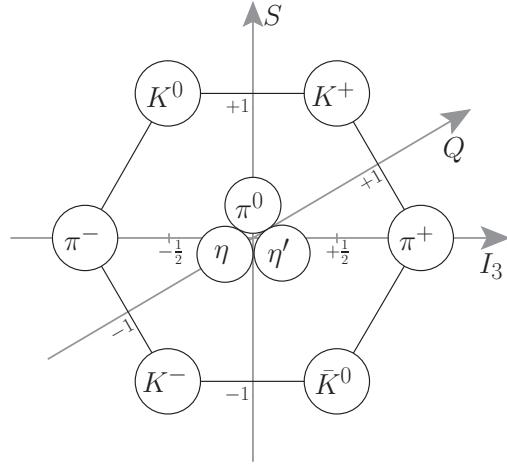
(a) pseudoscalar mesons
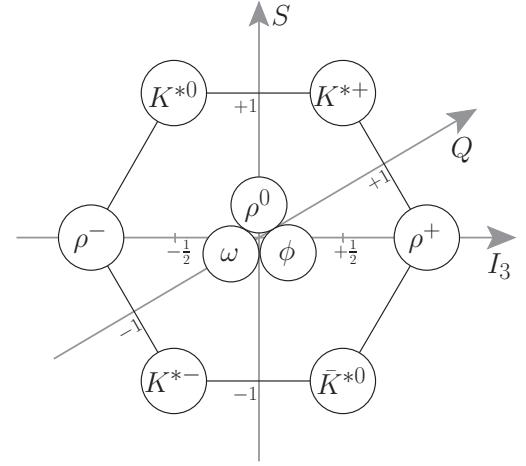
(b) vector mesons
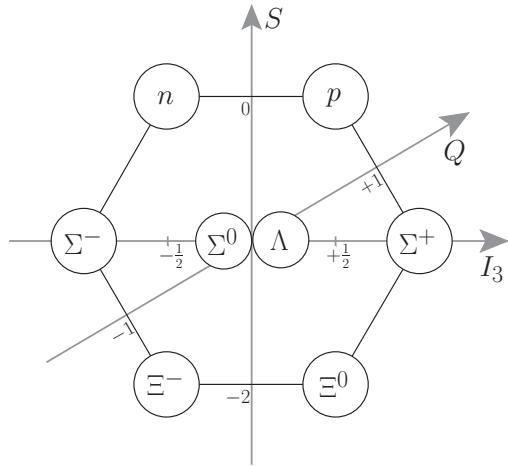
(c) spin 1/2 baryons
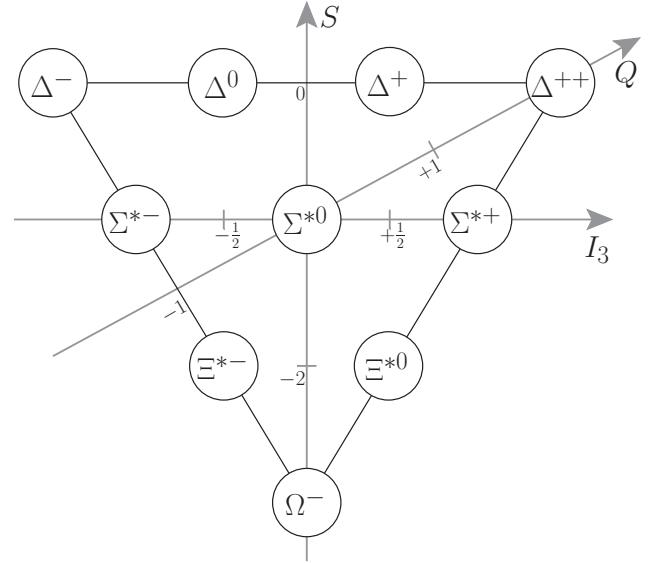
(d) spin 3/2 baryons

**Figure 1.1:** $SU(3)_f$ *pseudoscalar* (a) *and vector* (b) *meson nonets as well as $SU(3)_f$ spin 1/2 baryon octet* (c) *and spin 3/2 baryon decuplet* (d). *The particles contain u, d and s quarks and the quantum numbers vary from negative to positive values as indicated by the axes: electric charge Q, third component of the isospin I_3 from left to right and strangeness S from bottom to top.*

1.2 Exotic hadrons

In the previous section we have argued that mesons, baryons and anti-baryons are the conventional color-singlets of the standard model (see Fig. 1.2). However, there are in principle more allowed states: combinations like qq are indeed forbidden as they are not color-neutral, but the present knowledge of QCD does not forbid states made of more than three quarks. In few words: the combination of two $SU(3)_c$ invariants is still invariant. Thus, one could construct so-called *exotic* hadrons via the combinations $\bar{q}q\bar{q}q$, $\bar{q}qqqq$, $\bar{q}q\bar{q}\bar{q}\bar{q}$, $qqqqqq$, $\bar{q}\bar{q}\bar{q}\bar{q}\bar{q}\bar{q}$ and $qqq\bar{q}\bar{q}\bar{q}$ or – in principle – via combinations with even more quarks, grouped in the same manner. Additionally, also $\bar{q}qg$ quark–gluon combinations or even pure gluonic bound states like gg are not forbidden since they are color-neutral. Although all these states are referred to as exotic hadrons there is a more precise classification which takes into account their substructure: starting from behind there are on the one hand *glueballs* (gg) [4–6] and *hybrids* ($\bar{q}qg$) [4,7] where gluons are explicit degrees of freedom and on the other hand pure quark states with various numbers of quarks and anti-quarks which are called *tetra-*, *penta-* or *hexaquarks* in case of four, five or six constituents, respectively. These multi-quark states are compact objects which are tightly bound by the strong force, i.e. by quark–gluon interactions.

A different type of exotic hadrons are *hadronic molecules* whose concept was introduced in Refs. [12–14,56,57]. They are defined as multi-quark states with at least four constituents ($\bar{q}q\bar{q}q$) where the quarks cluster into conventional mesons, baryons or anti-baryons which are clearly separated and only bound by the nuclear force, i.e. light-meson (dominantly pion) exchange. Thus, hadronic molecules are extended objects in contrast to the compact tetraquarks, pentaquarks, etc. A sketch of the substructure of exotic hadrons is shown in Fig. 1.3.

Obviously, a particle which contains for example four quarks could be either a tetraquark or a hadronic molecule. Moreover, even the decay products of such a particle do – in general – not exclude one or the other substructure. A fully solved QCD would answer the question which explanation for a given particle is the right one. However, we are far from understanding this theory completely. Hence, there is still an intensive discussion about the exact substructure of the up to now discovered exotic hadrons.

Figure 1.2: *Conventional hadrons.*

1.2.1 Experimental results

Considering the results of a large number of finished or still running experiments like BaBar, Belle, BES or LHCb, it is clear that the mentioned discussion about the substructure of exotic hadrons is not an academic one. The former three experiments are located at so-called B-factories, i.e. e^+e^- colliders originally built for charmonium ($c\bar{c}$) and bottomonium ($b\bar{b}$) spectroscopy
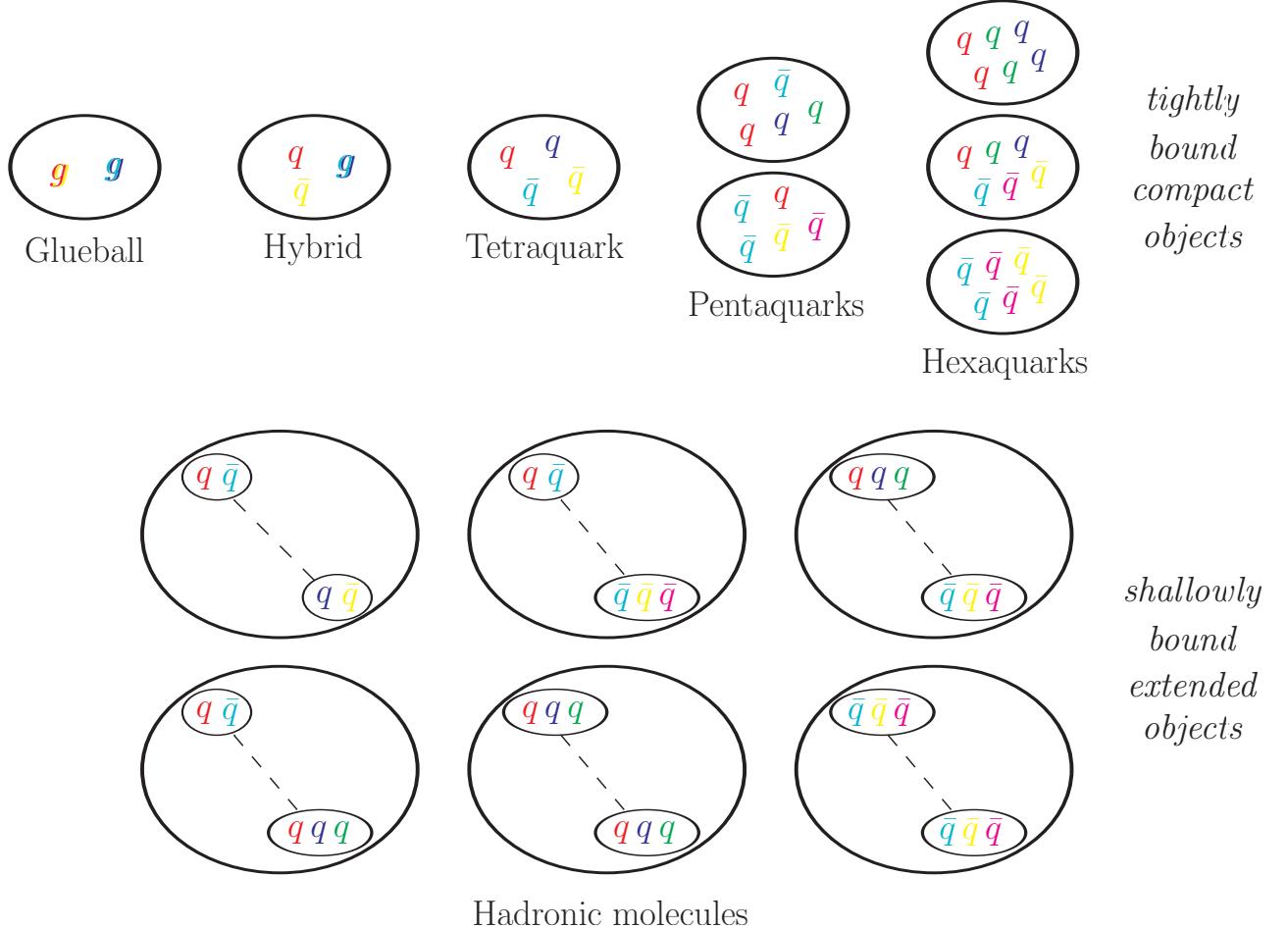
Figure 1.3: *Exotic hadrons.*

[15–17]. The latter analyzes the collision of protons or lead nuclei at extremely high center-of-mass energies of up to 8 TeV [18]. Contrary to the expectations the B-factories discovered a large number of states which do not fit into the scheme of conventional hadrons. Furthermore, some of these states are even charged so that it is clear (because of their definitely present $c\bar{c}$ content) that they must contain more than two quarks and thus must be exotic. The experiments observed among other states the electrically neutral $X(3872)$ [20] and $Y(4260)$ [21] as well as the charged $Z_c(3900)$ [22–24], $Z_c(4020)$ [25–28], $Z_c(4430)$ [29–31] and the somewhat controversial states $Z_1(4051)$, $Z_2(4250)$ [32] in the charmonium sector. Also in the bottomonium sector two states were found, the $Z_b(10610)$ and the $Z_b'(10650)$ [33]. An overview of the experimentally observed XYZ states and their possible substructure in the charmonium and bottomonium sector can be found in [34, 35] and most recent in Ref. [19] where Cleven *et al.* reviewed the particles above in full detail.

Besides these relatively heavy states without any open strangeness, charm or bottomness there additionally are at least two particles, $D_{s0}^*(2317)$ and $D_{s1}(2460)$, whose masses are only half as large and which carry open charm and strangeness. They were seen by BaBar [36] and by CLEO-c [37]. Soon after their discovery it was proposed that they could be $D^{(*)}K$ molecules [43, 44]

rather than conventional mesons with $c\bar{s}$ quark content (later the molecule interpretation was also discussed in Refs. [45–48]). A possible third candidate of this type is the $D^*_{s1}(2700)$ seen by Belle [59] which could be a $D_1 K$ molecule, however, due to its decay into $D^0 K^+$ it is more likely a radial excited D^*_{s0} as already indicated by its name.

As last candidates for exotic hadrons we mention the long known resonances $a_0(980)$ and $f_0(980)$ whose exact substructure is a widely discussed subject. A theoretical discussion of the possible substructure of all these states can be found in Refs. [60–62] and from an experimental view in Ref. [63].

1.3 Effective field theory

In section 1.1 we have already noted that QCD is not perturbative in the energy region of hadron formation. The self-energy corrections to the gluons which are present due to the fact that gluons themselves carry color, yield a strong coupling constant α_s which becomes weaker for higher energies (*asymptotic freedom*). In fact, for energies around 1 GeV, α_s is of order 1 and hence perturbation theory breaks down (for a review of experimental results concerning the running of α_s see Ref. [2]). It is thus necessary to find a different approach to describe QCD at low energies. Indeed, effective field theories (EFT's) are a perfectly suitable choice since they are designed to perform calculations in the low-energy regime of quantum field theory (QFT). The basic idea is that a physical phenomenon at a given energy scale E_0 is not affected by the details of high-energy effects at $E \gg E_0$. A semi-classical example would be the scattering of a low-energy photon with wavelength 500 nm off a crystal. Since the wavelength is too large to resolve the lattice structure of the crystal, Bragg-scattering as a high-energy phenomenon is not relevant. Following the ideas of Weinberg [64] one needs to identify two scales in a system to construct an EFT: one low-energy scale m and one high-energy scale Λ which define an expansion parameter m/Λ. It allows to arrange all – in general infinitely many – terms in the *effective* Lagrangian density which are allowed by the symmetries of the original full theory in powers of this expansion parameter. As long as $m/\Lambda < 1$ one knows that a term proportional to $(m/\Lambda)^{n+1}$ is less important than a term proportional to $(m/\Lambda)^n$. In this way one obtains a power-counting scheme which allows to calculate observables up to a certain order, knowing that the contribution of all higher order terms is small and thus can be neglected. Moreover, it is possible to use experimental data to fit unknown constants (so-called low-energy constants, LEC's). These constants are prefactors of the terms in the effective Lagrangian, so it is in principle possible to calculate observables to an arbitrary high order as long as enough experimental input is available. However, one subtlety of EFT's is that they are not renormalizable because new parameters (the LEC'S) appear in them if one goes to higher orders. In fact, this distinguishes an EFT from conventional perturbation theory in which independently of the considered order, the number of parameters is fixed in a way that the theory can be renormalized. Furthermore, the expansion parameter in an EFT is not assumed to be fundamental in physics. Consequently, the effective scales can change in a way that the expansion parameter m/Λ becomes of order 1 or even larger and hence a series in m/Λ does not converge anymore and the EFT breaks down. In terms of the photon example this means that the effective theory breaks down if the photon wavelength becomes of the order of the lattice spacing in the crystal since Bragg-scattering cannot be neglected anymore. Thus,

one has to ensure that the expansion parameter of an EFT is indeed small in the energy region where one wants to investigate the original theory.

1.3.1 EFT's for QCD

After this rather general introduction to EFT's two important effective theories for low-energy QCD are presented.

Chiral perturbation theory

For energies up to few GeV where the interaction of nucleons (or other heavier hadrons) can be described by pion exchange (or if one includes the strange quark, also by kaon or eta-meson exchange) a famous EFT called *chiral perturbation theory* (ChPT) based on the work of Weinberg in Ref. [65] and of Gasser and Leutwyler in Refs. [66, 67] exists. This theory uses the fact that the masses of the two (or, including s, three) lightest quarks are relatively small (see Eq. (1.1)). Hence, one can impose chiral symmetry which states that in a vector gauge theory with massless fermions, the right- and left-handed components of the latter can be transformed independently. Due to the existence of the quark condensate and the pion decay constant $f_\pi \neq 0$ the chiral symmetry is spontaneously broken with the three pions being the corresponding Goldstone bosons. To be more precisely the pions are pseudo-Goldstone bosons since the chiral symmetry is only an approximate symmetry because u and d quarks are not massless in nature. In fact, the mass of the quarks is also the reason why one cannot construct a ChPT including the very massive charm, bottom or top quarks. A extensive overview of ChPT can be found in Ref. [68].

Pionless effective field theory

As mentioned above ChPT is valid up to a few GeV, but especially in nuclear physics the energies are often even smaller, i.e. in the range of a few MeV. The question arises if one can construct an effective field theory also in this energy region. Indeed, this is possible: for energies where the pion exchange between heavy hadrons like nucleons is not relevant one can treat these hadrons as non-relativistic point-like particles which only interact via contact interactions. Such a theory with an expansion parameter Q/m_π with Q being the internal momentum is called *pionless effective field theory* (EFT($\not\pi$)). This was introduced in Refs. [69, 71–73]. It was derived for nucleon–nucleon interactions and is a commonly used tool in nuclear physics [74–77]. Especially the deuteron was very successfully analyzed in EFT($\not\pi$) (see Refs. [71–73, 78, 79, 93]). Since one can treat the deuteron as a simple hadronic molecule made of two nucleons, one concludes that EFT($\not\pi$) might also be suitable for the description of other hadronic molecules as long as their constituents have masses of the order of 1 GeV or more and their binding momenta (which define the internal momenta) are much smaller than the pion mass. However, also for lighter (but still reasonably heavier than pions) constituents or binding momenta around m_π one could use a pionless effective field theory to obtain at least some first insights into such a system. A comprehensive review of EFT($\not\pi$) and the universality in few-body physics can be found in Ref. [80].

1.4 Basic scattering theory

In this section we derive all relations of the effective range expansion (ERE) in scattering theory up to second order (next-to leading order, NLO) for the sake of completeness. However, throughout this thesis we will only consider the leading order (LO) effective range expansion.

We start with quantum mechanics and follow Refs. [81,82]: for a finite range potential V which falls off sufficiently fast for large r the wave function of a scattered particle has the following asymptotic form:

$$\psi(\mathbf{r}) = e^{i\mathbf{k}_i \cdot \mathbf{r}} + f(k,\theta)\frac{e^{ik_f r}}{r} \,, \tag{1.6}$$

where the scattering amplitude $f(k,\theta)$ depends on the modulus $k := |\mathbf{k}_i| = |\mathbf{k}_f|$ of the incoming (index i) and outgoing momenta (index f) and on the angle θ between those, i.e. $k^2 \cos\theta = \mathbf{k}_i \cdot \mathbf{k}_f$. If one considers a central potential $V(\mathbf{r}) \equiv V(r)$ one can expand the scattering amplitude in partial waves with angular momentum L. Ignoring additional spin for the moment the expansion has the form [81,82]:

$$f(k,\theta) = \frac{1}{k}\sum_L (2L+1)e^{i\delta_L}\sin(\delta_L)\mathcal{P}_L(\cos\theta) \,, \tag{1.7}$$

with the Legendre polynomials $\mathcal{P}_L$ and a phase shift δ_L. As we will deal with S-wave states in the following chapters we will now assume $L = 0$ and conclude from Eq. (1.7) that the scattering amplitude f is independent of θ and thus only a function of k. Since S-wave scattering is spherical symmetric one can furthermore integrate the incoming plane wave in Eq. (1.6) over the solid angle to find

$$\int \frac{d\Omega}{4\pi} e^{i\mathbf{k}_i \cdot \mathbf{r}} = \frac{\sin(kr)}{kr} \,, \tag{1.8}$$

which can be written as

$$\frac{\sin(kr)}{kr} = -\frac{1}{2i}\frac{e^{-ikr}}{kr} + \frac{1}{2i}\frac{e^{ikr}}{kr} \,. \tag{1.9}$$

As only the outgoing wave e^{ikr} can be affected by the scattering process and since the particle number is conserved, the only possible modification due to the potential is a phase shift $\delta \equiv \delta_{L=0}$ in the outgoing wave, that is,

$$e^{ikr} \overset{\text{scattering}}{\longrightarrow} e^{i(kr+2\delta)} = e^{ikr} + 2ie^{ikr}e^{i\delta}\sin\delta \,. \tag{1.10}$$

Hence, the asymptotic wave function in case of S-wave scattering has the form:

$$\psi = e^{i\mathbf{k}_i \cdot \mathbf{r}} + \frac{e^{i\delta}\sin\delta}{k}\frac{e^{ik_f r}}{r} \,, \tag{1.11}$$

which has to be compared with Eq. (1.6) in order to find a relation for the scattering amplitude in terms of the phase shift:

$$f(k) = \frac{e^{i\delta}\sin\delta}{k} = \frac{1}{k}\frac{\sin\delta}{\cos\delta - i\sin\delta} = \frac{1}{k\cot\delta - ik} \,. \tag{1.12}$$

Bethe has shown in Refs. [83, 84] that for a finite range potential one can expand $k \cot \delta$ about $k = 0$ yielding up to second order

$$k \cot \delta = -\frac{1}{a} + \frac{1}{2}r_0 k^2 + \dots , \tag{1.13}$$

where the scattering length a and the effective range r_0 were introduced (the reason why this approximation is called *effective range expansion*). Next, one can use Eq. (1.13) and Eq. (1.12) to finally get the second order ERE S-wave scattering amplitude:

$$f_{\mathrm{NLO}}^{\mathrm{ERE}}(k) = \frac{1}{-\frac{1}{a} + \frac{1}{2}r_0 k^2 - ik} . \tag{1.14}$$

In the next step we identify a relation between the quantum mechanical scattering amplitude f and the interaction part T of the scattering matrix $S = \mathbb{1} + T$ in quantum field theory. For this purpose we again consider the wave function ψ of Eq. (1.6) which obeys the Schrödinger equation

$$(H_0 - E)\psi = -V\psi , \tag{1.15}$$

where H_0 is the free Hamilton operator. Inverting this equation one finds that for $1/(H_0 - E)$ being the inverse of the operator $H_0 - E$ it holds:

$$\psi = -\frac{1}{H_0 - E}V\psi = \phi - \frac{1}{H_0 - E}V\psi , \tag{1.16}$$

with the free particle solution $\phi = e^{i\mathbf{k}_i \cdot \mathbf{r}}$ which fulfills $(H_0 - E)\phi = 0$. However, a particle can in general undergo multiple scattering processes. In order to take this fact into account one has to extend Eq. (1.16) using the T-matrix:

$$\psi = \phi - \frac{1}{H_0 - E}T\phi . \tag{1.17}$$

The derivation of this relation can be found for example in Ref. [1] where it is shown that one can schematically define the T-matrix in the following way:

$$T := V + V\frac{1}{H_0 - E}V + V\frac{1}{H_0 - E}V\frac{1}{H_0 - E}V + \dots = V + V\frac{1}{H_0 - E}T . \tag{1.18}$$

With this knowledge one can show that the quantum mechanical S-wave scattering amplitude f is related to the QFT T-matrix by

$$T(k) = \frac{2\pi}{\mu}f(k) , \tag{1.19}$$

where the prefactor $2\pi/\mu$ (with reduced mass μ) accounts for the different normalization in QFT. According to Eq. (1.19) the T-matrix itself is often called scattering amplitude and we will do so, too. Using what we have found in Eq. (1.14) one obtains the following first and second order ERE S-wave scattering amplitude (or T-matrix):

$$T_{\mathrm{LO}}^{\mathrm{ERE}}(k) = -\frac{2\pi}{\mu}\frac{1}{\frac{1}{a} + ik} \tag{1.20}$$

$$T_{\mathrm{NLO}}^{\mathrm{ERE}}(k) = -\frac{2\pi}{\mu}\frac{1}{\frac{1}{a} - \frac{1}{2}r_0 k^2 + ik} , \tag{1.21}$$

from which, however, we need in the following chapters only the LO relation.

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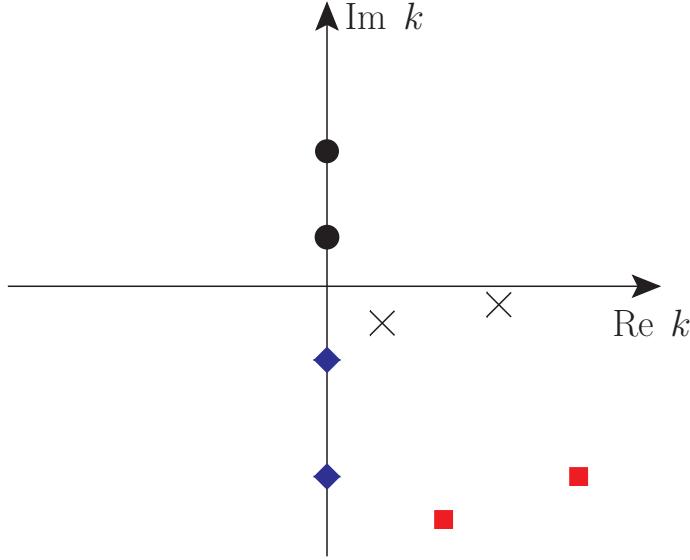
Figure 1.4: *Poles of the S-matrix in the complex momentum plane of k. Bound states corres-pond to poles marked by a dot on the positive imaginary axis, virtual states to the ones marked by diamonds on the negative imaginary axis. The remaining crosses and squares indicate poles on the second sheet, i.e. in the lower half of the k-plane. They are either identified as resonances (being states with definite quantum numbers so that one can interpret them as particles) if they are close to the positive real axis (crosses) or else as non-resonant background scattering effects (which could not be interpreted as particles) if they are far away from the positive real axis (squares).*

1.4.1 Some remarks on resonances, bound and virtual states

The scattering of two particles might be affected by inelastic effects due to intermediate two-body states which are either resonances, bound or virtual states. These states correspond to poles in the S-matrix (or equivalently in the T-matrix since $S = \mathbb{1} + T$) and are classified by the position of the respective pole in the complex momentum plane as it is for example explained in Ref. [82]. A bound state of two particles with masses m_1 and m_2 (and reduced mass μ) corresponds to a pole at $k = i\gamma$ with $\gamma > 0$, namely, to a pole with vanishing real part, but non-zero imaginary part. The energy of this state is $E_B = k^2/(2\mu) = -\gamma^2/(2\mu) < 0$ which is identified with the (negative) binding energy. Moreover, this relation also motivates the term *binding momentum* for γ. Note, that we will define (using the bound state mass M_{12}) a quantity $B := m_1 + m_2 - M_{12}$ which has positive values for bound states so that $B = -E_B$ and call B as well "binding energy". Using B instead of E_B one can write γ as $\gamma = \sqrt{-2\mu E_B} = \sqrt{2\mu B}$ which will be repeatedly used in this work. For energies below its threshold $m_1 + m_2 + E_B = m_1 + m_2 - B$ a bound state is stable according to the force which has generated it (which is the strong force for hadron–hadron scattering). However, there could be other forces in nature which can cause a decay of a bound state also below threshold via the decay of its constituents (in our case of hadrons these forces would be the electromagnetic or weak force). Next, we consider virtual states which are in some sense the counterparts of bound states with the difference that they are located on the

second sheet, i.e. in the lower half of the k-plane. They lie on the negative imaginary axis again corresponding to a pole at $k = i\gamma$, but with $\gamma < 0$. Their energy $E_B = k^2/(2\mu) = -\gamma^2/(2\mu) < 0$ is the same as for bound states. However, the quantity $B = m_1 + m_2 - M_{12}$ is negative for virtual states which thus are sometimes called *anti-bound*. Since the sum of the constituent masses $(m_1 + m_2)$ is smaller than the mass of the two-body state (M_{12}), virtual states cannot be observed in nature. But due to the corresponding pole in the S-matrix they affect the scattering of the two particles. The "binding momentum" γ of virtual states is negative, but also the binding energy B is smaller than zero. Hence, one must modify the relation between them according to

$$\gamma = \text{sgn}(B)\sqrt{2\mu|B|}\,, \tag{1.22}$$

which is valid for both bound and virtual states. Finally, resonances are dynamically generated states located on the second sheet somewhere below, but close to the positive real axis. They thus have a rather small imaginary part $(-k_i,\, k_i > 0)$ of the momentum, but also a non-vanishing real part (k_r) corresponding to a kinetic energy which is the reason for their *widths* as it can be seen from

$$E = \frac{k^2}{2\mu} = \frac{(k_r - ik_i)^2}{2\mu} = \frac{k_r^2 - k_i^2 - 2ik_rk_i}{2\mu} := E_r - i\frac{\Gamma}{2}\,, \tag{1.23}$$

where E_r is the resonance position and Γ the resonance width. From Eq. (1.23) it is also clear that the widths grows as the imaginary part of k becomes larger. Hence, a resonance far away from the real axis (i.e. with a large imaginary part k_i) is so broad that it cannot be interpreted as particle anymore, but rather as some non-resonant background in the scattering process. In Fig. 1.4 we have summarized the classification of resonances, bound and virtual states by sketching the corresponding poles of the S-matrix in the complex momentum plane of k.

In general hadronic molecules can be either bound or virtual states or resonances since all of them can be considered as particles with definite quantum numbers. However, we will below use the binding momentum as variable in the scattering amplitudes of processes including hadronic molecules. Hence, the particles we consider as molecular states should be either bound or virtual states where γ is well-defined. Nevertheless, one can – motivated by the often large experimental errors in the masses – argue that one could also assign to resonances which are located not too far away from the imaginary axis at least an approximate binding momentum.

1.4.2   Pole position and effective range expansion

In the first part of this section the ERE amplitude was derived. Now we will relate the pole position $k = i\gamma$ of bound states to this expansion. Considering Eq. (1.14) or equivalently Eq. (1.20) one concludes that the scattering amplitude $f(k)$ or $T(k)$ has a pole for a vanishing denominator. At LO this leads to

$$-\frac{1}{a} - ik = -\frac{1}{a} + \gamma \overset{!}{=} 0\,, \tag{1.24}$$

which yields

$$\gamma = \frac{1}{a}\,. \tag{1.25}$$

At NLO the denominator of the scattering amplitude is a function of k^2 and hence one finds

$$-\frac{1}{a} + \frac{1}{2}r_0 k^2 - ik = -\frac{1}{a} - \frac{1}{2}r_0 \gamma^2 + \gamma \overset{!}{=} 0 \,. \tag{1.26}$$

This quadratic equation has two solutions

$$\gamma_{1,2} = \frac{1}{r_0} \pm \frac{1}{r_0}\sqrt{1 - 2\frac{r_0}{a}} \,. \tag{1.27}$$

For a large scattering length $a \gg r_0$ being much larger than the effective range r_0 one can expand the square root in Eq. (1.27) using $\sqrt{1-x} \approx 1 - x/2 + \mathcal{O}(x^2)$ (valid for small x) to obtain

$$\gamma_{1,2} = \frac{1}{r_0} \pm \frac{1}{r_0}\left(1 - \frac{r_0}{a}\right) = \begin{cases} \frac{2}{r_0} - \frac{1}{a} \overset{a \gg r_0}{\approx} \frac{2}{r_0} \\ \frac{1}{a} \end{cases} \,, \tag{1.28}$$

where $\gamma_1 \approx 2/r_0$ corresponds to a deeply bound pole and where $\gamma_2 = 1/a$ reproduces the result Eq. (1.25) found at LO which corresponds to a shallow bound state with energy $E_B = \gamma_2^2/(2\mu) = 1/(2\mu a^2)$ which only depends on the scattering length.

Note, that the same derivation can be done for virtual states with a pole also at $k = i\gamma$, but with $\gamma < 0$ and hence one concludes from Eq. (1.25) that virtual states have a negative scattering length.

Chapter 2

Universality and the Efimov effect

So far we have only considered two-body systems. In the following we move on to three-body physics, in particular we discuss the phenomenon of the emergence of a three-body bound state spectrum in systems with large scattering length. This effect is an example of the universal scaling behavior of systems with large scattering length (*universality*) and known as *Efimov effect*. It was proposed by Efimov [85], theoretically proven in Ref. [86, 87] and experimentally observed, first in Ref. [100] (which is also reviewed in Ref. [88]). In a system of three identical spin- and isospinless bosons with divergent scattering length the Efimov effect leads to an infinite number of three-body bound states (*trimers*) whose binding energies B_3 are geometrically spaced and which accumulate at the three-body scattering threshold. The spacing is defined by the so-called *scaling factor*:

$$\frac{B_3^{(n+1)}}{B_3^{(n)}} \approx 515.03 \, . \tag{2.1}$$

Naively, one expects that in a scattering process the parameters of the ERE are all of the same order which is defined by the interaction range R of the corresponding potential $V(r) \to 0$ for $r > R$. Up to second order (i.e. NLO) this would mean that the S-wave scattering length a and effective range r_0 would scale as R. However, there could be systems where the scattering length is large, that is, much larger than the interaction range $a \gg R$ and hence much larger than the effective range $a \gg r_0 \sim R$. Such systems with a large scattering length $a \to \infty$ (i.e. systems in the *resonant limit*) exhibit a so-called *universal scaling behavior*, meaning that observables can be given in terms of the scattering length only. Besides in system where the scattering length $a \gg r_0 \sim R$ itself is large one finds an universal behavior also in systems where the interaction range goes to zero (*scaling* or *zero-range limit*).

Although there are no systems with divergent scattering length in nature, there are systems with reasonably large a. For example the nucleon–nucleon ${}^1S_0$ system with scattering length $a_{\rm NN} = -18.7(6)$ fm [89] which is approximately an order of magnitude larger than the effective range $r_{0,\rm NN} = 2.75(11)$ fm [90] in this channel. Moreover, systems treated in pionless EFT only interact via contact interactions (see section 1.3) which means that the interaction range is zero. Therefore every system that can be described in EFT($\not\pi$) is universal up to higher order corrections. Hence, the scattering of two particles with large, but finite scattering length has an universal scaling behavior as long as it is allowed ($\gamma < m_\pi$) to analyze it in EFT($\not\pi$) where the

interaction range is zero. Consequently, an accordant three particle system could be affected by the Efimov effect with an exact scaling behavior like that in Eq. (2.1). However, although the number is constant for all n it must not necessarily be 515.03. A further difference for systems with finite scattering length is that the trimer spectrum is cut off at the two-body (*dimer*) threshold B_2. Hence, there are not infinitely many Efimov trimers, but only as much as are fitting into the energy region $B_3^{(0)} < E < B_2$.

The Efimov effect and universality in few-body systems in general are discussed in Ref. [80] of Hammer and Braaten which in particular also contains a review of many important methods in EFT($\not{\pi}$) like the concept of dimer auxiliary fields. The following summary is based on this work.

2.1 Efimov effect for identical bosons

We consider a system of three identical particles ψ of mass m, but without spin or isospin degrees of freedom. Additionally, we assume that they have a two-body interaction with large scattering length and a three-body interaction. In the scaling limit, namely, for a vanishing interaction range, such a system can be described by the non-relativistic Lagrangian density [80]

$$\mathcal{L} = \psi^\dagger \left(i\partial_t + \frac{\nabla^2}{2m} \right) \psi - g_2 \left(\psi^\dagger \psi \right)^2 - g_3 \left(\psi^\dagger \psi \right)^3 . \tag{2.2}$$

Here, we use the so-called *dimer field trick* [91, 92] instead of the Lagrangian introduced in Eq. (2.2). Therefore we introduce an auxiliary dimer field d which represents a two-body bound state of two ψ particles. With this new field one can write down a new Lagrangian density,

$$\mathcal{L}_d = \psi^\dagger \left(i\partial_t + \frac{\nabla^2}{2m} \right) \psi + g_2 \, d^\dagger d - g_2 \left(d^\dagger \psi^2 + \left(\psi^\dagger \right)^2 d \right) - g_3 \, d^\dagger d \psi^\dagger \psi , \tag{2.3}$$

which is equivalent to Eq. (2.2). This equivalence can be seen by eliminating the d field from Eq. (2.3) using its equation of motion as it is shown in Ref. [80]. We note, that there is no kinetic term for the dimer field (which would contain time derivatives) and thus it is not dynamic. Nevertheless, it interacts with the field ψ via the last two terms in Eq. (2.3). The first interaction term proportional to g_2 thereby describes the decay of the dimer field into two boson fields ψ. Thus, it leads to self-energy corrections to the constant bare propagator i/g_2 of the field d. At the end, these corrections allow the dimer field to propagate in space and time. To show this explicitly we consider the *Dyson equation* shown in Fig. 2.1(a) as an infinite series of Feynman diagrams where a thick solid line represents the bare and a double line the full propagator of the dimer field.

Naming the full one as $iD(p_0, \mathbf{p})$ and the bare one $iD^0 = i/g_2$ one can – according to Fig. 2.1(a) – write down a relation for the full propagator:

$$iD(p_0, \mathbf{p}) = iD^0 + iD^0 \, i\Sigma \, iD^0 + iD^0 \, i\Sigma \, iD^0 \, i\Sigma \, iD^0 \, + \, ...$$

$$= iD^0 \left[\sum_{n=0}^{\infty} \left(-\Sigma D^0 \right)^n \right] = \frac{iD^0}{1 - (-\Sigma D^0)} = \frac{i}{(D^0)^{-1} + \Sigma} , \tag{2.4}$$

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(a)

$$i\Sigma(p_0, \mathbf{p}) =$$

(b)
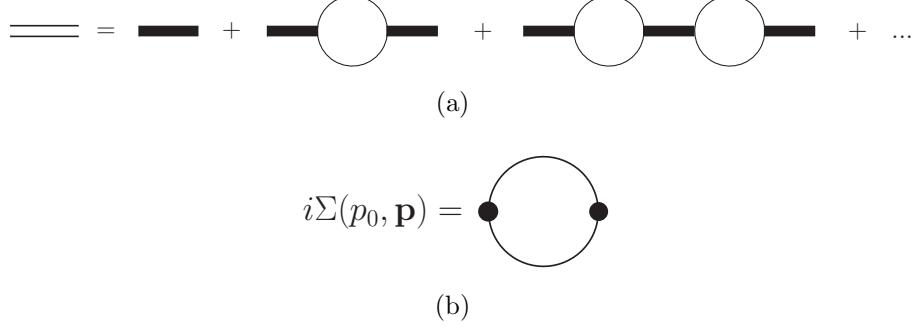
Figure 2.1: *Dyson equation for the dimer field in the three identical boson system as infinite series of boson loops. A thick solid line represents the bare and a double line the full dimer field propagator while a single line stands for the ψ propagator (a). Self-energy diagram for the identical boson system. The single lines in the loop represent the boson fields ψ (b).*

where we have introduced the self-energy Σ and used the geometric series to simplify the infinite sum. In order to find an expression for the self-energy we firstly identify the corresponding Feynman diagram (Fig. 2.1(b)) and then use the Feynman rules following from the Lagrangian in Eq. (2.3) to obtain

$$\Sigma(p_0, \mathbf{p}) = 2g_2^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{p_0 + q_0 - \frac{1}{2m}\left(\frac{\mathbf{p}}{2} + \mathbf{q}\right)^2 + i\varepsilon} \frac{1}{-q_0 - \frac{1}{2m}\left(\frac{\mathbf{p}}{2} - \mathbf{q}\right)^2 + i\varepsilon} , \qquad (2.5)$$

with the loop momentum $(q_0, \mathbf{q})$. We have used that the non-relativistic propagator of the field ψ is given by

$$iS(p_0, \mathbf{p}) = \frac{i}{p_0 - \frac{|\mathbf{p}|^2}{2m} + i\varepsilon} . \qquad (2.6)$$

Note, that the extra factor of 2 in Eq. (2.5) is a symmetry factor due to the fact that the bosons are identical. Applying the residue theorem one can carry out the integral over dq_0 yielding

$$\Sigma(p_0, \mathbf{p}) = 2g_2^2 \int \frac{d^3q}{(2\pi)^3} \frac{i\,m}{q^2 - mp_0 + \frac{p^2}{4} - i\varepsilon} , \qquad (2.7)$$

which can now be calculated in dimensional regularization with a scale μ using the relation

$$\int d^dq \frac{1}{(q^2 + 2\,q \cdot k - b^2)^\alpha} = (-1)^{\frac{d}{2}} i\pi^{\frac{d}{2}} \frac{\Gamma\left(\alpha - \frac{d}{2}\right)}{\Gamma(\alpha)} \left[-k^2 - b^2\right]^{\frac{d}{2}-\alpha} , \qquad (2.8)$$

which can be found in many QFT textbooks, for example in Ref. [1]. In $D = 4$ dimensions (i.e. $d = D - 1$), with $k = 0$ and $\Gamma(-1/2) = -2\sqrt{\pi}$ we thus end up with:

$$\Sigma(p_0, \mathbf{p}) = \lim_{D \to 4} \left(2g_2^2 \mu^{D-4} \int \frac{d^{D-1}q}{(2\pi)^{D-1}} \frac{i\,m}{q^2 - mp_0 + \frac{p^2}{4} - i\varepsilon}\right) = -\frac{mg_2^2}{2\pi}\sqrt{-mp_0 + \frac{p^2}{4} - i\varepsilon}. \quad (2.9)$$

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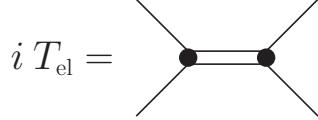
Figure 2.2: *Elastic scattering amplitude of two identical bosons ψ with an intermediate dimer state d.*

Hence, the full propagator of the dimer field can according to Eq. (2.4) be written as

$$iD(p_0, \mathbf{p}) = \frac{i}{g_2 - \frac{mg_2^2}{2\pi}\sqrt{-mp_0 + \frac{p^2}{4} - i\varepsilon}} = -\frac{2\pi}{mg_2^2} \frac{i}{-\frac{2\pi}{mg_2} + \sqrt{-mp_0 + \frac{p^2}{4} - i\varepsilon}} \ . \tag{2.10}$$

In the next step we relate the full propagator to the binding momentum γ of the dimer. Hence, we consider the elastic scattering of two boson with an intermediate dimer state (Fig. 2.2). The corresponding amplitude is given by

$$T_{\mathrm{el}}(k) = -4g_2^2 D\left(p_0 = E = \frac{k^2}{m}, \mathbf{p} = 0\right) = \frac{8\pi}{m} \frac{1}{-\frac{2\pi}{mg_2} + \sqrt{-k^2 - i\varepsilon}} = -\frac{8\pi}{m} \frac{1}{\frac{2\pi}{mg_2} + ik} \ , \tag{2.11}$$

with k being the modulus of the incoming and outgoing momenta which are equal in an elastic process. Additionally, we have taken into account a symmetry factor of 4 due to the identical bosons. Comparing this result with the leading order ERE amplitude in Eq. (1.20) we deduce that the scattering length and thus the binding momentum (cf. LO version of Eq. (1.25)) can be identified as

$$\frac{1}{a} \equiv \gamma = \frac{2\pi}{mg_2} \ . \tag{2.12}$$

This allows us to finally write the full propagator of the dimer field as

$$iD(p_0, \mathbf{p}) = -\frac{2\pi}{mg_2^2} \frac{i}{-\gamma + \sqrt{-mp_0 + \frac{p^2}{4} - i\varepsilon}} \ . \tag{2.13}$$

Moreover, this leads to a wave function renormalization constant Z defined as the residue of the pole in the full propagator [1] and given by

$$Z = \frac{4\pi\,\gamma}{g_2^2 m^2} \ . \tag{2.14}$$

2.1.1 Three-body scattering amplitude

The considerations above provide in some sense a tool box for the following. Namely, the determination of the three-body scattering amplitude, known as Skorniakov–Ter-Martirosian (STM) equation [94, 95]. Firstly, we note that this task is reasonably simpler using an auxiliary dimer field instead of using the Lagrangian in Eq. (2.2). Still following Ref. [80] we have to solve the

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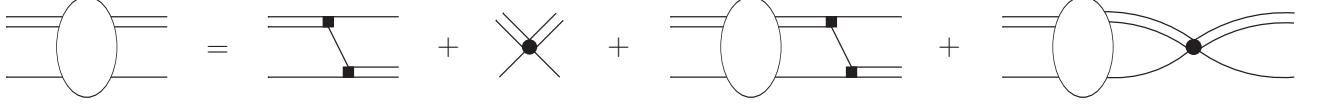
Figure 2.3: *Integral equation for the three-body scattering of identical bosons using the dimer field trick. A black square represent a two-body dimer–boson vertex and the black dot the three-body vertex.*

integral equation shown in Fig. 2.3 in terms of Feynman diagrams. This representation can be understood in the sense that the repeatedly insertion of the right-hand-side into itself will generate all allowed diagrams with an arbitrary number of bosons. The corresponding scattering amplitude is – using the Feynman rules according to the Lagrangian density Eq. (2.2) – in the center-of-mass system given by

$$
t(E, \mathbf{k}, \mathbf{p}) = - \left[\frac{4g_2^2}{E - \frac{k^2}{2m} - \frac{p^2}{2m} - \frac{(\mathbf{k}+\mathbf{p})^2}{2m} + i\varepsilon} + g_3 \right]
$$
$$
+ i \int \frac{d^4q}{(2\pi)^4} \, t(E, \mathbf{k}, \mathbf{q}) \, \frac{D(E + q_0, \mathbf{q})}{-q_0 - \frac{q^2}{2m} + i\varepsilon} \left[\frac{4g_2^2}{E + q_0 - \frac{p^2}{2m} - \frac{(\mathbf{p}+\mathbf{q})^2}{2m} + i\varepsilon} + g_3 \right] , \quad (2.15)
$$

with incoming 4-momentum k, outgoing one p, the center-of-mass energy E and a symmetry factor of 4 in front of g_2^2. Applying the residue theorem to solve the dq_0 integration sets $q_0 = -q^2/(2m)$. After multiplying with the wave function renormalization derived in Eq. (2.14), one ends up with the following equation for the renormalized amplitude $T := Z\, t$:

$$
T(E, \mathbf{k}, \mathbf{p}) = - Z \left[\frac{4g_2^2}{E - \frac{k^2}{2m} - \frac{p^2}{2m} - \frac{(\mathbf{k}+\mathbf{p})^2}{2m} + i\varepsilon} + g_3 \right]
$$
$$
+ \int \frac{d^3q}{(2\pi)^3} \, T(E, \mathbf{k}, \mathbf{q}) \, D\left(E - \frac{q^2}{2m}, \mathbf{q}\right) \left[\frac{4g_2^2}{E - \frac{q^2}{2m} - \frac{p^2}{2m} - \frac{(\mathbf{p}+\mathbf{q})^2}{2m} + i\varepsilon} + g_3 \right] .
$$
$$
(2.16)
$$

We will now restrict ourselves to the analysis of the S-wave three-body system. As explained in appendix D one can project out the $L = 0$ partial wave by applying the projection operator

$$
\frac{1}{2} \int_{-1}^{1} d\cos(\theta) \, \mathcal{P}_L(\cos\theta) \, .
$$

The remaining integral over d^3q can be rewritten in spherical coordinates and hence one ends up – after carrying out the integrals over the angles – with an expression proportional to the Legendre function of the second kind Q_L (see appendix D). In Eq. (D.15) it was derived a representation of $Q_{L=0}$ in terms of the logarithm. Hence, the three-body scattering amplitude in Eq. (2.16) only

depends on the moduli of the momenta and reads:

$$T(E,k,p) = \frac{16\pi\gamma}{m} \left[\frac{1}{2kp} \ln\left(\frac{p^2 + pk + k^2 - E - i\varepsilon}{p^2 - pk + k^2 - E - i\varepsilon} \right) - \frac{1}{4m}\frac{g_3}{g_2^2} \right]$$
$$+ \frac{4}{\pi} \int_0^\Lambda dq \frac{q^2\, T(E,k,q)}{-\gamma + \sqrt{-mE + \frac{3}{4}q^2 - i\varepsilon}} \left[\frac{1}{2qp} \ln\left(\frac{p^2 + pq + q^2 - E - i\varepsilon}{p^2 - pq + q^2 - E - i\varepsilon} \right) - \frac{1}{4m}\frac{g_3}{g_2^2} \right] ,$$

$$(2.17)$$

where the remaining integral over dq is regularized by a cutoff Λ. To account for a correct renormalization it is now convenient to introduce a cutoff dependent coupling constant $H(\Lambda)$ [80],

$$g_3 = -\frac{4mg_2^2}{\Lambda^2} H(\Lambda) , \qquad (2.18)$$

which yields

$$T(E,k,p) = \frac{16\pi\gamma}{m} \left[\frac{1}{2kp} \ln\left(\frac{p^2 + pk + k^2 - E - i\varepsilon}{p^2 - pk + k^2 - E - i\varepsilon} \right) + \frac{H(\Lambda)}{\Lambda^2} \right]$$
$$+ \frac{4}{\pi} \int_0^\Lambda dq \frac{q^2\, T(E,k,q)}{-\gamma + \sqrt{-mE + \frac{3}{4}q^2 - i\varepsilon}} \left[\frac{1}{2qp} \ln\left(\frac{p^2 + pq + q^2 - E - i\varepsilon}{p^2 - pq + q^2 - E - i\varepsilon} \right) + \frac{H(\Lambda)}{\Lambda^2} \right] .$$

$$(2.19)$$

Note, that the equation above is equivalent to Eq. (336) in Ref. [80] up to the fact that the mass of the bosons is not set to $m = 1$. As discussed in Ref. [80] the function $H(\Lambda)$ must compensate every change in the cutoff Λ so that the amplitude itself is well-behaved in the limit of Λ going to infinity. However, $H(\Lambda)$ is proportional to the three-body coupling g_3. Hence, one concludes that in a system without a three-body interaction it holds $H = 0$ and the divergent part of the amplitude drops out. In the other case where $g_3 \neq 0$ one thus needs a three-body observable in order to find the physical value of $H(\Lambda)$ which ensures the right behavior of the amplitude in the limit $\Lambda \to \infty$. In fact, a three-body bound state with binding energy $E_3 = -B_3$ corresponds to a pole in Eq. (2.19) exactly at the energy $E = -B_3$. Thus, one can fix the cutoff dependence of $H(\Lambda)$ by determining the poles in Eq. (2.19) for a varying cutoff and searching for the right pole position $E = -B_3$. If this is done one can calculate for every cutoff Λ the corresponding value of H. In Ref. [80] it is shown that a numerical determination of H yields

$$H(\Lambda) = \frac{\cos\left[s_0 \ln\left(\frac{\Lambda}{\Lambda_*} \right) + \arctan(s_0) \right]}{\cos\left[s_0 \ln\left(\frac{\Lambda}{\Lambda_*} \right) - \arctan(s_0) \right]} , \qquad (2.20)$$

which depends on two parameters: on the one hand Λ_* which must be fixed using a three-body observable and on the other hand the scaling parameter $s_0 = 1.00624$ which is independent of the three-body interaction and which can be determined as discussed in the next subsection.

Transcendental equation for the scaling parameter

As mentioned previously the scaling parameter $s_0 = 1.00624$ is not affected by any three-body physics. Therefore one can assume for simplicity that $g_3 = 0$ and hence, that $H(\Lambda)$ vanishes (according to the fact that there always exists a cutoff Λ so that $H(\Lambda) = 0$). In Eq. (2.19) one can then take the limit of $\Lambda \to \infty$ and one finds

$$
T(E, k, p) = \frac{16\pi}{m} \frac{\gamma}{2kp} \ln\left(\frac{p^2 + pk + k^2 - E - i\varepsilon}{p^2 - pk + k^2 - E - i\varepsilon}\right)
$$
$$
+ \frac{4}{\pi} \int_0^\infty dq \frac{q^2 \, T(E, k, q)}{-\gamma + \sqrt{-mE + \frac{3}{4}q^2 - i\varepsilon}} \frac{1}{2qp} \ln\left(\frac{p^2 + pq + q^2 - E - i\varepsilon}{p^2 - pq + q^2 - E - i\varepsilon}\right) . \quad (2.21)
$$

This equation is scale invariant and symmetric under the change $q \to 1/q$ ("inversion invariant"). In Ref. [80] it is argued that this invariance causes that in the limit of asymptotic large outgoing momenta p the amplitude $T(E, k, p)$ has a solution in form of a power law p^s. Furthermore, the authors motivate that in the limit $p \to \infty$ one can neglect the inhomogeneous term and in addition all variables E and γ which are proportional to the incoming momentum $k \ll p \to \infty$. Consequently, one ends up with

$$
T(p) = \frac{4}{\sqrt{3}\pi} \int_0^\infty \frac{dq}{p} T(q) \ln\left(\frac{p^2 + pq + q^2}{p^2 - pq + q^2}\right) . \quad (2.22)
$$

Redefining the amplitude via $\widetilde{T}(p) = pT(p)$,

$$
\widetilde{T}(p) = \frac{4}{\sqrt{3}\pi} \int_0^\infty \frac{dq}{q} \widetilde{T}(q) \ln\left(\frac{p^2 + pq + q^2}{p^2 - pq + q^2}\right) , \quad (2.23)
$$

and inserting the power law solution $\widetilde{T}(p) \sim p^s$ yields in the asymptotic momentum limit

$$
p^s = \frac{4}{\sqrt{3}\pi} \int_0^\infty dq \, q^{s-1} \ln\left(\frac{p^2 + pq + q^2}{p^2 - pq + q^2}\right) . \quad (2.24)
$$

Following Ref. [80] it is useful to substitute $X = q/p$ so that Eq. (2.24) finally simplifies to the relation

$$
1 = \frac{4}{\sqrt{3}\pi} \int_0^\infty dX \, X^{s-1} \ln\left(\frac{X + \frac{1}{X} + 1}{X + \frac{1}{X} - 1}\right) . \quad (2.25)
$$

The remaining integral is related to a Mellin transform (see e.g. Ref. [168]) and can be analytically solved. Hence, one obtains a transcendental equation for the scaling parameter s [80]:

$$
1 = \frac{8}{\sqrt{3}\pi} \frac{1}{s} \frac{\sin\left(\frac{\pi}{6}s\right)}{\cos\left(\frac{\pi}{2}s\right)} . \quad (2.26)
$$

This equation has a purely imaginary solution $s = is_0$ with $s_0 = 1.00624$. Thus, Eq. (2.26) is the relation which defines the scaling parameter s_0 in the three-body problem. As a remark, note

that one could have found Eq. (2.26) also by rewriting the logarithm in terms of the Legendre function of the second kind using Eq. (D.15),

$$\ln\left(\frac{X + \frac{1}{X} + 1}{X + \frac{1}{X} - 1}\right) = 2\, Q_0\left(X + \frac{1}{X}\right)\,,$$

and considering the results discussed in appendix E.

The purely imaginary solution of Eq. (2.26) tells us that the amplitude $\widetilde{T}(p) \sim p^s$ has not one uniquely determined real solution, but instead two linearly independent complex solutions (see also Ref. [167]). Consequently, there is a *discrete scaling invariance*, that is, a discrete scaling factor $\exp(\pi/s_0)$ in the system of three identical bosons. This discrete scaling invariance is manifestly an effect of Efimov physics [80] and one concludes that a different system of three particles which has *not* a purely imaginary exponent in the power law solution will not be affected by the Efimov effect, meaning that there is *no* three-body bound state (the Efimov trimer) in such a system. It is therefore important to note on the one hand that one does *not* need an explicit three-body force in the Lagrangian density in order to check if a three particle system is affected by the Efimov effect. On the other hand Eq. (2.26) is not an universal relation; it was derived for a system of three identical, spin- and isospinless bosons, but it will most probably change if one or more of these properties are changed. These facts motivate the following work: we will derive an analogous transcendental equation depending on some parameters in order to describe a large number of different three particle systems (including spin, isospin, identical or distinguishable bosons and fermions, etc.). Such an equation allows to simply check for an (almost) arbitrary three particle system if the scaling parameter has a purely imaginary solution, that is, to check if the Efimov effect is present in the considered system. Moreover, this can be done *without* any knowledge of possible three-body physics (corresponding to the available experimental data of most hadronic molecule candidates) since in the above derivation we have set the three-body coupling constant H equal to zero anyway. Only to obtain cutoff independent three-body observables would require to fix $H(\Lambda)$ to its physical value.

2.2 Efimov physics in cold atoms and halo nuclei

Up to now the discussion of the Efimov effect as an universality phenomenon was restricted to nuclear physics and – where this work focuses on – to hadronic molecules, i.e. particle physics. Besides these applications the Efimov effect is also an important phenomenon in cold atoms and halo nuclei. The former are atoms which are trapped (e.g. in a magneto-optical trap) and cooled down by different techniques like laser cooling until they reach temperatures very close to zero Kelvin (for an overview of this topic see for example Ref. [96]). At such low temperatures – which are equivalent to low energies – quantum effects become important and thus cold atoms are a perfect testing ground for phenomena like Bose-Einstein condensation, but in fact also for the Efimov effect. Especially, the existence of *Feshbach resonances* [97–99] which allow to fine-tune the scattering length of a two-atom state to be "large", leads to many experimental observations of the Efimov effect. In bosonic systems it was found using cesium [100, 101], potassium [102] or lithium [103, 104] isotopes, but also in a fermionic system of a different lithium isotope Efimov physics were observed [105, 106]. The mentioned experiments have in common that they use the

indirect detection method of an enhanced recombination rate [109–111], but there are also direct observations of Efimov trimers [107, 108].

The second field where universality effects are important are halo nuclei. The idea behind this term is that some isotopes of a few different elements can be interpreted as a compact, deeply bound core and one or more orbiting nucleons (*halo-nucleons*) [112–115]. This picture has led to the development of the so-called *Halo EFT* where one treats the core and the halo-nucleons as shallowly bound few-body system. The expansion parameter is R/a with R being the range of the core–halo-nucleon interaction and its scattering length a [116, 117]. Both, core and halo-nucleons are described as non-relativistic fields and hence the degrees of freedom in the theory are drastically reduced (for more details on Halo EFT see the reviews in Refs. [118, 119]). Examples for halo nuclei are – besides others – ^{11}Li with two surrounding neutrons, ^{11}Be with one neutron or ^{8}B with a halo-proton [120]. Thus, one concludes that there are in particular also three-body systems made of a core and two nucleons. Since the core–nucleon two-body scattering length a is expected to be large compared to the range R of the core–nucleon interaction, it is likely that such a system might be affected by the Efimov effect. Indeed, there was much effort to quantify this assumption about the Efimov effect in halo nuclei [121–127] (see the review in Ref. [128]).

The final conclusion of this subsection is that Efimov physics are an important part of many different fields in physics which – at least on the first sight – deal with very different subjects. Already at this point we want to emphasize that the considerations in the following chapters are not restricted to hadronic molecules and in particular that the final transcendental equation for the scaling parameter (which tells us if the Efimov effect exists in a system) can also be applied to a system of cold atoms or to a three-body halo nucleus.

Chapter 3

Efimov effect in a general three particle system

In section 2.1 we have explained how the Efimov effect occurs in a system of three identical spin- and isospinless bosons. At the end we showed a transcendental equation for the exponent s of the power law solution in the limit of asymptotic large momenta. In case of S-wave scattering we have found:

$$1 = \frac{8}{\sqrt{3}} \frac{1}{s} \frac{\sin\left(\frac{\pi}{6}s\right)}{\cos\left(\frac{\pi}{2}s\right)} . \tag{3.1}$$

From the derivation of this equation we know that its structure depends on the following particle properties: species (bosons, fermions or mixture), relation of masses as well as angular momentum, spin and isospin quantum numbers of the dimer and its constituents. The question is now: how will this equation change if one changes one or more of these properties? To answer this question we will derive a general transcendental equation with a number of parameters representing various three particle configurations. However, choosing one specific three particle system with known dimer states the parameters straightforwardly reduce to numbers and at the end one simply has to find the eigenvalues of a real matrix.

3.1 General dimer states

Let us consider three isospin multiplets A_1, A_2, A_3 and their corresponding anti-multiplets $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$ which are related via the G-parity operator $\mathcal{G}$,

$$\bar{A}_i = \mathcal{G}A_i = \mathcal{C}e^{i\pi I_2}A_i , \tag{3.2}$$

where $\mathcal{C}$ is the charge conjugation operator and $e^{i\pi I_2}$ represents a rotation in isospin space around the I_2 axis, that is, a change from $I_3 = \pm 1/2$ to $I_3 = \mp 1/2$. Because $\bar{A}_i = \mathcal{G}A_i$ their masses are equal and hence there are only three mass parameters m_1, m_2 and m_3. Nevertheless, we treat $\bar{A}_i$ and A_i as independent states since we work in a non-relativistic theory.

A general system of three particles P_1, P_2, P_3 can contain both particles and anti-particles. We define these particles is terms of the multiplets introduced above:

$$P_1 := a_1 A_1 + b_1 \bar{A}_1$$
$$P_2 := a_2 A_2 + b_2 \bar{A}_2$$
$$P_3 := a_3 A_3 + b_3 \bar{A}_3 \, , \tag{3.3}$$

with $a_i = \{0,1\}$ and $b_i = \{1,0\} \, \forall \, i \in \{1,2,3\}$. Hence, choosing e.g. the system $P_1 = \bar{N}$, $P_2 = N$, $P_3 = \Lambda$ this would fix the parameters to be $b_1 = a_2 = a_3 = 1$ and $a_1 = b_2 = b_3 = 0$.

We assume that these particles have shallow two-body bound or virtual states d_{ij} which we will for simplicity both call "dimers" (and which in general are isospin multiplets as well). Since the dimer d_{12} between particle P_1 and P_2 is physically identical to the dimer d_{21} between P_2 and P_1 there only are three different dimers in the system. Note, that mathematically $d_{ij} \neq d_{ji}$ due to possible fermion minus signs from interchanging P_i and P_j (physically such a minus sign can be absorbed into the coupling constant which does not affect the observables). Thus, one has to choose one convention for the order of appearance of P_i and P_j, but which of them is irrelevant since the observables are independent of this choice (of course if one does not mix both conventions). We will use the convention

$$d_{ij} \sim P_j P_i \qquad \text{with} \quad i < j \in \{1,2,3\} \, , \tag{3.4}$$

and the three possible dimers are thus given as d_{12}, d_{13} and d_{23}.

3.1.1 Dimer flavor wave function

The possible flavor wave functions for a dimer d_{ij} are

$$d_{ij} = A_j A_i \tag{3.5}$$
$$d_{ij} = \bar{A}_j \bar{A}_i \tag{3.6}$$

$$d_{ij} = \begin{cases} \frac{1}{\sqrt{2}} \left(\bar{A}_j A_i + \eta_{ij} A_j \bar{A}_i \right) \, , & \text{if baryon number and flavor are 0 and } A_i \neq A_j \\ \eta_{ij} \bar{A}_j A_i \, , & \text{if baryon number and flavor are 0 and } A_i = A_j \\ \bar{A}_j A_i \, , & \text{else} \end{cases} \tag{3.7}$$

In the third equation above the dimer has well-defined G-parity with G-parity quantum number $\eta_{ij} = \pm 1$ if the baryon number and the flavor meaning strangeness, charm, beauty (bottomness) and – for completeness although hadrons containing top quarks does not exist due to the short lifetime of t quarks – topness are all equal to zero. This condition implies that for $A_i \neq A_j$ (e.g. $\bar{B}^* B$) the combinations $\bar{A}_j A_i$ and $A_j \bar{A}_i$ have the same quark content with the same quantum numbers and masses. Thus, a possible dimer must be a superposition of both states. The G-parity quantum number η_{ij} in Eq. (3.7) fixes the sign within the superposition in the following manner: the G-parity eigenstate d_{ij} must fulfill $\mathcal{G} d_{ij} = \eta_{ij} d_{ij}$ and since quantum mechanics tell us that there could be a phase $e^{i\varphi}$ between the two states of the superposition, we start with the general wave function

$$d_{ij} = \frac{1}{\sqrt{2}} \left(\bar{A}_j A_i + e^{i\varphi} A_j \bar{A}_i \right) = \frac{1}{\sqrt{2}} \left(\bar{A}_j A_i + e^{i\varphi} \mathcal{G} \left(\bar{A}_j A_i \right) \right) \, , \tag{3.8}$$

and apply $\mathcal{G}$ to both sides:

$$\mathcal{G}d_{ij} = \mathcal{G}\frac{1}{\sqrt{2}}\left(\bar{A}_j A_i + e^{i\varphi}\mathcal{G}\left(\bar{A}_j A_i\right)\right) = \frac{1}{\sqrt{2}}\left(\mathcal{G}\bar{A}_j A_i + e^{i\varphi}\left(\bar{A}_j A_i\right)\right)$$
$$\overset{!}{=} \eta_{ij}d_{ij} = \frac{1}{\sqrt{2}}\left(\eta_{ij}\bar{A}_j A_i + \eta_{ij}e^{i\varphi}\mathcal{G}\left(\bar{A}_j A_i\right)\right) . \tag{3.9}$$

Comparing the coefficients one finds that for $\eta_{ij} = \pm 1$ the phase factor must be $e^{i\varphi} = \pm 1$. Therefore one can replace in Eq. (3.8) the phase factor by the G-parity quantum number which leads to the wave function in Eq. (3.7). In the case with $A_i = A_j$ (e.g. $\bar{B}^* B^*$) the prefactor η_{ij} does not affect observables since later on it could be absorbed in the coupling constant. However, for the sake of consistency and to distinguish this case from the "else" case where G-parity is not a good quantum number we will keep it.

Using a_i and b_i one can parametrize the flavor wave function in a way that for a given system P_1, P_2, P_3 (which fixes all a_i's and b_i's to be either 0 or 1) the correct wave function is automatically generated. For this we introduce two short-hand notations. Firstly, we need a parameter which tells us if the considered dimer has a well-defined G-parity. This is the case if the following quantum numbers of the dimer are zero: baryon number, strangeness, charm, bottomness (beauty) and topness must vanish because only hadrons consisting either solely of up and down quarks or an arbitrary number of $q\bar{q}$ pairs of heavier quarks are up to a sign invariant under the application of the G-parity operator $\mathcal{G}$. In order to avoid the lengthy combination of Kronecker-deltas which ensure the vanishing of all the quantum numbers we instead define:

$$\delta_{\eta|1|} := \delta_{\text{baryon number } 0} \times \delta_{\text{strangeness } 0} \times \delta_{\text{charm } 0} \times \delta_{\text{beauty } 0} \times \delta_{\text{topness } 0} , \tag{3.10}$$

motivated by the fact that if the right-hand-side is 1, i.e. if all mentioned quantum numbers indeed vanish, then the dimer is a G-parity eigenstate with corresponding quantum number $\eta = \pm 1 = |1|$. The second short-hand notation is:

$$\delta_{A_i A_j} = \begin{cases} 1 , & \text{if } A_i \text{ is identical to } A_j, \text{ i.e. } A_i = A_j \\ 0 , & \text{else} \end{cases} . \tag{3.11}$$

Note here, that

$$\delta_{P_i P_j} = 1 \quad \Rightarrow \quad \delta_{A_i A_j} = 1 ,$$
$$\delta_{A_i A_j} = 1 \quad \not\Rightarrow \quad \delta_{P_i P_j} = 1 . \tag{3.12}$$

With this notation one can parametrize the dimer flavor wave function as follows:

$$d_{ij} = a_i a_j A_j A_i + b_i b_j \bar{A}_j \bar{A}_i$$
$$+ \delta_{\eta_{ij}|1|}\left[\frac{1}{\sqrt{2}} + \delta_{A_i A_j}\left(1 - \frac{1}{\sqrt{2}}\right)\right](b_i a_j + a_i b_j)\left[(\eta_{ij})^{\delta_{A_i A_j}}\bar{A}_j A_i + \left(1 - \delta_{A_i A_j}\right)\eta_{ij}A_j\bar{A}_i\right]$$
$$+ \left(1 - \delta_{\eta_{ij}|1|}\right)b_i a_j \bar{A}_j A_i \qquad \forall\, i < j \in \{1, 2, 3\} . \tag{3.13}$$

To clarify this notation let us consider the three particle system $\bar{B}^* B B^*$. Consequently, we have $P_1 = \bar{B}^*$, $P_2 = B$, $P_3 = B^*$ and thus $A_1 = A_3 = B^*$, $A_2 = B$ and $b_1 = a_2 = a_3 = 1$,

$a_1 = b_2 = b_3 = 0$. Furthermore, we conclude from the quark content of the three particles that $\delta_{\eta_{12}|1|} = \delta_{\eta_{13}|1|} = 1$, but $\delta_{\eta_{23}|1|} = 0$. With this information we can write down the possible dimers:

$$d_{12} = 1 \times \left[\frac{1}{\sqrt{2}} + 0 \times \left(1 - \frac{1}{\sqrt{2}} \right) \right] (1 \times 1 + 0 \times 0) \left[(\eta_{12})^0 \bar{B} B^* + (1 - 0) \eta_{12} B \bar{B}^* \right]$$

$$= \frac{1}{\sqrt{2}} \left(\bar{B} B^* + \eta_{12} B \bar{B}^* \right) \tag{3.14}$$

$$d_{13} = 1 \times \left[\frac{1}{\sqrt{2}} + 1 \times \left(1 - \frac{1}{\sqrt{2}} \right) \right] (1 \times 1 + 0 \times 0) \left[(\eta_{13})^1 \bar{B}^* B^* + (1 - 1) \eta_{13} B^* \bar{B}^* \right]$$

$$= \eta_{13} \bar{B}^* B^* \tag{3.15}$$

$$d_{23} = 1 \times 1 \times B^* B = B^* B . \tag{3.16}$$

The next step is now to identify the *physical* dimers in the system, i.e. those bound or virtual states which exist in nature (or in the used theory). In the case above one finds on the one hand that there is the $Z_b(10610)$ as $\bar{B}^* B$ bound state and on the other hand the $Z_b'(10650)$ as a $\bar{B}^* B^*$ bound state [33]. A BB^* state is not known so far. Therefore one has to erase d_{23} from the theory and only keeps d_{12} and d_{13} in a further analysis.

3.1.2   Spin and isospin part of the dimer wave function

Up to now we have ignored the spin and isospin degrees of freedom. Therefore we define projection operators $\left(\mathcal{O}_{ij}^\dagger \right)_{\alpha,\beta\gamma}$ with combined spin and isospin indices α, β, γ which couple the spin/isospin of $(A_i)_\beta$ and $(A_j)_\gamma$ to a total dimer spin/isospin of $(d_{ij})_\alpha$. Since the anti-multiplet $\bar{A}_i$ has the same spin and isospin structure as the multiplet A_i one finds for the dimer wave function including flavor, spin and isospin:

$$(d_{ij})_\alpha = a_i a_j (A_j)_\beta \left(\mathcal{O}_{ij}^\dagger \right)_{\alpha,\beta\gamma} (A_i)_\gamma + b_i b_j (\bar{A}_j)_\beta \left(\mathcal{O}_{ij}^\dagger \right)_{\alpha,\beta\gamma} (\bar{A}_i)_\gamma$$

$$+ \delta_{\eta_{ij}|1|} \left[\frac{1}{\sqrt{2}} + \delta_{A_i A_j} \left(1 - \frac{1}{\sqrt{2}} \right) \right] (b_i a_j + a_i b_j)$$

$$\times \left[(\eta_{ij})^{\delta_{A_i A_j}} (\bar{A}_j)_\beta \left(\mathcal{O}_{ij}^\dagger \right)_{\alpha,\beta\gamma} (A_i)_\gamma + (1 - \delta_{A_i A_j}) \eta_{ij} (A_j)_\beta \left(\mathcal{O}_{ij}^\dagger \right)_{\alpha,\beta\gamma} (\bar{A}_i)_\gamma \right]$$

$$+ (1 - \delta_{\eta_{ij}|1|}) b_i a_j (\bar{A}_j)_\beta \left(\mathcal{O}_{ij}^\dagger \right)_{\alpha,\beta\gamma} (A_i)_\gamma \qquad \forall\, i < j \in \{1, 2, 3\} . \tag{3.17}$$

If we again consider the Z_b, Z_b' example from the previous subsection to clarify the notation, we would need the two projection operators $\mathcal{O}_{12}^\dagger$ and $\mathcal{O}_{13}^\dagger$ which are given by

$$\left(\mathcal{O}_{12}^\dagger \right)_{\{\alpha=aA\},\{\beta=\widetilde{\beta}\}\{\gamma=c\widetilde{\gamma}\}} = \delta_{ca} \frac{-i}{\sqrt{2}} (\tau_2 \tau_A)_{\widetilde{\beta}\widetilde{\gamma}} ,$$

$$\left(\mathcal{O}_{13}^\dagger \right)_{\{\alpha=aA\},\{\beta=b\widetilde{\beta}\}\{\gamma=c\widetilde{\gamma}\}} = -\frac{1}{\sqrt{2}} (U_a)_{bc} \frac{-i}{\sqrt{2}} (\tau_2 \tau_A)_{\widetilde{\beta}\widetilde{\gamma}} , \tag{3.18}$$

where $\{a, b, c\}$ are spin 1 indices, $\{\widetilde{\beta}, \widetilde{\gamma}\}$ are isospin 1/2 indices and A is an isospin 1 index. τ_A are the Pauli matrices and U_a are the generators of the $SO(3)$ rotation group. Hence, $\mathcal{O}_{ij}$ is

the product of the spin projector and the isospin projector which are completely independent because they act in different spaces. Consequently, they commute and the trace in this combined spin/isospin space simply means "trace in spin space" times "trace in isospin space". More details on and the derivation of these operators and of all other projectors from $0 \otimes 0$ up to $1 \otimes 1$ can be found in appendix A.

Following the usual coupling of spins and isospins like $1/2 \otimes 1/2 = 0 \oplus 1$ we notice that there might be more than just one dimer with the same particle content. In order to take this fact into account we introduce a second dimer d'_{ij} with identical constituents, but different spin/isospin quantum numbers than d_{ij}. Since also the G-parity could change in this case we need in advance a new parameter η'_{ij} and write:

$$
\begin{aligned}
\left(d'_{ij}\right)_\alpha =\ & a_i a_j \left(A_j\right)_\beta \left(\mathcal{O}'^\dagger_{ij}\right)_{\alpha,\beta\gamma} \left(A_i\right)_\gamma + b_i b_j \left(\bar{A}_j\right)_\beta \left(\mathcal{O}'^\dagger_{ij}\right)_{\alpha,\beta\gamma} \left(\bar{A}_i\right)_\gamma \\
& + \delta_{\eta'_{ij}|1|} \left[\frac{1}{\sqrt{2}} + \delta_{A_i A_j}\left(1 - \frac{1}{\sqrt{2}}\right)\right] \left(b_i a_j + a_i b_j\right) \\
& \times \left[\left(\eta'_{ij}\right)^{\delta_{A_i A_j}} \left(\bar{A}_j\right)_\beta \left(\mathcal{O}'^\dagger_{ij}\right)_{\alpha,\beta\gamma} \left(A_i\right)_\gamma + \left(1 - \delta_{A_i A_j}\right) \eta'_{ij} \left(A_j\right)_\beta \left(\mathcal{O}'^\dagger_{ij}\right)_{\alpha,\beta\gamma} \left(\bar{A}_i\right)_\gamma\right] \\
& + \left(1 - \delta_{\eta'_{ij}|1|}\right) b_i a_j \left(\bar{A}_j\right)_\beta \left(\mathcal{O}'^\dagger_{ij}\right)_{\alpha,\beta\gamma} \left(A_i\right)_\gamma \qquad \forall\, i < j \in \{1,2,3\}. \quad (3.19)
\end{aligned}
$$

In general one could continue and define d''_{ij}, d'''_{ij} and so on. However, for most two particle systems two states should be enough since in the end not every possible spin/isospin configuration has a bound or virtual state (e.g. in the NN system there are four possible configurations, but only the $S = 1$, $I = 0$ bound state (the deuteron) and the $S = 0$, $I = 1$ virtual state exist in nature). Therefore we restrict ourselves to only one extra dimer d'_{ij}. However, one could straightforwardly extend this work to a system with three or more states.

3.1.3 Spatial part of the dimer wave function

Finally, we want to make some comments on the spatial part of the dimer wave function. It depends on the angular momentum of the constituent particles. Thus, we must specify in which partial wave the constituent particles are. Since the energy within a bound state rises with the total angular momentum it is more likely that a dimer is formed in the lowest partial wave. In fact, as long as there is no physical property which forbids a $L = 0$ interaction, it is justified to assume that all dimers are S-wave states. In this work we will ignore the special cases where L must be unequal to zero and state that:

$$\text{all dimers are } S\text{-wave states.} \qquad (3.20)$$

In this case the spatial structure stays trivial:

$$d_{ij}(\mathbf{x}) = a_i a_j\, A_j(\mathbf{x})\, A_i(\mathbf{x}) + b_i b_j\, \bar{A}_j(\mathbf{x})\, \bar{A}_i(\mathbf{x}) + \ldots, \qquad (3.21)$$

and we will not write the $\mathbf{x}$ dependence explicitly in our equations. Note, that already for a P-wave dimer there would appear spatial derivatives:

$$d_{ij}(\mathbf{x}) = a_i a_j \left[\left(i\vec{\nabla} A_j(\mathbf{x})\right) A_i(\mathbf{x}) + A_j(\mathbf{x}) \left(i\vec{\nabla} A_i(\mathbf{x})\right)\right] + \ldots. \qquad (3.22)$$

More details on P-wave dimers in EFT can be found in Ref. [129].

3.2   Lagrangian density, vertices and propagators

We are working in a non-relativistic theory. Hence, particles and their corresponding anti-particles are point-like and treated as independent fields. Therefore all particles have the same non-relativistic kinetic term in the Lagrangian density independently of their spin. Furthermore, there only are contact interactions between the fields due to their point-like character. As explained in section 2.1 one can construct an effective Lagrangian density using auxiliary fields. To get a general expression which takes into account all different three particle systems, we need six auxiliary fields representing the six possible dimers d_{ij} and d'_{ij} for $i < j \in \{1, 2, 3\}$. At this point one has to choose the order of the Lagrangian density: we will only consider leading order (LO) because already next-to-leading order (NLO) terms depend on the effective range of the dimers which is very poorly known for hadronic molecules so far. In fact, the two nucleon system is more or less the only system where effective ranges are measured (one exception is the $N\Lambda$ system [130,131]). At LO we need six new parameters $\Delta_{ij}^{(\prime)} \in \mathbb{R}$ and six coupling constants $g_{ij}^{(\prime)}$ for $i < j \in \{1, 2, 3\}$. The Δ parameters can be related to the scattering length of the corresponding dimer which is at LO related to the – easier measurable – binding momentum via $a^{-1} = \gamma$ (see section 1.3). We name the auxiliary fields like the dimers and find Eq. (3.23) where the Lagrangian density is given in terms of the isospin multiplet fields A and $\bar{A}$. Lines 8 to 12 are the Hermitian conjugation of lines 3 to 7 and the last line stands for the interaction Lagrangian of d'_{ij} which is achieved by replacing in line 3 to 12 all d_{ij} by d'_{ij}, $\mathcal{O}_{ij}$ by $\mathcal{O}'_{ij}$, g_{ij} by g'_{ij} and η_{ij} by η'_{ij}. From the Lagrangian density one can deduce the vertex factors describing the interactions between the dimers and their constituents as well as the propagators of all included particles.

$$\mathcal{L}_{\text{LO}} = \sum_{i=1}^{3} (A_i)_\alpha^\dagger \left(i\partial_t + \frac{\nabla^2}{2m_i} \right) (A_i)_\alpha \ + \ \sum_{i=1}^{3} (\bar{A}_i)_\alpha^\dagger \left(i\partial_t + \frac{\nabla^2}{2m_i} \right) (\bar{A}_i)_\alpha$$

$$+ \sum_{\substack{i,j=1 \\ i<j}}^{3} (d_{ij})_\alpha^\dagger \, \Delta_{ij} \, (d_{ij})_\alpha \ + \ \sum_{\substack{i,j=1 \\ i<j}}^{3} (d'_{ij})_\alpha^\dagger \, \Delta'_{ij} \, (d'_{ij})_\alpha$$

$$- \sum_{\substack{i,j=1 \\ i<j}}^{3} \left(g_{ij} \left\{ a_i a_j \left(A_i^\dagger \right)_\beta (\mathcal{O}_{ij})_{\alpha,\beta\gamma} \left(A_j^\dagger \right)_\gamma (d_{ij})_\alpha \ + \ b_i b_j \left(\bar{A}_i^\dagger \right)_\beta (\mathcal{O}_{ij})_{\alpha,\beta\gamma} \left(\bar{A}_j^\dagger \right)_\gamma (d_{ij})_\alpha \right. \right.$$

$$+ \ \delta_{\eta_{ij}|1|} \left[\frac{1}{\sqrt{2}} + \delta_{A_i A_j} \left(1 - \frac{1}{\sqrt{2}} \right) \right] (b_i a_j + a_i b_j)$$

$$\times \left[(\eta_{ij})^{\delta_{A_i A_j}} \left(\bar{A}_i^\dagger \right)_\beta (\mathcal{O}_{ij})_{\alpha,\beta\gamma} \left(A_j^\dagger \right)_\gamma (d_{ij})_\alpha \right.$$

$$+ \ (1 - \delta_{A_i A_j}) \, \eta_{ij} \left(A_i^\dagger \right)_\beta (\mathcal{O}_{ij})_{\alpha,\beta\gamma} \left(\bar{A}_j^\dagger \right)_\gamma (d_{ij})_\alpha \Bigg]$$

$$+ \ (1 - \delta_{\eta_{ij}|1|}) \, b_i a_j \left(\bar{A}_i^\dagger \right)_\beta (\mathcal{O}_{ij})_{\alpha,\beta\gamma} \left(A_j^\dagger \right)_\gamma (d_{ij})_\alpha \Bigg\}$$

$$+ \ g_{ij} \left\{ a_i a_j \left(d_{ij}^\dagger \right)_\alpha (A_j)_\gamma \left(\mathcal{O}_{ij}^\dagger \right)_{\alpha,\gamma\beta} (A_i)_\beta \ + \ b_i b_j \left(d_{ij}^\dagger \right)_\alpha (\bar{A}_j)_\gamma \left(\mathcal{O}_{ij}^\dagger \right)_{\alpha,\gamma\beta} (\bar{A}_i)_\beta \right.$$

$$+ \ \delta_{\eta_{ij}|1|} \left[\frac{1}{\sqrt{2}} + \delta_{A_i A_j} \left(1 - \frac{1}{\sqrt{2}} \right) \right] (b_i a_j + a_i b_j)$$

$$\times \left[(\eta_{ij})^{\delta_{A_i A_j}} \left(d_{ij}^\dagger \right)_\alpha (A_j)_\gamma \left(\mathcal{O}_{ij}^\dagger \right)_{\alpha,\gamma\beta} (\bar{A}_i)_\beta \right.$$

$$+ \ (1 - \delta_{A_i A_j}) \, \eta_{ij} \left(d_{ij}^\dagger \right)_\alpha (\bar{A}_j)_\gamma \left(\mathcal{O}_{ij}^\dagger \right)_{\alpha,\gamma\beta} (A_i)_\beta \Bigg]$$

$$+ \ (1 - \delta_{\eta_{ij}|1|}) \, b_i a_j \left(d_{ij}^\dagger \right)_\alpha (A_j)_\gamma \left(\mathcal{O}_{ij}^\dagger \right)_{\alpha,\gamma\beta} (\bar{A}_i)_\beta \Bigg\} \right)$$

$$- \sum_{\substack{i,j=1 \\ i<j}}^{3} \left(g_{ij} \to g'_{ij}, \ d_{ij} \to d'_{ij}, \ \eta_{ij} \to \eta'_{ij}, \ \mathcal{O}_{ij} \to \mathcal{O}'_{ij} \right) \tag{3.23}$$

3.2.1 Vertices

The interaction vertices are obtained by collecting all terms with the same number of fields. Erasing the fields from these terms and multiplying them with a factor i yields the Feynman rules shown in Fig. 3.1 valid for all indices $i, j \in \{1, 2, 3\}$ with $i < j$. The two parameters $v_{ij}^{(\prime)}$ and $w_{ij}^{(\prime)}$ appearing in the vertex factors are defined as

$$v_{ij}^{(\prime)} := \delta_{\eta_{ij}^{(\prime)}|1|} \left[\frac{1}{\sqrt{2}} + \delta_{A_i A_j} \left(1 - \frac{1}{\sqrt{2}} \right) \right] (b_i a_j + a_i b_j) \left(\eta_{ij}^{(\prime)} \right)^{\delta_{A_i A_j}} + \left(1 - \delta_{\eta_{ij}^{(\prime)}|1|} \right) b_i a_j , \qquad (3.24)$$

and

$$w_{ij}^{(\prime)} := \delta_{\eta_{ij}^{(\prime)}|1|} \left[\frac{1}{\sqrt{2}} + \delta_{A_i A_j} \left(1 - \frac{1}{\sqrt{2}} \right) \right] (b_i a_j + a_i b_j) \left(1 - \delta_{A_i A_j} \right) \eta_{ij}^{(\prime)} . \qquad (3.25)$$

At this point we have to make two remarks. Firstly, note that we write $\left[(\mathcal{O}_{ij})_{\alpha,\gamma\beta} \right]^\dagger = \left(\mathcal{O}_{ij}^\dagger \right)_{\alpha,\beta\gamma}$ according to projection operators like $\left[(\tau_A \tau_2)_{\alpha\beta} \right]^\dagger = \left((\tau_A \tau_2)^\dagger \right)_{\beta\alpha} = (\tau_2 \tau_A)_{\beta\alpha}$. The second remark is on the coupling constants: we use just one symbol g_{ij} in front of each interaction term although the coupling strength between e.g. d_{ij} and $A_j A_i$ must not be same as between d_{ij} and $\bar{A}_j \bar{A}_i$ because the dimer is different in both cases. However, in nature $d_{ij} \sim P_j P_i$ is always unique even if bound states between $A_j A_i$ and between $\bar{A}_j \bar{A}_i$ exist, since A_i and $\bar{A}_i$ are not the same particle. Thus, it is justified to only use one coupling constant which could have different values for different systems.

