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On classical gravitational corrections to the functional Schrödinger equation

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Abstract.

A full theory of quantum gravity is not yet available, and an approximation in which spacetime remains classical while matter is described by quantum fields is often physically and computationally appropriate. It is therefore of interest to investigate hybrid systems which describe the interaction of classical gravity with quantum matter. Such systems may provide valuable clues relevant to the search of a quantum theory of gravity. Furthermore, one should also consider the possibility that the gravitational field may not be quantum in nature; in that case, it would become necessary to search for a consistent hybrid description.

It is known that the Wheeler-De Witt equation with coupling to quantum fields results in quantum gravitational corrections to the functional Schrödinger equation. A similar result can be obtained for some hybrid models where a classical gravitational field interacts with quantum matter fields. I use the approach of ensembles on configuration space to look at a hybrid model where matter is in the form of a quantized scalar field and determine the corresponding classical gravitational corrections to the functional Schrödinger equation.

1. Introduction

Hybrid systems are models in which two subsystems, one of them classical and the other one quantum mechanical, interact. They can be useful for practical applications: For many calculations, it is convenient or even necessary to describe parts of a complex system classically, while the rest of the system is described quantum mechanically. In addition, there are theoretical reasons for studying hybrid systems: A full theory of quantum gravity is not yet available and a detailed description of classical gravity (within the framework of general relativity) interacting with quantum matter (described by quantum field theory) may provide valuable clues relevant to the search of a quantum theory of gravity. Furthermore, one should also consider the possibility that the gravitational field may not be quantum in nature; if that is the case, it would become necessary to come up with a consistent hybrid description. Hybrid systems are also of interest for foundations of quantum mechanics: In the Copenhagen interpretation, a measuring apparatus is described in classical terms. This implies a coupling (typically left undefined) between the quantum system and the classical apparatus.

A consistent description of a hybrid systems is nontrivial. Since classical mechanics and quantum mechanics are formulated using very different mathematical structures, it is first necessary to find a common framework that can accommodate both. Furthermore, there are conceptual issues; e.g., one needs to consider which features of classical and quantum mechanics ought to be preserved when describing a mixed system, which brings into the play the issue of defining "classicality" – that is, what makes a system classical? There are many different ways of addressing these questions. As results, there are many "variations on a theme," many proposals which differ by the way in which these issues are handled.

It will be useful to give a brief overview, necessarily incomplete, of some of the approaches to modeling hybrid systems that appear to be most relevant to foundations of physics. In the mean-field approach, the phase space coordinates of the classical system appear in the quantum Hamiltonian operator as parameters. The operator determines the evolution of the quantum system while its average over the quantum degrees of freedom specifies a classical Hamiltonian for the classical parameters [1, 2]. As the classical system evolves deterministically, it is not possible to couple any quantum uncertainties into the classical parameters. Nevertheless, Elze, using his formulation, has shown that the mean-field approach satisfies several fundamental consistency criteria of importance [3]. In the phase space approach, the classical system is described by a set of mutually commuting phase space observables on a Hilbert space [4], which allows a unitary interaction with the quantum system. In these models [5, 6, 7, 8], the classical generators of transformations do not coincide in general with the observables; as a consequence, the Hamiltonian will depend on non-observable classical operators. Difficulties arise in that classical observables remain 'classical' for a limited class of interactions, which does not include for example the case of a Stern-Gerlach measurement [7]. Peres and Terno have further shown that this approach does not reproduce the correct classical limit for quantum-classical oscillators [9, 10]. Nevertheless, in the past few years models of this type have been further developed and some of the issues that affected the original formulations have been resolved [11, 12, 13]. In the approach of Markovian master equations [14, 15, 16], the dynamics of the hybrid system is formulated in analogy to the dynamics of a purely quantum composite system but the classical observables are restricted to a commuting set of diagonal operators in a fixed basis. The hybrid formalism that results is equivalent to the standard Markovian theory of time-continuous quantum measurement [14, 15]. Finally, in the approach of ensembles on configuration space, physical systems are described using a Hamiltonian formalism that provides a common mathematical framework for classical, quantum, and hybrid systems [17, 18, 19]. The dynamics is derived from an ensemble Hamiltonian $H[P, S]$ that is a functional of P and S, where P is the probability density over configuration space of the system and S a canonically conjugate variable. Observables are functionals of P and S. The approach satisfies a number of consistency requirements [18, 19]. However, it has been shown that when the interaction can be 'switched off,' noninteracting ensembles of quantum and classical particles can be associated with nonlocal signaling [19, 20]. These are effects that are suppressed when a requirement of 'classicality' is imposed: we must restrict to macroscopic systems that can not be decoupled from the environment and have a large number of degrees of freedom in order to describe a system as 'classical.' Violations of locality are then suppressed by essentially the same mechanism that ensures measurement irreversibility [19].

