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Schur Averages in Random Matrix Ensembles

"Documento Definitivo"

Doutoramento em Matemática
Especialidade de Física Matemática e Mecânica dos Meios Contínuos

David García-García

Tese orientada por:
Dr. Miguel Tierz

Documento especialmente elaborado para a obtenção do grau de doutor

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La presencia y el apoyo de varias personas en estos años me han hecho crecer y llegar hasta aquí. Me siento muy afortunado por ello. Gracias,

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Ya es para siempre.

A J.

Abstract

The main focus of this PhD thesis is the study of minors of Toeplitz, Hankel and Toeplitz±Hankel matrices. These can be expressed as matrix models over the classical Lie groups $G(N) = U(N), Sp(2N), O(2N), O(2N+1)$, with the insertion of irreducible characters associated to each of the groups. In order to approach this topic, we consider matrices generated by formal power series in terms of symmetric functions.

We exploit these connections to obtain several relations between the models over the different groups $G(N)$, and to investigate some of their structural properties. We compute explicitly several objects of interest, including a variety of matrix models, evaluations of certain skew Schur polynomials, partition functions and Wilson loops of $G(N)$ Chern-Simons theory on S^3, and fermion quantum models with matrix degrees of freedom. We also explore the connection with orthogonal polynomials, and study the large N behaviour of the average of a characteristic polynomial in the Laguerre Unitary Ensemble by means of the associated Riemann-Hilbert problem.

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Keywords: Random matrix theory; Toeplitz determinant; Schur polynomial; Chern-Simons theory; Riemann-Hilbert problem.

Resumo

A teoria das matrizes aleatórias, e das propriedades dos seus autovalores, é um âmbito de estudo com grande atividade desde os trabalhos de Wigner e Dyson dos anos 1950 e 1960. De certa maneira, a área deve a sua existência às aplicações, e a quantidade de conexões com vários ambitos da matemática e da física é ainda uma das suas qualidades mais importantes.

As matrizes estruturadas, como as matrizes de Toeplitz ou Hankel, têm um papel fundamental no estudo das matrizes aleatórias. Por exemplo, modelos unitários com suporte no círculo unidade ou na reta real podem ser expressados como determinantes de Toeplitz ou Hankel. Entre outras aplicações, esta relação é relevante na área da combinatória, pois permitiu a resolução de problemas abertos de importância. Assim, resultados da teoria das funções simétricas foram essenciais no problema da maior subsequência crescente numa permutação aleatória, estudado por Baik, Deift e Johansson.

Desenvolvimentos deste tipo mostram o alcance da teoria das matrizes aleatórias, onde a aparição de técnicas de áreas diversas da matemática é a norma, e não uma exeção.

O principal objetivo desta tese é o estudo dos menores das matrizes de Toeplitz, Hankel, e Toeplitz±Hankel. Parte do nosso interesse neste tópico deve-se ao fato de que estos menores podem ser expressados como as integrais

$$\int_{G(N)} \chi^{\lambda}_{G(N)}(U^{-1})\chi^{\mu}_{G(N)}(U)f(U)dU,$$

onde dU é a medida de Haar num dos grupos de Lie clássicos

$$G(N) = U(N), Sp(2N), O(2N), O(2N+1),$$

e os $\chi^{\mu}_{G(N)}(U)$ são os caracteres associados às representações irredutíveis destes grupos. Estas integrais supõem uma generalização natural dos ensembles clássicos de matrizes aleatórias, pois envolvem o uso de técnicas algébricas e analíticas no seu estudo. Além disso, estes modelos têm também expressões em termos de funções simétricas. Outras motivações para o nosso estudo incluem

- O estudo de inserções generalizadas em modelos de matrizes aleatórias, em particular por meio de expansões em caracteres,
- A obtenção de propriedades estruturais de ensembles de matrizes nos grupos de Lie clássicos, mediante o uso de técnicas da teoria das representações, assim como o cálculo de vários objetos de interesse no campo da combinatória,
- A computação das funções de partição e de observáveis de teorias gauge com grupo de simetria $G(N)$, no contexto finito e infinito,

- A exploração das aplicações da formulação en termos de funções simétricas e modelos de matrizes na teoria de polinómios ortogonais.

Na primeira parte da tese, focalizamo-nos no desenvolvimento do formalismo de menores de Toeplitz, e explicamos a sua relação com as integrais unitárias. Depois de revisar alguns resultados da teoria das funções simétricas, expressamos estes menores em termos de polinómios de Schur e obtemos o seu comportamento assintótico em termos dos determinantes de Toeplitz associados. Depois, calculamos as inversas de várias matrizes de Toeplitz, utilizando polinómios de Chebyshev, a fórmula de Duduchava-Roch e o kernel associado a duas sequências de polinómios biortogonais no círculo unidade. Comparando as nossas fórmulas para menores de Toeplitz com estas inversas, obtemos evaluações explícitas de uma integral de Selberg-Morris e de certos polinómios skew Schur. Utilizamos também a fórmula de Laplace num determinante de Toeplitz geral para deduzir un conjunto de relações verificadas por polinómios skew Schur.