3.2.2 Propagators

We can read off from Eq. (3.23) the non-relativistic propagators of the point-like multiplets A_i and $\bar{A}_i$:

$$i \left(S_i \right)_{\alpha\beta} (p_0, \mathbf{p}) = \frac{i \delta_{\alpha\beta}}{p_0 - \frac{|\mathbf{p}|^2}{2m_i} + i\varepsilon} \qquad \text{for } i = 1, 2, 3 . \qquad (3.26)$$

Hence, all particles propagate forward in time and it is no further work needed. For the dimer fields the procedure is more complicated since they have a coupling to their constituent fields. Firstly, we note that at LO the dimer fields themselves are not dynamic. Their bare propagators,

$$i \left(D_{ij}^{(\prime)0} \right)_{\alpha\beta} (p_0, \mathbf{p}) = i \frac{\delta_{\alpha\beta}}{\Delta_{ij}^{(\prime)}} , \qquad (3.27)$$

are constant, but they are dressed by loops of their constituent particles. This leads to the Dyson equation in Fig. 3.2(a) and hence to a possible propagation.
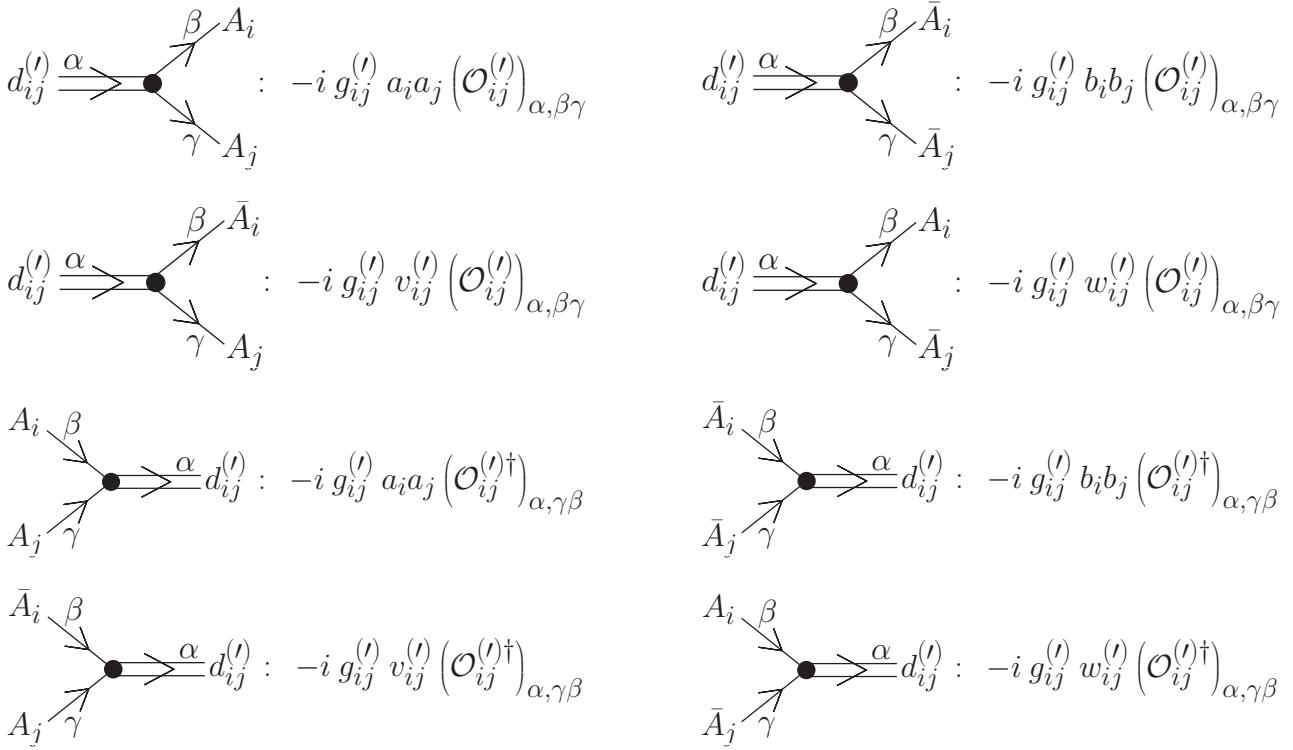
Figure 3.1: *Vertex factors for all possible interaction terms depending on the parameters a and b valid $\forall\, i, j \in \{1, 2, 3\}$ with $i < j$. Time and momentum flow from left to right and furthermore $v_{ij}^{(\prime)}$ and $w_{ij}^{(\prime)}$ are given in Eq. (3.24) and Eq. (3.25).*

(a)

(b)
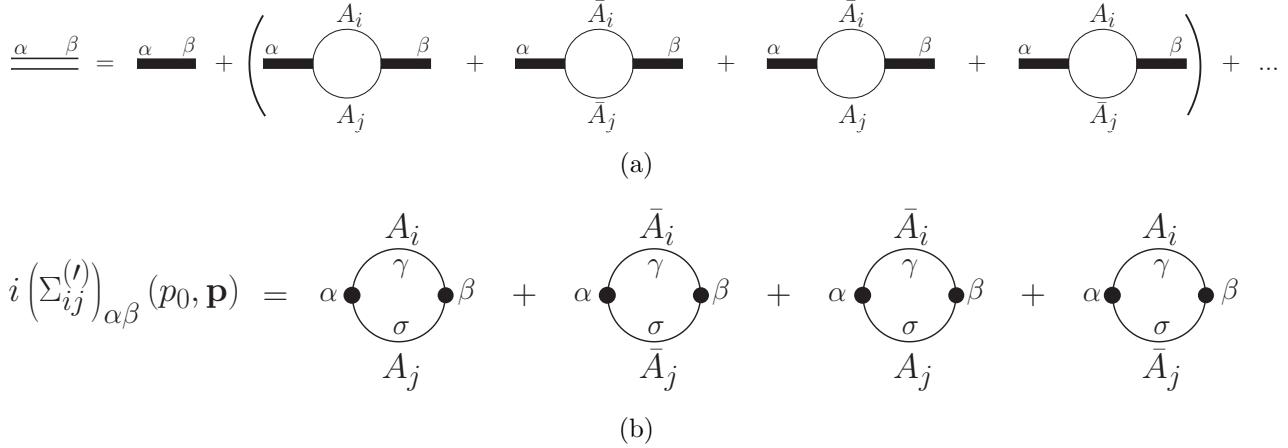
Figure 3.2: *Diagrammatic representation of the Dyson equation. The full dimer propagator is depicted as double line and the bare ones as thick solid lines (a). Diagrammatic representation of the dimer self-energy (b).*

Introducing the self-energy $\left(\Sigma_{ij}^{(\prime)}\right)_{\alpha\beta}(p_0, \mathbf{p})$ the full dimer propagator $i\left(D_{ij}^{(\prime)}\right)_{\alpha\beta}(p_0, \mathbf{p})$ is given by

$$
\begin{aligned}
i\left(D_{ij}^{(\prime)}\right)_{\alpha\beta} &= i\left(D_{ij}^{(\prime)0}\right)_{\alpha\beta} + i\left(D_{ij}^{(\prime)0}\right)_{\alpha\gamma} i\left(\Sigma_{ij}^{(\prime)}\right)_{\gamma\rho} i\left(D_{ij}^{(\prime)0}\right)_{\rho\beta} \\
&\quad + i\left(D_{ij}^{(\prime)0}\right)_{\alpha\gamma} i\left(\Sigma_{ij}^{(\prime)}\right)_{\gamma\rho} i\left(D_{ij}^{(\prime)0}\right)_{\rho\mu} i\left(\Sigma_{ij}^{(\prime)}\right)_{\mu\nu} i\left(D_{ij}^{(\prime)0}\right)_{\nu\beta} + \dots \\
&= i\left(D_{ij}^{(\prime)0}\right)_{\alpha\gamma} \left[(\mathbb{1})_{\gamma\beta} + i\left(\Sigma_{ij}^{(\prime)}\right)_{\gamma\rho} i\left(D_{ij}^{(\prime)0}\right)_{\rho\beta} \right. \\
&\quad \left. + i\left(\Sigma_{ij}^{(\prime)}\right)_{\gamma\rho} i\left(D_{ij}^{(\prime)0}\right)_{\rho\mu} i\left(\Sigma_{ij}^{(\prime)}\right)_{\mu\nu} i\left(D_{ij}^{(\prime)0}\right)_{\nu\beta} + \dots \right] \\
&= i\left(D_{ij}^{(\prime)0}\right)_{\alpha\gamma} \left\{ \sum_{n=0}^{\infty} \left[-\left(\Sigma_{ij}^{(\prime)}\right)_{\gamma\rho} \left(D_{ij}^{(\prime)0}\right)_{\rho\beta} \right]^n \right\} .
\end{aligned}
\tag{3.28}
$$

Self-energy

In the next step one needs to determine the self-energy. Therefore we consider all Feynman diagrams which – depending on the parameters a_i and b_i – contribute to it (Fig. 3.2(b)) and conclude:

$$
\begin{aligned}
i\left(\Sigma_{ij}^{(\prime)}\right)_{\alpha\beta}(p_0, \mathbf{p}) &= \left(g_{ij}^{(\prime)}\right)^2 S_{ij} \left(\mathcal{O}_{ij}^{(\prime)}\right)_{\alpha,\gamma\sigma} \left(\mathcal{O}_{ij}^{(\prime)\dagger}\right)_{\beta,\sigma\gamma} \left[a_i a_j a_i a_j + v_{ij}^{(\prime)} v_{ij}^{(\prime)} + w_{ij}^{(\prime)} w_{ij}^{(\prime)} + b_i b_j b_i b_j \right] \\
&\quad \times \int \frac{d^4 q}{(2\pi)^4} \frac{1}{p_0 + q_0 - \frac{1}{2m_i}\left(\frac{\mathbf{p}}{2} + \mathbf{q}\right)^2 + i\varepsilon} \frac{1}{-q_0 - \frac{1}{2m_j}\left(\frac{\mathbf{p}}{2} - \mathbf{q}\right)^2 + i\varepsilon} ,
\end{aligned}
\tag{3.29}
$$

with loop momentum $(q_0, \mathbf{q})$ and S_{ij} being the symmetry factor of the diagrams shown in Fig. 3.2(b). Depending on whether P_i is identical to P_j or not the latter is either 1 or 2 (see appendix B):

$$
S_{ij} = \begin{cases} 2 , & \text{if } P_i = P_j \\ 1 , & \text{if } P_i \neq P_j \end{cases} .
\tag{3.30}
$$

Before we continue it is useful to have a closer look on the term in square brackets: since P_i cannot be equal to A_i and $\bar{A}_i$ at once, we know that there only are two possible parameter sets $\{a_i = 1, b_i = 0\}$ or $\{a_i = 0, b_i = 1\}$. Combining this with the set of particle P_j and taking into account a possible non-vanishing G-parity quantum number, five physical combinations for the term in square brackets remain:

- $a_i = a_j = 1 \ \wedge \ b_i = b_j = 0 \ \wedge \ \eta_{ij}^{(\prime)} = 0$
  $\Rightarrow \left[(a_i a_j) + \left(v_{ij}^{(\prime)}\right)^2 + \left(w_{ij}^{(\prime)}\right)^2 + (b_i b_j)^2 \right] = a_i a_j = 1$

- $a_i = a_j = 0 \ \wedge \ b_i = b_j = 1 \ \wedge \ \eta_{ij}^{(\prime)} = 0$
  $\Rightarrow \left[(a_i a_j) + \left(v_{ij}^{(\prime)}\right)^2 + \left(w_{ij}^{(\prime)}\right)^2 + (b_i b_j)^2 \right] = b_i b_j = 1$

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- $b_i = a_j = 1 \;\wedge\; a_i = b_j = 0 \;\wedge\; \eta_{ij}^{(\prime)} = 0$

  $\Rightarrow \quad \left[(a_i a_j) + \left(v_{ij}^{(\prime)} \right)^2 + \left(w_{ij}^{(\prime)} \right)^2 + (b_i b_j)^2 \right] = b_i a_j = 1$

- $b_i = a_j = 1 \;\wedge\; a_i = b_j = 0 \;\wedge\; \eta_{ij}^{(\prime)} = \pm 1$

  $\Rightarrow \quad \left[(a_i a_j) + \left(v_{ij}^{(\prime)} \right)^2 + \left(w_{ij}^{(\prime)} \right)^2 + (b_i b_j)^2 \right] = b_i a_j = 1$

- $b_i = a_j = 0 \;\wedge\; a_i = b_j = 1 \;\wedge\; \eta_{ij}^{(\prime)} = \pm 1$

  $\Rightarrow \quad \left[(a_i a_j) + \left(v_{ij}^{(\prime)} \right)^2 + \left(w_{ij}^{(\prime)} \right)^2 + (b_i b_j)^2 \right] = \delta_{\eta_{ij}^{(\prime)} |1|} a_i b_j = 1$

Hence, one finds

$$\left[(a_i a_j) + \left(v_{ij}^{(\prime)} \right)^2 + \left(w_{ij}^{(\prime)} \right)^2 + (b_i b_j)^2 \right] = 1 \qquad \forall \, a_i, b_i \,. \tag{3.31}$$

Consequently, the self-energy and therefore the full propagator of the dimer $d_{ij}^{(\prime)}$ are independent of the species of its constituents. Thus, the self-energy simplifies to

$$i \left(\Sigma_{ij}^{(\prime)} \right)_{\alpha\beta} (p_0, \mathbf{p}) = \left(g_{ij}^{(\prime)} \right)^2 S_{ij} \left(\mathcal{O}_{ij}^{(\prime)} \right)_{\alpha,\gamma\sigma} \left(\mathcal{O}_{ij}^{(\prime)\dagger} \right)_{\beta,\sigma\gamma}$$
$$\times \int \frac{d^4 q}{(2\pi)^4} \frac{1}{p_0 + q_0 - \frac{1}{2m_i} \left(\frac{\mathbf{P}}{2} + \mathbf{q} \right)^2 + i\varepsilon} \frac{-1}{q_0 + \frac{1}{2m_j} \left(\frac{\mathbf{P}}{2} - \mathbf{q} \right)^2 - i\varepsilon} \,. \tag{3.32}$$

Using the residue theorem one can perform the integration over q_0 which yields after some algebra

$$i \left(\Sigma_{ij}^{(\prime)} \right)_{\alpha\beta} (p_0, \mathbf{p}) = \left(g_{ij}^{(\prime)} \right)^2 S_{ij} \left(\mathcal{O}_{ij}^{(\prime)} \right)_{\alpha,\gamma\sigma} \left(\mathcal{O}_{ij}^{(\prime)\dagger} \right)_{\beta,\sigma\gamma}$$
$$\times \int \frac{d^3 q}{(2\pi)^3} \frac{2\mu_{ij} i}{q^2 - 2\mu_{ij} \, p_0 + \frac{p^2}{4} + \sqrt{1 - \frac{4\mu_{ij}}{m_i + m_j}} \, \mathbf{p} \cdot \mathbf{q} - i\varepsilon} \,, \tag{3.33}$$

where we have written the modulus of a 3-vector as $|\mathbf{x}| := x$ and introduced the reduced mass

$$\mu_{ij} := \frac{m_i \, m_j}{m_i + m_j} \,, \tag{3.34}$$

which is obviously equal for both unprimed and primed dimers. The remaining integral can be calculated in dimensional regularization using a scale μ. As one can found in many textbooks on QFT (e.g. Ref. [1]) it holds

$$\int d^d q \frac{1}{(q^2 + 2 \, q \cdot k - b^2)^{\alpha}} = (-1)^{\frac{d}{2}} \, i\pi^{\frac{d}{2}} \frac{\Gamma \left(\alpha - \frac{d}{2} \right)}{\Gamma(\alpha)} \left[-k^2 - b^2 \right]^{\frac{d}{2} - \alpha} \,, \tag{3.35}$$

which was also applied to the self-energy integral in section 2.1. Thus, in $D = 4$ dimensions we have with $d = D - 1$:

$$
\left(\Sigma_{ij}^{(\prime)}\right)_{\alpha\beta} = \left(g_{ij}^{(\prime)}\right)^2 S_{ij} \left(\mathcal{O}_{ij}^{(\prime)}\right)_{\alpha,\gamma\sigma} \left(\mathcal{O}_{ij}^{(\prime)\dagger}\right)_{\beta,\sigma\gamma}
$$

$$
\times \mu^{D-4} \int \frac{d^{D-1}q}{(2\pi)^{D-1}} \frac{2\mu_{ij}}{q^2 - 2\mu_{ij}\, p_0 + \frac{p^2}{4} + \sqrt{1 - \frac{4\mu_{ij}}{m_i+m_j}}\, \mathbf{p} \cdot \mathbf{q} - i\varepsilon}
$$

$$
= 2\left(g_{ij}^{(\prime)}\right)^2 \mu_{ij} S_{ij} \left(\mathcal{O}_{ij}^{(\prime)}\right)_{\alpha,\gamma\sigma} \left(\mathcal{O}_{ij}^{(\prime)\dagger}\right)_{\beta,\sigma\gamma} \frac{\mu^{D-4}}{(2\pi)^{D-1}} \left[(-1)^{\frac{D-1}{2}}\, i\, \pi^{\frac{D-1}{2}} \frac{\Gamma\left(\alpha - \frac{D-1}{2}\right)}{\Gamma(1)} \right.
$$

$$
\times \left. \left\{ -\left(1 - \frac{4\mu_{ij}}{m_i+m_j}\right) \frac{p^2}{4} - \left(2\mu_{ij}\, p_0 - \frac{p^2}{4} + i\varepsilon\right) \right\}^{\frac{D-1}{2}-1} \right]
$$

$$
\xrightarrow{D\to 4} -\frac{1}{2\pi} \left(g_{ij}^{(\prime)}\right)^2 \mu_{ij} S_{ij} \left(\mathcal{O}_{ij}^{(\prime)}\right)_{\alpha,\gamma\sigma} \left(\mathcal{O}_{ij}^{(\prime)\dagger}\right)_{\beta,\sigma\gamma} \sqrt{-2\mu_{ij}\left(p_0 - \frac{p^2}{2(m_i+m_j)}\right) - i\varepsilon}\,.
$$

$$(3.36)$$

In the last step we took the limit $D \to 4$. This is possible because there is no pole in $D = 4$ dimensions. However, for $D = 3$ it is and therefore the S-wave scattering length has an unnatural scaling behavior as it is discussed for nucleon–nucleon interactions in Ref. [70]. As the same authors pointed out in Ref. [72], a scattering length of natural size scales as $a_{\mathrm{nat}} \sim 1/\mu$ since the scale μ is set by the pion mass where EFT($\not\pi$) breaks down. However, the unnatural large scattering length a_{NN} in the NN system yields $a_{\mathrm{NN}} \gg 1/\mu$. This fact will lead to problems in the power counting scheme of the EFT. Hence, they introduced in Refs. [71, 72] instead of the minimal subtraction scheme which only removes poles in four dimensions, a new scheme called *power divergence subtraction* (PDS) scheme which subtracts the poles in both three and four dimensions by a counter term $\delta\Sigma_{ij}^{(\prime)}$ so that a correct power counting is restored. Although the authors only assumed NN interactions the same argument is true for any other system with unnatural large scattering length. Hence, one should use the PDS scheme also for hadronic molecules whose scattering length is expected to be rather large. With the PDS scale μ_{PDS} (which we use instead of μ) the corresponding PDS counter term for our problem is given by

$$
\left(\delta\Sigma_{ij}^{(\prime)}\right)_{\alpha\beta} = \frac{1}{2\pi} \left(g_{ij}^{(\prime)}\right)^2 \mu_{ij} S_{ij} \left(\mathcal{O}_{ij}^{(\prime)}\right)_{\alpha,\gamma\sigma} \left(\mathcal{O}_{ij}^{(\prime)\dagger}\right)_{\beta,\sigma\gamma} \frac{\mu_{\mathrm{PDS}}}{D-3}\,, \tag{3.37}
$$

and the self-energy is changed to

$$
\left(\Sigma_{ij}^{(\prime)}\right)_{\alpha\beta} \xrightarrow{\mathrm{PDS}} \left(\Sigma_{ij}^{(\prime)}\right)_{\alpha\beta}(p_0, \mathbf{p}) + \left(\delta\Sigma_{ij}^{(\prime)}\right)_{\alpha\beta}
$$

$$
= 2\left(g_{ij}^{(\prime)}\right)^2 \mu_{ij} S_{ij} \left(\mathcal{O}_{ij}^{(\prime)}\right)_{\alpha,\gamma\sigma} \left(\mathcal{O}_{ij}^{(\prime)\dagger}\right)_{\beta,\sigma\gamma} \frac{\mu_{\mathrm{PDS}}^{D-4}}{(2\pi)^{D-1}} \left[(-1)^{\frac{D-1}{2}}\, i\, \pi^{\frac{D-1}{2}} \frac{\Gamma\left(\alpha - \frac{D-1}{2}\right)}{\Gamma(1)} \right.
$$

$$
\times \left. \left\{ -\left(1 - \frac{4\mu_{ij}}{m_i+m_j}\right) \frac{p^2}{4} - \left(2\mu_{ij}\, p_0 - \frac{p^2}{4} + i\varepsilon\right) \right\}^{\frac{D-1}{2}-1} \right]
$$

$$
+ \frac{1}{2\pi} \left(g_{ij}^{(\prime)}\right)^2 \mu_{ij} S_{ij} \left(\mathcal{O}_{ij}^{(\prime)}\right)_{\alpha,\gamma\sigma} \left(\mathcal{O}_{ij}^{(\prime)\dagger}\right)_{\beta,\sigma\gamma} \frac{\mu_{\mathrm{PDS}}}{D-3}\,, \tag{3.38}
$$

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which yields after taking the limit $D \to 4$

$$\left(\Sigma_{ij}^{(\prime)}\right)_{\alpha\beta} \xrightarrow{D \to 4} -\frac{1}{2\pi} \left(g_{ij}^{(\prime)}\right)^2 \mu_{ij} \, S_{ij} \left(\mathcal{O}_{ij}^{(\prime)}\right)_{\alpha,\gamma\sigma} \left(\mathcal{O}_{ij}^{(\prime)\dagger}\right)_{\beta,\sigma\gamma} \sqrt{-2\mu_{ij}\left(p_0 - \frac{p^2}{2(m_i + m_j)}\right) - i\varepsilon}$$

$$+ \frac{1}{2\pi}\left(g_{ij}^{(\prime)}\right)^2 \mu_{ij} \, S_{ij} \left(\mathcal{O}_{ij}^{(\prime)}\right)_{\alpha,\gamma\sigma} \left(\mathcal{O}_{ij}^{(\prime)\dagger}\right)_{\beta,\sigma\gamma} \mu_{\mathrm{PDS}}$$

$$= \frac{1}{2\pi}\left(g_{ij}^{(\prime)}\right)^2 \mu_{ij} \, S_{ij} \left(\mathcal{O}_{ij}^{(\prime)}\right)_{\alpha,\gamma\sigma} \left(\mathcal{O}_{ij}^{(\prime)\dagger}\right)_{\beta,\sigma\gamma}$$

$$\times \left[-\sqrt{-2\mu_{ij}\left(p_0 - \frac{p^2}{2(m_i + m_j)}\right) - i\varepsilon} + \mu_{\mathrm{PDS}}\right] . \tag{3.39}$$

Before we proceed we shorten this result via the definition

$$\left(\widetilde{\Sigma}_{ij}^{(\prime)}\right)(p_0, \mathbf{p}) := \frac{1}{2\pi}\left(g_{ij}^{(\prime)}\right)^2 \mu_{ij} \, S_{ij} \left[-\sqrt{-2\mu_{ij}\left(p_0 - \frac{p^2}{2(m_i + m_j)}\right) - i\varepsilon} + \mu_{\mathrm{PDS}}\right] , \tag{3.40}$$

and write:

$$\left(\Sigma_{ij}^{(\prime)}\right)_{\alpha\beta}(p_0, \mathbf{p}) = \left(\mathcal{O}_{ij}^{(\prime)}\right)_{\alpha,\gamma\sigma} \left(\mathcal{O}_{ij}^{(\prime)\dagger}\right)_{\beta,\sigma\gamma} \left(\widetilde{\Sigma}_{ij}^{(\prime)}\right)(p_0, \mathbf{p}) . \tag{3.41}$$

Now consider the (iso)spin dependent part $\left(\mathcal{O}_{ij}^{(\prime)}\right)_{\alpha,\gamma\sigma} \left(\mathcal{O}_{ij}^{(\prime)\dagger}\right)_{\beta,\sigma\gamma}$. It is not necessary to know the projection operators in detail: as shown in appendix A all projectors are orthonormal (or at least orthogonal if one has shifted the normalization factor into the coupling constant within the Lagrangian Eq. (3.23)). Consequently, it holds

$$\left(\mathcal{O}_{ij}^{(\prime)}\right)_{\alpha,\gamma\sigma} \left(\mathcal{O}_{ij}^{(\prime)\dagger}\right)_{\beta,\sigma\gamma} = \left(\left(\mathcal{O}_{ij}^{(\prime)}\right)_{\alpha} \left(\mathcal{O}_{ij}^{(\prime)\dagger}\right)_{\beta}\right)_{\gamma\gamma} = \mathrm{Tr}\left(\left(\mathcal{O}_{ij}^{(\prime)}\right)_{\alpha} \left(\mathcal{O}_{ij}^{(\prime)\dagger}\right)_{\beta}\right) = c_{ij}^{(\prime)} \, \delta_{\alpha\beta} , \tag{3.42}$$

with $c_{ij}^{(\prime)} \in \mathbb{R}$. If – as in our case (see appendix A) – all projectors are correctly normalized one finds that

$$\text{if } \mathcal{O}_{ij}^{(\prime)} \text{ is normalized: } \quad c_{ij}^{(\prime)} = 1 \qquad \forall \, i < j \in \{1, 2, 3\} . \tag{3.43}$$

Otherwise, $c_{ij}^{(\prime)}$ depends on the in this case changed prefactors in the projection operators and must be calculated separately. Note, that such a different normalization would be canceled at another point in the calculation and does not change any observable. Considering again our example of $Z_b(10610)$ and $Z_b'(10650)$ we find (cf. Eq. (3.18)):

$$\left(\mathcal{O}_{12}\right)_{\{\alpha=aA\},\{\gamma=c\widetilde{\gamma}\}\{\sigma=\widetilde{\sigma}\}} \left(\mathcal{O}_{12}^\dagger\right)_{\{\beta=bB\},\{\sigma=\widetilde{\sigma}\}\{\gamma=c\widetilde{\gamma}\}} = \delta_{ac}\frac{i}{\sqrt{2}} (\tau_A \tau_2)_{\widetilde{\gamma}\widetilde{\sigma}} \, \delta_{cb}\frac{-i}{\sqrt{2}} (\tau_2 \tau_B)_{\widetilde{\sigma}\widetilde{\gamma}}$$

$$= \frac{1}{2} \, \delta_{ab} \, \mathrm{Tr}\left(\tau_A \tau_2 \tau_2 \tau_B\right)$$

$$= \frac{1}{2} \, \delta_{ab} \, \mathrm{Tr}\left(\tau_A \tau_B\right)$$

$$= \delta_{ab} \, \delta_{AB} , \tag{3.44}$$

$$\left(\mathcal{O}_{13}\right)_{\{\alpha=aA\},\{\gamma=c\widetilde{\gamma}\}\{\sigma=s\widetilde{\sigma}\}} \left(\mathcal{O}_{13}^{\dagger}\right)_{\{\beta=bB\},\{\sigma=s\widetilde{\sigma}\}\{\gamma=c\widetilde{\gamma}\}} = \frac{-i}{2} \left(U_a\right)_{cs} \left(\tau_A\tau_2\right)_{\widetilde{\gamma}\widetilde{\sigma}} \frac{i}{2} \left(U_b\right)_{sc} \left(\tau_2\tau_B\right)_{\widetilde{\sigma}\widetilde{\gamma}}$$

$$= \frac{1}{4} \operatorname{Tr}\left(U_a U_b\right) \operatorname{Tr}\left(\tau_A\tau_2\tau_2\tau_B\right)$$

$$= \delta_{ab}\,\delta_{AB}\,, \tag{3.45}$$

as expected since $\mathcal{O}_{12}$ and $\mathcal{O}_{13}$ are indeed normalized. However, we will keep $c_{ij}^{(\prime)}$ in our work to be as general as possible. Inserting Eq. (3.42) into Eq. (3.41) yields

$$\left(\Sigma_{ij}^{(\prime)}\right)_{\alpha\beta}(p_0,\mathbf{p}) = \delta_{\alpha\beta}\, c_{ij}^{(\prime)}\, \widetilde{\Sigma}_{ij}^{(\prime)}(p_0,\mathbf{p})\,. \tag{3.46}$$

Together with Eq. (3.27) we can use this result to simplify Eq. (3.28) to

$$i\left(D_{ij}^{(\prime)}\right)_{\alpha\beta}(p_0,\mathbf{p}) = i\frac{\delta_{\alpha\gamma}}{\Delta_{ij}^{(\prime)}} \left\{\sum_{n=0}^{\infty}\left[-\delta_{\gamma\rho}\, c_{ij}^{(\prime)}\, \widetilde{\Sigma}_{ij}^{(\prime)}(p_0,\mathbf{p})\,\frac{\delta_{\rho\beta}}{\Delta_{ij}^{(\prime)}}\right]^n\right\}$$

$$= i\frac{\delta_{\alpha\gamma}}{\Delta_{ij}^{(\prime)}} \left\{\sum_{n=0}^{\infty}(\delta_{\gamma\beta})^n\left[-c_{ij}^{(\prime)}\, \widetilde{\Sigma}_{ij}^{(\prime)}(p_0,\mathbf{p})\,\frac{1}{\Delta_{ij}^{(\prime)}}\right]^n\right\}$$

$$= i\frac{\delta_{\alpha\gamma}}{\Delta_{ij}^{(\prime)}} \left\{\sum_{n=0}^{\infty}\delta_{\gamma\beta}\left[-c_{ij}^{(\prime)}\, \widetilde{\Sigma}_{ij}^{(\prime)}(p_0,\mathbf{p})\,\frac{1}{\Delta_{ij}^{(\prime)}}\right]^n\right\}$$

$$= \delta_{\alpha\beta}\frac{i}{\Delta_{ij}^{(\prime)}} \left\{\sum_{n=0}^{\infty}\left[-c_{ij}^{(\prime)}\,\left(\widetilde{\Sigma}_{ij}^{(\prime)}\right)(p_0,\mathbf{p})\,\frac{1}{\Delta_{ij}^{(\prime)}}\right]^n\right\}$$

$$= \delta_{\alpha\beta}\frac{i}{\Delta_{ij}^{(\prime)}}\frac{1}{1-\left(-c_{ij}^{(\prime)}\,\widetilde{\Sigma}_{ij}^{(\prime)}(p_0,\mathbf{p})\,\frac{1}{\Delta_{ij}^{(\prime)}}\right)}$$

$$= \frac{i\delta_{\alpha\beta}}{\Delta_{ij}^{(\prime)} + c_{ij}^{(\prime)}\,\widetilde{\Sigma}_{ij}(p_0,\mathbf{p})}\,, \tag{3.47}$$

where we have used the geometric series in the second to last step. Using the definition of $\widetilde{\Sigma}_{ij}$ in Eq. (3.40) we finally find that the full dimer propagator is given by

$$i\left(D_{ij}^{(\prime)}\right)_{\alpha\beta}(p_0,\mathbf{p}) = -\frac{2\pi\,i}{\left(g_{ij}^{(\prime)}\right)^2\mu_{ij}\,S_{ij}\,c_{ij}^{(\prime)}}$$

$$\times\,\frac{\delta_{\alpha\beta}}{-\left(\frac{2\pi\,\Delta_{ij}^{(\prime)}}{\left(g_{ij}^{(\prime)}\right)^2\mu_{ij}\,S_{ij}\,c_{ij}^{(\prime)}}+\mu_{\mathrm{PDS}}\right)+\sqrt{-2\mu_{ij}\left(p_0-\frac{p^2}{2(m_i+m_j)}\right)}-i\varepsilon}\,. \tag{3.48}$$

This result can be used to determine the elastic two-body scattering amplitude T_{ij}^{el} of the particles P_i and P_j. The advantage is then that one can compare T_{ij}^{el} with the LO effective range expansion

amplitude T_{ij}^{ERE} in order to write the full dimer in terms of an observable, namely the binding momentum $\gamma_{ij}^{(\prime)}$ instead of the parameter $\Delta_{ij}^{(\prime)}$.

Elastic scattering amplitude

In Fig. 3.3 we consider all possible diagrams which could – depending on the parameters a and b – contribute to the elastic on-shell scattering amplitude $T_{ij}^{(\prime)\text{el}}(E = k^2/(2\mu_{ij}), 0)$ of the two particles P_i and P_j. Written as an equation we get

$$i\left(T_{ij}^{(\prime)\text{el}}\right)_{\alpha\beta}^{\mu\nu}\left(E = \frac{k^2}{2\mu_{ij}}, 0\right) = -\left(g_{ij}^{(\prime)}\right)^2 S_{\text{el}}\left[a_i a_j a_i a_j + v_{ij}^{(\prime)} v_{ij}^{(\prime)} + w_{ij}^{(\prime)} w_{ij}^{(\prime)} + b_i b_j b_i b_j\right]$$

$$\times \left(\mathcal{O}_{ij}^{(\prime)}\right)_{\sigma,\mu\nu} i\left(D_{ij}^{(\prime)}\right)_{\sigma\gamma}\left(E = \frac{k^2}{2\mu_{ij}}, 0\right)\left(\mathcal{O}_{ij}^{(\prime)\dagger}\right)_{\gamma,\beta\alpha}$$

$$= \frac{2\pi i}{\mu_{ij} c_{ij}^{(\prime)}} \frac{S_{\text{el}}}{S_{ij}}\left(\mathcal{O}_{ij}^{(\prime)}\right)_{\gamma,\mu\nu}\left(\mathcal{O}_{ij}^{(\prime)\dagger}\right)_{\gamma,\beta\alpha}$$

$$\times \frac{1}{-\left(\dfrac{2\pi \Delta_{ij}^{(\prime)}}{\left(g_{ij}^{(\prime)}\right)^2 \mu_{ij} S_{ij} c_{ij}^{(\prime)}} + \mu_{\text{PDS}}\right) + \sqrt{-k^2 - i\varepsilon}}, \qquad (3.49)$$

with symmetry factor S_{el} (see appendix B),

$$S_{\text{el}} = \begin{cases} 4, & \text{if } P_i = P_j \\ 1, & \text{if } P_i \neq P_j \end{cases}, \qquad (3.50)$$

and in the second step we have used that the term in square brackets is equal to 1 (cf. Eq. (3.31)). Due to the $-i\varepsilon$ in the square root one has to choose its negative branch cut and we end up with a relation only depending on k:

$$\left(T_{ij}^{(\prime)\text{el}}\right)_{\alpha\beta}^{\mu\nu}(k) = -\frac{2\pi}{\mu_{ij} c_{ij}^{(\prime)}} \frac{S_{\text{el}}}{S_{ij}}\left(\mathcal{O}_{ij}^{(\prime)}\right)_{\gamma,\mu\nu}\left(\mathcal{O}_{ij}^{(\prime)\dagger}\right)_{\gamma,\beta\alpha} \frac{1}{\left(\dfrac{2\pi \Delta_{ij}^{(\prime)}}{\left(g_{ij}^{(\prime)}\right)^2 \mu_{ij} S_{ij} c_{ij}^{(\prime)}} + \mu_{\text{PDS}}\right) + ik}. \qquad (3.51)$$

However, it still has some free (iso)spin indices so we apply projection operators to it. In an elastic scattering process we know that initial and final state (iso)spin must be equal to the

$$i\left(T_{ij}^{(\prime)\text{el}}\right)_{\alpha\beta}^{\mu\nu}\left(E = \frac{k^2}{2\mu_{ij}}, 0\right) =$$

Figure 3.3: *Diagrams contributing to the elastic on-shell scattering amplitude of the two particles P_i and P_j which correspond to one of the four possible combinations of the multiplets A, characterized by the parameters a and b.*

(iso)spin of the intermediate dimer. Therefore we conclude that the needed projectors are the same as those for the vertices within the diagrams in Fig. 3.3. From Eq. (3.42) we know that $\mathcal{O}_{ij}^{(\prime)}$ is not necessarily normalized and thus we have to do this now via a factor $1/\sqrt{c_{ij}^{(\prime)}}$. Note, that for already normalized projectors $c_{ij}^{(\prime)} = 1$ and thus nothing is changed by the extra factor. After projection onto the dimer (iso)spin one still has to average over initial and sum over final (iso)spins in order to obtain an index-free amplitude:

$$T_{ij}^{(\prime)\text{el}} = \frac{1}{\text{dof}} \sum_{\eta,\rho} \left(T_{ij}^{(\prime)\text{el}} \right)_{\eta}^{\rho} = \frac{1}{\text{dof}} \sum_{\eta,\rho} \frac{1}{\sqrt{c_{ij}^{(\prime)}}} \left(\mathcal{O}_{ij}^{(\prime)\dagger} \right)_{\rho,\nu\mu} \left(T_{ij}^{(\prime)\text{el}} \right)_{\alpha\beta}^{\mu\nu} \frac{1}{\sqrt{c_{ij}^{(\prime)}}} \left(\mathcal{O}_{ij}^{(\prime)} \right)_{\eta,\alpha\beta} . \quad (3.52)$$

Here dof stands for "spin degrees of freedom $\times$ isospin degrees of freedom", i.e. dof $= (2J+1) \times (2I+1)$. Inserting what we calculated in Eq. (3.51) leads to

$$T_{ij}^{(\prime)\text{el}} = \frac{1}{\text{dof}} \sum_{\eta,\rho} \frac{-2\pi}{\mu_{ij} \left(c_{ij}^{(\prime)} \right)^2} \frac{S_{\text{el}}}{S_{ij}} \text{Tr}\left(\left(\mathcal{O}_{ij}^{(\prime)\dagger} \right)_{\rho} \left(\mathcal{O}_{ij}^{(\prime)} \right)_{\gamma} \right) \text{Tr}\left(\left(\mathcal{O}_{ij}^{(\prime)\dagger} \right)_{\gamma} \left(\mathcal{O}_{ij}^{(\prime)} \right)_{\eta} \right)$$
$$\times \frac{1}{\left(\frac{2\pi \Delta_{ij}^{(\prime)}}{\left(g_{ij}^{(\prime)} \right)^2 \mu_{ij} S_{ij} c_{ij}^{(\prime)}} + \mu_{\text{PDS}} \right) + ik} . \quad (3.53)$$

Following Eq. (3.42) both traces yield a factor $c_{ij}^{(\prime)}$ and a Kronecker-delta for the (iso)spin indices:

$$T_{ij}^{(\prime)\text{el}} = -\frac{2\pi}{\mu_{ij}} \frac{S_{\text{el}}}{S_{ij}} \frac{1}{\left(\frac{2\pi \Delta_{ij}^{(\prime)}}{\left(g_{ij}^{(\prime)} \right)^2 \mu_{ij} S_{ij} c_{ij}^{(\prime)}} + \mu_{\text{PDS}} \right) + ik} \frac{1}{\text{dof}} \sum_{\eta,\rho} \delta_{\rho\eta} . \quad (3.54)$$

The sum over the (iso)spin indices ρ and η reduces to

$$\sum_{\eta,\rho} \delta_{\rho\eta} = \delta_{\eta\eta} = \text{Tr}\left(\mathbb{1}_{\text{dim. spin space}} \right) \text{Tr}\left(\mathbb{1}_{\text{dim. isospin space}} \right)$$
$$= \text{dim. spin space} \times \text{dim. isospin space}$$
$$= \text{spin degrees of freedom} \times \text{isospin degrees of freedom}$$
$$= \text{dof} . \quad (3.55)$$

Thus, we end up with

$$T_{ij}^{(\prime)\text{el}} = -\frac{2\pi}{\mu_{ij}} \frac{S_{\text{el}}}{S_{ij}} \frac{1}{\left(\frac{2\pi \Delta_{ij}^{(\prime)}}{\left(g_{ij}^{(\prime)} \right)^2 \mu_{ij} S_{ij} c_{ij}^{(\prime)}} + \mu_{\text{PDS}} \right) + ik} , \quad (3.56)$$

which can be compared to the ERE amplitude (cf. section 1.4),

$$T_{ij}^{(\prime)\text{ERE}} = -\frac{\left(2 + 2\,\delta_{P_i P_j} \right) \pi}{\mu_{ij}} \frac{1}{\frac{1}{a_{ij}^{(\prime)}} + ik} , \quad (3.57)$$

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where the additional term $2\,\delta_{P_i P_j}$ ensures the right normalization in case of identical particles. Using the definitions of S_{el} (Eq. (3.50)) and S_{ij} (Eq. (3.30)) we observe that also

$$\frac{S_{\mathrm{el}}}{S_{ij}} = \begin{cases} \frac{4}{2} = 2\,, & \text{if } P_i = P_j \\ \frac{1}{1} = 1\,, & \text{if } P_i \neq P_j \end{cases} \tag{3.58}$$

reproduces the right prefactor in both cases. Therefore we can identify the scattering length $a_{ij}^{(\prime)}$ which is at LO equivalent to the binding momentum $\gamma_{ij}^{(\prime)}$ as

$$\frac{1}{a_{ij}^{(\prime)}} \equiv \gamma_{ij}^{(\prime)} = \frac{2\pi\,\Delta_{ij}^{(\prime)}}{\left(g_{ij}^{(\prime)}\right)^2\,\mu_{ij}\,S_{ij}\,c_{ij}^{(\prime)}} + \mu_{\mathrm{PDS}}\,. \tag{3.59}$$

Note, that the explicit dependence on the PDS scale μ_{PDS} may be shifted into the coupling constant, but for our purpose this does not cause any problems. Hence, we leave the equation above unchanged. Furthermore, we can plug it into Eq. (3.48) to find an expression for the LO full dimer propagator which only depends on observables except for the prefactor $\left(g_{ij}^{(\prime)}\right)^{-2}$ which is canceled anyway in all following calculations. We find:

$$i\left(D_{ij}^{(\prime)}\right)_{\alpha\beta}(p_0, \mathbf{p}) = -\frac{2\pi\,i}{\left(g_{ij}^{(\prime)}\right)^2\,\mu_{ij}\,S_{ij}\,c_{ij}^{(\prime)}}\,\frac{\delta_{\alpha\beta}}{-\gamma_{ij}^{(\prime)} + \sqrt{-2\mu_{ij}\left(p_0 - \frac{p^2}{2(m_i + m_j)}\right)} - i\varepsilon}$$

$$:= \delta_{\alpha\beta}\,iD_{ij}^{(\prime)}(p_0, \mathbf{p})\,, \tag{3.60}$$

where we have defined an expression without (iso)spin indices in the last step.

3.2.3 Wave function renormalization constant

The only missing relation before one can derive a general $d_{ij}^{(\prime)}$–P_k scattering amplitude is the wave function renormalization constant $Z_{ij}^{(\prime)}$. Firstly, we note that the fields A_i and $\bar{A}_i$ have a wave function renormalization constant equal to one since they are considered as point-like particles. Only the dimer fields must be renormalized. From basic QFT we know that $Z_{ij}^{(\prime)}$ is the residue of the corresponding (full) propagator [1]:

$$\left(Z_{ij}^{(\prime)}\right)^{-1} = i\frac{\partial}{\partial p_0}\left[\left(iD_{ij}^{(\prime)}(p_0, \mathbf{p})\right)^{-1}\right]\Bigg|_{p_0 = -\frac{\left(\gamma_{ij}^{(\prime)}\right)^2}{2\mu_{ij}}\,,\,\mathbf{p}=0}\,, \tag{3.61}$$

where p_0 is set to the binding energy $B_{ij}^{(\prime)}$ of the bound or virtual state characterized by the dimer. As explained in section 1.4.1 we use the definition

$$\gamma_{ij}^{(\prime)} \equiv \mathrm{sgn}\left(\mathrm{B}_{ij}^{(\prime)}\right)\sqrt{2\mu_{ij}\left|\mathrm{B}_{ij}^{(\prime)}\right|}\,, \tag{3.62}$$

for the binding momentum $\gamma_{ij}^{(\prime)}$. With the result of Eq. (3.60) we find

$$Z_{ij}^{(\prime)} = \frac{2\pi\,\gamma_{ij}^{(\prime)}}{\left(g_{ij}^{(\prime)}\right)^2\,\mu_{ij}^2\,S_{ij}\,c_{ij}^{(\prime)}}\,. \tag{3.63}$$

3.3 General three-body scattering amplitude

In principle one has to analyze d_{12}–P_3, d_{13}–P_2, d_{23}–P_1, d'_{12}–P_3, d'_{13}–P_2 and d'_{23}–P_1 scattering in order to account for all possible three particle states. Alternatively, but much simpler one can only calculate d_{12}–P_3 scattering and instead change – if necessary – the particle allocation in the following way:

Particle allocation

1. Choose the three particles of the system ($\rightarrow$ e.g. B, B^*, $\bar{B}^*$).

2. Choose the dimer–particle scattering process you are interested in within this system ($\rightarrow$ e.g. $(B\bar{B}^*)$–B^*).

3. Allocate the multiplets according to that choice, but *always* in a way that P_3 is scattered off the dimer d_{12} ($\rightarrow$ e.g. $A_1 = B$, $A_2 = B^*$, $A_3 = B^*$).

4. Set a_i, b_i in P_i ($i \in \{1, 2, 3\}$) to either 0 or 1 so that the desired three particle system is reproduced ($\rightarrow$ e.g. $a_1 = b_2 = a_3 = 1$, $b_1 = a_2 = b_3 = 0 \Rightarrow P_1 = B$, $P_2 = \bar{B}^*$, $P_3 = B^*$).

$$iT^{(')}_{ij} = \begin{matrix} d^{(')}_{ij} & \!\!\!\!\!\!\!\!\!\!\!\!\! & d^{(')}_{ij} \\ & T^{(')}_{ij} & \\ P_k & \!\!\!\!\!\!\!\!\!\!\!\!\! & P_k \end{matrix}$$
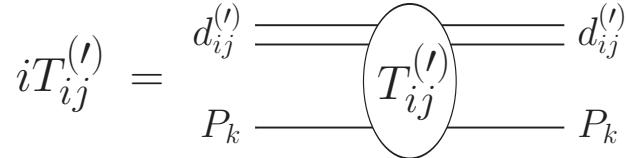
Figure 3.4: *Fixing the notation for the dimer–particle scattering amplitudes $T^{(')}_{ij}$ with $i, j, k \in \{1, 2, 3\}$ and $i < j$, $k \neq i, j$.*

The notation for the mentioned amplitudes is shown in Fig. 3.4. Although with this scheme we only need to determine d_{12}–P_3 scattering it is in general possible that all other scattering amplitudes contribute to this scattering. Hence, in most cases one has to solve a coupled integral equation system of all amplitudes. The Feynman diagrams for all six coupled amplitudes are shown in Fig. C.2 and explained in appendix C. The straightforward, but lengthy task is now to use the Feynman rules derived in the previous section 3.2 and to determine the coupled integral equation system. To do so we once more have to fix our notation for (iso)spin indices and momenta. We will work in the center-of-mass system with incoming 4-momenta k, $-k$ and outgoing 4-momenta p, $-p$. Omitting possible "primes" for the moment the center-of-mass energy is

$$E = \frac{k^2}{2(m_i + m_j)} + \frac{k^2}{2m_k} - \frac{\gamma_{ij}}{2\mu_{ij}} \ . \tag{3.64}$$

Furthermore, we assign the combined spin and isospin indices as follows:

- initial state dimer: α,

- initial state single particle: β,

- final state dimer: γ,

- final state single particle: σ,

- intermediate dimer: μ,

- intermediate single particle: ν,

- intermediate exchanged particle: ρ.

Since there only are two different topologies for the Feynman diagrams the notation above leads to the assignment shown in Fig. 3.5 which is valid for every diagram we need to determine. In appendix B we have derived the symmetry factor for diagrams like those in Figure 3.5.

Figure 3.5: *Definition of (iso)spin indices and momenta in the center-of-mass system for both diagram topologies appearing in the scattering amplitudes.*

We found

$$S_{ijk} = \zeta_{ijk} \left(1 + \delta_{P_i P_j} + \delta_{P_j P_k} + \delta_{P_i P_j} \, \delta_{P_j P_k} \right) , \tag{3.65}$$

with ζ_{ijk} defined in Eq. (B.14).

With all these ingredients one finally can write down the coupled integral equation system of the six amplitudes contributing to the general d_{12}–P_3 scattering amplitude corresponding to the diagrams shown in Figure C.2. For details on the equations see appendix F. In Eq. (F.1) we have combined all diagrams with the same momentum structure. Furthermore, we added terms which only differ by a "primed" or "unprimed" final state dimer. In order to have more compact

equations we have introduced two short-hand notations $f^{(h)}_{(ij)^{(l)}(kn)^{[l]}}$ and $\widetilde{f}^{(h)}_{(ij)^{(l)}(kn)^{[l]}}$ for the sum over the a_i, b_i dependent vertex factors:

$$
f^{(h)}_{(ij^{(l)})(kn^{[l]})} := \begin{cases} a_i a_j a_k a_n + v^{(l)}_{ij} a_k a_n + a_h \widehat{w}^{(l)}_{ij} v^{[l]}_{kn} + b_i b_j v^{[l]}_{kn} + a_i a_j \widehat{w}^{[l]}_{kn} + b_h v^{(l)}_{ij} \widehat{w}^{[l]}_{kn} + \widehat{w}^{(l)}_{ij} b_k b_n \\ a_i a_j a_k a_n + v^{(l)}_{ij} a_k a_n + a_h \widehat{w}^{(l)}_{ij} \widehat{w}^{[l]}_{kn} + b_i b_j \widehat{w}^{[l]}_{kn} + a_i a_j v^{[l]}_{kn} + b_h v^{(l)}_{ij} v^{[l]}_{kn} + \widehat{w}^{(l)}_{ij} b_k b_n \end{cases}
$$

$$
\begin{aligned}
&+ b_i b_j b_k b_n\,, \quad \text{for } i < k \,\wedge\, \{ j = 2, k = 3 \,\vee\, j = 3, k = 2 \} \\
&+ b_i b_j b_k b_n\,, \quad \text{else}
\end{aligned}
\qquad (3.66)
$$

$$
\widetilde{f}^{(h)}_{(ij^{(l)})(kn^{[l]})} := \begin{cases} a_i a_j a_k a_n + v^{(l)}_{ij} b_k b_n + b_h \widehat{w}^{(l)}_{ij} \widehat{w}^{[l]}_{kn} + a_i a_j \widehat{w}^{[l]}_{kn} + b_i b_j v^{[l]}_{kn} + a_h v^{(l)}_{ij} v^{[l]}_{kn} + \widehat{w}^{(l)}_{ij} a_k a_n \\ a_i a_j a_k a_n + v^{(l)}_{ij} b_k b_n + b_h \widehat{w}^{(l)}_{ij} v^{[l]}_{kn} + a_i a_j v^{[l]}_{kn} + b_i b_j \widehat{w}^{[l]}_{kn} + a_h v^{(l)}_{ij} \widehat{w}^{[l]}_{kn} + \widehat{w}^{(l)}_{ij} a_k a_n \end{cases}
$$

$$
\begin{aligned}
&+ b_i b_j b_k b_n\,, \quad \text{for } \left\{ i < k \,\wedge\, \{ j = 2, k = 3 \,\vee\, j = 3, k = 2 \} \right\} \vee \{ i = k \,\wedge\, j = n \} \\
&+ b_i b_j b_k b_n\,, \quad \text{else}
\end{aligned}
\qquad (3.67)
$$

where it was necessary to replace the vertex factor $w^{(l)}_{ij}$ by $\widehat{w}^{(l)}_{ij} := w^{(l)}_{ij} + \delta_{A_i A_j} v^{(l)}_{ij}$ because a diagram like that in Fig. 3.6 yields in general a vertex factor proportional to w_{13}. However, if the constituents A_1 and A_3 are identical (although $P_1 = A_1 \neq \bar{A}_3 = P_3$) one obtains according to the vertex rules in Fig. 3.1 a factor v_{13} instead. In Eq. (F.1) the q_0 integration is already performed
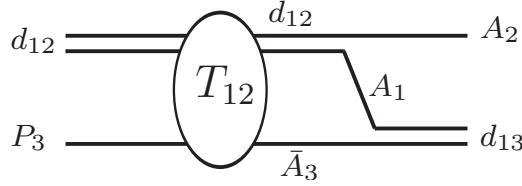
Figure 3.6: *Exemplary diagram to clarify the change of the vertex factor $w^{(l)}_{ij} \to \widehat{w}^{(l)}_{ij} := w^{(l)}_{ij} + \delta_{A_i A_j} v^{(l)}_{ij}$ which is necessary for not missing out contributions to the molecule–particle scattering amplitudes.*

using the residue theorem: after applying the Feynman rules the terms have the form

$$
\int \frac{d^4 q}{(2\pi)^4} \, t(E, \mathbf{k}, \mathbf{q}) \, \frac{1}{-q_0 - \frac{q^2}{2m_i} + i\varepsilon} \, \frac{D(E + q_0, \mathbf{q})}{E + q_0 - \frac{p^2}{2m_j} - \frac{(\mathbf{p} + \mathbf{q})^2}{2m_k} + i\varepsilon} =: \int \frac{d^3 q}{(2\pi)^3} \int \frac{dq_0}{2\pi} \varphi(q_0)\,. \quad (3.68)
$$

They are simplified by noting that $\varphi(q_0)$ has a $(\ell = 1)$-fold pole at $q_0 = -q^2/2m_i + i\varepsilon$. Therefore we used the residue theorem,

$$
\int \frac{dq_0}{2\pi} \varphi(q_0) = i\xi \mathrm{Res}_\varphi \left(-\frac{q^2}{2m_i} + i\varepsilon \right)\,, \quad \text{where} \quad \mathrm{Res}_{g(z)}(c) = \frac{1}{(\ell - 1)!} \frac{\partial^{\ell-1}}{\partial z^{\ell-1}} \left[(z - c)^\ell g(z) \right] \Big|_{z=c}\,,
$$

with positive winding number $\xi = +1$ to obtain that Eq. (3.68) is reduced to

$$
-i \int \frac{d^3 q}{(2\pi)^3} \, t(E, \mathbf{k}, \mathbf{q}) \, D\left(E - \frac{q^2}{2m_i}, \mathbf{q} \right) \frac{1}{E - \frac{q^2}{2m_i} - \frac{p^2}{2m_j} - \frac{(\mathbf{p} + \mathbf{q})^2}{2m_k} + i\varepsilon}\,, \quad (3.69)
$$

which is the form appearing in Eq. (F.1). However, neither (iso)spin projection nor wave function renormalization is applied to the amplitudes $\left(t_{ij}^{(\prime)}\right)_{\alpha\beta}^{\gamma\sigma}$. As explained in appendix C there are factors $(1 - \delta_{P_i P_j}/2)$ in front of each diagram and thus in front of each amplitude in Eq. (F.1) which are chosen according to the rules concerning identical particles explained in appendix C and reviewed in the box on page 51.

3.3.1 Wave function renormalization

We proceed by applying wave function renormalization to the amplitudes in Eq. (F.1) by multiplying each amplitude with the square root of the wave function renormalization constant of the incoming dimer and with the square root of the wave function renormalization constant of the outgoing dimer:

$$
\begin{pmatrix}
\left(\widetilde{T}_{12}\right)_{\alpha\beta}^{\gamma\sigma} \\
\left(\widetilde{T}_{13}\right)_{\alpha\beta}^{\gamma\sigma} \\
\left(\widetilde{T}_{23}\right)_{\alpha\beta}^{\gamma\sigma} \\
\left(\widetilde{T}'_{12}\right)_{\alpha\beta}^{\gamma\sigma} \\
\left(\widetilde{T}'_{13}\right)_{\alpha\beta}^{\gamma\sigma} \\
\left(\widetilde{T}'_{23}\right)_{\alpha\beta}^{\gamma\sigma}
\end{pmatrix}
:= \sqrt{Z_{12}}
\begin{pmatrix}
(t_{12})_{\alpha\beta}^{\gamma\sigma} \sqrt{Z_{12}} \\
(t_{13})_{\alpha\beta}^{\gamma\sigma} \sqrt{Z_{13}} \\
(t_{23})_{\alpha\beta}^{\gamma\sigma} \sqrt{Z_{23}} \\
(t'_{12})_{\alpha\beta}^{\gamma\sigma} \sqrt{Z'_{12}} \\
(t'_{13})_{\alpha\beta}^{\gamma\sigma} \sqrt{Z'_{13}} \\
(t'_{23})_{\alpha\beta}^{\gamma\sigma} \sqrt{Z'_{23}}
\end{pmatrix} ,
\tag{3.70}
$$

with $Z_{ij}^{(\prime)}$ given in Eq. (3.63). The renormalized (i.e. physical) amplitudes $\widetilde{T}_{ij}^{(\prime)}$ are then proportional to $\sqrt{\gamma_{12}/\gamma_{ij}^{(\prime)}}$. To get rid of these factors one can redefine all amplitudes via

$$
T_{ij}^{(\prime)} := \sqrt{\frac{\gamma_{12}}{\gamma_{ij}^{(\prime)}}} \, \widetilde{T}_{ij}^{(\prime)} .
\tag{3.71}
$$

Note, that the d_{12}–P_3 scattering amplitude $\widetilde{T}_{12} = T_{12}$ in which we are interested is not changed by this definition. Thus, we can divide both sides of Eq. (F.1) by i and plug in the full dimer propagator Eq. (3.60) to find Eq. (F.2) given in appendix F.

3.3.2 Projection on partial wave amplitude

One would expect that the scattering of a particle P_3 off the dimer d_{12} preferentially occurs in the partial wave with $L = 0$ because there is no general reason which forbids a S-wave interaction and one expects the energy of the scattering state to be minimal. Nevertheless, we keep the equations general and project out the L-th partial wave amplitude $T_{ij}^{(\prime)(L)}$ in Eq. (F.2). For details on this projection see appendix D, especially for the properties of the Legendre function of the second kind Q_L which we use in the following analysis. As a remainder we once more give

its definition:

$$Q_L(\beta - i\varepsilon) := (-1)^L \frac{1}{2} \int_{-1}^{1} dx \, \frac{P_L(x)}{(\beta - i\varepsilon) + x} \qquad \text{with } \beta \in \mathbb{R} \text{ and } |\beta| \neq 1 \,. \qquad (3.72)$$

Using Eq. (D.18) one can rewrite all six amplitudes $T_{ij}^{(\prime)}(E, \mathbf{k}, \mathbf{p})$ in terms of the partial wave projected amplitudes $T_{ij}^{(\prime)(L)}(E, k, p)$. The corresponding results can be found in appendix F in Eq. (F.3) where we have introduced a short-hand notation for the repeatedly appearing Legendre function:

$$Q_L^{ijk}(q, p; E) := Q_L \left(\frac{m_i}{qp} \left(\frac{q^2}{2\mu_{ij}} + \frac{p^2}{2\mu_{jk}} - E \right) - i\varepsilon \right) \,, \qquad (3.73)$$

where one has to keep in mind that $\mu_{ij} = \mu_{ji}$ holds by definition.

3.3.3 Spin and isospin projection onto specific scattering channel

Now we can concentrate on the spin and isospin structure: we start by applying projection operators $\mathcal{O}_{T_{ij}^{(\prime)}}$ to the amplitudes on the left-hand-side of Eq. (F.3) which couple the (iso)spins of the dimer $d_{ij}^{(\prime)}$ and the third particle P_k to an incoming (iso)spin state η and an outgoing one λ:

$$\left(T_{ij}^{(\prime)} \right)_{\eta}^{\lambda} = (\mathcal{O}_{T_{12}})_{\eta, \alpha\beta} \left(T_{ij}^{(\prime)} \right)_{\alpha\beta}^{\gamma\sigma} \left(\mathcal{O}_{T_{ij}^{(\prime)}}^{\dagger} \right)_{\lambda, \sigma\gamma} \,. \qquad (3.74)$$

Note here, that in general the projection operators $\mathcal{O}_{T_{ij}}$ and $\mathcal{O}_{ij}$ are not the same because the former couples dimer and particle (iso)spin while the latter couples two particle (iso)spins to the dimer (iso)spin. To get the full scattering amplitude one still has to average over initial (iso)spin and to sum over the final one:

$$T_{ij}^{(\prime)} = \frac{1}{\text{dof}(\eta)} \sum_{\eta, \lambda} \left(T_{ij}^{(\prime)} \right)_{\eta}^{\lambda} = \frac{1}{\text{dof}(\eta)} \sum_{\eta, \lambda} (\mathcal{O}_{T_{12}})_{\eta, \alpha\beta} \left(T_{ij}^{(\prime)} \right)_{\alpha\beta}^{\gamma\sigma} \left(\mathcal{O}_{T_{ij}^{(\prime)}}^{\dagger} \right)_{\lambda, \sigma\gamma} \,. \qquad (3.75)$$

Similar to what we have done in section 3.2.2 "dof(η)" represents the spin degrees of freedom times the isospin degrees of freedom in the initial state. If we only consider the (iso)spin dependent terms in Eq. (F.3) the application of the projection operators yields according to Eq. (3.75) the

structure below:

$$
\begin{pmatrix} T_{12} \\ T_{13} \\ T_{23} \\ T'_{12} \\ T'_{13} \\ T'_{23} \end{pmatrix} = \frac{1}{\mathrm{dof}(\eta)} \sum_{\eta,\lambda} (\mathcal{O}_{T_{12}})_{\eta,\alpha\beta} \begin{pmatrix} (T_{12})^{\gamma\sigma}_{\alpha\beta} \\ (T_{13})^{\gamma\sigma}_{\alpha\beta} \\ (T_{23})^{\gamma\sigma}_{\alpha\beta} \\ (T'_{12})^{\gamma\sigma}_{\alpha\beta} \\ (T'_{13})^{\gamma\sigma}_{\alpha\beta} \\ (T'_{23})^{\gamma\sigma}_{\alpha\beta} \end{pmatrix} \begin{pmatrix} (\mathcal{O}^{\dagger}_{T_{12}})_{\lambda,\sigma\gamma} \\ (\mathcal{O}^{\dagger}_{T_{13}})_{\lambda,\sigma\gamma} \\ (\mathcal{O}^{\dagger}_{T_{23}})_{\lambda,\sigma\gamma} \\ (\mathcal{O}^{\dagger}_{T'_{12}})_{\lambda,\sigma\gamma} \\ (\mathcal{O}^{\dagger}_{T'_{13}})_{\lambda,\sigma\gamma} \\ (\mathcal{O}^{\dagger}_{T'_{23}})_{\lambda,\sigma\gamma} \end{pmatrix} \sim
$$

$$
\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ x'_1 \\ y'_1 \\ z'_1 \end{pmatrix} + \begin{pmatrix} \widetilde{x}_1 \\ \widetilde{y}_1 \\ \widetilde{z}_1 \\ \widetilde{x}'_1 \\ \widetilde{y}'_1 \\ \widetilde{z}'_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ x'_2 \\ y'_2 \\ z'_2 \end{pmatrix} T_{12} + \begin{pmatrix} \widetilde{x}_2 \\ \widetilde{y}_2 \\ \widetilde{z}_2 \\ \widetilde{x}'_2 \\ \widetilde{y}'_2 \\ \widetilde{z}'_2 \end{pmatrix} T_{12} + \begin{pmatrix} x_3 \\ y_3 \\ z_3 \\ x'_3 \\ y'_3 \\ z'_3 \end{pmatrix} T_{13} + \begin{pmatrix} \widetilde{x}_3 \\ \widetilde{y}_3 \\ \widetilde{z}_3 \\ \widetilde{x}'_3 \\ \widetilde{y}'_3 \\ \widetilde{z}'_3 \end{pmatrix} T_{13} + \begin{pmatrix} x_4 \\ y_4 \\ z_4 \\ x'_4 \\ y'_4 \\ z'_4 \end{pmatrix} T_{23} + \begin{pmatrix} \widetilde{x}_4 \\ \widetilde{y}_4 \\ \widetilde{z}_4 \\ \widetilde{x}'_4 \\ \widetilde{y}'_4 \\ \widetilde{z}'_4 \end{pmatrix} T_{23}
$$

$$
+ \begin{pmatrix} x_5 \\ y_5 \\ z_5 \\ x'_5 \\ y'_5 \\ z'_5 \end{pmatrix} T'_{12} + \begin{pmatrix} \widetilde{x}_5 \\ \widetilde{y}_5 \\ \widetilde{z}_5 \\ \widetilde{x}'_5 \\ \widetilde{y}'_5 \\ \widetilde{z}'_5 \end{pmatrix} T'_{12} + \begin{pmatrix} x_6 \\ y_6 \\ z_6 \\ x'_6 \\ y'_6 \\ z'_6 \end{pmatrix} T'_{13} + \begin{pmatrix} \widetilde{x}_6 \\ \widetilde{y}_6 \\ \widetilde{z}_6 \\ \widetilde{x}'_6 \\ \widetilde{y}'_6 \\ \widetilde{z}'_6 \end{pmatrix} T'_{13} + \begin{pmatrix} x_7 \\ y_7 \\ z_7 \\ x'_7 \\ y'_7 \\ z'_7 \end{pmatrix} T'_{23} + \begin{pmatrix} \widetilde{x}_7 \\ \widetilde{y}_7 \\ \widetilde{z}_7 \\ \widetilde{x}'_7 \\ \widetilde{y}'_7 \\ \widetilde{z}'_7 \end{pmatrix} T'_{23} , \tag{3.76}
$$

where we have introduced new parameters $x_i^{(\prime)}$, $y_i^{(\prime)}$, $z_i^{(\prime)}$, $\widetilde{x}_i^{(\prime)}$, $\widetilde{y}_i^{(\prime)}$ and $\widetilde{z}_i^{(\prime)}$ for $i \in \{1, 2, 3, 4, 5, 6, 7\}$. Their definition (Eq. (A.40) - Eq. (A.81)) and the derivation of the equation above can be found in appendix A.2. In the last step we want to combine the pairs of terms proportional to the same amplitude $T_{ij}^{(\prime)}$. We know if $\delta_{P_i P_j} = 1$ then

- $m_i = m_j \quad \Rightarrow \quad \mu_{ik} = \mu_{jk}$,

- $v_{ij}^{(\prime)} = w_{ij}^{(\prime)} = 0$ because $P_i = P_j$ directly induces that there are no mixed terms of a_i and b_j, i.e. only $a_i = a_j = 1$ or $b_i = b_j = 1$ are possible combinations,

- $\delta_{P_i P_j} f^{(m)}_{(ij)(kn)} = \delta_{P_i P_j} \widetilde{f}^{(m)}_{(ij)(kn)}$ since for $i < k \wedge \{j = 2, k = 3 \vee j = 3, k = 2\}$ (and similarly for the "else" case in the definition of f and $\widetilde{f}$) it holds:
$$
\delta_{P_i P_j} \Big(a_i a_j a_k a_n + v_{ij} a_k a_n + a_m \widehat{w}_{ij} v_{kn} + b_i b_j v_{kn} + a_i a_j \widehat{w}_{kn} + b_m v_{ij} \widehat{w}_{kn} + \widehat{w}_{ij} b_k b_n + b_i b_j b_k b_n \Big)
$$
$$
= \delta_{P_i P_j} \Big(a_i a_j a_k a_n + b_i b_j v_{kn} + a_i a_j \widehat{w}_{kn} + b_i b_j b_k b_n \Big)
$$
$$
= \delta_{P_i P_j} \Big(a_i a_j a_k a_n + a_m v_{ij} v_{kn} + \widehat{w}_{ij} a_k a_n + b_i b_j v_{kn} + a_i a_j \widehat{w}_{kn} + v_{ij} b_k b_n + b_m \widehat{w}_{ij} \widehat{w}_{kn} + b_i b_j b_k b_n \Big).
$$

Therefore we can factor out not only the amplitude itself, but also the a, b parameter dependent term and write the amplitudes of the coupled integral equation system for d_{12}–P_3 scattering as shown in Eqs. (F.4 - F.9).

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3.4 Asymptotic behavior

The momentum integrals in the previously mentioned Eqs. (F.4 - F.9) are divergent. Consequently, we introduce a momentum cutoff Λ_C to regularize them. In principle, one could proceed and solve the coupled integral equation system numerically for different three particle systems in order to find a possible Efimov effect. However, this would mean that we loose all generality in our equations. More favorably would be a method which keeps this generality. Indeed, one can consider the approach of asymptotic large off-shell momenta $\Lambda_C \gg q, p \gg \gamma_{ij}^{(l)} \sim k \sim E$ for $i < j \in \{1, 2, 3\}$ which has this property. In Ref. [80] and in the references therein it is explained how one can use asymptotic Faddeev wave functions to derive the conditions under which the Efimov effect can occur in a system. However, this method does not provide an instruction to – more or less – directly read off from the properties of the three particles whether the Efimov effect occurs if the particles have more general spin and isospin quantum numbers and different scattering channels. The equations themselves are not changed that much considering these additional degrees of freedom, but to determine all the parameters therein becomes more involved for higher spins and isospins. In contrast one can – still using the asymptotic momentum approach – consider the work in Ref. [166] and follow its applications in Refs. [92, 167]. Here, the integral equation system is decoupled in order to reduce the problem to the determination of eigenvalues of a matrix whose entries are spin and isospin dependent. To do so we will see that one needs to make some assumptions concerning the ratio of masses within the three particle system. However, these assumptions will match with the given systems if we consider particles scattering off hadronic molecules. Therefore we will use the second method.

In the mentioned limit of large asymptotic off-shell momenta both, the denominator of the integrand and the argument of the Legendre function Q_L considerably simplify. Taking into account that one has to take the limit $\varepsilon \to 0$ in the end, it holds

$$
\lim_{\varepsilon \to 0} \; -\gamma_{ij}^{(l)} + \sqrt{-2\mu_{ij}\left(E - \frac{q^2}{2m_k} - \frac{q^2}{2(m_i + m_j)}\right) - i\varepsilon}
$$

$$
\xrightarrow{q \gg E} \lim_{\varepsilon \to 0} \; -\gamma_{ij}^{(l)} + \sqrt{q^2\mu_{ij}\left(\frac{1}{m_k} + \frac{1}{m_i + m_j}\right) - i\varepsilon}
$$

$$
= -\gamma_{ij}^{(l)} + q\sqrt{\mu_{ij}\left(\frac{1}{m_k} + \frac{1}{m_i + m_j}\right)}
$$

$$
\xrightarrow{q \gg \gamma_{ij}} \sqrt{\frac{\mu_{ij}}{\mu_{(ij)k}}} \; q \;, \tag{3.77}
$$

where we have introduced the reduced mass of the dimer $d_{ij}^{(l)}$ and particle P_k system,

$$
\mu_{(ij)k} = \frac{(m_i + m_j)m_k}{m_i + m_j + m_k} \;. \tag{3.78}
$$

Moreover, – omitting the particle indices for the moment – the argument of the Legendre function reduces to

$$\lim_{\varepsilon\to 0}\frac{m}{qp}\,Q_L\left(\frac{m}{qp}\left(\frac{q^2}{2\widetilde{\mu}}+\frac{p^2}{2\mu}-E\right)-i\varepsilon\right)\xrightarrow{p,q\gg E}\lim_{\varepsilon\to 0}\frac{m}{qp}\,Q_L\left(\frac{m}{qp}\left(\frac{q^2}{\widetilde{\mu}}+\frac{p^2}{2\mu}\right)-i\varepsilon\right)$$

$$=\frac{m}{qp}\,Q_L\left(\frac{m}{qp}\left(\frac{q^2}{2\widetilde{\mu}}+\frac{p^2}{2\mu}\right)\right),\qquad(3.79)$$

if it depends on both q and p. The short-hand notation in Eq. (3.73) can thus be extended via the definition

$$Q_L^{ijk}(q,p):=\lim_{\varepsilon\to 0}Q_L^{ijk}(q,p;0)=Q_L\left(\frac{m_i}{qp}\left(\frac{q^2}{2\mu_{ij}}+\frac{p^2}{2\mu_{jk}}\right)\right).\qquad(3.80)$$

In the inhomogeneous term where the Legendre function only depends on p and $k\ll p$ one finds in the same way

$$\lim_{\varepsilon\to 0}\frac{m}{kp}\,Q_L\left(\frac{m}{kp}\left(\frac{k^2}{2\mu'}+\frac{p^2}{2\mu}-E\right)-i\varepsilon\right)\xrightarrow{p\gg E,k}\lim_{\varepsilon\to 0}\frac{m}{kp}\,Q_L\left(\frac{m}{kp}\left(\frac{p^2}{2\mu}\right)-i\varepsilon\right)$$

$$=\frac{m}{kp}\,Q_L\left(\frac{m}{2\mu}\frac{p}{k}\right).\qquad(3.81)$$

In Ref. [165] one can find a formal argument that one can neglect the contribution of the inhomogeneous term compared to the the homogeneous part. As a less formal but more straightforward method one can instead use that $Q_L(z)$ can be written in terms of a hypergeometric function $_2F_1$ which is proportional to $\sum_{n=0}^{\infty}z^{-(2n+L+1)}$ [164, 165]. Thus, Eq. (3.81) has the following proportionality:

$$\frac{m}{kp}\,Q_L\left(\frac{m}{2\mu}\frac{p}{k}\right)\sim\frac{m}{kp}\sum_{n=0}^{\infty}\left(\frac{m}{2\mu}\frac{p}{k}\right)^{-(2n+L+1)}$$

$$=m\sum_{n=0}^{\infty}\left(\frac{2\mu}{m}\right)^{(2n+L+1)}\frac{k^{(2n+L)}}{p^{(2n+L+2)}}$$

$$\xrightarrow{p\gg k}m\sum_{n=0}^{\infty}\left(\frac{2\mu}{m}\right)^{(2n+L+1)}\frac{1}{p^{(2n+L+2)}}\ll 1.\qquad(3.82)$$

Consequently, in our further analysis we can for asymptotic large off-shell momenta neglect the inhomogeneous terms in Eqs. (F.4 - F.9) and replace the denominators of the integrands using Eq. (3.77) and the Legendre functions via Eq. (3.80). Due to the vanishing inhomogeneities all amplitudes become independent of k and since E does not appear anymore, too, we simply write $T_{ij}^{(l)}(p)\sim\int dq\,T_{ij}^{(l)}(q)$ in the following. Since p is unequal to zero one can additionally define new amplitudes according to

$$\widetilde{T}_{ij}^{(l)}(p):=p\,T_{ij}^{(l)}(p),\quad\forall\,i<j\in\{1,2,3\},\qquad(3.83)$$

in order to find Eqs. (F.10 - F.15) written in terms of these new amplitudes and shown in appendix F.

3.4.1 Scale and inversion invariance and decoupling of amplitudes

The coupled integral equation system of Eqs. (F.10 - F.15) is scale invariant, but not invariant under the inversion $q \to 1/q$ for large loop momenta. The reason for this is that in each term Q_L depends on the momentum q, but with different prefactors in front of q and in front of q^{-1}. The prefactors themselves are not problematic. As long as in each Q_L, q and q^{-1} have the same one, the equation system *is* inversion invariant, but if only one of them is different in at least one Q_L the invariance is destroyed. However, we know from Refs. [92, 167] that this is an essential property of a three particle system if one wants to decouple the integral equation system. From a mathematical point of view the necessity of inversion invariance can be motivated by the fact that one can factor out the Q_L's (which are then equal in each term), write the equations system as a matrix equation, decouple the amplitudes and insert power law solutions for these decoupled amplitudes (a power law is the most straightforward choice for a scale and inversion invariant and physically meaningful amplitude [80]). After these steps one can use a Mellin transform to calculate the dq integral which leads to transcendental equations for the angular momentum dependent exponent of the power law $s_i^{(L)}$ (one equation per amplitude i). These equations have purely complex solutions, that is, the Efimov effect is present, if a parameter λ (which depends on spin / isospin factors, symmetry factors, etc.) is larger than a critical value. To apply this method to our coupled integral equation system and to discuss the related issues in more detail will be the main task for the rest of this section.

We deduce from Eqs. (F.10 - F.15) that the problematic prefactors mentioned above are caused by the fact that in general the masses m_1, m_2 and m_3 are different. Thus, only for (approximately) equal masses the equation system is (approximately) inversion invariant and a (so-called *approximate*) Efimov effect could be present in the system. However, if there is not a dimer state for all combinations of particles (e.g. no d_{23} and thus no $\widetilde{T}_{23}^{(l)}$) it may exist a scale and inversion invariant equation subsystem with different masses. To be more precisely a not-existing dimer state means that there is no resonant interaction with large scattering length between the two corresponding particles. Nevertheless, they could have a deeply bound two-body state which is not described by EFT($\not{\pi}$).

As a measure for the mass difference between particles P_i and P_j we define a parameter

$$\varepsilon_{ij} := \frac{m_i - m_j}{m_i + m_j} \quad \forall\, i, j \in \{1, 2, 3\}\,, \tag{3.84}$$

with the following properties:

$$1 - \varepsilon_{ij} = \frac{2m_j}{m_i + m_j}\,, \tag{3.85}$$

$$1 + \varepsilon_{ij} = \frac{2m_i}{m_i + m_j}\,, \tag{3.86}$$

$$\frac{m_i}{\mu_{ij}} = \frac{2}{1 - \varepsilon_{ij}}\,, \tag{3.87}$$

$$\frac{m_j}{\mu_{ij}} = \frac{2}{1 + \varepsilon_{ij}}\,, \tag{3.88}$$

$$\sqrt{\frac{\mu_{ij}}{\mu_{(ij)k}}} = \sqrt{\frac{1 - \varepsilon_{ij}}{1 - \varepsilon_{ik}} - \frac{1}{4}(1 - \varepsilon_{ij})^2} \ . \tag{3.89}$$

It can be used to eliminate one mass parameter from the coupled integral equations via

$$m_j = \frac{1 + \varepsilon_{ij}}{1 - \varepsilon_{ij}} m_i \ . \tag{3.90}$$

To ensure at least approximate scale and inversion invariance we will see that it is necessary to have $\varepsilon_{ij} \ll 1$ so that one can neglect it and write $m_j \approx m_i$. The error due to this replacement depends on the exact value of ε_{ij}. However, we will assume that it is at most of the order of the error coming from effective range corrections which are anyway neglected in our derivation of the coupled integral equations. Which mass can be replaced depends on the properties of the specific system. In fact, one can identify three possible types of systems:

System types

- **Type 1**: The full system with shallow P_1P_2, P_1P_3 and P_2P_3 bound or virtual states.

- **Type 2**: The subsystem with only P_1P_2 and P_1P_3 bound or virtual states. Note, that the subsystem with only P_1P_2 and P_2P_3 bound states instead can be transformed into the first one by interchanging the particle allocation of P_1 and P_2 which does not change the dimer d_{12} in a physical sense. Thus, it is enough to consider the first subsystem.

- **Type 3**: The subsystem with only P_1P_2 bound or virtual states (i.e. d_{12} and d'_{12}). Note, that one allocates the three particles in the system so that the scattering one is interested in is always between particle P_1 and P_2. Since in our framework the scattering of two particles is only possible if there exists a dimer state (i.e. a two-body state with large scattering length) between these two particles, this subsystem is the only choice if there are bound states only between two of the three particles.

At this point we have to make some important remarks: firstly, about not existing P_iP_j bound states and secondly, regarding the counting scheme of identical diagrams explained in appendix C. In general one would transform the coupled integral equation system into a 6×6 matrix equation because we have started with six amplitudes. However, if there is no P_iP_j bound or virtual state in the system one has to erase all terms proportional to $T_{ij}^{(\prime)(L)}$ and thus to $\widetilde{T}_{ij}^{(\prime)(L)}$ from the analysis since the corresponding Feynman diagrams do not exist in this case. Mathematically this is allowed since the characteristic polynomial of a 6×6 matrix with two rows and columns filled with zeros is identical to the characteristic polynomial of the same matrix where one has erased the zero-filled rows and columns. If there is just one P_iP_j bound or virtual state, i.e. d_{ij} exists, but d'_{ij} not, then one erases all terms proportional to $T_{ij}^{\prime(L)}$ and thus to $\widetilde{T}_{ij}^{\prime(L)}$ from the analysis, but keeps the unprimed amplitudes whose Feynman diagrams are still present. Consequently, there is just one row and column filled with zeros in the 6×6 matrix which can thus be reduced to a 5×5 matrix. The procedure how one deals with not existing dimers is thus clear. But what happens in case of two identical particles? If P_i and P_j are identical it is clear that also the amplitudes $T_{ik}^{(\prime)(L)}$ and $T_{jk}^{(\prime)(L)}$ are mathematically and physically identical. Therefore one has to erase

the second amplitude $T_{jk}^{(\prime)(L)}$ from the analysis since the just by hand inserted (cf. appendix C) factors of $(1 - \delta_{P_i P_j}/2)$ in front of the Feynman diagrams are chosen exactly in the way that one can also neglect these non-zero amplitudes. Consequently, it remain only $T_{12}^{(\prime)(L)}$ amplitudes if all three particles are identical. However, this *does not* mean that one deals with a type 3 system since *there are* bound states between e.g. P_1 and P_3, but they are equal to $P_1 P_2$ bound states. In summary we have the following rules (remember that a not present dimer means that there is no resonant interaction with large scattering length between the two particles):

Rules on how to treat not-existing dimers and identical particles in a system

- no d_{ij} and no d'_{ij} dimer $\quad \Rightarrow \quad$ erase all terms proportional to $T_{ij}^{(L)}$ and $T_{ij}^{\prime(L)}$.

- d_{ij} dimer present, but no d'_{ij} dimer $\quad \Rightarrow \quad$ erase all terms proportional to $T_{ij}^{\prime(L)}$.

- if $P_i = P_j \quad \Rightarrow \quad$ keep amplitude $T_{ik}^{(\prime)(L)}$, but erase all terms proportional to $T_{jk}^{(\prime)(L)}$.

The advantage of the less straightforward counting scheme of identical diagrams is that one also needs less parameters which have to be calculated.

3.4.2 Type 1 systems

Consider the full system where bound or virtual states between all three particles exist. Namely, at least d_{12}, d_{13} and d_{23} exist. To have a general expression we also assume the primed dimers to be present and consequently no amplitude is erased from Eqs. (F.10 - F.15). As we have already noticed these equations are not inversion invariant for $q \to 1/q$. To achieve this property one has to ensure that the argument of the Legendre function Q_L is equal in each term. Thus, it is required that all three masses are at least approximately equal in order to have a system where the Efimov effect could at all be present. This means that the mass difference parameters ε_{12}, ε_{13} and ε_{23} must be small and from their definition in Eq. (3.84) follows that one can replace m_3 and m_2 via

$$m_3 = \frac{1 + \varepsilon_{13}}{1 - \varepsilon_{13}} \, m_1 \approx m_1 \qquad \text{(for } \varepsilon_{13} \ll 1) , \tag{3.91}$$

$$m_2 = \frac{1 + \varepsilon_{12}}{1 - \varepsilon_{12}} \, m_1 \approx m_1 \qquad \text{(for } \varepsilon_{12} \ll 1) . \tag{3.92}$$

Furthermore, the reduced masses μ_{13} and μ_{23} can be replaced by μ_{12}, respectively, since

$$\mu_{13} = \frac{m_1 m_3}{m_1 + m_3} = \frac{m_1 m_2}{\frac{1 - \varepsilon_{23}}{1 + \varepsilon_{23}} m_1 + m_2} \approx \mu_{12} \qquad \text{(for } \varepsilon_{23} \ll 1) , \tag{3.93}$$

$$\mu_{23} = \frac{m_2 m_3}{m_2 + m_3} = \frac{m_1 m_2}{\frac{1 - \varepsilon_{13}}{1 + \varepsilon_{13}} m_1 + m_2} \approx \mu_{12} \qquad \text{(for } \varepsilon_{13} \ll 1) . \tag{3.94}$$

Applying these approximations to Eqs. (F.10 - F.15) one can factor out

$$\int_0^{\Lambda_C} \frac{dq}{q} Q_L^{122}(q, p) = \int_0^{\Lambda_C} \frac{dq}{q} Q_L \left(\frac{m_1}{qp} \left(\frac{q^2}{2\mu_{12}} + \frac{p^2}{2\mu_{12}} \right) \right) ,$$

and write the coupled integral equation system as a 6×6 matrix equation:

$$
\begin{pmatrix}
\widetilde{T}_{12}^{(L)}(p) \\
\widetilde{T}_{13}^{(L)}(p) \\
\widetilde{T}_{23}^{(L)}(p) \\
\widetilde{T}_{12}^{\prime(L)}(p) \\
\widetilde{T}_{13}^{\prime(L)}(p) \\
\widetilde{T}_{23}^{\prime(L)}(p)
\end{pmatrix}
= \frac{(-1)^L}{\pi} \frac{m_1}{\mu_{12}} \frac{1}{\sqrt{\frac{\mu_{12}}{\mu_{(12)2}}}} \mathcal{A}_1 \int_0^{\Lambda_C} \frac{dq}{q} Q_L \left(\frac{m_1}{qp} \left(\frac{q^2}{2\mu_{12}} + \frac{p^2}{2\mu_{12}} \right) \right)
\begin{pmatrix}
\widetilde{T}_{12}^{(L)}(q) \\
\widetilde{T}_{13}^{(L)}(q) \\
\widetilde{T}_{23}^{(L)}(q) \\
\widetilde{T}_{12}^{\prime(L)}(q) \\
\widetilde{T}_{13}^{\prime(L)}(q) \\
\widetilde{T}_{23}^{\prime(L)}(q)
\end{pmatrix} .
\tag{3.95}
$$

Since $m_1 = m_2$ or at least $m_1 \approx m_2$ one finds

$$
\mu_{12} \approx \frac{m_1 m_1}{m_1 + m_1} = \frac{m_1}{2} ,
\tag{3.96}
$$

and thus

$$
\frac{\mu_{12}}{\mu_{(12)2}} \approx \frac{m_1}{2} \frac{(m_1 + m_1) + m_1}{(m_1 + m_1)m_1} = \frac{m_1}{2} \frac{3m_1}{2m_1^2} = \frac{3}{4} .
\tag{3.97}
$$

Therefore the matrix equation simplifies to

$$
\begin{pmatrix}
\widetilde{T}_{12}^{(L)}(p) \\
\widetilde{T}_{13}^{(L)}(p) \\
\widetilde{T}_{23}^{(L)}(p) \\
\widetilde{T}_{12}^{\prime(L)}(p) \\
\widetilde{T}_{13}^{\prime(L)}(p) \\
\widetilde{T}_{23}^{\prime(L)}(p)
\end{pmatrix}
= (-1)^L \frac{4}{\sqrt{3}\,\pi} \mathcal{A}_1 \int_0^{\Lambda_C} \frac{dq}{q} Q_L \left(\frac{q}{p} + \frac{p}{q} \right)
\begin{pmatrix}
\widetilde{T}_{12}^{(L)}(q) \\
\widetilde{T}_{13}^{(L)}(q) \\
\widetilde{T}_{23}^{(L)}(q) \\
\widetilde{T}_{12}^{\prime(L)}(q) \\
\widetilde{T}_{13}^{\prime(L)}(q) \\
\widetilde{T}_{23}^{\prime(L)}(q)
\end{pmatrix} ,
\tag{3.98}
$$

with $\mathcal{A}_1$ being a 6×6 matrix whose columns are given by Eqs. (3.99 - 3.104).

$$(\mathcal{A}_1)_{i1} = \begin{pmatrix} \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{S_{12}\, c_{12}} \left[x_2\, \delta^{(12)}_{P_1 P_3}\, f^{(3)}_{(12)(12)}\, S_{123} + \widetilde{x}_2\, \delta^{(12)}_{P_2 P_3}\, \widetilde{f}^{(3)}_{(12)(12)}\, S_{213} \right] \\[2ex] \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{\left(y_2\, \delta_{P_1 P_2}\, S_{123} + \widetilde{y}_2\, S_{213} \right)}{\sqrt{S_{13}\, S_{12}\, c_{13}\, c_{12}}}\, \widetilde{f}^{(2)}_{(12)(13)} \\[2ex] \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{\left(z_2\, S_{123} + \widetilde{z}_2\, \delta_{P_1 P_2}\, S_{213} \right)}{\sqrt{S_{23}\, S_{12}\, c_{23}\, c_{12}}}\, f^{(1)}_{(12)(23)} \\[2ex] \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{S_{12}\, \sqrt{c'_{12}\, c_{12}}} \left[x'_2\, \delta^{(12)}_{P_1 P_3}\, f^{(3)}_{(12)(12')}\, S_{123} + \widetilde{x}'_2\, \delta^{(12)}_{P_2 P_3}\, \widetilde{f}^{(3)}_{(12)(12')}\, S_{213} \right] \\[2ex] \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{\left(y'_2\, \delta_{P_1 P_2}\, S_{123} + \widetilde{y}'_2\, S_{213} \right)}{\sqrt{S_{13}\, S_{12}\, c'_{13}\, c_{12}}}\, \widetilde{f}^{(2)}_{(12)(13')} \\[2ex] \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{\left(z'_2\, S_{123} + \widetilde{z}'_2\, \delta_{P_1 P_2}\, S_{213} \right)}{\sqrt{S_{23}\, S_{12}\, c'_{23}\, c_{12}}}\, f^{(1)}_{(12)(23')} \end{pmatrix}, \quad (3.99)$$

$$(\mathcal{A}_1)_{i2} = \begin{pmatrix} \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{\left(x_3\, \delta_{P_1 P_3}\, S_{132} + \widetilde{x}_3\, S_{312} \right)}{\sqrt{S_{12}\, S_{13}\, c_{12}\, c_{13}}}\, \widetilde{f}^{(3)}_{(13)(12)} \\[2ex] \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{S_{13}\, c_{13}} \left[y_3\, \delta^{(13)}_{P_1 P_2}\, f^{(1)}_{(13)(13)}\, S_{132} + \widetilde{y}_3\, \delta^{(13)}_{P_2 P_3}\, \widetilde{f}^{(3)}_{(13)(13)}\, S_{312} \right] \\[2ex] \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{\left(z_3\, S_{132} + \widetilde{z}_3\, \delta_{P_1 P_3}\, S_{312} \right)}{\sqrt{S_{23}\, S_{13}\, c_{23}\, c_{13}}}\, f^{(1)}_{(13)(23)} \\[2ex] \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{\left(x'_3\, \delta_{P_1 P_3}\, S_{132} + \widetilde{x}'_3\, S_{312} \right)}{\sqrt{S_{12}\, S_{13}\, c'_{12}\, c_{13}}}\, \widetilde{f}^{(3)}_{(13)(12')} \\[2ex] \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{S_{13}\, \sqrt{c'_{13}\, c_{13}}} \left[y'_3\, \delta^{(13)}_{P_1 P_2}\, f^{(1)}_{(13)(13')}\, S_{132} + \widetilde{y}'_3\, \delta^{(13)}_{P_2 P_3}\, \widetilde{f}^{(3)}_{(13)(13')}\, S_{312} \right] \\[2ex] \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{\left(z'_3\, S_{132} + \widetilde{z}'_3\, \delta_{P_1 P_3}\, S_{312} \right)}{\sqrt{S_{23}\, S_{13}\, c'_{23}\, c_{13}}}\, f^{(1)}_{(13)(23')} \end{pmatrix}, \quad (3.100)$$

$$(\mathcal{A}_1)_{i3} = \begin{pmatrix} \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) \dfrac{\left(x_4\, \delta_{P_2 P_3}\, S_{231} + \widetilde{x}_4\, S_{321}\right)}{\sqrt{S_{12}\, S_{23}\, c_{12}\, c_{23}}}\, \widetilde{f}^{(3)}_{(23)(12)} \\[4mm] \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) \dfrac{\left(y_4\, S_{231} + \widetilde{y}_4\, \delta_{P_2 P_3}\, S_{321}\right)}{\sqrt{S_{13}\, S_{23}\, c_{13}\, c_{23}}}\, f^{(2)}_{(23)(13)} \\[4mm] \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) \dfrac{1}{S_{23}\, c_{23}} \left[z_4\, \delta^{(23)}_{P_1 P_2}\, f^{(2)}_{(23)(23)}\, S_{231} + \widetilde{z}_4\, \delta^{(23)}_{P_1 P_3}\, \widetilde{f}^{(3)}_{(23)(23)}\, S_{321} \right] \\[4mm] \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) \dfrac{\left(x'_4\, \delta_{P_2 P_3}\, S_{231} + \widetilde{x}'_4\, S_{321}\right)}{\sqrt{S_{12}\, S_{23}\, c'_{12}\, c_{23}}}\, \widetilde{f}^{(3)}_{(23)(12')} \\[4mm] \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) \dfrac{\left(y'_4\, S_{231} + \widetilde{y}'_4\, \delta_{P_2 P_3}\, S_{321}\right)}{\sqrt{S_{13}\, S_{23}\, c'_{13}\, c_{23}}}\, f^{(2)}_{(23)(13')} \\[4mm] \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) \dfrac{1}{S_{23}\, \sqrt{c'_{23}\, c_{23}}} \left[z'_4\, \delta^{(23)}_{P_1 P_2}\, f^{(2)}_{(23)(23')}\, S_{231} + \widetilde{z}'_4\, \delta^{(23)}_{P_1 P_3}\, \widetilde{f}^{(3)}_{(23)(23')}\, S_{321} \right] \end{pmatrix}, \quad (3.101)$$

$$(\mathcal{A}_1)_{i4} = \begin{pmatrix} \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \dfrac{1}{S_{12}\, \sqrt{c_{12}\, c'_{12}}} \left[x_5\, \delta^{(12')}_{P_1 P_3}\, f^{(3)}_{(12')(12)}\, S_{123} + \widetilde{x}_5\, \delta^{(12')}_{P_2 P_3}\, \widetilde{f}^{(3)}_{(12')(12)}\, S_{213} \right] \\[4mm] \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \dfrac{\left(y_5\, \delta_{P_1 P_2}\, S_{123} + \widetilde{y}_5\, S_{213}\right)}{\sqrt{S_{13}\, S_{12}\, c_{13}\, c'_{12}}}\, \widetilde{f}^{(2)}_{(12')(13)} \\[4mm] \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \dfrac{\left(z_5\, S_{123} + \widetilde{z}_5\, \delta_{P_1 P_2}\, S_{213}\right)}{\sqrt{S_{23}\, S_{12}\, c_{23}\, c'_{12}}}\, f^{(1)}_{(12')(23)} \\[4mm] \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \dfrac{1}{S_{12}\, c'_{12}} \left[x'_5\, \delta^{(12')}_{P_1 P_3}\, f^{(3)}_{(12')(12')}\, S_{123} + \widetilde{x}'_5\, \delta^{(12')}_{P_2 P_3}\, \widetilde{f}^{(3)}_{(12')(12')}\, S_{213} \right] \\[4mm] \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \dfrac{\left(y'_5\, \delta_{P_1 P_2}\, S_{123} + \widetilde{y}'_5\, S_{213}\right)}{\sqrt{S_{13}\, S_{12}\, c'_{13}\, c'_{12}}}\, \widetilde{f}^{(2)}_{(12')(13')} \\[4mm] \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \dfrac{\left(z'_5\, S_{123} + \widetilde{z}'_5\, \delta_{P_1 P_2}\, S_{213}\right)}{\sqrt{S_{23}\, S_{12}\, c'_{23}\, c'_{12}}}\, f^{(1)}_{(12')(23')} \end{pmatrix}, \quad (3.102)$$

$$(\mathcal{A}_1)_{i5} = \begin{pmatrix} \left(1 - \frac{\delta_{P_1P_3}}{2}\right) \dfrac{\left(x_6\,\delta_{P_1P_3}\,S_{132} + \widetilde{x}_6\,S_{312}\right)}{\sqrt{S_{12}\,S_{13}\,c_{12}\,c'_{13}}}\,\widetilde{f}^{(3)}_{(13')(12)} \\[4pt] \left(1 - \frac{\delta_{P_1P_3}}{2}\right) \dfrac{1}{S_{13}\,\sqrt{c_{13}\,c'_{13}}}\left[y_6\,\delta^{(13')}_{P_1P_2}\,f^{(1)}_{(13')(13)}\,S_{132} + \widetilde{y}_6\,\delta^{(13')}_{P_2P_3}\,\widetilde{f}^{(3)}_{(13')(13)}\,S_{312}\right] \\[4pt] \left(1 - \frac{\delta_{P_1P_3}}{2}\right) \dfrac{\left(z_6\,S_{132} + \widetilde{z}_6\,\delta_{P_1P_3}\,S_{312}\right)}{\sqrt{S_{23}\,S_{13}\,c_{23}\,c'_{13}}}\,f^{(1)}_{(13')(23)} \\[4pt] \left(1 - \frac{\delta_{P_1P_3}}{2}\right) \dfrac{\left(x'_6\,\delta_{P_1P_3}\,S_{132} + \widetilde{x}'_6\,S_{312}\right)}{\sqrt{S_{12}\,S_{13}\,c'_{12}\,c'_{13}}}\,\widetilde{f}^{(3)}_{(13')(12')} \\[4pt] \left(1 - \frac{\delta_{P_1P_3}}{2}\right) \dfrac{1}{S_{13}\,c'_{13}}\left[y'_6\,\delta^{(13')}_{P_1P_2}\,f^{(1)}_{(13')(13')}\,S_{132} + \widetilde{y}'_6\,\delta^{(13')}_{P_2P_3}\,\widetilde{f}^{(3)}_{(13')(13')}\,S_{312}\right] \\[4pt] \left(1 - \frac{\delta_{P_1P_3}}{2}\right) \dfrac{\left(z'_6\,S_{132} + \widetilde{z}'_6\,\delta_{P_1P_3}\,S_{312}\right)}{\sqrt{S_{23}\,S_{13}\,c'_{23}\,c'_{13}}}\,f^{(1)}_{(13')(23')} \end{pmatrix}, \quad (3.103)$$

$$(\mathcal{A}_1)_{i6} = \begin{pmatrix} \left(1 - \frac{\delta_{P_2P_3}}{2}\right) \dfrac{\left(x_7\,\delta_{P_2P_3}\,S_{231} + \widetilde{x}_7\,S_{321}\right)}{\sqrt{S_{12}\,S_{23}\,c_{12}\,c'_{23}}}\,\widetilde{f}^{(3)}_{(23')(12)} \\[4pt] \left(1 - \frac{\delta_{P_2P_3}}{2}\right) \dfrac{\left(y_7\,S_{231} + \widetilde{y}_7\,\delta_{P_2P_3}\,S_{321}\right)}{\sqrt{S_{13}\,S_{23}\,c_{13}\,c'_{23}}}\,f^{(2)}_{(23')(13)} \\[4pt] \left(1 - \frac{\delta_{P_2P_3}}{2}\right) \dfrac{1}{S_{23}\,\sqrt{c_{23}\,c'_{23}}}\left[z_7\,\delta^{(23')}_{P_1P_2}\,f^{(2)}_{(23')(23)}\,S_{231} + \widetilde{z}_7\,\delta^{(23')}_{P_1P_3}\,\widetilde{f}^{(3)}_{(23')(23)}\,S_{321}\right] \\[4pt] \left(1 - \frac{\delta_{P_2P_3}}{2}\right) \dfrac{\left(x'_7\,\delta_{P_2P_3}\,S_{231} + \widetilde{x}'_7\,S_{321}\right)}{\sqrt{S_{12}\,S_{23}\,c'_{12}\,c'_{23}}}\,\widetilde{f}^{(3)}_{(23')(12')} \\[4pt] \left(1 - \frac{\delta_{P_2P_3}}{2}\right) \dfrac{\left(y'_7\,S_{231} + \widetilde{y}'_7\,\delta_{P_2P_3}\,S_{321}\right)}{\sqrt{S_{13}\,S_{23}\,c'_{13}\,c'_{23}}}\,f^{(2)}_{(23')(13')} \\[4pt] \left(1 - \frac{\delta_{P_2P_3}}{2}\right) \dfrac{1}{S_{23}\,c'_{23}}\left[z'_7\,\delta^{(23')}_{P_1P_2}\,f^{(2)}_{(23')(23')}\,S_{231} + \widetilde{z}'_7\,\delta^{(23')}_{P_1P_3}\,\widetilde{f}^{(3)}_{(23')(23')}\,S_{321}\right] \end{pmatrix}. \quad (3.104)$$

As a remainder we once more give the definition of all parameters appearing in the matrix $\mathcal{A}_1$:

- $P_i := a_i A_i + b_i \bar{A}_i$.

- $\delta_{P_i P_j} = \begin{cases} 1, & \text{if } P_i = P_j \\ 0, & \text{else} \end{cases}$ and similarly for $\delta_{A_i A_j}$.

- $\delta_{P_i P_j}^{(ab^{(l)})} := \delta_{P_i P_j} + \left(\delta_{A_i A_j} - \delta_{P_i P_j} \right) \delta_{\eta_{ab}^{(l)}|1|} \left(\delta_{\eta_{ab}^{(l)}|1|} - \delta_{A_a A_b} \right)$,
  with $\delta_{\eta_{ij}^{(l)}|1|} := \delta_{\text{baryon number } 0} \times \delta_{\text{strangeness } 0} \times \delta_{\text{charm } 0} \times \delta_{\text{beauty } 0} \times \delta_{\text{topness } 0}$.

- $f^{(h)}_{(ij^{(l)})(kn^{[l]})} := \begin{cases} a_i a_j a_k a_n + v_{ij}^{(l)} a_k a_n + a_h \widehat{w}_{ij}^{(l)} v_{kn}^{[l]} + b_i b_j v_{kn}^{[l]} + a_i a_j \widehat{w}_{kn}^{[l]} + b_h v_{ij}^{(l)} \widehat{w}_{kn}^{[l]} + \widehat{w}_{ij}^{(l)} b_k b_n \\ a_i a_j a_k a_n + v_{ij}^{(l)} a_k a_n + a_h \widehat{w}_{ij}^{(l)} \widehat{w}_{kn}^{[l]} + b_i b_j \widehat{w}_{kn}^{[l]} + a_i a_j v_{kn}^{[l]} + b_h v_{ij}^{(l)} v_{kn}^{[l]} + \widehat{w}_{ij}^{(l)} b_k b_n \\ \quad + b_i b_j b_k b_n, \quad \text{for } i < k \wedge \{j=2, k=3 \vee j=3, k=2\} \\ \quad + b_i b_j b_k b_n, \quad \text{else} \end{cases}$.

- $\widetilde{f}^{(h)}_{(ij^{(l)})(kn^{[l]})} := \begin{cases} a_i a_j a_k a_n + v_{ij}^{(l)} b_k b_n + b_h \widehat{w}_{ij}^{(l)} \widehat{w}_{kn}^{[l]} + a_i a_j \widehat{w}_{kn}^{[l]} + b_i b_j v_{kn}^{[l]} + a_h v_{ij}^{(l)} v_{kn}^{[l]} + \widehat{w}_{ij}^{(l)} a_k a_n \\ a_i a_j a_k a_n + v_{ij}^{(l)} b_k b_n + b_h \widehat{w}_{ij}^{(l)} v_{kn}^{[l]} + a_i a_j v_{kn}^{[l]} + b_i b_j \widehat{w}_{kn}^{[l]} + a_h v_{ij}^{(l)} \widehat{w}_{kn}^{[l]} + \widehat{w}_{ij}^{(l)} a_k a_n \\ \quad + b_i b_j b_k b_n, \quad \text{for } \left\{ i < k \wedge \{j=2, k=3 \vee j=3, k=2\} \right\} \vee \{i=k \wedge j=n\} \\ \quad + b_i b_j b_k b_n, \quad \text{else} \end{cases}$.

- $v_{ij}^{(l)} := \delta_{\eta_{ij}^{(l)}|1|} \left[\frac{1}{\sqrt{2}} + \delta_{A_i A_j} \left(1 - \frac{1}{\sqrt{2}} \right) \right] \left(b_i a_j + a_i b_j \right) \left(\eta_{ij}^{(l)} \right)^{\delta_{A_i A_j}} + \left(1 - \delta_{\eta_{ij}^{(l)}|1|} \right) b_i a_j$.

- $w_{ij}^{(l)} := \delta_{\eta_{ij}^{(l)}|1|} \left[\frac{1}{\sqrt{2}} + \delta_{A_i A_j} \left(1 - \frac{1}{\sqrt{2}} \right) \right] \left(b_i a_j + a_i b_j \right) \left(1 - \delta_{A_i A_j} \right) \eta_{ij}^{(l)}$.

- $\widehat{w}_{ij}^{(l)} := w_{ij}^{(l)} + \delta_{A_i A_j} v_{ij}^{(l)}$.

- first symmetry factor $S_{ij} = \begin{cases} 2, & \text{if } P_i = P_j \\ 1, & \text{else} \end{cases}$.

- second symmetry factor: $S_{ijk} = \zeta_{ijk} \left(1 + \delta_{P_i P_j} + \delta_{P_j P_k} + \delta_{P_i P_j} \delta_{P_j P_k} \right)$,

with fermion minus sign ζ_{ijk}:

$$
\zeta_{ijk} = \begin{cases}
-1 , & \text{if} \begin{cases}
\bullet \ P_i \neq P_j \neq P_k \ \wedge \ \text{only } P_i, P_j \text{ fermions} \ \wedge \ \left\{ i > j < k \ \vee \ i > j > k \right\} \\
\bullet \ P_i \neq P_j \neq P_k \ \wedge \ \text{only } P_j, P_k \text{ fermions} \ \wedge \ \left\{ i < j > k \ \vee \ i > j > k \right\} \\
\bullet \ P_i \neq P_j \neq P_k \ \wedge \ P_i, P_j, P_k \text{ fermions} \ \wedge \ \left\{ i > j < k \ \vee \ i < j > k \right\} \\
\bullet \ P_i = P_j \neq P_k \ \wedge \ P_i, P_j, P_k \text{ fermions} \ \wedge \ i > k \\
\bullet \ P_i = P_k \neq P_j \ \wedge \ P_i, P_j, P_k \text{ fermions} \\
\bullet \ P_i \neq P_j = P_k \ \wedge \ \text{only } P_j \text{ fermion} \\
\bullet \ P_i \neq P_j = P_k \ \wedge \ P_i, P_j, P_k \text{ fermions} \ \wedge \ i < j \\
\bullet \ P_i = P_j = P_k \ \wedge \ P_i, P_j, P_k \text{ fermions}
\end{cases} \\
+1 , & \text{else}
\end{cases} .
$$

(3.105)

- $c_{ij}^{(\prime)}$ is the normalization factor of the (iso)spin operators defined via

$$
\left(\mathcal{O}_{ij}^{(\prime)} \right)_{\alpha,\gamma\sigma} \left(\mathcal{O}_{ij}^{(\prime)\dagger} \right)_{\beta,\sigma\gamma} = \left(\left(\mathcal{O}_{ij}^{(\prime)} \right)_{\alpha} \left(\mathcal{O}_{ij}^{(\prime)\dagger} \right)_{\beta} \right)_{\gamma\gamma} = \mathrm{Tr} \left(\left(\mathcal{O}_{ij}^{(\prime)} \right)_{\alpha} \left(\mathcal{O}_{ij}^{(\prime)\dagger} \right)_{\beta} \right) =: c_{ij}^{(\prime)} \, \delta_{\alpha\beta}.
$$

Note, that all projectors given explicitly in appendix A are normalized which then implies that $c_{ij}^{(\prime)} = 1 \quad \forall \, i < j \in \{1, 2, 3\}$.

- x, y, z are spin and isospin dependent real numbers defined in Eqs. (A.40 - A.81) which can be calculated by hand using the tables in appendix A. A second method would be to use `Mathematica` and the 6-J symbol representation explained in the same appendix. Note, that using one of these two suggested methods implies $c_{ij}^{(\prime)} = 1$ since both are based on normalized projectors (see previous point).

In the next step we decouple the amplitudes: the matrix $\mathcal{A}_1$ can be diagonalized using an unitary transformation matrix $\mathcal{S}$ which fulfills

$$
\mathcal{S}^{-1} \mathcal{A}_1 \mathcal{S} = \mathrm{diag} \left(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \right) ,
$$

(3.106)

with λ_i ($i \in \{1, 2, 3, 4, 5, 6\}$) being the eigenvalues of the matrix $\mathcal{A}_1$. Applying the transformation matrix $\mathcal{S}$ to Eq. (3.98) and defining a new set of amplitudes,

$$
\begin{pmatrix}
\widetilde{T}_1^{(L)}(p) \\
\widetilde{T}_2^{(L)}(p) \\
\widetilde{T}_3^{(L)}(p) \\
\widetilde{T}_4^{(L)}(p) \\
\widetilde{T}_5^{(L)}(p) \\
\widetilde{T}_6^{(L)}(p)
\end{pmatrix}
:= \mathcal{S}^{-1}
\begin{pmatrix}
\widetilde{T}_{12}^{(L)}(p) \\
\widetilde{T}_{13}^{(L)}(p) \\
\widetilde{T}_{23}^{(L)}(p) \\
\widetilde{T}_{12}^{\prime(L)}(p) \\
\widetilde{T}_{13}^{\prime(L)}(p) \\
\widetilde{T}_{23}^{\prime(L)}(p)
\end{pmatrix} ,
$$

(3.107)

one can write Eq. (3.98) in terms of these decoupled amplitudes:

$$
\begin{pmatrix}
\widetilde{T}_1^{(L)}(p) \\
\widetilde{T}_2^{(L)}(p) \\
\widetilde{T}_3^{(L)}(p) \\
\widetilde{T}_4^{(L)}(p) \\
\widetilde{T}_5^{(L)}(p) \\
\widetilde{T}_6^{(L)}(p)
\end{pmatrix}
= (-1)^L \frac{4}{\sqrt{3}\,\pi} \mathrm{diag}\left(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\right) \int_0^{\Lambda_C} \frac{dq}{q}\, Q_L\left(\frac{q}{p} + \frac{p}{q}\right)
\begin{pmatrix}
\widetilde{T}_1^{(L)}(q) \\
\widetilde{T}_2^{(L)}(q) \\
\widetilde{T}_3^{(L)}(q) \\
\widetilde{T}_4^{(L)}(q) \\
\widetilde{T}_5^{(L)}(q) \\
\widetilde{T}_6^{(L)}(q)
\end{pmatrix}.
$$

$$(3.108)$$

Each row of this matrix equation leads to one (decoupled) integral equation for $\widetilde{T}_i^{(L)}(p)$ which is in the limit of Λ_C going to infinity *scale invariant* and *symmetric under the inversion* $q \to 1/q$, i.e. inversion invariant. As explained in Refs. [80,92] the solutions to the integral equations thus have – for $\Lambda_C \to \infty$ – the form of a power law:

$$
\widetilde{T}_i^{(L)}(p) = p^{s_i^{(L)}}, \quad s_i^{(L)} \in \mathbb{C} \quad \forall\, i \in \{1, 2, 3, 4, 5, 6\}, \tag{3.109}
$$

which is the most straightforward and physically meaningful solution obeying the mentioned symmetries. Replacing all amplitudes by their corresponding power law solution one finds

$$
\begin{pmatrix}
p^{s_1^{(L)}} \\
p^{s_2^{(L)}} \\
p^{s_3^{(L)}} \\
p^{s_4^{(L)}} \\
p^{s_5^{(L)}} \\
p^{s_6^{(L)}}
\end{pmatrix}
= (-1)^L \frac{4}{\sqrt{3}\,\pi} \mathrm{diag}\left(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\right) \int_0^{\infty} \frac{dq}{q}\, Q_L\left(\frac{q}{p} + \frac{p}{q}\right)
\begin{pmatrix}
q^{s_1^{(L)}} \\
q^{s_2^{(L)}} \\
q^{s_3^{(L)}} \\
q^{s_4^{(L)}} \\
q^{s_5^{(L)}} \\
q^{s_6^{(L)}}
\end{pmatrix}.
$$

$$(3.110)$$

Note, that we have taken the limit $\Lambda_C \to \infty$ which does not cause problems since the remaining integrals are not divergent anymore. It is now useful to substitute $q = Xp$ [165],

$$
\begin{pmatrix}
p^{s_1^{(L)}} \\
p^{s_2^{(L)}} \\
p^{s_3^{(L)}} \\
p^{s_4^{(L)}} \\
p^{s_5^{(L)}} \\
p^{s_6^{(L)}}
\end{pmatrix}
= (-1)^L \frac{4}{\sqrt{3}\,\pi} \mathrm{diag}\left(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\right) \int_0^{\infty} \frac{dX}{X}\, Q_L\left(X + \frac{1}{X}\right)
\begin{pmatrix}
(Xp)^{s_1^{(L)}} \\
(Xp)^{s_2^{(L)}} \\
(Xp)^{s_3^{(L)}} \\
(Xp)^{s_4^{(L)}} \\
(Xp)^{s_5^{(L)}} \\
(Xp)^{s_6^{(L)}}
\end{pmatrix},
$$

$$(3.111)$$

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and – as the integral equations are decoupled – to divide the i-th row by $p^{s_i^{(L)}}$ to get

$$\begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix} = (-1)^L \, \frac{4}{\sqrt{3}\,\pi} \mathrm{diag}\Big(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\Big) \int_0^\infty dX \begin{pmatrix}X^{s_1^{(L)}-1}\\X^{s_2^{(L)}-1}\\X^{s_3^{(L)}-1}\\X^{s_4^{(L)}-1}\\X^{s_5^{(L)}-1}\\X^{s_6^{(L)}-1}\end{pmatrix} Q_L\left(X+\frac{1}{X}\right), \quad (3.112)$$

which finally is equivalent to

$$1 = (-1)^L \, \frac{4}{\sqrt{3}\,\pi} \, \lambda_i \int_0^\infty dX \, X^{s_i^{(L)}-1} \, Q_L\left(X+\frac{1}{X}\right), \quad \text{for } i = 1, 2, 3, 4, 5, 6. \quad (3.113)$$

To achieve our main goal, namely to verify whether the Efimov effect is present in a given three particle system or not, we need to check if the parameter $s_i^{(L)}$ is real or purely imaginary. Since we know that only in the latter case it is present (see section 2.1 or Refs. [80, 167] and references therein). In order to derive the needed transcendental equation for each $s_i^{(L)}$ one has to solve the remaining integral in Eq. (3.113). One method would be to insert for arbitrary L the explicit form of Q_L. For $L = 0$ this is perfectly suitable (see for example Refs. [80, 92, 167]), but already for $L = 1$ one ends up with an integral which cannot be solved in an easy way. Therefore we follow the work of Grießhammer [165] who derived a general expression for the integral

$$\int_0^\infty dX \, X^{s_i^{(L)}-1} \, Q_L\left(X+\frac{1}{X}\right)$$

using hypergeometric functions and the Mellin transform. We use his derivation to find a solution to the slightly more general integral

$$\int_0^\infty dX \, X^{s_i^{(L)}-1} \, Q_L\left(\alpha X + \beta \frac{1}{X}\right),$$

where the coefficients of q and $1/q$ are arbitrary (although we need $\alpha = \beta$ for the inversion invariance). In appendix E we deduced in Eq. (E.18) the following result for the Mellin transform $\mathcal{M}$ of the Legendre function of the second kind with the mentioned structure of its argument:

$$\mathcal{M}\left[Q_L\left(\alpha X + \beta \frac{1}{X}\right), s_i^{(L)}\right] := \int_0^\infty dX \, X^{s_i^{(L)}-1} Q_L\left(\alpha X + \beta \frac{1}{X}\right)$$

$$= \sqrt{\pi}\, 2^{-(L+2)} \left(\frac{\beta}{\alpha}\right)^{\frac{s_i^{(L)}}{2}} \left(\frac{1}{\sqrt{\alpha\beta}}\right)^{L+1} \frac{\Gamma\left(\frac{L+s_i^{(L)}+1}{2}\right)\Gamma\left(\frac{L-s_i^{(L)}+1}{2}\right)}{\Gamma\left(\frac{2L+3}{2}\right)}$$

$$\times \; {}_2F_1\left(\frac{L+s_i^{(L)}+1}{2}, \frac{L-s_i^{(L)}+1}{2}; \frac{2L+3}{2}; \frac{1}{4\alpha\beta}\right). \quad (3.114)$$

Given in terms of the Γ-function and the hypergeometric function

$$
{}_pF_q(a_1, ..., a_p; b_1, ..., b_q; z) = \sum_{k=0}^{\infty} \prod_{i=1}^{p} \frac{\Gamma(k+a_i)}{\Gamma(a_i)} \prod_{j=1}^{q} \frac{\Gamma(b_j)}{\Gamma(k+b_j)} \frac{z^k}{k!}, \quad p, q \in \mathbb{N}.
\tag{3.115}
$$

It is known that due to parity conservation only even or odd partial waves can mix. Together with the argument that one considers low-energy scattering it is thus justified to assume S-wave scattering or – if $L = 0$ interactions are forbidden by some symmetry – D-wave scattering alone and neglect all higher partial wave contributions. In appendix E it is shown that Eq. (3.114) yields for $L = 0$

$$
\int_0^{\infty} dX \, X^{s_i^{(0)}-1} Q_0 \left(\alpha X + \beta \frac{1}{X} \right) = \left(\sqrt{\frac{\beta}{\alpha}} \right)^{s_i^{(0)}} \frac{\pi}{s_i^{(0)}} \frac{\sin\left(s_i^{(0)} \arcsin\left(\frac{1}{2}\sqrt{\frac{1}{\alpha\beta}} \right) \right)}{\cos\left(\frac{\pi}{2} s_i^{(0)} \right)},
\tag{3.116}
$$

and for $L = 1$

$$
\int_0^{\infty} dX \, X^{s_i^{(1)}-1} Q_1 \left(\alpha X + \beta \frac{1}{X} \right) = \left(\sqrt{\frac{\beta}{\alpha}} \right)^{s_i^{(1)}} \frac{\pi s_i^{(1)}}{\left[\left(s_i^{(1)} \right)^2 - 1 \right]} \frac{1}{\sin\left(\frac{\pi}{2} s_i^{(1)} \right)}
$$
$$
\times \left[\frac{\sqrt{4\alpha\beta - 1}}{s_i^{(1)}} \sin\left(s_i^{(1)} \arcsin\left(\frac{1}{2}\sqrt{\frac{1}{\alpha\beta}} \right) \right) \right.
$$
$$
\left. - \cos\left(s_i^{(1)} \arcsin\left(\frac{1}{2}\sqrt{\frac{1}{\alpha\beta}} \right) \right) \right].
\tag{3.117}
$$

Plugging these results into Eq. (3.113) one finds with $\alpha = \beta = 1$ (since all particles have (roughly) the same mass) for each parameter $s_i^{(0)}$ a transcendental equation of the form

$$
1 = \frac{4\lambda_i}{\sqrt{3}} \frac{1}{s_i^{(0)}} \frac{\sin\left(\frac{\pi}{6} s_i^{(0)} \right)}{\cos\left(\frac{\pi}{2} s_i^{(0)} \right)},
\tag{3.118}
$$

and for D-wave scattering a similar one for $s_i^{(1)}$:

$$
1 = \frac{4\lambda_i}{\sqrt{3}} \frac{s_i^{(1)}}{\left[\left(s_i^{(1)} \right)^2 - 1 \right]} \frac{\cos\left(\frac{\pi}{6} s_i^{(1)} \right)}{\sin\left(\frac{\pi}{2} s_i^{(1)} \right)} - 4\lambda_i \frac{1}{\left[\left(s_i^{(1)} \right)^2 - 1 \right]} \frac{\sin\left(\frac{\pi}{6} s_i^{(1)} \right)}{\sin\left(\frac{\pi}{2} s_i^{(1)} \right)},
\tag{3.119}
$$

where we have used that $\arcsin(1/2) = \pi/6$.