Hybrid models which describe a classical general relativistic gravitational field interacting with a quantized matter field are few in number. The first model that was proposed is semi-classical gravity [21, 22, 23, 24], were the energy momentum tensor that serves as the source in the Einstein equations is set equal to the expectation value of the energy momentum operator of a given quantum state. However, it is well known that this approach presents a number of difficulties, which is the main motivation for searching for alternatives. The approach of ensembles on configuration space can be extended to field theories and it has been applied to the description of a classical gravitational field interacting with quantized matter [17, 19, 25, 26, 27]. Since details will be presented in the next sections, I will not discuss this further here except to point out that the nonlocal signaling effect discussed previously becomes irrelevant in this case, as there is no sense in which the interaction between systems can be 'switched off' because there is a direct multiplicative coupling of the metric tensor to the fields in the corresponding ensemble

Hamiltonian, as will be shown in the next section. The Markovian master equations approach has also been extended to encompass gravity and this has resulted in the postquantum theory of classical gravity developed by Oppenheim and coworkers [28, 29, 30]. It has been recently pointed out, however, that such an extension of the formalism to special and general relativity presents technical difficulties that are at present unresolved, and it is not clear at this time how they should be addressed [31, 32].

The approach of ensembles on configuration space [17, 18, 19] provides one way of describing a hybrid model where a classical relativistic gravitational field interacts with quantum matter fields [17, 19, 25, 26, 27], as was mentioned above. It is used here to examine a hybrid model with matter in the form of a quantized scalar field, with the aim of determining the classical gravitational corrections to the functional Schrödinger equation. In addition, it is found that the backreaction of the quantum field affects the evolution of the classical gravitational field. The approach followed is similar to the one that has been developed to study the quantum gravitational corrections to the functional Schrödinger equation that result from a semi-classical expansion of the Wheeler-DeWitt equation [33, 34, 35, 36].

2. Ensembles on configuration space for a quantum scalar field coupled to classical gravity

The approach of ensembles on configuration space allows the introduction of a hybrid model where a classical gravitational field interacts with a scalar quantum field [17, 19] . It is convenient to develop this approach by first considering ensembles of classical gravitational fields, which are given in terms of the Einstein-Hamilton-Jacobi equation together with a continuity equation for the probability associated with the ensemble. This formulation is analogous to the description of hybrid ensembles of non-relativistic particles on configuration space [19]. For the purpose of this paper, it will be sufficient to consider the case of vacuum gravity. One then introduces the quantum matter fields that interact with the classical gravitational field via the functional Schrödinger equation formulation of quantum field theory, written in terms of Madelung variables.

2.1. Ensembles on configuration space for vacuum gravity

In the 3+1 decomposition of spacetime, the line element is written as

$$g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + h_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right), \tag{1}$$

where N is the lapse function, N^i (with $i, j = 1, 2, 3$) is the shift function, and h_{ij} is the metric on the spatial hypersurface.

The Einstein-Hamilton-Jacobi equation for vacuum gravity is a functional equation for $S[h_{ij}]$ given by [35, 37, 38, 39, 40]

$$H_h^C = \frac{1}{2M}G_{ijkl}\frac{\delta S}{\delta h_{ij}}\frac{\delta S}{\delta h_{kl}} - 2Mc^2\sqrt{h}\,(R-2\Lambda) = 0, \tag{2}$$

where $M = \frac{c^2}{32\pi G}$, h is the determinant of h_{ij}, G is the gravitational constant, c the speed of light, Λ the cosmological constant, and R the curvature scalar. The DeWitt supermetric is given by $G_{ijkl} = \frac{1}{2\sqrt{h}}\left(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}\right)$. The parameter M can also be written in terms of the Planck mass m_p and the Planck length l_p as $M = \frac{1}{32\pi}\frac{m_p}{l_p}$, where $m_p = \sqrt{\hbar c/G}$ and $l_p = \sqrt{\hbar G/c^3}$. The functional $S[h_{ij}]$ is assumed to be invariant under the gauge group of spatial coordinate transformations, which is equivalent to satisfying the momentum constraints of the canonical formulation of general relativity. Furthermore, it satisfies $\frac{\partial S}{\partial t} = 0$.

Equation (2) provides a Hamilton-Jacobi formulation of the Einstein equations [38]. To define ensembles of gravitational fields, one introduces a probability $P[h_{ij}]$ over the space of the h_{ij} and an appropriate measure Dh over the space of metrics (technical issues are discussed in [19]). The functional P, like S, is also required to be invariant under the gauge group of spatial coordinate transformations and satisfy $\frac{\partial P}{\partial t} = 0$ [19]. The probability obeys a continuity equation of the form

$$C_h = \frac{1}{M}\frac{\delta}{\delta h_{ij}}\left(P\, G_{ijkl}\frac{\delta S}{\delta h_{kl}}\right) = 0. \tag{3}$$

The interpretation of Eq. (3) as a continuity equation leads to an equation for the time derivative of h_{ij}, which can be put in the standard form

$$\dot{h}_{ij} = N G_{ijkl}\frac{\delta S}{\delta h_{kl}} + D_i N_j + D_j N_i, \tag{4}$$

as discussed in Appendix A.

Eq. (4) plays an important role in the Einstein-Hamilton-Jacobi formulation because it allows a reconstruction of the four-dimensional spacetime given a solution of the Einstein-Hamilton-Jacobi equation on the three-dimensional space-like hypersurface. It is necessary for deriving all ten Einstein field equations and, as shown by Gerlach, it can be derived directly from the Einstein-Hamilton-Jacobi formalism [38]. Eq. (4) coincides with the equation derived from the ADM canonical formalism [41]. It also follows from the formalism of ensembles on configuration space in a natural way [19].

2.2. Ensembles on configuration space for a quantum scalar field and classical gravity

A mixed classical-quantum system where a quantum scalar field ϕ couples to the classical metric h_{kl} requires a generalization of Eqs. (2-3) which corresponds to adding a quantized scalar field as a source term to the gravitational field. The equations take the form [19]

$$H_{h\phi} = H_h^C + \frac{1}{2\sqrt{h}}\left(\frac{\delta S}{\delta \phi}\right)^2 + \sqrt{h}\left[\frac{1}{2}h^{ij}\frac{\partial \phi}{\partial x^i}\frac{\partial \phi}{\partial x^j} + V(\phi)\right] + \frac{\hbar^2}{2\sqrt{h}}\left(\frac{1}{\sqrt{P}}\frac{\delta^2\sqrt{P}}{\delta\phi^2}\right) = 0 \tag{5}$$

and

$$C_{h\phi} = C_h + \frac{1}{\sqrt{h}}\frac{\delta}{\delta\phi}\left(P\frac{\delta S}{\delta\phi}\right) = 0. \tag{6}$$

As shown in section 4, the last term in Eq. (5) gives rise to a "Bohm quantum potential" for the functional Schrödinger equation of the quantized scalar field, which appears in Eqs. (5-6) written in terms of Madelung variables. If this term is omitted, Eq. (5) becomes the Einstein-Hamilton-Jacobi equation with a classical scalar field as source.

3. Solving the equations of the mixed classical-quantum system

It is convenient to express S and P in the form

$$S[h_{ij},\phi] = S_0[h_{ij}] + S_1[h_{ij},\phi], \qquad P[h_{ij},\phi] = P_0[h_{ij}]P_1[h_{ij},\phi]. \tag{7}$$

This *ansatz* does not restrict the possible solutions. One can interpret P_0 as a prior probability for h_{ij} and P_1 as a conditional probability of ϕ given h_{ij}.

This form for S and P are particularly useful to show that quantum field theory in curved spacetime appears as an approximation to the mixed classical-quantum theory. This is shown in the next section. It also seem appropriate for the case in which the solutions are, in some sense, close to the approximate solutions of quantum field theory in curved space time. This is

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the case that we discuss in the example of this paper. It is also the case that applies to a semi-classical expansion of the Wheeler-DeWitt equation where *quantum* gravitational corrections to the functional Schrödinger equation have been derived [33, 34, 35, 36].