We note that Eq. (3.118) and Eq. (3.119) have exactly the same form as in the three spinless boson case. This comes not as a surprise because all particles and dimers in a non-relativistic effective theory are described in the same way. Hence, their momentum and energy dependence

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and their behavior for asymptotic large momenta is independent of their spin, isospin, species and mass ratios. The latter would in principle influence the overall prefactor of the matrix $\mathcal{A}_1$, but as we must require all masses to be equal, even this prefactor is unchanged compared to the three boson case. However, all other additional degrees of freedom affect the eigenvalue λ_i since they determine the entries of the matrix $\mathcal{A}_1$ (for three spinless boson $\lambda_1 \equiv \lambda = 2$ is simply the element of an 1-dimensional matrix). To answer the question whether or not the Efimov effect appears in a system of three particles P_1, P_2, P_3 one just has to determine the eigenvalues of the matrix $\mathcal{A}_1$ whose elements can be read off from the properties of the three particles. Only the calculation of the spin and isospin dependent parameters x, y and z needs some effort if one does it by hand and not with `Mathematica` using the 6-J symbol notation.

Before we apply this result to different three particle systems we discuss the two remaining types of three particle systems.

3.4.3 Type 2 systems

In a system without shallow P_2P_3 bound or virtual states one has – according to the rules derived at the beginning of this section (see page 51) – to erase all terms proportional to $T_{23}^{(L)}$ and $T_{23}'^{(L)}$ from Eqs. (F.10 - F.15). On the one hand a $\delta_{13}^{\left(12^{(\prime)}\right)} = 1$ factor in terms proportional to $Q_L(\sim m_2)$ sets $m_1 = m_3$ and on the other hand $\delta_{12}^{\left(13^{(\prime)}\right)} = 1$ yields $m_1 = m_2$ in terms with $Q_L(\sim m_3)$. Due to this fact it is sufficient to set m_2 at least approximately equal to m_3 in order to get a scale and inversion invariant system necessary for the Efimov effect. What does this statement mean in few words? In fact, it means that we are able to analyze a system with two particles of equal mass and one particle with different mass where only two of the three pairs have a large scatting length (i.e. a dimer state), if *the two equal mass particles are the pair without dimer*. Otherwise, it is not possible to decouple the amplitudes which is necessary for our method.

If we indeed assume $\varepsilon_{23} \ll 1$ we can replace in the remaining equations all m_3 and μ_{13} via

$$
\begin{aligned}
m_3 &= \frac{1 + \varepsilon_{23}}{1 - \varepsilon_{23}} m_2 \approx m_2 \,, \\
\mu_{13} &= \frac{m_1 m_3}{m_1 + m_3} = \frac{m_1 m_2}{\frac{1 - \varepsilon_{23}}{1 + \varepsilon_{23}} m_1 + m_2} \approx \mu_{12} \,.
\end{aligned}
\tag{3.120}
$$

Together with the modified Kronecker-deltas this leads to a set of equations all proportional to $Q_L^{122}(q,p) = Q_L\left(\frac{m_1}{qp}\left(\frac{q^2}{2\mu_{12}} + \frac{p^2}{2\mu_{12}}\right)\right)$ which can be found in Eqs. (3.122 - 3.125). In the same way as we did for type 1 systems one can now factor out the Legendre function Q_L and write Eqs. (3.122 - 3.125) as a 4×4 matrix equation:

$$
\begin{pmatrix} \widetilde{T}_{12}^{(L)}(p) \\ \widetilde{T}_{13}^{(L)}(p) \\ \widetilde{T}_{12}'^{(L)}(p) \\ \widetilde{T}_{13}'^{(L)}(p) \end{pmatrix} = \frac{(-1)^L}{\pi} \frac{m_1}{\mu_{12}} \frac{1}{\sqrt{\frac{\mu_{12}}{\mu_{(12)2}}}} \mathcal{A}_2 \int_0^{\Lambda_C} \frac{dq}{q} \, Q_L\left(\frac{m_1}{qp}\left(\frac{q^2}{2\mu_{12}} + \frac{p^2}{2\mu_{12}}\right)\right) \begin{pmatrix} \widetilde{T}_{12}^{(L)}(q) \\ \widetilde{T}_{13}^{(L)}(q) \\ \widetilde{T}_{12}'^{(L)}(q) \\ \widetilde{T}_{13}'^{(L)}(q) \end{pmatrix} \,, \tag{3.121}
$$

where the matrix $\mathcal{A}_2$ has in principle the same entries as for a type 1 system, but with erased third and sixth row and column as it is shown in Eqs. (3.126 - 3.129).

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$$\widetilde{T}_{12}^{(L)}(p) =$$

$$(-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{12} \, S_{12} \, c_{12}} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{12}^{(L)}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)2}}}}$$

$$\times \left[x_2 \, \delta_{P_1 P_3}^{(12)} \, f_{(12)(12)}^{(3)} \, S_{123} \, m_1 Q_L^{122}(q,p) + \widetilde{x}_2 \, \delta_{P_2 P_3}^{(12)} \, \widetilde{f}_{(12)(12)}^{(3)} \, S_{213} \, m_1 Q_L^{122}(q,p) \right]$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(x_3 \, \delta_{P_1 P_3} \, S_{132} + \widetilde{x}_3 \, S_{312}\right)}{\mu_{12} \sqrt{S_{12} \, S_{13} \, c_{12} \, c_{13}}} \, \widetilde{f}_{(13)(12)}^{(3)} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{13}^{(L)}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)2}}}} \, m_1 Q_L^{122}(q,p)$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{12} \, S_{12} \, \sqrt{c_{12} \, c_{12}'}} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{12}'^{(L)}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)2}}}}$$

$$\times \left[x_5 \, \delta_{P_1 P_3}^{(12')} \, f_{(12')(12)}^{(3)} \, S_{123} \, m_1 Q_L^{122}(q,p) + \widetilde{x}_5 \, \delta_{P_2 P_3}^{(12')} \, \widetilde{f}_{(12')(12)}^{(3)} \, S_{213} \, m_1 Q_L^{122}(q,p) \right]$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(x_6 \, \delta_{P_1 P_3} \, S_{132} + \widetilde{x}_6 \, S_{312}\right)}{\mu_{12} \sqrt{S_{12} \, S_{13} \, c_{12} \, c_{13}'}} \, \widetilde{f}_{(13')(12)}^{(3)} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{13}'^{(L)}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)2}}}} \, m_1 Q_L^{122}(q,p) \,,$$

$$(3.122)$$

$$\widetilde{T}_{13}^{(L)}(p) =$$

$$(-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi} \frac{\left(y_2 \, \delta_{P_1 P_2} \, S_{123} + \widetilde{y}_2 \, S_{213}\right)}{\mu_{12} \sqrt{S_{13} \, S_{12} \, c_{13} \, c_{12}}} \, \widetilde{f}_{(12)(13)}^{(2)} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{12}^{(L)}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)2}}}} \, m_1 Q_L^{122}(q,p)$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{12} \, S_{13} \, c_{13}} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{13}^{(L)}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)2}}}}$$

$$\times \left[y_3 \, \delta_{P_1 P_2}^{(13)} \, f_{(13)(13)}^{(1)} \, S_{132} \, m_1 Q_L^{122}(q,p) + \widetilde{y}_3 \, \delta_{P_2 P_3}^{(13)} \, \widetilde{f}_{(13)(13)}^{(3)} \, S_{312} \, m_1 Q_L^{122}(q,p) \right]$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi} \frac{\left(y_5 \, \delta_{P_1 P_2} \, S_{123} + \widetilde{y}_5 \, S_{213}\right)}{\mu_{12} \sqrt{S_{13} \, S_{12} \, c_{13} \, c_{12}'}} \, \widetilde{f}_{(12')(13)}^{(2)} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{12}'^{(L)}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)2}}}} \, m_1 Q_L^{122}(q,p)$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{12} \, S_{13} \, \sqrt{c_{13} \, c_{13}'}} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{13}'^{(L)}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)2}}}}$$

$$\times \left[y_6 \, \delta_{P_1 P_2}^{(13')} \, f_{(13')(13)}^{(1)} \, S_{132} \, m_1 Q_L^{122}(q,p) + \widetilde{y}_6 \, \delta_{P_2 P_3}^{(13')} \, \widetilde{f}_{(13')(13)}^{(3)} \, S_{312} \, m_1 Q_L^{122}(q,p) \right], \quad (3.123)$$

$$\widetilde{T}_{12}^{\prime(L)}(p) =$$

$$(-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{12}\, S_{12}\, \sqrt{c_{12}'\, c_{12}}} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{12}^{(L)}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)2}}}}$$

$$\times \left[x_2'\, \delta_{P_1 P_3}^{(12)}\, f_{(12)(12')}^{(3)}\, S_{123}\, m_1 Q_L^{122}(q,p) + \widetilde{x}_2'\, \delta_{P_2 P_3}^{(12)}\, \widetilde{f}_{(12)(12')}^{(3)}\, S_{213}\, m_1 Q_L^{122}(q,p) \right]$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(x_3'\, \delta_{P_1 P_3}\, S_{132} + \widetilde{x}_3'\, S_{312}\right)}{\mu_{12}\, \sqrt{S_{12}\, S_{13}\, c_{12}'\, c_{13}}}\, \widetilde{f}_{(13)(12')}^{(3)} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{13}^{(L)}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)2}}}}\, m_1 Q_L^{122}(q,p)$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{12}\, S_{12}\, c_{12}'} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{12}^{\prime(L)}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)2}}}}$$

$$\times \left[x_5'\, \delta_{P_1 P_3}^{(12')}\, f_{(12')(23')}^{(3)}\, S_{123}\, m_1 Q_L^{122}(q,p) + \widetilde{x}_5'\, \delta_{P_2 P_3}^{(12')}\, \widetilde{f}_{(12')(12')}^{(3)}\, S_{213}\, m_1 Q_L^{122}(q,p) \right]$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(x_6'\, \delta_{P_1 P_3}\, S_{132} + \widetilde{x}_6'\, S_{312}\right)}{\mu_{12}\, \sqrt{S_{12}\, S_{13}\, c_{12}'\, c_{13}'}}\, \widetilde{f}_{(13')(12')}^{(3)} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{13}^{\prime(L)}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)2}}}}\, m_1 Q_L^{122}(q,p)\,,$$

$$(3.124)$$

$$\widetilde{T}_{13}^{\prime(L)}(p) =$$

$$(-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi} \frac{\left(y_2'\, \delta_{P_1 P_2}\, S_{123} + \widetilde{y}_2'\, S_{213}\right)}{\mu_{12}\, \sqrt{S_{13}\, S_{12}\, c_{13}'\, c_{12}}}\, \widetilde{f}_{(12)(13')}^{(2)} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{12}^{(L)}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)2}}}}\, m_1 Q_L^{122}(q,p)$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{12}\, S_{13}\, \sqrt{c_{13}'\, c_{13}}} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{13}^{(L)}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)2}}}}$$

$$\times \left[y_3'\, \delta_{P_1 P_2}^{(13)}\, f_{(13)(13')}^{(1)}\, S_{132}\, m_1 Q_L^{122}(q,p) + \widetilde{y}_3'\, \delta_{P_2 P_3}^{(13)}\, \widetilde{f}_{(13)(13')}^{(3)}\, S_{312}\, m_1 Q_L^{122}(q,p) \right]$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi} \frac{\left(y_5'\, \delta_{P_1 P_2}\, S_{123} + \widetilde{y}_5'\, S_{213}\right)}{\mu_{12}\, \sqrt{S_{13}\, S_{12}\, c_{13}'\, c_{12}'}}\, \widetilde{f}_{(12')(13')}^{(2)} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{12}^{\prime(L)}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)2}}}}\, m_1 Q_L^{122}(q,p)$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{12}\, S_{13}\, c_{13}'} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{13}^{\prime(L)}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)2}}}}$$

$$\times \left[y_6'\, \delta_{P_1 P_2}^{(13')}\, f_{(13')(13')}^{(1)}\, S_{132}\, m_1 Q_L^{122}(q,p) + \widetilde{y}_6'\, \delta_{P_2 P_3}^{(13')}\, \widetilde{f}_{(13')(13')}^{(3)}\, S_{312}\, m_1 Q_L^{122}(q,p) \right]. \quad (3.125)$$

$$(\mathcal{A}_2)_{i1} = \begin{pmatrix} \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{S_{12}\, c_{12}} \left[x_2\, \delta^{(12)}_{P_1 P_3}\, f^{(3)}_{(12)(12)}\, S_{123} + \widetilde{x}_2\, \delta^{(12)}_{P_2 P_3}\, \widetilde{f}^{(3)}_{(12)(12)}\, S_{213} \right] \\ \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{\left(y_2\, \delta_{P_1 P_2}\, S_{123} + \widetilde{y}_2\, S_{213}\right)}{\sqrt{S_{13}\, S_{12}\, c_{13}\, c_{12}}}\, \widetilde{f}^{(2)}_{(12)(13)} \\ \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{S_{12}\, \sqrt{c'_{12}\, c_{12}}} \left[x'_2\, \delta^{(12)}_{P_1 P_3}\, f^{(3)}_{(12)(12')}\, S_{123} + \widetilde{x}'_2\, \delta^{(12)}_{P_2 P_3}\, \widetilde{f}^{(3)}_{(12)(12')}\, S_{213} \right] \\ \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{\left(y'_2\, \delta_{P_1 P_2}\, S_{123} + \widetilde{y}'_2\, S_{213}\right)}{\sqrt{S_{13}\, S_{12}\, c'_{13}\, c_{12}}}\, \widetilde{f}^{(2)}_{(12)(13')} \end{pmatrix}, \quad (3.126)$$

$$(\mathcal{A}_2)_{i2} = \begin{pmatrix} \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{\left(x_3\, \delta_{P_1 P_3}\, S_{132} + \widetilde{x}_3\, S_{312}\right)}{\sqrt{S_{12}\, S_{13}\, c_{12}\, c_{13}}}\, \widetilde{f}^{(3)}_{(13)(12)} \\ \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{S_{13}\, c_{13}} \left[y_3\, \delta^{(13)}_{P_1 P_2}\, f^{(1)}_{(13)(13)}\, S_{132} + \widetilde{y}_3\, \delta^{(13)}_{P_2 P_3}\, \widetilde{f}^{(3)}_{(13)(13)}\, S_{312} \right] \\ \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{\left(x'_3\, \delta_{P_1 P_3}\, S_{132} + \widetilde{x}'_3\, S_{312}\right)}{\sqrt{S_{12}\, S_{13}\, c'_{12}\, c_{13}}}\, \widetilde{f}^{(3)}_{(13)(12')} \\ \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{S_{13}\, \sqrt{c'_{13}\, c_{13}}} \left[y'_3\, \delta^{(13)}_{P_1 P_2}\, f^{(1)}_{(13)(13')}\, S_{132} + \widetilde{y}'_3\, \delta^{(13)}_{P_2 P_3}\, \widetilde{f}^{(3)}_{(13)(13')}\, S_{312} \right] \end{pmatrix}, \quad (3.127)$$

$$(\mathcal{A}_2)_{i3} = \begin{pmatrix} \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{S_{12}\, \sqrt{c_{12}\, c'_{12}}} \left[x_5\, \delta^{(12')}_{P_1 P_3}\, f^{(3)}_{(12')(12)}\, S_{123} + \widetilde{x}_5\, \delta^{(12')}_{P_2 P_3}\, \widetilde{f}^{(3)}_{(12')(12)}\, S_{213} \right] \\ \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{\left(y_5\, \delta_{P_1 P_2}\, S_{123} + \widetilde{y}_5\, S_{213}\right)}{\sqrt{S_{13}\, S_{12}\, c_{13}\, c'_{12}}}\, \widetilde{f}^{(2)}_{(12')(13)} \\ \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{S_{12}\, c'_{12}} \left[x'_5\, \delta^{(12')}_{P_1 P_3}\, f^{(3)}_{(12')(12')}\, S_{123} + \widetilde{x}'_5\, \delta^{(12')}_{P_2 P_3}\, \widetilde{f}^{(3)}_{(12')(12')}\, S_{213} \right] \\ \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{\left(y'_5\, \delta_{P_1 P_2}\, S_{123} + \widetilde{y}'_5\, S_{213}\right)}{\sqrt{S_{13}\, S_{12}\, c'_{13}\, c'_{12}}}\, \widetilde{f}^{(2)}_{(12')(13')} \end{pmatrix}, \quad (3.128)$$

$$(\mathcal{A}_2)_{i4} = \begin{pmatrix} \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{\left(x_6\, \delta_{P_1 P_3}\, S_{132} + \widetilde{x}_6\, S_{312}\right)}{\sqrt{S_{12}\, S_{13}\, c_{12}\, c'_{13}}}\, \widetilde{f}^{(3)}_{(13')(12)} \\ \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{S_{13}\, \sqrt{c_{13}\, c'_{13}}} \left[y_6\, \delta^{(13')}_{P_1 P_2}\, f^{(1)}_{(13')(13)}\, S_{132} + \widetilde{y}_6\, \delta^{(13')}_{P_2 P_3}\, \widetilde{f}^{(3)}_{(13')(13)}\, S_{312} \right] \\ \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{\left(x'_6\, \delta_{P_1 P_3}\, S_{132} + \widetilde{x}'_6\, S_{312}\right)}{\sqrt{S_{12}\, S_{13}\, c'_{12}\, c'_{13}}}\, \widetilde{f}^{(3)}_{(13')(12')} \\ \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{S_{13}\, c'_{13}} \left[y'_6\, \delta^{(13')}_{P_1 P_2}\, f^{(1)}_{(13')(13')}\, S_{132} + \widetilde{y}'_6\, \delta^{(13')}_{P_2 P_3}\, \widetilde{f}^{(3)}_{(13')(13')}\, S_{312} \right] \end{pmatrix}, \quad (3.129)$$

Although the matrices $\mathcal{A}_1$ and $\mathcal{A}_2$ are almost identical (they just differ by their dimension, but not by their entries) Eq. (3.98) and Eq. (3.121) have a different, mass dependent prefactor, respectively, which can affect the result for $s_i^{(L)}$ in the end.

To find the corresponding transcendental equation for the parameter $s_i^{(L)}$ we do the same as in the type 1 case. One applies an unitary transformation under which the amplitudes decouple, namely, under which $\mathcal{A}_2$ becomes diagonal. Since the decoupled amplitudes fulfill scale and inversion invariant integral equations their replacement by a power law solution $\sim p^{s_i^{(L)}}$ ($i \in \{1, 2, 3, 4\}$) is justified and hence one finds in the limit $\Lambda_C \to \infty$ a similar equation to Eq. (3.113):

$$1 = \frac{(-1)^L}{\pi} \frac{m_1}{\mu_{12}} \frac{1}{\sqrt{\frac{\mu_{12}}{\mu_{(12)2}}}} \lambda_i \int_0^\infty dX \, X^{s_i^{(L)}-1} \, Q_L\left(\frac{m_1}{2\mu_{12}}\left(X + \frac{1}{X}\right)\right) , \qquad (3.130)$$

where λ_i are now the eigenvalues of the matrix $\mathcal{A}_2$. Considering the argument of the Legendre function it becomes clear why we had to find the Mellin transform Eq. (3.114) of the more general Q_L. Using Eq. (3.116) one finds with $\alpha = \beta = m_1/(2\mu_{12})$ the corresponding transcendental equation for $L = 0$

$$1 = \frac{m_1}{\mu_{12}} \sqrt{\frac{\mu_{(12)2}}{\mu_{12}}} \lambda_i \frac{1}{s_i^{(0)}} \frac{\sin\left(s_i^{(0)} \arcsin\left(\frac{\mu_{12}}{m_1}\right)\right)}{\cos\left(\frac{\pi}{2} s_i^{(0)}\right)} \qquad (3.131)$$

$$\Longleftrightarrow \qquad 1 = \frac{4\,\lambda_i}{\sqrt{4 - [(1 - \varepsilon_{12})^2 - 2]^2}} \frac{1}{s_i^{(0)}} \frac{\sin\left(s_i^{(0)} \arcsin\left(\frac{(1-\varepsilon_{12})}{2}\right)\right)}{\cos\left(\frac{\pi}{2} s_i^{(0)}\right)} . \qquad (3.132)$$

In the second representation (Eq. (3.132)) we have with Eq. (3.89) and

$$\frac{4}{x\sqrt{4 - x^2}} = \frac{4}{\sqrt{4x^2 - x^4}} = \frac{4}{\sqrt{-(-4x^2 + x^4)}} = \frac{4}{\sqrt{-[(x^2 - 2)^2 - 4]}} = \frac{4}{\sqrt{4 - (x^2 - 2)^2}} , \quad (3.133)$$

rewritten it in terms of the mass difference parameter $-1 < \varepsilon_{12} < 1$ which *must not* tend to zero in a system of type 2. However, if it is very close to zero one directly observers that Eq. (3.132) looks the same as the determining equation of $s_i^{(0)}$ for a type 1 system (Eq. (3.118)), but as the eigenvalues λ_i can be different for $\mathcal{A}_1$ and $\mathcal{A}_2$ they lead in general to different results for $s_i^{(0)}$. Only in the case of a type 1 system without a $P_2 P_3$ dimer where one has to erase all $T_{23}^{(\prime)}$ amplitudes so that $\mathcal{A}_1$ becomes equal to $\mathcal{A}_2$ both equations are equivalent. But this is not surprising since such a system (type 1 without $d_{23}^{(\prime)}$) is what we have called type 2. Nevertheless, in general where ε_{12} is not close to zero it can have a large effect on $s_i^{(0)}$: if one mass is much larger than the other it tends to ± 1. In the limit of $\varepsilon_{12} \to +1$ the right-hand-side of Eq. (3.132) yields $0/0$ and hence one can use the rule of L'Hospital and $\arcsin(0) = 0$ to find

$$1 = \lim_{\varepsilon_{12} \to +1} \frac{4\,\lambda_i}{\sqrt{4 - [(1 - \varepsilon_{12})^2 - 2]^2}} \frac{1}{s_i^{(0)}} \frac{\sin\left(s_i^{(0)} \arcsin\left(\frac{(1-\varepsilon_{12})}{2}\right)\right)}{\cos\left(\frac{\pi}{2} s_i^{(0)}\right)}$$

$$= -\frac{\lambda_i}{\cos\left(\frac{\pi}{2} s_i^{(0)}\right)} , \qquad (3.134)$$

which can be solved for $s_i^{(0)}$:

$$s_i^{(0)} = \frac{2}{\pi} \arccos\left(-\lambda_i\right) = \frac{2}{\pi}\left[\pi - \arccos\left(\lambda_i\right)\right]$$

$$= 2 + \frac{2i}{\pi} \ln\left(\lambda_i + i\sqrt{1 - \lambda_i^2}\right) , \tag{3.135}$$

where we have used the complex logarithm to continue the arc-cosine to the region outside the interval $[-1, 1]$. For eigenvalues $|\lambda_i| > 1$, $s_i^{(0)}$ can be purely imaginary and hence the existence of the Efimov effect is not excluded and one cannot make a general statement about its occurrence. In the other limit, i.e. $\varepsilon_{12} \to -1$, Eq. (3.132) goes to

$$1 = \lim_{\varepsilon_{12} \to -1} \frac{4\,\lambda_i}{\sqrt{4 - [(1-\varepsilon_{12})^2 - 2]^2}} \frac{1}{s_i^{(0)}} \frac{\sin\left(s_i^{(0)} \arcsin\left(\frac{(1-\varepsilon_{12})}{2}\right)\right)}{\cos\left(\frac{\pi}{2}s_i^{(0)}\right)}$$

$$= \frac{\lambda_i}{0} \frac{\tan\left(\frac{\pi}{2}s^{(0)}\right)}{s_i^{(0)}} , \tag{3.136}$$

with $\arcsin(x) \xrightarrow{x\to 1} \pi/2$. Thus, $s_i^{(0)}$ tends to infinity, but its sign and in particular the fact whether it is real or purely imaginary depends on the sign of λ_i. Using the definition of $\varepsilon_{12} = (m_1 - m_2)/(m_1 + m_2)$ we conclude that on the one hand the limit of it going to $+1$ is equivalent to $m_1 \gg m_2 = m_3$ and on the other hand $\varepsilon_{12} \to -1$ corresponds to $m_1 \ll m_2 = m_3$. The conclusion is thus that one cannot deduce solely from the mass ratio of the particles if there is an Efimov effect in the system or not.

For completeness we state also the result of Eq. (3.130) for angular momentum $L = 1$ (cf. Eq. (3.117)):

$$1 = -\frac{m_1}{\mu_{12}} \sqrt{\frac{\mu_{(12)2}}{\mu_{12}}} \lambda_i \frac{s_i^{(1)}}{\left[\left(s_i^{(1)}\right)^2 - 1\right]} \frac{1}{\sin\left(\frac{\pi}{2}s_i^{(1)}\right)}$$

$$\times \left[\frac{\sqrt{\left(\frac{m_1}{\mu_{12}}\right)^2 - 1}}{s_i^{(1)}} \sin\left(s_i^{(1)} \arcsin\left(\frac{\mu_{12}}{m_1}\right)\right) - \cos\left(s_i^{(1)} \arcsin\left(\frac{\mu_{12}}{m_1}\right)\right)\right]$$

$$\iff \quad 1 = \frac{4\,\lambda_i}{\sqrt{4 - [(1-\varepsilon_{12})^2 - 2]^2}} \frac{s_i^{(1)}}{\left[\left(s_i^{(1)}\right)^2 - 1\right]} \frac{\cos\left(s_i^{(1)} \arcsin\left(\frac{(1-\varepsilon_{12})}{2}\right)\right)}{\sin\left(\frac{\pi}{2}s_i^{(1)}\right)}$$

$$- \frac{4\,\lambda_i}{(1-\varepsilon_{12})^2} \frac{1}{\left[\left(s_i^{(1)}\right)^2 - 1\right]} \frac{\sin\left(s_i^{(1)} \arcsin\left(\frac{(1-\varepsilon_{12})}{2}\right)\right)}{\sin\left(\frac{\pi}{2}s_i^{(1)}\right)} . \tag{3.137}$$

Taking the limit $\varepsilon_{12} \to \pm 1$ does not lead to further insights regarding the Efimov effect. Hence, we omit its derivation. However, besides these subtleties the search for a possible Efimov effect

is like in the type 1 system reduced to the determination of eigenvalues of a matrix whose entries can straightforwardly be read off from the properties of the three particles in the system.

Before we continue with the last type of systems we can – as a check – apply the type 2 results above to a system of either two (identical or distinguishable) bosons or two identical fermions and a third boson in order to reproduce (in parts) Fig. (53) in the work of Hammer and Braaten in Ref. [80]. To do so one has to change the particle allocation via $P_1 = P_3^{\mathrm{HB}}$, $P_2 = P_2^{\mathrm{HB}}$ and $P_3 = P_1^{\mathrm{HB}}$ since in Ref. [80] a system with $m_1^{\mathrm{HB}} = m_2^{\mathrm{HB}} \neq m_3^{\mathrm{HB}}$ is considered which does not fit in our scheme with $m_1 \neq m_2 = m_3$. Assuming the bosons to be scalars and the fermions to have spin $1/2$ one has the following dimers in the system: d_{12} and d_{13}. Both with spin $1/2$ (there are no primed dimers) so that the spin 0 scattering channel is the only allowed channel for bosonic and fermionic configurations. From the particle allocation we read off that $S_{12} = S_{13} = 1$, $\delta_{P_i P_j} = \delta_{P_i P_j}^{(ab)} = 0$ for $i = 1 \wedge j = 2, 3$ and that $f_{(12)(12)}^{(3)} = \widetilde{f}_{(12)(12)}^{(3)} = f_{(12)(13)}^{(2)} = \widetilde{f}_{(12)(13)}^{(2)} = 1$. Furthermore, we know from the symmetry factor S_{ijk} derived in appendix B and given in Eq. (B.13) that for all above mentioned combinations of bosons and fermions it holds $S_{213} = S_{312} = 1$. The matrix $\mathcal{A}_2$ (cf. its first two columns Eq. (3.126) and Eq. (3.127)) simplifies to

$$\mathcal{A}_2 = \begin{pmatrix} \widetilde{x}_2 \, \delta_{P_2 P_3}^{(12)} & \widetilde{x}_3 \\ \widetilde{y}_2 & \widetilde{y}_3 \, \delta_{P_2 P_3}^{(13)} \end{pmatrix} , \tag{3.138}$$

which is a 2×2 matrix since we erased all primed variables because there are no primed dimers. All in all there are three cases left:

- $P_2 \neq P_3$ bosons: the modified Kronecker-deltas both vanish. Thus, one finds

$$\mathcal{A}_2 = \begin{pmatrix} 0 & \widetilde{x}_3 \\ \widetilde{y}_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

  where the parameters $\widetilde{x}_3$ and $\widetilde{y}_2$ are calculated using one of the methods explained in the last two sections of appendix A. The matrix $\mathcal{A}_2$ has the eigenvalues $\lambda_{1,2} = \pm 1$.

- $P_2 = P_3$ bosons: the modified Kronecker-deltas yield 1 and in addition it holds $\widetilde{T}_{13} = \widetilde{T}_{12}$ since $P_2 = P_3$. Hence, according to the rules in the box on page 51 one erases $\widetilde{T}_{13}$ from the equations. The result is a number $\mathcal{A}_2 = \widetilde{x}_2 = 1$ so that $\lambda_1 = 1$.

- $P_2 = P_3$ fermions: the modified Kronecker-delta yield 1 and as above the matrix $\mathcal{A}_2$ again has a 1×1 structure, but now the $\widetilde{x}$ parameter is -1 so that one finds: $\mathcal{A}_2 = \widetilde{x}_2 = -1$ and thus $\lambda_1 = -1$.

The non-positive eigenvalues in the fermionic case are already a hint to what is shown in Ref. [172] and used in a similar framework in Ref. [173]. Namely, that in a system as above the Efimov effect can only occur in S-wave scattering if the particles are (identical or distinguishable) bosons or in P-wave scattering if at least two of the particles are fermions. Indeed, one does not find purely imaginary solutions to Eq. (3.131) for $\lambda_1 = -1$, independently of the mass factors. In contrast, for $\lambda_1 = 1$ the same equation (3.131) yields purely imaginary solutions for arbitrary
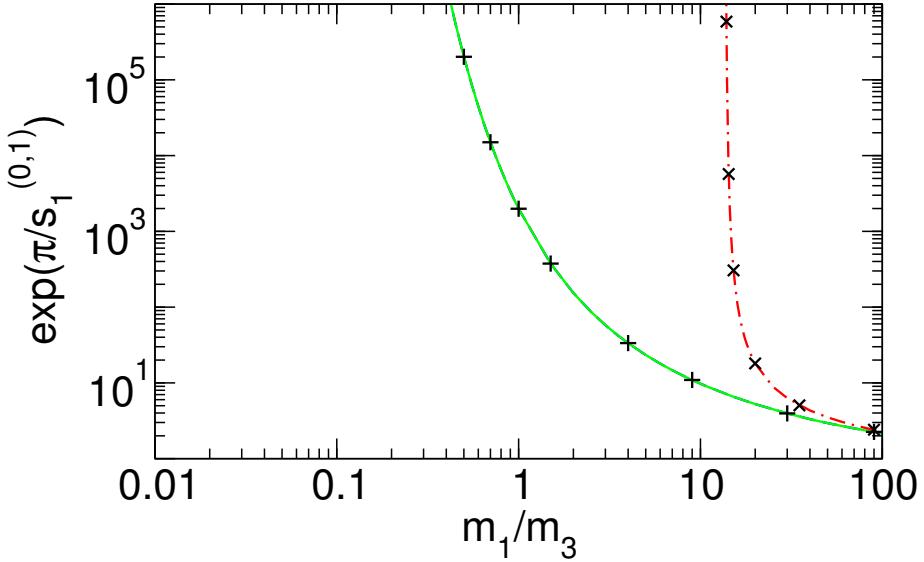
Figure 3.7: *Results of type-2-system calculations for two either identical or distinguishable bosons (pluses) or identical fermions (crosses) P_2, P_3 and a third boson P_1 in comparison with Fig. (53) in [80] where the solid line describes the scaling factor $\exp(\pi/s_1^{(0)})$ for P_2, P_3 being bosons and the dash-dotted line that for P_2, P_3 being identical fermions, i.e. $\exp(\pi/s_1^{(1)})$, as functions of the mass ratio m_1/m_3.*

mass ratios. Rewriting the mass dependent factors in Eq. (3.131) in terms of $m_1/m_3 = m_1/m_2$ one can reproduce the solid line of Fig. (53) in Ref. [80] which belongs to the two bosonic cases: we took this figure and used our method to find the "pluses" shown in Fig. 3.7 which match with the solid line as they should. In the same way one can reformulate the P-wave relation Eq. (3.137) in terms of the mass ratio m_1/m_3 and search for purely imaginary solutions of $s_1^{(1)}$ for $\lambda_1 = -1$ in the fermionic case. One finds that for mass ratios $m_1/m_3 \gtrsim 13.6$ such solutions exist and thus the Efimov effect is present, which agrees with the statements in Refs. [80, 172, 173]. In Fig. 3.7 we have marked the results of our method by crosses which match with the dash-dotted line for fermions as expected. Note, that the dashed line of Fig. (53) in Ref. [80] which corresponds to a similar (though different) system of two bosons / fermions and a third boson, but with a bound state d_{23} between the particles with equal masses, cannot be described by our method (see opening words of this section 3.4.3) and is thus not shown in Fig. 3.7.

3.4.4 Type 3 systems

We will now focus on three particle systems of type 3 where only shallow P_1P_2 dimers exist. As there is just one pair of particles with large scattering length one would expect that Efimov physics are not relevant in such a system. Indeed, we will show that the analysis of type 3 systems only yields an Efimov effect if the system can fluctuate into a type 2 system due to a superposition in the flavor wave function of the respective dimer.

Following the rules on page 51 one has to erase in a type 3 system all terms proportional to

$T_{13}^{(\prime)(L)}$ and $T_{23}^{(\prime)(L)}$ from Eqs. (F.10 - F.15). It remain just two coupled integral equations:

$$\widetilde{T}_{12}^{(L)}(p) = (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{12}\, S_{12}\, c_{12}} \int_0^\infty \frac{dq}{q} \frac{\widetilde{T}_{12}^{(L)}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)3}}}}$$

$$\times \left[x_2\, \delta_{P_1 P_3}^{(12)}\, f_{(12)(12)}^{(3)}\, S_{123}\, m_2 Q_L^{211}(q,p) + \widetilde{x}_2\, \delta_{P_2 P_3}^{(12)}\, \widetilde{f}_{(12)(12)}^{(3)}\, S_{213}\, m_1 Q_L^{122}(q,p) \right]$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{12}\, S_{12}\, \sqrt{c_{12}\, c_{12}'}} \int_0^\infty \frac{dq}{q} \frac{\widetilde{T}_{12}'^{(L)}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)3}}}}$$

$$\times \left[x_5\, \delta_{P_1 P_3}^{(12')}\, f_{(12')(12)}^{(3)}\, S_{123}\, m_2 Q_L^{211}(q,p) + \widetilde{x}_5\, \delta_{P_2 P_3}^{(12')}\, \widetilde{f}_{(12')(12)}^{(3)}\, S_{213}\, m_1 Q_L^{122}(q,p) \right],$$

(3.139)

$$\widetilde{T}_{12}'^{(L)}(p) = (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{12}\, S_{12}\, \sqrt{c_{12}'\, c_{12}}} \int_0^\infty \frac{dq}{q} \frac{\widetilde{T}_{12}^{(L)}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)3}}}}$$

$$\times \left[x_2'\, \delta_{P_1 P_3}^{(12)}\, f_{(12)(12')}^{(3)}\, S_{123}\, m_2 Q_L^{211}(q,p) + \widetilde{x}_2'\, \delta_{P_2 P_3}^{(12)}\, \widetilde{f}_{(12)(12')}^{(3)}\, S_{213}\, m_1 Q_L^{122}(q,p) \right]$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{12}\, S_{12}\, c_{12}'} \int_0^\infty \frac{dq}{q} \frac{\widetilde{T}_{12}'^{(L)}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)3}}}}$$

$$\times \left[x_5'\, \delta_{P_1 P_3}^{(12')}\, f_{(12')(23')}^{(3)}\, S_{123}\, m_2 Q_L^{211}(q,p) + \widetilde{x}_5'\, \delta_{P_2 P_3}^{(12')}\, \widetilde{f}_{(12')(12')}^{(3)}\, S_{213}\, m_1 Q_L^{122}(q,p) \right].$$

(3.140)

We note that each term is proportional either to $\delta_{P_1 P_3}^{\left(12^{(\prime)}\right)}$ or to $\delta_{P_2 P_3}^{\left(12^{(\prime)}\right)}$, but if $P_1 = P_3$ ($P_2 = P_3$) there must be an extra $P_2 P_3$ ($P_1 P_3$) bound state which is a contradiction to the classification of type 3 systems. Thus, we conclude that a type 3 system only exists for the special case (cf. Eq. (C.2))

$$\delta_{P_1/P_2, P_3}^{\left(12^{(\prime)}\right)} \overset{P_1/P_2 \neq P_3}{=} \delta_{A_1/A_2, A_3}\, \delta_{\eta_{12}^{(\prime)}|1|} \left(\delta_{\eta_{12}^{(\prime)}|1|} - \delta_{A_1 A_2} \right) = 1 .$$

(3.141)

Furthermore, it also leads to a contradiction if we assume $\delta_{P_1 P_3}^{\left(12^{(\prime)}\right)} = \delta_{P_2 P_3}^{\left(12^{(\prime)}\right)} = 1$ because this would imply $A_1 = A_2 = A_3$, but with $A_1 = A_2$ the equation above always yields zero. Therefore one has to distinguish two cases for which a type 3 system can exist: firstly, $\delta_{P_1 P_3}^{\left(12^{(\prime)}\right)} = 1$ and $\delta_{P_2 P_3}^{\left(12^{(\prime)}\right)} = 0$. And secondly, the vice versa case $\delta_{P_1 P_3}^{\left(12^{(\prime)}\right)} = 0$ and $\delta_{P_2 P_3}^{\left(12^{(\prime)}\right)} = 1$. However, by interchanging the particle allocation of P_1 and P_2 one can transform these two cases into each other and we only have to consider one of them. We chose the first one.

Case 1: $\delta_{\mathbf{P_1 P_3}}^{\left(\mathbf{12^{(\prime)}}\right)} = \mathbf{1}$ and $\delta_{\mathbf{P_2 P_3}}^{\left(\mathbf{12^{(\prime)}}\right)} = \mathbf{0}$

From the condition $\delta_{P_1 P_3}^{\left(12^{(\prime)}\right)} = 1$ for $P_1 \neq P_3$ it follows that $\delta_{\eta_{12}^{(\prime)}|1|} = 1$, $\delta_{A_1 A_2} = 0$ and $\delta_{A_1 A_3} = 1$. Therefore one can conclude that P_1 and P_2 both are bosons which implies $\zeta_{123} = +1$ and thus $S_{123} = 1$ as well as $S_{12} = 1$. Also concerning the a, b parameters one finds some constraints: $a_1 a_2 = b_1 b_2 = 0$, $v_{12}^{(\prime)} = w_{12}^{(\prime)} = 1/\sqrt{2}$ and $a_3 = 1$ or $b_3 = 1$. Altogether this leads to

$$f_{\left(12^{(\prime)}\right)\left(12^{[\prime]}\right)}^{(3)} = \left(a_1 a_2 a_1 a_2 + a_3 w_{12}^{(\prime)} w_{12}^{[\prime]} + b_3 v_{12}^{(\prime)} v_{12}^{[\prime]} + b_1 b_2 b_1 b_2\right) = \frac{1}{2}, \tag{3.142}$$

for all combinations of primed and unprimed variables. Finally, this yields with $\delta_{P_1 P_2} = 0$ (follows from $\delta_{A_1 A_2} = 0$) the matrix equation below:

$$\begin{pmatrix} \widetilde{T}_{12}^{(L)}(p) \\ \widetilde{T}_{12}^{\prime(L)}(p) \end{pmatrix} = \frac{(-1)^L}{\pi} \frac{m_2}{\mu_{12}} \sqrt{\frac{\mu_{(12)1}}{\mu_{12}}} \begin{pmatrix} \frac{x_2}{c_{12}} & \frac{x_5}{\sqrt{c_{12} c_{12}'}} \\ \frac{x_2'}{\sqrt{c_{12} c_{12}'}} & \frac{x_5'}{c_{12}'} \end{pmatrix} \int_0^{\Lambda_C} \frac{dq}{q} Q_L \left(\frac{m_2}{2\mu_{12}} \left(\frac{q}{p} + \frac{p}{q} \right) \right) \begin{pmatrix} \widetilde{T}_{12}^{(L)}(q) \\ \widetilde{T}_{12}^{\prime(L)}(q) \end{pmatrix}. \tag{3.143}$$

Decoupling the amplitudes and inserting the power-law solution $\sim p^{s_i^{(L)}}$ one finds in the limit $\Lambda_C \to \infty$ with $X = q/p$:

$$1 = \frac{(-1)^L}{\pi} \frac{m_2}{\mu_{12}} \sqrt{\frac{\mu_{(12)1}}{\mu_{12}}} \lambda_i \int_0^\infty \frac{dX}{X} X^{s_i^{(L)}-1} Q_L \left(\frac{m_2}{2\mu_{12}} \left(X + \frac{1}{X} \right) \right), \quad \text{for } i \in \{1, 2\}, \tag{3.144}$$

with λ_i being the eigenvalues of the matrix

$$\mathcal{A}_{3a} = \begin{pmatrix} \frac{x_2}{c_{12}} & \frac{x_5}{\sqrt{c_{12} c_{12}'}} \\ \frac{x_2'}{\sqrt{c_{12} c_{12}'}} & \frac{x_5'}{c_{12}'} \end{pmatrix}. \tag{3.145}$$

For S-wave scattering we deduce from Eq. (3.116) with $\alpha = \beta = m_2/(2\mu_{12})$ the following transcendental equation for $s_i^{(0)}$:

$$1 = \frac{m_2}{\mu_{12}} \sqrt{\frac{\mu_{(12)1}}{\mu_{12}}} \lambda_i \frac{1}{s_i^{(0)}} \frac{\sin\left(s_i^{(0)} \arcsin\left(\frac{\mu_{12}}{m_2}\right)\right)}{\cos\left(\frac{\pi}{2} s_i^{(0)}\right)}. \tag{3.146}$$

This result is very similar to what we have found for the type 2 system. However, this is not surprising because a system of type 3 only exists in a situation where one deals with a superposition-dimer $d_{12}^{(\prime)} \sim \bar{A}_2 A_1 + \eta_{12}^{(\prime)} A_2 \bar{A}_1$ (cf. the properties of P_1 and P_2 derived above). In fact, it accounts for the case that one considers a three particle system $\bar{A}_2 A_1 A_3$ where neither $A_1 A_3$ nor $\bar{A}_2 A_3$ have a bound or virtual state and which seems to not contribute at all. But as it can fluctuate into the system $A_2 \bar{A}_1 A_3$ where the latter two fields have a dimer state, it does contribute. Then, we finally note that the fluctuated system is because of $A_1 = A_3$ nothing else then a type 2 system with particles 1 and 2 interchanged compared to what we have discusses in section 3.4.3. This explains the mentioned similarity and we conclude that type 3 systems are just a remnant of type 2 ones. Hence, they can be ignored in further analyses.

3.4.5 Implementation with `Mathematica`

Considering the matrices $\mathcal{A}_1$ and $\mathcal{A}_2$ corresponding to type 1 and type 2 three particle systems we observe that they only depend on a rather small number of parameters like spin and isospin. Hence, it is relatively straightforward to implement the full calculation of the two mentioned matrices into a computer program. For this purpose we choose `Mathematica` since it already contains a command `SixJSymbol` which determines the Wigner 6-J symbols for given spins. Furthermore, one can – after determining the eigenvalues of $\mathcal{A}_{1,2}$ – use the `FindRoot` command to find a solution of the transcendental equation for the scaling parameter s which depends on these eigenvalues. To use such a program in the right way one has, however, to remember the rules in the box on page 51 in order to assign to the parameters the right input values. Considering for example the NNN system one has to erase the amplitudes T_{13}, T_{23} and the respective primed counterparts. Consequently, one has to set the spins $J_{13} = J_{23} = no$ (see comments within the program) to get the right results. The corresponding `Mathematica` notebook "`Efimov_Analysis.nb`" can be found on the attached CD. On start the input parameters are set to values which correspond to the $NN\Lambda$ system with the particle allocation $P_1 = P_2 = N$, $P_3 = \Lambda$ and with the hypertriton as Efimov trimer.

Chapter 4

Efimov effect in hadronic molecules

After the discussion of the different types of three particle systems in the previous section one can continue with two possible applications. Firstly, one can apply the derived methods for the determination of the parameter $s^{(L)}$ to various three particle systems where some particle scatters off a – more or less – established hadronic molecule, in order to check if the Efimov effect is present in such a system. The nature of many states appearing below is not completely understood due to lacking experimental data. We will interpret all of them as S-wave hadronic molecules. Additionally, we assume that they at least approximately fulfill the conditions that are necessary to treat them in a pionless EFT (see also section 4.3). The second application will be that we consider specific combinations of charm and bottom mesons (which are the constituents of a large number of molecules, cf. Tab. 4.1) and check which spin and isospin configurations lead to Efimov trimers in the scattering off a third charm or bottom meson. This procedure is also applied to hypothetical dibaryon systems.

Before we start with this task we want to emphasize that the ideas of treating hadronic molecules like the $X(3872)$ in EFT($\not{\pi}$) [132] or to search for possible Efimov trimers in $X(3872)$–D scattering [133] were already invented by Hammer and Braaten and respective co-workers. Moreover, one should also mention the work regarding hadronic molecules done in Refs. [134–162] which is in many aspects the basis for this thesis.

4.1 Established systems

We start with the first application where we discuss the Efimov effect in established molecule systems. For this purpose we have collected a set of experimentally known bound or virtual states in Tab. 4.1 which have in common that one can interpret them as hadronic molecules. Furthermore, one can find the constituent particles of these molecules in the same table where the given mass of an isospin multiplet has to be understood as the charged/uncharged mean mass. A short discussion of the applicability of EFT($\not{\pi}$) to the particles in this table can be found in section 4.3. Before we discus the results of our search for Efimov trimers we start with two demonstrative examples to clarify the derived method: on the one hand the well-known three nucleon system with the triton as three-body bound state (discussed e.g. in Refs. [80, 92, 93]) and on the other hand we consider as a "new" system the one made of three kaons.

particle / molecule	wave function	$\mathbf{I^G(J^P)}$	isospin av. mass [MeV]
K [2]	/	$\frac{1}{2}(0^-)$	495.6
D [2]	/	$\frac{1}{2}(0^-)$	1867.2
D^* [2]	/	$\frac{1}{2}(1^-)$	2008.6
D_1 [2]	/	$\frac{1}{2}(1^+)$	2421.4
B [2]	/	$\frac{1}{2}(0^-)$	5279.4
B^* [2]	/	$\frac{1}{2}(1^-)$	5325.2
N [2]	/	$\frac{1}{2}(\frac{1}{2}^+)$	938.9
Λ [2]	/	$0(\frac{1}{2}^+)$	1115.7
$a_0(980)$ [2]	$\bar{K}K$ [40–42,61]	$1^-(0^+)$	980
$f_0(980)$ [2]	$\bar{K}K$ [40–42,61]	$0^+(0^+)$	990
$D^*_{s0}(2317)$ [36,37]	DK [43,44,47]	$0(0^+)$	2317.8
$D_{s1}(2460)$ [36,37]	D^*K [43,44,47]	$0(1^+)$	2459.6
$D^*_{s1}(2700)$ [59]	D_1K (?)	$0(1^-)$	2709
$X(3872)$ [20]	$\frac{1}{\sqrt{2}}\left(\bar{D}^*D + D^*\bar{D}\right)$ [58]	$0^+(1^+)$	3871.7
$Z_c(3900)$ [22–24]	$\frac{1}{\sqrt{2}}\left(\bar{D}^*D + D^*\bar{D}\right)$ [49,50]	$1^+(1^+)$	3899.0
$Z_1(4051)$ [32]	$-\bar{D}^*D^*$ [51,52]	$1^-(1^+)$	4051
$Z_2(4250)$ [32]	$\frac{1}{\sqrt{2}}\left(\bar{D}_1D - D_1\bar{D}\right)$ [51,53]	$1^-(1^-)$	4248
$Z(4430)$ [29–31]	$\frac{1}{\sqrt{2}}\left(\bar{D}^*D_1 + D^*\bar{D}_1\right)$ [51,54]	$1^+(1^-)$	4443
$Z_b(10610)$ [33]	$\frac{1}{\sqrt{2}}\left(\bar{B}^*B + B^*\bar{B}\right)$ [55]	$1^+(1^+)$	10608.4
$Z'_b(10650)$ [33]	$\bar{B}^*B^*$ [55]	$1^+(1^+)$	10653.2
$\Lambda(1405)$ [2]	$\bar{K}N$ [56,57]	$0(\frac{1}{2}^-)$	1405.1
$^3S_1$ NN PW (deuteron)	NN	$0(1^+)$	1875.6
$^1S_0$ NN PW	NN	$1(0^-)$	/
$^3S_1$ ΛN PW [38,39]	ΛN	$\frac{1}{2}(1^+)$	/
$^1S_0$ ΛN PW [38,39]	ΛN	$\frac{1}{2}(0^-)$	/
$^1S_0$ $\Lambda\Lambda$ PW (?)	$\Lambda\Lambda$ [175]	$0(0^+)$	/

Table 4.1: *Considered hadrons and their two-body molecule states (PW: partial wave). The isospin averaged mass is the mean one of the iso-multiplet members. Note, that most of the molecule wave functions are hypotheses (their first proposal is referenced); especially, the interpretation of $D^*_{s1} = KD_1$ and the existence of the H-dibaryon $\Lambda\Lambda$ are very hypothetical as indicated by the question marks. Note also, the remarks on the applicability of EFT($\not\pi$) in section 4.3. For each particle the reference for its discovery and properties is given in the first column.*

4.1.1 Three nucleon system

In the NN system it is known that there is one $^3S_1$ partial wave bound state with isospin 0 (the deuteron) and a $^1S_0$ partial wave virtual state being an iso-triplet. Since the three particles in the system are identical their allocation to the generic particles P_1, P_2 and P_3 is obviously $P_i = N$ for $i = 1, 2, 3$. The deuteron will be assigned to the dimer d_{12} and the virtual $^1S_0$ state to d'_{12} with quantum numbers (cf. Tab. 4.1)

$$
\begin{aligned}
I\left(J^P\right)(d_{12}) &= 0(1^+)\,, \\
I\left(J^P\right)(d'_{12}) &= 1(0^-)\,.
\end{aligned}
\tag{4.1}
$$

Keeping in mind that nucleons have both spin and isospin given by $1/2$ one concludes that one only has – according to the rules in the box on page 51 – to deal with T_{12} and T'_{12}. Since all particles have the same mass we consider a type 1 system where the matrix $\mathcal{A}_1$ is a 2×2 matrix with elements

$$
(\mathcal{A}_1)_{11} = \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{S_{12}\, c_{12}} \left[x_2\, \delta^{(12)}_{P_1 P_3}\, f^{(3)}_{(12)(12)}\, S_{123} + \widetilde{x}_2\, \delta^{(12)}_{P_2 P_3}\, \widetilde{f}^{(3)}_{(12)(12)}\, S_{213}\right],
$$

$$
(\mathcal{A}_1)_{21} = \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{S_{12}\, \sqrt{c'_{12}\, c_{12}}} \left[x'_2\, \delta^{(12)}_{P_1 P_3}\, f^{(3)}_{(12)(12')}\, S_{123} + \widetilde{x}'_2\, \delta^{(12)}_{P_2 P_3}\, \widetilde{f}^{(3)}_{(12)(12')}\, S_{213}\right],
$$

$$
(\mathcal{A}_1)_{12} = \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{S_{12}\, \sqrt{c_{12}\, c'_{12}}} \left[x_5\, \delta^{(12')}_{P_1 P_3}\, f^{(3)}_{(12')(12)}\, S_{123} + \widetilde{x}_5\, \delta^{(12')}_{P_2 P_3}\, \widetilde{f}^{(3)}_{(12')(12)}\, S_{213}\right],
$$

$$
(\mathcal{A}_1)_{22} = \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{S_{12}\, c'_{12}} \left[x'_5\, \delta^{(12')}_{P_1 P_3}\, f^{(3)}_{(12')(12')}\, S_{123} + \widetilde{x}'_5\, \delta^{(12')}_{P_2 P_3}\, \widetilde{f}^{(3)}_{(12')(12')}\, S_{213}\right].
$$

The particle allocation yields that $a_1 = a_2 = a_3 = 1$, but all b's are zero and consequently the functions f and $\widetilde{f}$ return "1" independently of their arguments. Furthermore, all Kronecker-deltas are 1 and the symmetry factors $S_{12} = 2$ and $S_{123} = S_{213} = -4$ can be found according to appendix B. The remaining spin and isospin dependent factors x are determined using one of the methods explained in appendix A. Depending on spin (S) and isospin (I) channel one finds using for example the 6-J symbols in Eqs. (A.111 - A.146) with $j_1 = j_2 = i_1 = i_2 = 1/2$, $J_{12} = I'_{12} = 1$ and $J'_{12} = I_{12} = 0$ the following values:

$$
x_2 = \widetilde{x}_2 = (-1)^{2(J+I)} \times 3 \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & S & 1 \end{Bmatrix} \times 1 \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & I & 0 \end{Bmatrix} = \begin{cases} -\frac{1}{4}\,, & \text{for } S = \frac{1}{2} \ \& \ I = \frac{1}{2} \\ \frac{1}{2}\,, & \text{for } S = \frac{3}{2} \ \& \ I = \frac{1}{2} \end{cases},
$$

$$
x'_2 = \widetilde{x}'_2 = (-1)^{2(J+I)} \times \sqrt{3} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & S & 1 \end{Bmatrix} \times \sqrt{3} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & I & 0 \end{Bmatrix} = \begin{cases} \frac{3}{4}\,, & \text{for } S = \frac{1}{2} \ \& \ I = \frac{1}{2} \\ 0\,, & \text{for } S = \frac{3}{2} \ \& \ I = \frac{1}{2} \end{cases},
$$

$$
x_5 = \widetilde{x}_5 = (-1)^{2(J+I)} \times \sqrt{3} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & S & 0 \end{Bmatrix} \times \sqrt{3} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & I & 1 \end{Bmatrix} = \begin{cases} \frac{3}{4}\,, & \text{for } S = \frac{1}{2} \ \& \ I = \frac{1}{2} \\ 0\,, & \text{for } S = \frac{3}{2} \ \& \ I = \frac{1}{2} \end{cases},
$$

$$
x'_5 = \widetilde{x}'_5 = (-1)^{2(J+I)} \times 1 \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & S & 0 \end{Bmatrix} \times 3 \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & I & 1 \end{Bmatrix} = \begin{cases} -\frac{1}{4}\,, & \text{for } S = \frac{1}{2} \ \& \ I = \frac{1}{2} \\ 0\,, & \text{for } S = \frac{3}{2} \ \& \ I = \frac{1}{2} \end{cases}.
$$

All in all the matrix $\mathcal{A}_1$ reduces to

$$\mathcal{A}_1^{S=\frac{1}{2},I=\frac{1}{2}} = \begin{pmatrix} \frac{1}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} , \tag{4.2}$$

in the spin and isospin $1/2$ channel (doublet channel) and to $\mathcal{A}_1^{S=\frac{3}{2},I=\frac{1}{2}} = \lambda_1^{S=\frac{3}{2},I=\frac{1}{2}} = -1/2$ in the quartet channel with $S = \frac{3}{2}$. The eigenvalues in the former case are $\lambda_{1,2}^{S=\frac{1}{2},I=\frac{1}{2}} = 2, -1$. Plugging them into the transcendental equation for the S-wave scaling parameter $s_i^{(0)}$ (Eq. (3.118)) one obtains that only in the doublet channel $s_1^{(0)} = 1.00624\,i$ is indeed purely imaginary. Hence, the Efimov effect is present in the system. This result perfectly coincides with the work done in Refs. [80, 92].

4.1.2 $\bar{K}KK$ system

In the second detailed example we consider the system $\bar{K}KK$. From experiment we know that there are two candidates for hadronic molecules in this system (cf. Tab. 4.1): $a_0(980)$ and $f_0(980)$. Both can be interpreted as a $\bar{K}K$ state, but with different quantum numbers [2]:

$$\begin{aligned} I^G\left(J^{PC}\right)(a_0) &= 1^-(0^{++}) \\ I^G\left(J^{PC}\right)(f_0) &= 0^+(0^{++}) . \end{aligned} \tag{4.3}$$

The kaons K themselves have $I(J^P) = 1/2(0^-)$. To answer the question whether or not an Efimov trimer exists in the system we will analyze a_0–K scattering. Namely, following the rules described in the previous section we allocate the three kaons to the generic three particles P_1, P_2 and P_3 as follows:

$$\begin{aligned} P_1 &= \bar{K} , \\ P_2 &= P_3 = K . \end{aligned} \tag{4.4}$$

Because of $P_2 = P_3$ we have to consider just two dimers (cf. rules in the box on page 51): d_{12} and d'_{12}. Since all three particles in the system have the same mass we consider the type 1 results, i.e. we have to find the matrix $\mathcal{A}_1$ and diagonalize it. From the allocation above we can directly read off almost all parameters needed to determine the elements of $\mathcal{A}_1$. Indeed, Eq. (4.4) sets $b_1 = a_2 = a_3 = 1$ and $a_1 = b_2 = b_3 = 0$. Furthermore, we have

$$\begin{aligned} \delta_{P_1 P_2} &= \delta_{P_1 P_3} = 0 , \\ \delta_{P_2 P_3} &= 1 , \\ \delta_{A_1 A_2} &= \delta_{A_1 A_3} = \delta_{A_2 A_3} = 1 , \end{aligned} \tag{4.5}$$

and thus

$$\delta_{P_i P_j}^{\left(ab^{(\prime)}\right)} \equiv \delta_{P_i P_j} , \quad \forall\, a < b \in \{1, 2, 3\} . \tag{4.6}$$

It is sufficient to calculate $v_{12}^{(\prime)}$ and $w_{12}^{(\prime)}$ since all elements with other indices are erased from $\mathcal{A}_1$. With G-parity quantum numbers $\eta_{12} = -1$ and $\eta'_{12} = +1$ we find

$$v_{12} = 1 \times \left[\frac{1}{\sqrt{2}} + 1 \times \left(1 - \frac{1}{\sqrt{2}} \right) \right] (1 \times 1 + 0 \times 0) \times (-1)^1 + (1 - 1) \times 1 \times 1 = -1 , \quad (4.7)$$

$$v'_{12} = +1 , \quad (4.8)$$

$$w_{12} = w'_{12} = 0 , \quad (4.9)$$

which is needed to determine f and $\widetilde{f}$. Also the symmetry factors $S_{12} = 1$ and $S_{123} = 2$ as well as $S_{213} = 1$ are a consequence of the particle allocation in Eq. (4.4). Obviously, there are no fermion minus signs (i.e. $\zeta_{123} = \zeta_{213} = +1$) since all three particles are bosons. The 2×2 matrix $\mathcal{A}_1$ is thus given by

$$\mathcal{A}_1 = \begin{pmatrix} \dfrac{\widetilde{x}_2}{c_{12}} & -\dfrac{\widetilde{x}_5}{\sqrt{c_{12}c'_{12}}} \\[2ex] -\dfrac{\widetilde{x}'_2}{\sqrt{c_{12}c'_{12}}} & \dfrac{\widetilde{x}'_5}{c'_{12}} \end{pmatrix} . \quad (4.10)$$

The remaining four spin and isospin dependent parameters $\widetilde{x}_2$, $\widetilde{x}_5$, $\widetilde{x}'_2$ and $\widetilde{x}'_5$ can be calculated using one of the methods explained in appendix A. All methods have in common that they use normalized projection operators and hence we know $c_{12} = c'_{12} = 1$. This holds independently of the scattering channel. However, the x parameters are of course different for different spin and isospin. While the spin channel is uniquely determined to be 0 the isospin one can be either $1/2$ or $3/2$. In order to find the x parameters we use in this example their defining equations in Eqs. (A.43 - A.52). Plugging in the right operator of Tab. A.2 to Tab. A.7 according to the spin and isospin quantum numbers of a_0, f_0, K and choose one of the two scattering channels one has to calculate in the spin 0 & isospin $1/2$ channel:

$$\widetilde{x}_2 = \frac{1}{2} \frac{-1}{\sqrt{3}} (\tau_g)_{\lambda\sigma} \frac{i}{\sqrt{2}} (\tau_m \tau_2)_{\rho\sigma} \frac{-i}{\sqrt{2}} (\tau_2 \tau_g)_{\nu\rho} \frac{-1}{\sqrt{3}} (\tau_m)_{\nu\lambda} = -\frac{1}{2} ,$$

$$\widetilde{x}'_2 = \frac{1}{2} \delta_{\lambda\sigma} \frac{i}{\sqrt{2}} (\tau_m \tau_2)_{\rho\sigma} \frac{-i}{\sqrt{2}} (\tau_2)_{\nu\rho} \frac{-1}{\sqrt{3}} (\tau_m)_{\nu\lambda} = \frac{\sqrt{3}}{2} ,$$

$$\widetilde{x}_5 = \frac{1}{2} \frac{-1}{\sqrt{3}} (\tau_g)_{\lambda\sigma} \frac{i}{\sqrt{2}} (\tau_2)_{\rho\sigma} \frac{-i}{\sqrt{2}} (\tau_2 \tau_g)_{\nu\rho} \delta_{\nu\lambda} = \frac{\sqrt{3}}{2} ,$$

$$\widetilde{x}'_5 = \frac{1}{2} \delta_{\lambda\sigma} \frac{i}{\sqrt{2}} (\tau_2)_{\rho\sigma} \frac{-i}{\sqrt{2}} (\tau_2)_{\nu\rho} \delta_{\nu\lambda} = \frac{1}{2} , \quad (4.11)$$

and in the other possible channel with isospin $3/2$ the task is to determine

$$\widetilde{x}_2 = \frac{1}{4} \frac{1}{3} \left[(\tau_g \tau_\ell)_{\lambda\sigma} + \delta_{g\ell} \delta_{\lambda\sigma} \right] \frac{i}{\sqrt{2}} (\tau_m \tau_2)_{\rho\sigma} \frac{-i}{\sqrt{2}} (\tau_2 \tau_g)_{\nu\rho} \frac{1}{3} \left[(\tau_\ell \tau_m)_{\nu\lambda} + \delta_{\ell m} \delta_{\nu\lambda} \right] = 1 , \quad (4.12)$$

since $\widetilde{x}'_2 = \widetilde{x}_5 = \widetilde{x}'_5 = 0$ because the isoscalar f_0 cannot contribute to the second channel. The numbers above are obtained using the well-known identities for the Pauli matrices summarized in appendix A.

The matrix $\mathcal{A}_1$ in the spin 0 & isospin 1/2 channel is given by

$$\mathcal{A}_1^{S=0,I=\frac{1}{2}} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} , \qquad (4.13)$$

and its eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 1$ so that the transcendental equation for $L = 0$ in Eq. (3.118) yields

$$s_1^{(0)} = 2 \qquad (4.14)$$
$$s_2^{(0)} = 0.413697 \, i \, . \qquad (4.15)$$

Since one solution is indeed purely imaginary we deduce that the Efimov effect is present, but its scaling factor $\exp(i\pi/s_2^{(0)}) = 1986.14$ is rather large.

In the $I = 3/2$ channel one obtains just a single number, i.e. an 1×1 matrix $\mathcal{A}_1^{S=0,I=\frac{3}{2}} = 1 = \lambda_1$. For S-wave scattering Eq. (3.118) has a purely imaginary solution $s_1^{(0)} = 0.413697i$ so the Efimov effect is again present with the same scaling factor of 1986.14 as for isospin 1/2 and we conclude that there must be in either channel a $\bar{K}KK$ trimer state in the system. As a remark, note that even if the $a_0(980)$ is not a molecule and hence the $f_0(980)$ (whose molecular substructure is more established [174]) is the only remaining dimer in the three kaon system, the scaling factor of 1986.14 is unchanged.

4.1.3 Summary of other established systems

After those two detailed examples we will now summarize the results for all physically promising combinations of molecules and single particles in Tab. 4.1. Namely, for all three particle systems where at least two of the three particles have a bound or virtual state. Depending on the considered particles we either apply the rules for type 1 or type 2 system. However, the in section 3 derived methods do not provide a straightforward analysis for the following kind of systems: a system of two particles with (approximately) equal mass and a third particle with different mass where the equal mass ones have a resonant interaction (i.e. a bound or virtual state) cannot be analyzed. Hence, we will use the mass difference parameters $\varepsilon_{ij} := (m_i - m_j)/(m_i + m_j)$ introduced in Eq. (3.84) to classify systems where the mass difference is still small enough so that type 1 system equations are applicable. Therefore one has to compare in each relevant system the mass difference parameter and the effective range of the corresponding molecule. The contribution of the latter is already neglected as we have only considered leading order terms in the effective range expansion. Unfortunately, in almost all two-body systems the effective range is not known (one exception is the NN system with the deuteron as bound state). Thus, we have to assume that the corrections from setting $\varepsilon_{ij} \approx 0$ are indeed at most of the order of the effective range corrections. This should be a justified assumption as it can be seen by comparison with Ref. [167] where the mass difference between nucleons and Λ particles ($\varepsilon_{N,\Lambda} \approx 0.1$) is also neglected.

In the same way as described above for the NNN and $\bar{K}KK$ system one finds for all other systems the matrices which needs to be diagonalized in order to find their eigenvalues λ_i. We plug these eigenvalues into the transcendental equation for the S- and P-wave scaling parameter

$s_i^{(0,1)}$ given in Eqs. (3.118, 3.119) for type 1 or in Eqs. (3.131, 3.137) for type 2 systems. Note, however, that it is shown in Refs. [80, 172, 173] that for symmetry reasons some boson/fermion configurations can only occur in even or odd partial wave channels, respectively. For instance in a three boson system the Efimov effect can only be present in the S-wave channel.

In Tab. 4.2 we have summarized all systems where the Efimov effect occurs. One observes that – concerning fully bosonic systems – only in the $\bar{K}KK$ and in the scattering of charm mesons off the not so well-established KD_1 molecule $D_{s1}^*(2700)$ the Efimov effect is present. Is the molecule in the latter scattering process a $D_{s0}^*(2317)$ or a $D_{s1}(2460)$ instead one finds no Efimov trimer since the mass ratio $\varepsilon_{12} = \varepsilon_{K,D^{(*)}}$ is not large enough to push the factor in front of the trigonometric functions in the transcendental equation for $s^{(0)}$ above a critical value so that $s^{(0)}$ becomes purely imaginary. Also in all (according to Tab. 4.1) possible combinations of three charm mesons or of three bottom mesons (both analyzed using the type 1 scheme since the mass difference is reasonable small) there is no Efimov effect. The conditions under which such meson–meson molecules would be affected by Efimov physics will be discussed in the next section.

Next, we focus on systems consisting of fermions only. Besides the already known results concerning the NNN [80, 92] and $NN\Lambda$ system [167] one finds evidence for Efimov trimers in the scattering of a nucleon off a H-dibaryon ($\Lambda\Lambda$) which was suggested by Jaffe in Ref. [175]. A further extension with more dibaryon states like $N\Sigma$, $N\Xi$, $\Lambda\Sigma$, $\Lambda\Xi$, $\Sigma\Sigma$, $\Sigma\Xi$, $\Xi\Xi$ or even with Ω combinations could be possible, but there is up to now no experimental evidence for such dimers. However, we will discus some issues concerning them in the next section.

As a third alternative there could be molecules consisting of a baryon and a meson. An example for such a two-body system is the $\Lambda(1405)$ lying just below the $\bar{K}N$ threshold so that its interpretation as a virtual kaon–nucleon state is justified (see also Refs. [176, 177]). The two corresponding three particle states are thus $N\bar{K}\bar{K}$ and $\bar{K}NN$. The former fits into the type 2 system scheme, but is not affected by Efimov physics. Unfortunately, the latter cannot be analyzed because the two equal mass particles (the nucleons) can form a bound ($^3S_1$ partial wave) and a virtual ($^1S_0$ partial wave) state. However, if one chooses one specific isospin channel, namely the one corresponding to the system K^-pp, the type 2 method is applicable to $\bar{K}NN$ since the two protons have no dimer state. This gives us the opportunity to check whether the theoretically predicted [178–180], but not yet experimentally confirmed [181, 182] K^-pp three particle bound state can be interpreted as Efimov trimer in the scattering of a proton off a $\Lambda(1405)$. Indeed, we found in the $\bar{K}NN$ isospin 1/2 channel with $I_3 = +1/2$ fixed that there is an Efimov effect which gives rise to a K^-pp trimer. Its existence would thus support the interpretation of the $\Lambda(1405)$ as hadronic molecule.

system	channel	scaling parameter $\mathrm{Im}\left(\mathbf{s^{(0)}}\right)$	scaling factor $\exp\left(\frac{\mathbf{i}\pi}{\mathbf{s^{(0)}}}\right)$
$\bar{K}KK$	$S=0$ & $I=\frac{1}{2}$	0.413697	1986.14
	$S=0$ & $I=\frac{3}{2}$	0.413697	1986.14
KD_1D (?)	$S=1$ & $I=\frac{1}{2}$	0.231624	777104.0
KD_1D^* (?)	$S=0$ & $I=\frac{1}{2}$	0.231624	777104.0
	$S=1$ & $I=\frac{1}{2}$	0.231624	777104.0
	$S=2$ & $I=\frac{1}{2}$	0.231624	777104.0
KD_1D_1 (?)	$S=0$ & $I=\frac{1}{2}$	0.231624	777104.0
	$S=1$ & $I=\frac{1}{2}$	0	/
	$S=2$ & $I=\frac{1}{2}$	0.231624	777104.0
NNN	$S=\frac{1}{2}$ & $I=\frac{1}{2}$	1.00624	22.69
	$S=\frac{1}{2}$ & $I=\frac{3}{2}$	0	/
	$S=\frac{3}{2}$ & $I=\frac{1}{2}$	0	/
$NN\Lambda$	$S=\frac{1}{2}$ & $I=0$	1.00624	22.69
	$S=\frac{1}{2}$ & $I=1$	1.00624	22.69
	$S=\frac{3}{2}$ & $I=0$	1.00624	22.69
	$S=\frac{3}{2}$ & $I=1$	0	/
$N\Lambda\Lambda$	$S=\frac{1}{2}$ & $I=\frac{1}{2}$	1.00624	22.69
	$S=\frac{3}{2}$ & $I=\frac{1}{2}$	0	/
K^-pp	$S=0$ & $I=\frac{1}{2}$	0.605355	179.41
	$S=1$ & $I=\frac{1}{2}$	0.605355	179.41

Table 4.2: *Overview of the considered three particle systems according to Tab. 4.1 in which the Efimov effect is present. Shown is the imaginary part of the (in this case purely imaginary) S-wave scaling parameter $s^{(0)}$ and the corresponding scaling factor $\exp\left(\frac{i\pi}{s^{(0)}}\right)$. Note, that there is no P-wave Efimov trimer because of symmetry reason such a state can only be present for specific boson / fermion configuration as it is shown in Refs. [80, 172, 173]. The question mark indicates that the $D_{s1}^* = KD_1$ molecule interpretation is* very *hypothetical.*

4.2 Hypothetical systems

After our analysis of a large number of more or less established molecules scattering off a third particle we now consider some hypothetical two-body bound states in order to identify the spin and isospin quantum numbers which would lead to Efimov trimers in corresponding scattering processes. For all these systems we assume that one indeed can treat them in EFT($\not{\pi}$). Namely, we assume that their binding momenta are at most of the order of the pion mass so that one can obtain at least some first estimates (see also section 4.3). We will focus on three different systems: firstly, one can assume that besides the known $\bar{D}D$ ($\bar{B}B$) charm and bottom mesons also DD (BB) bound states exist (a possible explanation for the fact that they are not yet discovered at one of the B-factories would be that one needs at least four charm or bottom mesons since they are produced in anti-particle–particle pairs). Secondly, one can consider the known charm and bottom molecules, but change their quantum numbers to other values. Thirdly, one can consider the large number of hypothetical dibaryon bound states. However, all these two-body systems have in common that – choosing an appropriate third particle scattering off the molecule – the corresponding three-body systems consists of identical particles or at least of multiplet–anti-multiplet combinations. Therefore the number of parameters in our type 1 system analyses reduces and additionally the remaining ones are more restricted.

4.2.1 Hypothetical charm and bottom meson systems

Assuming that there is a shallow dimer in the system of two charm or bottom mesons, respectively, one can consider the systems DDD, $D^*D^*D^*$ or $D_1D_1D_1$ in the charm sector and BBB, $B^*B^*B^*$ or $B_1B_1B_1$ in the bottom meson sector. In each system one deals with identical particles. According to the rules in the box on page 51 one thus has to erase the amplitudes $\widetilde{T}_{13}$ and $\widetilde{T}_{23}$ from type 1 system equations. The matrix $\mathcal{A}_1$ defined in Eqs. (3.99 - 3.104) is thus reduced to a 2×2 matrix $\mathcal{A}_1^{\mathrm{id}}$ whose elements are:

$$
\left(\mathcal{A}_1^{\mathrm{id}}\right)_{11} = \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{S_{12}\, c_{12}} \left[x_2\, \delta_{P_1 P_3}^{(12)}\, f_{(12)(12)}^{(3)}\, S_{123} + \widetilde{x}_2\, \delta_{P_2 P_3}^{(12)}\, \widetilde{f}_{(12)(12)}^{(3)}\, S_{213} \right],
$$

$$
\left(\mathcal{A}_1^{\mathrm{id}}\right)_{21} = \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{S_{12}\, \sqrt{c_{12}'\, c_{12}}} \left[x_2'\, \delta_{P_1 P_3}^{(12)}\, f_{(12)(12')}^{(3)}\, S_{123} + \widetilde{x}_2'\, \delta_{P_2 P_3}^{(12)}\, \widetilde{f}_{(12)(12')}^{(3)}\, S_{213} \right],
$$

$$
\left(\mathcal{A}_1^{\mathrm{id}}\right)_{12} = \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{S_{12}\, \sqrt{c_{12}\, c_{12}'}} \left[x_5\, \delta_{P_1 P_3}^{(12')}\, f_{(12')(12)}^{(3)}\, S_{123} + \widetilde{x}_5\, \delta_{P_2 P_3}^{(12')}\, \widetilde{f}_{(12')(12)}^{(3)}\, S_{213} \right],
$$

$$
\left(\mathcal{A}_1^{\mathrm{id}}\right)_{22} = \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{S_{12}\, c_{12}'} \left[x_5'\, \delta_{P_1 P_3}^{(12')}\, f_{(12')(12')}^{(3)}\, S_{123} + \widetilde{x}_5'\, \delta_{P_2 P_3}^{(12')}\, \widetilde{f}_{(12')(12')}^{(3)}\, S_{213} \right]. \tag{4.16}
$$

From this result it is straightforward to deduce the entries of $\mathcal{A}_1^{\mathrm{id}}$ if there are more than two different spin/isospin configurations, that is, if there are dimers d_{12}'', d_{12}''', etc. with other quantum numbers. But firstly, we notice that all (normal and modified) Kronecker-deltas yield 1 for three identical particles $P_1 = P_2 = P_3$. Furthermore, $f_{(..)(..)}^{(.)} = \widetilde{f}_{(..)(..)}^{(.)} = 1$ independently of the arguments since we have $a_i = 1$ for $i = 1, 2, 3$. Also the symmetry factor $S_{12} = 2$ is already fixed.

Only the second symmetry factor $S_{123} = S_{213} = \pm 4$ is not unique; it is positive in the case that the three identical particles are bosons, but negative for fermions. One can thus write for an arbitrary number of dimer states:

$$\left(\mathcal{A}_1^{\text{id}}\right)_{ab} = \pm \frac{1}{\sqrt{c_{12}^{[(a-1)\prime]} c_{12}^{[(b-1)\prime]}}} \left[x_{2b+(b-1)}^{[(a-1)\prime]} + \widetilde{x}_{2b+(b-1)}^{[(a-1)\prime]} \right], \tag{4.17}$$

where the plus (minus) sign is for bosons (fermions) and the notation $[(a-1)\prime]$ has to be understood in the manner of $[3\ \prime] \,\widehat{=}\, \prime\prime\prime$ and so on. In order to get rid of the remaining spin/isospin parameters we use the 6-J symbol notation explained in appendix A.3. From Eqs. (A.111 - A.146) we deduce with $\delta_{P_1 P_2} = 1$ that the parameters $x_{2b+(b-1)}^{[(a-1)\prime]}$ and $\widetilde{x}_{2b+(b-1)}^{[(a-1)\prime]}$ are equal and given by

$$x_{2b+(b-1)}^{[(a-1)\prime]} = \widetilde{x}_{2b+(b-1)}^{[(a-1)\prime]} = (-1)^{2S} \sqrt{\left(2J_{12}^{[(a-1)\prime]} + 1\right)\left(2J_{12}^{[(b-1)\prime]} + 1\right)} \begin{Bmatrix} j_1 & j_1 & J_{12}^{[(a-1)\prime]} \\ j_1 & S & J_{12}^{[(b-1)\prime]} \end{Bmatrix}$$

$$\times (-1)^{2I} \sqrt{\left(2I_{12}^{[(a-1)\prime]} + 1\right)\left(2I_{12}^{[(b-1)\prime]} + 1\right)} \begin{Bmatrix} i_1 & i_1 & I_{12}^{[(a-1)\prime]} \\ i_1 & I & I_{12}^{[(b-1)\prime]} \end{Bmatrix}, \tag{4.18}$$

with j_1 (i_1) being the spin (isospin) of the identical particles $P_1 = P_2 = P_3$ and S (I) is the considered spin (isospin) channel. Since we have derived the equation above using normalized spin and isospin projectors we know that the factors c_{12} are equal to 1 independently of the number of primes. Hence, Eq. (4.17) can be written as

$$\left(\mathcal{A}_1^{\text{id}}\right)_{ab} = \pm 2(-1)^{2(S+I)} \sqrt{\left(2J_{12}^{[(a-1)\prime]} + 1\right)\left(2J_{12}^{[(b-1)\prime]} + 1\right)} \sqrt{\left(2I_{12}^{[(a-1)\prime]} + 1\right)\left(2I_{12}^{[(b-1)\prime]} + 1\right)}$$

$$\times \begin{Bmatrix} j_1 & j_1 & J_{12}^{[(a-1)\prime]} \\ j_1 & S & J_{12}^{[(b-1)\prime]} \end{Bmatrix} \begin{Bmatrix} i_1 & i_1 & I_{12}^{[(a-1)\prime]} \\ i_1 & I & I_{12}^{[(b-1)\prime]} \end{Bmatrix}. \tag{4.19}$$

Consequently, the existence of the Efimov effect is completely determined by the spin and isospin structure of the particles and the dimers. The question is now: what do we know about these degrees of freedom? Firstly, we know from Tab. 4.1 that all charm and bottom mesons are isospin $1/2$ particles and secondly, the spin is either 0 for D, B or 1 for D^*, D_1, B^*, B_1. Thus, one concludes that on the one hand both pseudoscalar charm (DDD) and bottom (BBB) meson systems behave exactly the same and on the other hand that the remaining vector ($D^*D^*D^*$, $B^*B^*B^*$) and axialvector ($D_1D_1D_1$, $B_1B_1B_1$) meson systems are equivalent in the sense of Efimov physics. To find the spin and isospin configurations leading to an Efimov effect it is thus necessary to identify all possible two-body quantum numbers J_{12} and I_{12} of the dimers. From the constituent particles we conclude that for all systems $i_1 \otimes i_1 = \mathbf{1} \oplus \mathbf{3}$, i.e. that there are two molecule states: one with $I_{12} = 1$ and one with $I_{12}' = 0$. Concerning spin one has to distinguish between D or B systems and D^*, B^*, D_1 or B_1 systems. All in all one finds the combinations shown in Tab. 4.3 where also the matrix $\mathcal{A}_1^{\text{id}}$ is given. Calculating the eigenvalues for both systems and for all channels one concludes that for all spin and isospin configurations except for the $S = 0$ & $I = 3/2$ channel of the (axial-)vector system there is always at least one eigenvalue

$\lambda^{\text{id}} = 2$. Plugging this into the transcendental equation for the S-wave scaling parameter $s^{(0)}$, Eq. (3.118),

$$1 = \frac{4\lambda^{\text{id}}}{\sqrt{3}} \frac{1}{s^{(0)}} \frac{\sin\left(\frac{\pi}{6} s^{(0)}\right)}{\cos\left(\frac{\pi}{2} s^{(0)}\right)} \ ,$$

one finds that $s^{(0)} = 1.00624 \, i$ is a solution. Hence, a system of three identical charm or bottom mesons behaves in all channels – except for the mentioned $S = 0$ & $I = 3/2$ channel of the (axial-)vector system – like a three identical boson system with spin- and isospinless particles. This statement is only true if all theoretically possible two-body spin and isospin configurations indeed exist in nature. Considering for example the DDD or BBB pseudoscalar systems one observes that in the case of only a $I_{12}(J_{12}) = 0(0)$ dimer the system has a scaling factor of $s^{(0)} = 0.413697 \, i$. In the case of only an $I_{12}(J_{12}) = 1(0)$ dimer one finds $s^{(0)} = 1.00624 \, i$ in the $S = 0$, $I = 3/2$ channel, but there is no Efimov effect in the other allowed $S = 0$, $I = 1/2$ channel. Since in the (axial-)vector system of charm and bottom mesons the number of present / missing two-body configurations is large we will not discuss all of them for every scattering channel. Instead, we give some qualitative results concerning the existence of the Efimov effect:

Qualitative results for hypothetical identical charm and bottom meson systems

DDD and BBB systems:

- **One or two dimers:** in a system of three identical pseudoscalar charm or bottom mesons the Efimov effect is always present in at least one scattering channel independently of the number and of the quantum numbers of the dimer or the two dimers, respectively.

$D^*D^*D^*$, $D_1D_1D_1$, $B^*B^*B^*$ and $B_1B_1B_1$ systems:

- **Exactly one dimer:** in a three identical spin 1 charm or bottom meson system the Efimov effect does *not* occur if this dimer has the quantum numbers $I_{12}(J_{12}) = 0(0)$, $1(0)$ or $0(1)$. For the other allowed configurations ($1(1)$, $0(2)$ and $1(2)$) one always finds at least one scattering channel where Efimov physics are relevant.

- **Exactly two dimers:** in such a system the Efimov effect does *not* occur if the two dimers have the quantum numbers $I_{12}(J_{12}) = 0(0)$ and $I'_{12}(J'_{12}) = 1(0)$ or vice versa. All other combinations lead in at least one scattering channel to an Efimov trimer in the system.

- **More than two dimers:** for every configuration of dimers and quantum numbers there will be an Efimov trimer in at least one scattering channel.

The conclusion is thus that – if these kind of molecules indeed exist in nature – it is very likely that the Efimov effect is an important property of such a system of three identical charm or bottom meson systems.

j_1	dimer QN	channel QN	matrix $\mathcal{A}_1^{\mathrm{id}}$
$j_1 = 0$	$J_{12} = 0$ & $I_{12} = 0$ $J'_{12} = 0$ & $I'_{12} = 1$	$S = 0$ & $I = \frac{1}{2}$	$\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$
		$S = 0$ & $I = \frac{3}{2}$	2
$j_1 = 1$	$J_{12} = 0$ & $I_{12} = 0$ $J'_{12} = 0$ & $I'_{12} = 1$ $J''_{12} = 1$ & $I''_{12} = 0$ $J'''_{12} = 1$ & $I'''_{12} = 1$ $J''''_{12} = 2$ & $I''''_{12} = 0$ $J'''''_{12} = 2$ & $I'''''_{12} = 1$	$S = 0$ & $I = \frac{1}{2}$	$\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$
		$S = 0$ & $I = \frac{3}{2}$	-2
		$S = 1$ & $I = \frac{1}{2}$	$\begin{pmatrix} \frac{1}{3} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 1 & \frac{\sqrt{5}}{3} & -\frac{\sqrt{5}}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{3} & 1 & \frac{1}{\sqrt{3}} & -\frac{\sqrt{5}}{\sqrt{3}} & -\frac{\sqrt{5}}{3} \\ -\frac{1}{\sqrt{3}} & 1 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & \frac{\sqrt{5}}{\sqrt{12}} & -\frac{\sqrt{5}}{2} \\ 1 & \frac{1}{\sqrt{3}} & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & -\frac{\sqrt{5}}{2} & -\frac{\sqrt{5}}{\sqrt{12}} \\ \frac{\sqrt{5}}{3} & -\frac{\sqrt{5}}{\sqrt{3}} & \frac{\sqrt{5}}{\sqrt{12}} & -\frac{\sqrt{5}}{2} & \frac{1}{6} & -\frac{1}{\sqrt{12}} \\ -\frac{\sqrt{5}}{\sqrt{3}} & -\frac{\sqrt{5}}{3} & -\frac{\sqrt{5}}{2} & -\frac{\sqrt{5}}{\sqrt{12}} & -\frac{1}{\sqrt{12}} & -\frac{1}{6} \end{pmatrix}$
		$S = 1$ & $I = \frac{3}{2}$	$\begin{pmatrix} \frac{2}{3} & -\frac{2}{\sqrt{3}} & \frac{2\sqrt{5}}{3} \\ -\frac{2}{\sqrt{3}} & 1 & \frac{\sqrt{5}}{\sqrt{3}} \\ \frac{2\sqrt{5}}{3} & \frac{\sqrt{5}}{\sqrt{3}} & \frac{1}{3} \end{pmatrix}$
		$S = 2$ & $I = \frac{1}{2}$	$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & \frac{3}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & \frac{3}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{3}{2} & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{3}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$
		$S = 2$ & $I = \frac{3}{2}$	$\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$
		$S = 3$ & $I = \frac{1}{2}$	$\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$
		$S = 3$ & $I = \frac{3}{2}$	2

Table 4.3: *All possible spin and isospin configurations (QN: quantum numbers) for the pseudoscalar ($j_1 = 0$) meson systems DDD, BBB and for the three spin $j_1 = 1$ charm and bottom meson systems $D^*D^*D^*$, $D_1D_1D_1$, $B^*B^*B^*$, $B_1B_1B_1$. The isospin of the identical particles $P_1 = P_2 = P_3$ is always $i_1 = 1/2$ and not explicitly given.*

4.2.2 More dimers in existing charm and bottom meson systems

After the discussion of the completely hypothetical three identical meson systems in the previous subsection we will now consider the experimentally confirmed states in Tab. 4.1. We assume that there are more bound states in the underlying two-body molecule systems with different spin and isospin configurations. This is motivated by the fact that the processes in which the molecules are produced in experiments may favor some or even forbid other configurations because of symmetry. However, this argument makes sense only for (axial-)vector charm and bottom mesons, i.e. D^*, D_1, B^* and B_1 because for the molecules with one pseudoscalar meson instead we have already checked in section 4.1.3 that this does not lead to an Efimov effect. This was explicitly done for the charm molecules $X(3872)$ and $Z_c(3900)$ and thus we know that – a not yet discovered – iso-singlet partner of the $Z_b(10610)$ would not change the situation in the bottom sector. This is clear because except for their masses all relevant quantum numbers for Efimov physics of D and B as well as of D^* and B^* are equal. Since there are no $\bar{D}_1 D_1$ or $\bar{B}_1 B_1$ dimers known so far we restrict ourselves to the vector charm and bottom mesons which behave in the sense of the Efimov effect completely the same. Hence, it will be sufficient to analyze only one system; the result will be valid for both $\bar{D}^* D^*$ and $\bar{B}^* B^*$.

Consequently, we consider the three particle system $P_1 P_2 P_3$ with $P_1 = \bar{A}_1$, $P_2 = P_3 = A_1$, where A_1 is an isospin $i_1 = 1/2$ and spin $j_1 = 1$ field (corresponding to both D^* and B^*). A two-body bound state between $\bar{A}_1$ and A_1 can thus have the following spin and isospin quantum numbers:

$$
\begin{array}{ccc}
J_{12} = 0 \ \& \ I_{12} = 0, & J'_{12} = 0 \ \& \ I'_{12} = 1, & J''_{12} = 1 \ \& \ I''_{12} = 0, \\
J'''_{12} = 1 \ \& \ I'''_{12} = 1, & J''''_{12} = 2 \ \& \ I''''_{12} = 0, & J'''''_{12} = 2 \ \& \ I'''''_{12} = 1 .
\end{array}
$$

Although these are the same combinations as in the previous section on three identical mesons one has to keep in mind that the situation is different. On the one hand particles P_1 and P_2 are not identical and on the other hand not all particles have a large scattering length (we do not consider a hypothetical $A_1 A_1$ dimer in contrast to what was done above). Indeed, our particle allocation tells us that $b_1 = a_2 = a_3 = 1$ and $\eta_{12} = \pm 1$ for all numbers of primes. Therefore one finds $v_{12} = \eta_{12}$ and $f^{(3)}_{(12)^{(l)}(12)^{[l]}} = \tilde{f}^{(3)}_{(12)^{(l)}(12)^{[l]}} = a_3 v^{(l)}_{12} v^{[l]}_{12} = \eta^{(l)}_{12} \eta^{[l]}_{12}$ which also holds for all numbers and combinations of primes. Concerning the modified Kronecker-deltas one gets $\delta^{(12)}_{P_1 P_3} = 0$ and $\delta_{P_2 P_3} = 1$ and similarly for primed symbols. Finally, the relevant symmetry factors are $S_{12} = 1$, $S_{123} = +2$ and $S_{213} = +1$ as we deal with bosons. All together and with the "multiple prime" notation from the previous section 4.2.1 this leads to a matrix $\mathcal{A}_1^{\bar{A}AA}$ whose elements are given by

$$
\begin{aligned}
\left(\mathcal{A}_1^{\bar{A}AA} \right)_{ab} &= \tilde{x}^{[(a-1)\prime]}_{2b+(b-1)} \, \eta^{[(a-1)\prime]}_{12} \, \eta^{[(b-1)\prime]}_{12} \\
&= \eta^{[(a-1)\prime]}_{12} \, \eta^{[(b-1)\prime]}_{12} \, (-1)^{2(S+I)} (-1)^{J^{[(a-1)\prime]}_{12} + J^{[(b-1)\prime]}_{12} + I^{[(a-1)\prime]}_{12} + I^{[(b-1)\prime]}_{12} - 4 - 2} \\
&\quad \times \sqrt{\left(2J^{[(a-1)\prime]}_{12} + 1 \right) \left(2J^{[(b-1)\prime]}_{12} + 1 \right)} \sqrt{\left(2I^{[(a-1)\prime]}_{12} + 1 \right) \left(2I^{[(b-1)\prime]}_{12} + 1 \right)} \\
&\quad \times \begin{Bmatrix} j_1 & j_1 & J^{[(a-1)\prime]}_{12} \\ j_1 & S & J^{[(b-1)\prime]}_{12} \end{Bmatrix} \begin{Bmatrix} i_1 & i_1 & I^{[(a-1)\prime]}_{12} \\ i_1 & I & I^{[(b-1)\prime]}_{12} \end{Bmatrix} .
\end{aligned} \tag{4.20}
$$

Here, Eqs. (A.111 - A.146) were used to replace the spin and isospin dependent $\widetilde{x}$ parameters by 6-J symbols. Besides a minus sign from the additional phase factor and/or the G-parity quantum numbers in front, the matrix elements are half as large as those in the identical charm or bottom meson system (cf. Eq. (4.19)). Hence, one could make a similar table like Tab. 4.3 in order to deduce that although some elements indeed change their sign the eigenvalues does not. However, they are divided by two, i.e. $\lambda^{\text{id}} = \lambda^{\bar{A}AA}/2$. In Refs. [80,166] it is shown that

$$1 = \frac{4\lambda}{\sqrt{3}} \frac{1}{s^{(0)}} \frac{\sin\left(\frac{\pi}{6} s^{(0)}\right)}{\cos\left(\frac{\pi}{2} s^{(0)}\right)}$$

has a purely imaginary solution (i.e. the Efimov effect occurs) if the eigenvalue λ is larger than a critical value $\lambda_C = \frac{3\sqrt{3}}{2\pi} \approx 0.826993$. Therefore the extra factor of $1/2$ compared to the eigenvalues in $D^*D^*D^*$ or $B^*B^*B^*$ systems is an important difference. Indeed, one observes that the qualitative results concerning the existence of the Efimov effect are changed in comparison to the previously discussed case:

Qualitative results for extended $\bar{\text{D}}^*\text{D}^*$ or $\bar{\text{B}}^*\text{B}^*$ hadronic molecules

- **Exactly one dimer:** in a $\bar{D}^*D^*D^*$ or $\bar{B}^*B^*B^*$ system the only dimer quantum numbers for which the Efimov effect is present (exclusively in the $S = 3$, $I = 3/2$ channel) are $I_{12}(J_{12}) = 1(2)$. For all other configurations $(0(0), 1(0), 0(1), 1(1)$ and $0(2))$ Efimov physics are *not* relevant.

- **Exactly two dimers:** an Efimov trimer can be found if the two dimers have the quantum numbers $I_{12}(J_{12}) = 1(1)$ and $I'_{12}(J'_{12}) = 0(1)$, $1(0)$ or vice versa. In all other cases there is *no* Efimov effect.

- **Exactly three dimers:** in such a system one finds an Efimov trimer in at least one scattering channel for the dimers with all possible quantum numbers except for $I_{12}(J_{12}) = 0(0)$, $I'_{12}(J'_{12}) = 0(1)$ and $I''_{12}(J''_{12}) = 0(2)$ or any permutation of these regarding the primes.

- **More than three dimers:** for all dimer configurations and for all quantum numbers the Efimov effect will be present in at least one scattering channel.

As it is written in Tab. 4.1 we know that both $Z'_b = \bar{B}^*B^*$ and its charm analog $Z_1 = \bar{D}^*D^*$ have the quantum numbers $I(J) = 1(1)$. Therefore one concludes that as soon as an additional $\bar{D}^*D^*$ or $\bar{B}^*B^*$ state with either $I'(J') = 0(1)$ or $1(0)$ or $1(2)$ is discovered, Efimov physics should be taken into account in Z_1–D^* and Z'_b–B^* scattering.

4.2.3 Hypothetical dibaryons

Now we focus on fermionic systems: besides the H-dibaryon called $\Lambda\Lambda$ bound state which was predicted by Jaffe in Ref. [175] there are much more candidates for dibaryon states like $\Sigma\Sigma$ or $\Xi\Xi$, but also in the charmed or bottom baryons sector there could be states like $\Lambda_c\Lambda_c$. See for example Refs. [183,184] for a theoretical discussion of some of the candidates and Refs. [185,186] for lattice calculations regarding dibaryons with Ω's. Thus, it may be worth the effort to analyze three identical fermion systems like $\Sigma\Sigma\Sigma$, $\Xi\Xi\Xi$ and so on. One has to deal with a system of identical particles which can be described using the type 1 method. Moreover, the corresponding matrix $\mathcal{A}_1$ is already known from section 4.2.1 where we have derived it for three identical particles independently of their species:

$$
\left(\mathcal{A}_1^{\mathrm{id}}\right)_{ab} = \pm 2(-1)^{2(S+I)} \sqrt{\left(2J_{12}^{[(a-1)\prime]}+1\right)\left(2J_{12}^{[(b-1)\prime]}+1\right)} \sqrt{\left(2I_{12}^{[(a-1)\prime]}+1\right)\left(2I_{12}^{[(b-1)\prime]}+1\right)}
$$
$$
\times \begin{Bmatrix} j_1 & j_1 & J_{12}^{[(a-1)\prime]} \\ j_1 & S & J_{12}^{[(b-1)\prime]} \end{Bmatrix} \begin{Bmatrix} i_1 & i_1 & I_{12}^{[(a-1)\prime]} \\ i_1 & I & I_{12}^{[(b-1)\prime]} \end{Bmatrix} , \tag{4.21}
$$

where the plus sign is valid for bosons and the minus sign for fermions. All dibaryon–baryon scattering processes can thus be described using the relation above with a minus sign in front. However, the spin and isospin quantum numbers are less restricted than in the three charm or bottom meson case in section 4.2.1. Independently of additional charm or bottom quarks (see for example Ref. [2]) $\Lambda_{(c,b)}$ ground state baryons have the quantum numbers $I(J) = 0(1/2)$, $\Sigma_{(c,b)}$ ground state baryons have $I(J) = 1(1/2)$ and $\Xi_{(c,b)}$ ground states have spin and isospin $I(J) = 1/2(1/2)$. Only Ω baryons have different quantum numbers if one strange quark is replaced by a charm or bottom one. Namely, for the ground state one has $I(J)\,(\Omega) = 0(3/2)$, but $I(J)\,(\Omega_{c,b}) = 0(1/2)$. Hence, we notice that the $\Omega_{c,b}\,\Omega_{c,b}\,\Omega_{c,b}$ system is in the sense of Efimov physics equivalent to the $\Lambda_{(c,b)}\,\Lambda_{(c,b)}\,\Lambda_{(c,b)}$ system. Additionally, we deduce that the spin and isospin quantum numbers of the three $\Sigma_{(c,b)}$ system are interchanged compared to the $D^*D^*D^*$, $B^*B^*B^*$, etc. systems. Consequently, the eigenvalues in the latter fermionic system have the same modulus, but the opposite sign as in the bosonic system. In the following we summarize the qualitative results in the same way as it was done before:

Qualitative results for hypothetical dibaryons

$\Lambda_{(c,b)}\Lambda_{(c,b)}\Lambda_{(c,b)}$ and $\Omega_{c,b}\Omega_{c,b}\Omega_{c,b}$ systems:

- **Exactly one dimer:** if spin and isospin of the dimer are $I_{12}(J_{12}) = 0(0)$ one finds *no* Efimov trimer while for $I_{12}(J_{12}) = 0(1)$ the Efimov effect occurs in the $S = 1/2$, $I = 0$ channel.