With this *ansatz*, the modified Einstein-Hamilton-Jacobi equation and the continuity equation can be written in the forms

$$\left[\frac{1}{2M}G_{ijkl}\frac{\delta S_0}{\delta h_{ij}}\frac{\delta S_0}{\delta h_{kl}} - 2c^2M\sqrt{h}\,(R-2\Lambda)\right] + \left\{\frac{1}{2M}G_{ijkl}\left(\frac{\delta S_0}{\delta h_{ij}}\frac{\delta S_1}{\delta h_{kl}} + \frac{\delta S_1}{\delta h_{ij}}\frac{\delta S_0}{\delta h_{kl}}\right)\right.$$
$$\left. + \frac{1}{2\sqrt{h}}\left(\frac{\delta S_1}{\delta \phi}\right)^2 + \sqrt{h}\left(\frac{1}{2}h^{ij}\frac{\partial\phi}{\partial x^i}\frac{\partial\phi}{\partial x^j} + V(\phi)\right) + \frac{\hbar^2}{2\sqrt{h}}\left(\frac{1}{\sqrt{P_1}}\frac{\delta^2\sqrt{P_1}}{\delta\phi^2}\right)\right\}$$
$$+\frac{1}{2M}G_{ijkl}\frac{\delta S_1}{\delta h_{ij}}\frac{\delta S_1}{\delta h_{kl}} = 0, \tag{8}$$

and

$$P_1\left[\frac{1}{M}\frac{\delta}{\delta h_{ij}}\left(P_0\,G_{ijkl}\left(\frac{\delta S_0}{\delta h_{kl}} + \frac{\delta S_1}{\delta h_{kl}}\right)\right)\right]$$
$$+P_0\left\{\frac{1}{M}\frac{\delta P_1}{\delta h_{ij}}\left(G_{ijkl}\left(\frac{\delta S_0}{\delta h_{kl}} + \frac{\delta S_1}{\delta h_{kl}}\right)\right) + \frac{1}{\sqrt{h}}\frac{\delta}{\delta\phi}\left(P_1\frac{\delta S_1}{\delta\phi}\right)\right\} = 0. \tag{9}$$

4. Quantum field theory in curved spacetime as an approximation to the mixed classical-quantum theory

Quantum field theory in curved spacetime is an approximation to Eqs. (8-9) in which (i) the corrections to the functional Schrödinger equation and (ii) the back reaction of the quantum field on the classical gravitational field are neglected. This amounts to neglecting the last term in Eq. (8) as well as the terms in Eq. (9) that are proportional to $\frac{\delta S_1}{\delta h_{kl}}$. One can then consider solutions where the terms in square brackets and in curly brackets in Eqs.(8-9) are each set equal to zero. This leads to two equations, one for the classical sector and one for the quantum sector.

From now on, to simplify the equations, we restrict to spacetimes in Eq. (1) that satisfy the gauge $N = 1$, $N_j = 0$.

4.1. Classical gravitational sector

Setting the terms in square brackets equal to zero leads to

$$\frac{1}{2M}G_{ijkl}\frac{\delta S_0}{\delta h_{ij}}\frac{\delta S_0}{\delta h_{kl}} - 2Mc^2\sqrt{h}\,(R-2\Lambda) = 0, \tag{10}$$
$$\frac{1}{M}\frac{\delta}{\delta h_{ij}}\left(P_0\,G_{ijkl}\frac{\delta S_0}{\delta h_{kl}}\right) = 0. \tag{11}$$

which are the equations that define an ensemble of classical spacetimes on configuration space for the case of vacuum gravity. The four-dimensional spacetime is reconstructed via Eq. (4) and the conditions $N = 1$, $N_i = 0$, leading to

$$\dot{h}_{ij} = G_{ijkl}\frac{\delta S_0}{\delta h_{kl}}. \tag{12}$$

4.2. Quantum scalar field sector

When the terms in curly brackets are set equal to zero, the following two equations result,

$$\frac{1}{2M}G_{ijkl}\left(\frac{\delta S_0}{\delta h_{ij}}\frac{\delta S_1}{\delta h_{kl}}+\frac{\delta S_1}{\delta h_{ij}}\frac{\delta S_0}{\delta h_{kl}}\right) +\frac{1}{2\sqrt{h}}\left(\frac{\delta S_1}{\delta\phi}\right)^2+\sqrt{h}\left(\frac{1}{2}h^{ij}\frac{\partial\phi}{\partial x^i}\frac{\partial\phi}{\partial x^j}+V(\phi)\right)+\frac{\hbar^2}{2\sqrt{h}}\left(\frac{1}{\sqrt{P_1}}\frac{\delta^2\sqrt{P_1}}{\delta\phi^2}\right)=0, \tag{13}$$

$$\frac{1}{M}G_{ijkl}\frac{\delta S_0}{\delta h_{kl}}\frac{\delta P_1}{\delta h_{ij}}+\frac{1}{\sqrt{h}}\frac{\delta}{\delta\phi}\left(P_1\frac{\delta S_1}{\delta\phi}\right)=0. \tag{14}$$

Using Eq. (12), we have

$$\frac{1}{2}G_{ijkl}\left(\frac{\delta S_0}{\delta h_{ij}}\frac{\delta S_1}{\delta h_{kl}}+\frac{\delta S_1}{\delta h_{ij}}\frac{\delta S_0}{\delta h_{kl}}\right) = \frac{\delta S_1}{\delta h_{kl}}\dot{h}_{kl}, \tag{15}$$

$$G_{ijkl}\frac{\delta S_0}{\delta h_{kl}}\frac{\delta P_1}{\delta h_{ij}} = \frac{\delta P_1}{\delta h_{ij}}\dot{h}_{ij}, \tag{16}$$

where the time t is the gravitational time associated with the corresponding spacetime. The integration of Eqs. (13-14) with respect to the spatial coordinates leads to

$$\dot{S}_1+\int d^3x\,\frac{1}{2\sqrt{h}}\left(\frac{\delta S_1}{\delta\phi}\right)^2+\sqrt{h}\left[\frac{1}{2}h^{ij}\frac{\partial\phi}{\partial x^i}\frac{\partial\phi}{\partial x^j}+V(\phi)\right]-\frac{\hbar^2}{2\sqrt{h}}\left(\frac{1}{\sqrt{P_1}}\frac{\delta^2\sqrt{P_1}}{\delta\phi^2}\right)=0, \tag{17}$$

$$\dot{P}_1+\int d^3x\,\frac{1}{\sqrt{h}}\frac{\delta}{\delta\phi}\left(P_1\frac{\delta S_1}{\delta\phi}\right)=0, \tag{18}$$

using Eqs. (15-16) and the relation $\dot{F}[h_{ij}]=\int d^3x\,\frac{\delta F}{\delta h_{ij}}\dot{h}_{ij}$.

Using equations (17-18), the complex wave functional defined by $\Psi=\sqrt{P_1}\,e^{iS_1/\hbar}$ satisfies

$$i\hbar\dot{\Psi}=\int d^3x\,\left[-\frac{\hbar^2}{2\sqrt{h}}\frac{\delta^2}{\delta\phi^2}+\sqrt{h}\left(\frac{1}{2}h^{ij}\frac{\partial\phi}{\partial x^i}\frac{\partial\phi}{\partial x^j}+V(\phi)\right)\right]\Psi, \tag{19}$$

which is the Schrödinger functional equation on a curved space time for $N=1$, $N_i=0$ [42].

4.3. Example: quantized scalar field on a de Sitter background

Detailed applications of the formalism to particular cases is beyond the scope of this paper. Thus, instead of providing a fully worked out example, I will restrict myself to describing how the approach can be applied to derive solutions. It will be sufficient to discuss in a general way a simple example, that of a quantized massive scalar field ($V(\phi)=\frac{1}{2}m\phi^2$) on a de Sitter background, which in this section will be treated in the approximation of quantum field theory on curved space time [23, 24]. The approach presented can be seen as a hybrid version of a similar but fully quantum analysis that takes as its starting point the semiclassical Wheeler-DeWitt equation [34].

Consider first the classical gravitational field. While equation (5) corresponds to an infinity of constraints, one at each point, an alternative point of view is possible in which the equation is regarded as an equation to be integrated with respect to a test function, in which case it represents one equation for each particular choice of lapse function N (foliation) [34]. The advantage of this point of view is that one can find in some cases solutions of Eq.(2) for particular foliations, even when the general solution (which requires solving the Einstein-Hamilton-Jacobi equation for all choices of lapse functions) is not known.