- **Exactly two dimers:** if both allowed dimer states with quantum numbers $I_{12}(J_{12}) = 0(0)$ and $I'_{12}(J'_{12}) = 0(1)$ are present in the system, Efimov physics are relevant in the $S = 1/2$, $I = 0$ scattering channel (cf. $\Lambda\Lambda\Lambda$ system in Tab. 4.2).

$\Sigma_{(c,b)}\Sigma_{(c,b)}\Sigma_{(c,b)}$ systems:

- **Exactly one dimer:** for the dimer quantum numbers $I_{12}(J_{12}) = 1(0)$, $1(1)$ and $2(1)$ there will be at least one scattering channel with an Efimov trimer while for dimers with $I_{12}(J_{12}) = 0(0)$, $0(1)$ and $2(0)$ Efimov physics are *not* relevant.

- **Exactly two dimers:** except for the two dimers with quantum numbers $I_{12}(J_{12}) = 0(0)$ and $I'_{12}(J'_{12}) = 0(1)$, $2(0)$ or the dimers with $I_{12}(J_{12}) = 0(1)$ and $I'_{12}(J'_{12}) = 2(0)$, one finds for all other combinations at least one channel where the Efimov effect occurs.

- **More than two dimers:** for every possible number of dimers larger than two there is always at least one scattering channel with an Efimov trimer.

$\Xi_{(c,b)}\Xi_{(c,b)}\Xi_{(c,b)}$ systems:

- **Exactly one dimer:** only for a dimer with quantum numbers $I_{12}(J_{12}) = 1(1)$ the Efimov effect is present in the $S = 1/2$, $I = 3/2$ and in the $S = 3/2$, $I = 1/2$ channel. For the three other possible dimers one finds *no* Efimov effect.

- **More than one dimer:** for an arbitrary number of dimers larger than two one finds independently of the quantum numbers at least one scattering channel with an Efimov trimer.

$\Omega\Omega\Omega$ systems:

- **Exactly one dimer:** for a dimer with spin and isospin $I_{12}(J_{12}) = 0(0)$ one finds *no* Efimov trimer, but for all other dimers with quantum numbers $I_{12}(J_{12}) = 0(1)$, $0(2)$ or $0(3)$ the Efimov effect occurs in the $S = 1/2$, $S = 3/2$ or $S = 7/2$ scattering channel (with $I = 0$ being unique).

- **More than one dimer:** if there is more than one dimer in the system Efimov physics are relevant in at least one scattering channel independently of the dimer quantum numbers.

In conclusion, one deduces that Efimov physics are an important phenomenon in dibaryon–baryon scattering and if dibaryon molecules are found in experiments it would be promising to search for three particle bound states in order to clarify their substructure.

4.3 Summary of the results

In this last part of the current section we will summarize and give some remarks concerning the results above. First of all we must once again emphasize that the substructure of many of the considered particles is not clear. Hence, their interpretation as hadronic molecules is just a hypotheses. However, if this interpretation is correct their treatment in a non-relativistic pionless effective field theory is justified as long as their binding momenta are smaller than the pion mass or at least of that order. Calculating the binding momentum $\gamma_{ij} = \text{sig}(B_{ij})\sqrt{2\mu_{ij}|B_{ij}|}$ using the binding energy determined with the relation $B_{ij} = (m_i + m_j) - M_{d_{ij}}$ one observes that the

mentioned condition is on the one hand clearly fulfilled for a number of particles in Tab. 4.1 like the $X(3872)$ with binding momentum $\gamma_X \sim 15\,\text{MeV}$. On the other hand there are also particles like the $Z_c(3900)$ whose binding momentum $|\gamma_{Z_c}| \sim 211\,\text{MeV}$ is even larger than the pion mass. However, especially due to the often large uncertainties regarding the masses of the molecule candidates it should be justified to use EFT($\not{\pi}$) also for the "problematic" particles in order to obtain at least some first insights to the respective molecule–particle scattering. In the same way we assumed that the binding momenta of the considered hypothetical states are also at most of the order of m_π so that their treatment in EFT($\not{\pi}$) is possible. Of course, future experiments could prove this assumption wrong. As a remark, note that one could add pions to the theory to get a more accurate description of the physical system like it was done for the $X(3872)$ using the so-called XEFT in Refs. [187,188]. Regarding the Efimov effect we found that only a rather small number of molecule–particle scattering processes is affected by an intermediate Efimov trimer (see Tab. 4.2). However, experimental setups possibly restrict the allowed quantum numbers of a possible molecular state. Thus, there might be a reasonable chance that more states with different quantum numbers exist in nature. Depending on the exact spin and isospin of these additional molecules we have shown that the Efimov effect might become important (see box "Qualitative results for extended $\bar{D}^*D^*D^*$ or $\bar{B}^*B^*B^*$ hadronic molecules" on page 85). Moreover, there might exist charm or bottom meson molecules with identical constituents (DDD, $B^*B^*B^*$, etc.) for which we have checked that their scattering off a third (identical) particle is most likely affected by Efimov physics. In fact, only if there is just one dimer with $I_{12}(J_{12}) = 0(0), 1(0), 0(1)$ this will not be the case (see box "Qualitative results for hypothetical identical charm and bottom meson systems" on page 82). From this point of view it would be very interesting to search for more molecule states of the "anti-charm/bottom meson – charm/bottom meson" type and for the completely new "charm/bottom meson – charm/bottom meson" states. While for the first type the existing e^+e^- collider B-factories are not favorable due to symmetry reasons which forbid or suppress the production of other quantum numbers, they are in principle suitable for the latter type. However, one would need rather high energies since the charm and bottom mesons are produced in particle–anti-particle pairs. Consequently, one would need four bottom mesons in total in order to observe e.g. a BB molecule. Furthermore, we have also found that dibaryon systems are – with just a few exceptional quantum numbers – affected by the Efimov effect (see box "qualitative results for hypothetical dibaryons" on page 86). Hence, more experimental effort to discover such states could be worthwhile.

A common feature in all cases is that experimentally observing a molecule–particle scattering process which is affected by an Efimov trimer would strongly support the molecule interpretation of the accordant particle. Moreover, discovering even a second Efimov trimer with binding energy $(\exp(i\pi/s))^2$ times larger could be seen as almost a proof of the molecular nature of the corresponding particle. However, it is not clear whether there are observable second trimer states at all since the Efimov spectrum is cut off at a certain point as explained in section 2.1.

An important remark concerning this work is that – although we exclusively have focused on them – it is not restricted to hadronic molecules. The derivation in section 3 was completely general and hence one can apply the presented methods to *every* particle scattering off a S-wave dimer as long as either for an arbitrary number of dimer states all masses are (approximately) equal (type 1 system) or for $m_1 \neq m_2 = m_3$ if there is *no* d_{23} dimer (type 2 system). Not

analyzable with our method are systems with three different particle masses and systems where exactly two of the three particles have equal mass (and the third an unequal one), but where the equal mass particles also have a dimer state. However, apart from these subtleties the derived matrices $\mathcal{A}_1$ (type 1) and $\mathcal{A}_2$ (type 2) are also valid in the physics of cold atoms or halo nuclei.

As a final remark, we want to emphasize that in all three-body systems of equal mass which we have discussed, the maximal eigenvalue was $\lambda = 2$ corresponding to a scaling parameter of $s = 1.00624\, i$. Thus, it seems to be the case that there is some kind of minimal scaling factor $\exp(\pi/1.00624)^2 \approx 515.03$ in Efimov physics. A further analysis in order to proof or falsify this conjecture and if it is true to check its possible extension to the unequal mass system should be worthwhile.

Chapter 5

Elastic molecule–particle scattering observables

As it was shown in the previous chapter the up to now known charm and bottom meson molecules are not affected by Efimov physics if they are scattered off a third charm or bottom meson. However, the absence of Efimov trimers makes the respective scattering processes elastic. Therefore one can extract from the given S-wave three-body scattering amplitude the scattering length and the phase shift as three-body observables.

At this point it is useful to recapitulate the cutoff independence of these observables: in chapter 2.1 it was explained that the existence of a three-body bound state is in some sense independent of the exact value of the three-body coupling constant $H(\Lambda)$ which thus can be set to zero (according to the argument that there always exists a cutoff Λ_0 so that $H(\Lambda_0) = 0$). However, some of the observables are – directly or indirectly – affected by a three-body bound state and therefore via H cutoff dependent. This means that if a three particle bound state is found, the three-body binding energy B_3 and scattering length a_3 are functions of Λ. This dependence can only be removed using an experimentally measured value of either B_3 or a_3 to fix $H(\Lambda)$ which then allows a prediction of the respective other. In contrast, if there is no three-body bound state then one concludes that there is no three-body force at all. For energies $|E| < \Lambda$ the remaining scattering observables are thus not cutoff dependent. In particular, this fact allows us to treat molecule–particle scattering at negative energies below the dimer threshold as a two-body scattering problem (see section 5.2.1). *With* an existing three-body bound state the situation would change. According to the Efimov effect such a state would lie below the two-body binding energy. Thus, the system of a molecule and a particle could form such an Efimov trimer and hence their scattering would not be elastic anymore.

As a selected example we will discuss the elastic scattering of B and B^* meson off the two molecules $Z_b(10610)$ and $Z_b'(10650)$ in more detail. All other scattering processes without Efimov trimer, i.e. all elastic processes, could be analyzed in the same way.

5.1 Elastic S-wave $Z_b^{(\prime)}$–$B^{(*)}$ scattering

In total there are four different scattering processes where the bottom mesons B and B^* are involved: following Tab. 4.1 one has to consider Z_b–B, Z_b–B^*, Z_b'–B and Z_b'–B^* scattering. Consequently, one has to deal with the three particle systems $B\bar{B}^*B$, $B\bar{B}^*B^*$, $B^*\bar{B}^*B$ and $B^*\bar{B}^*B^*$ which will be discussed below. Regarding the notation we will use m and m_* for the masses of B, $\bar{B}$ and B^*, $\bar{B}^*$ mesons, respectively. Furthermore, μ (γ) is used for the reduced mass (the binding momentum) of the molecule $Z_b(10610)$ and μ' (γ') for the reduced mass (the binding momentum) of $Z_b'(10650)$.

5.1.1 Z_b–B scattering amplitude

According to the method derived in chapter 3 we allocate the particles P_i as follows: $P_1 = P_3 = B$ and $P_2 = \bar{B}^*$. Hence, we deduce $a_1 = b_2 = a_3 = 1$ as well as

$$\delta_{P_1 P_2} = \delta_{P_2 P_3} = \delta_{A_1 A_2} = \delta_{A_2 A_3} = 0 \,,$$
$$\delta_{P_1 P_3} = \delta_{A_1 A_3} = 1 \,. \tag{5.1}$$

Furthermore (cf. Tab. 4.1), the G-parity quantum number is $\eta_{12} = +1$. Firstly, there is no resonant interaction with large scattering length between the two B mesons and secondly, the two dimers d_{12} and d_{23} are identical. Thus, one has to – according to the rules in the box on page 51 – erase the latter from the equations. Therefore it remains just one dimer in the system (cf. Eq. (3.13)),

$$d_{12} = \frac{1}{\sqrt{2}} \left(\bar{B}^*B + B^*\bar{B} \right) \,, \tag{5.2}$$

which represents the molecule $Z_b(10610)$ with $I^G(J^P) = 1^+(1^+)$. The only remaining amplitude is thus $T_{12}^{(L)}$ and since we are interested in the scattering process one has to consider the amplitude in Eq. (F.4) where the asymptotic momentum limit is not yet applied. For S-wave scattering one finds

$$
T_{12}^{(0)}(E,k,p) = \frac{2\pi\gamma_{12}}{\mu_{12}^2 S_{12} c_{12}}
$$
$$
\times \left[x_1\, \delta_{P_1 P_3}^{(12)}\, f_{(12)(12)}^{(3)}\, S_{123} \frac{m_2}{kp} Q_0^{211}(k,p;E) + \tilde{x}_1\, \delta_{P_2 P_3}^{(12)}\, \tilde{f}_{(12)(12)}^{(3)}\, S_{213} \frac{m_1}{kp} Q_0^{122}(k,p;E) \right]
$$
$$
+ \frac{1}{\pi} \frac{1}{\mu_{12} S_{12} c_{12}} \int_0^\infty dq \frac{q^2\, T_{12}^{(0)}(E,k,q)}{-\gamma_{12} + \sqrt{-2\mu_{12}\left(E - \frac{q^2}{2m_3} - \frac{q^2}{2(m_1+m_2)} \right)} - i\varepsilon}
$$
$$
\times \left[x_2\, \delta_{P_1 P_3}^{(12)}\, f_{(12)(12)}^{(3)}\, S_{123} \frac{m_2}{qp} Q_0^{211}(q,p;E) + \tilde{x}_2\, \delta_{P_2 P_3}^{(12)}\, \tilde{f}_{(12)(12)}^{(3)}\, S_{213} \frac{m_1}{qp} Q_0^{122}(q,p;E) \right] . \tag{5.3}
$$

Using the summary on pages 56 and 57 all other parameters in $T_{12}^{(0)}$ are obtained to be given as

$$\delta_{P_1 P_3}^{(12)} = 1 \,, \qquad\qquad \delta_{P_2 P_3}^{(12)} = 0 \,,$$

$$v_{12} = \tfrac{1}{\sqrt{2}} \,, \qquad\qquad w_{12} = \tfrac{1}{\sqrt{2}} \,,$$

$$f_{(12)(12)}^{(3)} = a_3 w_{12} w_{12} = \tfrac{1}{2} \,, \qquad\qquad \widetilde{f}_{(12)(12)}^{(3)} = a_3 v_{12} v_{12} = \tfrac{1}{2} \,,$$

$$S_{12} = 1 \,, \qquad\qquad c_{12} = 1 \,,$$

$$S_{123} = 1 \,, \qquad\qquad S_{213} = 2 \,,$$

$$\text{spin 1 \& isospin 1/2 channel:} \quad x_1 = x_2 = -\tfrac{1}{2} \,, \qquad\qquad \widetilde{x}_1 = \widetilde{x}_2 = +\tfrac{1}{2} \,,$$

$$\text{spin 1 \& isospin 3/2 channel:} \quad x_1 = x_2 = +1 \,, \qquad\qquad \widetilde{x}_1 = \widetilde{x}_2 = -1 \,,$$

where we have used in the last two lines Eqs. (A.111, A.129) together with the fact that B is a pseudoscalar and B^* is vector isospin doublet so that the allowed channels in Z_b–B scattering are spin 1 and either isospin 1/2 or isospin 3/2. Plugging the results into Eq. (5.3) yields

$$
T_{12}^{(0)}(E,k,p) = \left\{ \begin{matrix} -\frac{1}{2} \\ 1 \end{matrix} \right\} \pi \, \frac{\gamma_{12}\, m_2}{\mu_{12}^2} \frac{1}{kp} \, Q_0^{211}(k,p;E)
$$
$$
+ \left\{ \begin{matrix} -\frac{1}{2} \\ 1 \end{matrix} \right\} \frac{1}{2\pi} \frac{m_2}{\mu_{12}} \int_0^\infty dq \frac{q^2\, T_{12}^{(0)}(E,k,q)\, \frac{1}{qp}\, Q_0^{211}(q,p;E)}{-\gamma_{12} + \sqrt{-2\mu_{12}\left(E - \frac{q^2}{2m_3} - \frac{q^2}{2(m_1+m_2)}\right)} - i\varepsilon} \,, \tag{5.4}
$$

where the upper entry corresponds to the isospin 1/2 and the lower one to the isospin 3/2 channel. One can now reinsert the short-hand notation for the Legendre function of the second kind defined in Eq. (3.73) and additionally use Eq. (D.10) which relates it to the logarithm (for the behavior in the limit $\varepsilon \to 0$ see Eq. (D.15) which was derived in appendix D) in order to find the representation of the amplitude below:

$$
T_{12}^{(0)}(E,k,p) = \left\{ \begin{matrix} -\frac{1}{2} \\ 1 \end{matrix} \right\} \pi \, \frac{\gamma\, m_*}{\mu^2} \frac{1}{2kp} \ln\left[\frac{\frac{k^2}{2\mu} + \frac{p^2}{2\mu} - E + \frac{kp}{m_*} - i\varepsilon}{\frac{k^2}{2\mu} + \frac{p^2}{2\mu} - E - \frac{kp}{m_*} - i\varepsilon} \right]
$$
$$
+ \left\{ \begin{matrix} -\frac{1}{2} \\ 1 \end{matrix} \right\} \frac{1}{2\pi} \frac{m_*}{\mu} \int_0^\infty dq \frac{q^2\, T_{12}^{(0)}(E,k,q)}{-\gamma + \sqrt{-2\mu\left(E - \frac{q^2}{2m} - \frac{q^2}{2(m+m_*)}\right)} - i\varepsilon}
$$
$$
\times \frac{1}{2pq} \ln\left[\frac{\frac{q^2}{2\mu} + \frac{p^2}{2\mu} - E + \frac{pq}{m_*} - i\varepsilon}{\frac{q^2}{2\mu} + \frac{p^2}{2\mu} - E - \frac{pq}{m_*} - i\varepsilon} \right] \,. \tag{5.5}
$$

Note, that the upper (lower) component still represents isospin 1/2 (isospin 3/2) as before and that we have replaced m_2 by m_*, $m_{1,3}$ by m, μ_{12} by μ being the reduced mass of the $Z_b(10610)$ and γ_{12} by γ being its binding momentum.

5.1.2 Z_b–B^* scattering amplitude

The second process we want to analyze is S-wave Z_b–B^* scattering. Hence, the scalar iso-doublet B is replaced by the vector B^*. The corresponding three particle system $B\bar{B}^*B^*$ is allocated to

the generic particles P_i as follows: $P_1 = B$, $P_2 = \bar{B}^*$ and $P_3 = B^*$. Thus, $a_1 = b_2 = a_3 = 1$ as before, but in contrast to the Z_b–B case it holds

$$\delta_{P_1 P_2} = \delta_{P_1 P_3} = \delta_{P_2 P_3} = \delta_{A_1 A_2} = \delta_{A_1 A_3} = 0 \,,$$
$$\delta_{A_2 A_3} = 1 \,. \tag{5.6}$$

While both two-body systems $B\bar{B}^*$ and $B^*\bar{B}^*$ are G-parity eigenstates (leading to $\delta_{\eta_{12}|1|} = \delta_{\eta_{23}|1|} = 1$), the combination BB^* is not and hence $\delta_{\eta_{13}|1|} = 0$. According to Eq. (3.13) there are three different dimers in the system:

$$d_{12} = \frac{1}{\sqrt{2}} \left(\bar{B}^* B + \eta_{12} B^* \bar{B} \right) \,, \tag{5.7}$$
$$d_{13} = BB^* \,, \tag{5.8}$$
$$d_{23} = \eta_{23} \bar{B}^* B^* \,. \tag{5.9}$$

From Tab. 4.1 we deduce that only $d_{12} \,\widehat{=}\, Z_b(10610)$ and $d_{23} \,\widehat{=}\, Z_b'(10650)$ are present in nature. Thus, one can on the one hand insert the quantum numbers $I^G(J^P) = 1^+(1^+)$ for both states and on the other hand erase all d_{13} contributions. Since there are no states with other quantum numbers than $1^+(1^+)$ one has to erase all primed amplitudes, too (see page 51). The coupled scattering amplitudes $T_{12}^{(L)}$ and $T_{23}^{(L)}$ are given by the S-wave projected versions of Eq. (F.4) and Eq. (F.6) without $T_{13}^{(L)}$ and without primed amplitudes $T_{ij}'^{(L)}$. Before we write them down we note that also the modified Kronecker-deltas are zero except for $\delta_{P_2 P_3}^{(12)} = 1$ and hence we find

$$
\begin{aligned}
T_{12}^{(0)}(E,k,p) = {} & \frac{2\pi\,\gamma_{12}\,\widetilde{x}_1\,\widetilde{f}_{(12)(12)}^{(3)}\,S_{213}}{\mu_{12}^2\,S_{12}\,c_{12}}\,\frac{m_1}{kp}\,Q_0^{122}(k,p;E) \\
& + \frac{1}{\pi}\frac{\widetilde{x}_2\,\widetilde{f}_{(12)(12)}^{(3)}\,S_{213}}{\mu_{12}\,S_{12}\,c_{12}}\int_0^\infty dq\,\frac{q^2\,T_{12}^{(0)}(E,k,q)\,\frac{m_1}{qp}\,Q_0^{122}(q,p;E)}{-\gamma_{12}+\sqrt{-2\mu_{12}\left(E-\frac{q^2}{2m_3}-\frac{q^2}{2(m_1+m_2)}\right)}-i\varepsilon} \\
& + \frac{1}{\pi}\frac{\widetilde{x}_4\,\widetilde{f}_{(23)(12)}^{(3)}\,S_{321}}{\mu_{12}\,\sqrt{S_{12}\,S_{23}}\,c_{12}\,c_{23}}\int_0^\infty dq\,\frac{q^2\,T_{23}^{(0)}(E,k,q)\,\frac{m_2}{qp}\,Q_0^{213}(q,p;E)}{-\gamma_{23}+\sqrt{-2\mu_{23}\left(E-\frac{q^2}{2m_1}-\frac{q^2}{2(m_2+m_3)}\right)}-i\varepsilon} \,,
\end{aligned}
\tag{5.10}
$$

and

$$
\begin{aligned}
T_{23}^{(0)}(E,k,p) = {} & \frac{2\pi\,\gamma_{12}\,z_1\,f_{(12)(23)}^{(1)}\,S_{123}}{\mu_{23}\,\mu_{12}\,\sqrt{S_{23}\,S_{12}}\,c_{23}\,c_{12}}\,\frac{m_2}{kp}\,Q_0^{231}(k,p;E) \\
& + \frac{1}{\pi}\frac{z_2\,f_{(12)(23)}^{(1)}\,S_{123}}{\mu_{23}\,\sqrt{S_{23}\,S_{12}}\,c_{23}\,c_{12}}\int_0^\infty dq\,\frac{q^2\,T_{12}^{(0)}(E,k,q)\,\frac{m_2}{qp}\,Q_0^{231}(q,p;E)}{-\gamma_{12}+\sqrt{-2\mu_{12}\left(E-\frac{q^2}{2m_3}-\frac{q^2}{2(m_1+m_2)}\right)}-i\varepsilon} \,.
\end{aligned}
\tag{5.11}
$$

The three particles P_1, P_2 and P_3 are distinguishable bosons in the Z_b–B^* system. Hence, all symmetry factors are equal to 1 independently of their indices. Using the summarized definitions on page 56 one obtains for the f and $\widetilde{f}$ factors

$$\widetilde{f}^{(3)}_{(12)(12)} = \frac{1}{2}\,, \qquad \widetilde{f}^{(3)}_{(23)(12)} = \frac{1}{\sqrt{2}}\,, \qquad f^{(1)}_{(12)(23)} = \frac{1}{\sqrt{2}}\,. \tag{5.12}$$

The remaining $\widetilde{x}$ and z parameters are spin and isospin dependent. Depending on the respective channel the results of appendix A.3 lead to the values summarized in Tab. 5.1. Since normalized projectors are considered in the mentioned method it is directly implied that all factors c_{ij} are equal to one. One observes that in the spin 0 and 2 channels only the amplitude $T^{(0)}_{12}$ contributes while for spin 1 both amplitudes are part of a coupled system. In the former case one finds

$$T^{(0)}_{12}(E,k,p) = \begin{Bmatrix} -\frac{1}{2} \\ 1 \end{Bmatrix} \pi \frac{\gamma\, m}{\mu^2} \frac{1}{2pk} \ln \left[\frac{\frac{k^2}{2\mu} + \frac{p^2}{2\mu'} - E + \frac{pk}{m} - i\varepsilon}{\frac{k^2}{2\mu} + \frac{p^2}{2\mu'} - E - \frac{pk}{m} - i\varepsilon} \right]$$
$$+ \begin{Bmatrix} -\frac{1}{2} \\ 1 \end{Bmatrix} \frac{1}{2\pi} \frac{m}{\mu} \int_0^\infty dq \frac{q^2\, T^{(0)}_{12}(E,k,q)}{-\gamma + \sqrt{-2\mu \left(E - \frac{q^2}{2m_*} - \frac{q^2}{2(m+m_*)} \right)} - i\varepsilon}$$
$$\times \frac{1}{2pq} \ln \left[\frac{\frac{q^2}{2\mu} + \frac{p^2}{2\mu'} - E + \frac{pq}{m} - i\varepsilon}{\frac{q^2}{2\mu} + \frac{p^2}{2\mu'} - E - \frac{pq}{m} - i\varepsilon} \right]\,, \tag{5.13}$$

where the upper components represent the spin 0 or spin 2 channel with isospin $1/2$ and the lower ones are valid for spin 0 or spin 2 with isospin $3/2$. Furthermore, the short-hand notation for the Legendre function was replaced according to Eq. (3.73) and Eq. (D.10) as in the previous section. The coupled integral equation system for the spin 1 channel is then given by

$$T^{(0)}_{12}(E,k,p) = \begin{Bmatrix} \frac{1}{2} \\ -1 \end{Bmatrix} \pi \frac{\gamma\, m}{\mu^2} \frac{1}{2pk} \ln \left[\frac{\frac{k^2}{2\mu} + \frac{p^2}{2\mu'} - E + \frac{pk}{m} - i\varepsilon}{\frac{k^2}{2\mu} + \frac{p^2}{2\mu'} - E - \frac{pk}{m} - i\varepsilon} \right]$$
$$+ \begin{Bmatrix} \frac{1}{2} \\ -1 \end{Bmatrix} \frac{1}{2\pi} \frac{m}{\mu} \int_0^\infty dq \frac{q^2\, T^{(0)}_{12}(E,k,q)}{-\gamma + \sqrt{-2\mu \left(E - \frac{q^2}{2m_*} - \frac{q^2}{2(m+m_*)} \right)} - i\varepsilon}$$
$$\times \frac{1}{2pq} \ln \left[\frac{\frac{q^2}{2\mu} + \frac{p^2}{2\mu'} - E + \frac{pq}{m} - i\varepsilon}{\frac{q^2}{2\mu} + \frac{p^2}{2\mu'} - E - \frac{pq}{m} - i\varepsilon} \right]$$
$$+ \begin{Bmatrix} -\frac{1}{2} \\ 1 \end{Bmatrix} \frac{1}{\sqrt{2}\pi} \frac{m_*}{\mu} \int_0^\infty dq \frac{q^2\, T^{(0)}_{23}(E,k,q)}{-\gamma' + \sqrt{-2\mu' \left(E - \frac{q^2}{2m} - \frac{q^2}{2(m_*+m_*)} \right)} - i\varepsilon}$$
$$\times \frac{1}{2pq} \ln \left[\frac{\frac{q^2}{2\mu} + \frac{p^2}{2\mu} - E + \frac{pq}{m_*} - i\varepsilon}{\frac{q^2}{2\mu} + \frac{p^2}{2\mu} - E - \frac{pq}{m_*} - i\varepsilon} \right]\,, \tag{5.14}$$

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$$T_{23}^{(0)}(E,k,p) = \begin{Bmatrix} -\frac{1}{2} \\ 1 \end{Bmatrix} \sqrt{2}\pi \frac{\gamma \, m_*}{\mu' \, \mu} \frac{1}{2pk} \ln \left[\frac{\frac{k^2}{2\mu'} + \frac{p^2}{2\mu} - E + \frac{pk}{m_*} - i\varepsilon}{\frac{k^2}{2\mu'} + \frac{p^2}{2\mu} - E - \frac{pk}{m_*} - i\varepsilon} \right]$$

$$+ \begin{Bmatrix} -\frac{1}{2} \\ 1 \end{Bmatrix} \frac{1}{\sqrt{2}\pi} \frac{m_*}{\mu'} \int_0^\infty dq \frac{q^2 \, T_{12}^{(0)}(E,k,q)}{-\gamma + \sqrt{-2\mu \left(E - \frac{q^2}{2m_*} - \frac{q^2}{2(m+m_*)} \right) - i\varepsilon}}$$

$$\times \frac{1}{2pq} \ln \left[\frac{\frac{q^2}{2\mu'} + \frac{p^2}{2\mu} - E + \frac{pq}{m_*} - i\varepsilon}{\frac{q^2}{2\mu'} + \frac{p^2}{2\mu} - E - \frac{pq}{m_*} - i\varepsilon} \right] , \tag{5.15}$$

where again the upper entries must be used for isospin $1/2$ and the lower ones for isospin $3/2$. Moreover, it holds $m_1 = m$, $m_2 = m_3 = m_*$, $\mu_{12} = \mu$, $\mu_{23} = \mu'$, $\gamma_{12} = \gamma$ and $\gamma_{23} = \gamma'$ with the notation introduced at the beginning of this section on elastic $Z_b^{(\prime)}$–$B^{(*)}$ scattering.

	$\widetilde{x}_1$	$\widetilde{x}_2$	$\widetilde{x}_4$	z_1	z_2
spin 0 & isospin 1/2	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0
spin 0 & isospin 3/2	1	1	0	0	0
spin 1 & isospin 1/2	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
spin 1 & isospin 3/2	-1	-1	1	1	1
spin 2 & isospin 1/2	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0
spin 2 & isospin 3/2	1	1	0	0	0

Table 5.1: *Spin and isospin channel dependent parameters for Z_b–B^* scattering.*

5.1.3 Z_b'–B scattering amplitude

The three particle system $P_1 = B^*$, $P_2 = \bar{B}^*$ and $P_3 = B$ again leads to $a_1 = b_2 = a_3 = 1$, but it holds

$$\delta_{P_1 P_2} = \delta_{P_1 P_3} = \delta_{P_2 P_3} = \delta_{A_1 A_3} = \delta_{A_2 A_3} = 0 \, ,$$
$$\delta_{A_1 A_2} = 1 \, . \tag{5.16}$$

This is not surprising since compared to the previous section on Z_b–B^* scattering we simply interchanged the particle allocation of P_1 and P_2. Therefore one has to deal with the same two dimers

$$d_{12} = (\eta_{12} = +1) \, \bar{B}^* B^* \, , \tag{5.17}$$

$$d_{23} = \frac{1}{\sqrt{2}} \left(\bar{B}^* B + (\eta_{23} = +1) \, B^* \bar{B} \right) \, , \tag{5.18}$$

representing the $Z_b(10610)$ and the $Z_b'(10650)$, but now d_{12} is interpreted as the latter. Again there is no $d_{13} = B^* B$ dimer and there are no primed ones at all in the system. However, in

contrast to the above discussed case there are less scattering channels because the spin $1 \otimes 0 = 1$ is fixed. Applying $\delta^{(12)}_{P_1 P_3} = \delta^{(12)}_{P_2 P_3} = \delta^{(23)}_{P_1 P_3} = 0$ and $\delta^{(23)}_{P_1 P_2} = 1$ to the corresponding amplitudes $T^{(L)}_{12}$ (Eq. (F.4)) and $T^{(L)}_{23}$ (Eq. (F.6)) and projecting both onto S-waves one ends up with

$$
T^{(0)}_{12}(E, k, p) = \frac{1}{\pi} \frac{\widetilde{x}_4 \, \widetilde{f}^{(3)}_{(23)(12)} \, S_{321}}{\mu_{12} \, \sqrt{S_{12} \, S_{23}} \, c_{12} \, c_{23}} \int_0^\infty dq \, \frac{q^2 \, T^{(L)}_{23}(E, k, q) \, \frac{m_2}{qp} \, Q^{213}_L(q, p; E)}{-\gamma_{23} + \sqrt{-2\mu_{23} \left(E - \frac{q^2}{2m_1} - \frac{q^2}{2(m_2 + m_3)} \right)} - i\varepsilon} \, ,
$$

(5.19)

and

$$
T^{(0)}_{23}(E, k, p) = \frac{2\pi \, \gamma_{12} \, z_1 \, f^{(1)}_{(12)(23)} \, S_{123}}{\mu_{23} \, \mu_{12} \, \sqrt{S_{23} \, S_{12}} \, c_{23} \, c_{12}} \frac{m_2}{kp} \, Q^{231}_L(k, p; E)
$$

$$
+ \frac{1}{\pi} \frac{z_2 \, f^{(1)}_{(12)(23)} \, S_{123}}{\mu_{23} \, \sqrt{S_{23} \, S_{12}} \, c_{23} \, c_{12}} \int_0^\infty dq \, \frac{q^2 \, T^{(L)}_{12}(E, k, q) \, \frac{m_2}{qp} \, Q^{231}_L(q, p; E)}{-\gamma_{12} + \sqrt{-2\mu_{12} \left(E - \frac{q^2}{2m_3} - \frac{q^2}{2(m_1 + m_2)} \right)} - i\varepsilon}
$$

$$
+ \frac{1}{\pi} \frac{z_4 \, f^{(2)}_{(23)(23)} \, S_{231}}{\mu_{23} \, S_{23} \, c_{23}} \int_0^\infty dq \, \frac{q^2 \, T^{(L)}_{23}(E, k, q) \, \frac{m_3}{qp} \, Q^{322}_L(q, p; E)}{-\gamma_{23} + \sqrt{-2\mu_{23} \left(E - \frac{q^2}{2m_1} - \frac{q^2}{2(m_2 + m_3)} \right)} - i\varepsilon} \, . \quad (5.20)
$$

The remaining parameters are according to their definitions on pages 56 and 57 given by

$$
S_{ij} = S_{ijk} = 1 \, , \quad \forall \, i, j, k \in \{1, 2, 3\} \, ,
$$

regarding the symmetry factors and by

$$
\widetilde{f}^{(3)}_{(23)(12)} = \frac{1}{\sqrt{2}} \, , \qquad f^{(1)}_{(12)(23)} = \frac{1}{\sqrt{2}} \, , \qquad f^{(1)}_{(13)(23)} = \frac{1}{\sqrt{2}} \, , \qquad f^{(2)}_{(23)(23)} = \frac{1}{2} \, .
$$

Moreover, one needs the values given in Tab. 5.2 which are obtained using the methods presented in appendix A.3, i.e. $c_{ij} = 1$ for all $i < j \in \{1, 2, 3\}$.

	$\widetilde{x}_4$	z_1	z_2	z_4
spin 1 & isospin 1/2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
spin 1 & isospin 3/2	-1	-1	-1	-1

Table 5.2: *Spin and isospin channel dependent parameters for Z'_b–B scattering.*

With the notation that upper entries are for isospin 1/2 and lower ones for isospin 3/2 this yields the coupled integral equation system below (with $m_1 = m_2 = m_*$, $m_3 = m$, $\mu_{12} = \mu'$, $\mu_{23} = \mu$,

$\gamma_{12} = \gamma'$ and $\gamma_{23} = \gamma$):

$$T_{12}^{(0)}(E,k,p) = \left\{ \begin{array}{c} \frac{1}{2} \\ -1 \end{array} \right\} \frac{1}{\sqrt{2\pi}} \frac{m_*}{\mu'} \int_0^\infty dq \, \frac{q^2 \, T_{23}^{(L)}(E,k,q)}{-\gamma + \sqrt{-2\mu \left(E - \frac{q^2}{2m_*} - \frac{q^2}{2(m_*+m)} \right) - i\varepsilon}}$$

$$\times \frac{1}{2pq} \ln \left[\frac{\frac{q^2}{2\mu'} + \frac{p^2}{2\mu} - E + \frac{pq}{m_*} - i\varepsilon}{\frac{q^2}{2\mu'} + \frac{p^2}{2\mu} - E - \frac{pq}{m_*} - i\varepsilon} \right] , \tag{5.21}$$

$$T_{23}^{(0)}(E,k,p) = \left\{ \begin{array}{c} \frac{1}{2} \\ -1 \end{array} \right\} \sqrt{2\pi} \frac{\gamma' \, m_*}{\mu \, \mu'} \frac{1}{2pk} \ln \left[\frac{\frac{k^2}{2\mu} + \frac{p^2}{2\mu} - E + \frac{pk}{m_*} - i\varepsilon}{\frac{k^2}{2\mu} + \frac{p^2}{2\mu} - E - \frac{pk}{m_*} - i\varepsilon} \right]$$

$$+ \left\{ \begin{array}{c} \frac{1}{2} \\ -1 \end{array} \right\} \frac{1}{\sqrt{2\pi}} \frac{m_*}{\mu} \int_0^\infty dq \, \frac{q^2 \, T_{12}^{(L)}(E,k,q)}{-\gamma' + \sqrt{-2\mu' \left(E - \frac{q^2}{2m} - \frac{q^2}{2(m_*+m_*)} \right) - i\varepsilon}}$$

$$\times \frac{1}{2pq} \ln \left[\frac{\frac{q^2}{2\mu'} + \frac{p^2}{2\mu} - E + \frac{pq}{m_*} - i\varepsilon}{\frac{q^2}{2\mu'} + \frac{p^2}{2\mu} - E - \frac{pq}{m_*} - i\varepsilon} \right]$$

$$+ \left\{ \begin{array}{c} \frac{1}{2} \\ -1 \end{array} \right\} \frac{1}{2\pi} \frac{m}{\mu} \int_0^\infty dq \, \frac{q^2 \, T_{23}^{(L)}(E,k,q)}{-\gamma + \sqrt{-2\mu \left(E - \frac{q^2}{2m_*} - \frac{q^2}{2(m_*+m)} \right) - i\varepsilon}}$$

$$\times \frac{1}{2pq} \ln \left[\frac{\frac{q^2}{2\mu} + \frac{p^2}{2\mu'} - E + \frac{pq}{m} - i\varepsilon}{\frac{q^2}{2\mu} + \frac{p^2}{2\mu'} - E - \frac{pq}{m} - i\varepsilon} \right] , \tag{5.22}$$

where it was again used that one can – according to Eq. (3.73) and Eq. (D.10) – write the $L = 0$ Legendre function of the second kind in terms of the logarithm.

5.1.4 Z_b'–B^* scattering amplitude

Finally, we consider Z_b'–B^* scattering leading to the three particle system $P_1 = P_3 = B^*$ and $P_2 = \bar{B}^*$ which corresponds to $a_1 = b_2 = a_3 = 1$ and

$$\delta_{P_1 P_2} = \delta_{P_2 P_3} = 0 \, ,$$
$$\delta_{P_1 P_3} = \delta_{A_1 A_2} = \delta_{A_1 A_3} = \delta_{A_2 A_3} = 1 \, . \tag{5.23}$$

From Eq. (3.13) it follows – keeping in mind that there is up to now no experimental evidence for a $B^* B^*$ state with large scattering length – that there is just one dimer $d_{12} = \eta_{12} \bar{B}^* B^*$ in the system (cf. rules on page 51) whose G-parity is according to Tab. 4.1 given by $\eta_{12} = +1$. The corresponding S-wave three-body amplitude $T_{12}^{(0)}$ is defined by Eq. (F.4) where one has erased

all contributions from the other amplitudes:

$$T_{12}^{(0)}(E,k,p) = \frac{2\,\pi\,\gamma_{12}}{\mu_{12}^2\,S_{12}\,c_{12}}$$

$$\times \left[x_1\,\delta_{P_1P_3}^{(12)}\,f_{(12)(12)}^{(3)}\,S_{123}\,\frac{m_2}{kp}Q_L^{211}(k,p;E) + \widetilde{x}_1\,\delta_{P_2P_3}^{(12)}\,\widetilde{f}_{(12)(12)}^{(3)}\,S_{213}\,\frac{m_1}{kp}Q_L^{122}(k,p;E) \right]$$

$$+ \frac{1}{\pi}\frac{1}{\mu_{12}\,S_{12}\,c_{12}}\int_0^\infty dq\,\frac{q^2\,T_{12}^{(L)}(E,k,q)}{-\gamma_{12} + \sqrt{-2\mu_{12}\left(E - \frac{q^2}{2m_3} - \frac{q^2}{2(m_1+m_2)}\right)} - i\varepsilon}$$

$$\times \left[x_2\,\delta_{P_1P_3}^{(12)}\,f_{(12)(12)}^{(3)}\,S_{123}\frac{m_2}{qp}Q_L^{211}(q,p;E) + \widetilde{x}_2\,\delta_{P_2P_3}^{(12)}\,\widetilde{f}_{(12)(12)}^{(3)}\,S_{213}\frac{m_1}{qp}Q_L^{122}(q,p;E) \right].$$

$$(5.24)$$

From the summarized definitions on pages 56 and 57 one deduces $\delta_{P_1P_3}^{(12)} = 1$, but $\delta_{P_2P_3}^{(12)} = 0$ and

$$S_{12} = S_{123} = 1\,,$$
$$S_{123} = 2\,,$$
$$f_{(12)(12)}^{(3)} = \widetilde{f}_{(12)(12)}^{(3)} = 1\,.$$

With Eqs. (A.111, A.129) derived in appendix A.3 for normalized projection operators (i.e. $c_{12} = 1$) one additionally finds the values shown in Tab. 5.3 for the spin and isospin dependent x factors. Hence, one ends up with the following amplitude (with $m_1 = m_2 = m_3 = m_*$, $\mu_{12} = \mu'$ and $\gamma_{12} = \gamma'$ as explained at the beginning of section 5.1):

$$T_{12}^{(0)}(E,k,p) = \begin{Bmatrix} \frac{1}{2} \\ -1 \\ -\frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{2} \\ -\frac{1}{4} \\ \frac{1}{2} \end{Bmatrix} 2\pi\frac{\gamma'\,m_*}{\mu'^2}\frac{1}{2pk}\ln\left[\frac{\frac{k^2}{2\mu'} + \frac{p^2}{2\mu'} - E + \frac{pk}{m_*} - i\varepsilon}{\frac{k^2}{2\mu'} + \frac{p^2}{2\mu'} - E - \frac{pk}{m_*} - i\varepsilon}\right]$$

$$+ \begin{Bmatrix} \frac{1}{2} \\ -1 \\ -\frac{1}{4} \\ \frac{1}{2} \\ -\frac{1}{4} \\ \frac{1}{2} \end{Bmatrix}\frac{1}{\pi}\frac{m_*}{\mu'}\int_0^\infty dq\,\frac{q^2\,T_{12}^{(L)}(E,k,q)}{-\gamma' + \sqrt{-2\mu'\left(E - \frac{q^2}{2m_*} - \frac{q^2}{2(m_*+m_*)}\right)} - i\varepsilon}$$

$$\times \frac{1}{2pq}\ln\left[\frac{\frac{q^2}{2\mu'} + \frac{p^2}{2\mu'} - E + \frac{pq}{m_*} - i\varepsilon}{\frac{q^2}{2\mu'} + \frac{p^2}{2\mu'} - E - \frac{pq}{m_*} - i\varepsilon}\right]\,, \qquad (5.25)$$

where the entries in curly brackets represent the channels from top to bottom in the same order as in Tab. 5.3 and where Eq. (3.73) together with Eq. (D.10) were used to write the Legendre functions in terms of the logarithm.

	x_1	x_2	$\widetilde{x}_1$	$\widetilde{x}_2$
spin 0 & isospin 1/2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
spin 0 & isospin 3/2	-1	-1	-1	-1
spin 1 & isospin 1/2	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$
spin 1 & isospin 3/2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
spin 2 & isospin 1/2	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$
spin 2 & isospin 3/2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

Table 5.3: *Spin and isospin channel dependent parameters for Z_b'–B^* scattering.*

5.2 Numerical determination of scattering length and phase shift

In this section it will be described how the three-body observables scattering length and phase shift can be deduced from the discretized scattering amplitude, given in form of a matrix equation $\mathbf{T} = \mathbf{R} + \mathcal{M}\mathbf{T}$. The discretization itself is explained in appendix G. For a single integral equation describing $Z_b^{(\prime)}$–$B^{(*)}$ scattering in an arbitrary channel there are three relevant regions of the center-of-mass energy E (the binding energy of the relevant molecule Z_b or Z_b' is denoted as $B(Z)$ which is equal to either B or B'):

- $-B(Z) \leq E \leq 0$: in terms of the momentum k this energy region translates to $0 \leq k \leq k_{\max}$ where $k_{\max}$ is the root of $E(k)$ which defines the momentum where the molecule breaks apart into its constituents. Hence, in this region the **elastic scattering of a bottom meson off a molecule** takes place.

- $-\infty < E < -B(Z)$: for smaller energies below the molecule threshold there would appear – if present – **three particle bound states** due to the Efimov effect.

- $0 < E < \infty$: for larger energies the molecule will break apart and one has to deal with a **system of three individual bottom mesons** which scatter off each other. However, this system will not be further discussed in this work.

In a system of two coupled integral equations where both Z_b and Z_b' are involved one has to replace in the second case $-\infty < E < -B(Z)$ by $-\infty < E < -\max(B, B')$ as a trimer state must lie below both dimer thresholds. In the first case one has to take care of the relation between the two binding energies B and B'. The elastic two-body scattering Z_b–B^* only takes place for $B \geq B'$, namely in the energy region $-B \leq E \leq -B'$. In the other case, i.e. for $B < B'$ both molecular states can be built of the three bottom mesons. Thus, there is no region where we would deal with a two-body problem alone. Since there is no straightforward way of describing such a three-body scattering problem in the used theory we will not analyze it in more detail. The second coupled system appears in Z_b'–B scattering. Here, the situation is interchanged: for $B < B'$ the valid energy region is $-B' \leq E \leq -B$ and the inaccessible case is given by $B \geq B'$.

5.2.1 Method

As we know already from section 4.1 that there is no Efimov effect in $Z_b^{(\prime)}$–$B^{(*)}$ scattering we can restrict the discussion to the elastic molecule–particle scattering region. According to the dimer auxiliary field trick (cf. section 2.1) the molecule is treated as one particle and hence one can describe molecule–particle scattering in terms of two-body scattering theory which was discussed in section 1.4. Consequently, one can write for the elastic S-wave amplitude $T_{12}^{(0)}$ in all four processes:

$$T_{12}^{(0)}(k) = \frac{2\pi}{\mu_{(12)3}} \frac{1}{k \, \cot\delta - ik} \, , \tag{5.26}$$

with leading order effective range expansion

$$T_{12}^{(0)}(k) = \frac{2\pi}{\mu_{(12)3}} \frac{1}{-\frac{1}{a_3} - ik} \, . \tag{5.27}$$

From the considerations above it is clear that one needs the explicit form of the amplitude $T_{12}^{(0)}(E,k,p)$. It can be found by numerically solving the inhomogeneous matrix equation $\mathbf{T} = \mathbf{R} + \mathcal{M}\mathbf{T}$ (see appendix G) for a given momentum k, i.e. for a given center-of-mass energy $E \sim k^2$. Here, the vector $\mathbf{T}$ is given by $\mathbf{T} = T_{12}^{(0)}$ for the single integral equations of Z_b–B and $Z_b^{\prime}$–B^* scattering and by $\mathbf{T} = \left(T_{12}^{(0)}, T_{23}^{(0)}\right)^T$ for the coupled systems of Z_b–B^* and $Z_b^{\prime}$–B scattering (cf. appendix G). These matrix equations are solved using the ZGESV routine of the LAPACK library which is documented in Ref. [190].

Thus, one concludes from Eq. (5.27) that the three-body molecule–particle scattering length a_3 can be determined via

$$a_3 = -\frac{\mu_{(12)3}}{2\pi} \, \mathrm{Re}\left[T_{12}^{(0)}(0)\right] \, . \tag{5.28}$$

In a numerical calculation one sets $k = p_1$ (i.e. k is equal to the first mesh point, cf. appendix G) instead of exactly zero and uses $T_{12}^{(0)}(k) \equiv T_{12}^{(0)}(E = k^2/(2\mu_{(12)3}), k, p = k)$. This approach is justified since for a typical number of 100 mesh points one gets $p_1 < 0.005$ even for large cutoffs $\Lambda \sim 10000 \, \mathrm{MeV}$. Therefore p_1 is sufficiently close to 0 and hence no extrapolation is needed. The second observable, namely the scattering phase shift, can be determined as a function of the momentum k by rearranging Eq. (5.26):

$$\delta(k) = \mathrm{arccot}\left(\frac{2\pi}{\mu_{(12)3}} \frac{1}{k} \mathrm{Re}\left[\frac{1}{T_{12}^{(0)}(k)}\right]\right) \, . \tag{5.29}$$

Note, that for each k one has to take – after solving the matrix equation – that part of the amplitude $T_{12}^{(0)}(k)$ which corresponds to the pole contribution, that is, in the discretized version T_{N-1} or T_{N-2} depending on the number of poles in the scattering process (cf. appendix G). Besides the predictions one can use Eq. (5.26) in the form

$$-\frac{2\pi}{\mu_{(12)3}} \mathrm{Im}\left[\frac{1}{T(k)}\right] = k \, , \tag{5.30}$$

as a check for the numerics used, in the sense that a linear dependence is from a numerical point of view very sensitive to any programming error.

5.2.2 Results

The (coupled) integral equations derived in section 5.1 will now be analyzed using the numerical methods explained in the previous section and in appendix G. The binding energies needed as input for these calculations are obtained by Cleven *et al.* via an analysis of bottom meson loops in the framework of hadronic molecules which is presented in Ref. [139]. The values are:

$$B = 4.7^{+2.3}_{-2.2}\,\text{MeV}\,, \tag{5.31}$$

$$B' = 0.11^{+0.14}_{-0.06}\,\text{MeV}\,, \tag{5.32}$$

which lead – according to Eq. (1.22) – to the binding momenta

$$\gamma = 157.9^{+38.6}_{-37.0}\,\text{MeV}\,, \tag{5.33}$$

$$\gamma' = 24.20^{+15.40}_{-6.60}\,\text{MeV}\,. \tag{5.34}$$

Hence, one observes that the value for the $Z_b(10610)$ is larger than the pion mass which in principle contradicts the application of EFT($\not{\pi}$), however, due to the large uncertainties of γ it is not excluded that the binding momentum of Z_b is below m_π (though close to it) so that one can use a pionless EFT at least to obtain some first insights.

Discussion of Z_b–B scattering

As a consequence of the fact that there is no Efimov effect in the system, the elastic scattering Z_b–B is completely described by the formulae in section 5.2.1 which lead to cutoff independent values for the two observables, Z_b–B scattering length a_3 and S-wave phase shift $\delta(k)$. The scattering length in the $I = 3/2$ & $S = 1$ channel is given by

$$a_3^{I=\frac{3}{2},S=1} = -15.13\,\text{fm}\,, \tag{5.35}$$

and the corresponding phase shift in this channel is shown as a function of k in Fig. 5.1. As it is known from basic scattering theory [81, 82] a positive phase shift corresponds to an attractive potential between the two scattering particles. However, such a potential can only induce a S-wave bound state if the scattering length is also positive. Since $a_3^{I=\frac{3}{2},S=1}$ is negative this observation agrees with the absence of the Efimov effect in the system. From Fig. 5.1 we conclude that the scattering length in the other channel with $I = 1/2$ & $S = 1$ must be positive since the phase shift is negative. This is indeed the case:

$$a_3^{I=\frac{1}{2},S=1} = 0.62\,\text{fm}\,. \tag{5.36}$$

Discussion of Z_b–B^* scattering

In the same way as for Z_b–B scattering one can analyze the coupled Z_b–B^* system. Again one can predict the scattering length and the phase shift in all six isospin-spin channels of the elastic scattering process. Since the projection onto some of the isospin and spin states yields identical

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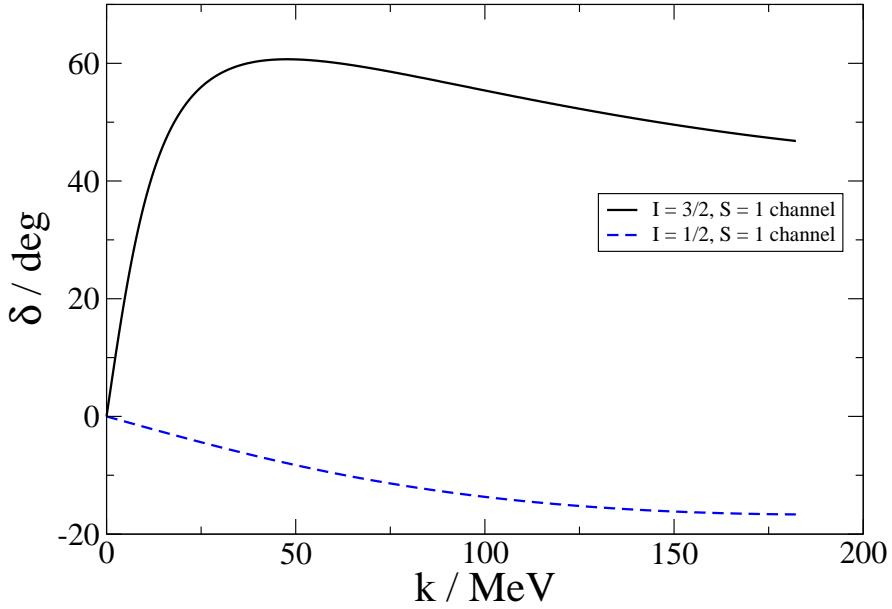
Figure 5.1: *S-wave phase shift δ as function of the momentum k for both channels in elastic Z_b–B scattering.*

prefactors, there only remain four different values and curves. The latter are shown in Fig. 5.2 and the scattering lengths are given by

$$a_3^{I=\frac{3}{2},S=0,2} = -15.66 \,\text{fm} \,, \tag{5.37}$$

$$a_3^{I=\frac{1}{2},S=0,2} = 0.62 \,\text{fm} \,, \tag{5.38}$$

$$a_3^{I=\frac{3}{2},S=1} = 0.88 \,\text{fm} \,, \tag{5.39}$$

$$a_3^{I=\frac{1}{2},S=1} = -1.97 \,\text{fm} \,. \tag{5.40}$$

Thus, the correlation between the signs of δ and a_3 agrees with the missing Efimov trimers.

Discussion of Z_b'–B scattering

It is already known that there is no Efimov effect in the Z_b'–B system. But moreover, it holds $B \simeq 4.7 \,\text{MeV} \geq 0.1 \,\text{MeV} \simeq B'$. Thus, it is not possible (according to section 5.2.1) to extract other observables for this process. A different approach would be necessary to deal with such a non-trivial three-body system which is beyond the aim of this work.
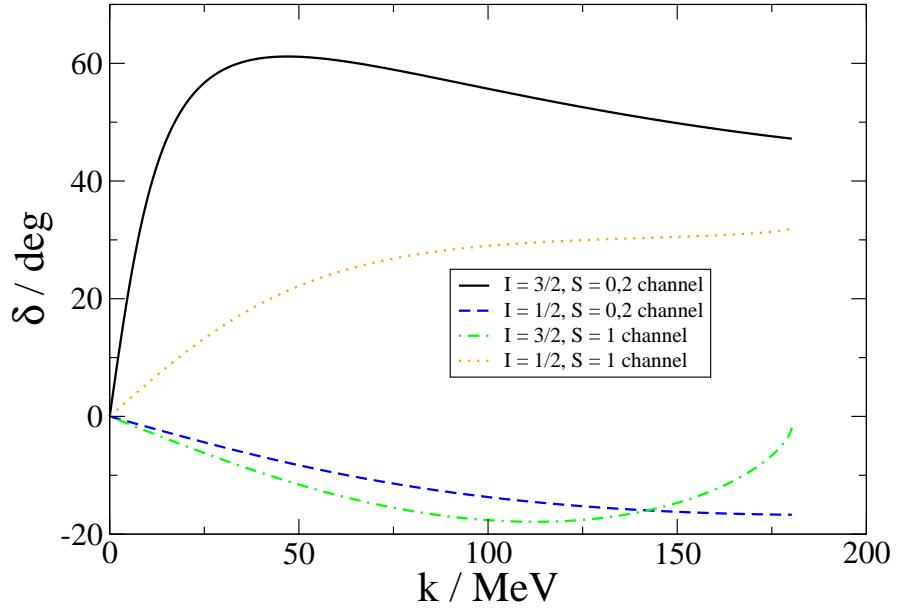
Figure 5.2: *S-wave phase shift δ as function of the momentum k for all six channels in elastic Z_b–B^* scattering. Note, that the $S = 0$ and $S = 2$ spin channels yield the same result.*

Discussion of Z'_b–B^* scattering

Finally, the analysis of Z'_b–B^* scattering allows us to obtain the scattering lengths and the phase shifts which are shown in Fig. 5.3 and given by

$$a_3^{I=\frac{3}{2},S=1,2} = a_3^{I=\frac{1}{2},S=0} = -99.02 \,\text{fm} , \tag{5.41}$$

$$a_3^{I=\frac{1}{2},S=1,2} = 4.03 \,\text{fm} , \tag{5.42}$$

$$a_3^{I=\frac{3}{2},S=0} = 9.61 \,\text{fm} . \tag{5.43}$$

One observes that the absolute value of the scattering length in the $I = 3/2$ & $S = 1, 2$ and in the $I = 1/2$ & $S = 0$ channel is almost two orders of magnitude larger than in all other processes and channels. In Ref. [82] it is explained that for a growing attractive potential, the scattering length a_3 tends to minus infinity until it appears a bound state in the system which causes at $B_3 = 0$, that a_3 changes its sign to positive infinity. Hence, for $B_3 > 0$, a_3 is positive and finite. Consequently, the large absolute value of the scattering length indicates that the interaction between Z'_b and B^* is just quite not strong enough to from a bound state.
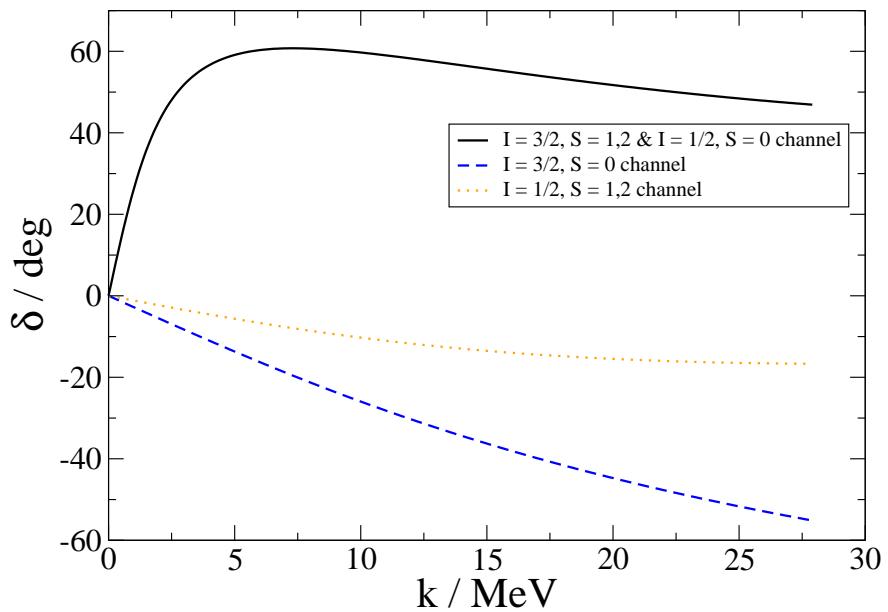
Figure 5.3: *S-wave phase shift δ as function of the momentum k for all six channels in elastic Z'_b–B^* scattering. Note, that for each isospin state the $S = 1$ and $S = 2$ spin channels yield the same result and additionally that the $I = 1/2$ & $S = 0$ result is equivalent to that of $I = 3/2$ & $S = 1, 2$.*

Chapter 6

Conclusion and outlook

In this work we considered the generic three-body scattering of a particle off a S-wave dimer in a pionless non-relativistic effective field theory. This was done in order to derive a transcendental equation whose solution tells us whether or not the corresponding three-body system is affected by Efimov physics. For this purpose we made the assumption that the constituents of the two-body bound or virtual state are heavy compared to pions, but that their binding momentum is rather small, i.e. at most of the order of the pion mass, so that EFT($\pi\!\!\!/$) is applicable. The method we have used to search for Efimov trimers in a three particle system was to decouple the molecule–particle scattering amplitudes within the coupled integral equation system in the limit of asymptotic large momenta, that is, to diagonalize the matrix whose elements contain all information about spin, isospin, flavor normalization factors and symmetry factors. Unfortunately, the decoupling / diagonalizing is only possible if one imposes some constraints on the masses and the existence of shallow dimer states between some particles. We found two types of systems which are suitable for this method: type 1 must have (approximately) equal masses (but no further constraints) and type 2 must have $m_2 = m_3$ or $m_1 = m_3$ (remember the freedom in the allocation of particles 1 and 2) with the restriction that the equal mass particles must have *no* bound or virtual state (i.e. they do *not* have a large scattering length). Only systems made of two equal mass particles with a dimer and a third particle of different mass and in addition systems where all particles have different masses cannot be described by the presented method. However, especially in a system with three unequal masses whose difference is so large that even the approximation of equal masses is not justified, it is relatively unlikely that such particles at all have a bound state explained by the strong force. Besides these subtleties there are no more constraints. Hence, one finally finds a transcendental equation for the scaling parameter s which can have – depending on the eigenvalues of the matrix and on the mass ratios – a purely imaginary solution which corresponds to the existence of the Efimov effect. Although this method could be used in many different fields of few-body physics (e.g. cold atoms or halo nuclei) we applied it to hadronic molecules scattering off a third particle which has a large scattering length with at least one of the constituents of the molecule. Indeed, we found that a couple of established particles – interpreted as hadronic molecules – are affected by Efimov physics. In particular, the three kaon system $\bar{K}KK$ has a reasonably small scaling factor of 1986.14. In the up to now known charm and bottom meson system no Efimov effect was found. Instead, we have analyzed the – due to the absence of Efimov trimers – elastic $Z_b^{(\prime)}$–$B^{(*)}$ scattering as a selected example in

order to obtain three-body observables like S-wave scattering length and phase shift in systems without Efimov effect using EFT($\not{\pi}$). We have also checked that hypothetical molecules made of identical bosons (with spin and isospin) or fermions have for most spin and isospin configurations an intermediate Efimov trimer state in their scattering off a third identical boson or fermion.

Although it might be experimentally challenging to find such new states it might also be worth the effort because the existence of trimer states would be hard to explain in the tightly bound diquark picture, but would emerge naturally in the hadronic molecule interpretation. Hence, in this sense this work provides in principle a further method to clarify the nature of – at least some – exotic particles in the charmonium and bottomonium sector.

As an outlook one can say that the extension of this work to P-wave or even higher L dimer states would be a nice feature, especially in cold atoms or halo nuclei where the existence of higher partial wave dimers is more common than in the hadronic molecule sector. Furthermore, an inclusion of higher order terms would obviously improve the quality of the results. However, it would require more experimental knowledge about the scattering length and the effective range of the particles. Also the extension of the used pionless EFT to a EFT with explicit pions (XEFT) would be useful, especially for the not so shallowly bound molecular states. Independently of the theoretical effort it is expected that the planned or already running experiments like Belle II, BES III or LHCb will provide more data which could help improving the theoretical predictions.

Appendix A

Spin and isospin projection operators

In the first part of the appendix our goal is to explain the structure of the combined spin and isospin projection operators $\mathcal{O}_{ij}$. In the second part we focus on the projection onto a specific scattering channel in order to derive the defining equations for the parameters x, y and z. Finally, in the last subsection we will discuss the relation between these parameters and the Wigner 6-J symbol which gives us the opportunity to determine them for arbitrary spin and isospin without explicit knowledge of the projection operators. Since spin and isospin are described equivalently we derive all relations only considering spin. To generalize the results to isospin is straightforward (in fact, it corresponds to simply renaming all variables).

A.1 Combined spin and isospin projection operators

We start with some conventions regarding the matrix elements defining a generic scattering amplitude $T \sim \langle \text{out}| \dots |\text{in}\rangle$ with ket and bra vectors defined according to the conventions of Ref. [1]:

$$\langle \mathbf{k}, \mathbf{p}| \sim \langle 0| \, \widehat{a}_p \widehat{a}_k \,,$$
$$|\mathbf{k}, \mathbf{p}\rangle \sim \widehat{a}_k^\dagger \widehat{a}_p^\dagger \, |0\rangle \,. \tag{A.1}$$

Hence, it holds

$$(\langle \mathbf{k}, \mathbf{p}|)^\dagger = (\langle 0| \, \widehat{a}_p \widehat{a}_k)^\dagger = \widehat{a}_k^\dagger \widehat{a}_p^\dagger \, |0\rangle = |\mathbf{k}, \mathbf{p}\rangle \,. \tag{A.2}$$

Here, $\widehat{a}_p$ is the annihilation and $\widehat{a}_p^\dagger$ is the creation operator which annihilates and creates a state with momentum $\mathbf{p}$, respectively. Again following the conventions of Ref. [1] the field used in the Lagrangian density A is related to $\widehat{a}$ and $A^\dagger$ to $\widehat{a}^\dagger$ by a Fourier transform. Thus, we will use the fields A and $A^\dagger$ as synonyms for annihilation and creation operators, respectively. Since we are working in a non-relativistic theory it holds that the contraction of a state with momentum $\mathbf{p}$ and the operator A yields 1, i.e.

$$\langle \mathbf{p}| \, A^\dagger = 1 \,,$$
$$A \, |\mathbf{p}\rangle = 1 \,. \tag{A.3}$$

$$\begin{array}{c}
\overline{\beta} \quad (A_i)_{\widetilde{\beta}} \\
\overline{\alpha} \quad \nearrow \mathbf{q} \\
(d_{ij})_{\widetilde{\alpha}} \xRightarrow{} \bullet \qquad = i\, T^{\widetilde{\beta}\,\widetilde{\gamma}}_{\widetilde{\alpha}}(\mathbf{k},\mathbf{p},\mathbf{q}) \\
\mathbf{k} \qquad \searrow \mathbf{p} \\
\widetilde{\gamma} \quad (A_j)_{\widetilde{\gamma}}
\end{array}$$
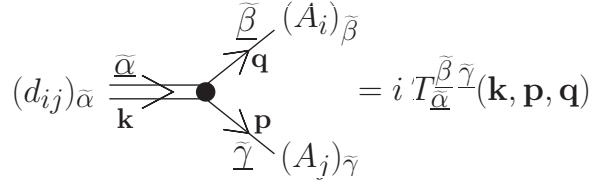
Figure A.1: *Diagrammatic representation of a generic dimer decay. The corresponding amplitude $T^{\widetilde{\beta}\,\widetilde{\gamma}}_{\widetilde{\alpha}}$ depends on the incoming and outgoing spin indices.*

Adding spin degrees of freedom to the theory the fields $A_{\widetilde{\alpha}}$ become proportional to polarization vectors $(\vec{\varepsilon}_{\widetilde{\alpha}})_{\widetilde{\alpha}}$ for which we use the same symbol independently of the exact spin. In fact, for $\widetilde{\alpha} = \diagup$ being a scalar spin "index" the polarization vector becomes $\vec{\varepsilon}_{\widetilde{\alpha}} = 1$. For $\widetilde{\alpha} = \alpha \in \{1,2\}$ being a spin $1/2$ index one gets a non-relativistic two dimensional spinor $\vec{\varepsilon}_{\widetilde{\alpha}} = \vec{\chi}_{\alpha}$ and for $\widetilde{\alpha} = \underline{i} \in \{1,2,3\}$ being a spin 1 index $\vec{\varepsilon}_{\widetilde{\alpha}} = \vec{\varepsilon}_i$ is the usual polarization vector of vector particles. The contractions in Eq. (A.3) are thus changed to contractions of a state with momentum $\mathbf{p}$ and spin $\widetilde{\alpha}$ with the operator $A_{\widetilde{\alpha}}$ where $\widetilde{\alpha}$ denotes the component of the polarization vector $\vec{\varepsilon}_{\widetilde{\alpha}}$:

$$\langle \mathbf{p}, \widetilde{\alpha}|\, A^{\dagger}_{\widetilde{\alpha}} = \left(\vec{\varepsilon}^{\,\dagger}_{\widetilde{\alpha}}\right)_{\widetilde{\alpha}}(\mathbf{p})\,, \tag{A.4}$$

$$A_{\widetilde{\alpha}}\,|\mathbf{p}, \widetilde{\alpha}\rangle = (\vec{\varepsilon}_{\widetilde{\alpha}})_{\widetilde{\alpha}}(\mathbf{p})\,. \tag{A.5}$$

To clarify the notation consider a state $\langle \mathbf{p}, (\underline{i} = 1)|$ which is contracted with an operator $A^{\dagger}_{\underline{i}}$ which is associated to a spin 1 field:

$$\langle \mathbf{p}, (\underline{i} = 1)|\, A^{\dagger}_{\underline{i}} = (\vec{\varepsilon}_1)_{\underline{i}}(\mathbf{p}) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{\underline{i}} = \begin{cases} 1\,, & \text{for } i = 1 \\ 0\,, & \text{for } i = 2 \\ 0\,, & \text{for } i = 3 \end{cases}\,, \tag{A.6}$$

where we have used the Cartesian polarization basis.

After this introduction we can continue and use the relations above to derive the projection operators $\mathcal{O}_{ij}$ which appear in the Lagrangian density Eq. (3.23). Note, that the results for projectors up to spin $1 \otimes 1$ are collected in Tab. A.1 in section A.4. In order to proof that these projectors $\mathcal{O}_{ij}$ in Tab. A.1 indeed are correct we consider the vertex function of a dimer decaying into its constituents which is in Fig. A.1 shown as a Feynman diagram (note, that we can restrict ourselves without loss of generality to the case $d_{ij} \sim A_j A_i$). We will compare the numerical value of the decay amplitude with the corresponding Clebsch-Gordan coefficient. If we ignore for the moment the non-trivial wave function renormalization of the dimer field d_{ij} the decay amplitude is according to the vertex functions in Fig 3.1 given by

$$T^{\widetilde{\beta}\,\widetilde{\gamma}}_{\widetilde{\alpha}}(\mathbf{k},\mathbf{p},\mathbf{q}) = \left\langle \mathbf{q}, \widetilde{\beta}; \mathbf{p}, \widetilde{\gamma}\right|\, a_i a_j (-g_{ij}) \left(A^{\dagger}_i\right)_{\widetilde{\beta}} (\mathcal{O}_{ij})_{\widetilde{\alpha},\widetilde{\beta}\widetilde{\gamma}} \left(A^{\dagger}_j\right)_{\widetilde{\gamma}} (d_{ij})_{\widetilde{\alpha}} \left|\mathbf{k}, \widetilde{\alpha}\right\rangle$$

$$= -g_{ij}\, a_i\, a_j \left(\vec{\varepsilon}^{\,\dagger}_{\widetilde{\beta}}\right)_{\widetilde{\beta}} (\mathcal{O}_{ij})_{\widetilde{\alpha},\widetilde{\beta}\widetilde{\gamma}} \left(\vec{\varepsilon}^{\,\dagger}_{\widetilde{\gamma}}\right)_{\widetilde{\gamma}} (\vec{\varepsilon}_{\widetilde{\alpha}})_{\widetilde{\alpha}}\, \delta^{(3)}(\mathbf{p} + \mathbf{q} - \mathbf{k})\,, \tag{A.7}$$

where we have in the second step contracted the operators with the external lines according to Fig. A.1. If we additionally set the momenta to $\mathbf{k} = \mathbf{p} + \mathbf{q}$ the δ-function yields 1 and we conclude from phenomenology that the expression

$$\mathcal{C} := \left(\vec{\varepsilon}^{\dagger}_{\underline{\widetilde{\beta}}} \right)_{\widetilde{\beta}} (\mathcal{O}_{ij})_{\widetilde{\alpha}, \widetilde{\beta}\widetilde{\gamma}} \left(\vec{\varepsilon}^{\dagger}_{\underline{\widetilde{\gamma}}} \right)_{\widetilde{\gamma}} (\vec{\varepsilon}_{\widetilde{\alpha}})_{\widetilde{\alpha}} \ , \tag{A.8}$$

as part of the amplitude $T^{\widetilde{\beta}\,\widetilde{\gamma}}_{\underline{\widetilde{\alpha}}}(\mathbf{p}, \mathbf{q})$ must be equal to the Clebsch-Gordan coefficient for coupling spin $\underline{\widetilde{\beta}}$ of A_i and spin $\widetilde{\gamma}$ of A_j to a total spin $\widetilde{\alpha}$ of d_{ij}. In the following we will check this condition exemplary for the spin coupling $1/2 \otimes 1 \to 1/2$, but first we give some remarks regarding the notation.

We write all projectors in terms of non-relativistic spinor indices $\alpha, \beta, \gamma, \ldots \in \{1, 2\}$ and in terms of the Cartesian polarization vector indices $i, j, k, \ldots \in \{1, 2, 3\}$ which describe spin $1/2$ and spin 1 interactions, respectively. The underlying group of spin $1/2$ is $SU(2)$ whose generators are the three Pauli-matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.9}$$

and for spin 1 it is the three dimensional rotation group $SO(3)$ which is generated by the matrices

$$U_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad U_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \qquad U_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{A.10}$$

For both groups the generators fulfill besides other properties the following useful relations. For the Pauli-matrices it holds:

$$\begin{aligned}
\sigma_i^2 &= \mathbb{1} \quad \forall\, i\,, \\
\mathrm{Tr}(\sigma_i) &= 0 \quad \forall\, i\,, \\
\mathrm{Tr}(\sigma_i \sigma_j) &= 2\, \delta_{ij} \quad \forall\, i, j\,, \\
\sigma_i \sigma_j &= \delta_{ij}\, \mathbb{1} + i\, \varepsilon_{ijk}\, \sigma_k \quad \forall\, i, j\,, \\
[\sigma_i, \sigma_j] &= 2i\, \varepsilon_{ijk}\, \sigma_k \quad \forall\, i, j\,, \\
\{\sigma_i, \sigma_j\} &= 2\delta_{ij}\, \mathbb{1} \quad \forall\, i, j\,.
\end{aligned} \tag{A.11}$$

And similarly the matrices U_i fulfill:

$$\begin{aligned}
(U_i)_{jk} &\equiv -i\, \varepsilon_{ijk} \quad \forall\, i, j, k\,, \\
\mathrm{Tr}(U_i) &= 0 \quad \forall\, i\,, \\
\mathrm{Tr}(U_i U_j) &= 2\, \delta_{ij} \quad \forall\, i, j\,, \\
(U_i U_j)_{k\ell} &= \delta_{ij}\, \delta_{k\ell} - \delta_{i\ell}\delta_{kj} \quad \forall\, i, j, k, \ell\,, \\
[U_i, U_j] &= i\, \varepsilon_{ijk}\, U_k \quad \forall\, i, j\,.
\end{aligned} \tag{A.12}$$

The spin indices we are working with are given in the Cartesian polarization basis, namely, they correspond to the spin $1/2$ polarization vectors $\vec{\chi}_\alpha$ and the spin 1 polarization vectors $\vec{\varepsilon}_{\underline{i}}$ given by

$$\left\{ \vec{\chi}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \,,\, \vec{\chi}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad \left\{ \vec{\varepsilon}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \,,\, \vec{\varepsilon}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \,,\, \vec{\varepsilon}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}\,, \qquad (\text{A.13})$$

respectively. Since $\vec{\chi}_{1,2}$ are the eigenvectors to the eigenvalues $\lambda_{1,2} = \pm 1/2$ of the matrix $\sigma_3/2$ one can directly identify $\vec{\chi}_\alpha$ with the physical spin states defined by their magnetic quantum number m:

$$\vec{\chi}_1 \,\widehat{=}\, m = +\frac{1}{2}\,,$$
$$\vec{\chi}_2 \,\widehat{=}\, m = -\frac{1}{2}\,. \qquad (\text{A.14})$$

In the spin 1 case one has to perform a change of basis from Cartesian ($\{\vec{\varepsilon}_1, \vec{\varepsilon}_2, \vec{\varepsilon}_3\}$) to spherical ($\{\vec{\varepsilon}_+, \vec{\varepsilon}_0, \vec{\varepsilon}_-\}$) polarization vectors where the momentum $\mathbf{p}$ points in z-direction. The latter can be identified with physical spin states $m = 0,\ \pm 1$ in the following manner:

$$\vec{\varepsilon}_+ = -\frac{1}{\sqrt{2}}(\vec{\varepsilon}_1 + i\vec{\varepsilon}_2) = -\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \,\widehat{=}\, m = +1\,,$$

$$\vec{\varepsilon}_0 = \vec{\varepsilon}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \,\widehat{=}\, m = 0\,,$$

$$\vec{\varepsilon}_- = \frac{1}{\sqrt{2}}(\vec{\varepsilon}_1 - i\vec{\varepsilon}_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \,\widehat{=}\, m = -1\,, \qquad (\text{A.15})$$

meaning that $\vec{\varepsilon}_+$, $\vec{\varepsilon}_0$ and $\vec{\varepsilon}_-$ are the normalized eigenvectors to the eigenvalues $+1$, 0 and -1 of the generator U_3. Alternatively, one can maintain the Cartesian basis indices in the Lagrangian density and insert

$$j = -\frac{1}{\sqrt{2}}(1 + i2) \quad \text{for a } m = +1 \text{ state,}$$

$$j = \frac{1}{\sqrt{2}}(1 - i2) \quad \text{for a } m = -1 \text{ state,}$$

$$j = 3 \quad \text{for a } m = 0 \text{ state,} \qquad (\text{A.16})$$

instead (note, that the i on the right-hand-side is the imaginary unit while j on the left is a spin index). This is the method we will use.

Ket spin vectors $|j, m\rangle$ are then related to the polarization vectors via

$$\vec{\chi}_{\underline{\alpha}} = \left|\frac{1}{2}, \frac{1}{2}\right\rangle, \quad \text{for } \underline{\alpha} = 1,$$

$$\vec{\chi}_{\underline{\alpha}} = \left|\frac{1}{2}, -\frac{1}{2}\right\rangle, \quad \text{for } \underline{\alpha} = 2,$$

$$\vec{\varepsilon}_{\underline{i}} = |1, 1\rangle, \quad \text{for } \underline{i} = -\tfrac{1}{\sqrt{2}}(1 + i2),$$

$$\vec{\varepsilon}_{\underline{i}} = |1, 0\rangle, \quad \text{for } \underline{i} = 3,$$

$$\vec{\varepsilon}_{\underline{i}} = |1, -1\rangle, \quad \text{for } \underline{i} = -\tfrac{1}{\sqrt{2}}(1 - i2). \tag{A.17}$$

From the Clebsch-Gordan coefficients it additionally follows that one can identify for spin $3/2$ the ket vectors with

$$\vec{\varepsilon}_{\underline{i}}\,\vec{\chi}_{\underline{\alpha}} = \left|\frac{3}{2}, \frac{3}{2}\right\rangle, \quad \text{for } \left\{\underline{i} = -\tfrac{1}{\sqrt{2}}(1 + i2), \underline{\alpha} = 1\right\},$$

$$\vec{\varepsilon}_{\underline{i}}\,\vec{\chi}_{\underline{\alpha}} = \left|\frac{3}{2}, \frac{1}{2}\right\rangle, \quad \text{for } \tfrac{1}{\sqrt{3}}\left\{\underline{i} = -\tfrac{1}{\sqrt{2}}(1 + i2), \underline{\alpha} = 2\right\} + \tfrac{\sqrt{2}}{\sqrt{3}}\left\{\underline{i} = 3, \underline{\alpha} = 1\right\},$$

$$\vec{\varepsilon}_{\underline{i}}\,\vec{\chi}_{\underline{\alpha}} = \left|\frac{3}{2}, -\frac{1}{2}\right\rangle, \quad \text{for } \tfrac{\sqrt{2}}{\sqrt{3}}\left\{\underline{i} = 3, \underline{\alpha} = 2\right\} + \tfrac{1}{\sqrt{3}}\left\{\underline{i} = \tfrac{1}{\sqrt{2}}(1 - i2), \underline{\alpha} = 1\right\},$$

$$\vec{\varepsilon}_{\underline{i}}\,\vec{\chi}_{\underline{\alpha}} = \left|\frac{3}{2}, -\frac{3}{2}\right\rangle, \quad \text{for } \left\{\underline{i} = \tfrac{1}{\sqrt{2}}(1 - i2), \underline{\alpha} = 2\right\}, \tag{A.18}$$

and similarly for spin 2:

$$\vec{\varepsilon}_{\underline{i}}\,\vec{\varepsilon}_{\underline{j}} = |2, 2\rangle, \quad \text{for } \left\{\underline{i} = -\tfrac{1}{\sqrt{2}}(1 + i2), \underline{j} = -\tfrac{1}{\sqrt{2}}(1 + i2)\right\},$$

$$\vec{\varepsilon}_{\underline{i}}\,\vec{\varepsilon}_{\underline{j}} = |2, 1\rangle, \quad \text{for } \tfrac{1}{\sqrt{2}}\left\{\underline{i} = -\tfrac{1}{\sqrt{2}}(1 + i2), \underline{j} = 3\right\} + \tfrac{1}{\sqrt{2}}\left\{\underline{i} = 3, \underline{j} = -\tfrac{1}{\sqrt{2}}(1 + i2)\right\},$$

$$\vec{\varepsilon}_{\underline{i}}\,\vec{\varepsilon}_{\underline{j}} = |2, 0\rangle, \quad \text{for } \tfrac{1}{\sqrt{6}}\left\{\underline{i} = -\tfrac{1}{\sqrt{2}}(1 + i2), \underline{j} = \tfrac{1}{\sqrt{2}}(1 - i2)\right\} + \tfrac{\sqrt{2}}{\sqrt{3}}\left\{\underline{i} = 3, \underline{j} = 3\right\},$$

$$+ \tfrac{1}{\sqrt{6}}\left\{\underline{i} = \tfrac{1}{\sqrt{2}}(1 - i2), \underline{j} = -\tfrac{1}{\sqrt{2}}(1 + i2)\right\},$$

$$\vec{\varepsilon}_{\underline{i}}\,\vec{\varepsilon}_{\underline{j}} = |2, -1\rangle, \quad \text{for } \tfrac{1}{\sqrt{2}}\left\{\underline{i} = 3, \underline{j} = \tfrac{1}{\sqrt{2}}(1 - i2)\right\} + \tfrac{1}{\sqrt{2}}\left\{\underline{i} = \tfrac{1}{\sqrt{2}}(1 - i2), \underline{j} = 3\right\},$$

$$\vec{\varepsilon}_{\underline{i}}\,\vec{\varepsilon}_{\underline{j}} = |2, -2\rangle, \quad \text{for } \left\{\underline{i} = \tfrac{1}{\sqrt{2}}(1 - i2), \underline{j} = \tfrac{1}{\sqrt{2}}(1 - i2)\right\}. \tag{A.19}$$

The corresponding bra spin vectors $\langle j, m| = (|j, m\rangle)^{\dagger}$ can be found by Hermitian conjugation of the relations above.

Let us now consider the coupling $\Big(J(A_i) = 1/2\Big) \otimes \Big(J(A_j) = 1\Big) \to \Big(J(d_{ij}) = 1/2\Big)$. From Tab. A.1 we take the corresponding projection operator

$$(\mathcal{O}_{ij})_{\widetilde{\alpha}, \widetilde{\beta}\widetilde{\gamma}} = \frac{1}{\sqrt{3}}\,(\sigma_g)_{\beta\alpha},$$

where $\widetilde{\alpha} = \alpha$ and $\widetilde{\beta} = \beta$ are spin 1/2 indices and $\widetilde{\gamma} = g$ is a spin 1 index. Plugging this into Eq. (A.8) we find

$$
\mathcal{C} = \left(\vec{\chi}_{\underline{\beta}}^{\dagger}\right)_{\beta} \frac{1}{\sqrt{3}} (\sigma_g)_{\alpha\beta} \left(\vec{\varepsilon}_{\underline{g}}^{\dagger}\right)_g (\vec{\chi}_{\underline{\alpha}})_{\alpha}
$$

$$
= \frac{1}{\sqrt{3}} \left[\left(\vec{\chi}_{\underline{\beta}}^{\dagger}\right)_1 \left(\left(\vec{\varepsilon}_{\underline{g}}^{\dagger}\right)_1 - i \left(\vec{\varepsilon}_{\underline{g}}^{\dagger}\right)_2 \right) (\vec{\chi}_{\underline{\alpha}})_2 + \left(\vec{\chi}_{\underline{\beta}}^{\dagger}\right)_2 \left(\left(\vec{\varepsilon}_{\underline{g}}^{\dagger}\right)_1 + i \left(\vec{\varepsilon}_{\underline{g}}^{\dagger}\right)_2 \right) (\vec{\chi}_{\underline{\alpha}})_1 \right.
$$

$$
\left. + \left(\vec{\chi}_{\underline{\beta}}^{\dagger}\right)_1 \left(\vec{\varepsilon}_{\underline{g}}^{\dagger}\right)_3 (\vec{\chi}_{\underline{\alpha}})_1 - \left(\vec{\chi}_{\underline{\beta}}^{\dagger}\right)_2 \left(\vec{\varepsilon}_{\underline{g}}^{\dagger}\right)_3 (\vec{\chi}_{\underline{\alpha}})_2 \right]. \tag{A.20}
$$

Depending on $\underline{\alpha}$, $\underline{\beta}$ and $\underline{g}$ the equation above yields

- for $\underline{\alpha} = 1$, $\underline{\beta} = 1$:

$$
\mathcal{C} = \frac{1}{\sqrt{3}} \left(\vec{\varepsilon}_{\underline{g}}^{\dagger}\right)_3 = \begin{cases} 0, & \text{for } \underline{g} = -\frac{1}{\sqrt{2}}(1 - i2) \\ \frac{1}{\sqrt{3}}, & \text{for } \underline{g} = 3 \\ 0, & \text{for } \underline{g} = \frac{1}{2}(1 + i2) \end{cases}, \tag{A.21}
$$

- for $\underline{\alpha} = 2$, $\underline{\beta} = 1$:

$$
\mathcal{C} = \frac{1}{\sqrt{3}} \left(\left(\vec{\varepsilon}_{\underline{g}}^{\dagger}\right)_1 - i \left(\vec{\varepsilon}_{\underline{g}}^{\dagger}\right)_2 \right) = \begin{cases} \frac{1}{\sqrt{3}} \left(-\frac{1}{\sqrt{2}} - i\frac{i}{\sqrt{2}} \right) = 0, & \text{for } \underline{g} = -\frac{1}{\sqrt{2}}(1 - i2) \\ 0, & \text{for } \underline{g} = 3 \\ \frac{1}{\sqrt{3}} \left(\frac{1}{\sqrt{2}} - i\frac{i}{\sqrt{2}} \right) = \frac{\sqrt{2}}{\sqrt{3}}, & \text{for } \underline{g} = \frac{1}{2}(1 + i2) \end{cases}, \tag{A.22}
$$

- for $\underline{\alpha} = 1$, $\underline{\beta} = 2$:

$$
\mathcal{C} = \frac{1}{\sqrt{3}} \left(\left(\vec{\varepsilon}_{\underline{g}}^{\dagger}\right)_1 + i \left(\vec{\varepsilon}_{\underline{g}}^{\dagger}\right)_2 \right) = \begin{cases} \frac{1}{\sqrt{3}} \left(-\frac{1}{\sqrt{2}} + i\frac{i}{\sqrt{2}} \right) = -\frac{\sqrt{2}}{\sqrt{3}}, & \text{for } \underline{g} = -\frac{1}{\sqrt{2}}(1 - i2) \\ 0, & \text{for } \underline{g} = 3 \\ \frac{1}{\sqrt{3}} \left(\frac{1}{\sqrt{2}} + i\frac{i}{\sqrt{2}} \right) = 0, & \text{for } \underline{g} = \frac{1}{2}(1 + i2) \end{cases}, \tag{A.23}
$$

- for $\underline{\alpha} = 2$, $\underline{\beta} = 2$:

$$
\mathcal{C} = -\frac{1}{\sqrt{3}} \left(\vec{\varepsilon}_{\underline{g}}^{\dagger}\right)_3 = \begin{cases} 0, & \text{for } \underline{g} = -\frac{1}{\sqrt{2}}(1 - i2) \\ -\frac{1}{\sqrt{3}}, & \text{for } \underline{g} = 3 \\ 0, & \text{for } \underline{g} = \frac{1}{2}(1 + i2) \end{cases}. \tag{A.24}
$$

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These results can be compared with the Clebsch-Gordan coefficients in Ref. [2]:

$$\left(\langle\tfrac{1}{2},\tfrac{1}{2}|\otimes\langle 1,1|\right)|\tfrac{1}{2},1;\tfrac{1}{2},\tfrac{1}{2}\rangle=(-1)^{\frac{1}{2}-1-\frac{1}{2}}\left(\langle 1,1|\otimes\langle\tfrac{1}{2},\tfrac{1}{2}|\right)|1,\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2}\rangle=0\,,$$

$$\left(\langle\tfrac{1}{2},\tfrac{1}{2}|\otimes\langle 1,0|\right)|\tfrac{1}{2},1;\tfrac{1}{2},\tfrac{1}{2}\rangle=(-1)^{\frac{1}{2}-1-\frac{1}{2}}\left(\langle 1,0|\otimes\langle\tfrac{1}{2},\tfrac{1}{2}|\right)|1,\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2}\rangle=\tfrac{1}{\sqrt{3}}\,,$$

$$\left(\langle\tfrac{1}{2},\tfrac{1}{2}|\otimes\langle 1,-1|\right)|\tfrac{1}{2},1;\tfrac{1}{2},\tfrac{1}{2}\rangle=(-1)^{\frac{1}{2}-1-\frac{1}{2}}\left(\langle 1,-1|\otimes\langle\tfrac{1}{2},\tfrac{1}{2}|\right)|1,\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2}\rangle=0\,, \qquad (A.25)$$

$$\left(\langle\tfrac{1}{2},\tfrac{1}{2}|\otimes\langle 1,1|\right)|\tfrac{1}{2},1;\tfrac{1}{2},-\tfrac{1}{2}\rangle=(-1)^{\frac{1}{2}-1-\frac{1}{2}}\left(\langle 1,1|\otimes\langle\tfrac{1}{2},\tfrac{1}{2}|\right)|1,\tfrac{1}{2};\tfrac{1}{2},-\tfrac{1}{2}\rangle=0\,,$$

$$\left(\langle\tfrac{1}{2},\tfrac{1}{2}|\otimes\langle 1,0|\right)|\tfrac{1}{2},1;\tfrac{1}{2},-\tfrac{1}{2}\rangle=(-1)^{\frac{1}{2}-1-\frac{1}{2}}\left(\langle 1,0|\otimes\langle\tfrac{1}{2},\tfrac{1}{2}|\right)|1,\tfrac{1}{2};\tfrac{1}{2},-\tfrac{1}{2}\rangle=0\,,$$

$$\left(\langle\tfrac{1}{2},\tfrac{1}{2}|\otimes\langle 1,-1|\right)|\tfrac{1}{2},1;\tfrac{1}{2},-\tfrac{1}{2}\rangle=(-1)^{\frac{1}{2}-1-\frac{1}{2}}\left(\langle 1,-1|\otimes\langle\tfrac{1}{2},\tfrac{1}{2}|\right)|1,\tfrac{1}{2};\tfrac{1}{2},-\tfrac{1}{2}\rangle=\tfrac{\sqrt{2}}{\sqrt{3}}\,, \qquad (A.26)$$

$$\left(\langle\tfrac{1}{2},-\tfrac{1}{2}|\otimes\langle 1,1|\right)|\tfrac{1}{2},1;\tfrac{1}{2},\tfrac{1}{2}\rangle=(-1)^{\frac{1}{2}-1-\frac{1}{2}}\left(\langle 1,1|\otimes\langle\tfrac{1}{2},-\tfrac{1}{2}|\right)|1,\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2}\rangle=-\tfrac{\sqrt{2}}{\sqrt{3}}\,,$$

$$\left(\langle\tfrac{1}{2},-\tfrac{1}{2}|\otimes\langle 1,0|\right)|\tfrac{1}{2},1;\tfrac{1}{2},\tfrac{1}{2}\rangle=(-1)^{\frac{1}{2}-1-\frac{1}{2}}\left(\langle 1,0|\otimes\langle\tfrac{1}{2},-\tfrac{1}{2}|\right)|1,\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2}\rangle=0\,,$$

$$\left(\langle\tfrac{1}{2},-\tfrac{1}{2}|\otimes\langle 1,-1|\right)|\tfrac{1}{2},1;\tfrac{1}{2},\tfrac{1}{2}\rangle=(-1)^{\frac{1}{2}-1-\frac{1}{2}}\left(\langle 1,-1|\otimes\langle\tfrac{1}{2},-\tfrac{1}{2}|\right)|1,\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2}\rangle=0\,, \qquad (A.27)$$

$$\left(\langle\tfrac{1}{2},-\tfrac{1}{2}|\otimes\langle 1,1|\right)|\tfrac{1}{2},1;\tfrac{1}{2},-\tfrac{1}{2}\rangle=(-1)^{\frac{1}{2}-1-\frac{1}{2}}\left(\langle 1,1|\otimes\langle\tfrac{1}{2},-\tfrac{1}{2}|\right)|1,\tfrac{1}{2};\tfrac{1}{2},-\tfrac{1}{2}\rangle=0\,,$$

$$\left(\langle\tfrac{1}{2},-\tfrac{1}{2}|\otimes\langle 1,0|\right)|\tfrac{1}{2},1;\tfrac{1}{2},-\tfrac{1}{2}\rangle=(-1)^{\frac{1}{2}-1-\frac{1}{2}}\left(\langle 1,0|\otimes\langle\tfrac{1}{2},-\tfrac{1}{2}|\right)|1,\tfrac{1}{2};\tfrac{1}{2},-\tfrac{1}{2}\rangle=-\tfrac{1}{\sqrt{3}}\,,$$

$$\left(\langle\tfrac{1}{2},-\tfrac{1}{2}|\otimes\langle 1,-1|\right)|\tfrac{1}{2},1;\tfrac{1}{2},-\tfrac{1}{2}\rangle=(-1)^{\frac{1}{2}-1-\frac{1}{2}}\left(\langle 1,-1|\otimes\langle\tfrac{1}{2},-\tfrac{1}{2}|\right)|1,\tfrac{1}{2};\tfrac{1}{2},-\tfrac{1}{2}\rangle=0\,, \qquad (A.28)$$

where it was used that

$$\left(\langle j_1,m_1|\otimes\langle j_2,m_2|\right)|j_1,j_2;J,M\rangle=(-1)^{J-j_1-j_2}\left(\langle j_2,m_2|\otimes\langle j_1,m_1|\right)|j_2,j_1;J,M\rangle \qquad (A.29)$$

holds with the standard notation of Ref. [2]. Indeed, one observes that $\mathcal{C}$ is equivalent to the Clebsch-Gordan coefficient of the respective coupling. In the same way one could proof that the other projectors given in Tab. A.1 are correct.

A.2 Projection onto scattering channel

Up to now the decay amplitude Eq. (A.7) still has free (underlined) indices. In fact, $\underline{\widetilde{\beta}}$ and $\underline{\widetilde{\gamma}}$ represent two polarization vectors $\vec{\varepsilon}_{\underline{\widetilde{\beta}}}$ and $\vec{\varepsilon}_{\underline{\widetilde{\gamma}}}$ which can be coupled to – in general – more then one total spin called $\underline{\widetilde{\rho}}$. Hence, the decay amplitude with initial spin state $\underline{\widetilde{\eta}}$ and final state spin $\underline{\widetilde{\rho}}$ can be written as

$$T_{\underline{\widetilde{\eta}}}^{\underline{\widetilde{\rho}}}=\frac{1}{\sqrt{c}}\left(\mathcal{O}_T^{\dagger}\right)_{\underline{\widetilde{\rho}},\underline{\widetilde{\gamma}\widetilde{\beta}}}T_{\underline{\widetilde{\alpha}}}^{\underline{\widetilde{\beta}\widetilde{\gamma}}}\,\delta_{\underline{\widetilde{\alpha}\widetilde{\eta}}}\,, \qquad (A.30)$$

where we have introduced an operator $\mathcal{O}_T$ which projects onto a specific final state $\widetilde{\rho}$, i.e. which couples the spins $\underline{\widetilde{\beta}}$ and $\widetilde{\gamma}$ to a total spin $\widetilde{\rho}$. Since this projector must be normalized in order to do not over-count some spin states, there is an normalization factor of $1/\sqrt{c}$ added. The Kronecker-delta $\delta_{\underline{\widetilde{\alpha}}\widetilde{\eta}}$ in Eq. (A.30) is nothing else than the (already normalized) projection operator which couples "nothing" and spin $\underline{\widetilde{\alpha}}$ to spin $\widetilde{\eta}$. To get the full index-free decay amplitude one has – in the usual way, cf. Ref. [1] – to average over initial and sum over final state spins leading to

$$T = \frac{1}{\mathrm{dof}(\widetilde{\eta})} \sum_{\widetilde{\eta},\widetilde{\rho}} T_{\widetilde{\eta}}^{\widetilde{\rho}} = \frac{1}{\mathrm{dof}(\widetilde{\eta})} \sum_{\widetilde{\eta},\widetilde{\rho}} \frac{1}{\sqrt{c}} \left(\mathcal{O}_T^{\dagger}\right)_{\widetilde{\rho},\widetilde{\gamma}\widetilde{\beta}} T_{\underline{\widetilde{\alpha}}}^{\widetilde{\beta}\widetilde{\gamma}} \delta_{\underline{\widetilde{\alpha}}\widetilde{\eta}} . \tag{A.31}$$

We now introduce a convention for scattering amplitudes, namely that initial spins are written as subscript and final state spins as superscript. Next, we consider a generic two-body scattering amplitude $T_{\underline{\widetilde{\alpha}}\underline{\widetilde{\beta}}}^{\widetilde{\gamma}\widetilde{\sigma}}$ and know from the considerations above that the full amplitude T is given by

$$T = \frac{1}{\mathrm{dof}(\widetilde{\eta})} \sum_{\widetilde{\eta},\widetilde{\lambda}} T_{\widetilde{\eta}}^{\widetilde{\lambda}} = \frac{1}{\mathrm{dof}(\widetilde{\eta})} \sum_{\widetilde{\eta},\widetilde{\lambda}} \frac{1}{\sqrt{cc'}} \left(\mathcal{O}_T^{\dagger}\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} T_{\underline{\widetilde{\alpha}}\underline{\widetilde{\beta}}}^{\widetilde{\gamma}\widetilde{\sigma}} \left(\mathcal{O}_T'\right)_{\widetilde{\eta},\widetilde{\alpha}\widetilde{\beta}} , \tag{A.32}$$

with $\widetilde{\eta}$ being the total initial and $\widetilde{\lambda}$ being the total final state spin onto which the two operators $\mathcal{O}_T'$ and $\mathcal{O}_T$ project. Note, that for elastic scattering one has to set $\widetilde{\eta} = \widetilde{\lambda}$ as initial and final state must be equal in this case. The projection operators $\mathcal{O}_T^{(\prime)}$ have the same structure as the operators $\mathcal{O}$ derived above since both couple two spins to a total spin. The only difference is that the former couple the dimer spin and the spin of the third particle to the scattering channel spin while the latter couple the spins of the two constituent particles to the spin of the corresponding dimer. It is thus not necessary to give $\mathcal{O}_T$ explicitly; it can be deduced from Tab. A.1.