In the case of a quantized scalar field on a de Sitter background, one can derive a solution for the case of a flat foliation (R=0) with $N = 1$, $N_i = 0$. It is given by [34]

$$S_0 = -8Mc\sqrt{\frac{\Lambda}{3}}\int d^3x\sqrt{h}, \tag{20}$$

with the corresponding line element

$$g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j. \tag{21}$$

Application of Eq. (12) leads to

$$\dot{a} = Mc\sqrt{\frac{\Lambda}{3}}a \tag{22}$$

or $a = \exp\{Mc\sqrt{\Lambda/3}\,t\}$, which is the expansion law for the scale factor in de Sitter space.

One can now consider the quantized scalar field. The metric of Eq. (21) depends on the time only and, furthermore, the foliation chosen is flat. Therefore, Eq. (19) simplifies, and it can be solved using standard methods for the functional Schrödinger equation on curved spacetime [34, 42, 43]. In particular, the vacuum state is a Gaussian functional which, while not unique, can be set equal to the Bunch-Davies background which is the preferred vacuum for de Sitte space [24].

5. Classical gravitational corrections to the functional Schrödinger equation and back reaction on the classical gravitational field

In the approximation discussed in the previous section, it was shown that Eqs. (5-6) lead to quantum field theory in curved spacetime provided certain terms are neglected. When these terms are *not* neglected, one can still talk about a "classical sector" and a "quantum sector" but now there are corrections to the Einstein-Hamilton-Jacobi equation, Eq. (10), and to the functional schrödinger equation, Eq. (19), due to additional effects that derive from the interaction of the classical gravitational field with the quantized scalar field.

5.1. Classical gravitational sector incorporating the back reaction from the quantum matter field

The full equations for the classical sector take the form

$$\frac{1}{2M}G_{ijkl}\frac{\delta S_0}{\delta h_{ij}}\frac{\delta S_0}{\delta h_{kl}} - 2Mc^2\sqrt{h}\,(R-2\Lambda) = 0, \tag{23}$$

$$\frac{1}{M}\frac{\delta}{\delta h_{ij}}\left(P_0\,G_{ijkl}\left(\frac{\delta S_0}{\delta h_{kl}}+\frac{\delta S_1}{\delta h_{kl}}\right)\right) = 0, \tag{24}$$

which differ from Eqs. (10) by the additional term $\frac{\delta S_1}{\delta h_{kl}}$ in Eq. (24). Thus the reconstruction of the four-dimensional spacetime will proceed now via the equation

$$\dot{h}_{ij} = G_{ijkl}\left(\frac{\delta S_0}{\delta h_{kl}}+\frac{\delta S_1}{\delta h_{kl}}\right). \tag{25}$$

As $\frac{\delta S_1}{\delta h_{kl}}$ depends on the quantum scalar field ϕ, the evolution of the classical gravitational field will also depend on ϕ and this amounts to taking into consideration the backreaction of the quantum field on the classical gravitational field.

5.2. Quantum scalar field sector including classical gravitational corrections to the functional Schrödinger equation

The full equations for the quantum sector take the form of a non-linear functional Schrödinger equation. It now includes the the last term in Eq. (8) that was neglected when considering the approximation of quantum field theory in curved space time, leading to

$$i\dot{\Psi} = \int d^3x \left[-\frac{\hbar^2}{2\sqrt{h}} \frac{\delta^2}{\delta \phi_a^2} + \sqrt{h} \left(\frac{1}{2} h^{ij} \frac{\partial \phi_a}{\partial x^i} \frac{\partial \phi_a}{\partial x^j} + V\left(\phi_a\right) \right) + \Delta \right] \Psi, \tag{26}$$

where the non-linear term Δ is given by

$$\Delta = \frac{1}{2M} G_{ijkl} \frac{\delta S_1}{\delta h_{ij}} \frac{\delta S_1}{\delta h_{kl}} = -\frac{1}{8M} G_{ijkl} \left(\frac{1}{\Psi} \frac{\delta \Psi}{\delta h_{ij}} - \frac{1}{\bar{\Psi}} \frac{\delta \bar{\Psi}}{\delta h_{ij}} \right) \left(\frac{1}{\Psi} \frac{\delta \Psi}{\delta h_{kl}} - \frac{1}{\bar{\Psi}} \frac{\delta \bar{\Psi}}{\delta h_{kl}} \right). \tag{27}$$

Note that the additional term Δ results from the variation of S_1 with respect to h_{ij}. It can therefore be interpreted as a correction to the functional Schrödinger equation that is due to additional effects of the classical gravitational field on the quantized scalar field (i.e., effects that go beyond quantum field theory on curved space time).

5.3. Example: Corrections to a quantized scalar field on a de Sitter background

One can now examine the corrections to quantum field theory on curved spacetime that result from applying the full hybrid model to the example of Section 4.3. A detailed analysis requires a detailed solution; nevertheless, even without an explicit solution it is possible to obtain qualitative results. If we assume that Ψ can be approximated by a Gaussian functional (by assuming, for example, a time dependent Gaussian approximation [44]), then $S_2 \sim \mathcal{O}(\phi^2)$. Therefore, the correction term Δ in Eq. (27) will be of the order $\Delta \sim \mathcal{O}(\phi^4)$ and one should be able to treat it, in a perturbative approach, as a modification of the potential $V(\phi)$. The correction term in Eq. (25) will be of the order $\frac{\delta S_1}{\delta h_{kl}} \sim \mathcal{O}(\phi^2)$ and represents the backreaction of the quantized scalar field on the classical gravitational field.

6. Discussion

The approach of ensembles on configuration space has been used to look at a hybrid model in which a classical gravitational field interacts with matter in the form of a quantized scalar field, with the aim of determining the classical gravitational corrections to the functional Schrödinger equation. It was shown that the hybrid model leads to a non-linear correction term in the functional Schrödinger equation. In addition, it was found that the backreaction of the quantum field affects the evolution of the classical gravitational field.

In the case of the approximation of quantum field theory in curved spacetime, the equations that determine the gravitational field are given by Eq. (10) and Eq. (12). As these equations are known to be equivalent to the Einstein equations in vacuum, one is assured that the time evolution of the metric given by Eq. (12) does not lead to inconsistencies. However, there is an open question regarding the consistency of Eq. (25) as a way of reconstructing the four-dimensional spacetime, because in this case the source is not classical but quantum. This issue will be taken up in a future publication. Nevertheless, the formalism should be at least approximately valid in the case in which the exact solution is close to the solution that results from applying quantum field theory in curved spacetime.

As the approach presented here contains quantum field theory in curved spacetime as an approximation, one can study Hawking radiation in the context of the hybrid model. This has been already done in this approximation for the case of a CGHS black hole [19]. A possible application of the full theory is to black hole evaporation in the hybrid model, provided the model remains consistent in the limit in which the black hole reaches its final stage.

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Appendix A. The equation for the time derivative of the metric h_{ij} from the approach of ensembles on configuration space

Consider Eq. (3), interpreted as a continuity equation. Keeping in mind the requirement that $\frac{\partial P}{\partial t} = 0$ [19], the most general equation for the time derivative of the metric h_{ij} given eq. (3) is of the form

$$\delta h_{ij} = \left(\alpha G_{ijkl} \frac{\delta S}{\delta h_{kl}} + \delta_\epsilon h_{ij} \right) \delta t \tag{A.1}$$

for some arbitrary function α. The term $\delta_\epsilon h_{ij} = -(D_i \epsilon_j + D_j \epsilon_i)$ has been included because it represents gauge transformations of h_{kl} (i.e., infinitesimal spatial coordiante transformations), which are permitted because gauge transformations are assumed to leave $\int \mathrm{D}hP$ invariant [19]. Therefore the most general infinitesimal change δh_{ij} of h_{ij} will be a combination of motion along the "velocity field" $G_{ijkl} \frac{\delta S}{\delta h_{kl}}$ and a gauge transformation. Then, the time derivative of h_{kl} can be put in the standard form

$$\dot{h}_{ij} = N G_{ijkl} \frac{\delta S}{\delta h_{kl}} + D_i N_j + D_j N_i, \tag{A.2}$$

writing N and N_j in place of α and $-\epsilon_j$ to agree with the usual notation. Eq. (A.2) is identical to the equation derived from the ADM canonical formalism when N is identified with the lapse function and N_k with the shift vector [41].