In the derivation of the transcendental equation for different types of systems (cf. section 3) the deduced scattering amplitudes on the one hand have the same index structure $(T_{ij})_{\underline{\widetilde{\alpha}}\underline{\widetilde{\beta}}}^{\widetilde{\gamma}\widetilde{\sigma}}$ as the generic one in Eq. (A.32). On the other hand they are given as integral equation, that is, they are proportional to themselves and to the other amplitudes T_{ik} and T_{jk}, but with different final state indices. To explain the method of projecting onto a specific spin channel it is sufficient to consider only one example as which we chose the $T_{23}(q)$ contribution to the $T_{13}(p)$ amplitude (cf. second row of the first diagram on page 158):

$$\begin{aligned}
(T_{13})_{\underline{\widetilde{\alpha}}\underline{\widetilde{\beta}}}^{\widetilde{\gamma}\widetilde{\sigma}} &\sim \left\langle \mathbf{p},\widetilde{\gamma}; -\mathbf{p},\underline{\widetilde{\sigma}} \right| \left(d_{13}^{\dagger}\right)_{\widetilde{\gamma}} (A_3)_{\widetilde{\rho}} \left(\mathcal{O}_{13}^{\dagger}\right)_{\widetilde{\gamma},\widetilde{\rho}\widetilde{\nu}} (A_1)_{\widetilde{\nu}} (T_{23})_{\widetilde{\alpha}\widetilde{\beta}}^{\widetilde{\mu}\widetilde{\nu}} \\
&\qquad \times \left(A_2^{\dagger}\right)_{\widetilde{\sigma}} (\mathcal{O}_{23})_{\widetilde{\mu},\widetilde{\sigma}\widetilde{\rho}} \left(A_3^{\dagger}\right)_{\widetilde{\rho}} (d_{23})_{\widetilde{\mu}} \left| \mathbf{k},\underline{\widetilde{\alpha}}; -\mathbf{k},\underline{\widetilde{\beta}} \right\rangle \\
&= \left(\vec{\varepsilon}_{\widetilde{\gamma}}^{\dagger}\right)_{\widetilde{\gamma}} \left(\vec{\varepsilon}_{\underline{\widetilde{\sigma}}}^{\dagger}\right)_{\widetilde{\sigma}} \left(\mathcal{O}_{13}^{\dagger}\right)_{\widetilde{\gamma},\widetilde{\rho}\widetilde{\nu}} (T_{23})_{\widetilde{\alpha}\widetilde{\beta}}^{\widetilde{\mu}\widetilde{\nu}} (\mathcal{O}_{23})_{\widetilde{\mu},\widetilde{\sigma}\widetilde{\rho}} (\vec{\varepsilon}_{\underline{\widetilde{\alpha}}})_{\widetilde{\alpha}} \left(\vec{\varepsilon}_{\underline{\widetilde{\beta}}}\right)_{\widetilde{\beta}} \\
&= \delta_{\widetilde{\gamma}\widetilde{\gamma}} \, \delta_{\underline{\widetilde{\sigma}}\widetilde{\sigma}} \left(\mathcal{O}_{13}^{\dagger}\right)_{\widetilde{\gamma},\widetilde{\rho}\widetilde{\nu}} (T_{23})_{\widetilde{\alpha}\widetilde{\beta}}^{\widetilde{\mu}\widetilde{\nu}} (\mathcal{O}_{23})_{\widetilde{\mu},\widetilde{\sigma}\widetilde{\rho}} \, \delta_{\underline{\widetilde{\alpha}}\widetilde{\alpha}} \, \delta_{\underline{\widetilde{\beta}}\widetilde{\beta}} \\
&= \left(\mathcal{O}_{13}^{\dagger}\right)_{\underline{\widetilde{\gamma}},\widetilde{\rho}\widetilde{\nu}} (T_{23})_{\underline{\widetilde{\alpha}}\underline{\widetilde{\beta}}}^{\widetilde{\mu}\widetilde{\nu}} (\mathcal{O}_{23})_{\widetilde{\mu},\underline{\widetilde{\sigma}}\widetilde{\rho}} ,
\end{aligned} \tag{A.33}$$

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where we have used in the third step that in Cartesian spin basis $(\vec{\varepsilon}_{\widetilde{\underline{\alpha}}})_{\widetilde{\alpha}} = \delta_{\widetilde{\underline{\alpha}}\widetilde{\alpha}}$ holds for all α which directly follows from the definition in Eq. (A.13). The full index-free amplitude therefore has the proportionality

$$T_{13} = \frac{1}{\mathrm{dof}(\widetilde{\underline{\eta}})} \sum_{\widetilde{\eta},\widetilde{\underline{\lambda}}} (T_{13})^{\widetilde{\underline{\lambda}}}_{\widetilde{\underline{\eta}}} \sim \frac{1}{\mathrm{dof}(\widetilde{\underline{\eta}})} \sum_{\widetilde{\eta},\widetilde{\underline{\lambda}}} \frac{1}{\sqrt{c_{T_{12}}c_{T_{13}}}} \left(\mathcal{O}^{\dagger}_{T_{13}}\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}^{\dagger}_{13}\right)_{\widetilde{\gamma},\widetilde{\rho}\widetilde{\nu}} (T_{23})^{\widetilde{\mu}\widetilde{\nu}}_{\widetilde{\underline{\alpha}}\widetilde{\beta}} (\mathcal{O}_{23})_{\widetilde{\mu},\widetilde{\sigma}\widetilde{\rho}} (\mathcal{O}_{T_{12}})_{\widetilde{\eta},\widetilde{\alpha}\widetilde{\beta}} \ .$$

(A.34)

What we want to achieve is that also the right-hand-side is written in terms of the full amplitude T_{23}. We know from Eq. (A.32) that

$$T_{23} = \frac{1}{\mathrm{dof}(\widetilde{\underline{\eta}})} \sum_{\widetilde{\eta},\widetilde{\underline{\lambda}}} \frac{1}{\sqrt{c_{T_{12}}c_{T_{23}}}} \left(\mathcal{O}^{\dagger}_{T_{23}}\right)_{\widetilde{\lambda},\widetilde{\nu}\widetilde{\mu}} (T_{23})^{\widetilde{\mu}\widetilde{\nu}}_{\widetilde{\underline{\alpha}}\widetilde{\beta}} (\mathcal{O}_{T_{12}})_{\widetilde{\eta},\widetilde{\alpha}\widetilde{\beta}} \ .$$

(A.35)

Since the projection operators are unitary, i.e.

$$\left(\mathcal{O}^{\dagger}_{T}\right)_{\widetilde{\underline{\lambda}},\widetilde{\nu}\widetilde{\mu}} (\mathcal{O}_{T})_{\widetilde{\underline{\lambda}},\widetilde{\mu}\widetilde{\nu}} = c_T \, \mathrm{dof}(\widetilde{\underline{\lambda}}) \ ,$$

(A.36)

we can write Eq. (A.34) as

$$T_{13} \sim y_4 \frac{1}{\mathrm{dof}(\widetilde{\eta})} \sum_{\widetilde{\eta},\widetilde{\underline{\lambda}}} \frac{1}{\sqrt{c_{T_{12}}c_{T_{23}}}} \left(\mathcal{O}^{\dagger}_{T_{23}}\right)_{\widetilde{\lambda},\widetilde{\nu}\widetilde{\mu}} (T_{23})^{\widetilde{\mu}\widetilde{\nu}}_{\widetilde{\underline{\alpha}}\widetilde{\beta}} (\mathcal{O}_{T_{12}})_{\widetilde{\eta},\widetilde{\alpha}\widetilde{\beta}} = y_4 \, T_{23} \ ,$$

(A.37)

with y_4 defined as

$$y_4 := \frac{1}{\sqrt{c_{T_{13}}c_{T_{23}}}} \frac{1}{\mathrm{dof}(\widetilde{\underline{\lambda}})} \left(\mathcal{O}^{\dagger}_{T_{13}}\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}^{\dagger}_{13}\right)_{\widetilde{\gamma},\widetilde{\rho}\widetilde{\nu}} (\mathcal{O}_{23})_{\widetilde{\mu},\widetilde{\sigma}\widetilde{\rho}} (\mathcal{O}_{T_{23}})_{\widetilde{\lambda},\widetilde{\nu}\widetilde{\mu}} \ .$$

(A.38)

In the same way all other x_i, y_i, z_i, $\widetilde{x}_i$, $\widetilde{y}_i$ and $\widetilde{z}_i$ parameters can be determined. Their defining equations are given in Eqs. (A.40 - A.81) where we have assumed that all operators $\mathcal{O}_T$ are normalized, i.e. $c_T = 1$, and furthermore changed the notation of indices by removing the underline. In section A.4 we have given some overview tables (Tab. A.2 to Tab. A.7) for the normalized projection operators in which the indices are named in the same way as they are needed to calculate the parameters. Note, however, that independently of the tables it holds

$$(\mathcal{O}_{ij})_{\mu,\rho\sigma} = \begin{cases} (\mathcal{O}_{ij})_{\mu,\rho\sigma} \ , & \text{if } P_i \neq P_j \\ (\mathcal{O}_{ij})_{\mu,\sigma\rho} \ , & \text{if } P_i = P_j \end{cases} \ .$$

(A.39)

because if $P_i = P_j$ there is just one possibility of assigning the indices ρ and σ to the two identical particles P_i (in contrast to the general case $P_i \neq P_j$ where – depending on the considered diagram – in a vertex either $A^{\dagger}_i$ or $A^{\dagger}_j$ is allocated with the index ρ so that according to Eq. (3.4) one has either a projector $(\mathcal{O}_{ij})_{\mu,\rho\sigma}$ or a projector $(\mathcal{O}_{ij})_{\mu,\sigma\rho}$ in the expression).

$$x^{(\prime)}_1 = \frac{1}{\mathrm{dof}(\widetilde{\eta})} \sum_{\widetilde{\eta},\widetilde{\lambda}} \left(\mathcal{O}^{\dagger}_{T^{(\prime)}_{12}}\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{12}\right)_{\widetilde{\alpha},\widetilde{\sigma}\widetilde{\rho}} \left(\mathcal{O}^{(\prime)\dagger}_{12}\right)_{\widetilde{\gamma},\widetilde{\rho}\widetilde{\beta}} \left(\mathcal{O}_{T_{12}}\right)_{\widetilde{\eta},\widetilde{\alpha}\widetilde{\beta}}$$

(A.40)

$$y_1^{(\prime)} = \frac{1}{\operatorname{dof}(\widetilde{\eta})} \sum_{\widetilde{\eta},\widetilde{\lambda}} \left(\mathcal{O}_{T_{13}^{(\prime)}}^{\dagger}\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{12}\right)_{\widetilde{\alpha},\widetilde{\sigma}\widetilde{\rho}} \left(\mathcal{O}_{13}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\beta}\widetilde{\rho}} \left(\mathcal{O}_{T_{12}}\right)_{\widetilde{\eta},\widetilde{\alpha}\widetilde{\beta}} \tag{A.41}$$

$$z_1^{(\prime)} = \frac{1}{\operatorname{dof}(\widetilde{\eta})} \sum_{\widetilde{\eta},\widetilde{\lambda}} \left(\mathcal{O}_{T_{23}^{(\prime)}}^{\dagger}\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{12}\right)_{\widetilde{\alpha},\widetilde{\sigma}\widetilde{\rho}} \left(\mathcal{O}_{23}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\beta}\widetilde{\rho}} \left(\mathcal{O}_{T_{12}}\right)_{\widetilde{\eta},\widetilde{\alpha}\widetilde{\beta}} \tag{A.42}$$

$$x_2^{(\prime)} = \frac{1}{\operatorname{dof}(\widetilde{\lambda})} \left(\mathcal{O}_{T_{12}^{(\prime)}}^{\dagger}\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{12}\right)_{\widetilde{\mu},\widetilde{\sigma}\widetilde{\rho}} \left(\mathcal{O}_{12}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\rho}\widetilde{\nu}} \left(\mathcal{O}_{T_{12}}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.43}$$

$$y_2^{(\prime)} = \frac{1}{\operatorname{dof}(\widetilde{\lambda})} \left(\mathcal{O}_{T_{13}^{(\prime)}}^{\dagger}\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{12}\right)_{\widetilde{\mu},\widetilde{\sigma}\widetilde{\rho}} \left(\mathcal{O}_{13}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\nu}\widetilde{\rho}} \left(\mathcal{O}_{T_{12}}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.44}$$

$$z_2^{(\prime)} = \frac{1}{\operatorname{dof}(\widetilde{\lambda})} \left(\mathcal{O}_{T_{23}^{(\prime)}}^{\dagger}\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{12}\right)_{\widetilde{\mu},\widetilde{\sigma}\widetilde{\rho}} \left(\mathcal{O}_{23}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\nu}\widetilde{\rho}} \left(\mathcal{O}_{T_{12}}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.45}$$

$$x_3^{(\prime)} = \frac{1}{\operatorname{dof}(\widetilde{\lambda})} \left(\mathcal{O}_{T_{12}^{(\prime)}}^{\dagger}\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{13}\right)_{\widetilde{\mu},\widetilde{\sigma}\widetilde{\rho}} \left(\mathcal{O}_{12}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\nu}\widetilde{\rho}} \left(\mathcal{O}_{T_{13}}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.46}$$

$$y_3^{(\prime)} = \frac{1}{\operatorname{dof}(\widetilde{\lambda})} \left(\mathcal{O}_{T_{13}^{(\prime)}}^{\dagger}\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{13}\right)_{\widetilde{\mu},\widetilde{\sigma}\widetilde{\rho}} \left(\mathcal{O}_{13}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\rho}\widetilde{\nu}} \left(\mathcal{O}_{T_{13}}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.47}$$

$$z_3^{(\prime)} = \frac{1}{\operatorname{dof}(\widetilde{\lambda})} \left(\mathcal{O}_{T_{23}^{(\prime)}}^{\dagger}\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{13}\right)_{\widetilde{\mu},\widetilde{\sigma}\widetilde{\rho}} \left(\mathcal{O}_{23}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\rho}\widetilde{\nu}} \left(\mathcal{O}_{T_{13}}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.48}$$

$$x_4^{(\prime)} = \frac{1}{\operatorname{dof}(\widetilde{\lambda})} \left(\mathcal{O}_{T_{12}^{(\prime)}}^{\dagger}\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{23}\right)_{\widetilde{\mu},\widetilde{\sigma}\widetilde{\rho}} \left(\mathcal{O}_{12}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\rho}\widetilde{\nu}} \left(\mathcal{O}_{T_{23}}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.49}$$

$$y_4^{(\prime)} = \frac{1}{\operatorname{dof}(\widetilde{\lambda})} \left(\mathcal{O}_{T_{13}^{(\prime)}}^{\dagger}\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{23}\right)_{\widetilde{\mu},\widetilde{\sigma}\widetilde{\rho}} \left(\mathcal{O}_{13}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\rho}\widetilde{\nu}} \left(\mathcal{O}_{T_{23}}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.50}$$

$$z_4^{(\prime)} = \frac{1}{\operatorname{dof}(\widetilde{\lambda})} \left(\mathcal{O}_{T_{23}^{(\prime)}}^{\dagger}\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{23}\right)_{\widetilde{\mu},\widetilde{\sigma}\widetilde{\rho}} \left(\mathcal{O}_{23}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\rho}\widetilde{\nu}} \left(\mathcal{O}_{T_{23}}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.51}$$

$$x_5^{(\prime)} = \frac{1}{\operatorname{dof}(\widetilde{\lambda})} \left(\mathcal{O}_{T_{12}^{(\prime)}}^{\dagger}\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{12}'\right)_{\widetilde{\mu},\widetilde{\sigma}\widetilde{\rho}} \left(\mathcal{O}_{12}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\rho}\widetilde{\nu}} \left(\mathcal{O}_{T_{12}'}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.52}$$

$$y_5^{(\prime)} = \frac{1}{\operatorname{dof}(\widetilde{\lambda})} \left(\mathcal{O}_{T_{13}^{(\prime)}}^{\dagger}\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{12}'\right)_{\widetilde{\mu},\widetilde{\sigma}\widetilde{\rho}} \left(\mathcal{O}_{13}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\nu}\widetilde{\rho}} \left(\mathcal{O}_{T_{12}'}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.53}$$

$$z_5^{(\prime)} = \frac{1}{\operatorname{dof}(\widetilde{\lambda})} \left(\mathcal{O}_{T_{23}^{(\prime)}}^{\dagger}\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{12}'\right)_{\widetilde{\mu},\widetilde{\sigma}\widetilde{\rho}} \left(\mathcal{O}_{23}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\nu}\widetilde{\rho}} \left(\mathcal{O}_{T_{12}'}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.54}$$

$$x_6^{(\prime)} = \frac{1}{\operatorname{dof}(\widetilde{\lambda})} \left(\mathcal{O}_{T_{12}^{(\prime)}}^{\dagger}\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{13}'\right)_{\widetilde{\mu},\widetilde{\sigma}\widetilde{\rho}} \left(\mathcal{O}_{12}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\nu}\widetilde{\rho}} \left(\mathcal{O}_{T_{13}'}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.55}$$

$$y_6^{(\prime)} = \frac{1}{\text{dof}(\widetilde{\lambda})} \left(\mathcal{O}_{T_{13}^{(\prime)}}^\dagger\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{13}'\right)_{\widetilde{\mu},\widetilde{\sigma}\widetilde{\rho}} \left(\mathcal{O}_{13}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\rho}\widetilde{\nu}} \left(\mathcal{O}_{T_{13}'}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.56}$$

$$z_6^{(\prime)} = \frac{1}{\text{dof}(\widetilde{\lambda})} \left(\mathcal{O}_{T_{23}^{(\prime)}}^\dagger\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{13}'\right)_{\widetilde{\mu},\widetilde{\sigma}\widetilde{\rho}} \left(\mathcal{O}_{23}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\rho}\widetilde{\nu}} \left(\mathcal{O}_{T_{13}'}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.57}$$

$$x_7^{(\prime)} = \frac{1}{\text{dof}(\widetilde{\lambda})} \left(\mathcal{O}_{T_{12}^{(\prime)}}^\dagger\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{23}'\right)_{\widetilde{\mu},\widetilde{\sigma}\widetilde{\rho}} \left(\mathcal{O}_{12}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\rho}\widetilde{\nu}} \left(\mathcal{O}_{T_{23}'}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.58}$$

$$y_7^{(\prime)} = \frac{1}{\text{dof}(\widetilde{\lambda})} \left(\mathcal{O}_{T_{13}^{(\prime)}}^\dagger\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{23}'\right)_{\widetilde{\mu},\widetilde{\sigma}\widetilde{\rho}} \left(\mathcal{O}_{13}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\rho}\widetilde{\nu}} \left(\mathcal{O}_{T_{23}'}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.59}$$

$$z_7^{(\prime)} = \frac{1}{\text{dof}(\widetilde{\lambda})} \left(\mathcal{O}_{T_{23}^{(\prime)}}^\dagger\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{23}'\right)_{\widetilde{\mu},\widetilde{\sigma}\widetilde{\rho}} \left(\mathcal{O}_{23}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\rho}\widetilde{\nu}} \left(\mathcal{O}_{T_{23}'}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.60}$$

$$\widetilde{x}_1^{(\prime)} = \frac{1}{\text{dof}(\widetilde{\eta})} \sum_{\widetilde{\eta},\widetilde{\lambda}} \left(\mathcal{O}_{T_{12}^{(\prime)}}^\dagger\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{12}\right)_{\widetilde{\alpha},\widetilde{\rho}\widetilde{\sigma}} \left(\mathcal{O}_{12}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\beta}\widetilde{\rho}} \left(\mathcal{O}_{T_{12}}\right)_{\widetilde{\eta},\widetilde{\alpha}\widetilde{\beta}} \tag{A.61}$$

$$\widetilde{y}_1^{(\prime)} = \frac{1}{\text{dof}(\widetilde{\eta})} \sum_{\widetilde{\eta},\widetilde{\lambda}} \left(\mathcal{O}_{T_{13}^{(\prime)}}^\dagger\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{12}\right)_{\widetilde{\alpha},\widetilde{\rho}\widetilde{\sigma}} \left(\mathcal{O}_{13}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\beta}\widetilde{\rho}} \left(\mathcal{O}_{T_{12}}\right)_{\widetilde{\eta},\widetilde{\alpha}\widetilde{\beta}} \tag{A.62}$$

$$\widetilde{z}_1^{(\prime)} = \frac{1}{\text{dof}(\widetilde{\eta})} \sum_{\widetilde{\eta},\widetilde{\lambda}} \left(\mathcal{O}_{T_{23}^{(\prime)}}^\dagger\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{12}\right)_{\widetilde{\alpha},\widetilde{\rho}\widetilde{\sigma}} \left(\mathcal{O}_{23}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\beta}\widetilde{\rho}} \left(\mathcal{O}_{T_{12}}\right)_{\widetilde{\eta},\widetilde{\alpha}\widetilde{\beta}} \tag{A.63}$$

$$\widetilde{x}_2^{(\prime)} = \frac{1}{\text{dof}(\widetilde{\lambda})} \left(\mathcal{O}_{T_{12}^{(\prime)}}^\dagger\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{12}\right)_{\widetilde{\mu},\widetilde{\rho}\widetilde{\sigma}} \left(\mathcal{O}_{12}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\nu}\widetilde{\rho}} \left(\mathcal{O}_{T_{12}}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.64}$$

$$\widetilde{y}_2^{(\prime)} = \frac{1}{\text{dof}(\widetilde{\lambda})} \left(\mathcal{O}_{T_{13}^{(\prime)}}^\dagger\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{12}\right)_{\widetilde{\mu},\widetilde{\rho}\widetilde{\sigma}} \left(\mathcal{O}_{13}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\nu}\widetilde{\rho}} \left(\mathcal{O}_{T_{12}}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.65}$$

$$\widetilde{z}_2^{(\prime)} = \frac{1}{\text{dof}(\widetilde{\lambda})} \left(\mathcal{O}_{T_{23}^{(\prime)}}^\dagger\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{12}\right)_{\widetilde{\mu},\widetilde{\rho}\widetilde{\sigma}} \left(\mathcal{O}_{23}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\nu}\widetilde{\rho}} \left(\mathcal{O}_{T_{12}}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.66}$$

$$\widetilde{x}_3^{(\prime)} = \frac{1}{\text{dof}(\widetilde{\lambda})} \left(\mathcal{O}_{T_{12}^{(\prime)}}^\dagger\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{13}\right)_{\widetilde{\mu},\widetilde{\rho}\widetilde{\sigma}} \left(\mathcal{O}_{12}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\nu}\widetilde{\rho}} \left(\mathcal{O}_{T_{13}}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.67}$$

$$\widetilde{y}_3^{(\prime)} = \frac{1}{\text{dof}(\widetilde{\lambda})} \left(\mathcal{O}_{T_{13}^{(\prime)}}^\dagger\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{13}\right)_{\widetilde{\mu},\widetilde{\rho}\widetilde{\sigma}} \left(\mathcal{O}_{13}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\nu}\widetilde{\rho}} \left(\mathcal{O}_{T_{13}}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.68}$$

$$\widetilde{z}_3^{(\prime)} = \frac{1}{\text{dof}(\widetilde{\lambda})} \left(\mathcal{O}_{T_{23}^{(\prime)}}^\dagger\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{13}\right)_{\widetilde{\mu},\widetilde{\rho}\widetilde{\sigma}} \left(\mathcal{O}_{23}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\rho}\widetilde{\nu}} \left(\mathcal{O}_{T_{13}}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.69}$$

$$\widetilde{x}_4^{(\prime)} = \frac{1}{\text{dof}(\widetilde{\lambda})} \left(\mathcal{O}_{T_{12}^{(\prime)}}^\dagger\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{23}\right)_{\widetilde{\mu},\widetilde{\rho}\widetilde{\sigma}} \left(\mathcal{O}_{12}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\rho}\widetilde{\nu}} \left(\mathcal{O}_{T_{23}}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.70}$$

$$\widetilde{y}_4^{(\prime)} = \frac{1}{\mathrm{dof}(\widetilde{\lambda})} \left(\mathcal{O}^\dagger_{T_{13}^{(\prime)}}\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{23}\right)_{\widetilde{\mu},\widetilde{\rho}\widetilde{\sigma}} \left(\mathcal{O}_{13}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\rho}\widetilde{\nu}} \left(\mathcal{O}_{T_{23}}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.71}$$

$$\widetilde{z}_4^{(\prime)} = \frac{1}{\mathrm{dof}(\widetilde{\lambda})} \left(\mathcal{O}^\dagger_{T_{23}^{(\prime)}}\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{23}\right)_{\widetilde{\mu},\widetilde{\rho}\widetilde{\sigma}} \left(\mathcal{O}_{23}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\nu}\widetilde{\rho}} \left(\mathcal{O}_{T_{23}}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.72}$$

$$\widetilde{x}_5^{(\prime)} = \frac{1}{\mathrm{dof}(\widetilde{\lambda})} \left(\mathcal{O}^\dagger_{T_{12}^{(\prime)}}\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}'_{12}\right)_{\widetilde{\mu},\widetilde{\rho}\widetilde{\sigma}} \left(\mathcal{O}_{12}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\nu}\widetilde{\rho}} \left(\mathcal{O}_{T'_{12}}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.73}$$

$$\widetilde{y}_5^{(\prime)} = \frac{1}{\mathrm{dof}(\widetilde{\lambda})} \left(\mathcal{O}^\dagger_{T_{13}^{(\prime)}}\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}'_{12}\right)_{\widetilde{\mu},\widetilde{\rho}\widetilde{\sigma}} \left(\mathcal{O}_{13}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\nu}\widetilde{\rho}} \left(\mathcal{O}_{T'_{12}}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.74}$$

$$\widetilde{z}_5^{(\prime)} = \frac{1}{\mathrm{dof}(\widetilde{\rho})} \left(\mathcal{O}^\dagger_{T_{23}^{(\prime)}}\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}'_{12}\right)_{\widetilde{\mu},\widetilde{\sigma}\widetilde{\sigma}} \left(\mathcal{O}_{23}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\nu}\widetilde{\rho}} \left(\mathcal{O}_{T'_{12}}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.75}$$

$$\widetilde{x}_6^{(\prime)} = \frac{1}{\mathrm{dof}(\widetilde{\lambda})} \left(\mathcal{O}^\dagger_{T_{12}^{(\prime)}}\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}'_{13}\right)_{\widetilde{\mu},\widetilde{\rho}\widetilde{\sigma}} \left(\mathcal{O}_{12}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\nu}\widetilde{\rho}} \left(\mathcal{O}_{T'_{13}}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.76}$$

$$\widetilde{y}_6^{(\prime)} = \frac{1}{\mathrm{dof}(\widetilde{\lambda})} \left(\mathcal{O}^\dagger_{T_{13}^{(\prime)}}\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}'_{13}\right)_{\widetilde{\mu},\widetilde{\rho}\widetilde{\sigma}} \left(\mathcal{O}_{13}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\nu}\widetilde{\rho}} \left(\mathcal{O}_{T'_{13}}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.77}$$

$$\widetilde{z}_6^{(\prime)} = \frac{1}{\mathrm{dof}(\widetilde{\lambda})} \left(\mathcal{O}^\dagger_{T_{23}^{(\prime)}}\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}'_{13}\right)_{\widetilde{\mu},\widetilde{\rho}\widetilde{\sigma}} \left(\mathcal{O}_{23}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\rho}\widetilde{\nu}} \left(\mathcal{O}_{T'_{13}}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.78}$$

$$\widetilde{x}_7^{(\prime)} = \frac{1}{\mathrm{dof}(\widetilde{\lambda})} \left(\mathcal{O}^\dagger_{T_{12}^{(\prime)}}\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}'_{23}\right)_{\widetilde{\mu},\widetilde{\rho}\widetilde{\sigma}} \left(\mathcal{O}_{12}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\rho}\widetilde{\nu}} \left(\mathcal{O}_{T'_{23}}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.79}$$

$$\widetilde{y}_7^{(\prime)} = \frac{1}{\mathrm{dof}(\widetilde{\lambda})} \left(\mathcal{O}^\dagger_{T_{13}^{(\prime)}}\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}'_{23}\right)_{\widetilde{\mu},\widetilde{\rho}\widetilde{\sigma}} \left(\mathcal{O}_{13}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\rho}\widetilde{\nu}} \left(\mathcal{O}_{T'_{23}}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.80}$$

$$\widetilde{z}_7^{(\prime)} = \frac{1}{\mathrm{dof}(\widetilde{\lambda})} \left(\mathcal{O}^\dagger_{T_{23}^{(\prime)}}\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}'_{23}\right)_{\widetilde{\mu},\widetilde{\rho}\widetilde{\sigma}} \left(\mathcal{O}_{23}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\nu}\widetilde{\rho}} \left(\mathcal{O}_{T'_{23}}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} \tag{A.81}$$

A.3 x, y, z parameters and the 6-J-symbol

Wigner introduced the 3-J symbol [169],

$$\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & M \end{pmatrix},$$

to describe the coupling of two angular momenta j_1 and j_2 to a total angular momentum J for given associated magnetic quantum numbers m_1, m_2 and M. It is related to the usual Clebsch-Gordan coefficients through [170]

$$\left(\langle j_1, m_1| \otimes \langle j_2, m_2| \right) |J, M\rangle = (-1)^{j_1-j_2+M}\sqrt{2J+1} \begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M \end{pmatrix}, \tag{A.82}$$

and only non-zero if the following properties are fulfilled [170]:

- $m_1 \in \{-|j_1|, ..., |j_1|\}$, $m_2 \in \{-|j_2|, ..., |j_2|\}$ and $M \in \{-|J|, ..., |J|\}$,

- $m_1 + m_2 = M$,

- $|j_1 - j_2| \leq J \leq j_1 + j_2$,

- $j_1 + j_2 + J \in \mathbb{N}$.

As it is discussed in Ref. [170] there are – besides a large number of Regge symmetries – three important symmetries of the 3-J symbol: it is invariant under even permutations of its columns, i.e.

$$
\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & M \end{pmatrix} = \begin{pmatrix} j_2 & J & j_1 \\ m_2 & M & m_1 \end{pmatrix} = \begin{pmatrix} J & j_1 & j_1 \\ M & m_1 & m_2 \end{pmatrix} ,
\tag{A.83}
$$

but receive a phase factor for odd permutations:

$$
\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & M \end{pmatrix} = (-1)^{j_1+j_2+J} \begin{pmatrix} j_2 & j_1 & J \\ m_2 & m_1 & M \end{pmatrix} = (-1)^{j_1+j_2+J} \begin{pmatrix} j_1 & J & j_2 \\ m_1 & M & m_2 \end{pmatrix} .
\tag{A.84}
$$

The third property is that also changing the sign of all magnetic quantum numbers yields a phase:

$$
\begin{pmatrix} j_1 & j_2 & J \\ -m_1 & -m_2 & -M \end{pmatrix} = (-1)^{j_1+j_2+J} \begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & M \end{pmatrix} .
\tag{A.85}
$$

If we consider Eq. (A.8) and again use that in the Cartesian basis $(\vec{\varepsilon}_{\underline{\widetilde{\alpha}}})_{\widetilde{\alpha}} = \delta_{\widetilde{\alpha}\widetilde{\alpha}}$ holds (cf. Eq. (A.13)) we conclude that $\mathcal{C}$ can be written as

$$
\mathcal{C} := \left(\vec{\varepsilon}^{\,\dagger}_{\underline{\widetilde{\beta}}} \right)_{\widetilde{\beta}} (\mathcal{O}_{ij})_{\widetilde{\alpha},\widetilde{\beta}\widetilde{\gamma}} \left(\vec{\varepsilon}^{\,\dagger}_{\underline{\widetilde{\gamma}}} \right)_{\widetilde{\gamma}} (\vec{\varepsilon}_{\underline{\widetilde{\alpha}}})_{\widetilde{\alpha}} = (\mathcal{O}_{ij})_{\underline{\widetilde{\alpha}},\underline{\widetilde{\beta}}\underline{\widetilde{\gamma}}} .
\tag{A.86}
$$

As it was done in the defining equations for the x, y, z parameters (Eqs. (A.40 - A.81)) we change the index notation by removing the underline and conclude that

$$
\mathcal{C} = (\mathcal{O}_{ij})_{\widetilde{\alpha},\widetilde{\beta}\widetilde{\gamma}} = \left(\langle j_i, m_i| \otimes \langle j_j, m_j| \right) |J_{ij}, M_{ij}\rangle ,
\tag{A.87}
$$

since phenomenology tells us that $\mathcal{C}$ is equal to the corresponding Clebsch-Gordan coefficient for coupling spin j_i of particle A_i and spin j_j of A_j to the spin J_{ij} of the dimer d_{ij}. Consequently, one can use Eq. (A.82) and write

$$
(\mathcal{O}_{ij})_{\widetilde{\alpha},\widetilde{\beta}\widetilde{\gamma}} = (-1)^{j_i-j_j+M_{ij}} \sqrt{2J_{ij}+1} \begin{pmatrix} j_i & j_j & J_{ij} \\ m_i & m_j & -M_{ij} \end{pmatrix} .
\tag{A.88}
$$

However, in the coupled integral equations appear at the same time projectors $(\mathcal{O}_{ij})_{\widetilde{\alpha},\widetilde{\gamma}\widetilde{\beta}}$ with interchanged second and third index (note, that "interchanged" does not mean "renamed" since in each scattering process the indices are fixed by the particle allocation). Thus, the question arises how the relation Eq. (A.88) is changed? The operator $\mathcal{O}_{ij}$ couples the fields $A_i^\dagger$ and $A_j^\dagger$

and according to the order of appearance of these particles within the dimer wave function (fixed in Eq. (3.4)), the second index corresponds to particle P_i and the third to P_j (cf. Lagrangian density in Eq. (3.23)). However, this is only true for distinguishable particles P_i and P_j. If they are identical the swap of them obviously does not change the Clebsch-Gordan coefficient. Therefore the interchange of second and third index of $\mathcal{O}_{ij}$ yields the relation

$$(\mathcal{O}_{ij})_{\widetilde{\alpha},\widetilde{\gamma}\widetilde{\beta}} = \begin{cases} \Big(\langle j_j, m_j| \otimes \langle j_i, m_i| \Big) |J_{ij}, M_{ij}\rangle \,, & \text{for } P_i \neq P_j \\ \Big(\langle j_i, m_i| \otimes \langle j_j, m_j| \Big) |J_{ij}, M_{ij}\rangle \,, & \text{for } P_i = P_j \end{cases}, \tag{A.89}$$

where the total spin J_{ij} is of course not changed. However, in the upper case this is not our convention regarding the order of the constituents within a dimer wave function. Hence, we remember the relation in Eq. (A.29) and write

$$(\mathcal{O}_{ij})_{\widetilde{\alpha},\widetilde{\gamma}\widetilde{\beta}} = (-1)^{(1-\delta_{P_i P_j})(J_{ij}-j_i-j_j)} \Big(\langle j_i, m_i| \otimes \langle j_j, m_j| \Big) |J_{ij}, M_{ij}\rangle \,, \tag{A.90}$$

which holds for the same set of indices fixed in Eq. (A.87) and where the factor $\delta_{P_i P_j}$ ensures that nothing is changed if the particles are identical. Since the right-hand-side yields a (real) Clebsch-Gordan coefficient it is Hermitian and thus the left-hand-side must be. Consequently, Hermitian conjugation does not change Eq. (A.87). In summary we thus find with the help of Eq. (A.88) for a (due to the particle allocation) *fixed set of indices* $\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}$ and $i < j \in \{1, 2, 3\}$:

$$(\mathcal{O}_{ij})_{\widetilde{\alpha},\widetilde{\beta}\widetilde{\gamma}} = \Big(\langle j_i, m_i| \otimes \langle j_j, m_j| \Big) |J_{ij}, M_{ij}\rangle = (-1)^{j_i-j_j+M_{ij}} \sqrt{2J_{ij}+1} \begin{pmatrix} j_i & j_j & J_{ij} \\ m_i & m_j & -M_{ij} \end{pmatrix}, \tag{A.91}$$

$$(\mathcal{O}_{ij})_{\widetilde{\alpha},\widetilde{\gamma}\widetilde{\beta}} = \Big(\langle j_j, m_j| \otimes \langle j_i, m_i| \Big) |J_{ij}, M_{ij}\rangle = (-1)^{(1-\delta_{P_i P_j})(J_{ij}-j_i-j_j)} \Big(\langle j_i, m_i| \otimes \langle j_j, m_j| \Big) |J_{ij}, M_{ij}\rangle$$

$$= (-1)^{(1-\delta_{P_i P_j})(J_{ij}-j_i-j_j)}(-1)^{j_i-j_j+M_{ij}} \sqrt{2J_{ij}+1} \begin{pmatrix} j_i & j_j & J_{ij} \\ m_i & m_j & -M_{ij} \end{pmatrix}$$

$$= (-1)^{j_i-j_j+M_{ij}+(1-\delta_{P_i P_j})(J_{ij}-j_i-j_j)} \sqrt{2J_{ij}+1} \begin{pmatrix} j_i & j_j & J_{ij} \\ m_i & m_j & -M_{ij} \end{pmatrix}, \tag{A.92}$$

$$\left(\mathcal{O}_{ij}^\dagger\right)_{\widetilde{\alpha},\widetilde{\gamma}\widetilde{\beta}} = \left[(\mathcal{O}_{ij})_{\widetilde{\alpha},\widetilde{\beta}\widetilde{\gamma}}\right]^\dagger = \left[(-1)^{j_i-j_j+M_{ij}} \sqrt{2J_{ij}+1} \begin{pmatrix} j_i & j_j & J_{ij} \\ m_i & m_j & -M_{ij} \end{pmatrix}\right]^\dagger$$

$$= (-1)^{j_i-j_j+M_{ij}} \sqrt{2J_{ij}+1} \begin{pmatrix} j_i & j_j & J_{ij} \\ m_i & m_j & -M_{ij} \end{pmatrix}, \tag{A.93}$$

$$\left(\mathcal{O}_{ij}^\dagger\right)_{\widetilde{\alpha},\widetilde{\beta}\widetilde{\gamma}} = \left[(\mathcal{O}_{ij})_{\widetilde{\alpha},\widetilde{\gamma}\widetilde{\beta}}\right]^\dagger = \left[(-1)^{J_{ij}-2j_j+M_{ij}} \sqrt{2J_{ij}+1} \begin{pmatrix} j_i & j_j & J_{ij} \\ m_i & m_j & -M_{ij} \end{pmatrix}\right]^\dagger$$

$$= (-1)^{j_i-j_j+M_{ij}+(1-\delta_{P_i P_j})(J_{ij}-j_i-j_j)} \sqrt{2J_{ij}+1} \begin{pmatrix} j_i & j_j & J_{ij} \\ m_i & m_j & -M_{ij} \end{pmatrix}. \tag{A.94}$$

These results allow us to write the x, y, z parameters in terms of 3-J symbols. Before we do so keep in mind that in all Feynman diagrams in appendix C the exchanged particle has the index ρ, i.e. $\widetilde{\rho}$ in the detailed notation of the current section. Hence, depending on the particle P_i, $\widetilde{\rho}$

is either the second or the third index of the projection operator $\mathcal{O}$. With our convention fixed in Eq. (3.4) it is then also fixed that

$$
\begin{aligned}
(\mathcal{O}_{ij})_{\widetilde{\cdot},\,\widetilde{\cdot}\,\widetilde{\rho}} &= \Big(\langle j_i, m_i | \otimes \langle j_j, m_j | \Big) | J_{ij}, M_{ij} \rangle \\
&= (-1)^{j_i - j_j + M_{ij}} \sqrt{2J_{ij}+1} \begin{pmatrix} j_i & j_j & J_{ij} \\ m_i & m_j & -M_{ij} \end{pmatrix}, \quad \text{for } i < j \in \{1,2,3\},
\end{aligned}
$$

and the relation for $(\mathcal{O}_{ij})_{\widetilde{\cdot}\,\widetilde{\cdot},\,\widetilde{\rho}\,\widetilde{\cdot}}$ is determined by Eq. (A.92). Note, that there is no convention regarding the channel projectors $\mathcal{O}_{T_{ij}}$; they are simply chosen in the way that the dimer spin comes first. Thus, one can apply Eq. (A.91) in the form

$$
\begin{aligned}
(\mathcal{O}_{T_{ij}})_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}} &= \Big(\langle J_{ij}, M_{ij} | \otimes \langle j_k, m_k | \Big) | J_{(ij)k}, M_{(ij)k} \rangle \\
&= (-1)^{J_{ij} - j_k + M} \sqrt{2J+1} \begin{pmatrix} J_{ij} & j_k & J \\ M_{ij} & m_k & -M \end{pmatrix}, \quad \text{for } i < j \in \{1,2,3\}.
\end{aligned}
$$

As an example for this method we again consider y_4 derived in the previous section:

$$
y_4 = \frac{1}{\mathrm{dof}(\widetilde{\lambda})} \sum_{\widetilde{\lambda},\widetilde{\sigma},\widetilde{\gamma},\widetilde{\mu},\widetilde{\nu},\widetilde{\rho}} \left(\mathcal{O}_{T_{13}}^\dagger \right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}} \left(\mathcal{O}_{13}^\dagger \right)_{\widetilde{\gamma},\widetilde{\rho}\widetilde{\nu}} (\mathcal{O}_{23})_{\widetilde{\mu},\widetilde{\sigma}\widetilde{\rho}} (\mathcal{O}_{T_{23}})_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}}.
$$

For simplicity we below assume that all particles are distinguishable. Nevertheless, everything works out in the same way if some particles are identical; one only has to keep in mind that the indices of some operators $\mathcal{O}$ might be interchanged to ensure the right order according to Eq. (3.4). In terms of 3-J symbols y_4 is given by

$$
\begin{aligned}
y_4 = \frac{1}{\mathrm{dof}(\widetilde{\lambda})} \sum_{\widetilde{\lambda},\widetilde{\sigma},\widetilde{\gamma},\widetilde{\mu},\widetilde{\nu},\widetilde{\rho}} & (-1)^{J_{13} - j_2 + M} \sqrt{2J+1} \begin{pmatrix} J_{13} & j_2 & J \\ M_{13}(\widetilde{\gamma}) & m_2(\widetilde{\sigma}) & -M(\widetilde{\lambda}) \end{pmatrix} \\
& \times (-1)^{j_1 - j_3 + M_{13}} \sqrt{2J_{13}+1} \begin{pmatrix} j_1 & j_3 & J_{13} \\ m_1(\widetilde{\nu}) & m_3(\widetilde{\rho}) & -M_{13}(\widetilde{\gamma}) \end{pmatrix} \\
& \times (-1)^{j_2 - j_3 + M_{23}} \sqrt{2J_{23}+1} \begin{pmatrix} j_2 & j_3 & J_{23} \\ m_2(\widetilde{\sigma}) & m_3(\widetilde{\rho}) & -M_{23}(\widetilde{\mu}) \end{pmatrix} \\
& \times (-1)^{J_{23} - j_1 + M} \sqrt{2J+1} \begin{pmatrix} J_{23} & j_1 & J \\ M_{23}(\widetilde{\mu}) & m_1(\widetilde{\nu}) & -M(\widetilde{\lambda}) \end{pmatrix}, \quad (A.95)
\end{aligned}
$$

where we have made the sum over spin indices explicit and used the variables J and M for the spin channel $\widetilde{\lambda}$ and its magnetic quantum number. Since each magnetic quantum number implicitly depends on the corresponding index we have written $m \equiv m(\widetilde{\cdot})$ to emphasize this fact. In this sense the summation over all spin indices can be expressed as a summation over all magnetic quantum numbers. Furthermore, we know from quantum mechanics that $(2J+1)$ is the dimension of the associated spin space and thus $(2J+1) = \mathrm{dof}(\widetilde{\lambda})$ holds. Thus, Eq. (A.95)

simplifies to

$$y_4 = \sqrt{(2J_{13}+1)(2J_{23}+1)} \sum_{m_1=-|j_1|}^{|j_1|} \sum_{m_2=-|j_2|}^{|j_2|} \sum_{m_3=-|j_3|}^{|j_3|} \sum_{M_{13}=-|J_{13}|}^{|J_{13}|} \sum_{M_{23}=-|J_{23}|}^{|J_{23}|} \sum_{M=-|J|}^{|J|}$$

$$\times (-1)^{J_{13}+J_{23}-2j_3+M_{13}+M_{23}+2M} \begin{pmatrix} J_{13} & j_2 & J \\ M_{13} & m_2 & -M \end{pmatrix} \begin{pmatrix} j_2 & j_3 & J_{23} \\ m_2 & m_3 & -M_{23} \end{pmatrix}$$

$$\times \begin{pmatrix} j_1 & j_3 & J_{13} \\ m_1 & m_3 & -M_{13} \end{pmatrix} \begin{pmatrix} J_{23} & j_1 & J \\ M_{23} & m_1 & -M \end{pmatrix} . \tag{A.96}$$

One observes that the substitution $M_{13} \to -M_{13}$ does not change the sum in y_4 since $M_{13} \in \{-|J_{13}|, ..., |J_{13}|\}$:

$$y_4 = \sqrt{(2J_{13}+1)(2J_{23}+1)} \sum_{m_1=-|j_1|}^{|j_1|} \sum_{m_2=-|j_2|}^{|j_2|} \sum_{m_3=-|j_3|}^{|j_3|} \sum_{M_{13}=-|J_{13}|}^{|J_{13}|} \sum_{M_{23}=-|J_{23}|}^{|J_{23}|} \sum_{M=-|J|}^{|J|}$$

$$\times (-1)^{J_{13}+J_{23}-2j_3-M_{13}+M_{23}+2M} \begin{pmatrix} J_{13} & j_2 & J \\ -M_{13} & m_2 & -M \end{pmatrix} \begin{pmatrix} j_2 & j_3 & J_{23} \\ m_2 & m_3 & -M_{23} \end{pmatrix}$$

$$\times \begin{pmatrix} j_1 & j_3 & J_{13} \\ m_1 & m_3 & M_{13} \end{pmatrix} \begin{pmatrix} J_{23} & j_1 & J \\ M_{23} & m_1 & -M \end{pmatrix} . \tag{A.97}$$

Using the symmetry properties of the 3-J symbol given in Eq. (A.83) and Eq. (A.85) yields

$$y_4 = \sqrt{(2J_{13}+1)(2J_{23}+1)} \sum_{m=-|j|}^{|j|} (-1)^{J_{13}+J_{23}-2j_3-M_{13}+M_{23}+2M}$$

$$\times (-1)^{j_2+J+J_{13}} \begin{pmatrix} j_2 & J & J_{13} \\ -m_2 & M & M_{13} \end{pmatrix} \begin{pmatrix} j_2 & j_3 & J_{23} \\ m_2 & m_3 & -M_{23} \end{pmatrix}$$

$$\times (-1)^{j_1+j_3+J_{13}} \begin{pmatrix} j_1 & j_3 & J_{13} \\ -m_1 & -m_3 & -M_{13} \end{pmatrix} \begin{pmatrix} j_1 & J & J_{23} \\ m_1 & -M & M_{23} \end{pmatrix} , \tag{A.98}$$

with $\sum_m$ being a placeholder for all sums over magnetic quantum numbers.

Now consider the so-called 6-J symbol,

$$\begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & J & J_{23} \end{Bmatrix} ,$$

introduced by Wigner in the same work of Ref. [169]. It is used to describe the coupling of three spins j_1, j_2 and j_3 to a total spin J with two intermediate spins J_{12} and J_{23} coming from the coupling of j_1 with j_2 and j_2 with j_3, respectively. More precisely, it is related to the Clebsch-Gordan coefficient

$$\langle (j_1, (j_2, j_3)J_{23}) J | ((j_1, j_2)J_{12}, j_3) J \rangle , \tag{A.99}$$

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which couples the *incoming* spins according to $j_1 \otimes j_2 \to J_{12}$ and $J_{12} \otimes j_3 \to J$ while in the "out" state one has $j_2 \otimes j_3 \to J_{23}$ and $j_1 \otimes J_{23} \to J$. Keep in mind that the order of the respective spins is important since due to Eq. (A.29) swapping leads to possible phase factors. Following Ref. [170] we know that the coefficient Eq. (A.99) is related to the Racah W-coefficient via

$$W(j_1 j_2 J j_3; J_{12} J_{23}) = \frac{1}{\sqrt{(2J_{12}+1)(2J_{23}+1)}} \langle (j_1,(j_2,j_3)J_{23})\, J| \,((j_1,j_2)J_{12},j_3)\, J\rangle \,, \quad \text{(A.100)}$$

which itself can be written in terms of the 6-J symbol

$$W(j_1 j_2 J j_3; J_{12} J_{23}) = (-1)^{j_1+j_2+j_3+J} \begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & J & J_{23} \end{Bmatrix} . \quad \text{(A.101)}$$

Consequently, one finds that the 6-J symbol above is proportional to the Clebsch-Gordan coefficient for the coupling of incoming spins $j_1 \otimes j_2 \to J_{12}$, $J_{12} \otimes j_3 \to J$ and outgoing spins $j_2 \otimes j_3 \to J_{23}$, $j_1 \otimes J_{23} \to J$:

$$\langle (j_1,(j_2,j_3)J_{23})\, J| \,((j_1,j_2)J_{12},j_3)\, J\rangle = (-1)^{j_1+j_2+j_3+J} \sqrt{(2J_{12}+1)(2J_{23}+1)} \begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & J & J_{23} \end{Bmatrix} . \quad \text{(A.102)}$$

In Ref. [170] it is shown that the 6-J symbol has certain symmetries: firstly, it is invariant under any permutation of columns,

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = \begin{Bmatrix} j_2 & j_3 & j_1 \\ j_5 & j_6 & j_4 \end{Bmatrix} = \begin{Bmatrix} j_3 & j_1 & j_2 \\ j_6 & j_4 & j_5 \end{Bmatrix} = \begin{Bmatrix} j_2 & j_1 & j_3 \\ j_5 & j_4 & j_6 \end{Bmatrix} = \begin{Bmatrix} j_1 & j_3 & j_2 \\ j_4 & j_6 & j_5 \end{Bmatrix} = \begin{Bmatrix} j_3 & j_2 & j_1 \\ j_6 & j_5 & j_4 \end{Bmatrix} , \quad \text{(A.103)}$$

and secondly, it is also invariant if one interchanges the upper and lower argument in each of any two columns [170]:

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = \begin{Bmatrix} j_1 & j_5 & j_6 \\ j_4 & j_2 & j_3 \end{Bmatrix} = \begin{Bmatrix} j_4 & j_5 & j_3 \\ j_1 & j_2 & j_6 \end{Bmatrix} = \begin{Bmatrix} j_4 & j_2 & j_6 \\ j_1 & j_5 & j_3 \end{Bmatrix} . \quad \text{(A.104)}$$

From the considerations above it is clear that the x, y, z parameters should be related to the 6-J symbol. Using again our example parameter y_4 one can indeed show that this is the case: from the diagram in Fig. A.2 to which y_4 is proportional, one observes that y_4 is the coefficient for the coupling of spins $(j_2 \otimes j_3 \to J_{23}) \otimes j_1 \to J$ in the "in" state and $(j_1 \otimes j_3 \to J_{13}) \otimes j_2 \to J$ in the outgoing one (remember our convention that the dimer is coupled with the left-over particle to a total spin J and not vice versa). Comparing this with the relation in Eq. (A.102) one concludes that one has to interchange j_1 with j_3 as well as j_2 with J_{13}. Thus, with the help of Eq. (A.29) one finds

$$\begin{aligned}
y_4 &= \langle ((j_1,j_3)J_{13},j_2)\, J| \,((j_2,j_3)J_{23},j_1)\, J\rangle \\
&= (-1)^{J-J_{13}-j_2}(-1)^{J_{13}-j_1-j_3} \langle (j_2,(j_3,j_1)J_{13})\, J| \,((j_2,j_3)J_{23},j_1)\, J\rangle \\
&= (-1)^{J-J_{13}-j_2}(-1)^{J_{13}-j_1-j_3}(-1)^{j_1+j_2+j_3+J} \sqrt{(2J_{23}+1)(2J_{13}+1)} \begin{Bmatrix} j_2 & j_3 & J_{23} \\ j_1 & J & J_{13} \end{Bmatrix} \\
&= (-1)^{2J} \sqrt{(2J_{23}+1)(2J_{13}+1)} \begin{Bmatrix} j_1 & j_3 & J_{13} \\ j_2 & J & J_{23} \end{Bmatrix} , \quad \text{(A.105)}
\end{aligned}$$

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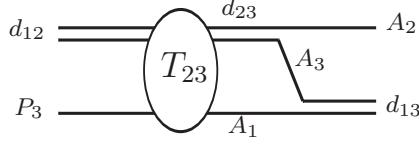
Figure A.2: *Diagram to which y_4 is proportional.*

where we have used Eq. (A.102) in the third and the symmetry properties in Eq. (A.104) in the fourth step.

In order to check that Eq. (A.105) is a correct representation of the parameter y_4 we will verify that it is equivalent to what we have found in Eq. (A.97). In Ref. [170] it is shown that the 6-J symbol can be defined in terms of 3-J symbols:

$$\begin{Bmatrix} j_1 & j_3 & J_{13} \\ j_2 & J & J_{23} \end{Bmatrix} = \sum_{m=-|j|}^{|j|} (-1)^{j_1-m_1+j_3-m_3+J_{13}-M_{13}+j_2-m_2+J-M+J_{23}-M_{23}} \begin{pmatrix} j_1 & j_3 & J_{13} \\ -m_1 & -m_3 & -M_{13} \end{pmatrix}$$
$$\times \begin{pmatrix} j_1 & J & J_{23} \\ m_1 & -M & M_{23} \end{pmatrix} \begin{pmatrix} j_2 & j_3 & J_{23} \\ m_2 & m_3 & -M_{23} \end{pmatrix} \begin{pmatrix} j_2 & J & J_{13} \\ -m_2 & M & M_{13} \end{pmatrix} . \tag{A.106}$$

Hence, we can replace the product of four 3-J symbols in Eq. (A.97) by the 6-J symbol times the corresponding phase factor in Eq. (A.106) to obtain

$$y_4 = \sqrt{(2J_{13}+1)(2J_{23}+1)} \begin{Bmatrix} j_1 & j_3 & J_{13} \\ j_2 & J & J_{23} \end{Bmatrix} \sum_{m=-|j|}^{|j|} (-1)^{J_{13}+J_{23}-2j_3-M_{13}+M_{23}+2M}$$
$$\times (-1)^{j_2+J+J_{13}} (-1)^{j_1+j_3+J_{13}} (-1)^{j_1-m_1+j_3-m_3+J_{13}-M_{13}+j_2-m_2+J-M+J_{23}-M_{23}} . \tag{A.107}$$

In order to simplify the sum over the phase factor

$$\sum_{m=-|j|}^{|j|} (-1)^{4J_{13}+2J_{23}+2J+2j_1+2j_2-2M_{13}+M-m_1-m_2-m_3} ,$$

we use that the second condition for a non-vanishing 3-J symbol given on page 118 yields in our case

$$-m_1 - m_3 = M_{13} \quad \Rightarrow \quad -(m_1 + m_3) = M_{13}$$
$$-m_2 + M = -M_{13} \quad \Rightarrow \quad M - m_2 = -M_{13} . \tag{A.108}$$

Furthermore, the first condition on page 118 implies that among others $(J_{13} - M_{13}) \in \mathbb{N}$ so that the factor $(-1)^{2(J_{13}-M_{13})} = 1$. Similarly, $(-1)^{2(J_{13}+j_2+J)} = (-1)^{2(J_{23}+j_1+J)} = 1$ since the fourth condition on page 118 tells us that the respective combinations in the exponent are also integers.

The phase factor is thus equal to

$$\sum_{m=-|j|}^{|j|} (-1)^{4J_{13}+2J_{23}+2J+2j_1+2j_2-2M_{13}+M-m_1-m_2-m_3} = \sum_{m=-|j|}^{|j|} (-1)^{2J_{13}+2J_{23}+2J+2j_1+2j_2}(-1)^{2(J_{13}-M_{13})}$$

$$= (-1)^{2J_{13}+2J_{23}+2J+2j_1+2j_2}$$

$$= (-1)^{2(J_{13}+j_2+J)}(-1)^{2(J_{23}+j_1+J)}(-1)^{-2J}$$

$$= (-1)^{-2J} = (-1)^{2J} . \tag{A.109}$$

Therefore the parameter y_4 in Eq. (A.107) gets the compact form

$$y_4 = (-1)^{2J} \sqrt{(2J_{13}+1)(2J_{23}+1)} \begin{Bmatrix} j_1 & j_3 & J_{13} \\ j_2 & J & J_{23} \end{Bmatrix} , \tag{A.110}$$

which is equivalent to Eq. (A.105). Hence, we conclude that indeed both representations Eq. (A.97) and Eq. (A.105) for the parameter y_4 are correct.

The advantage of the 6-J symbol notation in y_4 is that in contrast to the definition of y_4 in the previous section (Eq. (A.38)) it is not necessary to have the projection operators $\mathcal{O}$ explicitly given. It is sufficient to know the spin of the three particles and corresponding dimers as well as the scattering channel. This allows us to analyze systems consisting of particles with spins higher than 1 which undergo scattering processes in channels higher than 2, where the projection operators are more and more complicated or even not known.

In the same way as we derived Eq. (A.105) (or equivalently Eq. (A.110)) one can deduce expressions for all other x, y, z parameters in terms of 6-J symbols. For instance, one can – using Eq. (A.102) – deduce $x_2^{(\prime)}$ from

$$x_2^{(\prime)} = \delta_{P_1 P_3}^{(12)} \left\langle \left((j_2, j_3)J_{23}^{(\prime)}, j_1\right) J \,\Big|\, ((j_1, j_2)J_{12}, j_3)\right\rangle = \left\langle \left((j_1, j_2)J_{12}^{(\prime)}, j_1\right) J \,\Big|\, ((j_1, j_2)J_{12}, j_1)\right\rangle$$

$$= (-1)^{2J} \sqrt{\left(2J_{12}^{(\prime)}+1\right)(2J_{12}+1)} \begin{Bmatrix} j_1 & j_2 & J_{12}^{(\prime)} \\ j_1 & J & J_{12} \end{Bmatrix} ,$$

and $y_2^{(\prime)}$ from

$$y_2^{(\prime)} = \delta_{P_1 P_2} \left\langle \left((j_2, j_3)J_{23}^{(\prime)}, j_1\right) J \,\Big|\, ((j_1, j_2)J_{12}, j_3)\right\rangle = \left\langle \left((j_1, j_3)J_{13}^{(\prime)}, j_1\right) J \,\Big|\, ((j_1, j_1)J_{12}, j_3)\right\rangle$$

$$= (-1)^{(1-\delta_{P_1 P_3})(J_{13}^{(\prime)}-j_1-j_3)} \left\langle \left((j_3, j_1)J_{13}^{(\prime)}, j_1\right) J \,\Big|\, ((j_1, j_1)J_{12}, j_3)\right\rangle$$

$$= (-1)^{2J+\left(1-\delta_{P_1 P_3}\right)\left(J_{13}^{(\prime)}-j_1-j_3\right)} \sqrt{\left(2J_{13}^{(\prime)}+1\right)(2J_{12}+1)} \begin{Bmatrix} j_3 & j_1 & J_{13}^{(\prime)} \\ j_1 & J & J_{12} \end{Bmatrix} .$$

Note, that the Kronecker-delta in front of the Clebsch-Gordan coefficient originally stands in front of the corresponding diagram. Therefore the diagram *itself* is changed before one applies the Feynman rules and thus one must *not* interchange the spins within the Clebsch-Gordan coefficient in the first line; they are interchanged due to the different vertex factor one has to apply for the changed diagram. However, in the second line of $y_2^{(\prime)}$ we indeed swap the spins in

the coefficient and thus pick up a phase factor (as long as $P_1 \neq P_3$).

The results are summarized in Eqs. (A.111 - A.146) where we notice that the parameters with index "1" are identical to those with index "2". Keep in mind that we have derived all relations in terms of spin only. Consequently, one has to multiply the results of the x, y, z, parameters for spin with the results for isospin to get the full spin *and* isospin dependent parameter. However, we do not give the isospin formulae explicitly since spin and isospin behave analogously. Hence, one simply has to replace all j's and J's by i's and I's for the latter. As a practical remark in the application of the 6-J symbol method, note that in `Mathematica` the command `SixJSymbol` provides an automated calculation of 6-J symbols for given spins.

$$x_{1,2}^{(\prime)} = (-1)^{2J} \sqrt{\left(2J_{12}^{(\prime)} + 1\right)(2J_{12} + 1)} \begin{Bmatrix} j_1 & j_2 & J_{12}^{(\prime)} \\ j_1 & J & J_{12} \end{Bmatrix} \tag{A.111}$$

$$y_{1,2}^{(\prime)} = (-1)^{2J + \left(1 - \delta_{P_1 P_3}\right)\left(J_{13}^{(\prime)} - j_1 - j_3\right)} \sqrt{\left(2J_{13}^{(\prime)} + 1\right)(2J_{12} + 1)} \begin{Bmatrix} j_3 & j_1 & J_{13}^{(\prime)} \\ j_1 & J & J_{12} \end{Bmatrix} \tag{A.112}$$

$$z_{1,2}^{(\prime)} = (-1)^{2J + \left(1 - \delta_{P_2 P_3}\right)\left(J_{23}^{(\prime)} - j_2 - j_3\right)} \sqrt{\left(2J_{23}^{(\prime)} + 1\right)(2J_{12} + 1)} \begin{Bmatrix} j_3 & j_2 & J_{23}^{(\prime)} \\ j_1 & J & J_{12} \end{Bmatrix} \tag{A.113}$$

$$x_3^{(\prime)} = (-1)^{2J + \left(1 - \delta_{P_1 P_2}\right)\left(J_{12}^{(\prime)} - j_1 - j_2\right)} \sqrt{\left(2J_{12}^{(\prime)} + 1\right)(2J_{13} + 1)} \begin{Bmatrix} j_2 & j_1 & J_{12}^{(\prime)} \\ j_1 & J & J_{13} \end{Bmatrix} \tag{A.114}$$

$$y_3^{(\prime)} = (-1)^{2J} \sqrt{\left(2J_{13}^{(\prime)} + 1\right)(2J_{13} + 1)} \begin{Bmatrix} j_1 & j_3 & J_{13}^{(\prime)} \\ j_1 & J & J_{13} \end{Bmatrix} \tag{A.115}$$

$$z_3^{(\prime)} = (-1)^{2J} \sqrt{\left(2J_{23}^{(\prime)} + 1\right)(2J_{13} + 1)} \begin{Bmatrix} j_2 & j_3 & J_{23}^{(\prime)} \\ j_1 & J & J_{13} \end{Bmatrix} \tag{A.116}$$

$$x_4^{(\prime)} = (-1)^{2J} \sqrt{\left(2J_{12}^{(\prime)} + 1\right)(2J_{23} + 1)} \begin{Bmatrix} j_1 & j_2 & J_{12}^{(\prime)} \\ j_2 & J & J_{23} \end{Bmatrix} \tag{A.117}$$

$$y_4^{(\prime)} = (-1)^{2J} \sqrt{\left(2J_{13}^{(\prime)} + 1\right)(2J_{23} + 1)} \begin{Bmatrix} j_1 & j_3 & J_{13}^{(\prime)} \\ j_2 & J & J_{23} \end{Bmatrix} \tag{A.118}$$

$$z_4^{(\prime)} = (-1)^{2J} \sqrt{\left(2J_{23}^{(\prime)} + 1\right)(2J_{23} + 1)} \begin{Bmatrix} j_2 & j_3 & J_{23}^{(\prime)} \\ j_2 & J & J_{23} \end{Bmatrix} \tag{A.119}$$

$$x_5^{(\prime)} = (-1)^{2J} \sqrt{\left(2J_{12}^{(\prime)} + 1\right)(2J_{12}' + 1)} \begin{Bmatrix} j_1 & j_2 & J_{12}^{(\prime)} \\ j_1 & J & J_{12}' \end{Bmatrix} \tag{A.120}$$

$$y_5^{(\prime)} = (-1)^{2J + \left(1 - \delta_{P_1 P_3}\right)\left(J_{13}^{(\prime)} - j_1 - j_3\right)} \sqrt{\left(2J_{13}^{(\prime)} + 1\right)(2J_{12}' + 1)} \begin{Bmatrix} j_3 & j_1 & J_{13}^{(\prime)} \\ j_1 & J & J_{12}' \end{Bmatrix} \tag{A.121}$$

$$z_5^{(\prime)} = (-1)^{2J + \left(1 - \delta_{P_2 P_3}\right)\left(J_{23}^{(\prime)} - j_2 - j_3\right)} \sqrt{\left(2J_{23}^{(\prime)} + 1\right)(2J_{12}' + 1)} \begin{Bmatrix} j_3 & j_2 & J_{23}^{(\prime)} \\ j_1 & J & J_{12}' \end{Bmatrix} \tag{A.122}$$

$$x_6^{(\prime)} = (-1)^{2J+\left(1-\delta_{P_1 P_2}\right)\left(J_{12}^{(\prime)}-j_1-j_2\right)}\sqrt{\left(2J_{12}^{(\prime)}+1\right)\left(2J_{13}'+1\right)}\begin{Bmatrix} j_2 & j_1 & J_{12}^{(\prime)} \\ j_1 & J & J_{13}' \end{Bmatrix} \tag{A.123}$$

$$y_6^{(\prime)} = (-1)^{2J}\sqrt{\left(2J_{13}^{(\prime)}+1\right)\left(2J_{13}'+1\right)}\begin{Bmatrix} j_1 & j_3 & J_{13}^{(\prime)} \\ j_1 & J & J_{13}' \end{Bmatrix} \tag{A.124}$$

$$z_6^{(\prime)} = (-1)^{2J}\sqrt{\left(2J_{23}^{(\prime)}+1\right)\left(2J_{13}'+1\right)}\begin{Bmatrix} j_2 & j_3 & J_{23}^{(\prime)} \\ j_1 & J & J_{13}' \end{Bmatrix} \tag{A.125}$$

$$x_7^{(\prime)} = (-1)^{2J}\sqrt{\left(2J_{12}^{(\prime)}+1\right)\left(2J_{23}'+1\right)}\begin{Bmatrix} j_1 & j_2 & J_{12}^{(\prime)} \\ j_2 & J & J_{23}' \end{Bmatrix} \tag{A.126}$$

$$y_7^{(\prime)} = (-1)^{2J}\sqrt{\left(2J_{13}^{(\prime)}+1\right)\left(2J_{23}'+1\right)}\begin{Bmatrix} j_1 & j_3 & J_{13}^{(\prime)} \\ j_2 & J & J_{23}' \end{Bmatrix} \tag{A.127}$$

$$z_7^{(\prime)} = (-1)^{2J}\sqrt{\left(2J_{23}^{(\prime)}+1\right)\left(2J_{23}'+1\right)}\begin{Bmatrix} j_2 & j_3 & J_{23}^{(\prime)} \\ j_2 & J & J_{23}' \end{Bmatrix} \tag{A.128}$$

$$\widetilde{x}_{1,2}^{(\prime)} = (-1)^{2J+\left(1-\delta_{P_1 P_2}\right)\left(J_{12}+J_{12}^{(\prime)}-2j_1-2j_2\right)}\sqrt{\left(2J_{12}^{(\prime)}+1\right)\left(2J_{12}+1\right)}\begin{Bmatrix} j_2 & j_1 & J_{12}^{(\prime)} \\ j_2 & J & J_{12} \end{Bmatrix} \tag{A.129}$$

$$\widetilde{y}_{1,2}^{(\prime)} = (-1)^{2J+\left(1-\delta_{P_1 P_2}\right)(J_{12}-j_1-j_2)+\left(1-\delta_{P_1 P_3}\right)\left(J_{13}^{(\prime)}-j_1-j_3\right)}\sqrt{\left(2J_{13}^{(\prime)}+1\right)\left(2J_{12}+1\right)}\begin{Bmatrix} j_3 & j_1 & J_{13}^{(\prime)} \\ j_2 & J & J_{12} \end{Bmatrix} \tag{A.130}$$

$$\widetilde{z}_{1,2}^{(\prime)} = (-1)^{2J+\left(1-\delta_{P_2 P_3}\right)\left(J_{23}^{(\prime)}-j_2-j_3\right)}\sqrt{\left(2J_{23}^{(\prime)}+1\right)\left(2J_{12}+1\right)}\begin{Bmatrix} j_3 & j_2 & J_{23}^{(\prime)} \\ j_2 & J & J_{12} \end{Bmatrix} \tag{A.131}$$

$$\widetilde{x}_3^{(\prime)} = (-1)^{2J+\left(1-\delta_{P_1 P_3}\right)(J_{13}-j_1-j_3)+\left(1-\delta_{P_1 P_2}\right)\left(J_{12}^{(\prime)}-j_1-j_2\right)}\sqrt{\left(2J_{12}^{(\prime)}+1\right)\left(2J_{13}+1\right)}\begin{Bmatrix} j_2 & j_1 & J_{12}^{(\prime)} \\ j_3 & J & J_{13} \end{Bmatrix} \tag{A.132}$$

$$\widetilde{y}_3^{(\prime)} = (-1)^{2J+\left(1-\delta_{P_1 P_3}\right)\left(J_{13}+J_{13}^{(\prime)}-2j_1-2j_3\right)}\sqrt{\left(2J_{13}^{(\prime)}+1\right)\left(2J_{13}+1\right)}\begin{Bmatrix} j_3 & j_1 & J_{13}^{(\prime)} \\ j_3 & J & J_{13} \end{Bmatrix} \tag{A.133}$$

$$\widetilde{z}_3^{(\prime)} = (-1)^{2J}\sqrt{\left(2J_{23}^{(\prime)}+1\right)\left(2J_{13}+1\right)}\begin{Bmatrix} j_2 & j_3 & J_{23}^{(\prime)} \\ j_3 & J & J_{13} \end{Bmatrix} \tag{A.134}$$

$$\widetilde{x}_4^{(\prime)} = (-1)^{2J+\left(1-\delta_{P_2 P_3}\right)(J_{23}-j_2-j_3)}\sqrt{\left(2J_{12}^{(\prime)}+1\right)\left(2J_{23}+1\right)}\begin{Bmatrix} j_1 & j_2 & J_{12}^{(\prime)} \\ j_3 & J & J_{23} \end{Bmatrix} \tag{A.135}$$

$$\widetilde{y}_4^{(\prime)} = (-1)^{2J}\sqrt{\left(2J_{13}^{(\prime)}+1\right)\left(2J_{23}+1\right)}\begin{Bmatrix} j_1 & j_3 & J_{13}^{(\prime)} \\ j_3 & J & J_{23} \end{Bmatrix} \tag{A.136}$$

$$\widetilde{z}_4^{(\prime)} = (-1)^{2J+\left(1-\delta_{P_2 P_3}\right)\left(J_{23}+J_{23}^{(\prime)}-2j_2-2j_3\right)}\sqrt{\left(2J_{23}^{(\prime)}+1\right)\left(2J_{23}+1\right)}\begin{Bmatrix} j_3 & j_2 & J_{23}^{(\prime)} \\ j_3 & J & J_{23} \end{Bmatrix} \tag{A.137}$$

$$\widetilde{x}_5^{(\prime)} = (-1)^{2J+\left(1-\delta_{P_1 P_2}\right)\left(J_{12}+J_{12}^{(\prime)}-2j_1-2j_2\right)}\sqrt{\left(2J_{12}^{(\prime)}+1\right)\left(2J_{12}'+1\right)}\begin{Bmatrix} j_2 & j_1 & J_{12}^{(\prime)} \\ j_2 & J & J_{12}' \end{Bmatrix} \tag{A.138}$$

$$\widetilde{y}_5^{(\prime)} = (-1)^{2J+\left(1-\delta_{P_1 P_2}\right)\left(J_{12}'-j_1-j_2\right)+\left(1-\delta_{P_1 P_3}\right)\left(J_{13}^{(\prime)}-j_1-j_3\right)}\sqrt{\left(2J_{13}^{(\prime)}+1\right)\left(2J_{12}'+1\right)}\begin{Bmatrix} j_3 & j_1 & J_{13}^{(\prime)} \\ j_2 & J & J_{12}' \end{Bmatrix} \tag{A.139}$$

$$\widetilde{z}_5^{(\prime)} = (-1)^{2J+\left(1-\delta_{P_2 P_3}\right)\left(J_{23}^{(\prime)}-j_2-j_3\right)}\sqrt{\left(2J_{23}^{(\prime)}+1\right)\left(2J_{12}'+1\right)}\begin{Bmatrix} j_3 & j_2 & J_{23}^{(\prime)} \\ j_2 & J & J_{12}' \end{Bmatrix} \tag{A.140}$$

$$\widetilde{x}_6^{(\prime)} = (-1)^{2J+\left(1-\delta_{P_1 P_3}\right)\left(J_{13}'-j_1-j_3\right)+\left(1-\delta_{P_1 P_2}\right)\left(J_{12}^{(\prime)}-j_1-j_2\right)}\sqrt{\left(2J_{12}^{(\prime)}+1\right)\left(2J_{13}'+1\right)}\begin{Bmatrix} j_2 & j_1 & J_{12}^{(\prime)} \\ j_3 & J & J_{13}' \end{Bmatrix} \tag{A.141}$$

$$\widetilde{y}_6^{(\prime)} = (-1)^{2J+\left(1-\delta_{P_1 P_3}\right)\left(J_{13}+J_{13}^{(\prime)}-2j_1-2j_3\right)}\sqrt{\left(2J_{13}^{(\prime)}+1\right)\left(2J_{13}'+1\right)}\begin{Bmatrix} j_1 & j_3 & J_{13}^{(\prime)} \\ j_1 & J & J_{13}' \end{Bmatrix} \tag{A.142}$$

$$\widetilde{z}_6^{(\prime)} = (-1)^{2J}\sqrt{\left(2J_{23}^{(\prime)}+1\right)\left(2J_{13}'+1\right)}\begin{Bmatrix} j_2 & j_3 & J_{23}^{(\prime)} \\ j_3 & J & J_{13}' \end{Bmatrix} \tag{A.143}$$

$$\widetilde{x}_7^{(\prime)} = (-1)^{2J+\left(1-\delta_{P_2 P_3}\right)\left(J_{23}'-j_2-j_3\right)}\sqrt{\left(2J_{12}^{(\prime)}+1\right)\left(2J_{23}'+1\right)}\begin{Bmatrix} j_1 & j_2 & J_{12}^{(\prime)} \\ j_3 & J & J_{23}' \end{Bmatrix} \tag{A.144}$$

$$\widetilde{y}_7^{(\prime)} = (-1)^{2J}\sqrt{\left(2J_{13}^{(\prime)}+1\right)\left(2J_{23}'+1\right)}\begin{Bmatrix} j_1 & j_3 & J_{13}^{(\prime)} \\ j_3 & J & J_{23}' \end{Bmatrix} \tag{A.145}$$

$$\widetilde{z}_7^{(\prime)} = (-1)^{2J+\left(1-\delta_{P_2 P_3}\right)\left(J_{23}+J_{23}^{(\prime)}-2j_2-2j_3\right)}\sqrt{\left(2J_{23}^{(\prime)}+1\right)\left(2J_{23}'+1\right)}\begin{Bmatrix} j_3 & j_2 & J_{23}^{(\prime)} \\ j_3 & J & J_{23}' \end{Bmatrix} \tag{A.146}$$

A.4   Projection operator summary tables

$J(A_i)_{\widetilde{\beta}} \otimes J(A_j)_{\widetilde{\gamma}} \to J(d_{ij})_{\widetilde{\alpha}}$	index assignment	projection operator $(\mathcal{O}_{ij})_{\widetilde{\alpha},\widetilde{\beta}\widetilde{\gamma}}$
$0 \otimes 0 \to 0$	$\widetilde{\alpha} = \diagup$ spin 0 index $\widetilde{\beta} = \diagup$ spin 0 index $\widetilde{\gamma} = \diagup$ spin 0 index	1
$0 \otimes \frac{1}{2} \to \frac{1}{2}$	$\widetilde{\alpha} = \alpha$ spin 1/2 index $\widetilde{\beta} = \diagup$ spin 0 index $\widetilde{\gamma} = \gamma$ spin 1/2 index	$\delta_{\alpha\gamma}$
$0 \otimes 1 \to 1$	$\widetilde{\alpha} = i$ spin 1 index $\widetilde{\beta} = \diagup$ spin 0 index $\widetilde{\gamma} = k$ spin 1 index	δ_{ik}
$\frac{1}{2} \otimes 0 \to \frac{1}{2}$	$\widetilde{\alpha} = \alpha$ spin 1/2 index $\widetilde{\beta} = \beta$ spin 1/2 index $\widetilde{\gamma} = \diagup$ spin 0 index	$\delta_{\alpha\beta}$
$\frac{1}{2} \otimes \frac{1}{2} \to 0$	$\widetilde{\alpha} = \diagup$ spin 0 index $\widetilde{\beta} = \beta$ spin 1/2 index $\widetilde{\gamma} = \gamma$ spin 1/2 index	$\frac{i}{\sqrt{2}} (\sigma_2)_{\beta\gamma}$
$\frac{1}{2} \otimes \frac{1}{2} \to 1$	$\widetilde{\alpha} = i$ spin 1 index $\widetilde{\beta} = \beta$ spin 1/2 index $\widetilde{\gamma} = \gamma$ spin 1/2 index	$\frac{i}{\sqrt{2}} (\sigma_i\sigma_2)_{\beta\gamma}$
$\frac{1}{2} \otimes 1 \to \frac{1}{2}$	$\widetilde{\alpha} = \alpha$ spin 1/2 index $\widetilde{\beta} = \beta$ spin 1/2 index $\widetilde{\gamma} = k$ spin 1 index	$\frac{1}{\sqrt{3}} (\sigma_k)_{\beta\alpha}$
$1 \otimes 0 \to 1$	$\widetilde{\alpha} = i$ spin 1 index $\widetilde{\beta} = j$ spin 1 index $\widetilde{\gamma} = \diagup$ spin 0 index	δ_{ij}
$1 \otimes \frac{1}{2} \to \frac{1}{2}$	$\widetilde{\alpha} = \alpha$ spin 1/2 index $\widetilde{\beta} = j$ spin 1 index $\widetilde{\gamma} = \gamma$ spin 1/2 index	$-\frac{1}{\sqrt{3}} (\sigma_j)_{\gamma\alpha}$
$1 \otimes 1 \to 0$	$\widetilde{\alpha} = \diagup$ spin 0 index $\widetilde{\beta} = j$ spin 1 index $\widetilde{\gamma} = k$ spin 1 index	$-\frac{1}{\sqrt{3}}\delta_{jk}$
$1 \otimes 1 \to 1$	$\widetilde{\alpha} = i$ spin 1 index $\widetilde{\beta} = j$ spin 1 index $\widetilde{\gamma} = k$ spin 1 index	$-\frac{1}{\sqrt{2}} (U_i)_{jk}$

Table A.1: *Summary of spin projection operators where the indices $\{\alpha, \beta, \gamma...\} \in \{1,2\}$ represent spin $\frac{1}{2}$, indices $\{i, j, k...\} \in \{1,2,3\}$ represent spin 1, σ_i are the $SU(2)$ generators (Pauli matrices) and U_i are the generators of $SO(3)$.*

$J\left(A_i\right)_{\widetilde{\sigma}} \otimes J\left(A_j\right)_{\widetilde{\rho}} \to J\left(d_{ij}^{(\prime)}\right)_{\widetilde{\mu}}$	index assignment	projection operator $\left(\mathcal{O}_{ij}^{(\prime)}\right)_{\widetilde{\mu},\widetilde{\sigma}\widetilde{\rho}}$
$0 \otimes 0 \to 0$	$\widetilde{\mu} = \diagup$ spin 0 index $\widetilde{\sigma} = \diagup$ spin 0 index $\widetilde{\rho} = \diagup$ spin 0 index	1
$0 \otimes \frac{1}{2} \to \frac{1}{2}$	$\widetilde{\mu} = \mu$ spin 1/2 index $\widetilde{\sigma} = \diagup$ spin 0 index $\widetilde{\rho} = \rho$ spin 1/2 index	$\delta_{\mu\rho}$
$0 \otimes 1 \to 1$	$\widetilde{\mu} = m$ spin 1 index $\widetilde{\sigma} = \diagup$ spin 0 index $\widetilde{\rho} = r$ spin 1 index	δ_{mr}
$\frac{1}{2} \otimes 0 \to \frac{1}{2}$	$\widetilde{\mu} = \mu$ spin 1/2 index $\widetilde{\sigma} = \sigma$ spin 1/2 index $\widetilde{\rho} = \diagup$ spin 0 index	$\delta_{\mu\sigma}$
$\frac{1}{2} \otimes \frac{1}{2} \to 0$	$\widetilde{\mu} = \diagup$ spin 0 index $\widetilde{\sigma} = \sigma$ spin 1/2 index $\widetilde{\rho} = \rho$ spin 1/2 index	$\frac{i}{\sqrt{2}}\left(\sigma_2\right)_{\sigma\rho}$
$\frac{1}{2} \otimes \frac{1}{2} \to 1$	$\widetilde{\mu} = m$ spin 1 index $\widetilde{\sigma} = \sigma$ spin 1/2 index $\widetilde{\rho} = \rho$ spin 1/2 index	$\frac{i}{\sqrt{2}}\left(\sigma_m\sigma_2\right)_{\sigma\rho}$
$\frac{1}{2} \otimes 1 \to \frac{1}{2}$	$\widetilde{\mu} = \mu$ spin 1/2 index $\widetilde{\sigma} = \sigma$ spin 1/2 index $\widetilde{\rho} = r$ spin 1 index	$\frac{1}{\sqrt{3}}\left(\sigma_r\right)_{\sigma\mu}$
$1 \otimes 0 \to 1$	$\widetilde{\mu} = m$ spin 1 index $\widetilde{\sigma} = s$ spin 1 index $\widetilde{\rho} = \diagup$ spin 0 index	δ_{ms}
$1 \otimes \frac{1}{2} \to \frac{1}{2}$	$\widetilde{\mu} = \mu$ spin 1/2 index $\widetilde{\sigma} = s$ spin 1 index $\widetilde{\rho} = \rho$ spin 1/2 index	$-\frac{1}{\sqrt{3}}\left(\sigma_s\right)_{\rho\mu}$
$1 \otimes 1 \to 0$	$\widetilde{\mu} = \diagup$ spin 0 index $\widetilde{\sigma} = s$ spin 1 index $\widetilde{\rho} = r$ spin 1 index	$-\frac{1}{\sqrt{3}}\delta_{sr}$
$1 \otimes 1 \to 1$	$\widetilde{\mu} = m$ spin 1 index $\widetilde{\sigma} = s$ spin 1 index $\widetilde{\rho} = r$ spin 1 index	$-\frac{1}{\sqrt{2}}\left(U_m\right)_{sr}$

Table A.2: *Summary of spin projection operators where the indices are chosen according to their appearance in the x, y, z parameters Eqs. (A.40 - A.81).*

$J\left(A_i\right)_{\widetilde{\rho}} \otimes J\left(A_j\right)_{\widetilde{\sigma}} \to J\left(d_{ij}^{(\prime)}\right)_{\widetilde{\mu}}$	index assignment	projection operator $\left(\mathcal{O}_{ij}^{(\prime)}\right)_{\widetilde{\mu},\widetilde{\rho}\widetilde{\sigma}}$
$0 \otimes 0 \to 0$	$\widetilde{\mu} = \diagup \quad$ spin 0 index $\widetilde{\rho} = \diagup \quad$ spin 0 index $\widetilde{\sigma} = \diagup \quad$ spin 0 index	1
$0 \otimes \frac{1}{2} \to \frac{1}{2}$	$\widetilde{\mu} = \mu \quad$ spin 1/2 index $\widetilde{\rho} = \diagup \quad$ spin 0 index $\widetilde{\sigma} = \sigma \quad$ spin 1/2 index	$\delta_{\mu\sigma}$
$0 \otimes 1 \to 1$	$\widetilde{\mu} = m \quad$ spin 1 index $\widetilde{\rho} = \diagup \quad$ spin 0 index $\widetilde{\sigma} = s \quad$ spin 1 index	δ_{ms}
$\frac{1}{2} \otimes 0 \to \frac{1}{2}$	$\widetilde{\mu} = \mu \quad$ spin 1/2 index $\widetilde{\rho} = \rho \quad$ spin 1/2 index $\widetilde{\sigma} = \diagup \quad$ spin 0 index	$\delta_{\mu\rho}$
$\frac{1}{2} \otimes \frac{1}{2} \to 0$	$\widetilde{\mu} = \diagup \quad$ spin 0 index $\widetilde{\rho} = \rho \quad$ spin 1/2 index $\widetilde{\sigma} = \sigma \quad$ spin 1/2 index	$\frac{i}{\sqrt{2}}\left(\sigma_2\right)_{\rho\sigma}$
$\frac{1}{2} \otimes \frac{1}{2} \to 1$	$\widetilde{\mu} = m \quad$ spin 1 index $\widetilde{\rho} = \rho \quad$ spin 1/2 index $\widetilde{\sigma} = \sigma \quad$ spin 1/2 index	$\frac{i}{\sqrt{2}}\left(\sigma_m\sigma_2\right)_{\rho\sigma}$
$\frac{1}{2} \otimes 1 \to \frac{1}{2}$	$\widetilde{\mu} = \mu \quad$ spin 1/2 index $\widetilde{\rho} = \rho \quad$ spin 1/2 index $\widetilde{\sigma} = s \quad$ spin 1 index	$\frac{1}{\sqrt{3}}\left(\sigma_s\right)_{\rho\mu}$
$1 \otimes 0 \to 1$	$\widetilde{\mu} = m \quad$ spin 1 index $\widetilde{\rho} = r \quad$ spin 1 index $\widetilde{\sigma} = \diagup \quad$ spin 0 index	δ_{mr}
$1 \otimes \frac{1}{2} \to \frac{1}{2}$	$\widetilde{\mu} = \mu \quad$ spin 1/2 index $\widetilde{\rho} = r \quad$ spin 1 index $\widetilde{\sigma} = \sigma \quad$ spin 1/2 index	$-\frac{1}{\sqrt{3}}\left(\sigma_r\right)_{\sigma\mu}$
$1 \otimes 1 \to 0$	$\widetilde{\mu} = \diagup \quad$ spin 0 index $\widetilde{\rho} = r \quad$ spin 1 index $\widetilde{\sigma} = s \quad$ spin 1 index	$-\frac{1}{\sqrt{3}}\delta_{rs}$
$1 \otimes 1 \to 1$	$\widetilde{\mu} = m \quad$ spin 1 index $\widetilde{\rho} = r \quad$ spin 1 index $\widetilde{\sigma} = s \quad$ spin 1 index	$-\frac{1}{\sqrt{2}}\left(U_m\right)_{rs}$

Table A.3: *Summary of spin projection operators where the indices are chosen according to their appearance in the x, y, z parameters Eqs. (A.40 - A.81).*

$J\left(A_i\right)_{\widetilde{\nu}} \otimes J\left(A_j\right)_{\widetilde{\rho}} \to J\left(d_{ij}^{(\prime)}\right)_{\widetilde{\gamma}}$	index assignment	projection operator $\left(\mathcal{O}_{ij}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\rho}\widetilde{\nu}}$
$0 \otimes 0 \to 0$	$\widetilde{\gamma} = \diagup$ spin 0 index $\widetilde{\nu} = \diagup$ spin 0 index $\widetilde{\rho} = \diagup$ spin 0 index	1
$0 \otimes \frac{1}{2} \to \frac{1}{2}$	$\widetilde{\gamma} = \gamma$ spin 1/2 index $\widetilde{\nu} = \diagup$ spin 0 index $\widetilde{\rho} = \rho$ spin 1/2 index	$\delta_{\rho\gamma}$
$0 \otimes 1 \to 1$	$\widetilde{\gamma} = g$ spin 1 index $\widetilde{\nu} = \diagup$ spin 0 index $\widetilde{\rho} = r$ spin 1 index	δ_{rg}
$\frac{1}{2} \otimes 0 \to \frac{1}{2}$	$\widetilde{\gamma} = \gamma$ spin 1/2 index $\widetilde{\nu} = \nu$ spin 1/2 index $\widetilde{\rho} = \diagup$ spin 0 index	$\delta_{\nu\gamma}$
$\frac{1}{2} \otimes \frac{1}{2} \to 0$	$\widetilde{\gamma} = \diagup$ spin 0 index $\widetilde{\nu} = \nu$ spin 1/2 index $\widetilde{\rho} = \rho$ spin 1/2 index	$-\frac{i}{\sqrt{2}}\left(\sigma_2\right)_{\rho\nu}$
$\frac{1}{2} \otimes \frac{1}{2} \to 1$	$\widetilde{\gamma} = g$ spin 1 index $\widetilde{\nu} = \nu$ spin 1/2 index $\widetilde{\rho} = \rho$ spin 1/2 index	$-\frac{i}{\sqrt{2}}\left(\sigma_2\sigma_g\right)_{\rho\nu}$
$\frac{1}{2} \otimes 1 \to \frac{1}{2}$	$\widetilde{\gamma} = \gamma$ spin 1/2 index $\widetilde{\nu} = \nu$ spin 1/2 index $\widetilde{\rho} = r$ spin 1 index	$\frac{1}{\sqrt{3}}\left(\sigma_r\right)_{\gamma\nu}$
$1 \otimes 0 \to 1$	$\widetilde{\gamma} = g$ spin 1 index $\widetilde{\nu} = n$ spin 1 index $\widetilde{\rho} = \diagup$ spin 0 index	δ_{ng}
$1 \otimes \frac{1}{2} \to \frac{1}{2}$	$\widetilde{\gamma} = \gamma$ spin 1/2 index $\widetilde{\nu} = n$ spin 1 index $\widetilde{\rho} = \rho$ spin 1/2 index	$-\frac{1}{\sqrt{3}}\left(\sigma_n\right)_{\gamma\rho}$
$1 \otimes 1 \to 0$	$\widetilde{\gamma} = \diagup$ spin 0 index $\widetilde{\nu} = n$ spin 1 index $\widetilde{\rho} = r$ spin 1 index	$-\frac{1}{\sqrt{3}}\delta_{rn}$
$1 \otimes 1 \to 1$	$\widetilde{\gamma} = g$ spin 1 index $\widetilde{\nu} = n$ spin 1 index $\widetilde{\rho} = r$ spin 1 index	$-\frac{1}{\sqrt{2}}\left(U_g\right)_{rn}$

Table A.4: *Summary of spin projection operators where the indices are chosen according to their appearance in the x, y, z parameters Eqs. (A.40 - A.81).*

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$J\left(A_i\right)_{\widetilde{\rho}} \otimes J\left(A_j\right)_{\widetilde{\nu}} \to J\left(d_{ij}^{(\prime)}\right)_{\widetilde{\gamma}}$	index assignment	projection operator $\left(\mathcal{O}_{ij}^{(\prime)\dagger}\right)_{\widetilde{\gamma},\widetilde{\nu}\widetilde{\rho}}$
$0 \otimes 0 \to 0$	$\widetilde{\gamma} = \diagup$ spin 0 index $\widetilde{\rho} = \diagup$ spin 0 index $\widetilde{\nu} = \diagup$ spin 0 index	1
$0 \otimes \frac{1}{2} \to \frac{1}{2}$	$\widetilde{\gamma} = \gamma$ spin 1/2 index $\widetilde{\rho} = \diagup$ spin 0 index $\widetilde{\nu} = \nu$ spin 1/2 index	$\delta_{\nu\gamma}$
$0 \otimes 1 \to 1$	$\widetilde{\gamma} = g$ spin 1 index $\widetilde{\rho} = \diagup$ spin 0 index $\widetilde{\nu} = n$ spin 1 index	δ_{ng}
$\frac{1}{2} \otimes 0 \to \frac{1}{2}$	$\widetilde{\gamma} = \gamma$ spin 1/2 index $\widetilde{\rho} = \rho$ spin 1/2 index $\widetilde{\nu} = \diagup$ spin 0 index	$\delta_{\rho\gamma}$
$\frac{1}{2} \otimes \frac{1}{2} \to 0$	$\widetilde{\gamma} = \diagup$ spin 0 index $\widetilde{\rho} = \rho$ spin 1/2 index $\widetilde{\nu} = \nu$ spin 1/2 index	$-\frac{i}{\sqrt{2}}\left(\sigma_2\right)_{\nu\rho}$
$\frac{1}{2} \otimes \frac{1}{2} \to 1$	$\widetilde{\gamma} = g$ spin 1 index $\widetilde{\rho} = \rho$ spin 1/2 index $\widetilde{\nu} = \nu$ spin 1/2 index	$-\frac{i}{\sqrt{2}}\left(\sigma_2\sigma_g\right)_{\nu\rho}$
$\frac{1}{2} \otimes 1 \to \frac{1}{2}$	$\widetilde{\gamma} = \gamma$ spin 1/2 index $\widetilde{\rho} = \rho$ spin 1/2 index $\widetilde{\nu} = n$ spin 1 index	$\frac{1}{\sqrt{3}}\left(\sigma_n\right)_{\gamma\rho}$
$1 \otimes 0 \to 1$	$\widetilde{\gamma} = g$ spin 1 index $\widetilde{\rho} = r$ spin 1 index $\widetilde{\nu} = \diagup$ spin 0 index	δ_{rg}
$1 \otimes \frac{1}{2} \to \frac{1}{2}$	$\widetilde{\gamma} = \gamma$ spin 1/2 index $\widetilde{\rho} = r$ spin 1 index $\widetilde{\nu} = \nu$ spin 1/2 index	$-\frac{1}{\sqrt{3}}\left(\sigma_r\right)_{\gamma\nu}$
$1 \otimes 1 \to 0$	$\widetilde{\gamma} = \diagup$ spin 0 index $\widetilde{\rho} = r$ spin 1 index $\widetilde{\nu} = n$ spin 1 index	$-\frac{1}{\sqrt{3}}\delta_{nr}$
$1 \otimes 1 \to 1$	$\widetilde{\gamma} = g$ spin 1 index $\widetilde{\rho} = r$ spin 1 index $\widetilde{\nu} = n$ spin 1 index	$-\frac{1}{\sqrt{2}}\left(U_g\right)_{nr}$

Table A.5: *Summary of spin projection operators where the indices are chosen according to their appearance in the x, y, z parameters Eqs. (A.40 - A.81).*

$J\left(d_{ij}^{(\prime)}\right)_{\widetilde{\mu}} \otimes J\left(A_k\right)_{\widetilde{\nu}} \to$ channel $\widetilde{\lambda}$	index assignment	projection operator $\left(\mathcal{O}_{T_{ij}^{(\prime)}}\right)_{\widetilde{\lambda},\widetilde{\mu}\widetilde{\nu}}$
$0 \otimes 0 \to 0$	$\widetilde{\lambda} = /$ spin 0 index $\widetilde{\mu} = /$ spin 0 index $\widetilde{\nu} = /$ spin 0 index	1
$0 \otimes \frac{1}{2} \to \frac{1}{2}$	$\widetilde{\lambda} = \lambda$ spin 1/2 index $\widetilde{\mu} = /$ spin 0 index $\widetilde{\nu} = \nu$ spin 1/2 index	$\delta_{\lambda\nu}$
$0 \otimes 1 \to 1$	$\widetilde{\lambda} = \ell$ spin 1 index $\widetilde{\mu} = /$ spin 0 index $\widetilde{\nu} = n$ spin 1 index	$\delta_{\ell n}$
$\frac{1}{2} \otimes 0 \to \frac{1}{2}$	$\widetilde{\lambda} = \lambda$ spin 1/2 index $\widetilde{\mu} = \mu$ spin 1/2 index $\widetilde{\nu} = /$ spin 0 index	$\delta_{\lambda\mu}$
$\frac{1}{2} \otimes \frac{1}{2} \to 0$	$\widetilde{\lambda} = /$ spin 0 index $\widetilde{\mu} = \mu$ spin 1/2 index $\widetilde{\nu} = \nu$ spin 1/2 index	$\frac{i}{\sqrt{2}}\left(\sigma_2\right)_{\mu\nu}$
$\frac{1}{2} \otimes \frac{1}{2} \to 1$	$\widetilde{\lambda} = \ell$ spin 1 index $\widetilde{\mu} = \mu$ spin 1/2 index $\widetilde{\nu} = \nu$ spin 1/2 index	$\frac{i}{\sqrt{2}}\left(\sigma_\ell \sigma_2\right)_{\mu\nu}$
$\frac{1}{2} \otimes 1 \to \frac{1}{2}$	$\widetilde{\lambda} = \lambda$ spin 1/2 index $\widetilde{\mu} = \mu$ spin 1/2 index $\widetilde{\nu} = n$ spin 1 index	$\frac{1}{\sqrt{3}}\left(\sigma_n\right)_{\mu\lambda}$
$\frac{1}{2} \otimes 1 \to \frac{3}{2}$	$\widetilde{\lambda} = \ell\lambda$ spin 3/2 index $\widetilde{\mu} = \mu$ spin 1/2 index $\widetilde{\nu} = n$ spin 1 index	$\frac{1}{3}\left[\left(\sigma_\ell \sigma_n\right)_{\mu\lambda} + \delta_{\ell n}\delta_{\mu\lambda}\right]$
$1 \otimes 0 \to 1$	$\widetilde{\lambda} = \ell$ spin 1 index $\widetilde{\mu} = m$ spin 1 index $\widetilde{\nu} = /$ spin 0 index	$\delta_{\ell m}$
$1 \otimes \frac{1}{2} \to \frac{1}{2}$	$\widetilde{\lambda} = \lambda$ spin 1/2 index $\widetilde{\mu} = m$ spin 1 index $\widetilde{\nu} = \nu$ spin 1/2 index	$-\frac{1}{\sqrt{3}}\left(\sigma_m\right)_{\nu\lambda}$
$1 \otimes \frac{1}{2} \to \frac{3}{2}$	$\widetilde{\lambda} = \ell\lambda$ spin 3/2 index $\widetilde{\mu} = m$ spin 1 index $\widetilde{\nu} = \nu$ spin 1/2 index	$\frac{1}{3}\left[\left(\sigma_\ell \sigma_m\right)_{\nu\lambda} + \delta_{\ell m}\delta_{\nu\lambda}\right]$
$1 \otimes 1 \to 0$	$\widetilde{\lambda} = /$ spin 0 index $\widetilde{\mu} = m$ spin 1 index $\widetilde{\nu} = n$ spin 1 index	$-\frac{1}{\sqrt{3}}\delta_{mn}$
$1 \otimes 1 \to 1$	$\widetilde{\lambda} = \ell$ spin 1 index $\widetilde{\mu} = m$ spin 1 index $\widetilde{\nu} = n$ spin 1 index	$-\frac{1}{\sqrt{2}}\left(U_\ell\right)_{mn}$
$1 \otimes 1 \to 2$	$\widetilde{\lambda} = \ell k$ spin 2 index $\widetilde{\mu} = m$ spin 1 index $\widetilde{\nu} = n$ spin 1 index	$\frac{1}{2}\left[\delta_{\ell m}\delta_{kn} + \delta_{\ell n}\delta_{km} - \frac{2}{3}\delta_{\ell k}\delta_{mn}\right]$

Table A.6: *Summary of spin projectors where the indices are chosen according to their appearance in the x, y, z parameters Eqs. (A.40 - A.81).*   134

$J\left(d_{ij}^{(\prime)}\right)_{\widetilde{\gamma}} \otimes J\left(A_k\right)_{\widetilde{\sigma}} \to$ **channel** $\widetilde{\lambda}$	**index assignment**	**projection operator** $\left(\mathcal{O}_{T_{ij}^{(\prime)}}^{\dagger}\right)_{\widetilde{\lambda},\widetilde{\sigma}\widetilde{\gamma}}$
$0 \otimes 0 \to 0$	$\widetilde{\lambda} = \diagup$ spin 0 index $\widetilde{\gamma} = \diagup$ spin 0 index $\widetilde{\sigma} = \diagup$ spin 0 index	1
$0 \otimes \frac{1}{2} \to \frac{1}{2}$	$\widetilde{\lambda} = \lambda$ spin 1/2 index $\widetilde{\gamma} = \diagup$ spin 0 index $\widetilde{\sigma} = \sigma$ spin 1/2 index	$\delta_{\sigma\lambda}$
$0 \otimes 1 \to 1$	$\widetilde{\lambda} = \ell$ spin 1 index $\widetilde{\gamma} = \diagup$ spin 0 index $\widetilde{\sigma} = s$ spin 1 index	$\delta_{s\ell}$
$\frac{1}{2} \otimes 0 \to \frac{1}{2}$	$\widetilde{\lambda} = \lambda$ spin 1/2 index $\widetilde{\gamma} = \gamma$ spin 1/2 index $\widetilde{\sigma} = \diagup$ spin 0 index	$\delta_{\gamma\lambda}$
$\frac{1}{2} \otimes \frac{1}{2} \to 0$	$\widetilde{\lambda} = \diagup$ spin 0 index $\widetilde{\gamma} = \gamma$ spin 1/2 index $\widetilde{\sigma} = \sigma$ spin 1/2 index	$-\frac{i}{\sqrt{2}}\left(\sigma_2\right)_{\sigma\gamma}$
$\frac{1}{2} \otimes \frac{1}{2} \to 1$	$\widetilde{\lambda} = \ell$ spin 1 index $\widetilde{\gamma} = \gamma$ spin 1/2 index $\widetilde{\sigma} = \sigma$ spin 1/2 index	$-\frac{i}{\sqrt{2}}\left(\sigma_2\sigma_\ell\right)_{\sigma\gamma}$
$\frac{1}{2} \otimes 1 \to \frac{1}{2}$	$\widetilde{\lambda} = \lambda$ spin 1/2 index $\widetilde{\gamma} = \gamma$ spin 1/2 index $\widetilde{\sigma} = s$ spin 1 index	$\frac{1}{\sqrt{3}}\left(\sigma_s\right)_{\lambda\gamma}$
$\frac{1}{2} \otimes 1 \to \frac{3}{2}$	$\widetilde{\lambda} = \ell\lambda$ spin 3/2 index $\widetilde{\gamma} = \gamma$ spin 1/2 index $\widetilde{\sigma} = s$ spin 1 index	$\frac{1}{3}\left[\left(\sigma_s\sigma_\ell\right)_{\lambda\gamma} + \delta_{s\ell}\delta_{\lambda\gamma}\right]$
$1 \otimes 0 \to 1$	$\widetilde{\lambda} = \ell$ spin 1 index $\widetilde{\gamma} = g$ spin 1 index $\widetilde{\sigma} = \diagup$ spin 0 index	$\delta_{g\ell}$
$1 \otimes \frac{1}{2} \to \frac{1}{2}$	$\widetilde{\lambda} = \lambda$ spin 1/2 index $\widetilde{\gamma} = g$ spin 1 index $\widetilde{\sigma} = \sigma$ spin 1/2 index	$-\frac{1}{\sqrt{3}}\left(\sigma_g\right)_{\lambda\sigma}$
$1 \otimes \frac{1}{2} \to \frac{3}{2}$	$\widetilde{\lambda} = \ell\lambda$ spin 3/2 index $\widetilde{\gamma} = g$ spin 1 index $\widetilde{\sigma} = \sigma$ spin 1/2 index	$\frac{1}{3}\left[\left(\sigma_g\sigma_\ell\right)_{\lambda\sigma} + \delta_{g\ell}\delta_{\lambda\sigma}\right]$
$1 \otimes 1 \to 0$	$\widetilde{\lambda} = \diagup$ spin 0 index $\widetilde{\gamma} = g$ spin 1 index $\widetilde{\sigma} = s$ spin 1 index	$-\frac{1}{\sqrt{3}}\delta_{sg}$
$1 \otimes 1 \to 1$	$\widetilde{\lambda} = \ell$ spin 1 index $\widetilde{\gamma} = g$ spin 1 index $\widetilde{\sigma} = s$ spin 1 index	$-\frac{1}{\sqrt{2}}\left(U_\ell\right)_{sg}$
$1 \otimes 1 \to 2$	$\widetilde{\lambda} = \ell k$ spin 2 index $\widetilde{\gamma} = g$ spin 1 index $\widetilde{\sigma} = s$ spin 1 index	$\frac{1}{2}\left[\delta_{g\ell}\delta_{sk} + \delta_{s\ell}\delta_{gk} - \frac{2}{3}\delta_{k\ell}\delta_{sg}\right]$

Table A.7: *Summary of spin projectors where the indices are chosen according to their appearance in the x, y, z parameters Eqs. (A.40 - A.81).*   135

Appendix B

Symmetry factors

In this section we will determine the symmetry factors of the different diagram topologies appearing in this work. For this purpose we completely ignore spin and isospin degrees of freedom and consider without loss of generality only the part $d_{ij}(\mathbf{x}) \sim A_j(\mathbf{x})A_i(\mathbf{x})$ of the dimer wave function. Namely, we ignore the terms proportional to $\bar{A}_j(\mathbf{x})\bar{A}_i(\mathbf{x})$ etc. since they lead to the same symmetry factors. The notation follows section A.1 where we have fixed in Eq. (A.1) how bra or ket vectors with arbitrary momentum are related to the vacuum state $|0\rangle$. Since the field $A^{(\dagger)}(\mathbf{x}) \sim \widehat{a}_{\mathbf{p}}^{(\dagger)}$ is related by a Fourier transform to the respective creation or annihilation operators we use the name of the fields $(A_i^{(\dagger)}, d_{ij}^{(\dagger)}, ...)$ as a representation for those (note, that one can distinguish them by their argument: the field depends on coordinate variables and the creation and annihilation operators depend on momenta). The only important property of a particle in this section is whether it is a fermion or a boson since the former can cause additional minus signs. Therefore it is useful to extend the notation and replace A_i by F_i if the corresponding particle is a fermion. In the same way we write for fermionic dimers f_{ij} instead of d_{ij} where the latter is still used for bosons. The equal time anticommutators of two fermion fields are then given by (cf. Ref. [1])

$$\left\{ F_i(\mathbf{x}), F_j^\dagger(\mathbf{y}) \right\} = \delta_{ij}\, \delta^{(3)}(\mathbf{x} - \mathbf{y})\,,$$
$$\left\{ F_i(\mathbf{x}), F_j(\mathbf{y}) \right\} = \left\{ F_i^\dagger(\mathbf{x}), F_j^\dagger(\mathbf{y}) \right\} = 0\,. \tag{B.1}$$

Consequently, the creation and annihilation operator (also for fermions there is just one, respectively, since we consider a non-relativistic theory without anti-particles) obey equal time anticommutators [1],

$$\left\{ \left(F_i\right)_\alpha^{(\mathbf{p})}, \left(F_j^\dagger\right)_\beta^{(\mathbf{q})} \right\} = \delta_{ij}\, \delta_{\alpha\beta}\, (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})\,,$$
$$\left\{ \left(F_i\right)_\alpha^{(\mathbf{p})}, \left(F_j\right)_\beta^{(\mathbf{q})} \right\} = \left\{ \left(F_i^\dagger\right)_\alpha^{(\mathbf{p})}, \left(F_j^\dagger\right)_\beta^{(\mathbf{q})} \right\} = 0\,, \tag{B.2}$$

where we have added combined spin and isospin indices α and β for completeness although we will neglect these degrees of freedom in the following. Furthermore, we have used – as explained

above – the same symbol F for the operators as for the fields themselves, but with explicit dependence on the momenta. Below we will neglect this explicit dependence and instead write in this section $F_i(\mathbf{x})$ if the field itself is meant.