References

[1] Boucher W and Traschen J 1988 *Phys. Rev. D* **37** 3522
[2] Makri N 1999 *Ann. Rev. Phys. Chem.* **50** 167
[3] Elze H-T 2012 *Phys. Rev. A* **85** 052109
[4] Koopman B O 1931 *Proc. Natl. Acad. Sci. U.S.A.* **17** 315
[5] Sudarshan E C G 1976 *Pramana* **6** 117
[6] Sherry T N and Sudarshan E C G 1978 *Phys. Rev. D* **18** 4580
[7] Sherry T N and Sudarshan E C G 1979 *Phys. Rev. D* **20** 857
[8] Gautam S R, Sherry T N and Sudarshan E C G 1979 *Phys. Rev. D* **20** 3081
[9] Peres A and Terno D R 2001 *Phys. Rev. A* **63** 022101
[10] Terno D R 2006 *Found. Phys.* **36** 102
[11] Bondar D I, Gay-Balmaz F and Tronci C 2019 *Proc. R. Soc. A* **475** 20180879
[12] Gay-Balmaz F and Tronci C 2020 *Nonlinearity* **33** 5383
[13] Gay-Balmaz F and Tronci C 2022 *Physics D* **440** 133450
[14] Diósi L 2014 *Phys. Scr.* **2014** 014004
[15] Diósi L 2023 *Phys. Rev. A* **107** 062206
[16] Oppenheim J, Sparaciari C, Šoda B and Weller-Davies Z 2023 *Quantum* **7** 891
[17] Hall M J W and Reginatto M 2005 *Phys. Rev. A* **72** 062109
[18] Hall M J W 2008 *Phys. Rev. A* **78** 042104
[19] Hall M J W and Reginatto M 2016 *Ensembles on Configuration Space: Classical, Quantum, and Beyond* (Switzerland: Springer International Publishing)
[20] Hall M J W, Reginatto M and Savage C M 2012 *Phys. Rev. A* **86** 054101
[21] Birrell N D and Davies P C W 1982 *Quantum fields in curved space* (Cambridge: Cambridge University Press)
[22] Fulling S A 1989 *Aspects of Quantum Field Theory in Curved Space-Time* (Cambridge: Cambridge University Press)
[23] Mukhanov V F and Winitzki S 2007 *Introduction to Quantum Effects in Gravity* (Cambridge: Cambridge University Press)
[24] Parker L and Toms D 2009 *Quantum field theory in curved spacetime: quantized fields and gravity* (Cambridge: Cambridge University Press)
[25] Reginatto M 2013 *J. Phys.: Conf. Ser.* **442** 012009

[26] Hall M J W and Reginatto M 2018 *J. Phys. A: Math. Theor.* **51** 085303
[27] Reginatto M and Hall M J W 2019 *J. Phys.: Conf. Ser.* **1275** 012039
[28] Oppenheim J and Weller-Davies Z 2022 *J. High Energ. Phys.* **2022** 80
[29] Oppenheim J, Sparaciari C, Šoda B and Weller-Davies Z 2023 *Nature Communications* **14** 7910
[30] Oppenheim J 2023 *Phys. Rev. X* **13** 041040
[31] Diosi J 2024 arXiv:2404.07723v1
[32] Tilloy A 2024 arXiv:2403.19748v3
[33] Kiefer C and Singh T P 1991 *Phys. Rev D* **44** 1067
[34] Kiefer C 1994 The semiclassical approximation to quantum gravity *Canonical gravity: from classical to quantum* ed J Ehlers and H Friedrich (Berlin: Springer) p 170
[35] Kiefer C 2012 *Quantum Gravity* (Oxford: Oxford University Press)
[36] Kiefer C and Wichmann D 2018 *Gen. Relativ. Gravit.* **50** 66
[37] Peres A 1962 *Nuovo Cim.* **XXVI** 53
[38] Gerlach U H 1969 *Phys. Rev.* **177** 1929
[39] Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (San Francisco: Freeman)
[40] Salopek D S and Stewart J M 1992 *Class. Quantum Grav.* **9** 1943
[41] Wald R 1984 *General Relativity* (Chicago: University of Chicago Press)
[42] Long D V and Shore G M 1998 *Nucl. Phys. B* **530** 247
[43] Guven J, Bennett L and Hill C T (1989) *Phys. Rev D* **39** 438
[44] Éboli o, Pi S-Y and Samiullah M (1989) *Ann. Phys.* **193** 102