In simple words, the interchange of any two fermion fields or operators yields a factor of (-1). Only in vertices it holds for operator combinations like $(d_{ij}^\dagger F_j F_i)$ that for $P_i = P_j$ the two now identical fermion operators F_i within this vertex commute since they are annihilated at the same point and time. As a remark, note that there are of course no spinless fermions in nature so the latter statement is a simplification: more precisely one observes that two identical fermions in such a vertex must be in different spin states due to the Pauli exclusion principle, that is, $(d_{ij}^\dagger)_\alpha (\mathcal{O}_{ij}^\dagger)_{\alpha,\gamma\beta} (F_i)_\gamma (F_i)_\beta$. Adding a third fermion $(F_k^\dagger)_\eta$ from the right to the system it is clear that one can directly contract $(F_i)_\beta$ with $(F_k^\dagger)_\eta$. For the other contraction (yielding the same diagram) where γ and η has to be coupled one picks up a minus sign from interchanging $(F_i)_\beta$ and $(F_i)_\gamma$. However, this minus sign is canceled by the projection operator $\mathcal{O}_{ij} \neq 1$ which is for particles with spin not trivial. At the end one always finds a factor of "2" for contractions like that. This factor would be absent if one particle in the vertex is $F_j \neq F_i$ so that the particles are distinguishable because the contractions of $(F_k^\dagger)_\eta$ with these fields lead to two different diagrams. Hence, we conclude that the simplification mentioned above leads to the correct result and is thus justified. Furthermore, one has to keep in mind that a contracted pair of two fermions, $F_i F_j^\dagger$, commutes with all other fields.

In the following we have to deal with three different topologies which define the symmetry factors S_{ij}, S_{el} and S_{ijk}. The procedure will be to write down the matrix element of the amplitude T for each diagram topology with regard to identical particles. Then one can analyze all combinations of fermionic and bosonic fields which are possible in the considered diagram so that one can determine the right contractions and thus its symmetry factor S via $T \sim \langle \text{out}| \dots |\text{in}\rangle \sim S$.

The LaTex stylefile `simplewick.sty` used to type the contractions in this work was provided in Ref. [171].

B.1   Symmetry factor S_{ij} of self-energy diagrams

The topology of self-energy diagrams is shown in Fig. B.1. Incoming and outgoing dimers d_{ij} have the same momentum $\mathbf{p}$ and the particles in the loop A_i and A_j are distinguishable if $P_i \neq P_j$, but in the special case where $P_i = P_j$ it holds that $A_j \equiv A_i$. Keep in mind that the order of the operators A_i and A_j is fixed by the convention $d_{ij} \sim A_j A_i$ for $i < j \in \{1,2,3\}$ in Eq. (3.4). Thus, for all dimers d_{ij} with $i < j$ the self-energy amplitude T_{SE} can be read off from Fig. B.1:

$$
\begin{aligned}
T_{SE} &\sim \langle \mathbf{p}| \left(d_{ij}^\dagger \mathcal{O}_{ij}^\dagger A_j A_i \right) \left(A_i^\dagger A_j^\dagger \mathcal{O}_{ij} d_{ij} \right) |\mathbf{p}\rangle \\
&= \langle 0| d_{ij} \left(d_{ij}^\dagger A_j A_i \right) \left(A_i^\dagger A_j^\dagger d_{ij} \right) d_{ij}^\dagger |0\rangle , \quad \forall\, i < j \in \{1,2,3\} ,
\end{aligned}
\tag{B.3}
$$

where we have used in the second step that the projection operators $\mathcal{O}_{ij} = 1$ for spin- and isospinless fields. Before we continue we have to distinguish two cases, namely whether P_i is identical to P_j or not. We start with the first.

$$i\,T_{\mathrm{SE}} \;=\; d_{ij}\ \mathbf{p}\ \bigcirc\ \mathbf{p}\ d_{ij}$$

with A_i on top and A_j on the bottom of the loop.
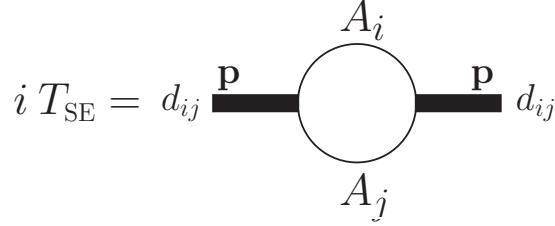
Figure B.1: *Topology of self-energy diagrams with equal incoming and outgoing momentum* $\mathbf{p}$. *The particles in the loop can be either distinguishable ($A_i \neq A_j$) or identical ($A_j \equiv A_i$).*

Case 1: $P_i \neq P_j$

The symmetry factor S_{ij} depends on the particle species. Thus, we have to consider four additional cases regarding the fermionic or bosonic nature of the particles:

- only bosons:

$$T_{\mathrm{SE}} \sim \langle 0| d_{ij}(d_{ij}^\dagger A_j A_i)(A_i^\dagger A_j^\dagger d_{ij}) d_{ij}^\dagger |0\rangle = \langle 0| d_{ij} d_{ij}^\dagger A_i A_i^\dagger A_j A_j^\dagger d_{ij} d_{ij}^\dagger |0\rangle \sim S_{ij} = +1\,,$$

- $A_i = F_i$ is a fermion $\quad\Rightarrow\quad d_{ij} = f_{ij}$ is a fermion:

$$T_{\mathrm{SE}} \sim \langle 0| f_{ij}(f_{ij}^\dagger A_j F_i)(F_i^\dagger A_j^\dagger f_{ij}) f_{ij}^\dagger |0\rangle = \langle 0| f_{ij} f_{ij}^\dagger F_i F_i^\dagger A_j A_j^\dagger f_{ij} f_{ij}^\dagger |0\rangle \sim S_{ij} = +1\,,$$

- $A_j = F_j$ is a fermion $\quad\Rightarrow\quad d_{ij} = f_{ij}$ is a fermion:

$$T_{\mathrm{SE}} \sim \langle 0| f_{ij}(f_{ij}^\dagger F_j A_i)(A_i^\dagger F_j^\dagger f_{ij}) f_{ij}^\dagger |0\rangle = \langle 0| f_{ij} f_{ij}^\dagger A_i A_i^\dagger F_j F_j^\dagger f_{ij} f_{ij}^\dagger |0\rangle \sim S_{ij} = +1\,,$$

- $A_i = F_i$, $A_j = F_j$ are fermions:

$$T_{\mathrm{SE}} \sim \langle 0| d_{ij}(d_{ij}^\dagger F_j F_i)(F_i^\dagger F_j^\dagger d_{ij}) d_{ij}^\dagger |0\rangle = \langle 0| d_{ij} d_{ij}^\dagger F_i F_i^\dagger F_j F_j^\dagger d_{ij} d_{ij}^\dagger |0\rangle \sim S_{ij} = +1\,.$$

Next, we consider the case with identical particles.

Case 2: $P_i = P_j$

Since $P_i = P_j$ the field A_j is identical to A_i. Thus, there are only two additional cases left: either A_i is a fermion or not. However, now there are more allowed contractions leading to the same diagram. We find:

- only bosons:

$$T_{\mathrm{SE}} \sim \langle 0| d_{ij} d_{ij}^\dagger [(A_i A_i)(A_i^\dagger A_i^\dagger) + (A_i A_i)(A_i^\dagger A_i^\dagger)] d_{ij} d_{ij}^\dagger |0\rangle$$
$$= 2\,\langle 0| d_{ij} d_{ij}^\dagger A_i A_i^\dagger A_i A_i^\dagger d_{ij} d_{ij}^\dagger |0\rangle \sim S_{ij} = +2\,,$$

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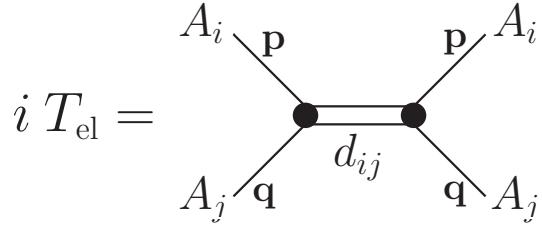
Figure B.2: *Topology of a generic two-body scattering process. Since elastic scattering is considered the momenta* **p** *and* **q** *are equal in initial and final state. The scattered particles can be either distinguishable ($A_i \neq A_j$) or identical ($A_j \equiv A_i$).*

- $A_i = F_i$ is a fermion:

$$T_{\text{SE}} \sim \langle 0| d_{ij} d_{ij}^\dagger [(F_i F_i)(F_i^\dagger F_i^\dagger) + (F_i F_i)(F_i^\dagger F_i^\dagger)] d_{ij} d_{ij}^\dagger |0\rangle$$
$$= 2 \langle 0| d_{ij} d_{ij}^\dagger F_i F_i^\dagger F_i F_i^\dagger d_{ij} d_{ij}^\dagger |0\rangle \sim S_{ij} = +2 .$$

In summary we conclude that there are no minus signs independently of the number of fermions in the system and that for identical particles one gets an additional factor of two:

$$S_{ij} = \begin{cases} 2 , & \text{if } P_i = P_j \\ 1 , & \text{if } P_i \neq P_j \end{cases} . \tag{B.4}$$

B.2 Symmetry factor S_{el} of two-body elastic scattering diagrams

We consider the elastic scattering of two particles P_i and P_j. A generic diagram for this process is shown in Fig. B.2. For $i < j \in \{1, 2, 3\}$ the corresponding amplitude T_{el} is proportional to

$$T_{\text{el}} \sim \langle \mathbf{p}, \mathbf{q}| \left(A_i^\dagger A_j^\dagger \mathcal{O}_{ij} d_{ij} \right) \left(d_{ij}^\dagger \mathcal{O}_{ij} A_j A_i \right) |\mathbf{p}, \mathbf{q}\rangle$$
$$= \langle 0| A_j A_i \left(A_i^\dagger A_j^\dagger d_{ij} \right) \left(d_{ij}^\dagger A_j A_i \right) A_i^\dagger A_j^\dagger |0\rangle , \quad \forall i < j \in \{1, 2, 3\} , \tag{B.5}$$

where the projection operators are set to 1 since we ignore spin and isospin. Again, one has to analyze the two cases with distinguishable or identical particles separately. However, in both cases one has to untangle the contractions to find the symmetry factor S_{el} for all possible diagrams.

Case 1: $P_i \neq P_j$

We have to deal with four combinations of bosons and/or fermions:

- only bosons:

$$T_{\text{el}} \sim \langle 0| A_j A_i (A_i^\dagger A_j^\dagger d_{ij})(d_{ij}^\dagger A_j A_i) A_i^\dagger A_j^\dagger |0\rangle = \langle 0| A_j A_j^\dagger A_i A_i^\dagger d_{ij} d_{ij}^\dagger A_i A_i^\dagger A_j A_j^\dagger |0\rangle \sim S_{\text{el}} = +1 ,$$

- $A_i = F_i$ is a fermion $\Rightarrow$ $d_{ij} = f_{ij}$ is a fermion:

$$T_{\text{el}} \sim \langle 0| A_j F_i (F_i^\dagger A_j^\dagger f_{ij})(f_{ij}^\dagger A_j F_i) F_i^\dagger A_j^\dagger |0\rangle = \langle 0| A_j A_j^\dagger F_i F_i^\dagger f_{ij} f_{ij}^\dagger F_i F_i^\dagger A_j A_j^\dagger |0\rangle \sim S_{\text{el}} = +1 \,,$$

- $A_j = F_j$ is a fermion $\Rightarrow$ $d_{ij} = f_{ij}$ is a fermion:

$$T_{\text{el}} \sim \langle 0| F_j A_i (A_i^\dagger F_j^\dagger f_{ij})(f_{ij}^\dagger F_j A_i) A_i^\dagger F_j^\dagger |0\rangle = \langle 0| F_j F_j^\dagger A_i A_i^\dagger f_{ij} f_{ij}^\dagger A_i A_i^\dagger F_j F_j^\dagger |0\rangle \sim S_{\text{el}} = +1 \,,$$

- $A_i = F_i$, $A_j = F_j$ are fermions:

$$T_{\text{el}} \sim \langle 0| F_j F_i (F_i^\dagger F_j^\dagger d_{ij})(d_{ij}^\dagger F_j F_i) F_i^\dagger F_j^\dagger |0\rangle = \langle 0| F_j F_j^\dagger F_i F_i^\dagger d_{ij} d_{ij}^\dagger F_i F_i^\dagger F_j F_j^\dagger |0\rangle \sim S_{\text{el}} = +1 \,.$$

And for $P_i = P_j$ one finds the results below.

Case 2: $P_i = P_j$

We have to deal with just two combinations, either only bosons or only fermions. However, there exist four different contractions:

- only bosons:

$$T_{\text{el}} \sim \langle 0| A_i A_i (A_i^\dagger A_i^\dagger d_{ij})(d_{ij}^\dagger A_i A_i) A_i^\dagger A_i^\dagger |0\rangle + \langle 0| A_i A_i (A_i^\dagger A_i^\dagger d_{ij})(d_{ij}^\dagger A_i A_i) A_i^\dagger A_i^\dagger |0\rangle$$

$$+ \langle 0| A_i A_i (A_i^\dagger A_i^\dagger d_{ij})(d_{ij}^\dagger A_i A_i) A_i^\dagger A_i^\dagger |0\rangle + \langle 0| A_i A_i (A_i^\dagger A_i^\dagger d_{ij})(d_{ij}^\dagger A_i A_i) A_i^\dagger A_i^\dagger |0\rangle$$

$$= 4 \, \langle 0| A_i A_i^\dagger A_i A_i^\dagger d_{ij} d_{ij}^\dagger A_i A_i^\dagger A_i A_i^\dagger |0\rangle \sim S_{\text{el}} = +4 \,,$$

- $A_i = F_i$ is a fermion:

$$T_{\text{el}} \sim \langle 0| F_i F_i (F_i^\dagger F_i^\dagger d_{ij})(d_{ij}^\dagger F_i F_i) F_i^\dagger F_i^\dagger |0\rangle + \langle 0| F_i F_i (F_i^\dagger F_i^\dagger d_{ij})(d_{ij}^\dagger F_i F_i) F_i^\dagger F_i^\dagger |0\rangle$$

$$+ \langle 0| F_i F_i (F_i^\dagger F_i^\dagger d_{ij})(d_{ij}^\dagger F_i F_i) F_i^\dagger F_i^\dagger |0\rangle + \langle 0| F_i F_i (F_i^\dagger F_i^\dagger d_{ij})(d_{ij}^\dagger F_i F_i) F_i^\dagger F_i^\dagger |0\rangle$$

$$= 4 \, \langle 0| F_i F_i^\dagger F_i F_i^\dagger d_{ij} d_{ij}^\dagger F_i F_i^\dagger F_i F_i^\dagger |0\rangle \sim S_{\text{el}} = +4 \,.$$

As for the self-energy diagram also here no additional minus signs appear due to fermionic constituents of the dimer d_{ij}. However, in contrast to the previous section identical particles yield a factor of "4" and thus one finds in summary:

$$S_{\text{el}} = \begin{cases} 4 \,, & \text{if } P_i = P_j \\ 1 \,, & \text{if } P_i \neq P_j \end{cases} . \tag{B.6}$$
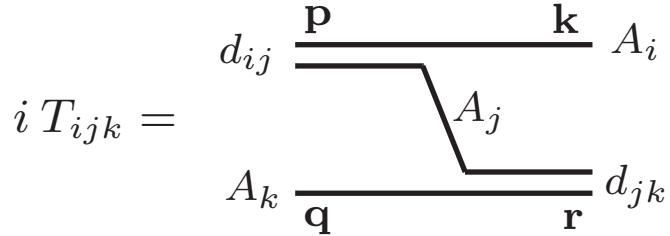
Figure B.3: *Topology of a generic exchange diagram in dimer–particle scattering with incoming momenta* **p**, **q** *and outgoing ones* **k**, **r**. *All three fields A_i, A_j and A_k are in general considered as distinguishable, but may be identical (e.g. $A_j \equiv A_i$) in some special cases. Note, that there are no constraints for the particle indices $i, j, k \in \{1, 2, 3\}$.*

B.3 Symmetry factor S_{ijk} of dimer–particle scattering diagrams

In dimer–particle scattering diagrams whose topology is shown in Fig. B.3 one has to be more carefully with the indices of the particles. In the previous two subsections it only appeared one dimer in each diagram and since we defined the dimer states d_{ij} only for $i < j \in \{1, 2, 3\}$ (cf. section 3.1) there was no doubt about the right order of the constituents. However, in the diagram Fig. B.3 and the corresponding amplitude T_{ijk} the hierarchy of the indices is not fixed, that is, $i, j, k \in \{1, 2, 3\}$ and there are *no* constraints like $i < j$. Consider for example the diagrams with on the one hand $i = 1$ and $j = 2$ and on the other hand with $i = 2$ and $j = 1$. Both topologies are allowed and do appear in our considerations. As the order of the constituents of the dimers $d_{21} \equiv d_{12}$ and d_{12} themselves is fixed to be $A_2 A_1$ (see Eq. (3.4)), we conclude that in this section the order of A_i and A_j depends on the hierarchy of their indices. For the dimer d_{ij} and the projection operator $\mathcal{O}_{ij}$ (which will be set to 1 as we ignore spin and isospin anyway) the order of their indices does not matter as indicated above. However, due the already mentioned convention in Eq. (3.4), $d_{ij} \sim A_j A_i$ for $i < j \in \{1, 2, 3\}$, the order of A_i and A_j is fixed. Hence, the operator with the *larger* index appears firstly. The amplitude T_{ijk} is thus different for varying hierarchy of the indices $i, j, k \in \{1, 2, 3\}$. One finds

$$
T_{ijk} \sim \langle \mathbf{k}, \mathbf{r} | \, d_{jk}^\dagger \mathcal{O}_{jk}^\dagger \begin{Bmatrix} (A_k A_j) \,, & j \le k \\ (A_j A_k) \,, & j > k \end{Bmatrix} \begin{Bmatrix} (A_i^\dagger A_j^\dagger) \,, & i \le j \\ (A_j^\dagger A_i^\dagger) \,, & i > j \end{Bmatrix} \mathcal{O}_{ij} d_{ij} | \mathbf{p}, \mathbf{q} \rangle
$$

$$
= \langle 0 | \, d_{jk} A_i d_{jk}^\dagger \begin{Bmatrix} (A_k A_j)(A_i^\dagger A_j^\dagger) \,, & i \le j \le k \\ (A_k A_j)(A_j^\dagger A_i^\dagger) \,, & i > j \le k \\ (A_j A_k)(A_i^\dagger A_j^\dagger) \,, & i \le j > k \\ (A_j A_k)(A_j^\dagger A_i^\dagger) \,, & i > j > k \end{Bmatrix} d_{ij} d_{ij}^\dagger A_k^\dagger | 0 \rangle \,, \tag{B.7}
$$

where we have set the projection operator $\mathcal{O}$ to 1 and combined the conditions of the vertex contributions. Furthermore, one can move $A_k^\dagger$ two steps to left which does not yield any minus sign independently of the particle species and already contract the two dimer operators $d_{ij}d_{ij}^\dagger$ on the right:

$$
T_{ijk} \sim \langle 0| \, d_{jk} A_i d_{jk}^\dagger
\begin{cases}
(A_k A_j)(A_i^\dagger A_j^\dagger) \,, & i \le j \le k \\
(A_k A_j)(A_j^\dagger A_i^\dagger) \,, & i > j \le k \\
(A_j A_k)(A_i^\dagger A_j^\dagger) \,, & i \le j > k \\
(A_j A_k)(A_j^\dagger A_i^\dagger) \,, & i > j > k
\end{cases}
A_k^\dagger \overline{d_{ij} d_{ij}^\dagger} \, |0\rangle \,.
\tag{B.8}
$$

One proceeds and analyzes all contractions for different combinations of fermions and bosons with regard to possibly identical particles. We start with the case where all particles are distinguishable.

Case 1: $P_i \ne P_j \ne P_k$

Here all operators A_i, A_j and A_k are distinguishable so one has to consider eight different boson / fermion configurations:

- only bosons:

$$
T_{ijk} \sim \langle 0| \, d_{jk} d_{jk}^\dagger
\begin{cases}
A_i(A_k A_j)(A_i^\dagger A_j^\dagger)A_k^\dagger \,, & i \le j \le k \\
A_i(A_k A_j)(A_j^\dagger A_i^\dagger)A_k^\dagger \,, & i > j \le k \\
A_i(A_j A_k)(A_i^\dagger A_j^\dagger)A_k^\dagger \,, & i \le j > k \\
A_i(A_j A_k)(A_j^\dagger A_i^\dagger)A_k^\dagger \,, & i > j > k
\end{cases}
d_{ij} d_{ij}^\dagger \, |0\rangle
$$

$$
= \langle 0| \, d_{jk} d_{jk}^\dagger
\begin{cases}
A_i A_i^\dagger A_j A_j^\dagger A_k A_k^\dagger \,, & i \le j \le k \\
A_i A_i^\dagger A_j A_j^\dagger A_k A_k^\dagger \,, & i > j \le k \\
A_i A_i^\dagger A_j A_j^\dagger A_k A_k^\dagger \,, & i \le j > k \\
A_i A_i^\dagger A_j A_j^\dagger A_k A_k^\dagger \,, & i > j > k
\end{cases}
d_{ij} d_{ij}^\dagger \, |0\rangle
$$

$$
\sim S_{ijk} = +1 \,,
$$

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- $A_i = F_i$ is a fermion $\quad\Rightarrow\quad d_{ij} = f_{ij}$ is a fermion:

$$
T_{ijk} \sim \langle 0| \overrightarrow{d_{jk}d_{jk}^\dagger}
\begin{Bmatrix}
F_i(A_kA_j)(F_i^\dagger A_j^\dagger)A_k^\dagger \,, & i \le j \le k \\
F_i(A_kA_j)(A_j^\dagger F_i^\dagger)A_k^\dagger \,, & i > j \le k \\
F_i(A_jA_k)(F_i^\dagger A_j^\dagger)A_k^\dagger \,, & i \le j > k \\
F_i(A_jA_k)(A_j^\dagger F_i^\dagger)A_k^\dagger \,, & i > j > k
\end{Bmatrix}
\overrightarrow{f_{ij}f_{ij}^\dagger} |0\rangle
$$

$$
= \langle 0| \overrightarrow{d_{jk}d_{jk}^\dagger}
\begin{Bmatrix}
F_iF_i^\dagger A_j A_j^\dagger A_k A_k^\dagger \,, & i \le j \le k \\
F_iF_i^\dagger A_j A_j^\dagger A_k A_k^\dagger \,, & i > j \le k \\
F_iF_i^\dagger A_j A_j^\dagger A_k A_k^\dagger \,, & i \le j > k \\
F_iF_i^\dagger A_j A_j^\dagger A_k A_k^\dagger \,, & i > j > k
\end{Bmatrix}
\overrightarrow{f_{ij}f_{ij}^\dagger} |0\rangle
$$

$$
\sim S_{ijk} = +1 \,,
$$

- $A_j = F_j$ is a fermion $\quad\Rightarrow\quad d_{ij} = f_{ij}, \, d_{jk} = f_{jk}$ are fermions:

$$
T_{ijk} \sim \langle 0| \overrightarrow{f_{jk}f_{jk}^\dagger}
\begin{Bmatrix}
A_i(A_kF_j)(A_i^\dagger F_j^\dagger)A_k^\dagger \,, & i \le j \le k \\
A_i(A_kF_j)(F_j^\dagger A_i^\dagger)A_k^\dagger \,, & i > j \le k \\
A_i(F_jA_k)(A_i^\dagger F_j^\dagger)A_k^\dagger \,, & i \le j > k \\
A_i(F_jA_k)(F_j^\dagger A_i^\dagger)A_k^\dagger \,, & i > j > k
\end{Bmatrix}
\overrightarrow{f_{ij}f_{ij}^\dagger} |0\rangle
$$

$$
= \langle 0| \overrightarrow{f_{jk}f_{jk}^\dagger}
\begin{Bmatrix}
A_iA_i^\dagger F_j F_j^\dagger A_k A_k^\dagger \,, & i \le j \le k \\
A_iA_i^\dagger F_j F_j^\dagger A_k A_k^\dagger \,, & i > j \le k \\
A_iA_i^\dagger F_j F_j^\dagger A_k A_k^\dagger \,, & i \le j > k \\
A_iA_i^\dagger F_j F_j^\dagger A_k A_k^\dagger \,, & i > j > k
\end{Bmatrix}
\overrightarrow{f_{ij}f_{ij}^\dagger} |0\rangle
$$

$$
\sim S_{ijk} = +1 \,,
$$

- $A_k = F_k$ is a fermion $\quad\Rightarrow\quad d_{jk} = f_{jk}$ is a fermion:

$$T_{ijk} \sim \langle 0| \, \overbrace{f_{jk} f_{jk}^\dagger} \left\{ \begin{array}{ll} A_i(F_k A_j)(A_i^\dagger A_j^\dagger)F_k^\dagger \,, & i \leq j \leq k \\[4pt] A_i(F_k A_j)(A_j^\dagger A_i^\dagger)F_k^\dagger \,, & i > j \leq k \\[4pt] A_i(A_j F_k)(A_i^\dagger A_j^\dagger)F_k^\dagger \,, & i \leq j > k \\[4pt] A_i(A_j F_k)(A_j^\dagger A_i^\dagger)F_k^\dagger \,, & i > j > k \end{array} \right\} \overbrace{d_{ij} d_{ij}^\dagger} |0\rangle$$

$$= \langle 0| \, \overbrace{f_{jk} f_{jk}^\dagger} \left\{ \begin{array}{ll} A_i A_i^\dagger A_j A_j^\dagger F_k F_k^\dagger \,, & i \leq j \leq k \\[4pt] A_i A_i^\dagger A_j A_j^\dagger F_k F_k^\dagger \,, & i > j \leq k \\[4pt] A_i A_i^\dagger A_j A_j^\dagger F_k F_k^\dagger \,, & i \leq j > k \\[4pt] A_i A_i^\dagger A_j A_j^\dagger F_k F_k^\dagger \,, & i > j > k \end{array} \right\} \overbrace{d_{ij} d_{ij}^\dagger} |0\rangle$$

$$\sim S_{ijk} = +1 \,,$$

- $A_i = F_i$, $A_j = F_j$ are fermions $\quad\Rightarrow\quad d_{jk} = f_{jk}$ is a fermion:

$$T_{ijk} \sim \langle 0| - \overbrace{f_{jk} f_{jk}^\dagger} \left\{ \begin{array}{ll} F_i(A_k F_j)(F_i^\dagger F_j^\dagger)A_k^\dagger \,, & i \leq j \leq k \\[4pt] F_i(A_k F_j)(F_j^\dagger F_i^\dagger)A_k^\dagger \,, & i > j \leq k \\[4pt] F_i(F_j A_k)(F_i^\dagger F_j^\dagger)A_k^\dagger \,, & i \leq j > k \\[4pt] F_i(F_j A_k)(F_j^\dagger F_i^\dagger)A_k^\dagger \,, & i > j > k \end{array} \right\} \overbrace{d_{ij} d_{ij}^\dagger} |0\rangle$$

$$= \langle 0| - \overbrace{f_{jk} f_{jk}^\dagger} \left\{ \begin{array}{ll} - F_i F_i^\dagger F_j F_j^\dagger A_k A_k^\dagger \,, & i \leq j \leq k \\[4pt] F_i F_i^\dagger F_j F_j^\dagger A_k A_k^\dagger \,, & i > j \leq k \\[4pt] - F_i F_i^\dagger F_j F_j^\dagger A_k A_k^\dagger \,, & i \leq j > k \\[4pt] F_i F_i^\dagger F_j F_j^\dagger A_k A_k^\dagger \,, & i > j > k \end{array} \right\} \overbrace{d_{ij} d_{ij}^\dagger} |0\rangle$$

$$\sim S_{ijk} = \left\{ \begin{array}{ll} +1 \,, & i \leq j \leq k \\ -1 \,, & i > j \leq k \\ +1 \,, & i \leq j > k \\ -1 \,, & i > j > k \end{array} \right. \,,$$

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- $A_i = F_i$, $A_k = F_k$ are fermions $\Rightarrow$ $d_{ij} = f_{ij}$, $d_{jk} = f_{jk}$ are fermions:

$$
T_{ijk} \sim \langle 0| - f_{jk}f_{jk}^\dagger
\begin{cases}
F_i(F_k A_j)(F_i^\dagger A_j^\dagger)F_k^\dagger \,, & i \le j \le k \\
F_i(F_k A_j)(A_j^\dagger F_i^\dagger)F_k^\dagger \,, & i > j \le k \\
F_i(A_j F_k)(F_i^\dagger A_j^\dagger)F_k^\dagger \,, & i \le j > k \\
F_i(A_j F_k)(A_j^\dagger F_i^\dagger)F_k^\dagger \,, & i > j > k
\end{cases}
f_{ij}f_{ij}^\dagger |0\rangle
$$

$$
= \langle 0| - f_{jk}f_{jk}^\dagger
\begin{cases}
- F_i F_i^\dagger A_j A_j^\dagger F_k F_k^\dagger \,, & i \le j \le k \\
- F_i F_i^\dagger A_j A_j^\dagger F_k F_k^\dagger \,, & i > j \le k \\
- F_i F_i^\dagger A_j A_j^\dagger F_k F_k^\dagger \,, & i \le j > k \\
- F_i F_i^\dagger A_j A_j^\dagger F_k F_k^\dagger \,, & i > j > k
\end{cases}
f_{ij}f_{ij}^\dagger |0\rangle
$$

$$
\sim S_{ijk} = +1 \,,
$$

- $A_j = F_j$, $A_k = F_k$ are fermions $\Rightarrow$ $d_{ij} = f_{ij}$ is a fermion:

$$
T_{ijk} \sim \langle 0| d_{jk}d_{jk}^\dagger
\begin{cases}
A_i(F_k F_j)(A_i^\dagger F_j^\dagger)F_k^\dagger \,, & i \le j \le k \\
A_i(F_k F_j)(F_j^\dagger A_i^\dagger)F_k^\dagger \,, & i > j \le k \\
A_i(F_j F_k)(A_i^\dagger F_j^\dagger)F_k^\dagger \,, & i \le j > k \\
A_i(F_j F_k)(F_j^\dagger A_i^\dagger)F_k^\dagger \,, & i > j > k
\end{cases}
f_{ij}f_{ij}^\dagger |0\rangle
$$

$$
= \langle 0| d_{jk}d_{jk}^\dagger
\begin{cases}
A_i A_i^\dagger F_j F_j^\dagger F_k F_k^\dagger \,, & i \le j \le k \\
A_i A_i^\dagger F_j F_j^\dagger F_k F_k^\dagger \,, & i > j \le k \\
- A_i A_i^\dagger F_j F_j^\dagger F_k F_k^\dagger \,, & i \le j > k \\
- A_i A_i^\dagger F_j F_j^\dagger F_k F_k^\dagger \,, & i > j > k
\end{cases}
f_{ij}f_{ij}^\dagger |0\rangle
$$

$$
\sim S_{ijk} =
\begin{cases}
+1 \,, & i \le j \le k \\
+1 \,, & i > j \le k \\
-1 \,, & i \le j > k \\
-1 \,, & i > j > k
\end{cases} \,,
$$

- $A_i = F_i$, $A_j = F_j$, $A_k = F_k$ are fermions:

$$T_{ijk} \sim \langle 0| \, d_{jk}d_{jk}^\dagger \begin{Bmatrix} F_i(F_kF_j)(F_i^\dagger F_j^\dagger)F_k^\dagger \,, & i \le j \le k \\ F_i(F_kF_j)(F_j^\dagger F_i^\dagger)F_k^\dagger \,, & i > j \le k \\ F_i(F_jF_k)(F_i^\dagger F_j^\dagger)F_k^\dagger \,, & i \le j > k \\ F_i(F_jF_k)(F_j^\dagger F_i^\dagger)F_k^\dagger \,, & i > j > k \end{Bmatrix} d_{ij}d_{ij}^\dagger |0\rangle$$

$$= \langle 0| \, d_{jk}d_{jk}^\dagger \begin{Bmatrix} F_iF_i^\dagger F_jF_j^\dagger F_kF_k^\dagger \,, & i \le j \le k \\ -\,F_iF_i^\dagger F_jF_j^\dagger F_kF_k^\dagger \,, & i > j \le k \\ -\,F_iF_i^\dagger F_jF_j^\dagger F_kF_k^\dagger \,, & i \le j > k \\ F_iF_i^\dagger F_jF_j^\dagger F_kF_k^\dagger \,, & i > j > k \end{Bmatrix} d_{ij}d_{ij}^\dagger |0\rangle$$

$$\sim S_{ijk} = \begin{cases} +1 \,, & i \le j \le k \\ -1 \,, & i > j \le k \\ -1 \,, & i \le j > k \\ +1 \,, & i > j > k \end{cases} .$$

Next, we consider the case $P_i = P_j \ne P_k$.

Case 2: $P_i = P_j \ne P_k$

The amplitude T_{ijk} in Eq. (B.8) is for $P_i = P_j \ne P_k$ reduced to

$$T_{ijk} \sim \langle 0| \, d_{ik}A_i d_{ik}^\dagger \begin{Bmatrix} (A_kA_i)(A_i^\dagger A_k^\dagger) \,, & i \le k \\ (A_iA_k)(A_i^\dagger A_k^\dagger) \,, & i > k \end{Bmatrix} A_k^\dagger d_{ij}d_{ij}^\dagger |0\rangle \,, \tag{B.9}$$

and the four different combinations of bosons and fermions yield the following symmetry factors:

- only bosons:

$$T_{ijk} \sim \langle 0| \, d_{ik}d_{ik}^\dagger \begin{Bmatrix} A_i(A_kA_i)(A_i^\dagger A_i^\dagger)A_k^\dagger + A_i(A_kA_i)(A_i^\dagger A_i^\dagger)A_k^\dagger \,, & i \le k \\ A_i(A_iA_k)(A_i^\dagger A_i^\dagger)A_k^\dagger + A_i(A_iA_k)(A_i^\dagger A_i^\dagger)A_k^\dagger \,, & i > k \end{Bmatrix} d_{ij}d_{ij}^\dagger |0\rangle$$

$$= 2\, \langle 0| \, d_{ik}d_{ik}^\dagger \begin{Bmatrix} A_iA_i^\dagger A_iA_i^\dagger A_kA_k^\dagger \,, & i \le k \\ A_iA_i^\dagger A_iA_i^\dagger A_kA_k^\dagger \,, & i > k \end{Bmatrix} d_{ij}d_{ij}^\dagger |0\rangle$$

$$\sim S_{ijk} = +2 \,,$$

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- $A_i = F_i$ is a fermion $\quad \Rightarrow \quad d_{ik} = f_{ik}$ is a fermion:

$$T_{ijk} \sim \langle 0| - f_{ik}f_{ik}^\dagger \left\{ \begin{array}{l} F_i(A_kF_i)(F_i^\dagger F_i^\dagger)A_k^\dagger + F_i(A_kF_i)(F_i^\dagger F_i^\dagger)A_k^\dagger , \ \ i \le k \\ F_i(F_iA_k)(F_i^\dagger F_i^\dagger)A_k^\dagger + F_i(F_iA_k)(F_i^\dagger F_i^\dagger)A_k^\dagger , \ \ i > k \end{array} \right\} d_{ij}d_{ij}^\dagger |0\rangle$$

$$= 2\,\langle 0| - f_{ik}f_{ik}^\dagger \left\{ \begin{array}{l} -\,F_iF_i^\dagger F_iF_i^\dagger A_kA_k^\dagger , \ \ i \le k \\ -\,F_iF_i^\dagger F_iF_i^\dagger A_kA_k^\dagger , \ \ i > k \end{array} \right\} d_{ij}d_{ij}^\dagger |0\rangle$$

$$\sim S_{ijk} = +2\,,$$

- $A_k = F_k$ is a fermion $\quad \Rightarrow \quad d_{ik} = f_{ik}$ is a fermion:

$$T_{ijk} \sim \langle 0| f_{ik}f_{ik}^\dagger \left\{ \begin{array}{l} A_i(F_kA_i)(A_i^\dagger A_i^\dagger)F_k^\dagger + A_i(F_kA_i)(A_i^\dagger A_i^\dagger)F_k^\dagger , \ \ i \le k \\ A_i(A_iF_k)(A_i^\dagger A_i^\dagger)F_k^\dagger + A_i(A_iF_k)(A_i^\dagger A_i^\dagger)F_k^\dagger , \ \ i > k \end{array} \right\} d_{ij}d_{ij}^\dagger |0\rangle$$

$$= 2\,\langle 0| f_{ik}f_{ik}^\dagger \left\{ \begin{array}{l} A_iA_i^\dagger A_iA_i^\dagger F_kF_k^\dagger , \ \ i \le k \\ A_iA_i^\dagger A_iA_i^\dagger F_kF_k^\dagger , \ \ i > k \end{array} \right\} d_{ij}d_{ij}^\dagger |0\rangle$$

$$\sim S_{ijk} = +2\,,$$

- $A_i = F_i$, $A_k = F_k$ are fermions:

$$T_{ijk} \sim \langle 0| d_{ik}d_{ik}^\dagger \left\{ \begin{array}{l} F_i(F_kF_i)(F_i^\dagger F_i^\dagger)F_k^\dagger + F_i(F_kF_i)(F_i^\dagger F_i^\dagger)F_k^\dagger , \ \ i \le k \\ F_i(F_iF_k)(F_i^\dagger F_i^\dagger)F_k^\dagger + F_i(F_iF_k)(F_i^\dagger F_i^\dagger)F_k^\dagger , \ \ i > k \end{array} \right\} d_{ij}d_{ij}^\dagger |0\rangle$$

$$= 2\,\langle 0| d_{ik}d_{ik}^\dagger \left\{ \begin{array}{l} F_iF_i^\dagger F_iF_i^\dagger F_kF_k^\dagger , \ \ i \le k \\ -\,F_iF_i^\dagger F_iF_i^\dagger F_kF_k^\dagger , \ \ i > k \end{array} \right\} d_{ij}d_{ij}^\dagger |0\rangle$$

$$\sim S_{ijk} = \begin{cases} +2\,, & i \le k \\ -2\,, & i > k \end{cases} .$$

Case 3: $P_i = P_k \neq P_j$

We continue with the analysis of the diagram in Fig. B.3 for $P_i = P_k \neq P_j$. The corresponding amplitude is given by (cf. Eq. (B.8))

$$T_{ijk} \sim \langle 0| d_{ij}A_id_{ij}^\dagger \left\{ \begin{array}{l} (A_jA_i)(A_i^\dagger A_j^\dagger)\,, \ \ i \le j \\ (A_iA_j)(A_j^\dagger A_i^\dagger)\,, \ \ i > j \end{array} \right\} A_i^\dagger d_{ij}d_{ij}^\dagger |0\rangle \,. \tag{B.10}$$

Consequently, we have to consider four different cases regarding the particle species:

- only bosons:

$$T_{ijk} \sim \langle 0| \, d_{ij}d_{ij}^\dagger \left\{ \begin{array}{ll} A_i(A_jA_i)(A_i^\dagger A_j^\dagger)A_i^\dagger\ , & i \le j \\[2mm] A_i(A_iA_j)(A_j^\dagger A_i^\dagger)A_i^\dagger\ , & i > j \end{array} \right\} d_{ij}d_{ij}^\dagger \, |0\rangle$$

$$= \langle 0| \, d_{ij}d_{ij}^\dagger \left\{ \begin{array}{ll} A_iA_i^\dagger A_jA_j^\dagger A_iA_i^\dagger\ , & i \le j \\[2mm] A_iA_i^\dagger A_jA_j^\dagger A_iA_i^\dagger\ , & i > j \end{array} \right\} d_{ij}d_{ij}^\dagger \, |0\rangle$$

$$\sim S_{ijk} = +1 \ ,$$

- $A_i = F_i$ is a fermion $\quad \Rightarrow \quad d_{ij} = f_{ij}$ is a fermion:

$$T_{ijk} \sim \langle 0| - f_{ij}f_{ij}^\dagger \left\{ \begin{array}{ll} F_i(A_jF_i)(F_i^\dagger A_j^\dagger)F_i^\dagger\ , & i \le j \\[2mm] F_i(F_iA_j)(A_j^\dagger F_i^\dagger)F_i^\dagger\ , & i > j \end{array} \right\} f_{ij}f_{ij}^\dagger \, |0\rangle$$

$$= \langle 0| - f_{ij}f_{ij}^\dagger \left\{ \begin{array}{ll} - F_iF_i^\dagger A_jA_j^\dagger F_iF_i^\dagger\ , & i \le j \\[2mm] - F_iF_i^\dagger A_jA_j^\dagger F_iF_i^\dagger\ , & i > j \end{array} \right\} f_{ij}f_{ij}^\dagger \, |0\rangle$$

$$\sim S_{ijk} = +1 \ ,$$

- $A_j = F_j$ is a fermion $\quad \Rightarrow \quad d_{ij} = f_{ij}$ is a fermion:

$$T_{ijk} \sim \langle 0| \, f_{ij}f_{ij}^\dagger \left\{ \begin{array}{ll} A_i(F_jA_i)(A_i^\dagger F_j^\dagger)A_i^\dagger\ , & i \le j \\[2mm] A_i(A_iF_j)(F_j^\dagger A_i^\dagger)A_i^\dagger\ , & i > j \end{array} \right\} f_{ij}f_{ij}^\dagger \, |0\rangle$$

$$= \langle 0| \, f_{ij}f_{ij}^\dagger \left\{ \begin{array}{ll} A_iA_i^\dagger F_jF_j^\dagger A_iA_i^\dagger\ , & i \le j \\[2mm] A_iA_i^\dagger F_jF_j^\dagger A_iA_i^\dagger\ , & i > j \end{array} \right\} f_{ij}f_{ij}^\dagger \, |0\rangle$$

$$\sim S_{ijk} = +1 \ ,$$

- $A_i = F_i$, $A_j = F_j$ are fermions:

$$T_{ijk} \sim \langle 0| \, d_{ij}d_{ij}^\dagger \left\{ \begin{array}{ll} F_i(F_jF_i)(F_i^\dagger F_j^\dagger)F_i^\dagger\ , & i \le j \\[2mm] F_i(F_iF_j)(F_j^\dagger F_i^\dagger)F_i^\dagger\ , & i > j \end{array} \right\} d_{ij}d_{ij}^\dagger \, |0\rangle$$

$$= \langle 0| \, d_{ij}d_{ij}^\dagger \left\{ \begin{array}{ll} - F_iF_i^\dagger F_jF_j^\dagger F_iF_i^\dagger\ , & i \le j \\[2mm] - F_iF_i^\dagger F_jF_j^\dagger F_iF_i^\dagger\ , & i > j \end{array} \right\} d_{ij}d_{ij}^\dagger \, |0\rangle$$

$$\sim S_{ijk} = -1 \ .$$

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Case 4: $P_i \neq P_j = P_k$

In the second to last possible configuration of identical particles where $P_i \neq P_j = P_k$ the amplitude of the diagram in Fig. B.3 yields

$$T_{ijk} \sim \langle 0| \, d_{jk} A_i d_{jk}^\dagger \begin{Bmatrix} (A_j A_j)(A_i^\dagger A_j^\dagger), & i \leq j \\ (A_j A_j)(A_j^\dagger A_i^\dagger), & i > j \end{Bmatrix} A_j^\dagger \, \overparen{d_{ij} d_{ij}^\dagger} \, |0\rangle \;, \tag{B.11}$$

and the symmetry factor S_{ijk} for different fermion / boson combinations is listed below:

- only bosons:

$$T_{ijk} \sim \langle 0| \, \overparen{d_{jk} d_{jk}^\dagger} \begin{Bmatrix} A_i(A_j A_j)(A_i^\dagger A_j^\dagger)A_j^\dagger + A_i(A_j A_j)(A_i^\dagger A_j^\dagger)A_j^\dagger, & i \leq j \\ A_i(A_j A_j)(A_j^\dagger A_i^\dagger)A_j^\dagger + A_i(A_j A_j)(A_j^\dagger A_i^\dagger)A_j^\dagger, & i > j \end{Bmatrix} \overparen{d_{ij} d_{ij}^\dagger} \, |0\rangle$$

$$= 2 \, \langle 0| \, \overparen{d_{jk} d_{jk}^\dagger} \begin{Bmatrix} A_i A_i^\dagger A_j A_j^\dagger A_j A_j^\dagger, & i \leq j \\ A_i A_i^\dagger A_j A_j^\dagger A_j A_j^\dagger, & i > j \end{Bmatrix} \overparen{d_{ij} d_{ij}^\dagger} \, |0\rangle$$

$$\sim S_{ijk} = +2 \;,$$

- $A_i = F_i$ is a fermion $\;\Rightarrow\; d_{ij} = f_{ij}$ is a fermion:

$$T_{ijk} \sim \langle 0| \, \overparen{d_{jk} d_{jk}^\dagger} \begin{Bmatrix} F_i(A_j A_j)(F_i^\dagger A_j^\dagger)A_j^\dagger + F_i(A_j A_j)(F_i^\dagger A_j^\dagger)A_j^\dagger, & i \leq j \\ F_i(A_j A_j)(A_j^\dagger F_i^\dagger)A_j^\dagger + F_i(A_j A_j)(A_j^\dagger F_i^\dagger)A_j^\dagger, & i > j \end{Bmatrix} \overparen{f_{ij} f_{ij}^\dagger} \, |0\rangle$$

$$= 2 \, \langle 0| \, \overparen{d_{jk} d_{jk}^\dagger} \begin{Bmatrix} F_i F_i^\dagger A_j A_j^\dagger A_j A_j^\dagger, & i \leq j \\ F_i F_i^\dagger A_j A_j^\dagger A_j A_j^\dagger, & i > j \end{Bmatrix} \overparen{f_{ij} f_{ij}^\dagger} \, |0\rangle$$

$$\sim S_{ijk} = +2 \;,$$

- $A_j = F_j$ is a fermion $\;\Rightarrow\; d_{ij} = f_{ij}$ is a fermion:

$$T_{ijk} \sim \langle 0| \, \overparen{d_{jk} d_{jk}^\dagger} \begin{Bmatrix} A_i(F_j F_j)(A_i^\dagger F_j^\dagger)F_j^\dagger + A_i(F_j F_j)(A_i^\dagger F_j^\dagger)F_j^\dagger, & i \leq j \\ A_i(F_j F_j)(F_j^\dagger A_i^\dagger)F_j^\dagger + A_i(F_j F_j)(F_j^\dagger A_i^\dagger)F_j^\dagger, & i > j \end{Bmatrix} \overparen{f_{ij} f_{ij}^\dagger} \, |0\rangle$$

$$= 2 \, \langle 0| \, \overparen{d_{jk} d_{jk}^\dagger} \begin{Bmatrix} - A_i A_i^\dagger F_j F_j^\dagger F_j F_j^\dagger, & i \leq j \\ - A_i A_i^\dagger F_j F_j^\dagger F_j F_j^\dagger, & i > j \end{Bmatrix} \overparen{f_{ij} f_{ij}^\dagger} \, |0\rangle$$

$$\sim S_{ijk} = -2 \;,$$

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- $A_i = F_i$, $A_j = F_j$ are fermions:

$$T_{ijk} \sim \langle 0| \, d_{jk}d_{jk}^\dagger \left\{ \begin{array}{l} F_i(F_jF_j)(F_i^\dagger F_j^\dagger)F_j^\dagger + F_i(F_jF_j)(F_i^\dagger F_j^\dagger)F_j^\dagger \, , \quad i \leq j \\ F_i(F_jF_j)(F_j^\dagger F_i^\dagger)F_j^\dagger + F_i(F_jF_j)(F_j^\dagger F_i^\dagger)F_j^\dagger \, , \quad i > j \end{array} \right\} d_{ij}d_{ij}^\dagger \, |0\rangle$$

$$= 2 \, \langle 0| \, d_{jk}d_{jk}^\dagger \left\{ \begin{array}{l} - F_iF_i^\dagger F_jF_j^\dagger F_jF_j^\dagger \, , \quad i \leq j \\ F_iF_i^\dagger F_jF_j^\dagger F_jF_j^\dagger \, , \quad i > j \end{array} \right\} d_{ij}d_{ij}^\dagger \, |0\rangle$$

$$\sim S_{ijk} = \begin{cases} -2 \, , & i \leq j \\ +2 \, , & i > j \end{cases} .$$

Case 5: $P_i = P_j = P_k$

Finally, we consider the case where all particles are identical, i.e. $P_i = P_j = P_k$. Hence, there only are two cases: all particles are either bosons or fermions. Furthermore, the corresponding amplitude,

$$T_{ijk} \sim \langle 0| \, dA_i d^\dagger (A_iA_i)(A_i^\dagger A_i^\dagger)A_i^\dagger dd^\dagger \, |0\rangle \, , \tag{B.12}$$

has no explicit index dependence since for only one index there is no hierarchy left in the system (note, that we have written $d_{ij} \equiv d_{jk} := d$ for simplicity). We find for

- only bosons:

$$T_{ijk} \sim \langle 0| \, dd^\dagger \Big[A_i(A_iA_i)(A_i^\dagger A_i^\dagger)A_i^\dagger + A_i(A_iA_i)(A_i^\dagger A_i^\dagger)A_i^\dagger$$

$$+ A_i(A_iA_i)(A_i^\dagger A_i^\dagger)A_i^\dagger + A_i(A_iA_i)(A_i^\dagger A_i^\dagger)A_i^\dagger \Big] dd^\dagger \, |0\rangle$$

$$= 4 \, \langle 0| \, dd^\dagger A_iA_i^\dagger A_iA_i^\dagger A_iA_i^\dagger dd^\dagger \, |0\rangle$$

$$\sim S_{ijk} = +4 \, ,$$

- only fermions ($A_i = F_i$):

$$T_{ijk} \sim \langle 0| \, dd^\dagger \Big[F_i(F_iF_i)(F_i^\dagger F_i^\dagger)F_i^\dagger + F_i(F_iF_i)(F_i^\dagger F_i^\dagger)F_i^\dagger$$

$$+ F_i(F_iF_i)(F_i^\dagger F_i^\dagger)F_i^\dagger + F_i(F_iF_i)(F_i^\dagger F_i^\dagger)F_i^\dagger \Big] dd^\dagger \, |0\rangle$$

$$= 4 \, \langle 0| \, dd^\dagger \Big[- F_iF_i^\dagger F_iF_i^\dagger F_iF_i^\dagger \Big] dd^\dagger \, |0\rangle$$

$$\sim S_{ijk} = -4 \, ,$$

Summary

Collecting the results for all different configurations one can summarize them by the following formula for the symmetry factor S_{ijk}:

$$S_{ijk} = \zeta_{ijk} \left(1 + \delta_{P_i P_j} + \delta_{P_j P_k} + \delta_{P_i P_j} \delta_{P_j P_k}\right) , \tag{B.13}$$

where ζ_{ijk} is either $+1$ or -1 depending on the particle species, the hierarchy between the indices and the fact whether some of the particles are identical or not. It is defined as:

$$\zeta_{ijk} := \begin{cases} -1 , & \text{if} \begin{cases} \bullet \ P_i \neq P_j \neq P_k \ \wedge \ \text{only } P_i, P_j \text{ fermions} \ \wedge \ \left\{i > j < k \ \vee \ i > j > k\right\} \\ \bullet \ P_i \neq P_j \neq P_k \ \wedge \ \text{only } P_j, P_k \text{ fermions} \ \wedge \ \left\{i < j > k \ \vee \ i > j > k\right\} \\ \bullet \ P_i \neq P_j \neq P_k \ \wedge \ P_i, P_j, P_k \text{ fermions} \ \wedge \ \left\{i > j < k \ \vee \ i < j > k\right\} \\ \bullet \ P_i = P_j \neq P_k \ \wedge \ P_i, P_j, P_k \text{ fermions} \ \wedge \ i > k \\ \bullet \ P_i = P_k \neq P_j \ \wedge \ P_i, P_j, P_k \text{ fermions} \\ \bullet \ P_i \neq P_j = P_k \ \wedge \ \text{only } P_j \text{ fermion} \\ \bullet \ P_i \neq P_j = P_k \ \wedge \ P_i, P_j, P_k \text{ fermions} \ \wedge \ i < j \\ \bullet \ P_i = P_j = P_k \ \wedge \ P_i, P_j, P_k \text{ fermions} \end{cases} \\ \\ +1 , & \text{else} \end{cases} .$$

$$\tag{B.14}$$

Note, that in each case the particles which are not mentioned to be fermions *must* be bosons, so the conditions are unique.

Appendix C

Feynman diagrams contributing to d_{12}–P_3 dimer–particle scattering

In order to find a completely general expression for the d_{12}–P_3 scattering amplitude it is necessary to identify all contributing Feynman diagrams. This is done in Figure C.2 and with the remarks below we want to motivate this result.

It is obvious that some diagrams only contribute if at least two of the three particles are identical. This is the reason for the Kronecker-delta $\delta_{P_i P_j}$,

$$
\delta_{P_i P_j} = \begin{cases} 1 \, , & \text{if } P_i = P_j, \text{ i.e. } P_i \text{ and } P_j \text{ are identical particles} \\ 0 \, , & \text{if } P_i \neq P_j, \text{ i.e. } P_i \text{ and } P_j \text{ are distinguishable particles} \end{cases} , \tag{C.1}
$$

in front of some diagrams. Moreover, we have to take into account the structure of the dimer wave function of G-parity eigenstates made of a superposition of distinguishable particles, e.g. $d_{12} = \frac{1}{\sqrt{2}}(\bar{A}_1 A_2 + A_1 \bar{A}_2)$. Choosing for example the three particle system $P_1 = \bar{A}_1, P_2 = A_2, P_3 = A_3 = A_1$ the normal $\delta_{P_1 P_3} = 0$ vanishes. However, since the two terms in the wave function can fluctuate into each other we know that the diagram in Fig. C.1 *does* contribute to the three particle system: within the blob d_{12} is in the state $\bar{A}_1 A_2$ and thus contributes together with $P_3 = A_3 = A_1$ to the chosen three particle system $P_1 = \bar{A}_1, P_2 = A_2, P_3 = A_3 = A_1$. After the blob and during the propagation to the next vertex it fluctuates into $A_1 \bar{A}_2$ and decays into A_1 and $\bar{A}_2$. The latter recombines with $P_3 = A_1$ to d_{12} which can change its state again so that in the final state one founds the same particle content as in the initial one. Thus, a prefactor $\delta_{P_1 P_3}$, which forbids such a contribution, cannot be right. For this we have introduced a modified Kronecker-delta $\delta_{P_i P_j}^{(ab^{(\prime)})}$ which solves this inconsistency:

$$
\delta_{P_i P_j}^{\left(ab^{(\prime)}\right)} := \delta_{P_i P_j} + \left(\delta_{A_i A_j} - \delta_{P_i P_j}\right) \delta_{\eta_{ab}^{(\prime)}|1|} \left(\delta_{\eta_{ab}^{(\prime)}|1|} - \delta_{A_a A_b}\right) , \tag{C.2}
$$

where the first factor of the additional term ensures that nothing must changed if $P_i = P_j$. The second factor checks if the dimer $d_{ab}^{(\prime)}$ is a G-parity eigenstate and if this is true the last factor finally checks if there is a superposition in the wave function ($A_a \neq A_b$) or just a single term
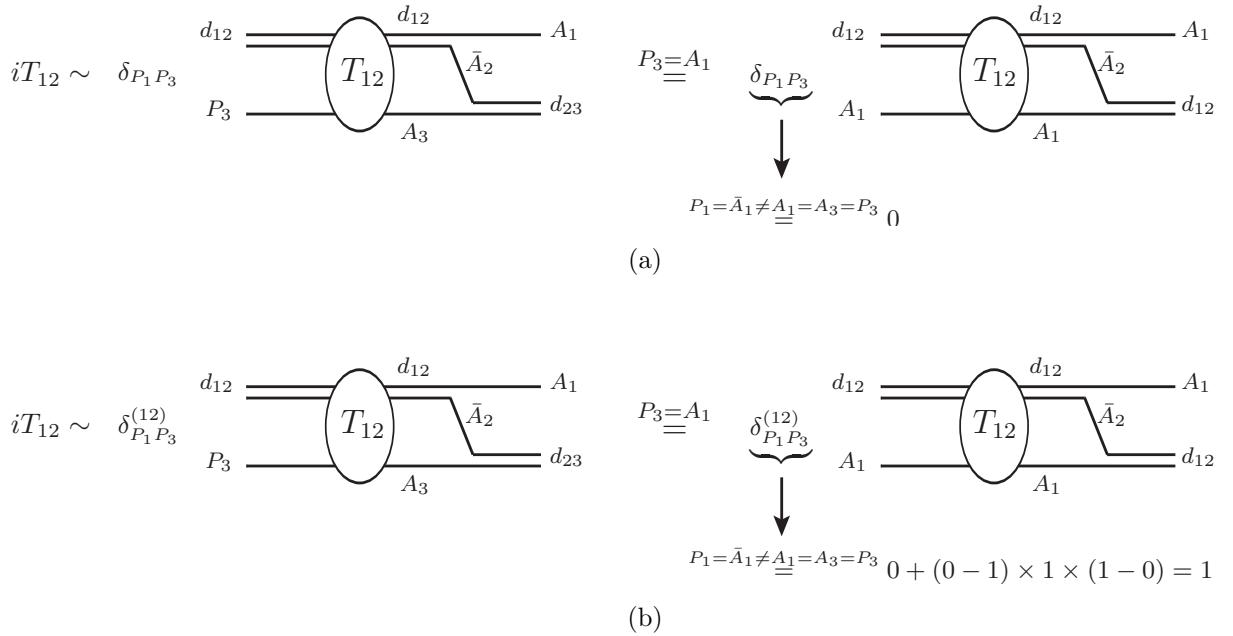
Figure C.1: *For d_{12} being a G-parity eigenstate with a superposition of states in the wave function, the shown diagram contributes to the amplitude T_{12}. However, with a "normal" Kronecker-delta, $\delta_{P_1 P_3} = 0$, it does not (a). Since this is wrong one must introduce a modified Kronecker-delta, $\delta_{P_1 P_3}^{(12)} = 1$, (cf. Eq. (C.2)) to get the right result (b).*

$(A_a = A_b)$. Note, however, that not every Kronecker-delta has to be modified.

Unfortunately, it is then not anymore ensured that the number of particles and of anti-particles is conserved separately. Considering Figure C.1(b) we see that $\delta_{P_1 P_3}^{(12)} = 1$ also for $P_3 = \bar{A}_1$ which leads to a contradiction of particle conservation. Thus, we need in advance a prefactor a_2 to restore it. In the same way we have added by hand a or b factors to some diagrams in Figure C.2.

As a second issue one observes that some diagrams become identical if two or all three particles are identical. Thus, one would count them twice (two identical particles) or even six times (three identical particles). Therefore we could define a new parameter n representing the number of identical particles,

$$n = \begin{cases} 1, & \text{if } P_1 \neq P_2 \neq P_3 \\ 2, & \text{if } P_1 = P_2 \neq P_3 \ \vee \ P_1 \neq P_2 = P_3 \ \vee \ P_1 = P_3 \neq P_2 \\ 3, & \text{if } P_1 = P_2 = P_3 \end{cases} \quad , \tag{C.3}$$

to avoid double-counting of diagrams. This straightforward procedure simply takes all diagrams into account and divides at the end by the number of identical ones. Hence, in some sense one does a lot of effort to determine diagrams which are equal to others. Therefore we introduce a different and a bit less straightforward approach to account for possibly identical diagrams. We do the following: whenever two amplitudes become identical because of identical particles

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(e.g. for $P_1 = P_2$ the amplitudes $T_{13}^{(\prime)}$ and $T_{23}^{(\prime)}$ are equal) one simply erases one of them from the equations. Thus, there is no double-counting except for the situation that in $T_{ij}^{(\prime)}$ the particles $P_i = P_j$ are identical. To correct the appearing factor of 2 we must put in by hand the additional factors of $\left(1 - \delta_{P_i P_j}/2\right)$ in front of the diagrams. The advantage of this more complicated notation is that one needs to calculate less parameters and thus has more clear equations.

Figure C.2: *All diagrams contributing to a general d_{12}–P_3 scattering amplitude. The extra factors of $(1-\delta_{P_iP_j}/2)$ correspond to the special counting scheme for diagrams becoming equal under some circumstances. The modified Kronecker-deltas $\delta_{P_iP_j}^{(ab^{(\prime)})}$ are defined in Eq. (C.2).*

$$+ \left(1 - \frac{\delta_{P_1 P_2}}{2}\right)\Bigg[$$

Row 1:
$$\begin{pmatrix}\delta^{(12)}_{P_1 P_3}\\ \delta_{P_1 P_2}\\ 1\\ 0\\ 0\\ 0\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to A_1,\,A_2,\,A_3;\; d_{12},\,d_{23})$$
$$+\begin{pmatrix}\delta^{(12)}_{P_1 P_3}\\ 1\\ \delta_{P_1 P_2}\\ 0\\ 0\\ 0\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to A_2,\,A_1,\,A_3;\; d_{12},\,d_{13})$$
$$+\begin{pmatrix}\delta^{(12)}_{P_1 P_3}\\ \delta_{P_1 P_2}\\ 1\\ 0\\ 0\\ 0\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to \bar{A}_1,\,A_2,\,A_3;\; d_{12},\,d_{23})$$
$$+\begin{pmatrix}\delta^{(12)}_{P_2 P_3}a_3\\ a_2\\ \delta_{P_1 P_2}a_2\\ 0\\ 0\\ 0\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to A_2,\,\bar{A}_1,\,A_3;\; d_{12},\,d_{13})$$

Row 2:
$$+\begin{pmatrix}\delta^{(12)}_{P_1 P_3}a_3\\ \delta_{P_1 P_2}a_1\\ a_1\\ 0\\ 0\\ 0\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to A_1,\,\bar{A}_2,\,A_3;\; d_{12},\,d_{23})$$
$$+\begin{pmatrix}\delta^{(12)}_{P_2 P_3}\\ 1\\ \delta_{P_1 P_2}\\ 0\\ 0\\ 0\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to \bar{A}_2,\,A_1,\,A_3;\; d_{12},\,d_{13})$$
$$+\begin{pmatrix}\delta^{(12)}_{P_1 P_3}\\ \delta_{P_1 P_2}\\ 1\\ 0\\ 0\\ 0\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to \bar{A}_1,\,\bar{A}_2,\,A_3;\; d_{12},\,d_{23})$$
$$+\begin{pmatrix}\delta^{(12)}_{P_2 P_3}\\ 1\\ \delta_{P_1 P_2}\\ 0\\ 0\\ 0\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to \bar{A}_2,\,\bar{A}_1,\,A_3;\; d_{12},\,d_{13})$$

Row 3:
$$+\begin{pmatrix}\delta^{(12)}_{P_1 P_3}\\ \delta_{P_1 P_2}\\ 1\\ 0\\ 0\\ 0\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to A_1,\,A_2,\,\bar{A}_3;\; d_{12},\,d_{23})$$
$$+\begin{pmatrix}\delta^{(12)}_{P_2 P_3}\\ 1\\ \delta_{P_1 P_2}\\ 0\\ 0\\ 0\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to A_2,\,A_1,\,\bar{A}_3;\; d_{12},\,d_{13})$$
$$+\begin{pmatrix}\delta^{(12)}_{P_1 P_3}b_3\\ \delta_{P_1 P_2}b_1\\ b_1\\ 0\\ 0\\ 0\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to \bar{A}_1,\,A_2,\,\bar{A}_3;\; d_{12},\,d_{23})$$
$$+\begin{pmatrix}\delta^{(12)}_{P_2 P_3}\\ 1\\ \delta_{P_1 P_2}\\ 0\\ 0\\ 0\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to A_2,\,\bar{A}_1,\,\bar{A}_3;\; d_{12},\,d_{13})$$

Row 4:
$$+\begin{pmatrix}\delta^{(12)}_{P_1 P_3}\\ \delta_{P_1 P_2}\\ 1\\ 0\\ 0\\ 0\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to A_1,\,\bar{A}_2,\,\bar{A}_3;\; d_{12},\,d_{23})$$
$$+\begin{pmatrix}\delta^{(12)}_{P_2 P_3}b_3\\ b_2\\ \delta_{P_1 P_2}b_2\\ 0\\ 0\\ 0\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to \bar{A}_2,\,A_1,\,\bar{A}_3;\; d_{12},\,d_{13})$$
$$+\begin{pmatrix}\delta^{(12)}_{P_1 P_3}\\ \delta_{P_1 P_2}\\ 1\\ 0\\ 0\\ 0\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to \bar{A}_1,\,\bar{A}_2,\,\bar{A}_3;\; d_{12},\,d_{23})$$
$$+\begin{pmatrix}\delta^{(12)}_{P_2 P_3}\\ 1\\ \delta_{P_1 P_2}\\ 0\\ 0\\ 0\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to \bar{A}_2,\,\bar{A}_1,\,\bar{A}_3;\; d_{12},\,d_{13})$$

Row 5:
$$+\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12)}_{P_1 P_3}\\ \delta_{P_1 P_2}\\ 1\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to A_1,\,A_2,\,A_3;\; d_{12},\,d'_{23})$$
$$+\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12)}_{P_2 P_3}\\ 1\\ \delta_{P_1 P_2}\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to A_2,\,A_1,\,A_3;\; d_{12},\,d'_{13})$$
$$+\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12)}_{P_1 P_3}\\ \delta_{P_1 P_2}\\ 1\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to \bar{A}_1,\,A_2,\,A_3;\; d_{12},\,d'_{23})$$
$$+\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12)}_{P_2 P_3}a_3\\ a_2\\ \delta_{P_1 P_2}a_2\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to A_2,\,\bar{A}_1,\,A_3;\; d_{12},\,d'_{13})$$

Row 6:
$$+\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12)}_{P_1 P_3}a_3\\ \delta_{P_1 P_2}a_1\\ a_1\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to A_1,\,\bar{A}_2,\,A_3;\; d_{12},\,d'_{23})$$
$$+\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12)}_{P_2 P_3}\\ 1\\ \delta_{P_1 P_2}\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to \bar{A}_2,\,A_1,\,A_3;\; d_{12},\,d'_{13})$$
$$+\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12)}_{P_1 P_3}\\ \delta_{P_1 P_2}\\ 1\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to \bar{A}_1,\,\bar{A}_2,\,A_3;\; d_{12},\,d'_{23})$$
$$+\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12)}_{P_2 P_3}\\ 1\\ \delta_{P_1 P_2}\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to \bar{A}_2,\,\bar{A}_1,\,A_3;\; d_{12},\,d'_{13})$$

Row 7:
$$+\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12)}_{P_1 P_3}\\ \delta_{P_1 P_2}\\ 1\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to A_1,\,A_2,\,\bar{A}_3;\; d_{12},\,d'_{23})$$
$$+\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12)}_{P_2 P_3}\\ 1\\ \delta_{P_1 P_2}\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to A_2,\,A_1,\,\bar{A}_3;\; d_{12},\,d'_{13})$$
$$+\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12)}_{P_1 P_3}b_3\\ \delta_{P_1 P_2}b_1\\ b_1\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to \bar{A}_1,\,A_2,\,\bar{A}_3;\; d_{12},\,d'_{23})$$
$$+\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12)}_{P_2 P_3}\\ 1\\ \delta_{P_1 P_2}\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to A_2,\,\bar{A}_1,\,\bar{A}_3;\; d_{12},\,d'_{13})$$

Row 8:
$$+\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12)}_{P_1 P_3}\\ \delta_{P_1 P_2}\\ 1\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to A_1,\,\bar{A}_2,\,\bar{A}_3;\; d_{12},\,d'_{23})$$
$$+\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12)}_{P_2 P_3}b_3\\ b_2\\ \delta_{P_1 P_2}b_2\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to \bar{A}_2,\,A_1,\,\bar{A}_3;\; d_{12},\,d'_{13})$$
$$+\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12)}_{P_1 P_3}\\ \delta_{P_1 P_2}\\ 1\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to \bar{A}_1,\,\bar{A}_2,\,\bar{A}_3;\; d_{12},\,d'_{23})$$
$$+\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12)}_{P_2 P_3}\\ 1\\ \delta_{P_1 P_2}\end{pmatrix}\; [T_{12}]\; (d_{12},\,P_3 \to \bar{A}_2,\,\bar{A}_1,\,\bar{A}_3;\; d_{12},\,d'_{13})\Bigg]$$

$$+\left(1-\frac{\delta_{P_1 P_3}}{2}\right)\Bigg[$$

Row 1:
$$\begin{pmatrix}\delta_{P_1P_3}\\\delta^{(13)}_{P_1P_2}\\1\\0\\0\\0\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},A_1,A_3,d_{23};\ A_2]\;+\;\begin{pmatrix}1\\\delta^{(13)}_{P_2P_3}\\\delta_{P_1P_3}\\0\\0\\0\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},A_3,A_1,d_{12};\ A_2]$$
$$+\;\begin{pmatrix}\delta_{P_1P_3}\\\delta^{(13)}_{P_1P_2}\\1\\0\\0\\0\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},\bar A_1,A_3,d_{23};\ A_2]\;+\;\begin{pmatrix}a_3\\\delta^{(13)}_{P_2P_3}a_3\\\delta_{P_1P_3}a_3\\0\\0\\0\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},A_3,\bar A_1,d_{12};\ A_2]$$

Row 2:
$$+\;\begin{pmatrix}\delta_{P_1P_3}a_3\\\delta^{(13)}_{P_1P_2}a_1\\a_1\\0\\0\\0\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},A_1,\bar A_3,d_{23};\ A_2]\;+\;\begin{pmatrix}1\\\delta^{(13)}_{P_2P_3}\\\delta_{P_1P_3}\\0\\0\\0\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},\bar A_3,A_1,d_{12};\ A_2]$$
$$+\;\begin{pmatrix}\delta_{P_1P_3}\\\delta^{(13)}_{P_1P_2}\\1\\0\\0\\0\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},\bar A_1,\bar A_3,d_{23};\ A_2]\;+\;\begin{pmatrix}1\\\delta^{(13)}_{P_2P_3}\\\delta_{P_1P_3}\\0\\0\\0\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},\bar A_3,\bar A_1,d_{12};\ A_2]$$

Row 3:
$$+\;\begin{pmatrix}\delta_{P_1P_3}\\\delta^{(13)}_{P_1P_2}\\1\\0\\0\\0\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},A_1,A_3,d_{23};\ \bar A_2]\;+\;\begin{pmatrix}1\\\delta^{(13)}_{P_2P_3}\\\delta_{P_1P_3}\\0\\0\\0\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},A_3,A_1,d_{12};\ \bar A_2]$$
$$+\;\begin{pmatrix}\delta_{P_1P_3}b_3\\\delta^{(13)}_{P_1P_2}b_1\\b_1\\0\\0\\0\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},\bar A_1,A_3,d_{23};\ \bar A_2]\;+\;\begin{pmatrix}1\\\delta^{(13)}_{P_2P_3}\\\delta_{P_1P_3}\\0\\0\\0\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},A_3,\bar A_1,d_{12};\ \bar A_2]$$

Row 4:
$$+\;\begin{pmatrix}\delta_{P_1P_3}\\\delta^{(13)}_{P_1P_2}\\1\\0\\0\\0\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},A_1,\bar A_3,d_{23};\ \bar A_2]\;+\;\begin{pmatrix}b_3\\\delta^{(13)}_{P_2P_3}b_3\\\delta_{P_1P_3}b_3\\0\\0\\0\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},\bar A_3,A_1,d_{12};\ \bar A_2]$$
$$+\;\begin{pmatrix}\delta_{P_1P_3}\\\delta^{(13)}_{P_1P_2}\\1\\0\\0\\0\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},\bar A_1,\bar A_3,d_{23};\ \bar A_2]\;+\;\begin{pmatrix}1\\\delta^{(13)}_{P_2P_3}\\\delta_{P_1P_3}\\0\\0\\0\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},\bar A_3,\bar A_1,d_{12};\ \bar A_2]$$

Row 5:
$$+\;\begin{pmatrix}0\\0\\0\\\delta_{P_1P_3}\\\delta^{(13)}_{P_1P_2}\\1\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},A_1,A_3,d'_{23};\ A_2]\;+\;\begin{pmatrix}0\\0\\0\\1\\\delta^{(13)}_{P_2P_3}\\\delta_{P_1P_3}\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},A_3,A_1,d'_{12};\ A_2]$$
$$+\;\begin{pmatrix}0\\0\\0\\\delta_{P_1P_3}\\\delta^{(13)}_{P_1P_2}\\1\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},\bar A_1,A_3,d'_{23};\ A_2]\;+\;\begin{pmatrix}0\\0\\0\\a_3\\\delta^{(13)}_{P_2P_3}a_3\\\delta_{P_1P_3}a_3\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},A_3,\bar A_1,d'_{12};\ A_2]$$

Row 6:
$$+\;\begin{pmatrix}0\\0\\0\\\delta_{P_1P_3}a_3\\\delta^{(13)}_{P_1P_2}a_1\\a_1\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},A_1,\bar A_3,d'_{23};\ A_2]\;+\;\begin{pmatrix}0\\0\\0\\1\\\delta^{(13)}_{P_2P_3}\\\delta_{P_1P_3}\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},\bar A_3,A_1,d'_{12};\ A_2]$$
$$+\;\begin{pmatrix}0\\0\\0\\\delta_{P_1P_3}\\\delta^{(13)}_{P_1P_2}\\1\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},\bar A_1,\bar A_3,d'_{23};\ A_2]\;+\;\begin{pmatrix}0\\0\\0\\1\\\delta^{(13)}_{P_2P_3}\\\delta_{P_1P_3}\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},\bar A_3,\bar A_1,d'_{12};\ A_2]$$

Row 7:
$$+\;\begin{pmatrix}0\\0\\0\\\delta_{P_1P_3}\\\delta^{(13)}_{P_1P_2}\\1\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},A_1,A_3,d'_{23};\ \bar A_2]\;+\;\begin{pmatrix}0\\0\\0\\1\\\delta^{(13)}_{P_2P_3}\\\delta_{P_1P_3}\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},A_3,A_1,d'_{12};\ \bar A_2]$$
$$+\;\begin{pmatrix}0\\0\\0\\\delta_{P_1P_3}b_3\\\delta^{(13)}_{P_1P_2}b_1\\b_1\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},\bar A_1,A_3,d'_{23};\ \bar A_2]\;+\;\begin{pmatrix}0\\0\\0\\1\\\delta^{(13)}_{P_2P_3}\\\delta_{P_1P_3}\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},A_3,\bar A_1,d'_{12};\ \bar A_2]$$

Row 8:
$$+\;\begin{pmatrix}0\\0\\0\\\delta_{P_1P_3}\\\delta^{(13)}_{P_1P_2}\\1\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},A_1,\bar A_3,d'_{23};\ \bar A_2]\;+\;\begin{pmatrix}0\\0\\0\\b_3\\\delta^{(13)}_{P_2P_3}b_3\\\delta_{P_1P_3}b_3\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},\bar A_3,A_1,d'_{12};\ \bar A_2]$$
$$+\;\begin{pmatrix}0\\0\\0\\\delta_{P_1P_3}\\\delta^{(13)}_{P_1P_2}\\1\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},\bar A_1,\bar A_3,d'_{23};\ \bar A_2]\;+\;\begin{pmatrix}0\\0\\0\\1\\\delta^{(13)}_{P_2P_3}\\\delta_{P_1P_3}\end{pmatrix}\;[d_{12},P_3 \to T_{13} \to d_{13},\bar A_3,\bar A_1,d'_{12};\ \bar A_2]\Bigg]$$

$$+ \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) \Bigg[\begin{pmatrix} \delta_{P_2 P_3} \\ 1 \\ \delta_{P_1 P_2}^{(23)} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{array}{c} d_{12} \\ \\ P_3 \end{array} T_{23} \begin{array}{c} A_2 \\ A_3 \\ d_{13} \\ A_1 \end{array} + \begin{pmatrix} 1 \\ \delta_{P_2 P_3} \\ \delta_{P_1 P_3}^{(23)} \\ 0 \\ 0 \\ 0 \end{pmatrix} T_{23} \begin{array}{c} A_3 \\ A_2 \\ d_{12} \\ A_1 \end{array} + \begin{pmatrix} \delta_{P_2 P_3} \\ 1 \\ \delta_{P_1 P_2}^{(23)} \\ 0 \\ 0 \\ 0 \end{pmatrix} T_{23} \begin{array}{c} \bar{A}_2 \\ A_3 \\ d_{13} \\ A_1 \end{array} + \begin{pmatrix} a_3 \\ \delta_{P_2 P_3} a_3 \\ \delta_{P_1 P_3}^{(23)} a_3 \\ 0 \\ 0 \\ 0 \end{pmatrix} T_{23} \begin{array}{c} A_3 \\ \bar{A}_2 \\ d_{12} \\ A_1 \end{array}$$

(Remaining rows of the array consist of analogous terms, each a six-component column vector multiplied by a T_{23} diagram with external legs labelled by combinations of $d_{12}, d_{23}, d_{13}, d'_{13}, d'_{12}, P_3$ and $A_1, A_2, A_3, \bar{A}_1, \bar{A}_2, \bar{A}_3$.)

$$
+\left(1-\frac{\delta_{P_1P_2}}{2}\right)\Bigg[
\begin{pmatrix}\delta^{(12')}_{P_1P_3}\\ \delta_{P_1P_2}\\ 1\\ 0\\ 0\\ 0\end{pmatrix}
\;T'_{12}\;
+
\begin{pmatrix}\delta^{(12')}_{P_2P_3}\\ 1\\ \delta_{P_1P_2}\\ 0\\ 0\\ 0\end{pmatrix}
\;T'_{12}\;
+
\begin{pmatrix}\delta^{(12')}_{P_1P_3}\\ \delta_{P_1P_2}\\ 1\\ 0\\ 0\\ 0\end{pmatrix}
\;T'_{12}\;
+
\begin{pmatrix}\delta^{(12')}_{P_2P_3}a_3\\ \delta_{P_1P_2}a_2\\ 0\\ 0\\ 0\\ 0\end{pmatrix}
\;T'_{12}
$$

$$
+
\begin{pmatrix}\delta^{(12')}_{P_1P_3}a_3\\ \delta_{P_1P_2}a_1\\ a_1\\ 0\\ 0\\ 0\end{pmatrix}
\;T'_{12}\;
+
\begin{pmatrix}\delta^{(12')}_{P_2P_3}\\ 1\\ \delta_{P_1P_2}\\ 0\\ 0\\ 0\end{pmatrix}
\;T'_{12}\;
+
\begin{pmatrix}\delta^{(12')}_{P_1P_3}\\ \delta_{P_1P_2}\\ 1\\ 0\\ 0\\ 0\end{pmatrix}
\;T'_{12}\;
+
\begin{pmatrix}\delta^{(12')}_{P_2P_3}\\ 1\\ \delta_{P_1P_2}\\ 0\\ 0\\ 0\end{pmatrix}
\;T'_{12}
$$

$$
+
\begin{pmatrix}\delta^{(12')}_{P_1P_3}\\ \delta_{P_1P_2}\\ 1\\ 0\\ 0\\ 0\end{pmatrix}
\;T'_{12}\;
+
\begin{pmatrix}\delta^{(12')}_{P_2P_3}\\ 1\\ \delta_{P_1P_2}\\ 0\\ 0\\ 0\end{pmatrix}
\;T'_{12}\;
+
\begin{pmatrix}\delta^{(12')}_{P_1P_3}b_3\\ \delta_{P_1P_2}b_1\\ b_1\\ 0\\ 0\\ 0\end{pmatrix}
\;T'_{12}\;
+
\begin{pmatrix}\delta^{(12')}_{P_2P_3}\\ 1\\ \delta_{P_1P_2}\\ 0\\ 0\\ 0\end{pmatrix}
\;T'_{12}
$$

$$
+
\begin{pmatrix}\delta^{(12')}_{P_1P_3}\\ \delta_{P_1P_2}\\ 1\\ 0\\ 0\\ 0\end{pmatrix}
\;T'_{12}\;
+
\begin{pmatrix}\delta^{(12')}_{P_2P_3}b_3\\ b_2\\ \delta_{P_1P_2}b_2\\ 0\\ 0\\ 0\end{pmatrix}
\;T'_{12}\;
+
\begin{pmatrix}\delta^{(12')}_{P_1P_3}\\ \delta_{P_1P_2}\\ 1\\ 0\\ 0\\ 0\end{pmatrix}
\;T'_{12}\;
+
\begin{pmatrix}\delta^{(12')}_{P_2P_3}\\ 1\\ \delta_{P_1P_2}\\ 0\\ 0\\ 0\end{pmatrix}
\;T'_{12}
$$

$$
+
\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12')}_{P_1P_3}\\ \delta_{P_1P_2}\\ 1\end{pmatrix}
\;T'_{12}\;
+
\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12')}_{P_2P_3}\\ 1\\ \delta_{P_1P_2}\end{pmatrix}
\;T'_{12}\;
+
\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12')}_{P_1P_3}\\ \delta_{P_1P_2}\\ 1\end{pmatrix}
\;T'_{12}\;
+
\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12')}_{P_2P_3}a_3\\ a_2\\ \delta_{P_1P_2}a_2\end{pmatrix}
\;T'_{12}
$$

$$
+
\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12')}_{P_1P_3}a_3\\ \delta_{P_1P_2}a_1\\ a_1\end{pmatrix}
\;T'_{12}\;
+
\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12')}_{P_2P_3}\\ 1\\ \delta_{P_1P_2}\end{pmatrix}
\;T'_{12}\;
+
\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12')}_{P_1P_3}\\ \delta_{P_1P_2}\\ 1\end{pmatrix}
\;T'_{12}\;
+
\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12')}_{P_2P_3}\\ 1\\ \delta_{P_1P_2}\end{pmatrix}
\;T'_{12}
$$

$$
+
\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12')}_{P_1P_3}\\ \delta_{P_1P_2}\\ 1\end{pmatrix}
\;T'_{12}\;
+
\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12')}_{P_2P_3}\\ 1\\ \delta_{P_1P_2}\end{pmatrix}
\;T'_{12}\;
+
\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12')}_{P_1P_3}b_3\\ \delta_{P_1P_2}b_1\\ b_1\end{pmatrix}
\;T'_{12}\;
+
\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12')}_{P_2P_3}\\ 1\\ \delta_{P_1P_2}\end{pmatrix}
\;T'_{12}
$$

$$
+
\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12')}_{P_1P_3}\\ \delta_{P_1P_2}\\ 1\end{pmatrix}
\;T'_{12}\;
+
\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12')}_{P_2P_3}b_3\\ b_2\\ \delta_{P_1P_2}b_2\end{pmatrix}
\;T'_{12}\;
+
\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12')}_{P_1P_3}\\ \delta_{P_1P_2}\\ 1\end{pmatrix}
\;T'_{12}\;
+
\begin{pmatrix}0\\ 0\\ 0\\ \delta^{(12')}_{P_2P_3}\\ 1\\ \delta_{P_1P_2}\end{pmatrix}
\;T'_{12}
\Bigg]
$$

$$+ \left(1 - \frac{\delta_{P_1 P_3}}{2}\right)\Bigg[$$

$$
+ \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) \Bigg[
\begin{pmatrix} \delta_{P_2 P_3} \\ 1 \\ \delta_{P_1 P_2}^{(23')} \\ 0 \\ 0 \\ 0 \end{pmatrix} \; [T'_{23}] \;
+ \begin{pmatrix} 1 \\ \delta_{P_2 P_3} \\ \delta_{P_1 P_3}^{(23')} \\ 0 \\ 0 \\ 0 \end{pmatrix} \; [T'_{23}] \;
+ \begin{pmatrix} \delta_{P_2 P_3} \\ 1 \\ \delta_{P_1 P_2}^{(23')} \\ 0 \\ 0 \\ 0 \end{pmatrix} \; [T'_{23}] \;
+ \begin{pmatrix} a_3 \\ \delta_{P_2 P_3} a_3 \\ \delta_{P_1 P_3}^{(23')} a_3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \; [T'_{23}]
$$

$$
+ \begin{pmatrix} \delta_{P_2 P_3} a_3 \\ a_2 \\ \delta_{P_1 P_2}^{(23')} a_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \; [T'_{23}] \;
+ \begin{pmatrix} 1 \\ \delta_{P_2 P_3} \\ \delta_{P_1 P_3}^{(23')} \\ 0 \\ 0 \\ 0 \end{pmatrix} \; [T'_{23}] \;
+ \begin{pmatrix} \delta_{P_2 P_3} \\ 1 \\ \delta_{P_1 P_2}^{(23')} \\ 0 \\ 0 \\ 0 \end{pmatrix} \; [T'_{23}] \;
+ \begin{pmatrix} 1 \\ \delta_{P_2 P_3} \\ \delta_{P_1 P_3}^{(23')} \\ 0 \\ 0 \\ 0 \end{pmatrix} \; [T'_{23}]
$$

$$
+ \begin{pmatrix} \delta_{P_2 P_3} \\ 1 \\ \delta_{P_1 P_2}^{(23')} \\ 0 \\ 0 \\ 0 \end{pmatrix} \; [T'_{23}] \;
+ \begin{pmatrix} 1 \\ \delta_{P_2 P_3} \\ \delta_{P_1 P_3}^{(23')} \\ 0 \\ 0 \\ 0 \end{pmatrix} \; [T'_{23}] \;
+ \begin{pmatrix} \delta_{P_2 P_3} b_3 \\ b_1 \\ \delta_{P_1 P_2}^{(23')} b_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \; [T'_{23}] \;
+ \begin{pmatrix} 1 \\ \delta_{P_2 P_3} \\ \delta_{P_1 P_3}^{(23')} \\ 0 \\ 0 \\ 0 \end{pmatrix} \; [T'_{23}]
$$

$$
+ \begin{pmatrix} \delta_{P_2 P_3} \\ 1 \\ \delta_{P_1 P_2}^{(23')} \\ 0 \\ 0 \\ 0 \end{pmatrix} \; [T'_{23}] \;
+ \begin{pmatrix} b_3 \\ \delta_{P_2 P_3} b_3 \\ \delta_{P_1 P_3}^{(23')} b_3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \; [T'_{23}] \;
+ \begin{pmatrix} \delta_{P_2 P_3} \\ 1 \\ \delta_{P_1 P_2}^{(23')} \\ 0 \\ 0 \\ 0 \end{pmatrix} \; [T'_{23}] \;
+ \begin{pmatrix} 1 \\ \delta_{P_2 P_3} \\ \delta_{P_1 P_3}^{(23')} \\ 0 \\ 0 \\ 0 \end{pmatrix} \; [T'_{23}]
$$

$$
+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_{P_2 P_3} \\ 1 \\ \delta_{P_1 P_2}^{(23')} \end{pmatrix} \; [T'_{23}] \;
+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \delta_{P_2 P_3} \\ \delta_{P_1 P_3}^{(23')} \end{pmatrix} \; [T'_{23}] \;
+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_{P_2 P_3} \\ 1 \\ \delta_{P_1 P_2}^{(23')} \end{pmatrix} \; [T'_{23}] \;
+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ a_3 \\ \delta_{P_2 P_3} a_3 \\ \delta_{P_1 P_3}^{(23')} a_3 \end{pmatrix} \; [T'_{23}]
$$

$$
+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_{P_2 P_3} a_3 \\ a_2 \\ \delta_{P_1 P_2}^{(23')} a_2 \end{pmatrix} \; [T'_{23}] \;
+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \delta_{P_2 P_3} \\ \delta_{P_1 P_3}^{(23')} \end{pmatrix} \; [T'_{23}] \;
+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_{P_2 P_3} \\ 1 \\ \delta_{P_1 P_2}^{(23')} \end{pmatrix} \; [T'_{23}] \;
+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \delta_{P_2 P_3} \\ \delta_{P_1 P_3}^{(23')} \end{pmatrix} \; [T'_{23}]
$$

$$
+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_{P_2 P_3} \\ 1 \\ \delta_{P_1 P_2}^{(23')} \end{pmatrix} \; [T'_{23}] \;
+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \delta_{P_2 P_3} \\ \delta_{P_1 P_3}^{(23')} \end{pmatrix} \; [T'_{23}] \;
+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_{P_2 P_3} b_3 \\ b_2 \\ \delta_{P_1 P_2}^{(23')} b_2 \end{pmatrix} \; [T'_{23}] \;
+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \delta_{P_2 P_3} \\ \delta_{P_1 P_3}^{(23')} \end{pmatrix} \; [T'_{23}]
$$

$$
+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_{P_2 P_3} \\ 1 \\ \delta_{P_1 P_2}^{(23')} \end{pmatrix} \; [T'_{23}] \;
+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ b_3 \\ \delta_{P_2 P_3} b_3 \\ \delta_{P_1 P_3}^{(23')} b_3 \end{pmatrix} \; [T'_{23}] \;
+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_{P_2 P_3} \\ 1 \\ \delta_{P_1 P_2}^{(23')} \end{pmatrix} \; [T'_{23}] \;
+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \delta_{P_2 P_3} \\ \delta_{P_1 P_3}^{(23')} \end{pmatrix} \; [T'_{23}] \Bigg]
$$

Appendix D

Projection on L-th partial wave

A scattering amplitude $T(E, \mathbf{k}, \mathbf{p})$ can be expanded in partial waves as

$$T(E, \mathbf{k}, \mathbf{p}) = \sum_{L=0}^{\infty} (2L+1) \, T^{(L)}(E, k, p) \mathcal{P}_L(\cos \theta) \,, \tag{D.1}$$

with the Legendre polynomial (i.e. Legendre function of the first kind) $\mathcal{P}_L$ and $\theta = \sphericalangle(\mathbf{k}, \mathbf{p})$ being the angle between incoming and outgoing momenta whose moduli we write as k and p. Since the Legendre polynomials are orthogonal,

$$\int_{-1}^{1} d\cos\theta \, \mathcal{P}_L(\cos\theta) \mathcal{P}_\ell(\cos\theta) = \frac{2}{2L+1} \delta_{L\ell} \,, \tag{D.2}$$

one can project out the L-th partial wave using an operator defined via

$$T^{(L)}(E, k, p) = \frac{1}{2} \int_{-1}^{1} d\cos(\theta) \, \mathcal{P}_L(\cos\theta) T(E, \mathbf{k}, \mathbf{p}) \,. \tag{D.3}$$

In Fig. D.1 we have defined the angles between the momenta of a scattering amplitude $T(E, \mathbf{k}, \mathbf{p})$. Following Ref. [163] in this system it holds the useful relation

$$\mathcal{P}_L(\cos\psi) = \frac{4\pi}{2L+1} \sum_{m=-L}^{L} Y_{Lm}^*(\theta', \varphi') \, Y_{Lm}(\theta, \varphi)$$

$$= \mathcal{P}_L(\cos\theta) \mathcal{P}_L(\cos\theta') + 2 \sum_{m=1}^{L} \frac{(L-m)!}{(L+m)!} \mathcal{P}_L^m(\cos\theta) \mathcal{P}_L^m(\cos\theta') \cos\left[m\left(\varphi - \varphi'\right)\right] \,, \tag{D.4}$$

where Y_{Lm} are the spherical harmonics and $\mathcal{P}_L^m$ are the associated Legendre polynomials.
In the following we will project out the L-th partial wave of the $d_{ij}^{(\prime)}-P_k$ scattering amplitudes contributing to Eq. (F.2). For the moment we will ignore all terms which do not depend on the

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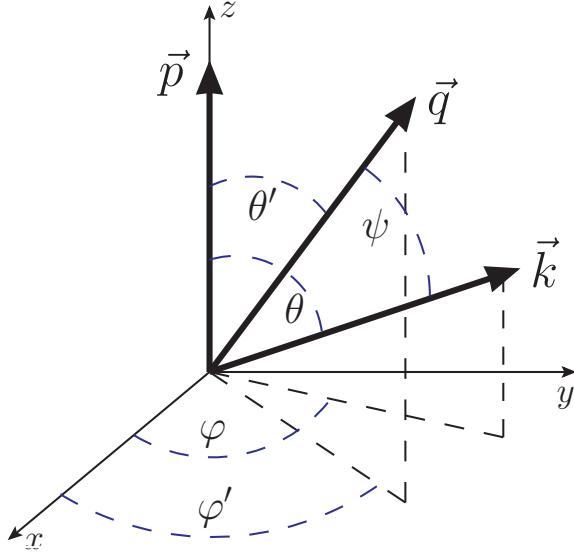
Figure D.1: *Definition of angles between the different momenta of a scattering amplitude $T(E, \mathbf{k}, \mathbf{p})$ where we have chosen $\mathbf{p}$ so that it points in z direction.*

momenta $\mathbf{k}$, $\mathbf{p}$, $\mathbf{q}$ or their moduli k, p, q and write

$$
\begin{aligned}
T_{ij}^{(\prime)}(E, \mathbf{k}, \mathbf{p}) \sim{}& \frac{1}{E - \frac{k^2}{2m_3} - \frac{p^2}{2m_1} - \frac{(\mathbf{k}+\mathbf{p})^2}{2m_2} + i\varepsilon} + \frac{1}{E - \frac{k^2}{2m_3} - \frac{p^2}{2m_2} - \frac{(\mathbf{k}+\mathbf{p})^2}{2m_1} + i\varepsilon} \\
&+ \sum_{\substack{i,j,k=1 \\ i<j \\ k \neq i,j}}^{3} \int \frac{d^3q}{(2\pi)^3} \frac{T_{ij}(E, \mathbf{k}, \mathbf{q})}{-\gamma_{ij} + \sqrt{-2\mu_{ij}\left(E - \frac{q^2}{2m_k} - \frac{q^2}{2(m_i+m_j)}\right)} - i\varepsilon} \\
&\quad \times \left(\frac{1}{E - \frac{q^2}{2m_k} - \frac{p^2}{2m_i} - \frac{(\mathbf{q}+\mathbf{p})^2}{2m_j} + i\varepsilon} + \frac{1}{E - \frac{q^2}{2m_k} - \frac{p^2}{2m_j} - \frac{(\mathbf{q}+\mathbf{p})^2}{2m_i} + i\varepsilon} \right) \\
&+ \sum_{\substack{i,j,k=1 \\ i<j \\ k \neq i,j}}^{3} \int \frac{d^3q}{(2\pi)^3} \frac{T_{ij}'(E, \mathbf{k}, \mathbf{q})}{-\gamma_{ij}' + \sqrt{-2\mu_{ij}\left(E - \frac{q^2}{2m_k} - \frac{q^2}{2(m_i+m_j)}\right)} - i\varepsilon} \\
&\quad \times \left(\frac{1}{E - \frac{q^2}{2m_k} - \frac{p^2}{2m_i} - \frac{(\mathbf{q}+\mathbf{p})^2}{2m_j} + i\varepsilon} + \frac{1}{E - \frac{q^2}{2m_k} - \frac{p^2}{2m_j} - \frac{(\mathbf{q}+\mathbf{p})^2}{2m_i} + i\varepsilon} \right).
\end{aligned}
\tag{D.5}
$$

If one applies the projection operator Eq. (D.3) and define according to Fig. D.1 the angles $\theta = \sphericalangle(\mathbf{k}, \mathbf{p})$, $\theta' = \sphericalangle(\mathbf{q}, \mathbf{p})$ and $\psi = \sphericalangle(\mathbf{k}, \mathbf{q})$ one can rewrite the d^3q integral in spherical

coordinates q, θ', φ' so that Eq. (D.5) reads

$$
T_{ij}^{(\prime)(L)}(E, k, p) = \frac{1}{2} \int_{-1}^{1} d\cos\theta \, \mathcal{P}_L(\cos\theta) T_{ij}^{(\prime)}(E, \mathbf{k}, \mathbf{p}) \sim
$$

$$
\frac{1}{2} \int_{-1}^{1} d\cos\theta \frac{\mathcal{P}_L(\cos\theta)}{E - \frac{k^2}{2\mu_{23}} - \frac{p^2}{2\mu_{12}} - \frac{kp}{m_2}\cos\theta + i\varepsilon} + \frac{1}{2} \int_{-1}^{1} d\cos\theta \frac{\mathcal{P}_L(\cos\theta)}{E - \frac{k^2}{2\mu_{13}} - \frac{p^2}{2\mu_{12}} - \frac{kp}{m_1}\cos\theta + i\varepsilon}
$$

$$
+ \sum_{\substack{i, j, k = 1 \\ i < j \\ k \neq i, j}}^{3} \frac{1}{(2\pi)^3} \frac{1}{2} \int_{-1}^{1} d\cos\theta \mathcal{P}_L(\cos\theta) \int_{0}^{\infty} dq \int_{-1}^{1} d\cos\theta' \int_{0}^{2\pi} d\varphi'
$$

$$
\times \left[\frac{q^2 \sum_{\ell=0}^{\infty}(2\ell+1)T_{ij}^{(\ell)}(E, k, q)\mathcal{P}_\ell(\cos\psi)}{-\gamma_{ij} + \sqrt{-2\mu_{ij}\left(E - \frac{q^2}{2m_k} - \frac{q^2}{2(m_i+m_j)}\right)} - i\varepsilon} \right.
$$

$$
\left. + \frac{q^2 \sum_{\ell=0}^{\infty}(2\ell+1)T_{ij}^{\prime(\ell)}(E, k, q)\mathcal{P}_\ell(\cos\psi)}{-\gamma_{ij}' + \sqrt{-2\mu_{ij}\left(E - \frac{q^2}{2m_k} - \frac{q^2}{2(m_i+m_j)}\right)} - i\varepsilon} \right]
$$

$$
\times \left(\frac{1}{E - \frac{q^2}{2\mu_{kj}} - \frac{p^2}{2\mu_{ij}} - \frac{qp}{m_j}\cos\theta' + i\varepsilon} + \frac{1}{E - \frac{q^2}{2\mu_{ki}} - \frac{p^2}{2\mu_{ij}} - \frac{qp}{m_i}\cos\theta' + i\varepsilon} \right), \quad \text{(D.6)}
$$

where Eq. (D.1) was used to replace the amplitudes on the right-hand-side by their partial wave expansions and μ_{ij} is the reduced mass defined in Eq. (3.34).

Consider the $d\varphi'$ integral. From Eq. (D.4) we know that $\mathcal{P}_\ell(\cos\psi)$ depends on φ'. In fact, it is the only φ' dependent term in the integral and hence we find

$$
\int_{0}^{2\pi} d\varphi' \mathcal{P}_\ell(\cos\psi) = \mathcal{P}_\ell(\cos\theta)\mathcal{P}_\ell(\cos\theta') \int_{0}^{2\pi} d\varphi'
$$

$$
+ 2\sum_{m=1}^{\ell} \frac{(\ell-m)!}{(\ell+m)!} \mathcal{P}_\ell^m(\cos\theta)\mathcal{P}_\ell^m(\cos\theta') \underbrace{\int_{0}^{2\pi} d\varphi' \cos\left[m\left(\varphi - \varphi'\right)\right]}_{= 0 \, \forall \, m \in \mathbb{N}}
$$

$$
= 2\pi \mathcal{P}_\ell(\cos\theta)\mathcal{P}_\ell(\cos\theta') . \quad \text{(D.7)}
$$

Thus, the second term of Eq. (D.6) is now proportional to the product $\mathcal{P}_L(\cos\theta)\mathcal{P}_\ell(\cos\theta)$. Together with the $d\cos\theta$ integration the mentioned product yields due to the orthogonality relation

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Eq. (D.2) just a factor of $2\delta_{L\ell}/(2L+1)$ and one ends up with

$$
\begin{aligned}
T_{ij}^{(\prime)(L)}(E,k,p) \sim \ & -\frac{m_2}{kp}\frac{1}{2}\int_{-1}^{1} d\cos\theta \frac{\mathcal{P}_L(\cos\theta)}{\frac{m_2}{kp}\left(\frac{k^2}{2\mu_{23}}+\frac{p^2}{2\mu_{12}}-E\right)+\cos\theta-i\varepsilon} \\
& -\frac{m_1}{kp}\frac{1}{2}\int_{-1}^{1} d\cos\theta \frac{\mathcal{P}_L(\cos\theta)}{\frac{m_1}{kp}\left(\frac{k^2}{2\mu_{13}}+\frac{p^2}{2\mu_{12}}-E\right)+\cos\theta-i\varepsilon} \\
& -\frac{1}{2\pi^2}\sum_{\substack{i,j,k=1\\ i<j\\ k\neq i,j}}^{3}\int_0^\infty dq \left[\frac{q^2 T_{ij}^{(L)}(E,k,q)}{-\gamma_{ij}+\sqrt{-2\mu_{ij}\left(E-\frac{q^2}{2m_k}-\frac{q^2}{2(m_i+m_j)}\right)}-i\varepsilon}\right. \\
& \left. \qquad\qquad + \frac{q^2 T_{ij}^{\prime(L)}(E,k,q)}{-\gamma_{ij}'+\sqrt{-2\mu_{ij}\left(E-\frac{q^2}{2m_k}-\frac{q^2}{2(m_i+m_j)}\right)}-i\varepsilon}\right] \\
& \times \left(\frac{m_j}{qp}\frac{1}{2}\int_{-1}^{1} d\cos\theta' \frac{\mathcal{P}_L(\cos\theta')}{\frac{m_j}{qp}\left(\frac{q^2}{2\mu_{kj}}+\frac{p^2}{2\mu_{ij}}-E\right)+\cos\theta'-i\varepsilon}\right. \\
& \left. \qquad + \frac{m_i}{qp}\frac{1}{2}\int_{-1}^{1} d\cos\theta' \frac{\mathcal{P}_L(\cos\theta')}{\frac{m_i}{qp}\left(\frac{q^2}{2\mu_{ki}}+\frac{p^2}{2\mu_{ij}}-E\right)+\cos\theta'-i\varepsilon}\right) . \quad \text{(D.8)}
\end{aligned}
$$

Now the question arises how one deals with the remaining $d\cos\theta^{(\prime)}$ integrals proportional to the Legendre polynomials? For this purpose we firstly define a short-hand-notation for this kind of integral:

$$
Q_L(\beta - i\varepsilon) := (-1)^L \frac{1}{2}\int_{-1}^{1} dx \, \frac{\mathcal{P}_L(x)}{(\beta - i\varepsilon)+x}, \quad \text{with } \beta \in \mathbb{R} \setminus \{-1,+1\}. \quad \text{(D.9)}
$$

The restriction that $|\beta| \neq 1$ is necessary since at these two points $Q_L(\beta - i\varepsilon)$ becomes singular. Note, that Eq. (D.9) coincides with the definition of the Legendre function of the second kind with complex argument introduced in Ref. [164]; we will point out this connection below in more detail.

Motivated by the fact that we assume low energy scattering of P_3 off the dimer d_{12}, one can argue – similarly to section 3.1.3 where the S-wave dimer approach was justified – that also d_{12}–P_3 scattering is dominated by S-wave interactions. Therefore it is a convenient method [80] to set $L = 0$ which simplifies Eq. (D.9) according to $\mathcal{P}_0(x) = 1 \, \forall \, x \in \mathbb{R}$ to

$$
Q_0(\beta - i\varepsilon) = \frac{1}{2}\int_{-1}^{1} dx \, \frac{1}{(\beta - i\varepsilon)+x} = \frac{1}{2}\left[\ln(\beta+1-i\varepsilon)-\ln(\beta-1-i\varepsilon)\right]. \quad \text{(D.10)}
$$

In the limit $\varepsilon \to 0$ we have to distinguish three cases for $\beta \neq \pm 1$:

- $\underline{-1 < \beta < 1}$: $\Rightarrow$ $\beta + 1 = |\beta + 1| \wedge \beta - 1 = -|\beta - 1|$,   with $|\beta \pm 1| \in \mathbb{R}^+$

$$\Rightarrow \quad \lim_{\varepsilon \to 0} Q_0(\beta - i\varepsilon) = \frac{1}{2}\left[\ln(|\beta + 1|) - \ln(-|\beta - 1|)\right]$$
$$= \frac{1}{2}\left[\ln(|\beta + 1|) - \ln(|\beta - 1|) - i\arg(-|\beta - 1|)\right]$$
$$= \frac{1}{2}\left[\ln(|\beta + 1|) - \ln(|\beta - 1|) - i\pi\right]$$
$$= \frac{1}{2}\left[\ln\left(\frac{|\beta + 1|}{|\beta - 1|}\right) - i\pi\right] = \frac{1}{2}\left[\ln\left(\left|\frac{\beta + 1}{\beta - 1}\right|\right) - i\pi\right], \tag{D.11}$$

- $\underline{\beta < 1}$: $\Rightarrow$ $\beta \pm 1 = -|\beta \pm 1|$,   with $|\beta \pm 1| \in \mathbb{R}^+$

$$\Rightarrow \quad \lim_{\varepsilon \to 0} Q_0(\beta - i\varepsilon) = \frac{1}{2}\left[\ln(-|\beta + 1|) - \ln(-|\beta - 1|)\right]$$
$$= \frac{1}{2}\left[\ln(|\beta + 1|) + i\arg(-|\beta + 1|) - \ln(|\beta - 1|) - i\arg(-|\beta - 1|)\right]$$
$$= \frac{1}{2}\left[\ln(|\beta + 1|) + i\pi - \ln(|\beta - 1|) - i\pi\right]$$
$$= \frac{1}{2}\left[\ln\left(\frac{|\beta + 1|}{|\beta - 1|}\right)\right] = \frac{1}{2}\left[\ln\left(\left|\frac{\beta + 1}{\beta - 1}\right|\right)\right], \tag{D.12}$$

- $\underline{\beta > 1}$: $\Rightarrow$ $\beta \pm 1 = |\beta \pm 1|$,   with $|\beta \pm 1| \in \mathbb{R}^+$

$$\Rightarrow \quad \lim_{\varepsilon \to 0} Q_0(\beta - i\varepsilon) = \frac{1}{2}\left[\ln(-|\beta + 1|) - \ln(-|\beta - 1|)\right]$$
$$= \frac{1}{2}\left[\ln(|\beta + 1|) - \ln(|\beta - 1|)\right]$$
$$= \frac{1}{2}\left[\ln\left(\frac{|\beta + 1|}{|\beta - 1|}\right)\right] = \frac{1}{2}\left[\ln\left(\left|\frac{\beta + 1}{\beta - 1}\right|\right)\right]. \tag{D.13}$$

Here, we have used that the logarithm of a complex number can be written as

$$\ln(-x) = \ln(|-x|) + i\arg(-x) = \ln(x) + i\pi, \quad x \in \mathbb{R}^+. \tag{D.14}$$

In summary we thus find

$$\lim_{\varepsilon \to 0} Q_0(\beta - i\varepsilon) = \begin{cases} \frac{1}{2}\left[\ln\left(\left|\frac{\beta+1}{\beta-1}\right|\right) - i\pi\right], & \text{if } |\beta| < 1 \\ \frac{1}{2}\ln\left(\left|\frac{\beta+1}{\beta-1}\right|\right), & \text{else} \end{cases}. \tag{D.15}$$

This result could then be plugged into Eq. (D.8) so that one could determine the d_{12}–P_3 S-wave scattering amplitude which corresponds to an approximation sufficient for most three particle

systems. However, ignoring this fact for the moment, one can in the same way determine the limit of $Q_1(\beta - i\varepsilon)$ for ε going to zero:

$$
\begin{aligned}
\lim_{\varepsilon \to 0} Q_1(\beta - i\varepsilon) &= \lim_{\varepsilon \to 0} \frac{1}{2} \int_{-1}^{1} dx \, \frac{x}{(\beta - i\varepsilon) + x} \\
&= \lim_{\varepsilon \to 0} \frac{1}{2}(\beta - i\varepsilon)\left[\ln\left(\beta + 1 - i\varepsilon\right) - \ln\left(\beta - 1 - i\varepsilon\right) \right] - 1 \\
&= \begin{cases} \frac{\beta}{2}\left[\ln\left(\left|\frac{\beta+1}{\beta-1}\right|\right) - i\pi\right] - 1 \,, & \text{if } |\beta| < 1 \\ \frac{\beta}{2}\ln\left(\left|\frac{\beta+1}{\beta-1}\right|\right) - 1 \,, & \text{else} \end{cases}
\end{aligned}
\tag{D.16}
$$

Already at this point we notice that – especially due to the $(-1)^L$ factor in Eq. (D.9) – both results for $L = 0$ and for $L = 1$ are quite similar to the first two Legendre functions of the second kind, which we call $\widetilde{Q}_L$. In fact, the only difference between Q_L and $\widetilde{Q}_L$ is that the denominator in the logarithm is not $|\beta - 1|$, but $|1 - \beta|$. From this we conclude that the standard Legendre functions of the second kind are real valued for $|\beta| < 1$, but complex outside the interval $[-1, 1]$; in contrast to Q_L where the situation is interchanged (note, that the singularities at ± 1 are not affected). However, this difference is not surprising since the Legendre differential equation,

$$
\frac{d}{dx}\left[\left(1 - x^2\right)\frac{dy(x)}{dx}\right] + L(L+1)y(x) = 0 \,, \quad \text{with } L \in \mathbb{N}^+ \text{ and } |x| < 1 \,,
\tag{D.17}
$$

with two linearly independent solutions $y(x) = A\mathcal{P}_L(x) + B\widetilde{Q}_L(x)$ is only defined on the interval $[-1, 1]$. Therefore one chooses the brunch cut of the complex logarithm in $\widetilde{Q}_L$ so that it is real inside this interval. Doing it the other way around (complex inside the interval $[-1, 1]$) one ends up with Q_L. Consequently, we conclude that Q_L are the Legendre functions of the second kind with a non-standard convention regarding the brunch cut of the complex logarithm [164]. We will thus also refer to Q_L as Legendre function of the second kind. If necessary we would add the term "standard" ("non-standard") Legendre function of the second kind if $\widetilde{Q}_L$ (Q_L) is meant (as it was done above).

Although we have already found an explicit expression for Q_0 and Q_1 and know from the low-energy scattering approximation that higher partial waves need not to be taken into account, we will keep the general Q_L in our scattering amplitude. This is done on the one hand because we will later on derive results where the explicit form of Q_L leads to integrals which cannot be straightforwardly calculated anyway. On the other hand because we simply want to claim a high level of generality in this work. Therefore we end up with a partial wave projected $d_{ij}^{(\prime)}-P_k$

scattering amplitude given by

$$
\begin{aligned}
T_{ij}^{(\prime)(L)}(E,k,p) \sim\ & \\
& -\frac{m_2}{kp}Q_L\left(\frac{m_2}{kp}\left(\frac{k^2}{2\mu_{23}}+\frac{p^2}{2\mu_{12}}-E\right)-i\varepsilon\right)-\frac{m_1}{kp}Q_L\left(\frac{m_1}{kp}\left(\frac{k^2}{2\mu_{13}}+\frac{p^2}{2\mu_{12}}-E\right)-i\varepsilon\right) \\
& -\frac{1}{2\pi^2}\sum_{\substack{i,j,k=1\\ i<j\\ k\neq i,j}}^{3}\int_0^\infty dq\left[\frac{q^2 T_{ij}^{(L)}(E,k,q)}{-\gamma_{ij}+\sqrt{-2\mu_{ij}\left(E-\frac{q^2}{2m_k}-\frac{q^2}{2(m_i+m_j)}\right)}-i\varepsilon}\right. \\
& \qquad\qquad\qquad +\frac{q^2 T_{ij}^{\prime(L)}(E,k,q)}{-\gamma'_{ij}+\sqrt{-2\mu_{ij}\left(E-\frac{q^2}{2m_k}-\frac{q^2}{2(m_i+m_j)}\right)}-i\varepsilon}\right] \\
& \times\left[\frac{m_j}{qp}Q_L\left(\frac{m_j}{qp}\left(\frac{q^2}{2\mu_{kj}}+\frac{p^2}{2\mu_{ij}}-E\right)-i\varepsilon\right)+\frac{m_i}{qp}Q_L\left(\frac{m_i}{qp}\left(\frac{q^2}{2\mu_{ki}}+\frac{p^2}{2\mu_{ij}}-E\right)-i\varepsilon\right)\right],
\end{aligned}
$$

$$(D.18)$$

and thus Eq. (F.2) simplifies to Eq. (F.3) shown in appendix F.

Appendix E

Mellin transform of Legendre functions of the second kind

In this section we want to calculate the integral

$$\int_0^\infty dX X^{s^{(L)}-1} Q_L \left(\alpha X + \beta \frac{1}{X} \right) , \tag{E.1}$$

which repeatedly appears in the derivation of the transcendental equations for type 1, type 2 and type 3 systems. For this purpose we closely follow the work of Grießhammer in Ref. [165]. Hence, we firstly notice that the Mellin transform of a function $f(z)$ is defined as [168]

$$\mathcal{M}[f(z), s] := \int_0^\infty dz z^{s-1} f(z) . \tag{E.2}$$

Therefore Eq. (E.1) is nothing else then the Mellin transform of the Legendre function of the second kind:

$$\mathcal{M}\left[Q_L \left(\alpha X + \beta \frac{1}{X} \right), s^{(L)} \right] := \int_0^\infty dX X^{(L)-1} Q_L \left(\alpha X + \beta \frac{1}{X} \right) . \tag{E.3}$$

Comparing our definition of Q_L (see Eq. (D.9)),

$$Q_L(z) := (-1)^L \frac{1}{2} \int_{-1}^1 dx \frac{\mathcal{P}_L(x)}{z + x} ,$$

with that of Grießhammer (cf. Eq. (2.4) in Ref. [165]),

$$Q_L^{(\mathrm{HG})}(z) := \frac{1}{2} \int_{-1}^1 dx \frac{\mathcal{P}_L(x)}{z - x} ,$$

we observe that

$$Q_L(z) = (-1)^{L+1} Q_L^{(\mathrm{HG})}(-z) . \tag{E.4}$$

However, in Eq. (A.7) in the appendix of Ref. [165] it is shown that $Q_L^{(\mathrm{HG})}$ can be expressed in terms of hypergeometric functions which additionally are expressed in a series representation [164]. Using the same method we find for Q_L the following result:

$$Q_L(z) = \frac{\sqrt{\pi}\Gamma(L+1)}{2^{L+1}\Gamma\left(\frac{L}{2}+1\right)\Gamma\left(\frac{L+1}{2}\right)} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{L}{2}+1+n\right)\Gamma\left(\frac{L+1}{2}+n\right)}{\Gamma\left(L+\frac{3}{2}+n\right)\Gamma(n+1)}(-1)^{-2n}z^{-(2n+L+1)}, \quad (\text{E.5})$$

which is indeed equivalent to the representation of $Q_L^{(\mathrm{HG})}(z)$ since $(-1)^{-2n}=1$ for all $n \in \mathbb{N}$. However, in contrast to Ref. [165] we consider the argument $z = \alpha X + \beta X^{-1}$. Thus, the Mellin transform in Eq. (E.3) is given as

$$\mathcal{M}\left[Q_L\left(\alpha X + \beta\frac{1}{X}\right), s^{(L)}\right] = \frac{\sqrt{\pi}\,\Gamma(L+1)}{2^{L+1}\Gamma\left(\frac{L}{2}+1\right)\Gamma\left(\frac{L+1}{2}\right)} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{L}{2}+1+n\right)\Gamma\left(\frac{L+1}{2}+n\right)}{\Gamma\left(L+\frac{3}{2}+n\right)\Gamma(n+1)}$$

$$\times \alpha^{-(2n+L+1)} \int_0^{\infty} dX\, X^{2n+L+s^{(L)}} \left(X^2 + \frac{\beta}{\alpha}\right)^{-(2n+L+1)}. \quad (\text{E.6})$$

As it is done in Ref. [165] the integral over X can be solved using Ref. [164] or even with Mathematica:

$$\int_0^{\infty} dX\, X^{2n+L+s^{(L)}} \left(X^2 + \frac{\beta}{\alpha}\right)^{-(2n+L+1)} = \frac{1}{2}\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}(-1-L-n2+s^{(L)})}$$

$$\times \frac{\Gamma\left(n + \frac{L+s^{(L)}+1}{2}\right)\Gamma\left(n + \frac{L-s^{(L)}+1}{2}\right)}{\Gamma(2n+L+1)}, \quad (\text{E.7})$$

if $\beta/\alpha > 0$, $1 + L + 2n > \mathrm{Re}(s^{(L)})$ and $1 + L + 2n + \mathrm{Re}(s^{(L)}) > 0$ holds. Note, that since $L, n \geq 0$ the second condition $(1 + L + 2n > \mathrm{Re}(s^{(L)}))$ directly implies the third and additionally we know that for μ and $\widetilde{\mu}$ being placeholders for various combinations of μ_{ij} and μ_{ik} it holds

$$\frac{\beta}{\alpha} = \frac{m}{2\mu}\frac{2\widetilde{\mu}}{m} = \frac{\widetilde{\mu}}{\mu} > 0, \quad \forall\, \mu, \widetilde{\mu}. \quad (\text{E.8})$$

Finally, it is shown in Ref. [165] that the remaining condition $1 + L + 2n > \mathrm{Re}(s^{(L)})$ is also fulfilled. Thus, it is indeed allowed to write the Mellin transform Eq. (E.6) as

$$\mathcal{M}\left[Q_L\left(\alpha X + \beta\frac{1}{X}\right), s^{(L)}\right] = \frac{1}{2}\frac{\sqrt{\pi}\,\Gamma(L+1)}{2^{L+1}\Gamma\left(\frac{L}{2}+1\right)\Gamma\left(\frac{L+1}{2}\right)} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{L}{2}+1+n\right)\Gamma\left(\frac{L+1}{2}+n\right)}{\Gamma\left(L+\frac{3}{2}+n\right)\Gamma(n+1)}$$

$$\times \left(\frac{1}{\sqrt{\alpha\beta}}\right)^{2n+L+1} \left(\frac{\beta}{\alpha}\right)^{\frac{s^{(L)}}{2}} \frac{\Gamma\left(n + \frac{L+s^{(L)}+1}{2}\right)\Gamma\left(n + \frac{L-s^{(L)}+1}{2}\right)}{\Gamma(2n+L+1)}. \quad (\text{E.9})$$

In the next step we use the well-known identity $\Gamma(z+1) = z\,\Gamma(z)$ and the Legendre doubling formula,

$$\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{z+1}{2}\right) = \frac{\pi}{2^{z-1}}\Gamma(z), \quad \forall\, z \in \mathbb{C} \setminus \{0, -1, -2, ...\}, \quad (\text{E.10})$$

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to simplify the term

$$\mathcal{I} := \frac{\Gamma(L+1)}{2^{L+1}\Gamma\left(\frac{L}{2}+1\right)\Gamma\left(\frac{L+1}{2}\right)}\frac{\Gamma\left(\frac{L}{2}+1+n\right)\Gamma\left(\frac{L+1}{2}+n\right)}{\Gamma\left(2n+L+1\right)}, \tag{E.11}$$

appearing in the Mellin transform above. Since Eq. (E.10) is only valid for $z < 0$ we can use it solely in the case of $L > 0$. Then it holds:

- $\Gamma\left(\frac{L}{2}+1\right)\Gamma\left(\frac{L+1}{2}\right) = \frac{L}{2}\Gamma\left(\frac{L}{2}\right)\Gamma\left(\frac{L+1}{2}\right) = L\frac{\sqrt{\pi}}{2^L}\Gamma(L)$,

- $\Gamma\left(\frac{L}{2}+1+n\right)\Gamma\left(\frac{L+1}{2}+n\right) = \frac{L+2n}{2}\Gamma\left(\frac{L+2n}{2}\right)\Gamma\left(\frac{L+2n+1}{2}\right) = (L+2n)\frac{\sqrt{\pi}}{2^{L+2n}}\Gamma(L+2n)$,

and thus $\mathcal{I} = 2^{-(L+1)}\,4^{-n}$. Plugging this result for $\mathcal{I}$ into Eq. (E.9) one finds for $L > 0$

$$\mathcal{M}\left[Q_L\left(\alpha X + \beta\frac{1}{X}\right), s^{(L)}\right] = \frac{\sqrt{\pi}}{2^{L+2}}\left(\frac{\beta}{\alpha}\right)^{\frac{s^{(L)}}{2}}\left(\frac{1}{\sqrt{\alpha\beta}}\right)^{L+1}$$
$$\times \sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{L+s^{(L)}+1}{2}\right)\Gamma\left(n+\frac{L-s^{(L)}+1}{2}\right)}{\Gamma\left(L+\frac{3}{2}+n\right)\Gamma\left(n+1\right)}(4\alpha\beta)^{-n}. \tag{E.12}$$

Using the series representation of the hypergeometric functions $_pF_q$ [164],

$$_pF_q(a_1,...,a_p;b_1,...,b_q;z) = \sum_{k=0}^{\infty}\prod_{i=1}^{p}\frac{\Gamma(k+a_i)}{\Gamma(a_i)}\prod_{j=1}^{q}\frac{\Gamma(b_j)}{\Gamma(k+b_j)}\frac{z^k}{k!}, \quad p,q\in\mathbb{N}, \tag{E.13}$$

one concludes that also Eq. (E.12) can be written in terms of $_2F_1$:

$$\mathcal{M}\left[Q_L\left(\alpha X + \beta\frac{1}{X}\right), s^{(L)}\right] = \frac{\sqrt{\pi}}{2^{L+2}}\left(\frac{\beta}{\alpha}\right)^{\frac{s^{(L)}}{2}}\left(\frac{1}{\sqrt{\alpha\beta}}\right)^{L+1}\frac{\Gamma\left(\frac{L+s^{(L)}+1}{2}\right)\Gamma\left(\frac{L-s^{(L)}+1}{2}\right)}{\Gamma\left(\frac{2L+3}{2}\right)}$$
$$\times\,_2F_1\left(\frac{L+s^{(L)}+1}{2},\frac{L-s^{(L)}+1}{2};\frac{2L+3}{2};\frac{1}{4\alpha\beta}\right), \tag{E.14}$$

valid for $L > 0$.

In order to show that this result also holds for S-waves we insert $L = 0$ explicitly into Eq. (E.9) and use $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(z+1) = z\,\Gamma(z)$ to deduce

$$\mathcal{M}\left[Q_0\left(\alpha X + \beta\frac{1}{X}\right), s^{(0)}\right] = \frac{\sqrt{\pi}}{2}\frac{1}{\sqrt{\alpha\beta}}\left(\frac{\beta}{\alpha}\right)^{\frac{s^{(0)}}{2}}\frac{\Gamma\left(\frac{s^{(0)}+1}{2}\right)\Gamma\left(\frac{-s^{(0)}+1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}$$
$$\times\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{s^{(0)}+1}{2}\right)\Gamma\left(n+\frac{-s^{(0)}+1}{2}\right)}{\Gamma\left(\frac{s^{(0)}+1}{2}\right)\Gamma\left(\frac{-s^{(0)}+1}{2}\right)}\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(n+\frac{3}{2}\right)}\left(\frac{1}{4\alpha\beta}\right)^n\frac{1}{n!}$$
$$\times\frac{4^n\Gamma(n+1)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(2n+1)\Gamma\left(\frac{1}{2}\right)}. \tag{E.15}$$

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For $n = 0$ we observe that the last term in the equation above yields 1, but also for $n > 0$ one can show with the Legendre doubling formula that

$$\frac{4^n \Gamma(n+1)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(2n+1)\Gamma\left(\frac{1}{2}\right)} \stackrel{n \neq 0}{=} \frac{4^n n\Gamma(n)\Gamma\left(n+\frac{1}{2}\right)}{2n\Gamma(2n)\Gamma\left(\frac{1}{2}\right)} = \frac{4^n \Gamma\left(\frac{2n}{2}\right)\Gamma\left(\frac{2n+1}{2}\right)}{2\Gamma(2n)\sqrt{\pi}}$$

$$\stackrel{n \neq 0}{=} \frac{4^n \sqrt{\pi}\, 2^{-(2n-1)}\Gamma(2n)}{2\Gamma(2n)\sqrt{\pi}} = \frac{4^n}{2^{2n-1} \times 2} = 1. \tag{E.16}$$

Therefore Eq. (E.15) is equivalent to

$$\mathcal{M}\left[Q_0\left(\alpha X + \beta\frac{1}{X}\right), s^{(0)}\right] = \frac{\sqrt{\pi}}{2}\frac{1}{\sqrt{\alpha\beta}}\left(\frac{\beta}{\alpha}\right)^{\frac{s^{(0)}}{2}}\frac{\Gamma\left(\frac{s^{(0)}+1}{2}\right)\Gamma\left(\frac{-s^{(0)}+1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}$$

$$\times \, {}_2F_1\left(\frac{s^{(0)}+1}{2}, \frac{-s^{(0)}+1}{2}; \frac{3}{2}; \frac{1}{4\alpha\beta}\right), \tag{E.17}$$

and we conclude that the relation

$$\mathcal{M}\left[Q_L\left(\alpha X + \beta\frac{1}{X}\right), s^{(L)}\right] = \frac{\sqrt{\pi}}{2^{L+2}}\left(\frac{\beta}{\alpha}\right)^{\frac{s^{(L)}}{2}}\left(\frac{1}{\sqrt{\alpha\beta}}\right)^{L+1}\frac{\Gamma\left(\frac{L+s^{(L)}+1}{2}\right)\Gamma\left(\frac{L-s^{(L)}+1}{2}\right)}{\Gamma\left(\frac{2L+3}{2}\right)}$$

$$\times \, {}_2F_1\left(\frac{L+s^{(L)}+1}{2}, \frac{L-s^{(L)}+1}{2}; \frac{2L+3}{2}; \frac{1}{4\alpha\beta}\right), \tag{E.18}$$

holds for all partial waves $L \geq 0$.

In particular, we can use the `FunctionExpand` command implemented in `Mathematica` to write Eq. (E.18) for $L = 0$ and for $L = 1$ in terms of trigonometric functions. The following identities hold in the S-wave case:

- $\Gamma\left(\frac{L+s^{(L)}+1}{2}\right)\Gamma\left(\frac{L-s^{(L)}+1}{2}\right)\Big|_{L=0} = \Gamma\left(\frac{s^{(0)}+1}{2}\right)\Gamma\left(\frac{-s^{(0)}+1}{2}\right) = \frac{\pi}{\cos\left(\frac{\pi}{2}s^{(0)}\right)}$,

- $\Gamma\left(\frac{2L+3}{2}\right)\Big|_{L=0} = \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$,

- ${}_2F_1\left(\frac{L+s^{(L)}+1}{2}, \frac{L-s^{(L)}+1}{2}; \frac{2L+3}{2}; z\right)\Big|_{L=0} = {}_2F_1\left(\frac{s^{(0)}+1}{2}, \frac{-s^{(0)}+1}{2}; \frac{3}{2}; z\right) = \frac{\sin\left(s^{(0)}\arcsin\left(\sqrt{z}\right)\right)}{s^{(0)}\sqrt{z}}$.

Also for the P-wave case the `FunctionExpand` command provides:

- $\Gamma\left(\frac{L+s^{(L)}+1}{2}\right)\Gamma\left(\frac{L-s^{(L)}+1}{2}\right)\Big|_{L=1} = \Gamma\left(\frac{1+s^{(1)}+1}{2}\right)\Gamma\left(\frac{1-s^{(1)}+1}{2}\right) = \frac{\pi s^{(1)}}{2}\frac{1}{\sin\left(\frac{\pi}{2}s^{(1)}\right)}$,

- $\Gamma\left(\frac{2L+3}{2}\right)\Big|_{L=1} = \Gamma\left(1+\frac{3}{2}\right) = \frac{3\sqrt{\pi}}{4}$,

- ${}_2F_1\left(\frac{L+s^{(L)}+1}{2}, \frac{L-s^{(L)}+1}{2}; \frac{2L+3}{2}; z\right)\Big|_{L=1} = {}_2F_1\left(\frac{1+s^{(1)}+1}{2}, \frac{1-s^{(1)}+1}{2}; \frac{2+3}{2}; z\right)$

$$= \frac{3}{\left[(s^{(1)})^2-1\right]z}\left[\frac{\sqrt{1-z}}{s^{(1)}\sqrt{z}}\sin\left(s^{(1)}\arcsin\left(\sqrt{z}\right)\right) - \cos\left(s^{(1)}\arcsin\left(\sqrt{z}\right)\right)\right].$$

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Therefore we find for $L = 0$ that the Mellin transform of Legendre function of the second kind is given by

$$\mathcal{M}\left[Q_0\left(\alpha X + \beta\frac{1}{X}\right), s^{(0)}\right] = \int_0^\infty dX\, X^{s_i^{(0)}-1} Q_0\left(\alpha X + \beta\frac{1}{X}\right)$$
$$= \left(\sqrt{\frac{\beta}{\alpha}}\right)^{s^{(0)}} \frac{\pi}{s^{(0)}} \frac{\sin\left(s^{(0)} \arcsin\left(\frac{1}{2}\sqrt{\frac{1}{\alpha\beta}}\right)\right)}{\cos\left(\frac{\pi}{2}s^{(0)}\right)}, \tag{E.19}$$

and similarly for $L = 1$:

$$\mathcal{M}\left[Q_1\left(\alpha X + \beta\frac{1}{X}\right), s^{(1)}\right] = \int_0^\infty dX\, X^{s^{(1)}-1} Q_1\left(\alpha X + \beta\frac{1}{X}\right)$$
$$= \left(\sqrt{\frac{\beta}{\alpha}}\right)^{s^{(1)}} \frac{\pi s^{(1)}}{\left[(s^{(1)})^2 - 1\right]} \frac{1}{\sin\left(\frac{\pi}{2}s^{(1)}\right)}$$
$$\times \left[\frac{\sqrt{4\alpha\beta - 1}}{s^{(1)}} \sin\left(s^{(1)} \arcsin\left(\frac{1}{2}\sqrt{\frac{1}{\alpha\beta}}\right)\right)\right.$$
$$\left. - \cos\left(s^{(1)} \arcsin\left(\frac{1}{2}\sqrt{\frac{1}{\alpha\beta}}\right)\right)\right]. \tag{E.20}$$

These two equations correspond to Eq. (3.116) and Eq. (3.117) which are repeatedly used in the derivation of the transcendental equation for systems of type 1, type 2 and type 3.

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Appendix F

Three-body scattering amplitude equations

The very long expressions of the three-body scattering amplitudes corresponding to the Feynman diagrams shown in appendix C are summarized here in order to keep the main part of this work more legible. In the respective parts of section 3.3 the according equation numbers are referenced.

Three-body scattering amplitudes where all terms with the same momentum structure are combined and where terms which only differ by a "primed" or "unprimed" final state dimer are added together. Furthermore, the dq_0 integration is already performed using the residue theorem as explained in section 3.3.

$$
i \begin{pmatrix} (t_{12})^{\gamma\sigma}_{\alpha\beta} \\ (t_{13})^{\gamma\sigma}_{\alpha\beta} \\ (t_{23})^{\gamma\sigma}_{\alpha\beta} \\ (t'_{12})^{\gamma\sigma}_{\alpha\beta} \\ (t'_{13})^{\gamma\sigma}_{\alpha\beta} \\ (t'_{23})^{\gamma\sigma}_{\alpha\beta} \end{pmatrix} (E, \mathbf{k}, \mathbf{p}) =
$$

$$
(-i)\left(1 - \frac{\delta_{P_1 P_2}}{2}\right) S_{123} \frac{1}{E - \frac{k^2}{2m_3} - \frac{p^2}{2m_1} - \frac{(\mathbf{k}+\mathbf{p})^2}{2m_2} + i\varepsilon} g_{12}(\mathcal{O}_{12})_{\alpha,\sigma\rho} \begin{pmatrix} g^*_{12}(\mathcal{O}^\dagger_{12})_{\gamma,\rho\beta} \\ g^*_{13}(\mathcal{O}^\dagger_{13})_{\gamma,\beta\rho} \\ g^*_{23}(\mathcal{O}^\dagger_{23})_{\gamma,\beta\rho} \\ g'^*_{12}(\mathcal{O}'^\dagger_{12})_{\gamma,\rho\beta} \\ g'^*_{13}(\mathcal{O}'^\dagger_{13})_{\gamma,\beta\rho} \\ g'^*_{23}(\mathcal{O}'^\dagger_{23})_{\gamma,\beta\rho} \end{pmatrix} \begin{pmatrix} \delta^{(12)}_{P_1 P_3} \\ \delta_{P_1 P_2} \\ 1 \\ \delta^{(12)}_{P_1 P_3} \\ \delta_{P_1 P_2} \\ 1 \end{pmatrix} \begin{pmatrix} f^{(3)}_{(12)(12)} \\ f^{(1)}_{(12)(13)} \\ f^{(1)}_{(12)(23)} \\ f^{(3)}_{(12)(12')} \\ f^{(1)}_{(12)(13')} \\ f^{(1)}_{(12)(23')} \end{pmatrix}
$$

$$
+ (-i)\left(1 - \frac{\delta_{P_1 P_2}}{2}\right) S_{213} \frac{1}{E - \frac{k^2}{2m_3} - \frac{p^2}{2m_2} - \frac{(\mathbf{k}+\mathbf{p})^2}{2m_1} + i\varepsilon} g_{12}(\mathcal{O}_{12})_{\alpha,\rho\sigma} \begin{pmatrix} g^*_{12}(\mathcal{O}^\dagger_{12})_{\gamma,\beta\rho} \\ g^*_{13}(\mathcal{O}^\dagger_{13})_{\gamma,\beta\rho} \\ g^*_{23}(\mathcal{O}^\dagger_{23})_{\gamma,\beta\rho} \\ g'^*_{12}(\mathcal{O}'^\dagger_{12})_{\gamma,\beta\rho} \\ g'^*_{13}(\mathcal{O}'^\dagger_{13})_{\gamma,\beta\rho} \\ g'^*_{23}(\mathcal{O}'^\dagger_{23})_{\gamma,\beta\rho} \end{pmatrix} \begin{pmatrix} \delta^{(12)}_{P_2 P_3} \\ 1 \\ \delta_{P_1 P_2} \\ \delta^{(12)}_{P_2 P_3} \\ 1 \\ \delta_{P_1 P_2} \end{pmatrix} \begin{pmatrix} \widetilde{f}^{(3)}_{(12)(12)} \\ \widetilde{f}^{(2)}_{(12)(13)} \\ \widetilde{f}^{(2)}_{(12)(23)} \\ \widetilde{f}^{(3)}_{(12)(12')} \\ \widetilde{f}^{(2)}_{(12)(13')} \\ \widetilde{f}^{(2)}_{(12)(23')} \end{pmatrix}
$$

$$
+ \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) S_{123} \int \frac{d^3 q}{(2\pi)^3} \, i \, (t_{12})^{\mu\nu}_{\alpha\beta}(E, \mathbf{k}, \mathbf{q}) \frac{D_{12}\left(E - \frac{q^2}{2m_3}, \mathbf{q}\right)}{E - \frac{q^2}{2m_3} - \frac{p^2}{2m_1} - \frac{(\mathbf{p}+\mathbf{q})^2}{2m_2} + i\varepsilon} g_{12}(\mathcal{O}_{12})_{\mu,\sigma\rho} \begin{pmatrix} g^*_{12}(\mathcal{O}^\dagger_{12})_{\gamma,\rho\nu} \\ g^*_{13}(\mathcal{O}^\dagger_{13})_{\gamma,\nu\rho} \\ g^*_{23}(\mathcal{O}^\dagger_{23})_{\gamma,\nu\rho} \\ g'^*_{12}(\mathcal{O}'^\dagger_{12})_{\gamma,\rho\nu} \\ g'^*_{13}(\mathcal{O}'^\dagger_{13})_{\gamma,\nu\rho} \\ g'^*_{23}(\mathcal{O}'^\dagger_{23})_{\gamma,\nu\rho} \end{pmatrix} \begin{pmatrix} \delta^{(12)}_{P_1 P_3} \\ \delta_{P_1 P_2} \\ 1 \\ \delta^{(12)}_{P_1 P_3} \\ \delta_{P_1 P_2} \\ 1 \end{pmatrix} \begin{pmatrix} f^{(3)}_{(12)(12)} \\ f^{(1)}_{(12)(13)} \\ f^{(1)}_{(12)(23)} \\ f^{(3)}_{(12)(12')} \\ f^{(1)}_{(12)(13')} \\ f^{(1)}_{(12)(23')} \end{pmatrix}
$$

$$
+ \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) S_{213} \int \frac{d^3 q}{(2\pi)^3} \, i \, (t_{12})^{\mu\nu}_{\alpha\beta} (E, \mathbf{k}, \mathbf{q}) \, \frac{D_{12}\left(E - \frac{q^2}{2m_3}, \mathbf{q}\right)}{E - \frac{q^2}{2m_3} - \frac{p^2}{2m_2} - \frac{(\mathbf{p}+\mathbf{q})^2}{2m_1} + i\varepsilon} \, g_{12}(\mathcal{O}_{12})_{\mu,\rho\sigma}
\begin{pmatrix} g_{12}^*(\mathcal{O}_{12}^\dagger)_{\gamma,\nu\rho} \\ g_{13}^*(\mathcal{O}_{13}^\dagger)_{\gamma,\nu\rho} \\ g_{23}^*(\mathcal{O}_{23}^\dagger)_{\gamma,\nu\rho} \\ g_{12}'^*(\mathcal{O}_{12}'^\dagger)_{\gamma,\nu\rho} \\ g_{13}'^*(\mathcal{O}_{13}'^\dagger)_{\gamma,\nu\rho} \\ g_{23}'^*(\mathcal{O}_{23}'^\dagger)_{\gamma,\nu\rho} \end{pmatrix}
\begin{pmatrix} \delta^{(12)}_{P_2 P_3} \\ 1 \\ \delta_{P_1 P_2} \\ \delta^{(12)}_{P_2 P_3} \\ 1 \\ \delta_{P_1 P_2} \end{pmatrix}
\begin{pmatrix} \widetilde{f}^{(3)}_{(12)(12)} \\ \widetilde{f}^{(2)}_{(12)(13)} \\ \widetilde{f}^{(2)}_{(12)(23)} \\ \widetilde{f}^{(3)}_{(12)(12')} \\ \widetilde{f}^{(2)}_{(12)(13')} \\ \widetilde{f}^{(2)}_{(12)(23')} \end{pmatrix}
$$

$$
+ \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) S_{132} \int \frac{d^3 q}{(2\pi)^3} \, i \, (t_{13})^{\mu\nu}_{\alpha\beta} (E, \mathbf{k}, \mathbf{q}) \, \frac{D_{13}\left(E - \frac{q^2}{2m_2}, \mathbf{q}\right)}{E - \frac{q^2}{2m_2} - \frac{p^2}{2m_1} - \frac{(\mathbf{p}+\mathbf{q})^2}{2m_3} + i\varepsilon} \, g_{13}(\mathcal{O}_{13})_{\mu,\sigma\rho}
\begin{pmatrix} g_{12}^*(\mathcal{O}_{12}^\dagger)_{\gamma,\nu\rho} \\ g_{13}^*(\mathcal{O}_{13}^\dagger)_{\gamma,\rho\nu} \\ g_{23}^*(\mathcal{O}_{23}^\dagger)_{\gamma,\rho\nu} \\ g_{12}'^*(\mathcal{O}_{12}'^\dagger)_{\gamma,\nu\rho} \\ g_{13}'^*(\mathcal{O}_{13}'^\dagger)_{\gamma,\rho\nu} \\ g_{23}'^*(\mathcal{O}_{23}'^\dagger)_{\gamma,\rho\nu} \end{pmatrix}
\begin{pmatrix} \delta_{P_1 P_3} \\ \delta^{(13)}_{P_1 P_2} \\ 1 \\ \delta_{P_1 P_3} \\ \delta^{(13)}_{P_1 P_2} \\ 1 \end{pmatrix}
\begin{pmatrix} f^{(3)}_{(13)(12)} \\ f^{(1)}_{(13)(13)} \\ f^{(1)}_{(13)(23)} \\ f^{(3)}_{(13)(12')} \\ f^{(1)}_{(13)(13')} \\ f^{(1)}_{(13)(23')} \end{pmatrix}
$$

$$
+ \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) S_{312} \int \frac{d^3 q}{(2\pi)^3} \, i \, (t_{13})^{\mu\nu}_{\alpha\beta} (E, \mathbf{k}, \mathbf{q}) \, \frac{D_{13}\left(E - \frac{q^2}{2m_2}, \mathbf{q}\right)}{E - \frac{q^2}{2m_2} - \frac{p^2}{2m_3} - \frac{(\mathbf{p}+\mathbf{q})^2}{2m_1} + i\varepsilon} \, g_{13}(\mathcal{O}_{13})_{\mu,\rho\sigma}
\begin{pmatrix} g_{12}^*(\mathcal{O}_{12}^\dagger)_{\gamma,\nu\rho} \\ g_{13}^*(\mathcal{O}_{13}^\dagger)_{\gamma,\nu\rho} \\ g_{23}^*(\mathcal{O}_{23}^\dagger)_{\gamma,\rho\nu} \\ g_{12}'^*(\mathcal{O}_{12}'^\dagger)_{\gamma,\nu\rho} \\ g_{13}'^*(\mathcal{O}_{13}'^\dagger)_{\gamma,\nu\rho} \\ g_{23}'^*(\mathcal{O}_{23}'^\dagger)_{\gamma,\rho\nu} \end{pmatrix}
\begin{pmatrix} 1 \\ \delta^{(13)}_{P_2 P_3} \\ \delta_{P_1 P_3} \\ 1 \\ \delta^{(13)}_{P_2 P_3} \\ \delta_{P_1 P_3} \end{pmatrix}
\begin{pmatrix} \widetilde{f}^{(3)}_{(13)(12)} \\ \widetilde{f}^{(3)}_{(13)(13)} \\ \widetilde{f}^{(3)}_{(13)(23)} \\ \widetilde{f}^{(3)}_{(13)(12')} \\ \widetilde{f}^{(3)}_{(13)(13')} \\ \widetilde{f}^{(3)}_{(13)(23')} \end{pmatrix}
$$

$$
+ \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) S_{231} \int \frac{d^3 q}{(2\pi)^3} \, i \, (t_{23})^{\mu\nu}_{\alpha\beta} (E, \mathbf{k}, \mathbf{q}) \, \frac{D_{23}\left(E - \frac{q^2}{2m_1}, \mathbf{q}\right)}{E - \frac{q^2}{2m_1} - \frac{p^2}{2m_2} - \frac{(\mathbf{p}+\mathbf{q})^2}{2m_3} + i\varepsilon} \, g_{23}(\mathcal{O}_{23})_{\mu,\sigma\rho}
\begin{pmatrix} g_{12}^*(\mathcal{O}_{12}^\dagger)_{\gamma,\rho\nu} \\ g_{13}^*(\mathcal{O}_{13}^\dagger)_{\gamma,\rho\nu} \\ g_{23}^*(\mathcal{O}_{23}^\dagger)_{\gamma,\rho\nu} \\ g_{12}'^*(\mathcal{O}_{12}'^\dagger)_{\gamma,\rho\nu} \\ g_{13}'^*(\mathcal{O}_{13}'^\dagger)_{\gamma,\rho\nu} \\ g_{23}'^*(\mathcal{O}_{23}'^\dagger)_{\gamma,\rho\nu} \end{pmatrix}
\begin{pmatrix} \delta_{P_2 P_3} \\ 1 \\ \delta^{(23)}_{P_1 P_2} \\ \delta_{P_2 P_3} \\ 1 \\ \delta^{(23)}_{P_1 P_2} \end{pmatrix}
\begin{pmatrix} f^{(3)}_{(23)(12)} \\ f^{(2)}_{(23)(13)} \\ f^{(3)}_{(23)(23)} \\ f^{(3)}_{(23)(12')} \\ f^{(2)}_{(23)(13')} \\ f^{(2)}_{(23)(23')} \end{pmatrix}
$$

$$
+ \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) S_{321} \int \frac{d^3 q}{(2\pi)^3} \, i \, (t_{23})^{\mu\nu}_{\alpha\beta}(E, \mathbf{k}, \mathbf{q}) \frac{D_{23}\left(E - \frac{q^2}{2m_1}, \mathbf{q}\right)}{E - \frac{q^2}{2m_1} - \frac{p^2}{2m_3} - \frac{(\mathbf{p}+\mathbf{q})^2}{2m_2} + i\varepsilon} \, g_{23}(\mathcal{O}_{23})_{\mu,\rho\sigma}
\begin{pmatrix}
g_{12}^*(\mathcal{O}_{12}^\dagger)_{\gamma,\rho\nu} \\
g_{13}^*(\mathcal{O}_{13}^\dagger)_{\gamma,\rho\nu} \\
g_{23}^*(\mathcal{O}_{23}^\dagger)_{\gamma,\nu\rho} \\
g_{12}'^*(\mathcal{O}_{12}'^\dagger)_{\gamma,\rho\nu} \\
g_{13}'^*(\mathcal{O}_{13}'^\dagger)_{\gamma,\rho\nu} \\
g_{23}'^*(\mathcal{O}_{23}'^\dagger)_{\gamma,\nu\rho}
\end{pmatrix}
\begin{pmatrix}
1 \\
\delta_{P_2 P_3} \\
\delta^{(23)}_{P_1 P_3} \\
1 \\
\delta_{P_2 P_3} \\
\delta^{(23)}_{P_1 P_3}
\end{pmatrix}
\begin{pmatrix}
\widehat{f}^{(3)}_{(23)(12)} \\
\widehat{f}^{(3)}_{(23)(13)} \\
\widehat{f}^{(3)}_{(23)(23)} \\
\widehat{f}^{(3)}_{(23)(12')} \\
\widehat{f}^{(3)}_{(23)(13')} \\
\widehat{f}^{(3)}_{(23)(23')}
\end{pmatrix}
$$

$$
+ \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) S_{123} \int \frac{d^3 q}{(2\pi)^3} \, i \, (t'_{12})^{\mu\nu}_{\alpha\beta}(E, \mathbf{k}, \mathbf{q}) \frac{D'_{12}\left(E - \frac{q^2}{2m_3}, \mathbf{q}\right)}{E - \frac{q^2}{2m_3} - \frac{p^2}{2m_1} - \frac{(\mathbf{p}+\mathbf{q})^2}{2m_2} + i\varepsilon} \, g'_{12}(\mathcal{O}'_{12})_{\mu,\sigma\rho}
\begin{pmatrix}
g_{12}^*(\mathcal{O}_{12}^\dagger)_{\gamma,\rho\nu} \\
g_{13}^*(\mathcal{O}_{13}^\dagger)_{\gamma,\nu\rho} \\
g_{23}^*(\mathcal{O}_{23}^\dagger)_{\gamma,\nu\rho} \\
g_{12}'^*(\mathcal{O}_{12}'^\dagger)_{\gamma,\rho\nu} \\
g_{13}'^*(\mathcal{O}_{13}'^\dagger)_{\gamma,\nu\rho} \\
g_{23}'^*(\mathcal{O}_{23}'^\dagger)_{\gamma,\nu\rho}
\end{pmatrix}
\begin{pmatrix}
\delta^{(12')}_{P_1 P_3} \\
\delta_{P_1 P_2} \\
1 \\
\delta^{(12')}_{P_1 P_3} \\
\delta_{P_1 P_2} \\
1
\end{pmatrix}
\begin{pmatrix}
f^{(3)}_{(12')(12)} \\
f^{(1)}_{(12')(13)} \\
f^{(1)}_{(12')(23)} \\
f^{(3)}_{(12')(23')} \\
f^{(1)}_{(12')(13')} \\
f^{(1)}_{(12')(23')}
\end{pmatrix}
$$

$$
+ \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) S_{213} \int \frac{d^3 q}{(2\pi)^3} \, i \, (t'_{12})^{\mu\nu}_{\alpha\beta}(E, \mathbf{k}, \mathbf{q}) \frac{D'_{12}\left(E - \frac{q^2}{2m_3}, \mathbf{q}\right)}{E - \frac{q^2}{2m_3} - \frac{p^2}{2m_2} - \frac{(\mathbf{p}+\mathbf{q})^2}{2m_1} + i\varepsilon} \, g'_{12}(\mathcal{O}'_{12})_{\mu,\rho\sigma}
\begin{pmatrix}
g_{12}^*(\mathcal{O}_{12}^\dagger)_{\gamma,\nu\rho} \\
g_{13}^*(\mathcal{O}_{13}^\dagger)_{\gamma,\nu\rho} \\
g_{23}^*(\mathcal{O}_{23}^\dagger)_{\gamma,\nu\rho} \\
g_{12}'^*(\mathcal{O}_{12}'^\dagger)_{\gamma,\nu\rho} \\
g_{13}'^*(\mathcal{O}_{13}'^\dagger)_{\gamma,\nu\rho} \\
g_{23}'^*(\mathcal{O}_{23}'^\dagger)_{\gamma,\nu\rho}
\end{pmatrix}
\begin{pmatrix}
\delta^{(12')}_{P_2 P_3} \\
1 \\
\delta_{P_1 P_2} \\
\delta^{(12')}_{P_2 P_3} \\
1 \\
\delta_{P_1 P_2}
\end{pmatrix}
\begin{pmatrix}
\widehat{f}^{(3)}_{(12')(12)} \\
\widehat{f}^{(2)}_{(12')(13)} \\
\widehat{f}^{(2)}_{(12')(23)} \\
\widehat{f}^{(3)}_{(12')(12')} \\
\widehat{f}^{(2)}_{(12')(13')} \\
\widehat{f}^{(2)}_{(12')(23')}
\end{pmatrix}
$$

$$
+ \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) S_{132} \int \frac{d^3 q}{(2\pi)^3} \, i \, (t'_{13})^{\mu\nu}_{\alpha\beta}(E, \mathbf{k}, \mathbf{q}) \frac{D'_{13}\left(E - \frac{q^2}{2m_2}, \mathbf{q}\right)}{E - \frac{q^2}{2m_2} - \frac{p^2}{2m_1} - \frac{(\mathbf{p}+\mathbf{q})^2}{2m_3} + i\varepsilon} \, g'_{13}(\mathcal{O}'_{13})_{\mu,\sigma\rho}
\begin{pmatrix}
g_{12}^*(\mathcal{O}_{12}^\dagger)_{\gamma,\nu\rho} \\
g_{13}^*(\mathcal{O}_{13}^\dagger)_{\gamma,\rho\nu} \\
g_{23}^*(\mathcal{O}_{23}^\dagger)_{\gamma,\rho\nu} \\
g_{12}'^*(\mathcal{O}_{12}'^\dagger)_{\gamma,\nu\rho} \\
g_{13}'^*(\mathcal{O}_{13}'^\dagger)_{\gamma,\rho\nu} \\
g_{23}'^*(\mathcal{O}_{23}'^\dagger)_{\gamma,\rho\nu}
\end{pmatrix}
\begin{pmatrix}
\delta_{P_1 P_3} \\
\delta^{(13')}_{P_1 P_2} \\
1 \\
\delta_{P_1 P_3} \\
\delta^{(13')}_{P_1 P_2} \\
1
\end{pmatrix}
\begin{pmatrix}
f^{(3)}_{(13')(12)} \\
f^{(1)}_{(13')(13)} \\
f^{(1)}_{(13')(23)} \\
f^{(3)}_{(13')(12')} \\
f^{(1)}_{(13')(13')} \\
f^{(1)}_{(13')(23')}
\end{pmatrix}
$$

$$+ \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) S_{312} \int \frac{d^3 q}{(2\pi)^3} \, i \, (t'_{13})^{\mu\nu}_{\alpha\beta} (E, \mathbf{k}, \mathbf{q}) \frac{D'_{13}\left(E - \frac{q^2}{2m_2}, \mathbf{q}\right)}{E - \frac{q^2}{2m_2} - \frac{p^2}{2m_3} - \frac{(\mathbf{p}+\mathbf{q})^2}{2m_1} + i\varepsilon} \, g'_{13}(\mathcal{O}'_{13})_{\mu,\rho\sigma} \begin{pmatrix} g^*_{12}(\mathcal{O}^\dagger_{12})_{\gamma,\nu\rho} \\ g^*_{13}(\mathcal{O}^\dagger_{13})_{\gamma,\nu\rho} \\ g^*_{23}(\mathcal{O}^\dagger_{23})_{\gamma,\rho\nu} \\ g'^*_{12}(\mathcal{O}'^\dagger_{12})_{\gamma,\nu\rho} \\ g'^*_{13}(\mathcal{O}'^\dagger_{13})_{\gamma,\nu\rho} \\ g'^*_{23}(\mathcal{O}'^\dagger_{23})_{\gamma,\rho\nu} \end{pmatrix} \begin{pmatrix} 1 \\ \delta^{(13')}_{P_2 P_3} \\ \delta_{P_1 P_3} \\ 1 \\ \delta^{(13')}_{P_2 P_3} \\ \delta_{P_1 P_3} \end{pmatrix} \begin{pmatrix} \widetilde{f}^{(3)}_{(13')(12)} \\ \widetilde{f}^{(3)}_{(13')(13)} \\ \widetilde{f}^{(3)}_{(13')(23)} \\ \widetilde{f}^{(3)}_{(13')(12')} \\ \widetilde{f}^{(3)}_{(13')(13')} \\ \widetilde{f}^{(3)}_{(13')(23')} \end{pmatrix}$$

$$+ \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) S_{231} \int \frac{d^3 q}{(2\pi)^3} \, i \, (t'_{23})^{\mu\nu}_{\alpha\beta} (E, \mathbf{k}, \mathbf{q}) \frac{D'_{23}\left(E - \frac{q^2}{2m_1}, \mathbf{q}\right)}{E - \frac{q^2}{2m_1} - \frac{p^2}{2m_2} - \frac{(\mathbf{p}+\mathbf{q})^2}{2m_3} + i\varepsilon} \, g'_{23}(\mathcal{O}'_{23})_{\mu,\sigma\rho} \begin{pmatrix} g^*_{12}(\mathcal{O}^\dagger_{12})_{\gamma,\rho\nu} \\ g^*_{13}(\mathcal{O}^\dagger_{13})_{\gamma,\rho\nu} \\ g^*_{23}(\mathcal{O}^\dagger_{23})_{\gamma,\rho\nu} \\ g'^*_{12}(\mathcal{O}'^\dagger_{12})_{\gamma,\rho\nu} \\ g'^*_{13}(\mathcal{O}'^\dagger_{13})_{\gamma,\rho\nu} \\ g'^*_{23}(\mathcal{O}'^\dagger_{23})_{\gamma,\rho\nu} \end{pmatrix} \begin{pmatrix} \delta_{P_2 P_3} \\ 1 \\ \delta^{(23')}_{P_1 P_2} \\ \delta_{P_2 P_3} \\ 1 \\ \delta^{(23')}_{P_1 P_2} \end{pmatrix} \begin{pmatrix} f^{(3)}_{(23')(12)} \\ f^{(2)}_{(23')(13)} \\ f^{(2)}_{(23')(23)} \\ f^{(3)}_{(23')(12')} \\ f^{(2)}_{(23')(13')} \\ f^{(2)}_{(23')(23')} \end{pmatrix}$$

$$+ \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) S_{321} \int \frac{d^3 q}{(2\pi)^3} \, i \, (t'_{23})^{\mu\nu}_{\alpha\beta} (E, \mathbf{k}, \mathbf{q}) \frac{D'_{23}\left(E - \frac{q^2}{2m_1}, \mathbf{q}\right)}{E - \frac{q^2}{2m_1} - \frac{p^2}{2m_3} - \frac{(\mathbf{p}+\mathbf{q})^2}{2m_2} + i\varepsilon} \, g'_{23}(\mathcal{O}'_{23})_{\mu,\rho\sigma} \begin{pmatrix} g^*_{12}(\mathcal{O}^\dagger_{12})_{\gamma,\rho\nu} \\ g^*_{13}(\mathcal{O}^\dagger_{13})_{\gamma,\rho\nu} \\ g^*_{23}(\mathcal{O}^\dagger_{23})_{\gamma,\nu\rho} \\ g'^*_{12}(\mathcal{O}'^\dagger_{12})_{\gamma,\rho\nu} \\ g'^*_{13}(\mathcal{O}'^\dagger_{13})_{\gamma,\rho\nu} \\ g'^*_{23}(\mathcal{O}'^\dagger_{23})_{\gamma,\nu\rho} \end{pmatrix} \begin{pmatrix} 1 \\ \delta_{P_2 P_3} \\ \delta^{(23')}_{P_1 P_3} \\ 1 \\ \delta_{P_2 P_3} \\ \delta^{(23')}_{P_1 P_3} \end{pmatrix} \begin{pmatrix} \widetilde{f}^{(3)}_{(23')(12)} \\ \widetilde{f}^{(3)}_{(23')(13)} \\ \widetilde{f}^{(3)}_{(23')(23)} \\ \widetilde{f}^{(3)}_{(23')(12')} \\ \widetilde{f}^{(3)}_{(23')(13')} \\ \widetilde{f}^{(3)}_{(23')(23')} \end{pmatrix}$$

$$\text{(F.1)}$$

Three-body scattering amplitudes after applying wave function renormalization (see section [3.3.1]). Additionally, the full dimer propagators are plugged in explicitly.

$$
\begin{pmatrix}
(T_{12})^{\gamma\sigma}_{\alpha\beta} \\
(T_{13})^{\gamma\sigma}_{\alpha\beta} \\
(T_{23})^{\gamma\sigma}_{\alpha\beta} \\
(T'_{12})^{\gamma\sigma}_{\alpha\beta} \\
(T'_{13})^{\gamma\sigma}_{\alpha\beta} \\
(T'_{23})^{\gamma\sigma}_{\alpha\beta}
\end{pmatrix}
(E, \mathbf{k}, \mathbf{p}) =
$$

$$
- \left(1 - \frac{\delta_{P_1 P_2}}{2}\right)
\begin{pmatrix}
(\mu_{12}^2\, S_{12}\, c_{12})^{-1} \\
(\mu_{13}\, \mu_{12}\, \sqrt{S_{13}\, S_{12}\, c_{13}\, c_{12}})^{-1} \\
(\mu_{23}\, \mu_{12}\, \sqrt{S_{23}\, S_{12}\, c_{23}\, c_{12}})^{-1} \\
(\mu_{12}^2\, S_{12}\, \sqrt{c'_{12}\, c_{12}})^{-1} \\
(\mu_{13}\, \mu_{12}\, \sqrt{S_{13}\, S_{12}\, c'_{13}\, c_{12}})^{-1} \\
(\mu_{23}\, \mu_{12}\, \sqrt{S_{23}\, S_{12}\, c'_{23}\, c_{12}})^{-1}
\end{pmatrix}
\frac{S_{123}\, 2\pi\, \gamma_{12}\, (\mathcal{O}_{12})_{\alpha,\sigma\rho}}{E - \frac{k^2}{2m_3} - \frac{p^2}{2m_1} - \frac{(\mathbf{k}+\mathbf{p})^2}{2m_2} + i\varepsilon}
\begin{pmatrix}
(\mathcal{O}^\dagger_{12})_{\gamma,\rho\beta} \\
(\mathcal{O}^\dagger_{13})_{\gamma,\beta\rho} \\
(\mathcal{O}^\dagger_{23})_{\gamma,\beta\rho} \\
(\mathcal{O}'^\dagger_{12})_{\gamma,\rho\beta} \\
(\mathcal{O}'^\dagger_{13})_{\gamma,\beta\rho} \\
(\mathcal{O}'^\dagger_{23})_{\gamma,\beta\rho}
\end{pmatrix}
\begin{pmatrix}
\delta^{(12)}_{P_1 P_3} \\
\delta_{P_1 P_2} \\
1 \\
\delta^{(12)}_{P_1 P_3} \\
\delta_{P_1 P_2} \\
1
\end{pmatrix}
\begin{pmatrix}
f^{(3)}_{(12)(12)} \\
f^{(1)}_{(12)(13)} \\
f^{(1)}_{(12)(23)} \\
f^{(3)}_{(12)(12')} \\
f^{(1)}_{(12)(13')} \\
f^{(1)}_{(12)(23')}
\end{pmatrix}
$$

$$
- \left(1 - \frac{\delta_{P_1 P_2}}{2}\right)
\begin{pmatrix}
(\mu_{12}^2\, S_{12}\, c_{12})^{-1} \\
(\mu_{13}\, \mu_{12}\, \sqrt{S_{13} S_{12}\, c_{13} c_{12}})^{-1} \\
(\mu_{23}\, \mu_{12}\, \sqrt{S_{23} S_{12}\, c_{23} c_{12}})^{-1} \\
(\mu_{12}^2\, S_{12}\, \sqrt{c'_{12} c_{12}})^{-1} \\
(\mu_{13}\, \mu_{12}\, \sqrt{S_{13} S_{12}\, c'_{13} c_{12}})^{-1} \\
(\mu_{23}\, \mu_{12}\, \sqrt{S_{23} S_{12}\, c'_{23} c_{12}})^{-1}
\end{pmatrix}
\frac{S_{213}\, 2\pi\, \gamma_{12}\, (\mathcal{O}_{12})_{\alpha,\rho\sigma}}{E - \frac{k^2}{2m_3} - \frac{p^2}{2m_2} - \frac{(\mathbf{k}+\mathbf{p})^2}{2m_1} + i\varepsilon}
\begin{pmatrix}
(\mathcal{O}^\dagger_{12})_{\gamma,\beta\rho} \\
(\mathcal{O}^\dagger_{13})_{\gamma,\beta\rho} \\
(\mathcal{O}^\dagger_{23})_{\gamma,\beta\rho} \\
(\mathcal{O}'^\dagger_{12})_{\gamma,\beta\rho} \\
(\mathcal{O}'^\dagger_{13})_{\gamma,\beta\rho} \\
(\mathcal{O}'^\dagger_{23})_{\gamma,\beta\rho}
\end{pmatrix}
\begin{pmatrix}
\delta^{(12)}_{P_2 P_3} \\
1 \\
\delta_{P_1 P_2} \\
\delta^{(12)}_{P_2 P_3} \\
1 \\
\delta_{P_1 P_2}
\end{pmatrix}
\begin{pmatrix}
\widetilde{f}^{(3)}_{(12)(12)} \\
\widetilde{f}^{(2)}_{(12)(13)} \\
\widetilde{f}^{(2)}_{(12)(23)} \\
\widetilde{f}^{(3)}_{(12)(12')} \\
\widetilde{f}^{(2)}_{(12)(13')} \\
\widetilde{f}^{(2)}_{(12)(23')}
\end{pmatrix}
$$

$$
-\left(1-\frac{\delta_{P_1 P_2}}{2}\right) S_{123}\, 2\pi
\begin{pmatrix}
(\mu_{12}\, S_{12}\, c_{12})^{-1} \\
\left(\mu_{13}\, \sqrt{S_{13}\, S_{12}}\, c_{13}\, c_{12}\right)^{-1} \\
\left(\mu_{23}\, \sqrt{S_{23}\, S_{12}}\, c_{23}\, c_{12}\right)^{-1} \\
\left(\mu_{12}\, S_{12}\, \sqrt{c'_{12}\, c_{12}}\right)^{-1} \\
\left(\mu_{13}\, \sqrt{S_{13}\, S_{12}}\, c'_{13}\, c_{12}\right)^{-1} \\
\left(\mu_{23}\, \sqrt{S_{23}\, S_{12}}\, c'_{23}\, c_{12}\right)^{-1}
\end{pmatrix}
(\mathcal{O}_{12})_{\mu,\sigma\rho}
\begin{pmatrix}
(\mathcal{O}^{\dagger}_{12})_{\gamma,\rho\nu} \\
(\mathcal{O}^{\dagger}_{13})_{\gamma,\nu\rho} \\
(\mathcal{O}^{\dagger}_{23})_{\gamma,\nu\rho} \\
(\mathcal{O}'^{\dagger}_{12})_{\gamma,\rho\nu} \\
(\mathcal{O}'^{\dagger}_{13})_{\gamma,\nu\rho} \\
(\mathcal{O}'^{\dagger}_{23})_{\gamma,\nu\rho}
\end{pmatrix}
\begin{pmatrix}
\delta^{(12)}_{P_1 P_3} \\
\delta_{P_1 P_2} \\
1 \\
\delta^{(12)}_{P_1 P_3} \\
\delta_{P_1 P_2} \\
1
\end{pmatrix}
\begin{pmatrix}
f^{(3)}_{(12)(12)} \\
f^{(1)}_{(12)(13)} \\
f^{(1)}_{(12)(23)} \\
f^{(3)}_{(12)(12')} \\
f^{(1)}_{(12)(13')} \\
f^{(1)}_{(12)(23')}
\end{pmatrix}
$$

$$
\times \int \frac{d^3 q}{(2\pi)^3}\, \frac{(T_{12})^{\mu\nu}_{\alpha\beta}(E,\mathbf{k},\mathbf{q})}{-\gamma_{12}+\sqrt{-2\mu_{12}\left(E-\frac{q^2}{2m_3}-\frac{q^2}{2(m_1+m_2)}\right)}-i\varepsilon}\, \frac{1}{E-\frac{q^2}{2m_3}-\frac{p^2}{2m_1}-\frac{(\mathbf{p}+\mathbf{q})^2}{2m_2}+i\varepsilon}
$$

$$
-\left(1-\frac{\delta_{P_1 P_2}}{2}\right) S_{213}\, 2\pi
\begin{pmatrix}
(\mu_{12}\, S_{12}\, c_{12})^{-1} \\
\left(\mu_{13}\, \sqrt{S_{13}\, S_{12}}\, c_{13}\, c_{12}\right)^{-1} \\
\left(\mu_{23}\, \sqrt{S_{23}\, S_{12}}\, c_{23}\, c_{12}\right)^{-1} \\
\left(\mu_{12}\, S_{12}\, \sqrt{c'_{12}\, c_{12}}\right)^{-1} \\
\left(\mu_{13}\, \sqrt{S_{13}\, S_{12}}\, c'_{13}\, c_{12}\right)^{-1} \\
\left(\mu_{23}\, \sqrt{S_{23}\, S_{12}}\, c'_{23}\, c_{12}\right)^{-1}
\end{pmatrix}
(\mathcal{O}_{12})_{\mu,\rho\sigma}
\begin{pmatrix}
(\mathcal{O}^{\dagger}_{12})_{\gamma,\nu\rho} \\
(\mathcal{O}^{\dagger}_{13})_{\gamma,\nu\rho} \\
(\mathcal{O}^{\dagger}_{23})_{\gamma,\nu\rho} \\
(\mathcal{O}'^{\dagger}_{12})_{\gamma,\nu\rho} \\
(\mathcal{O}'^{\dagger}_{13})_{\gamma,\nu\rho} \\
(\mathcal{O}'^{\dagger}_{23})_{\gamma,\nu\rho}
\end{pmatrix}
\begin{pmatrix}
\delta^{(12)}_{P_2 P_3} \\
1 \\
\delta_{P_1 P_2} \\
\delta^{(12)}_{P_2 P_3} \\
1 \\
\delta_{P_1 P_2}
\end{pmatrix}
\begin{pmatrix}
\widetilde{f}^{(3)}_{(12)(12)} \\
\widetilde{f}^{(2)}_{(12)(13)} \\
\widetilde{f}^{(2)}_{(12)(23)} \\
\widetilde{f}^{(3)}_{(12)(12')} \\
\widetilde{f}^{(2)}_{(12)(13')} \\
\widetilde{f}^{(2)}_{(12)(23')}
\end{pmatrix}
$$

$$
\times \int \frac{d^3 q}{(2\pi)^3}\, \frac{(T_{12})^{\mu\nu}_{\alpha\beta}(E,\mathbf{k},\mathbf{q})}{-\gamma_{12}+\sqrt{-2\mu_{12}\left(E-\frac{q^2}{2m_3}-\frac{q^2}{2(m_1+m_2)}\right)}-i\varepsilon}\, \frac{1}{E-\frac{q^2}{2m_3}-\frac{p^2}{2m_2}-\frac{(\mathbf{p}+\mathbf{q})^2}{2m_1}+i\varepsilon}
$$

$$
-\left(1-\frac{\delta_{P_1P_3}}{2}\right) S_{132}\, 2\pi
\begin{pmatrix}
\left(\mu_{12}\,\sqrt{S_{12}\,S_{13}}\,c_{12}\,c_{13}\right)^{-1} \\
\left(\mu_{13}\,S_{13}\,c_{13}\right)^{-1} \\
\left(\mu_{23}\,\sqrt{S_{23}\,S_{13}}\,c_{23}\,c_{13}\right)^{-1} \\
\left(\mu_{12}\,\sqrt{S_{12}\,S_{13}}\,c'_{12}\,c_{13}\right)^{-1} \\
\left(\mu_{13}\,S_{13}\,\sqrt{c'_{13}\,c_{13}}\right)^{-1} \\
\left(\mu_{23}\,\sqrt{S_{23}\,S_{13}}\,c'_{23}\,c_{13}\right)^{-1}
\end{pmatrix}
(\mathcal{O}_{13})_{\mu,\sigma\rho}
\begin{pmatrix}
(\mathcal{O}_{12}^{\dagger})_{\gamma,\nu\rho} \\
(\mathcal{O}_{13}^{\dagger})_{\gamma,\rho\nu} \\
(\mathcal{O}_{23}^{\dagger})_{\gamma,\rho\nu} \\
(\mathcal{O}_{12}'^{\dagger})_{\gamma,\nu\rho} \\
(\mathcal{O}_{13}'^{\dagger})_{\gamma,\rho\nu} \\
(\mathcal{O}_{23}'^{\dagger})_{\gamma,\rho\nu}
\end{pmatrix}
\begin{pmatrix}
\delta_{P_1P_3} \\
\delta^{(13)}_{P_1P_2} \\
1 \\
\delta_{P_1P_3} \\
\delta^{(13)}_{P_1P_2} \\
1
\end{pmatrix}
\begin{pmatrix}
f^{(3)}_{(13)(12)} \\
f^{(1)}_{(13)(13)} \\
f^{(1)}_{(13)(23)} \\
f^{(3)}_{(13)(12')} \\
f^{(1)}_{(13)(13')} \\
f^{(1)}_{(13)(23')}
\end{pmatrix}
$$

$$
\times \int \frac{d^3q}{(2\pi)^3}\,\frac{(T_{13})^{\mu\nu}_{\alpha\beta}\,(E,\mathbf{k},\mathbf{q})}{-\gamma_{13}+\sqrt{-2\mu_{13}\left(E-\frac{q^2}{2m_2}-\frac{q^2}{2(m_1+m_3)}\right)}-i\varepsilon}\,\frac{1}{E-\frac{q^2}{2m_2}-\frac{p^2}{2m_1}-\frac{(\mathbf{p}+\mathbf{q})^2}{2m_3}+i\varepsilon}
$$

$$
-\left(1-\frac{\delta_{P_1P_3}}{2}\right) S_{312}\, 2\pi
\begin{pmatrix}
\left(\mu_{12}\,\sqrt{S_{12}\,S_{13}}\,c_{12}\,c_{13}\right)^{-1} \\
\left(\mu_{13}\,S_{13}\,c_{13}\right)^{-1} \\
\left(\mu_{23}\,\sqrt{S_{23}\,S_{13}}\,c_{23}\,c_{13}\right)^{-1} \\
\left(\mu_{12}\,\sqrt{S_{12}\,S_{13}}\,c'_{12}\,c_{13}\right)^{-1} \\
\left(\mu_{13}\,S_{13}\,\sqrt{c'_{13}\,c_{13}}\right)^{-1} \\
\left(\mu_{23}\,\sqrt{S_{23}\,S_{13}}\,c'_{23}\,c_{13}\right)^{-1}
\end{pmatrix}
(\mathcal{O}_{13})_{\mu,\rho\sigma}
\begin{pmatrix}
(\mathcal{O}_{12}^{\dagger})_{\gamma,\nu\rho} \\
(\mathcal{O}_{13}^{\dagger})_{\gamma,\nu\rho} \\
(\mathcal{O}_{23}^{\dagger})_{\gamma,\rho\nu} \\
(\mathcal{O}_{12}'^{\dagger})_{\gamma,\nu\rho} \\
(\mathcal{O}_{13}'^{\dagger})_{\gamma,\nu\rho} \\
(\mathcal{O}_{23}'^{\dagger})_{\gamma,\rho\nu}
\end{pmatrix}
\begin{pmatrix}
1 \\
\delta^{(13)}_{P_2P_3} \\
\delta_{P_1P_3} \\
1 \\
\delta^{(13)}_{P_2P_3} \\
\delta_{P_1P_3}
\end{pmatrix}
\begin{pmatrix}
\widetilde{f}^{(3)}_{(13)(12)} \\
\widetilde{f}^{(3)}_{(13)(13)} \\
\widetilde{f}^{(3)}_{(13)(23)} \\
\widetilde{f}^{(3)}_{(13)(12')} \\
\widetilde{f}^{(3)}_{(13)(13')} \\
\widetilde{f}^{(3)}_{(13)(23')}
\end{pmatrix}
$$

$$
\times \int \frac{d^3q}{(2\pi)^3}\,\frac{(T_{13})^{\mu\nu}_{\alpha\beta}\,(E,\mathbf{k},\mathbf{q})}{-\gamma_{13}+\sqrt{-2\mu_{13}\left(E-\frac{q^2}{2m_2}-\frac{q^2}{2(m_1+m_3)}\right)}-i\varepsilon}\,\frac{1}{E-\frac{q^2}{2m_2}-\frac{p^2}{2m_3}-\frac{(\mathbf{p}+\mathbf{q})^2}{2m_1}+i\varepsilon}
$$

$$-\left(1 - \frac{\delta_{P_2 P_3}}{2}\right) S_{231}\, 2\pi
\begin{pmatrix}
\left(\mu_{12}\sqrt{S_{12}\,S_{23}\,c_{12}\,c_{23}}\right)^{-1} \\
\left(\mu_{13}\sqrt{S_{13}\,S_{23}\,c_{13}\,c_{23}}\right)^{-1} \\
\left(\mu_{23}\,S_{23}\,c_{23}\right)^{-1} \\
\left(\mu_{12}\sqrt{S_{12}\,S_{23}\,c'_{12}\,c_{23}}\right)^{-1} \\
\left(\mu_{13}\sqrt{S_{13}\,S_{23}\,c'_{13}\,c_{23}}\right)^{-1} \\
\left(\mu_{23}\,S_{23}\,\sqrt{c'_{23}\,c_{23}}\right)^{-1}
\end{pmatrix}
(\mathcal{O}_{23})_{\mu,\sigma\rho}
\begin{pmatrix}
(\mathcal{O}^\dagger_{12})_{\gamma,\rho\nu} \\
(\mathcal{O}^\dagger_{13})_{\gamma,\rho\nu} \\
(\mathcal{O}^\dagger_{23})_{\gamma,\rho\nu} \\
(\mathcal{O}'^\dagger_{12})_{\gamma,\rho\nu} \\
(\mathcal{O}'^\dagger_{13})_{\gamma,\rho\nu} \\
(\mathcal{O}'^\dagger_{23})_{\gamma,\rho\nu}
\end{pmatrix}
\begin{pmatrix}
\delta_{P_2 P_3} \\
1 \\
\delta^{(23)}_{P_1 P_2} \\
\delta_{P_2 P_3} \\
1 \\
\delta^{(23)}_{P_1 P_2}
\end{pmatrix}
\begin{pmatrix}
f^{(3)}_{(23)(12)} \\
f^{(2)}_{(23)(13)} \\
f^{(2)}_{(23)(23)} \\
f^{(3)}_{(23)(12')} \\
f^{(2)}_{(23)(13')} \\
f^{(2)}_{(23)(23')}
\end{pmatrix}$$

$$\times \int \frac{d^3 q}{(2\pi)^3}\,
\frac{(T_{23})^{\mu\nu}_{\alpha\beta}(E,\mathbf{k},\mathbf{q})}{-\gamma_{23} + \sqrt{-2\mu_{23}\left(E - \frac{q^2}{2m_1} - \frac{q^2}{2(m_2+m_3)}\right)} - i\varepsilon}\,
\frac{1}{E - \frac{q^2}{2m_1} - \frac{p^2}{2m_2} - \frac{(\mathbf{p}+\mathbf{q})^2}{2m_3} + i\varepsilon}$$

$$-\left(1 - \frac{\delta_{P_2 P_3}}{2}\right) S_{321}\, 2\pi
\begin{pmatrix}
\left(\mu_{12}\sqrt{S_{12}\,S_{23}\,c_{12}\,c_{23}}\right)^{-1} \\
\left(\mu_{13}\sqrt{S_{13}\,S_{23}\,c_{13}\,c_{23}}\right)^{-1} \\
\left(\mu_{23}\,S_{23}\,c_{23}\right)^{-1} \\
\left(\mu_{12}\sqrt{S_{12}\,S_{23}\,c'_{12}\,c_{23}}\right)^{-1} \\
\left(\mu_{13}\sqrt{S_{13}\,S_{23}\,c'_{13}\,c_{23}}\right)^{-1} \\
\left(\mu_{23}\,S_{23}\,\sqrt{c'_{23}\,c_{23}}\right)^{-1}
\end{pmatrix}
(\mathcal{O}_{23})_{\mu,\rho\sigma}
\begin{pmatrix}
(\mathcal{O}^\dagger_{12})_{\gamma,\rho\nu} \\
(\mathcal{O}^\dagger_{13})_{\gamma,\rho\nu} \\
(\mathcal{O}^\dagger_{23})_{\gamma,\nu\rho} \\
(\mathcal{O}'^\dagger_{12})_{\gamma,\rho\nu} \\
(\mathcal{O}'^\dagger_{13})_{\gamma,\rho\nu} \\
(\mathcal{O}'^\dagger_{23})_{\gamma,\nu\rho}
\end{pmatrix}
\begin{pmatrix}
1 \\
\delta_{P_2 P_3} \\
\delta^{(23)}_{P_1 P_3} \\
1 \\
\delta_{P_2 P_3} \\
\delta^{(23)}_{P_1 P_3}
\end{pmatrix}
\begin{pmatrix}
\widetilde{f}^{(3)}_{(23)(12)} \\
\widetilde{f}^{(3)}_{(23)(13)} \\
\widetilde{f}^{(3)}_{(23)(23)} \\
\widetilde{f}^{(3)}_{(23)(12')} \\
\widetilde{f}^{(3)}_{(23)(13')} \\
\widetilde{f}^{(3)}_{(23)(23')}
\end{pmatrix}$$

$$\times \int \frac{d^3 q}{(2\pi)^3}\,
\frac{(T_{23})^{\mu\nu}_{\alpha\beta}(E,\mathbf{k},\mathbf{q})}{-\gamma_{23} + \sqrt{-2\mu_{23}\left(E - \frac{q^2}{2m_1} - \frac{q^2}{2(m_2+m_3)}\right)} - i\varepsilon}\,
\frac{1}{E - \frac{q^2}{2m_1} - \frac{p^2}{2m_3} - \frac{(\mathbf{p}+\mathbf{q})^2}{2m_2} + i\varepsilon}$$

$$- \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) S_{123} \, 2\pi \begin{pmatrix} \left(\mu_{12} \, S_{12} \, \sqrt{c_{12} \, c'_{12}}\right)^{-1} \\ \left(\mu_{13} \, \sqrt{S_{13} \, S_{12} \, c_{13} \, c'_{12}}\right)^{-1} \\ \left(\mu_{23} \, \sqrt{S_{23} \, S_{12} \, c_{23} \, c'_{12}}\right)^{-1} \\ \left(\mu_{12} \, S_{12} \, c'_{12}\right)^{-1} \\ \left(\mu_{13} \, \sqrt{S_{13} \, S_{12} \, c'_{13} \, c'_{12}}\right)^{-1} \\ \left(\mu_{23} \, \sqrt{S_{23} \, S_{12} \, c'_{23} \, c'_{12}}\right)^{-1} \end{pmatrix} (\mathcal{O}'_{12})_{\mu,\sigma\rho} \begin{pmatrix} (\mathcal{O}^\dagger_{12})_{\gamma,\rho\nu} \\ (\mathcal{O}^\dagger_{13})_{\gamma,\nu\rho} \\ (\mathcal{O}^\dagger_{23})_{\gamma,\nu\rho} \\ (\mathcal{O}'^\dagger_{12})_{\gamma,\rho\nu} \\ (\mathcal{O}'^\dagger_{13})_{\gamma,\nu\rho} \\ (\mathcal{O}'^\dagger_{23})_{\gamma,\nu\rho} \end{pmatrix} \begin{pmatrix} \delta^{(12')}_{P_1 P_3} \\ \delta_{P_1 P_2} \\ 1 \\ \delta^{(12')}_{P_1 P_3} \\ \delta_{P_1 P_2} \\ 1 \end{pmatrix} \begin{pmatrix} f^{(3)}_{(12')(12)} \\ f^{(1)}_{(12')(13)} \\ f^{(1)}_{(12')(23)} \\ f^{(3)}_{(12')(23')} \\ f^{(1)}_{(12')(13')} \\ f^{(1)}_{(12')(23')} \end{pmatrix}$$

$$\times \int \frac{d^3 q}{(2\pi)^3} \frac{(T'_{12})^{\mu\nu}_{\alpha\beta}(E, \mathbf{k}, \mathbf{q})}{-\gamma'_{12} + \sqrt{-2\mu_{12}\left(E - \frac{q^2}{2m_3} - \frac{q^2}{2(m_1+m_2)}\right)} - i\varepsilon} \frac{1}{E - \frac{q^2}{2m_3} - \frac{p^2}{2m_1} - \frac{(\mathbf{p}+\mathbf{q})^2}{2m_2} + i\varepsilon}$$

$$- \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) S_{213} \, 2\pi \begin{pmatrix} \left(\mu_{12} \, S_{12} \, \sqrt{c_{12} \, c'_{12}}\right)^{-1} \\ \left(\mu_{13} \, \sqrt{S_{13} \, S_{12} \, c_{13} \, c'_{12}}\right)^{-1} \\ \left(\mu_{23} \, \sqrt{S_{23} \, S_{12} \, c_{23} \, c'_{12}}\right)^{-1} \\ \left(\mu_{12} \, S_{12} \, c'_{12}\right)^{-1} \\ \left(\mu_{13} \, \sqrt{S_{13} \, S_{12} \, c'_{13} \, c'_{12}}\right)^{-1} \\ \left(\mu_{23} \, \sqrt{S_{23} \, S_{12} \, c'_{23} \, c'_{12}}\right)^{-1} \end{pmatrix} (\mathcal{O}'_{12})_{\mu,\rho\sigma} \begin{pmatrix} (\mathcal{O}^\dagger_{12})_{\gamma,\nu\rho} \\ (\mathcal{O}^\dagger_{13})_{\gamma,\nu\rho} \\ (\mathcal{O}^\dagger_{23})_{\gamma,\nu\rho} \\ (\mathcal{O}'^\dagger_{12})_{\gamma,\nu\rho} \\ (\mathcal{O}'^\dagger_{13})_{\gamma,\nu\rho} \\ (\mathcal{O}'^\dagger_{23})_{\gamma,\nu\rho} \end{pmatrix} \begin{pmatrix} \delta^{(12')}_{P_2 P_3} \\ 1 \\ \delta_{P_1 P_2} \\ \delta^{(12')}_{P_2 P_3} \\ 1 \\ \delta_{P_1 P_2} \end{pmatrix} \begin{pmatrix} \widetilde{f}^{(3)}_{(12')(12)} \\ \widetilde{f}^{(2)}_{(12')(13)} \\ \widetilde{f}^{(2)}_{(12')(23)} \\ \widetilde{f}^{(3)}_{(12')(12')} \\ \widehat{f}^{(2)}_{(12')(13')} \\ \widetilde{f}^{(2)}_{(12')(23')} \end{pmatrix}$$

$$\times \int \frac{d^3 q}{(2\pi)^3} \frac{(T'_{12})^{\mu\nu}_{\alpha\beta}(E, \mathbf{k}, \mathbf{q})}{-\gamma'_{12} + \sqrt{-2\mu_{12}\left(E - \frac{q^2}{2m_3} - \frac{q^2}{2(m_1+m_2)}\right)} - i\varepsilon} \frac{1}{E - \frac{q^2}{2m_3} - \frac{p^2}{2m_2} - \frac{(\mathbf{p}+\mathbf{q})^2}{2m_1} + i\varepsilon}$$

$$
- \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) S_{132} \, 2\pi \begin{pmatrix} \left(\mu_{12} \sqrt{S_{12} S_{13} c_{12} c'_{13}}\right)^{-1} \\ \left(\mu_{13} S_{13} \sqrt{c_{13} c'_{13}}\right)^{-1} \\ \left(\mu_{23} \sqrt{S_{23} S_{13} c_{23} c'_{13}}\right)^{-1} \\ \left(\mu_{12} \sqrt{S_{12} S_{13} c'_{12} c'_{13}}\right)^{-1} \\ \left(\mu_{13} S_{13} c'_{13}\right)^{-1} \\ \left(\mu_{23} \sqrt{S_{23} S_{13} c'_{23} c'_{13}}\right)^{-1} \end{pmatrix} (\mathcal{O}'_{13})_{\mu,\sigma\rho} \begin{pmatrix} (\mathcal{O}^\dagger_{12})_{\gamma,\nu\rho} \\ (\mathcal{O}^\dagger_{13})_{\gamma,\rho\nu} \\ (\mathcal{O}^\dagger_{23})_{\gamma,\rho\nu} \\ (\mathcal{O}'^\dagger_{12})_{\gamma,\nu\rho} \\ (\mathcal{O}'^\dagger_{13})_{\gamma,\rho\nu} \\ (\mathcal{O}'^\dagger_{23})_{\gamma,\rho\nu} \end{pmatrix} \begin{pmatrix} \delta_{P_1 P_3} \\ \delta^{(13')}_{P_1 P_2} \\ 1 \\ \delta_{P_1 P_3} \\ \delta^{(13')}_{P_1 P_2} \\ 1 \end{pmatrix} \begin{pmatrix} f^{(3)}_{(13')(12)} \\ f^{(1)}_{(13')(13)} \\ f^{(1)}_{(13')(23)} \\ f^{(3)}_{(13')(12')} \\ f^{(1)}_{(13')(13')} \\ f^{(1)}_{(13')(23')} \end{pmatrix}
$$

$$
\times \int \frac{d^3 q}{(2\pi)^3} \frac{(T'_{13})^{\mu\nu}_{\alpha\beta}(E,\mathbf{k},\mathbf{q})}{-\gamma'_{13} + \sqrt{-2\mu_{13}\left(E - \frac{q^2}{2m_2} - \frac{q^2}{2(m_1+m_3)}\right)} - i\varepsilon} \frac{1}{E - \frac{q^2}{2m_2} - \frac{p^2}{2m_1} - \frac{(\mathbf{p}+\mathbf{q})^2}{2m_3} + i\varepsilon}
$$

$$
- \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) S_{312} \, 2\pi \begin{pmatrix} \left(\mu_{12} \sqrt{S_{12} S_{13} c_{12} c'_{13}}\right)^{-1} \\ \left(\mu_{13} S_{13} \sqrt{c_{13} c'_{13}}\right)^{-1} \\ \left(\mu_{23} \sqrt{S_{23} S_{13} c_{23} c'_{13}}\right)^{-1} \\ \left(\mu_{12} \sqrt{S_{12} S_{13} c'_{12} c'_{13}}\right)^{-1} \\ \left(\mu_{13} S_{13} c'_{13}\right)^{-1} \\ \left(\mu_{23} \sqrt{S_{23} S_{13} c'_{23} c'_{13}}\right)^{-1} \end{pmatrix} (\mathcal{O}'_{13})_{\mu,\rho\sigma} \begin{pmatrix} (\mathcal{O}^\dagger_{12})_{\gamma,\nu\rho} \\ (\mathcal{O}^\dagger_{13})_{\gamma,\nu\rho} \\ (\mathcal{O}^\dagger_{23})_{\gamma,\rho\nu} \\ (\mathcal{O}'^\dagger_{12})_{\gamma,\nu\rho} \\ (\mathcal{O}'^\dagger_{13})_{\gamma,\nu\rho} \\ (\mathcal{O}'^\dagger_{23})_{\gamma,\rho\nu} \end{pmatrix} \begin{pmatrix} 1 \\ \delta^{(13')}_{P_2 P_3} \\ \delta_{P_1 P_3} \\ 1 \\ \delta^{(13')}_{P_2 P_3} \\ \delta_{P_1 P_3} \end{pmatrix} \begin{pmatrix} \widetilde{f}^{(3)}_{(13')(12)} \\ \widetilde{f}^{(3)}_{(13')(13)} \\ \widetilde{f}^{(3)}_{(13')(23)} \\ \widetilde{f}^{(3)}_{(13')(12')} \\ \widetilde{f}^{(3)}_{(13')(13')} \\ \widehat{f}^{(3)}_{(13')(23')} \end{pmatrix}
$$

$$
\times \int \frac{d^3 q}{(2\pi)^3} \frac{(T'_{13})^{\mu\nu}_{\alpha\beta}(E,\mathbf{k},\mathbf{q})}{-\gamma'_{13} + \sqrt{-2\mu_{13}\left(E - \frac{q^2}{2m_2} - \frac{q^2}{2(m_1+m_3)}\right)} - i\varepsilon} \frac{1}{E - \frac{q^2}{2m_2} - \frac{p^2}{2m_3} - \frac{(\mathbf{p}+\mathbf{q})^2}{2m_1} + i\varepsilon}
$$

$$
-\left(1-\frac{\delta_{P_2 P_3}}{2}\right) S_{231} \, 2\pi
\begin{pmatrix}
\left(\mu_{12}\sqrt{S_{12}\,S_{23}\,c_{12}\,c'_{23}}\right)^{-1} \\
\left(\mu_{13}\sqrt{S_{13}\,S_{23}\,c_{13}\,c'_{23}}\right)^{-1} \\
\left(\mu_{23}\,S_{23}\sqrt{c_{23}\,c'_{23}}\right)^{-1} \\
\left(\mu_{12}\sqrt{S_{12}\,S_{23}\,c'_{12}\,c'_{23}}\right)^{-1} \\
\left(\mu_{13}\sqrt{S_{13}\,S_{23}\,c'_{13}\,c'_{23}}\right)^{-1} \\
\left(\mu_{23}\,S_{23}\,c'_{23}\right)^{-1}
\end{pmatrix}
(\mathcal{O}'_{23})_{\mu,\sigma\rho}
\begin{pmatrix}
(\mathcal{O}^\dagger_{12})_{\gamma,\rho\nu} \\
(\mathcal{O}^\dagger_{13})_{\gamma,\rho\nu} \\
(\mathcal{O}^\dagger_{23})_{\gamma,\rho\nu} \\
(\mathcal{O}'^\dagger_{12})_{\gamma,\rho\nu} \\
(\mathcal{O}'^\dagger_{13})_{\gamma,\rho\nu} \\
(\mathcal{O}'^\dagger_{23})_{\gamma,\rho\nu}
\end{pmatrix}
\begin{pmatrix}
\delta_{P_2 P_3} \\
1 \\
\delta^{(23')}_{P_1 P_2} \\
\delta_{P_2 P_3} \\
1 \\
\delta^{(23')}_{P_1 P_2}
\end{pmatrix}
\begin{pmatrix}
f^{(3)}_{(23')(12)} \\
f^{(2)}_{(23')(13)} \\
f^{(2)}_{(23')(23)} \\
f^{(3)}_{(23')(12')} \\
f^{(2)}_{(23')(13')} \\
f^{(2)}_{(23')(23')}
\end{pmatrix}
$$

$$
\times \int \frac{d^3 q}{(2\pi)^3} \frac{(T'_{23})^{\mu\nu}_{\alpha\beta}(E,\mathbf{k},\mathbf{q})}{-\gamma'_{23}+\sqrt{-2\mu_{23}\left(E-\frac{q^2}{2m_1}-\frac{q^2}{2(m_2+m_3)}\right)}-i\varepsilon} \frac{1}{E-\frac{q^2}{2m_1}-\frac{p^2}{2m_2}-\frac{(\mathbf{p}+\mathbf{q})^2}{2m_3}+i\varepsilon}
$$

$$
-\left(1-\frac{\delta_{P_2 P_3}}{2}\right) S_{321} \, 2\pi
\begin{pmatrix}
\left(\mu_{12}\sqrt{S_{12}\,S_{23}\,c_{12}\,c'_{23}}\right)^{-1} \\
\left(\mu_{13}\sqrt{S_{13}\,S_{23}\,c_{13}\,c'_{23}}\right)^{-1} \\
\left(\mu_{23}\,S_{23}\sqrt{c_{23}\,c'_{23}}\right)^{-1} \\
\left(\mu_{12}\sqrt{S_{12}\,S_{23}\,c'_{12}\,c'_{23}}\right)^{-1} \\
\left(\mu_{13}\sqrt{S_{13}\,S_{23}\,c'_{13}\,c'_{23}}\right)^{-1} \\
\left(\mu_{23}\,S_{23}\,c'_{23}\right)^{-1}
\end{pmatrix}
(\mathcal{O}'_{23})_{\mu,\rho\sigma}
\begin{pmatrix}
(\mathcal{O}^\dagger_{12})_{\gamma,\rho\nu} \\
(\mathcal{O}^\dagger_{13})_{\gamma,\rho\nu} \\
(\mathcal{O}^\dagger_{23})_{\gamma,\nu\rho} \\
(\mathcal{O}'^\dagger_{12})_{\gamma,\rho\nu} \\
(\mathcal{O}'^\dagger_{13})_{\gamma,\rho\nu} \\
(\mathcal{O}'^\dagger_{23})_{\gamma,\nu\rho}
\end{pmatrix}
\begin{pmatrix}
1 \\
\delta_{P_2 P_3} \\
\delta^{(23')}_{P_1 P_3} \\
1 \\
\delta_{P_2 P_3} \\
\delta^{(23')}_{P_1 P_3}
\end{pmatrix}
\begin{pmatrix}
\widetilde{f}^{(3)}_{(23')(12)} \\
\widetilde{f}^{(3)}_{(23')(13)} \\
\widetilde{f}^{(3)}_{(23')(23)} \\
\widetilde{f}^{(3)}_{(23')(12')} \\
\widetilde{f}^{(3)}_{(23')(13')} \\
\widetilde{f}^{(3)}_{(23')(23')}
\end{pmatrix}
$$

$$
\times \int \frac{d^3 q}{(2\pi)^3} \frac{(T'_{23})^{\mu\nu}_{\alpha\beta}(E,\mathbf{k},\mathbf{q})}{-\gamma'_{23}+\sqrt{-2\mu_{23}\left(E-\frac{q^2}{2m_1}-\frac{q^2}{2(m_2+m_3)}\right)}-i\varepsilon} \frac{1}{E-\frac{q^2}{2m_1}-\frac{p^2}{2m_3}-\frac{(\mathbf{p}+\mathbf{q})^2}{2m_2}+i\varepsilon}
\tag{F.2}
$$

Three-body scattering amplitudes after projection onto the L-th partial wave. For details (especially on the notation Q_L^{ijk}) see section 3.3.2 and appendix D.

$$
\begin{pmatrix}
\left(T_{12}^{(L)}\right)^{\gamma\sigma}_{\alpha\beta} \\
\left(T_{13}^{(L)}\right)^{\gamma\sigma}_{\alpha\beta} \\
\left(T_{23}^{(L)}\right)^{\gamma\sigma}_{\alpha\beta} \\
\left(T_{12}^{\prime(L)}\right)^{\gamma\sigma}_{\alpha\beta} \\
\left(T_{13}^{\prime(L)}\right)^{\gamma\sigma}_{\alpha\beta} \\
\left(T_{23}^{\prime(L)}\right)^{\gamma\sigma}_{\alpha\beta}
\end{pmatrix}
(E,k,p) =
$$

$$
(-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) S_{123}\, 2\pi\, \gamma_{12}
\begin{pmatrix}
\left(\mu_{12}^2\, S_{12}\, c_{12}\right)^{-1} \\
\left(\mu_{13}\, \mu_{12}\, \sqrt{S_{13}\, S_{12}\, c_{13}\, c_{12}}\right)^{-1} \\
\left(\mu_{23}\, \mu_{12}\, \sqrt{S_{23}\, S_{12}\, c_{23}\, c_{12}}\right)^{-1} \\
\left(\mu_{12}^2\, S_{12}\, \sqrt{c_{12}'\, c_{12}}\right)^{-1} \\
\left(\mu_{13}\, \mu_{12}\, \sqrt{S_{13}\, S_{12}\, c_{13}'\, c_{12}}\right)^{-1} \\
\left(\mu_{23}\, \mu_{12}\, \sqrt{S_{23}\, S_{12}\, c_{23}'\, c_{12}}\right)^{-1}
\end{pmatrix}
(\mathcal{O}_{12})_{\alpha,\sigma\rho}
\begin{pmatrix}
(\mathcal{O}_{12}^\dagger)_{\gamma,\rho\beta} \\
(\mathcal{O}_{13}^\dagger)_{\gamma,\beta\rho} \\
(\mathcal{O}_{23}^\dagger)_{\gamma,\beta\rho} \\
(\mathcal{O}_{12}'^\dagger)_{\gamma,\rho\beta} \\
(\mathcal{O}_{13}'^\dagger)_{\gamma,\beta\rho} \\
(\mathcal{O}_{23}'^\dagger)_{\gamma,\beta\rho}
\end{pmatrix}
\begin{pmatrix}
\delta_{P_1 P_3}^{(12)} \\
\delta_{P_1 P_2} \\
1 \\
\delta_{P_1 P_3}^{(12)} \\
\delta_{P_1 P_2} \\
1
\end{pmatrix}
\begin{pmatrix}
f_{(12)(12)}^{(3)} \\
f_{(12)(13)}^{(1)} \\
f_{(12)(23)}^{(1)} \\
f_{(12)(12')}^{(3)} \\
f_{(12)(13')}^{(1)} \\
f_{(12)(23')}^{(1)}
\end{pmatrix}
\frac{m_2}{kp} Q_L^{231}(k,p;E)
$$

$$
+ (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) S_{213}\, 2\pi\, \gamma_{12}
\begin{pmatrix}
\left(\mu_{12}^2\, S_{12}\, c_{12}\right)^{-1} \\
\left(\mu_{13}\, \mu_{12} \sqrt{S_{13} S_{12}\, c_{13} c_{12}}\right)^{-1} \\
\left(\mu_{23}\, \mu_{12} \sqrt{S_{23} S_{12}\, c_{23} c_{12}}\right)^{-1} \\
\left(\mu_{12}^2 S_{12} \sqrt{c_{12}' c_{12}}\right)^{-1} \\
\left(\mu_{13}\, \mu_{12} \sqrt{S_{13} S_{12}\, c_{13}' c_{12}}\right)^{-1} \\
\left(\mu_{23}\, \mu_{12} \sqrt{S_{23} S_{12}\, c_{23}' c_{12}}\right)^{-1}
\end{pmatrix}
(\mathcal{O}_{12})_{\alpha,\rho\sigma}
\begin{pmatrix}
(\mathcal{O}_{12}^\dagger)_{\gamma,\beta\rho} \\
(\mathcal{O}_{13}^\dagger)_{\gamma,\beta\rho} \\
(\mathcal{O}_{23}^\dagger)_{\gamma,\beta\rho} \\
(\mathcal{O}_{12}'^\dagger)_{\gamma,\beta\rho} \\
(\mathcal{O}_{13}'^\dagger)_{\gamma,\beta\rho} \\
(\mathcal{O}_{23}'^\dagger)_{\gamma,\beta\rho}
\end{pmatrix}
\begin{pmatrix}
\delta_{P_2 P_3}^{(12)} \\
1 \\
\delta_{P_1 P_2} \\
\delta_{P_2 P_3}^{(12)} \\
1 \\
\delta_{P_1 P_2}
\end{pmatrix}
\begin{pmatrix}
\widetilde{f}_{(12)(12)}^{(3)} \\
\widetilde{f}_{(12)(13)}^{(2)} \\
\widetilde{f}_{(12)(23)}^{(2)} \\
\widetilde{f}_{(12)(12')}^{(3)} \\
\widetilde{f}_{(12)(13')}^{(2)} \\
\widetilde{f}_{(12)(23')}^{(2)}
\end{pmatrix}
\frac{m_1}{kp} Q_L^{132}(k,p;E)
$$

$$
+ (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) S_{123} \frac{1}{\pi}
\begin{pmatrix}
(\mu_{12} \, S_{12} \, c_{12})^{-1} \\
\left(\mu_{13} \sqrt{S_{13} \, S_{12} \, c_{13} \, c_{12}}\right)^{-1} \\
\left(\mu_{23} \sqrt{S_{23} \, S_{12} \, c_{23} \, c_{12}}\right)^{-1} \\
\left(\mu_{12} \, S_{12} \sqrt{c'_{12} \, c_{12}}\right)^{-1} \\
\left(\mu_{13} \sqrt{S_{13} \, S_{12} \, c'_{13} \, c_{12}}\right)^{-1} \\
\left(\mu_{23} \sqrt{S_{23} \, S_{12} \, c'_{23} \, c_{12}}\right)^{-1}
\end{pmatrix}
(\mathcal{O}_{12})_{\mu,\sigma\rho}
\begin{pmatrix}
(\mathcal{O}_{12}^\dagger)_{\gamma,\rho\nu} \\
(\mathcal{O}_{13}^\dagger)_{\gamma,\nu\rho} \\
(\mathcal{O}_{23}^\dagger)_{\gamma,\nu\rho} \\
(\mathcal{O}_{12}'^\dagger)_{\gamma,\rho\nu} \\
(\mathcal{O}_{13}'^\dagger)_{\gamma,\nu\rho} \\
(\mathcal{O}_{23}'^\dagger)_{\gamma,\nu\rho}
\end{pmatrix}
\begin{pmatrix}
\delta_{P_1 P_3}^{(12)} \\
\delta_{P_1 P_2} \\
1 \\
\delta_{P_1 P_3}^{(12)} \\
\delta_{P_1 P_2} \\
1
\end{pmatrix}
\begin{pmatrix}
f_{(12)(12)}^{(3)} \\
f_{(12)(13)}^{(1)} \\
f_{(12)(23)}^{(1)} \\
f_{(12)(12')}^{(3)} \\
f_{(12)(13')}^{(1)} \\
f_{(12)(23')}^{(1)}
\end{pmatrix}
$$

$$
\times \int_0^\infty dq \, \frac{q^2 \left(T_{12}^{(L)}\right)^{\mu\nu}_{\alpha\beta}(E,k,q)}{-\gamma_{12} + \sqrt{-2\mu_{12}\left(E - \frac{q^2}{2m_3} - \frac{q^2}{2(m_1+m_2)}\right)} - i\varepsilon} \, \frac{m_2}{qp} Q_L^{231}(q,p;E)
$$

$$
+ (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) S_{213} \frac{1}{\pi}
\begin{pmatrix}
(\mu_{12} \, S_{12} \, c_{12})^{-1} \\
\left(\mu_{13} \sqrt{S_{13} \, S_{12} \, c_{13} \, c_{12}}\right)^{-1} \\
\left(\mu_{23} \sqrt{S_{23} \, S_{12} \, c_{23} \, c_{12}}\right)^{-1} \\
\left(\mu_{12} \, S_{12} \sqrt{c'_{12} \, c_{12}}\right)^{-1} \\
\left(\mu_{13} \sqrt{S_{13} \, S_{12} \, c'_{13} \, c_{12}}\right)^{-1} \\
\left(\mu_{23} \sqrt{S_{23} \, S_{12} \, c'_{23} \, c_{12}}\right)^{-1}
\end{pmatrix}
(\mathcal{O}_{12})_{\mu,\rho\sigma}
\begin{pmatrix}
(\mathcal{O}_{12}^\dagger)_{\gamma,\nu\rho} \\
(\mathcal{O}_{13}^\dagger)_{\gamma,\nu\rho} \\
(\mathcal{O}_{23}^\dagger)_{\gamma,\nu\rho} \\
(\mathcal{O}_{12}'^\dagger)_{\gamma,\nu\rho} \\
(\mathcal{O}_{13}'^\dagger)_{\gamma,\nu\rho} \\
(\mathcal{O}_{23}'^\dagger)_{\gamma,\nu\rho}
\end{pmatrix}
\begin{pmatrix}
\delta_{P_2 P_3}^{(12)} \\
1 \\
\delta_{P_1 P_2} \\
\delta_{P_2 P_3}^{(12)} \\
1 \\
\delta_{P_1 P_2}
\end{pmatrix}
\begin{pmatrix}
\widetilde{f}_{(12)(12)}^{(3)} \\
\widetilde{f}_{(12)(13)}^{(2)} \\
\widetilde{f}_{(12)(23)}^{(2)} \\
\widetilde{f}_{(12)(12')}^{(3)} \\
\widetilde{f}_{(12)(13')}^{(2)} \\
\widetilde{f}_{(12)(23')}^{(2)}
\end{pmatrix}
$$

$$
\times \int_0^\infty dq \, \frac{q^2 \left(T_{12}^{(L)}\right)^{\mu\nu}_{\alpha\beta}(E,k,q)}{-\gamma_{12} + \sqrt{-2\mu_{12}\left(E - \frac{q^2}{2m_3} - \frac{q^2}{2(m_1+m_2)}\right)} - i\varepsilon} \, \frac{m_1}{qp} Q_L^{132}(q,p;E)
$$

$$
+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) S_{132} \frac{1}{\pi}
\begin{pmatrix}
\left(\mu_{12} \sqrt{S_{12} S_{13}} \, c_{12} \, c_{13}\right)^{-1} \\
\left(\mu_{13} S_{13} c_{13}\right)^{-1} \\
\left(\mu_{23} \sqrt{S_{23} S_{13}} \, c_{23} \, c_{13}\right)^{-1} \\
\left(\mu_{12} \sqrt{S_{12} S_{13}} \, c'_{12} \, c_{13}\right)^{-1} \\
\left(\mu_{13} S_{13} \sqrt{c'_{13} \, c_{13}}\right)^{-1} \\
\left(\mu_{23} \sqrt{S_{23} S_{13}} \, c'_{23} \, c_{13}\right)^{-1}
\end{pmatrix}
(\mathcal{O}_{13})_{\mu,\sigma\rho}
\begin{pmatrix}
(\mathcal{O}_{12}^\dagger)_{\gamma,\nu\rho} \\
(\mathcal{O}_{13}^\dagger)_{\gamma,\rho\nu} \\
(\mathcal{O}_{23}^\dagger)_{\gamma,\rho\nu} \\
(\mathcal{O}_{12}'^\dagger)_{\gamma,\nu\rho} \\
(\mathcal{O}_{13}'^\dagger)_{\gamma,\rho\nu} \\
(\mathcal{O}_{23}'^\dagger)_{\gamma,\rho\nu}
\end{pmatrix}
\begin{pmatrix}
\delta_{P_1 P_3} \\
\delta^{(13)}_{P_1 P_2} \\
1 \\
\delta_{P_1 P_3} \\
\delta^{(13)}_{P_1 P_2} \\
1
\end{pmatrix}
\begin{pmatrix}
f^{(3)}_{(13)(12)} \\
f^{(1)}_{(13)(13)} \\
f^{(1)}_{(13)(23)} \\
f^{(3)}_{(13)(12')} \\
f^{(1)}_{(13)(13')} \\
f^{(1)}_{(13)(23')}
\end{pmatrix}
$$

$$
\times \int_0^\infty dq \, \frac{q^2 \left(T_{13}^{(L)}\right)^{\mu\nu}_{\alpha\beta}(E,k,q)}{-\gamma_{13} + \sqrt{-2\mu_{13}\left(E - \frac{q^2}{2m_2} - \frac{q^2}{2(m_1+m_3)}\right) - i\varepsilon}} \frac{m_3}{qp} Q_L^{321}(q,p;E)
$$

$$
+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) S_{312} \frac{1}{\pi}
\begin{pmatrix}
\left(\mu_{12} \sqrt{S_{12} S_{13}} \, c_{12} \, c_{13}\right)^{-1} \\
\left(\mu_{13} S_{13} c_{13}\right)^{-1} \\
\left(\mu_{23} \sqrt{S_{23} S_{13}} \, c_{23} \, c_{13}\right)^{-1} \\
\left(\mu_{12} \sqrt{S_{12} S_{13}} \, c'_{12} \, c_{13}\right)^{-1} \\
\left(\mu_{13} S_{13} \sqrt{c'_{13} \, c_{13}}\right)^{-1} \\
\left(\mu_{23} \sqrt{S_{23} S_{13}} \, c'_{23} \, c_{13}\right)^{-1}
\end{pmatrix}
(\mathcal{O}_{13})_{\mu,\rho\sigma}
\begin{pmatrix}
(\mathcal{O}_{12}^\dagger)_{\gamma,\nu\rho} \\
(\mathcal{O}_{13}^\dagger)_{\gamma,\nu\rho} \\
(\mathcal{O}_{23}^\dagger)_{\gamma,\rho\nu} \\
(\mathcal{O}_{12}'^\dagger)_{\gamma,\nu\rho} \\
(\mathcal{O}_{13}'^\dagger)_{\gamma,\nu\rho} \\
(\mathcal{O}_{23}'^\dagger)_{\gamma,\rho\nu}
\end{pmatrix}
\begin{pmatrix}
1 \\
\delta^{(13)}_{P_2 P_3} \\
\delta_{P_1 P_3} \\
1 \\
\delta^{(13)}_{P_2 P_3} \\
\delta_{P_1 P_3}
\end{pmatrix}
\begin{pmatrix}
\widetilde{f}^{(3)}_{(13)(12)} \\
\widetilde{f}^{(3)}_{(13)(13)} \\
\widetilde{f}^{(3)}_{(13)(23)} \\
\widetilde{f}^{(3)}_{(13)(12')} \\
\widetilde{f}^{(3)}_{(13)(13')} \\
\widetilde{f}^{(3)}_{(13)(23')}
\end{pmatrix}
$$

$$
\times \int_0^\infty dq \, \frac{q^2 \left(T_{13}^{(L)}\right)^{\mu\nu}_{\alpha\beta}(E,k,q)}{-\gamma_{13} + \sqrt{-2\mu_{13}\left(E - \frac{q^2}{2m_2} - \frac{q^2}{2(m_1+m_3)}\right) - i\varepsilon}} \frac{m_1}{qp} Q_L^{123}(q,p;E)
$$

$$
+ (-1)^L \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) S_{231} \frac{1}{\pi}
\begin{pmatrix}
\left(\mu_{12} \sqrt{S_{12}\, S_{23}\, c_{12}\, c_{23}}\right)^{-1} \\
\left(\mu_{13} \sqrt{S_{13}\, S_{23}\, c_{13}\, c_{23}}\right)^{-1} \\
\left(\mu_{23}\, S_{23}\, c_{23}\right)^{-1} \\
\left(\mu_{12} \sqrt{S_{12}\, S_{23}\, c'_{12}\, c_{23}}\right)^{-1} \\
\left(\mu_{13} \sqrt{S_{13}\, S_{23}\, c'_{13}\, c_{23}}\right)^{-1} \\
\left(\mu_{23}\, S_{23}\, \sqrt{c'_{23}\, c_{23}}\right)^{-1}
\end{pmatrix}
(\mathcal{O}_{23})_{\mu,\sigma\rho}
\begin{pmatrix}
(\mathcal{O}_{12}^{\dagger})_{\gamma,\rho\nu} \\
(\mathcal{O}_{13}^{\dagger})_{\gamma,\rho\nu} \\
(\mathcal{O}_{23}^{\dagger})_{\gamma,\rho\nu} \\
(\mathcal{O}_{12}'^{\dagger})_{\gamma,\rho\nu} \\
(\mathcal{O}_{13}'^{\dagger})_{\gamma,\rho\nu} \\
(\mathcal{O}_{23}'^{\dagger})_{\gamma,\rho\nu}
\end{pmatrix}
\begin{pmatrix}
\delta_{P_2 P_3} \\
1 \\
\delta_{P_1 P_2}^{(23)} \\
\delta_{P_2 P_3} \\
1 \\
\delta_{P_1 P_2}^{(23)}
\end{pmatrix}
\begin{pmatrix}
f_{(23)(12)}^{(3)} \\
f_{(23)(13)}^{(2)} \\
f_{(23)(23)}^{(2)} \\
f_{(23)(12')}^{(3)} \\
f_{(23)(13')}^{(2)} \\
f_{(23)(23')}^{(2)}
\end{pmatrix}
$$

$$
\times \int_0^\infty dq \, \frac{q^2 \left(T_{23}^{(L)}\right)_{\alpha\beta}^{\mu\nu}(E,k,q)}{-\gamma_{23} + \sqrt{-2\mu_{23}\left(E - \frac{q^2}{2m_1} - \frac{q^2}{2(m_2+m_3)}\right) - i\varepsilon}} \, \frac{m_3}{qp} Q_L^{312}(q,p;E)
$$

$$
+ (-1)^L \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) S_{321} \frac{1}{\pi}
\begin{pmatrix}
\left(\mu_{12} \sqrt{S_{12}\, S_{23}\, c_{12}\, c_{23}}\right)^{-1} \\
\left(\mu_{13} \sqrt{S_{13}\, S_{23}\, c_{13}\, c_{23}}\right)^{-1} \\
\left(\mu_{23}\, S_{23}\, c_{23}\right)^{-1} \\
\left(\mu_{12} \sqrt{S_{12}\, S_{23}\, c'_{12}\, c_{23}}\right)^{-1} \\
\left(\mu_{13} \sqrt{S_{13}\, S_{23}\, c'_{13}\, c_{23}}\right)^{-1} \\
\left(\mu_{23}\, S_{23}\, \sqrt{c'_{23}\, c_{23}}\right)^{-1}
\end{pmatrix}
(\mathcal{O}_{23})_{\mu,\rho\sigma}
\begin{pmatrix}
(\mathcal{O}_{12}^{\dagger})_{\gamma,\rho\nu} \\
(\mathcal{O}_{13}^{\dagger})_{\gamma,\rho\nu} \\
(\mathcal{O}_{23}^{\dagger})_{\gamma,\nu\rho} \\
(\mathcal{O}_{12}'^{\dagger})_{\gamma,\rho\nu} \\
(\mathcal{O}_{13}'^{\dagger})_{\gamma,\rho\nu} \\
(\mathcal{O}_{23}'^{\dagger})_{\gamma,\nu\rho}
\end{pmatrix}
\begin{pmatrix}
1 \\
\delta_{P_2 P_3} \\
\delta_{P_1 P_3}^{(23)} \\
1 \\
\delta_{P_2 P_3} \\
\delta_{P_1 P_3}^{(23)}
\end{pmatrix}
\begin{pmatrix}
\widetilde{f}_{(23)(12)}^{(3)} \\
\widetilde{f}_{(23)(13)}^{(3)} \\
\widetilde{f}_{(23)(23)}^{(3)} \\
\widetilde{f}_{(23)(12')}^{(3)} \\
\widetilde{f}_{(23)(13')}^{(3)} \\
\widetilde{f}_{(23)(23')}^{(3)}
\end{pmatrix}
$$

$$
\times \int_0^\infty dq \, \frac{q^2 \left(T_{23}^{(L)}\right)_{\alpha\beta}^{\mu\nu}(E,k,q)}{-\gamma_{23} + \sqrt{-2\mu_{23}\left(E - \frac{q^2}{2m_1} - \frac{q^2}{2(m_2+m_3)}\right) - i\varepsilon}} \, \frac{m_2}{qp} Q_L^{213}(q,p;E)
$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) S_{123} \frac{1}{\pi} \begin{pmatrix} \left(\mu_{12} \, S_{12} \, \sqrt{c_{12} \, c'_{12}}\right)^{-1} \\ \left(\mu_{13} \, \sqrt{S_{13} \, S_{12} \, c_{13} \, c'_{12}}\right)^{-1} \\ \left(\mu_{23} \, \sqrt{S_{23} \, S_{12} \, c_{23} \, c'_{12}}\right)^{-1} \\ \left(\mu_{12} \, S_{12} \, c'_{12}\right)^{-1} \\ \left(\mu_{13} \, \sqrt{S_{13} \, S_{12} \, c'_{13} \, c'_{12}}\right)^{-1} \\ \left(\mu_{23} \, \sqrt{S_{23} \, S_{12} \, c'_{23} \, c'_{12}}\right)^{-1} \end{pmatrix} (\mathcal{O}'_{12})_{\mu,\sigma\rho} \begin{pmatrix} (\mathcal{O}_{12}^\dagger)_{\gamma,\rho\nu} \\ (\mathcal{O}_{13}^\dagger)_{\gamma,\nu\rho} \\ (\mathcal{O}_{23}^\dagger)_{\gamma,\nu\rho} \\ (\mathcal{O}_{12}'^\dagger)_{\gamma,\rho\nu} \\ (\mathcal{O}_{13}'^\dagger)_{\gamma,\nu\rho} \\ (\mathcal{O}_{23}'^\dagger)_{\gamma,\nu\rho} \end{pmatrix} \begin{pmatrix} \delta_{P_1 P_3}^{(12')} \\ \delta_{P_1 P_2} \\ 1 \\ \delta_{P_1 P_3}^{(12')} \\ \delta_{P_1 P_2} \\ 1 \end{pmatrix} \begin{pmatrix} f_{(12')(12)}^{(3)} \\ f_{(12')(13)}^{(1)} \\ f_{(12')(23)}^{(1)} \\ f_{(12')(23')}^{(3)} \\ f_{(12')(13')}^{(1)} \\ f_{(12')(23')}^{(1)} \end{pmatrix}$$

$$\times \int_0^\infty dq \, \frac{q^2 \left(T_{12}'^{(L)}\right)_{\alpha\beta}^{\mu\nu}(E,k,q)}{-\gamma'_{12} + \sqrt{-2\mu_{12}\left(E - \frac{q^2}{2m_3} - \frac{q^2}{2(m_1+m_2)}\right)} - i\varepsilon} \, \frac{m_2}{qp} Q_L^{231}(q,p;E)$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) S_{213} \frac{1}{\pi} \begin{pmatrix} \left(\mu_{12} \, S_{12} \, \sqrt{c_{12} \, c'_{12}}\right)^{-1} \\ \left(\mu_{13} \, \sqrt{S_{13} \, S_{12} \, c_{13} \, c'_{12}}\right)^{-1} \\ \left(\mu_{23} \, \sqrt{S_{23} \, S_{12} \, c_{23} \, c'_{12}}\right)^{-1} \\ \left(\mu_{12} \, S_{12} \, c'_{12}\right)^{-1} \\ \left(\mu_{13} \, \sqrt{S_{13} \, S_{12} \, c'_{13} \, c'_{12}}\right)^{-1} \\ \left(\mu_{23} \, \sqrt{S_{23} \, S_{12} \, c'_{23} \, c'_{12}}\right)^{-1} \end{pmatrix} (\mathcal{O}'_{12})_{\mu,\rho\sigma} \begin{pmatrix} (\mathcal{O}_{12}^\dagger)_{\gamma,\nu\rho} \\ (\mathcal{O}_{13}^\dagger)_{\gamma,\nu\rho} \\ (\mathcal{O}_{23}^\dagger)_{\gamma,\nu\rho} \\ (\mathcal{O}_{12}'^\dagger)_{\gamma,\nu\rho} \\ (\mathcal{O}_{13}'^\dagger)_{\gamma,\nu\rho} \\ (\mathcal{O}_{23}'^\dagger)_{\gamma,\nu\rho} \end{pmatrix} \begin{pmatrix} \delta_{P_2 P_3}^{(12')} \\ 1 \\ \delta_{P_1 P_2} \\ \delta_{P_2 P_3}^{(12')} \\ 1 \\ \delta_{P_1 P_2} \end{pmatrix} \begin{pmatrix} \widetilde{f}_{(12')(12)}^{(3)} \\ \widetilde{f}_{(12')(13)}^{(2)} \\ \widetilde{f}_{(12')(23)}^{(2)} \\ \widetilde{f}_{(12')(12')}^{(3)} \\ \widetilde{f}_{(12')(13')}^{(2)} \\ \widetilde{f}_{(12')(23')}^{(2)} \end{pmatrix}$$

$$\times \int_0^\infty dq \, \frac{q^2 \left(T_{12}'^{(L)}\right)_{\alpha\beta}^{\mu\nu}(E,k,q)}{-\gamma'_{12} + \sqrt{-2\mu_{12}\left(E - \frac{q^2}{2m_3} - \frac{q^2}{2(m_1+m_2)}\right)} - i\varepsilon} \, \frac{m_1}{qp} Q_L^{132}(q,p;E)$$

$$
+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) S_{132} \frac{1}{\pi}
\begin{pmatrix}
\left(\mu_{12} \sqrt{S_{12} S_{13} c_{12} c'_{13}}\right)^{-1} \\
\left(\mu_{13} S_{13} \sqrt{c_{13} c'_{13}}\right)^{-1} \\
\left(\mu_{23} \sqrt{S_{23} S_{13} c_{23} c'_{13}}\right)^{-1} \\
\left(\mu_{12} \sqrt{S_{12} S_{13} c'_{12} c'_{13}}\right)^{-1} \\
(\mu_{13} S_{13} c'_{13})^{-1} \\
\left(\mu_{23} \sqrt{S_{23} S_{13} c'_{23} c'_{13}}\right)^{-1}
\end{pmatrix}
(\mathcal{O}'_{13})_{\mu,\sigma\rho}
\begin{pmatrix}
(\mathcal{O}^\dagger_{12})_{\gamma,\nu\rho} \\
(\mathcal{O}^\dagger_{13})_{\gamma,\rho\nu} \\
(\mathcal{O}^\dagger_{23})_{\gamma,\rho\nu} \\
(\mathcal{O}'^\dagger_{12})_{\gamma,\nu\rho} \\
(\mathcal{O}'^\dagger_{13})_{\gamma,\rho\nu} \\
(\mathcal{O}'^\dagger_{23})_{\gamma,\rho\nu}
\end{pmatrix}
\begin{pmatrix}
\delta_{P_1 P_3} \\
\delta^{(13')}_{P_1 P_2} \\
1 \\
\delta_{P_1 P_3} \\
\delta^{(13')}_{P_1 P_2} \\
1
\end{pmatrix}
\begin{pmatrix}
f^{(3)}_{(13')(12)} \\
f^{(1)}_{(13')(13)} \\
f^{(1)}_{(13')(23)} \\
f^{(3)}_{(13')(12')} \\
f^{(1)}_{(13')(13')} \\
f^{(1)}_{(13')(23')}
\end{pmatrix}
$$

$$
\times \int_0^\infty dq \, \frac{q^2 \left(T'^{(L)}_{13}\right)^{\mu\nu}_{\alpha\beta}(E,k,q)}{-\gamma'_{13} + \sqrt{-2\mu_{13}\left(E - \frac{q^2}{2m_2} - \frac{q^2}{2(m_1+m_3)}\right)} - i\varepsilon} \frac{m_3}{qp} Q_L^{321}(q,p;E)
$$

$$
+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) S_{312} \frac{1}{\pi}
\begin{pmatrix}
\left(\mu_{12} \sqrt{S_{12} S_{13} c_{12} c'_{13}}\right)^{-1} \\
\left(\mu_{13} S_{13} \sqrt{c_{13} c'_{13}}\right)^{-1} \\
\left(\mu_{23} \sqrt{S_{23} S_{13} c_{23} c'_{13}}\right)^{-1} \\
\left(\mu_{12} \sqrt{S_{12} S_{13} c'_{12} c'_{13}}\right)^{-1} \\
(\mu_{13} S_{13} c'_{13})^{-1} \\
\left(\mu_{23} \sqrt{S_{23} S_{13} c'_{23} c'_{13}}\right)^{-1}
\end{pmatrix}
(\mathcal{O}'_{13})_{\mu,\rho\sigma}
\begin{pmatrix}
(\mathcal{O}^\dagger_{12})_{\gamma,\nu\rho} \\
(\mathcal{O}^\dagger_{13})_{\gamma,\nu\rho} \\
(\mathcal{O}^\dagger_{23})_{\gamma,\rho\nu} \\
(\mathcal{O}'^\dagger_{12})_{\gamma,\nu\rho} \\
(\mathcal{O}'^\dagger_{13})_{\gamma,\nu\rho} \\
(\mathcal{O}'^\dagger_{23})_{\gamma,\rho\nu}
\end{pmatrix}
\begin{pmatrix}
1 \\
\delta^{(13')}_{P_2 P_3} \\
\delta_{P_1 P_3} \\
1 \\
\delta^{(13')}_{P_2 P_3} \\
\delta_{P_1 P_3}
\end{pmatrix}
\begin{pmatrix}
\widetilde{f}^{(3)}_{(13')(12)} \\
\widetilde{f}^{(3)}_{(13')(13)} \\
\widetilde{f}^{(3)}_{(13')(23)} \\
\widetilde{f}^{(3)}_{(13')(12')} \\
\widetilde{f}^{(3)}_{(13')(13')} \\
\widetilde{f}^{(3)}_{(13')(23')}
\end{pmatrix}
$$

$$
\times \int_0^\infty dq \, \frac{q^2 \left(T'^{(L)}_{13}\right)^{\mu\nu}_{\alpha\beta}(E,k,q)}{-\gamma'_{13} + \sqrt{-2\mu_{13}\left(E - \frac{q^2}{2m_2} - \frac{q^2}{2(m_1+m_3)}\right)} - i\varepsilon} \frac{m_1}{qp} Q_L^{123}(q,p;E)
$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) S_{231} \frac{1}{\pi} \begin{pmatrix} \left(\mu_{12} \sqrt{S_{12} S_{23}} c_{12} c'_{23}\right)^{-1} \\ \left(\mu_{13} \sqrt{S_{13} S_{23}} c_{13} c'_{23}\right)^{-1} \\ \left(\mu_{23} S_{23} \sqrt{c_{23} c'_{23}}\right)^{-1} \\ \left(\mu_{12} \sqrt{S_{12} S_{23}} c'_{12} c'_{23}\right)^{-1} \\ \left(\mu_{13} \sqrt{S_{13} S_{23}} c'_{13} c'_{23}\right)^{-1} \\ \left(\mu_{23} S_{23} c'_{23}\right)^{-1} \end{pmatrix} (\mathcal{O}'_{23})_{\mu,\sigma\rho} \begin{pmatrix} (\mathcal{O}^\dagger_{12})_{\gamma,\rho\nu} \\ (\mathcal{O}^\dagger_{13})_{\gamma,\rho\nu} \\ (\mathcal{O}^\dagger_{23})_{\gamma,\rho\nu} \\ (\mathcal{O}'^\dagger_{12})_{\gamma,\rho\nu} \\ (\mathcal{O}'^\dagger_{13})_{\gamma,\rho\nu} \\ (\mathcal{O}'^\dagger_{23})_{\gamma,\rho\nu} \end{pmatrix} \begin{pmatrix} \delta_{P_2 P_3} \\ 1 \\ \delta^{(23')}_{P_1 P_2} \\ \delta_{P_2 P_3} \\ 1 \\ \delta^{(23')}_{P_1 P_2} \end{pmatrix} \begin{pmatrix} f^{(3)}_{(23')(12)} \\ f^{(2)}_{(23')(13)} \\ f^{(2)}_{(23')(23)} \\ f^{(3)}_{(23')(12')} \\ f^{(2)}_{(23')(13')} \\ f^{(2)}_{(23')(23')} \end{pmatrix}$$

$$\times \int_0^\infty dq \, \frac{q^2 \left(T'^{(L)}_{23}\right)^{\mu\nu}_{\alpha\beta} (E,k,q)}{-\gamma'_{23} + \sqrt{-2\mu_{23}\left(E - \frac{q^2}{2m_1} - \frac{q^2}{2(m_2+m_3)}\right)} - i\varepsilon} \frac{m_3}{qp} Q^{312}_L(q,p;E)$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) S_{321} \frac{1}{\pi} \begin{pmatrix} \left(\mu_{12} \sqrt{S_{12} S_{23}} c_{12} c'_{23}\right)^{-1} \\ \left(\mu_{13} \sqrt{S_{13} S_{23}} c_{13} c'_{23}\right)^{-1} \\ \left(\mu_{23} S_{23} \sqrt{c_{23} c'_{23}}\right)^{-1} \\ \left(\mu_{12} \sqrt{S_{12} S_{23}} c'_{12} c'_{23}\right)^{-1} \\ \left(\mu_{13} \sqrt{S_{13} S_{23}} c'_{13} c'_{23}\right)^{-1} \\ \left(\mu_{23} S_{23} c'_{23}\right)^{-1} \end{pmatrix} (\mathcal{O}'_{23})_{\mu,\rho\sigma} \begin{pmatrix} (\mathcal{O}^\dagger_{12})_{\gamma,\rho\nu} \\ (\mathcal{O}^\dagger_{13})_{\gamma,\rho\nu} \\ (\mathcal{O}^\dagger_{23})_{\gamma,\nu\rho} \\ (\mathcal{O}'^\dagger_{12})_{\gamma,\rho\nu} \\ (\mathcal{O}'^\dagger_{13})_{\gamma,\rho\nu} \\ (\mathcal{O}'^\dagger_{23})_{\gamma,\nu\rho} \end{pmatrix} \begin{pmatrix} 1 \\ \delta_{P_2 P_3} \\ \delta^{(23')}_{P_1 P_3} \\ 1 \\ \delta_{P_2 P_3} \\ \delta^{(23')}_{P_1 P_3} \end{pmatrix} \begin{pmatrix} \widetilde{f}^{(3)}_{(23')(12)} \\ \widetilde{f}^{(3)}_{(23')(13)} \\ \widetilde{f}^{(3)}_{(23')(23)} \\ \widetilde{f}^{(3)}_{(23')(12')} \\ \widetilde{f}^{(3)}_{(23')(13')} \\ \widetilde{f}^{(3)}_{(23')(23')} \end{pmatrix}$$

$$\times \int_0^\infty dq \, \frac{q^2 \left(T'^{(L)}_{23}\right)^{\mu\nu}_{\alpha\beta} (E,k,q)}{-\gamma'_{23} + \sqrt{-2\mu_{23}\left(E - \frac{q^2}{2m_1} - \frac{q^2}{2(m_2+m_3)}\right)} - i\varepsilon} \frac{m_2}{qp} Q^{213}_L(q,p;E) \tag{F.3}$$

Three-body scattering amplitudes after projection onto a specific isospin and spin channel as explained in section 3.3.3. For the definition of the x_i, y_i and z_i parameters see appendix A.2.

$$T_{12}^{(L)}(E,k,p) =$$

$$(-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{2\pi\gamma_{12}}{\mu_{12}^2 S_{12} c_{12}} \left[x_1 \, \delta_{P_1 P_3}^{(12)} \, f_{(12)(12)}^{(3)} \, S_{123} \, \frac{m_2}{kp} Q_L^{211}(k,p;E) + \widetilde{x}_1 \, \delta_{P_2 P_3}^{(12)} \, \widetilde{f}_{(12)(12)}^{(3)} \, S_{213} \, \frac{m_1}{kp} Q_L^{122}(k,p;E)\right]$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{12} S_{12} c_{12}} \int_0^\infty dq \, \frac{q^2 \, T_{12}^{(L)}(E,k,q)}{-\gamma_{12} + \sqrt{-2\mu_{12}\left(E - \frac{q^2}{2m_3} - \frac{q^2}{2(m_1+m_2)}\right)} - i\varepsilon}$$

$$\times \left[x_2 \, \delta_{P_1 P_3}^{(12)} \, f_{(12)(12)}^{(3)} \, S_{123} \, \frac{m_2}{qp} Q_L^{211}(q,p;E) + \widetilde{x}_2 \, \delta_{P_2 P_3}^{(12)} \, \widetilde{f}_{(12)(12)}^{(3)} \, S_{213} \, \frac{m_1}{qp} Q_L^{122}(q,p;E)\right]$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(x_3 \, \delta_{P_1 P_3} \, S_{132} + \widetilde{x}_3 \, S_{312}\right)}{\mu_{12} \sqrt{S_{12} S_{13}} \, c_{12} c_{13}} \, \widetilde{f}_{(13)(12)}^{(3)} \int_0^\infty dq \, \frac{q^2 \, T_{13}^{(L)}(E,k,q) \, \frac{m_1}{qp} Q_L^{123}(q,p;E)}{-\gamma_{13} + \sqrt{-2\mu_{13}\left(E - \frac{q^2}{2m_2} - \frac{q^2}{2(m_1+m_3)}\right)} - i\varepsilon}$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(x_4 \, \delta_{P_2 P_3} \, S_{231} + \widetilde{x}_4 \, S_{321}\right)}{\mu_{12} \sqrt{S_{12} S_{23}} \, c_{12} c_{23}} \, \widetilde{f}_{(23)(12)}^{(3)} \int_0^\infty dq \, \frac{q^2 \, T_{23}^{(L)}(E,k,q) \, \frac{m_2}{qp} Q_L^{213}(q,p;E)}{-\gamma_{23} + \sqrt{-2\mu_{23}\left(E - \frac{q^2}{2m_1} - \frac{q^2}{2(m_2+m_3)}\right)} - i\varepsilon}$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{12} S_{12} \sqrt{c_{12} c_{12}'}} \int_0^\infty dq \, \frac{q^2 \, T_{12}'^{(L)}(E,k,q)}{-\gamma_{12}' + \sqrt{-2\mu_{12}\left(E - \frac{q^2}{2m_3} - \frac{q^2}{2(m_1+m_2)}\right)} - i\varepsilon}$$

$$\times \left[x_5 \, \delta_{P_1 P_3}^{(12')} \, f_{(12')(12)}^{(3)} \, S_{123} \, \frac{m_2}{qp} Q_L^{211}(q,p;E) + \widetilde{x}_5 \, \delta_{P_2 P_3}^{(12')} \, \widetilde{f}_{(12')(12)}^{(3)} \, S_{213} \, \frac{m_1}{qp} Q_L^{122}(q,p;E)\right]$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(x_6 \, \delta_{P_1 P_3} \, S_{132} + \widetilde{x}_6 \, S_{312}\right)}{\mu_{12} \sqrt{S_{12} S_{13}} \, c_{12} c_{13}'} \, \widetilde{f}_{(13')(12)}^{(3)} \int_0^\infty dq \, \frac{q^2 \, T_{13}'^{(L)}(E,k,q) \, \frac{m_1}{qp} Q_L^{123}(q,p;E)}{-\gamma_{13}' + \sqrt{-2\mu_{13}\left(E - \frac{q^2}{2m_2} - \frac{q^2}{2(m_1+m_3)}\right)} - i\varepsilon}$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(x_7 \, \delta_{P_2 P_3} \, S_{231} + \widetilde{x}_7 \, S_{321}\right)}{\mu_{12} \sqrt{S_{12} S_{23}} \, c_{12} c_{23}'} \, \widetilde{f}_{(23')(12)}^{(3)} \int_0^\infty dq \, \frac{q^2 \, T_{23}'^{(L)}(E,k,q) \, \frac{m_2}{qp} Q_L^{213}(q,p;E)}{-\gamma_{23}' + \sqrt{-2\mu_{23}\left(E - \frac{q^2}{2m_1} - \frac{q^2}{2(m_2+m_3)}\right)} - i\varepsilon} \tag{F.4}$$

$$T_{13}^{(L)}(E,k,p) =$$

$$(-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{2\pi\,\gamma_{12}\left(y_1\,\delta_{P_1 P_2}\,S_{123} + \widetilde{y}_1\,S_{213}\right)}{\mu_{13}\,\mu_{12}\,\sqrt{S_{13}\,S_{12}\,c_{13}\,c_{12}}}\,\widetilde{f}_{(12)(13)}^{(2)}\,\frac{m_1}{kp}Q_L^{132}(k,p;E)$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi} \frac{\left(y_2\,\delta_{P_1 P_2}\,S_{123} + \widetilde{y}_2\,S_{213}\right)}{\mu_{13}\,\sqrt{S_{13}\,S_{12}\,c_{13}\,c_{12}}}\,\widetilde{f}_{(12)(13)}^{(2)} \int_0^\infty dq \frac{q^2\,T_{12}^{(L)}(E,k,q)\,\frac{m_1}{qp}Q_L^{132}(q,p;E)}{-\gamma_{12} + \sqrt{-2\mu_{12}\left(E - \frac{q^2}{2m_3} - \frac{q^2}{2(m_1+m_2)}\right)} - i\varepsilon}$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{13}\,S_{13}\,c_{13}} \int_0^\infty dq \frac{q^2\,T_{13}^{(L)}(E,k,q)}{-\gamma_{13} + \sqrt{-2\mu_{13}\left(E - \frac{q^2}{2m_2} - \frac{q^2}{2(m_1+m_3)}\right)} - i\varepsilon}$$

$$\times \left[y_3\,\delta_{P_1 P_2}^{(13)}\,f_{(13)(13)}^{(1)}\,S_{132}\,\frac{m_3}{qp}Q_L^{311}(q,p;E) + \widetilde{y}_3\,\delta_{P_2 P_3}^{(13)}\,\widetilde{f}_{(13)(13)}^{(3)}\,S_{312}\,\frac{m_1}{qp}Q_L^{133}(q,p;E) \right]$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(y_4\,S_{231} + \widetilde{y}_4\,\delta_{P_2 P_3}\,S_{321}\right)}{\mu_{13}\,\sqrt{S_{13}\,S_{23}\,c_{13}\,c_{23}}}\,f_{(23)(13)}^{(2)} \int_0^\infty dq \frac{q^2\,T_{23}^{(L)}(E,k,q)\,\frac{m_3}{qp}Q_L^{312}(q,p;E)}{-\gamma_{23} + \sqrt{-2\mu_{23}\left(E - \frac{q^2}{2m_1} - \frac{q^2}{2(m_2+m_3)}\right)} - i\varepsilon}$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi} \frac{\left(y_5\,\delta_{P_1 P_2}\,S_{123} + \widetilde{y}_5\,S_{213}\right)}{\mu_{13}\,\sqrt{S_{13}\,S_{12}\,c_{13}\,c_{12}'}}\,\widetilde{f}_{(12')(13)}^{(2)} \int_0^\infty dq \frac{q^2\,T_{12}'^{(L)}(E,k,q)\,\frac{m_1}{qp}Q_L^{132}(q,p;E)}{-\gamma_{12}' + \sqrt{-2\mu_{12}\left(E - \frac{q^2}{2m_3} - \frac{q^2}{2(m_1+m_2)}\right)} - i\varepsilon}$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{13}\,S_{13}\,\sqrt{c_{13}\,c_{13}'}} \int_0^\infty dq \frac{q^2\,T_{13}'^{(L)}(E,k,q)}{-\gamma_{13}' + \sqrt{-2\mu_{13}\left(E - \frac{q^2}{2m_2} - \frac{q^2}{2(m_1+m_3)}\right)} - i\varepsilon}$$

$$\times \left[y_6\,\delta_{P_1 P_2}^{(13')}\,f_{(13')(13)}^{(1)}\,S_{132}\,\frac{m_3}{qp}Q_L^{311}(q,p;E) + \widetilde{y}_6\,\delta_{P_2 P_3}^{(13')}\,\widetilde{f}_{(13')(13)}^{(3)}\,S_{312}\,\frac{m_1}{qp}Q_L^{133}(q,p;E) \right]$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(y_7\,S_{231} + \widetilde{y}_7\,\delta_{P_2 P_3}\,S_{321}\right)}{\mu_{13}\,\sqrt{S_{13}\,S_{23}\,c_{13}\,c_{23}'}}\,f_{(23')(13)}^{(2)} \int_0^\infty dq \frac{q^2\,T_{23}'^{(L)}(E,k,q)\,\frac{m_3}{qp}Q_L^{312}(q,p;E)}{-\gamma_{23}' + \sqrt{-2\mu_{23}\left(E - \frac{q^2}{2m_1} - \frac{q^2}{2(m_2+m_3)}\right)} - i\varepsilon} \tag{F.5}$$

$$T_{23}^{(L)}(E,k,p) =$$

$$(-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{2\pi\,\gamma_{12}\left(z_1\,S_{123} + \widetilde{z}_1\,\delta_{P_1 P_2}\,S_{213}\right)}{\mu_{23}\,\mu_{12}\,\sqrt{S_{23}\,S_{12}\,c_{23}\,c_{12}}}\,f_{(12)(23)}^{(1)}\,\frac{m_2}{kp}Q_L^{231}(k,p;E)$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi}\frac{\left(z_2\,S_{123} + \widetilde{z}_2\,\delta_{P_1 P_2}\,S_{213}\right)}{\mu_{23}\,\sqrt{S_{23}\,S_{12}\,c_{23}\,c_{12}}}\,f_{(12)(23)}^{(1)}\int_0^\infty dq\,\frac{q^2\,T_{12}^{(L)}(E,k,q)\,\frac{m_2}{qp}Q_L^{231}(q,p;E)}{-\gamma_{12} + \sqrt{-2\mu_{12}\left(E - \frac{q^2}{2m_3} - \frac{q^2}{2(m_1+m_2)}\right)} - i\varepsilon}$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{\pi}\frac{\left(z_3\,S_{132} + \widetilde{z}_3\,\delta_{P_1 P_3}\,S_{312}\right)}{\mu_{23}\,\sqrt{S_{23}\,S_{13}\,c_{23}\,c_{13}}}\,f_{(13)(23)}^{(1)}\int_0^\infty dq\,\frac{q^2\,T_{13}^{(L)}(E,k,q)\,\frac{m_3}{qp}Q_L^{321}(q,p;E)}{-\gamma_{13} + \sqrt{-2\mu_{13}\left(E - \frac{q^2}{2m_2} - \frac{q^2}{2(m_1+m_3)}\right)} - i\varepsilon}$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) \frac{1}{\pi}\frac{1}{\mu_{23}\,S_{23}\,c_{23}}\int_0^\infty dq\,\frac{q^2\,T_{23}^{(L)}(E,k,q)}{-\gamma_{23} + \sqrt{-2\mu_{23}\left(E - \frac{q^2}{2m_1} - \frac{q^2}{2(m_2+m_3)}\right)} - i\varepsilon}$$

$$\times \left[z_4\,\delta_{P_1 P_2}^{(23)}\,f_{(23)(23)}^{(2)}\,S_{231}\,\frac{m_3}{qp}Q_L^{322}(q,p;E) + \widetilde{z}_4\,\delta_{P_1 P_3}^{(23)}\,\widetilde{f}_{(23)(23)}^{(3)}\,S_{321}\,\frac{m_2}{qp}Q_L^{233}(q,p;E)\right]$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi}\frac{\left(z_5\,S_{123} + \widetilde{z}_5\,\delta_{P_1 P_2}\,S_{213}\right)}{\mu_{23}\,\sqrt{S_{23}\,S_{12}\,c_{23}\,c_{12}'}}\,f_{(12')(23)}^{(1)}\int_0^\infty dq\,\frac{q^2\,T_{12}'^{(L)}(E,k,q)\,\frac{m_2}{qp}Q_L^{231}(q,p;E)}{-\gamma_{12}' + \sqrt{-2\mu_{12}\left(E - \frac{q^2}{2m_3} - \frac{q^2}{2(m_1+m_2)}\right)} - i\varepsilon}$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{\pi}\frac{\left(z_6\,S_{132} + \widetilde{z}_6\,\delta_{P_1 P_3}\,S_{312}\right)}{\mu_{23}\,\sqrt{S_{23}\,S_{13}\,c_{23}\,c_{13}'}}\,f_{(13')(23)}^{(1)}\int_0^\infty dq\,\frac{q^2\,T_{13}'^{(L)}(E,k,q)\,\frac{m_3}{qp}Q_L^{321}(q,p;E)}{-\gamma_{13}' + \sqrt{-2\mu_{13}\left(E - \frac{q^2}{2m_2} - \frac{q^2}{2(m_1+m_3)}\right)} - i\varepsilon}$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) \frac{1}{\pi}\frac{1}{\mu_{23}\,S_{23}\,\sqrt{c_{23}\,c_{23}'}}\int_0^\infty dq\,\frac{q^2\,T_{23}'^{(L)}(E,k,q)}{-\gamma_{23}' + \sqrt{-2\mu_{23}\left(E - \frac{q^2}{2m_1} - \frac{q^2}{2(m_2+m_3)}\right)} - i\varepsilon}$$

$$\times \left[z_7\,\delta_{P_1 P_2}^{(23')}\,f_{(23')(23)}^{(2)}\,S_{231}\,\frac{m_3}{qp}Q_L^{322}(q,p;E) + \widetilde{z}_7\,\delta_{P_1 P_3}^{(23')}\,\widetilde{f}_{(23')(23)}^{(3)}\,S_{321}\,\frac{m_2}{qp}Q_L^{233}(q,p;E)\right] \tag{F.6}$$

$$T'^{(L)}_{12}(E,k,p) =$$

$$(-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{2\pi\,\gamma_{12}}{\mu_{12}^2\, S_{12}\,\sqrt{c'_{12}\,c_{12}}} \left[x'_1\, \delta^{(12)}_{P_1 P_3}\, f^{(3)}_{(12)(12')}\, S_{123}\, \frac{m_2}{kp} Q^{211}_L(k,p;E) + \widetilde{x}'_1\, \delta^{(12)}_{P_2 P_3}\, \widetilde{f}^{(3)}_{(12)(12')}\, S_{213}\, \frac{m_1}{kp} Q^{122}_L(k,p;E) \right]$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{12}\, S_{12}\, \sqrt{c'_{12}\,c_{12}}} \int_0^\infty dq\, \frac{q^2\, T^{(L)}_{12}(E,k,q)}{-\gamma_{12} + \sqrt{-2\mu_{12}\left(E - \frac{q^2}{2m_3} - \frac{q^2}{2(m_1+m_2)}\right)} - i\varepsilon}$$

$$\times \left[x'_2\, \delta^{(12)}_{P_1 P_3}\, f^{(3)}_{(12)(12')}\, S_{123}\, \frac{m_2}{qp} Q^{211}_L(q,p;E) + \widetilde{x}'_2\, \delta^{(12)}_{P_2 P_3}\, \widetilde{f}^{(3)}_{(12)(12')}\, S_{213}\, \frac{m_1}{qp} Q^{122}_L(q,p;E) \right]$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(x'_3\, \delta_{P_1 P_3}\, S_{132} + \widetilde{x}'_3\, S_{312}\right)}{\mu_{12}\, \sqrt{S_{12}\, S_{13}\, c'_{12}\, c_{13}}}\, \widetilde{f}^{(3)}_{(13)(12')} \int_0^\infty dq\, \frac{q^2\, T^{(L)}_{13}(E,k,q)\, \frac{m_1}{qp} Q^{123}_L(q,p;E)}{-\gamma_{13} + \sqrt{-2\mu_{13}\left(E - \frac{q^2}{2m_2} - \frac{q^2}{2(m_1+m_3)}\right)} - i\varepsilon}$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(x'_4\, \delta_{P_2 P_3}\, S_{231} + \widetilde{x}'_4\, S_{321}\right)}{\mu_{12}\, \sqrt{S_{12}\, S_{23}\, c'_{12}\, c_{23}}}\, \widetilde{f}^{(3)}_{(23)(12')} \int_0^\infty dq\, \frac{q^2\, T^{(L)}_{23}(E,k,q)\, \frac{m_2}{qp} Q^{213}_L(q,p;E)}{-\gamma_{23} + \sqrt{-2\mu_{23}\left(E - \frac{q^2}{2m_1} - \frac{q^2}{2(m_2+m_3)}\right)} - i\varepsilon}$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi\, \mu_{12}\, S_{12}\, c'_{12}} \int_0^\infty dq\, \frac{q^2\, T'^{(L)}_{12}(E,k,q)}{-\gamma'_{12} + \sqrt{-2\mu_{12}\left(E - \frac{q^2}{2m_3} - \frac{q^2}{2(m_1+m_2)}\right)} - i\varepsilon}$$

$$\times \left[x'_5\, \delta^{(12')}_{P_1 P_3}\, f^{(3)}_{(12')(23')}\, S_{123}\, \frac{m_2}{qp} Q^{211}_L(q,p;E) + \widetilde{x}'_5\, \delta^{(12')}_{P_2 P_3}\, \widetilde{f}^{(3)}_{(12')(12')}\, S_{213}\, \frac{m_1}{qp} Q^{122}_L(q,p;E) \right]$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(x'_6\, \delta_{P_1 P_3}\, S_{132} + \widetilde{x}'_6\, S_{312}\right)}{\mu_{12}\, \sqrt{S_{12}\, S_{13}\, c'_{12}\, c'_{13}}}\, \widetilde{f}^{(3)}_{(13')(12')} \int_0^\infty dq\, \frac{q^2\, T'^{(L)}_{13}(E,k,q)\, \frac{m_1}{qp} Q^{123}_L(q,p;E)}{-\gamma'_{13} + \sqrt{-2\mu_{13}\left(E - \frac{q^2}{2m_2} - \frac{q^2}{2(m_1+m_3)}\right)} - i\varepsilon}$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(x'_7\, \delta_{P_2 P_3}\, S_{231} + \widetilde{x}'_7\, S_{321}\right)}{\mu_{12}\, \sqrt{S_{12}\, S_{23}\, c'_{12}\, c'_{23}}}\, \widetilde{f}^{(3)}_{(23')(12')} \int_0^\infty dq\, \frac{q^2\, T'^{(L)}_{23}(E,k,q)\, \frac{m_2}{qp} Q^{213}_L(q,p;E)}{-\gamma'_{23} + \sqrt{-2\mu_{23}\left(E - \frac{q^2}{2m_1} - \frac{q^2}{2(m_2+m_3)}\right)} - i\varepsilon} \tag{F.7}$$

$$T_{13}^{\prime(L)}(E,k,p) =$$

$$(-1)^L \left(1 - \frac{\delta_{P_1P_2}}{2}\right) \frac{2\pi\,\gamma_{12}\left(y_1'\,\delta_{P_1P_2}\,S_{123} + \widetilde{y}_1'\,S_{213}\right)}{\mu_{13}\,\mu_{12}\,\sqrt{S_{13}\,S_{12}\,c_{13}'\,c_{12}}}\,\widetilde{f}_{(12)(13')}^{(2)}\,\frac{m_1}{kp}Q_L^{132}(k,p;E)$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1P_2}}{2}\right) \frac{1}{\pi} \frac{\left(y_2'\,\delta_{P_1P_2}\,S_{123} + \widetilde{y}_2'\,S_{213}\right)}{\mu_{13}\,\sqrt{S_{13}\,S_{12}\,c_{13}'\,c_{12}}}\,\widetilde{f}_{(12)(13')}^{(2)}\int_0^\infty dq \frac{q^2\,T_{12}^{(L)}(E,k,q)\,\frac{m_1}{qp}Q_L^{132}(q,p;E)}{-\gamma_{12} + \sqrt{-2\mu_{12}\left(E - \frac{q^2}{2m_3} - \frac{q^2}{2(m_1+m_2)}\right)} - i\varepsilon}$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1P_3}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{13}\,S_{13}\,\sqrt{c_{13}'\,c_{13}}}\int_0^\infty dq \frac{q^2\,T_{13}^{(L)}(E,k,q)}{-\gamma_{13} + \sqrt{-2\mu_{13}\left(E - \frac{q^2}{2m_2} - \frac{q^2}{2(m_1+m_3)}\right)} - i\varepsilon}$$

$$\times \left[y_3'\,\delta_{P_1P_2}^{(13)}\,f_{(13)(13')}^{(1)}\,S_{132}\,\frac{m_3}{qp}Q_L^{311}(q,p;E) + \widetilde{y}_3'\,\delta_{P_2P_3}^{(13)}\,\widetilde{f}_{(13)(13')}^{(3)}\,S_{312}\,\frac{m_1}{qp}Q_L^{133}(q,p;E) \right]$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_2P_3}}{2}\right) \frac{1}{\pi} \frac{\left(y_4'\,S_{231} + \widetilde{y}_4'\,\delta_{P_2P_3}\,S_{321}\right)}{\mu_{13}\,\sqrt{S_{13}\,S_{23}\,c_{13}'\,c_{23}}}\,f_{(23)(13')}^{(2)}\int_0^\infty dq \frac{q^2\,T_{23}^{(L)}(E,k,q)\,\frac{m_3}{qp}Q_L^{312}(q,p;E)}{-\gamma_{23} + \sqrt{-2\mu_{23}\left(E - \frac{q^2}{2m_1} - \frac{q^2}{2(m_2+m_3)}\right)} - i\varepsilon}$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1P_2}}{2}\right) \frac{1}{\pi} \frac{\left(y_5'\,\delta_{P_1P_2}\,S_{123} + \widetilde{y}_5'\,S_{213}\right)}{\mu_{13}\,\sqrt{S_{13}\,S_{12}\,c_{13}'\,c_{12}'}}\,\widetilde{f}_{(12')(13')}^{(2)}\int_0^\infty dq \frac{q^2\,T_{12}^{\prime(L)}(E,k,q)\,\frac{m_1}{qp}Q_L^{132}(q,p;E)}{-\gamma_{12}' + \sqrt{-2\mu_{12}\left(E - \frac{q^2}{2m_3} - \frac{q^2}{2(m_1+m_2)}\right)} - i\varepsilon}$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1P_3}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{13}\,S_{13}\,c_{13}'}\int_0^\infty dq \frac{q^2\,T_{13}^{\prime(L)}(E,k,q)}{-\gamma_{13}' + \sqrt{-2\mu_{13}\left(E - \frac{q^2}{2m_2} - \frac{q^2}{2(m_1+m_3)}\right)} - i\varepsilon}$$

$$\times \left[y_6'\,\delta_{P_1P_2}^{(13')}\,f_{(13')(13')}^{(1)}\,S_{132}\,\frac{m_3}{qp}Q_L^{311}(q,p;E) + \widetilde{y}_6'\,\delta_{P_2P_3}^{(13')}\,\widetilde{f}_{(13')(13')}^{(3)}\,S_{312}\,\frac{m_1}{qp}Q_L^{133}(q,p;E) \right]$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_2P_3}}{2}\right) \frac{1}{\pi} \frac{\left(y_7'\,S_{231} + \widetilde{y}_7'\,\delta_{P_2P_3}\,S_{321}\right)}{\mu_{13}\,\sqrt{S_{13}\,S_{23}\,c_{13}'\,c_{23}'}}\,f_{(23')(13')}^{(2)}\int_0^\infty dq \frac{q^2\,T_{23}^{\prime(L)}(E,k,q)\,\frac{m_3}{qp}Q_L^{312}(q,p;E)}{-\gamma_{23}' + \sqrt{-2\mu_{23}\left(E - \frac{q^2}{2m_1} - \frac{q^2}{2(m_2+m_3)}\right)} - i\varepsilon} \qquad \text{(F.8)}$$

$$T_{23}^{\prime(L)}(E,k,p) =$$

$$(-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{2\pi\,\gamma_{12}\left(z_1' \, S_{123} + \widetilde{z}_1' \, \delta_{P_1 P_2} \, S_{213}\right)}{\mu_{23}\,\mu_{12}\,\sqrt{S_{23}\,S_{12}}\,c_{23}'\,c_{12}}\, f_{(12)(23')}^{(1)}\, \frac{m_2}{kp} Q_L^{231}(k,p;E)$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi} \frac{\left(z_2' \, S_{123} + \widetilde{z}_2' \, \delta_{P_1 P_2}\, S_{213}\right)}{\mu_{23}\,\sqrt{S_{23}\,S_{12}}\,c_{23}'\,c_{12}}\, f_{(12)(23')}^{(1)} \int_0^\infty dq\, \frac{q^2\,T_{12}^{(L)}(E,k,q)\,\frac{m_2}{qp}Q_L^{231}(q,p;E)}{-\gamma_{12} + \sqrt{-2\mu_{12}\left(E - \frac{q^2}{2m_3} - \frac{q^2}{2(m_1+m_2)}\right)} - i\varepsilon}$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(z_3' \, S_{132} + \widetilde{z}_3' \, \delta_{P_1 P_3}\, S_{312}\right)}{\mu_{23}\,\sqrt{S_{23}\,S_{13}}\,c_{23}'\,c_{13}}\, f_{(13)(23')}^{(1)} \int_0^\infty dq\, \frac{q^2\,T_{13}^{(L)}(E,k,q)\,\frac{m_3}{qp}Q_L^{321}(q,p;E)}{-\gamma_{13} + \sqrt{-2\mu_{13}\left(E - \frac{q^2}{2m_2} - \frac{q^2}{2(m_1+m_3)}\right)} - i\varepsilon}$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{23}\,S_{23}\,\sqrt{c_{23}'\,c_{23}}} \int_0^\infty dq\, \frac{q^2\,T_{23}^{(L)}(E,k,q)}{-\gamma_{23} + \sqrt{-2\mu_{23}\left(E - \frac{q^2}{2m_1} - \frac{q^2}{2(m_2+m_3)}\right)} - i\varepsilon}$$

$$\times \left[z_4' \, \delta_{P_1 P_2}^{(23)} \, f_{(23)(23')}^{(2)} \, S_{231}\, \frac{m_3}{qp}Q_L^{322}(q,p;E) + \widetilde{z}_4' \, \delta_{P_1 P_3}^{(23)} \, \widetilde{f}_{(23)(23')}^{(3)} \, S_{321}\, \frac{m_2}{qp}Q_L^{233}(q,p;E) \right]$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi} \frac{\left(z_5' \, S_{123} + \widetilde{z}_5' \, \delta_{P_1 P_2}\, S_{213}\right)}{\mu_{23}\,\sqrt{S_{23}\,S_{12}}\,c_{23}'\,c_{12}'}\, f_{(12')(23')}^{(1)} \int_0^\infty dq\, \frac{q^2\,T_{12}^{\prime(L)}(E,k,q)\,\frac{m_2}{qp}Q_L^{231}(q,p;E)}{-\gamma_{12}' + \sqrt{-2\mu_{12}\left(E - \frac{q^2}{2m_3} - \frac{q^2}{2(m_1+m_2)}\right)} - i\varepsilon}$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(z_6' \, S_{132} + \widetilde{z}_6' \, \delta_{P_1 P_3}\, S_{312}\right)}{\mu_{23}\,\sqrt{S_{23}\,S_{13}}\,c_{23}'\,c_{13}'}\, f_{(13')(23')}^{(1)} \int_0^\infty dq\, \frac{q^2\,T_{13}^{\prime(L)}(E,k,q)\,\frac{m_3}{qp}Q_L^{321}(q,p;E)}{-\gamma_{13}' + \sqrt{-2\mu_{13}\left(E - \frac{q^2}{2m_2} - \frac{q^2}{2(m_1+m_3)}\right)} - i\varepsilon}$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{23}\,S_{23}\,c_{23}'} \int_0^\infty dq\, \frac{q^2\,T_{23}^{\prime(L)}(E,k,q)}{-\gamma_{23}' + \sqrt{-2\mu_{23}\left(E - \frac{q^2}{2m_1} - \frac{q^2}{2(m_2+m_3)}\right)} - i\varepsilon}$$

$$\times \left[z_7' \, \delta_{P_1 P_2}^{(23')} \, f_{(23')(23')}^{(2)} \, S_{231}\, \frac{m_3}{qp}Q_L^{322}(q,p;E) + \widetilde{z}_7' \, \delta_{P_1 P_3}^{(23')} \, \widetilde{f}_{(23')(23')}^{(3)} \, S_{321}\, \frac{m_2}{qp}Q_L^{233}(q,p;E) \right] \tag{F.9}$$

Three-body scattering amplitudes in the limit of asymptotic large off-shell momenta (see section 3.4).

$$
\widetilde{T}_{12}^{(L)}(p) = (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{12} \, S_{12} \, c_{12}} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{12}^{(L)}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)3}}}}
$$

$$
\times \left[x_2 \, \delta_{P_1 P_3}^{(12)} \, f_{(12)(12)}^{(3)} \, S_{123} \, m_2 Q_L^{211}(q,p) + \widetilde{x}_2 \, \delta_{P_2 P_3}^{(12)} \, \widetilde{f}_{(12)(12)}^{(3)} \, S_{213} \, m_1 Q_L^{122}(q,p) \right]
$$

$$
+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(x_3 \, \delta_{P_1 P_3} \, S_{132} + \widetilde{x}_3 \, S_{312}\right)}{\mu_{12} \, \sqrt{S_{12} \, S_{13} \, c_{12} \, c_{13}}} \, \widetilde{f}_{(13)(12)}^{(3)} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{13}^{(L)}(q)}{\sqrt{\frac{\mu_{13}}{\mu_{(13)2}}}} \, m_1 Q_L^{123}(q,p)
$$

$$
+ (-1)^L \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(x_4 \, \delta_{P_2 P_3} \, S_{231} + \widetilde{x}_4 \, S_{321}\right)}{\mu_{12} \, \sqrt{S_{12} \, S_{23} \, c_{12} \, c_{23}}} \, \widetilde{f}_{(23)(12)}^{(3)} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{23}^{(L)}(q)}{\sqrt{\frac{\mu_{23}}{\mu_{(23)1}}}} \, m_2 Q_L^{213}(q,p)
$$

$$
+ (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{12} \, S_{12} \, \sqrt{c_{12} \, c'_{12}}} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{12}^{\prime (L)}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)3}}}}
$$

$$
\times \left[x_5 \, \delta_{P_1 P_3}^{(12')} \, f_{(12')(12)}^{(3)} \, S_{123} \, m_2 Q_L^{211}(q,p) + \widetilde{x}_5 \, \delta_{P_2 P_3}^{(12')} \, \widetilde{f}_{(12')(12)}^{(3)} \, S_{213} \, m_1 Q_L^{122}(q,p) \right]
$$

$$
+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(x_6 \, \delta_{P_1 P_3} \, S_{132} + \widetilde{x}_6 \, S_{312}\right)}{\mu_{12} \, \sqrt{S_{12} \, S_{13} \, c_{12} \, c'_{13}}} \, \widetilde{f}_{(13')(12)}^{(3)} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{13}^{\prime (L)}(q)}{\sqrt{\frac{\mu_{13}}{\mu_{(13)2}}}} \, m_1 Q_L^{123}(q,p)
$$

$$
+ (-1)^L \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(x_7 \, \delta_{P_2 P_3} \, S_{231} + \widetilde{x}_7 \, S_{321}\right)}{\mu_{12} \, \sqrt{S_{12} \, S_{23} \, c_{12} \, c'_{23}}} \, \widetilde{f}_{(23')(12)}^{(3)} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{23}^{\prime (L)}(q)}{\sqrt{\frac{\mu_{23}}{\mu_{(23)1}}}} \, m_2 Q_L^{213}(q,p) \tag{F.10}
$$

$$
\begin{aligned}
\widetilde{T}_{13}^{(L)}(p) = {} & (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi} \frac{\left(y_2\, \delta_{P_1 P_2}\, S_{123} + \widetilde{y}_2\, S_{213}\right)}{\mu_{13}\, \sqrt{S_{13}\, S_{12}\, c_{13}\, c_{12}}}\, \widetilde{f}_{(12)(13)}^{(2)} \int_0^{\Lambda_C} \frac{dq}{q}\, \frac{\widetilde{T}_{12}^{(L)}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)3}}}}\, m_1 Q_L^{132}(q,p) \\[2mm]
& + (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{13}\, S_{13}\, c_{13}} \int_0^{\Lambda_C} \frac{dq}{q}\, \frac{\widetilde{T}_{13}^{(L)}(q)}{\sqrt{\frac{\mu_{13}}{\mu_{(13)2}}}} \\[2mm]
& \quad \times \left[y_3\, \delta_{P_1 P_2}^{(13)}\, f_{(13)(13)}^{(1)}\, S_{132}\, m_3 Q_L^{311}(q,p) + \widetilde{y}_3\, \delta_{P_2 P_3}^{(13)}\, \widetilde{f}_{(13)(13)}^{(3)}\, S_{312}\, m_1 Q_L^{133}(q,p) \right] \\[2mm]
& + (-1)^L \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(y_4\, S_{231} + \widetilde{y}_4\, \delta_{P_2 P_3}\, S_{321}\right)}{\mu_{13}\, \sqrt{S_{13}\, S_{23}\, c_{13}\, c_{23}}}\, f_{(23)(13)}^{(2)} \int_0^{\Lambda_C} \frac{dq}{q}\, \frac{\widetilde{T}_{23}^{(L)}(q)}{\sqrt{\frac{\mu_{23}}{\mu_{(23)1}}}}\, m_3 Q_L^{312}(q,p) \\[2mm]
& + (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi} \frac{\left(y_5\, \delta_{P_1 P_2}\, S_{123} + \widetilde{y}_5\, S_{213}\right)}{\mu_{13}\, \sqrt{S_{13}\, S_{12}\, c_{13}\, c'_{12}}}\, \widetilde{f}_{(12')(13)}^{(2)} \int_0^{\Lambda_C} \frac{dq}{q}\, \frac{\widetilde{T}_{12}^{\prime(L)}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)3}}}}\, m_1 Q_L^{132}(q,p) \\[2mm]
& + (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{13}\, S_{13}\, \sqrt{c_{13}\, c'_{13}}} \int_0^{\Lambda_C} \frac{dq}{q}\, \frac{\widetilde{T}_{13}^{\prime(L)}(q)}{\sqrt{\frac{\mu_{13}}{\mu_{(13)2}}}} \\[2mm]
& \quad \times \left[y_6\, \delta_{P_1 P_2}^{(13')}\, f_{(13')(13)}^{(1)}\, S_{132}\, m_3 Q_L^{311}(q,p) + \widetilde{y}_6\, \delta_{P_2 P_3}^{(13')}\, \widetilde{f}_{(13')(13)}^{(3)}\, S_{312}\, m_1 Q_L^{133}(q,p) \right] \\[2mm]
& + (-1)^L \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(y_7\, S_{231} + \widetilde{y}_7\, \delta_{P_2 P_3}\, S_{321}\right)}{\mu_{13}\, \sqrt{S_{13}\, S_{23}\, c_{13}\, c'_{23}}}\, f_{(23')(13)}^{(2)} \int_0^{\Lambda_C} \frac{dq}{q}\, \frac{\widetilde{T}_{23}^{\prime(L)}(q)}{\sqrt{\frac{\mu_{23}}{\mu_{(23)1}}}}\, m_3 Q_L^{312}(q,p) \qquad \text{(F.11)}
\end{aligned}
$$

$$\widetilde{T}_{23}^{(L)}(p) = (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi} \frac{\left(z_2 \, S_{123} + \widetilde{z}_2 \, \delta_{P_1 P_2} \, S_{213}\right)}{\mu_{23} \, \sqrt{S_{23} \, S_{12} \, c_{23} \, c_{12}}} \, f_{(12)(23)}^{(1)} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{12}^{(L)}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)3}}}} \, m_2 Q_L^{231}(q,p)$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(z_3 \, S_{132} + \widetilde{z}_3 \, \delta_{P_1 P_3} \, S_{312}\right)}{\mu_{23} \, \sqrt{S_{23} \, S_{13} \, c_{23} \, c_{13}}} \, f_{(13)(23)}^{(1)} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{13}^{(L)}(q)}{\sqrt{\frac{\mu_{13}}{\mu_{(13)2}}}} \, m_3 Q_L^{321}(q,p)$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{23} \, S_{23} \, c_{23}} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{23}^{(L)}(q)}{\sqrt{\frac{\mu_{23}}{\mu_{(23)1}}}}$$

$$\times \left[z_4 \, \delta_{P_1 P_2}^{(23)} \, f_{(23)(23)}^{(2)} \, S_{231} \, m_3 Q_L^{322}(q,p) + \widetilde{z}_4 \, \delta_{P_1 P_3}^{(23)} \, \widetilde{f}_{(23)(23)}^{(3)} \, S_{321} \, m_2 Q_L^{233}(q,p) \right]$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi} \frac{\left(z_5 \, S_{123} + \widetilde{z}_5 \, \delta_{P_1 P_2} \, S_{213}\right)}{\mu_{23} \, \sqrt{S_{23} \, S_{12} \, c_{23} \, c_{12}'}} \, f_{(12')(23)}^{(1)} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{12}'^{(L)}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)3}}}} \, m_2 Q_L^{231}(q,p)$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(z_6 \, S_{132} + \widetilde{z}_6 \, \delta_{P_1 P_3} \, S_{312}\right)}{\mu_{23} \, \sqrt{S_{23} \, S_{13} \, c_{23} \, c_{13}'}} \, f_{(13')(23)}^{(1)} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{13}'^{(L)}(q)}{\sqrt{\frac{\mu_{13}}{\mu_{(13)2}}}} \, m_3 Q_L^{321}(q,p)$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{23} \, S_{23} \, \sqrt{c_{23} \, c_{23}'}} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{23}'^{(L)}(q)}{\sqrt{\frac{\mu_{23}}{\mu_{(23)1}}}}$$

$$\times \left[z_7 \, \delta_{P_1 P_2}^{(23')} \, f_{(23')(23)}^{(2)} \, S_{231} \, m_3 Q_L^{322}(q,p) + \widetilde{z}_7 \, \delta_{P_1 P_3}^{(23')} \, \widetilde{f}_{(23')(23)}^{(3)} \, S_{321} \, m_2 Q_L^{233}(q,p) \right] \tag{F.12}$$

$$
\widetilde{T}_{12}^{\prime(L)}(p) = (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{12} \, S_{12} \, \sqrt{c'_{12} \, c_{12}}} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{12}^{(L)}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)3}}}}
$$

$$
\times \left[x'_2 \, \delta_{P_1 P_3}^{(12)} \, f_{(12)(12')}^{(3)} \, S_{123} \, m_2 Q_L^{211}(q,p) + \widetilde{x}'_2 \, \delta_{P_2 P_3}^{(12)} \, \widetilde{f}_{(12)(12')}^{(3)} \, S_{213} \, m_1 Q_L^{122}(q,p) \right]
$$

$$
+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(x'_3 \, \delta_{P_1 P_3} \, S_{132} + \widetilde{x}'_3 \, S_{312}\right)}{\mu_{12} \, \sqrt{S_{12} \, S_{13} \, c'_{12} \, c_{13}}} \, \widetilde{f}_{(13)(12')}^{(3)} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{13}^{(L)}(q)}{\sqrt{\frac{\mu_{13}}{\mu_{(13)2}}}} \, m_1 Q_L^{123}(q,p)
$$

$$
+ (-1)^L \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(x'_4 \, \delta_{P_2 P_3} \, S_{231} + \widetilde{x}'_4 \, S_{321}\right)}{\mu_{12} \, \sqrt{S_{12} \, S_{23} \, c'_{12} \, c_{23}}} \, \widetilde{f}_{(23)(12')}^{(3)} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{23}^{(L)}(q)}{\sqrt{\frac{\mu_{23}}{\mu_{(23)1}}}} \, m_2 Q_L^{213}(q,p)
$$

$$
+ (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{12} \, S_{12} \, c'_{12}} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{12}^{\prime(L)}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)3}}}}
$$

$$
\times \left[x'_5 \, \delta_{P_1 P_3}^{(12')} \, f_{(12')(23')}^{(3)} \, S_{123} \, m_2 Q_L^{211}(q,p) + \widetilde{x}'_5 \, \delta_{P_2 P_3}^{(12')} \, \widetilde{f}_{(12')(12')}^{(3)} \, S_{213} \, m_1 Q_L^{122}(q,p) \right]
$$

$$
+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(x'_6 \, \delta_{P_1 P_3} \, S_{132} + \widetilde{x}'_6 \, S_{312}\right)}{\mu_{12} \, \sqrt{S_{12} \, S_{13} \, c'_{12} \, c'_{13}}} \, \widetilde{f}_{(13')(12')}^{(3)} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{13}^{\prime(L)}(q)}{\sqrt{\frac{\mu_{13}}{\mu_{(13)2}}}} \, m_1 Q_L^{123}(q,p)
$$

$$
+ (-1)^L \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(x'_7 \, \delta_{P_2 P_3} \, S_{231} + \widetilde{x}'_7 \, S_{321}\right)}{\mu_{12} \, \sqrt{S_{12} \, S_{23} \, c'_{12} \, c'_{23}}} \, \widetilde{f}_{(23')(12')}^{(3)} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{23}^{\prime(L)}(q)}{\sqrt{\frac{\mu_{23}}{\mu_{(23)1}}}} \, m_2 Q_L^{213}(q,p) \tag{F.13}
$$

$$\widetilde{T}'^{(L)}_{13}(p) = (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi} \frac{\left(y'_2 \, \delta_{P_1 P_2} \, S_{123} + \widetilde{y}'_2 \, S_{213}\right)}{\mu_{13} \, \sqrt{S_{13} \, S_{12} \, c'_{13} \, c_{12}}} \, \widetilde{f}^{(2)}_{(12)(13')} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}^{(L)}_{12}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)3}}}} \, m_1 Q_L^{132}(q,p)$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{13} \, S_{13} \, \sqrt{c'_{13} \, c_{13}}} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}^{(L)}_{13}(q)}{\sqrt{\frac{\mu_{13}}{\mu_{(13)2}}}}$$

$$\times \left[y'_3 \, \delta^{(13)}_{P_1 P_2} \, f^{(1)}_{(13)(13')} \, S_{132} \, m_3 Q_L^{311}(q,p) + \widetilde{y}'_3 \, \delta^{(13)}_{P_2 P_3} \, \widetilde{f}^{(3)}_{(13)(13')} \, S_{312} \, m_1 Q_L^{133}(q,p) \right]$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(y'_4 \, S_{231} + \widetilde{y}'_4 \, \delta_{P_2 P_3} \, S_{321}\right)}{\mu_{13} \, \sqrt{S_{13} \, S_{23} \, c'_{13} \, c_{23}}} \, f^{(2)}_{(23)(13')} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}^{(L)}_{23}(q)}{\sqrt{\frac{\mu_{23}}{\mu_{(23)1}}}} \, m_3 Q_L^{312}(q,p)$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi} \frac{\left(y'_5 \, \delta_{P_1 P_2} \, S_{123} + \widetilde{y}'_5 \, S_{213}\right)}{\mu_{13} \, \sqrt{S_{13} \, S_{12} \, c'_{13} \, c'_{12}}} \, \widetilde{f}^{(2)}_{(12')(13')} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}'^{(L)}_{12}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)3}}}} \, m_1 Q_L^{132}(q,p)$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{13} \, S_{13} \, c'_{13}} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}'^{(L)}_{13}(q)}{\sqrt{\frac{\mu_{13}}{\mu_{(13)2}}}}$$

$$\times \left[y'_6 \, \delta^{(13')}_{P_1 P_2} \, f^{(1)}_{(13')(13')} \, S_{132} \, m_3 Q_L^{311}(q,p) + \widetilde{y}'_6 \, \delta^{(13')}_{P_2 P_3} \, \widetilde{f}^{(3)}_{(13')(13')} \, S_{312} \, m_1 Q_L^{133}(q,p) \right]$$

$$+ (-1)^L \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(y'_7 \, S_{231} + \widetilde{y}'_7 \, \delta_{P_2 P_3} \, S_{321}\right)}{\mu_{13} \, \sqrt{S_{13} \, S_{23} \, c'_{13} \, c'_{23}}} \, f^{(2)}_{(23')(13')} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}'^{(L)}_{23}(q)}{\sqrt{\frac{\mu_{23}}{\mu_{(23)1}}}} \, m_3 Q_L^{312}(q,p) \qquad \text{(F.14)}$$

$$
\widetilde{T}_{23}^{\prime(L)}(p) = (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi} \frac{\left(z_2' \, S_{123} + \widetilde{z}_2 \, \delta_{P_1 P_2} \, S_{213}\right)}{\mu_{23} \, \sqrt{S_{23} \, S_{12} \, c_{23}' \, c_{12}}} f_{(12)(23')}^{(1)} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{12}^{(L)}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)3}}}} \, m_2 Q_L^{231}(q,p)
$$

$$
+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(z_3' \, S_{132} + \widetilde{z}_3' \, \delta_{P_1 P_3} \, S_{312}\right)}{\mu_{23} \, \sqrt{S_{23} \, S_{13} \, c_{23}' \, c_{13}}} f_{(13)(23')}^{(1)} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{13}^{(L)}(q)}{\sqrt{\frac{\mu_{13}}{\mu_{(13)2}}}} \, m_3 Q_L^{321}(q,p)
$$

$$
+ (-1)^L \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{23} \, S_{23} \, \sqrt{c_{23}' \, c_{23}}} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{23}^{(L)}(q)}{\sqrt{\frac{\mu_{23}}{\mu_{(23)1}}}}
$$

$$
\times \left[z_4' \, \delta_{P_1 P_2}^{(23)} \, f_{(23)(23')}^{(2)} \, S_{231} \, m_3 Q_L^{322}(q,p) + \widetilde{z}_4 \, \delta_{P_1 P_3}^{(23)} \, \widetilde{f}_{(23)(23')}^{(3)} \, S_{321} \, m_2 Q_L^{233}(q,p) \right]
$$

$$
+ (-1)^L \left(1 - \frac{\delta_{P_1 P_2}}{2}\right) \frac{1}{\pi} \frac{\left(z_5' \, S_{123} + \widetilde{z}_5 \, \delta_{P_1 P_2} \, S_{213}\right)}{\mu_{23} \, \sqrt{S_{23} \, S_{12} \, c_{23}' \, c_{12}'}} f_{(12')(23')}^{(1)} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{12}^{\prime(L)}(q)}{\sqrt{\frac{\mu_{12}}{\mu_{(12)3}}}} \, m_2 Q_L^{231}(q,p)
$$

$$
+ (-1)^L \left(1 - \frac{\delta_{P_1 P_3}}{2}\right) \frac{1}{\pi} \frac{\left(z_6' \, S_{132} + \widetilde{z}_6 \, \delta_{P_1 P_3} \, S_{312}\right)}{\mu_{23} \, \sqrt{S_{23} \, S_{13} \, c_{23}' \, c_{13}'}} f_{(13')(23')}^{(1)} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{13}^{\prime(L)}(q)}{\sqrt{\frac{\mu_{13}}{\mu_{(13)2}}}} \, m_3 Q_L^{321}(q,p)
$$

$$
+ (-1)^L \left(1 - \frac{\delta_{P_2 P_3}}{2}\right) \frac{1}{\pi} \frac{1}{\mu_{23} \, S_{23} \, c_{23}'} \int_0^{\Lambda_C} \frac{dq}{q} \frac{\widetilde{T}_{23}^{\prime(L)}(q)}{\sqrt{\frac{\mu_{23}}{\mu_{(23)1}}}}
$$

$$
\times \left[z_7' \, \delta_{P_1 P_2}^{(23')} \, f_{(23')(23')}^{(2)} \, S_{231} \, m_3 Q_L^{322}(q,p) + \widetilde{z}_7 \, \delta_{P_1 P_3}^{(23')} \, \widetilde{f}_{(23')(23')}^{(3)} \, S_{321} \, m_2 Q_L^{233}(q,p) \right] \tag{F.15}
$$

Appendix G

Numerical implementation of scattering amplitudes

In section 5.1 we have derived the elastic S-wave three-body scattering amplitudes for all possible $Z_b^{(\prime)}$–$B^{(*)}$ scattering processes. In this appendix it will be discussed how one can solve them numerically in order to extract the three-body observables scattering length and phase shift.

Comparing all scattering processes and their corresponding channels one concludes that all integral equations have a similar structure. However, there are some differences: on the one hand they differ in the prefactors determined by the spin and isospin projections. On the other hand there is a difference due to the different masses and two-body binding momenta of the scattered particles which affects the momentum independent prefactors as well as the integrand. However, the numerical treatment of the (coupled) integral equations is universal in the sense that the E, k, p and q dependence of the amplitudes is always of the same form. The numerical implementation used to solve these equations follows the method described in Ref. [79].

Firstly, one observes that each integral equation holds for all E and thus k. Therefore we will suppress their dependence and treat them as parameters which leads to integral equations of the form

$$T(p) = R(p) + \int_0^\infty dq \, \widetilde{K}(p,q) T(q) \, , \tag{G.1}$$

where the kernel $\widetilde{K}(p,q)$ could contain a pole in q depending on the center-of-mass energy and the binding momentum γ. Thus, we write it as $\widetilde{K}(p,q) = K(p,q)/(q - q_0 - i\varepsilon)$. The position of the pole (i.e. q_0) can be found by rewriting the scattering amplitudes. It is different in each scattering process, but does not depend on the specific scattering channel. Furthermore, one has to replace the upper integration boundary by a momentum cutoff Λ in numerical calculations. To get rid of the $i\varepsilon$ prescription in the denominator of $\widetilde{K}(p,q)$ we carry out the limit $\varepsilon \to 0$ using the Sokhatsky-Weierstrass theorem,

$$\frac{1}{q - q_0 \pm i\varepsilon} = \mathrm{PV}\frac{1}{q - q_0} \mp i\pi \, \delta(q - q_0) \, , \tag{G.2}$$

with the principal value PV. This changes the integral equation Eq. (G.1) to

$$T(p) = R(p) + \int_0^\Lambda dq \, \frac{K(p,q)\,T(q) - K(p,q_0)\,T(q_0)}{q - q_0} + K(p,q_0)\,T(q_0)\left(i\pi + \ln\left[\frac{\Lambda - q_0}{q_0}\right]\right), \tag{G.3}$$

where the principal value integral is already evaluated:

$$\mathrm{PV} \int_0^\Lambda dq \, \frac{1}{q - q_0} = \ln\left[\frac{\Lambda - q_0}{q_0}\right]. \tag{G.4}$$

Eq. (G.3) can be discretized by replacing the integration over q by a summation over discrete momenta $q =: p_j$ with weights w_j and setting $p_i := p$. Introducing the notation $R_i := R(p_i)$, $T_i := T(p_i)$ and $K_{ij} := K(p_i, p_j)$ one could write Eq. (G.3) in a discretized form, but as there are poles in the integral kernel one has to take into account their contribution, too. For this one has to distinguish three cases, corresponding to the number of poles appearing in the (coupled) integral equations.

In the trivial case without pole the procedure is straightforward. One uses N mesh points generated e.g. by the Gauss-Legendre quadrature [189] to numerically calculate the integral

$$T_i = R_i + \sum_{j=1}^N w_j \, \widetilde{K}_{ij} \, T_j \quad \text{for } i = 1, ..., N, \tag{G.5}$$

which holds for all $i = 1, ..., N$ and thus can be described by a matrix relation

$$\mathbf{T} = \mathbf{R} + \mathcal{M}_1^{(0)} \, \mathbf{T}, \tag{G.6}$$

with $\left(\mathcal{M}_1^{(0)}\right)_{ij} = w_j \widetilde{K}_{ij}$ being a $N \times N$ matrix.

If there is one pole we let $N-1$ be the number of mesh points and set $p_N := q_0$ with corresponding "weight" $w_N = 1$ which allows us to write Eq. (G.3) as

$$T_i = R_i + \sum_{j=1}^{N-1} w_j \, \frac{K_{ij}\,T_j - K_{iN}\,T_N}{p_j - p_N} + K_{iN}\,T_N\left(i\pi + \ln\left[\frac{\Lambda - p_N}{p_N}\right]\right)$$

$$= R_i + \sum_{j=1}^{N-1} w_j \, \widetilde{K}_{ij} \, T_j + K_{iN}\,T_N\left(i\pi + \ln\left[\frac{\Lambda - p_N}{p_N}\right] - \sum_{k=1}^{N-1} \frac{w_k}{p_k - p_N}\right), \quad \text{for } i = 1, ..., N. \tag{G.7}$$

Note, that for $i = N$ there is no special treatment necessary. One has just added an extra mesh point with a fixed value. Defining a $N \times N$ matrix $\mathcal{M}_1^{(1)}$,

$$\left(\mathcal{M}_1^{(1)}\right)_{ij} = \begin{cases} w_j \widetilde{K}_{ij}, & \text{for } j < N \\ w_j \, K_{ij}\left(i\pi + \ln\left[\frac{\Lambda - p_N}{p_N}\right] - \sum_{k=1}^{N-1} \frac{w_k}{p_k - p_N}\right), & \text{for } j = N \end{cases}, \tag{G.8}$$

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Eq. (G.7) can be written as simple matrix relation similarly to the case without pole

$$\mathbf{T} = \mathbf{R} + \mathcal{M}_1^{(1)}\mathbf{T} \,. \tag{G.9}$$

The third case with two poles only occurs in the coupled integral equations. Thus, one has to generalize the scheme above to a system of two coupled equations. Consider the following system of such equations:

$$T_1(p) = R_1(p) + \int_0^\Lambda dq \, \widetilde{K}_{11}(p,q) \, T_1(q) + \int_0^\Lambda dq \, \widetilde{K}_{12}(p,q) \, T_2(q) \,, \tag{G.10}$$

$$T_2(p) = R_2(p) + \int_0^\Lambda dq \, \widetilde{K}_{21}(p,q) \, T_1(q) + \int_0^\Lambda dq \, \widetilde{K}_{22}(p,q) \, T_2(q) \,, \tag{G.11}$$

where it is assumed that the kernels $\widetilde{K}_{i1}$, $i = 1, 2$, have the same pole at $q = q_{0,1}$ and also that $\widetilde{K}_{i2}$, $i = 1, 2$, have a common pole at $q = q_{0,2} \neq q_{0,1}$. Comparing this assumption with the coupled integral equations in section 5.1 one notices that it is indeed justified. Each integral is discretized using the same set of mesh points $p_j := q$ with weights w_j for $j = 1, ..., (N - \text{number of poles})$. Hence, with the notation introduced above $(T_1(p) = (T_1)_i, \, K_{12}(p,q) = (K_{12})_{ij}$, etc.) one finds

$$(T_1)_i = (R_1)_i + \sum_{j=1}^{N-\text{poles}} w_j \, (\widetilde{K}_{11})_{ij} \, (T_1)_j + \sum_{j=1}^{N-\text{poles}} w_j \, (\widetilde{K}_{12})_{ij} \, (T_2)_j \,, \quad \text{for } i = 1, ..., N \tag{G.12}$$

$$(T_2)_i = (R_2)_i + \sum_{j=1}^{N-\text{poles}} w_j \, (\widetilde{K}_{21})_{ij} \, (T_1)_j + \sum_{j=1}^{N-\text{poles}} w_j \, (\widetilde{K}_{22})_{ij} \, (T_2)_j \,, \quad \text{for } i = 1, ..., N \,. \tag{G.13}$$

These two coupled equations can be combined into one matrix equation of the form $\mathbf{T} = \mathbf{R} + \mathcal{M}_2 \, \mathbf{T}$, where $\mathbf{T}$ is a vector with $2N$ entries,

$$T = \begin{pmatrix} (T_1)_1 \\ \vdots \\ (T_1)_N \\ (T_2)_1 \\ \vdots \\ (T_2)_N \end{pmatrix} \,, \tag{G.14}$$

and $\mathcal{M}_2$ is a $2N \times 2N$ matrix whose elements $(\mathcal{M}_2)_{ij}$ are defined by the integral kernels and depend on the number of poles in those. If there is no pole present in the kernels one finds $\mathcal{M}_2 \equiv \mathcal{M}_2^{(0)}$ which is given in Eq. (G.18).

In case of a pole in $\widetilde{K}_{i1}$ and no one in $\widetilde{K}_{i2}$ we define $p_N := q_{0,1}$ and $w_N := 1$ in order to take care of the pole contribution to the integral. The matrix $\mathcal{M}_2$ then has the form of $\mathcal{M}_2^{(1a)}$ given in Eq. (G.19) where

$$z(p_N) = \left(i\pi + \ln\left[\frac{\Lambda - p_N}{p_N} \right] - \sum_{k=1}^{N-\text{poles}} \frac{w_k}{p_k - p_N} \right) \,, \tag{G.15}$$

is the pole contribution. Consequently, the $(2N)$-th column of $\mathcal{M}_2^{(1a)}$ is filled with zeros since there is no such contribution for the kernels $\widetilde{K}_{i2}$ which have no pole.

If the pole distribution is interchanged (i.e. one pole in $\widetilde{K}_{i2}$, no pole in $\widetilde{K}_{i1}$), the extra mesh point is given by $p_N := q_{0,2}$ with $w_N = 1$ and the structure of the N-th and the $(2N)$-th column is interchanged, too. Therefore $\mathcal{M}_2 \equiv \mathcal{M}_2^{(1b)}$ which is defined in Eq. (G.20). Note, that $z(p_N)$ is different from that in $\mathcal{M}_2^{(1a)}$ because the position of the pole p_N is different.

In the last case where both kernels $\widetilde{K}_{i1}$ and $\widetilde{K}_{i2}$ have a pole, one has to combine the results for just one pole in the right way. Since there are now $N-2$ mesh points one fills up the N element set with the additional points $p_{N-1} := q_{0,1}$ and $p_N := q_{0,2}$. The "wights" are both equal to one, i.e. $w_{N-1} = w_N = 1$. The pole contributions depend as before on the function $z(p_N)$ in which the sum over k now runs from 1 to $N-2$ instead of $N-1$. This leads to the matrix in Eq. (G.21) which we will call $\mathcal{M}_2^{(2)}$.

Note, that it is in the same way possible to extend the numerical implementation to an arbitrary number of coupled integral equations. Finally, we will make some remarks on the distribution of mesh points. Since the wave function falls off very quickly for large momenta it is useful to have more mesh points in the region of small momenta and less in that of large momenta instead of an equidistantly distribution. This leads to a faster convergence in the number of mesh points and can be accomplished in the following way: one calculates a set of $(N - \text{poles})$ mesh points x_j with weights ξ_j in the interval $[0, \ln(\Lambda+1)]$ using the Gauss-Legendre quadrature implementation described in Ref. [189] (note, that one could also use a different quadrature scheme). These mesh points and weights are used to define the mesh points p_j and weights w_j in the discussion above:

$$w_j = \exp(x_j)\,\xi_j\,, \tag{G.16}$$

$$p_j = \exp(x_j) - 1\,. \tag{G.17}$$

In the continuum this definition corresponds to the substitution $q(x) = e^x - 1$ in the integral $\int_0^\Lambda dq = \int_0^{\ln(\Lambda+1)} e^x dx$ with $x \in [0, \ln(\Lambda + 1)]$.

$$\mathcal{M}_2^{(0)} = \begin{pmatrix} w_1(\widetilde{K}_{11})_{11} & \cdots & w_N(\widetilde{K}_{11})_{1N} & w_1(\widetilde{K}_{12})_{11} & \cdots & w_N(\widetilde{K}_{12})_{1N} \\ \vdots & & \vdots & \vdots & & \vdots \\ w_1(\widetilde{K}_{11})_{N1} & \cdots & w_N(\widetilde{K}_{11})_{NN} & w_1(\widetilde{K}_{12})_{N1} & \cdots & w_N(\widetilde{K}_{12})_{NN} \\ w_1(\widetilde{K}_{21})_{11} & \cdots & w_N(\widetilde{K}_{21})_{1N} & w_1(\widetilde{K}_{22})_{11} & \cdots & w_N(K_{22})_{1N} \\ \vdots & & \vdots & \vdots & & \vdots \\ w_1(\widetilde{K}_{21})_{N1} & \cdots & w_N(\widetilde{K}_{21})_{NN} & w_1(\widetilde{K}_{22})_{N1} & \cdots & w_N(\widetilde{K}_{22})_{NN} \end{pmatrix}, \tag{G.18}$$

$$\mathcal{M}_2^{(1a)} =$$
$$\begin{pmatrix} w_1(\widetilde{K}_{11})_{11} & \cdots & w_{N-1}(\widetilde{K}_{11})_{1,N-1} & w_N(K_{11})_{1N}\, z(p_N) & w_1(\widetilde{K}_{12})_{11} & \cdots & w_{N-1}(\widetilde{K}_{12})_{1,N-1} & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ w_1(\widetilde{K}_{11})_{N1} & \cdots & w_{N-1}(\widetilde{K}_{11})_{N,N-1} & w_N(K_{11})_{NN}\, z(p_N) & w_1(\widetilde{K}_{12})_{N1} & \cdots & w_{N-1}(\widetilde{K}_{12})_{N,N-1} & 0 \\ w_1(\widetilde{K}_{21})_{11} & \cdots & w_{N-1}(\widetilde{K}_{21})_{1,N-1} & w_N(K_{21})_{1N}\, z(p_N) & w_1(\widetilde{K}_{22})_{11} & \cdots & w_{N-1}(\widetilde{K}_{22})_{1,N-1} & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ w_1(\widetilde{K}_{21})_{N1} & \cdots & w_{N-1}(\widetilde{K}_{21})_{N,N-1} & w_N(K_{21})_{1N}\, z(p_N) & w_1(\widetilde{K}_{22})_{N1} & \cdots & w_{N-1}(\widetilde{K}_{22})_{N,N-1} & 0 \end{pmatrix},$$
$$\tag{G.19}$$

$$\mathcal{M}_2^{(1b)} =$$
$$\begin{pmatrix} w_1(\widetilde{K}_{11})_{11} & \cdots & w_{N-1}(\widetilde{K}_{11})_{1,N-1} & 0 & w_1(\widetilde{K}_{12})_{11} & \cdots & w_{N-1}(\widetilde{K}_{12})_{1,N-1} & w_N(K_{12})_{1N}\, z(p_N) \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ w_1(\widetilde{K}_{11})_{N1} & \cdots & w_{N-1}(\widetilde{K}_{11})_{N,N-1} & 0 & w_1(\widetilde{K}_{12})_{N1} & \cdots & w_{N-1}(\widetilde{K}_{12})_{N,N-1} & w_N(K_{12})_{1N}\, z(p_N) \\ w_1(\widetilde{K}_{21})_{11} & \cdots & w_{N-1}(\widetilde{K}_{21})_{1,N-1} & 0 & w_1(\widetilde{K}_{22})_{11} & \cdots & w_{N-1}(\widetilde{K}_{22})_{1,N-1} & w_N(K_{22})_{1N}\, z(p_N) \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ w_1(\widetilde{K}_{21})_{N1} & \cdots & w_{N-1}(\widetilde{K}_{21})_{N,N-1} & 0 & w_1(\widetilde{K}_{22})_{N1} & \cdots & w_{N-1}(\widetilde{K}_{22})_{N,N-1} & w_N(K_{22})_{1N}\, z(p_N) \end{pmatrix},$$
$$\tag{G.20}$$

$$\mathcal{M}_2^{(2)} =$$
$$\left(\begin{array}{cccc} w_1(\widetilde{K}_{11})_{11} & \cdots & w_{N-2}(\widetilde{K}_{11})_{1,N-2} & w_{N-1}(K_{11})_{1,N-1}\, z(p_{N-1}) & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ w_1(\widetilde{K}_{11})_{N1} & \cdots & w_{N-2}(\widetilde{K}_{11})_{N,N-2} & w_{N-1}(K_{11})_{N,N-1}\, z(p_{N-1}) & 0 \\ w_1(\widetilde{K}_{21})_{11} & \cdots & w_{N-2}(\widetilde{K}_{21})_{1,N-2} & w_{N-1}(K_{21})_{1,N-1}\, z(p_{N-1}) & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ w_1(\widetilde{K}_{21})_{N1} & \cdots & w_{N-2}(\widetilde{K}_{21})_{N,N-2} & w_{N-1}(K_{21})_{1,N-1}\, z(p_{N-1}) & 0 \end{array} \right.$$
$$\left. \begin{array}{ccccc} & w_1(\widetilde{K}_{12})_{11} & \cdots & w_{N-2}(\widetilde{K}_{12})_{1,N-2} & 0 & w_N(K_{12})_{1,N}\, z(p_N) \\ & \vdots & & \vdots & \vdots & \vdots \\ & w_1(\widetilde{K}_{12})_{N1} & \cdots & w_{N-2}(\widetilde{K}_{12})_{N,N-2} & 0 & w_N(K_{12})_{1,N}\, z(p_N) \\ & w_1(\widetilde{K}_{22})_{11} & \cdots & w_{N-2}(\widetilde{K}_{22})_{1,N-2} & 0 & w_N(K_{22})_{1,N}\, z(p_N) \\ & \vdots & & \vdots & \vdots & \vdots \\ & w_1(\widetilde{K}_{22})_{N1} & \cdots & w_{N-2}(\widetilde{K}_{22})_{N,N-2} & 0 & w_N(K_{22})_{1,N}\, z(p_N) \end{array} \right). \tag{G.21}$$

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