A continuação, estudamos integrais de matrizes nos grupos de Lie clássicos $G(N) = U(N), Sp(2N), O(2N)$ e $O(2N+1)$, por meio de funções simétricas e a formulação equivalente em termos de determinantes e menores de matrizes Toeplitz$\pm$Hankel. Isto permite-nos obter relações entre estas integrais, incluindo

1. Fatorações de integrais unitárias como produtos e somas de produtos de integrais simpléticas e ortogonais,
2. A expressão de uma classe de modelos como a especialização de um único caracter associado ao grupo de simetria correspondente,
3. Expansões de integrais simpléticas e ortogonais como somas ponderadas de integrais unitárias com caracteres ou, equivalentemente, expansões de determinantes de matrizes Toeplitz$\pm$ Hankel como somas ponderadas de menores de matrizes de Toeplitz,
4. Generalizações da identidade de Gessel, expressando as integrais em estudo como séries de funções de Schur,
5. O comportamento assintótico das médias de caracteres irredutíveis sobre os modelos de matrizes mencionados.

Consideramos então o modelo associado à terceira função teta de Jacobi, que modeliza a teoria de Chern-Simons em S^3. Calculamos as funções de partição, os Wilson loops e os Hopf links das teorias com grupos de simetria $G(N)$, e mostramos que os modelos são Giambelli-compatíveis. Neste contexto, as relações gerais antes encontradas traduzem-se em identidades entre observáveis das teorias com diferentes grupos de simetria. Finalmente, usamos expansões em caracteres e o comportamento assintótico dos determinantes associados para estudar inserções particulares no modelo de Chern-Simons, descrevendo espectros de modelos fermiônicos com graus de liberdade matriciais.

Finalmente, tratamos os menores de matrizes de Hankel, e estabelecemos algumas conexões com a teoria de polinómios ortogonais. Em particular, expressamos o kernel de Christoffel-Darboux associado a um conjunto de polinómios ortogonais em forma de soma ponderada sobre polinómios de Chebyshev, cujos coeficientes são menores da matriz de Hankel

associada. Depois, estudamos como exemplo a inserção de um polinómio característico no Laguerre Unitary Ensemble. Analisamos o modelo correspondente, tanto no contexto finito, utilizando expansões em polinómios de Schur, como no contexto infinito, resolvendo o problema de Riemann-Hilbert associado.

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Palavras-chave: Teoria das matrizes aleatórias; determinante de Toeplitz; polinómio de Schur; teoria de Chern-Simons; problema de Riemann-Hilbert.

Contents

Chapter 1

Context and general overview

1.1 Introduction

The study of random matrices, and in particular the properties of their eigenvalues, has been an active field of research since the seminal works of Wigner and Dyson in the 1950s and 1960s[1]. In a sense, the area owes its existence to applications, and one of its main appeals is the large number of connections it possesses with different branches of mathematics and physics.

As the field continues to evolve, the statistical properties of a surprising number of mathematical objects and physical systems are found to be modeled by the eigenvalues of matrices belonging to random ensembles. Moreover, diverse techniques are naturally involved in the study of these ensembles, including tools from linear algebra, functional analysis, combinatorics, classical analysis and representation theory, among others.

Specially structured matrices play a central role in random matrix theory. Indeed, unitary models supported on the unit circle or the real line can be expressed as Toeplitz or Hankel determinants, respectively. Also the determinants of matrices that are the sum or difference of a Toeplitz and a Hankel matrix arise in this context, as they express integrals over the classical Lie groups with respect to Haar measure.

Numerous properties of these matrices have been studied over the years[2]. A fundamental result is the strong Szegő limit theorem, which describes the asymptotic behaviour of Toeplitz determinants generated by a sufficiently smooth function. While being a less investigated topic, several developments concerning Toeplitz±Hankel determinants have also been accomplished, including generalizations of Szegő's theorem.

Combinatorics is one of the areas that has benefited from the appearance of random matrix theory. Several problems in the field, some of which were long standing, have been solved after recognizing that a matrix model formulation is available for them. The application of techniques from the theory of random matrices has then sometimes lead to a solution for these problems[3].

[1]We do not attempt to provide a comprehensive historical review of the vast field of random matrix theory here, but refer to [93] for an excellent survey on the topic.

[2]Once again, since reviewing in detail the history and advances in the area would be an unfeasible task, we rather point to the outstanding survey [68] and references therein.

[3]A detailed exposition of two such examples can be found in [17], including the longest increasing subsequence problem outlined below.

Reciprocally, tools from combinatorics have also been found to be useful for studying matrix ensembles.

A key example of this premise is due to Baik, Deift and Johansson [15]. These authors found that the distribution of the longest increasing subsequence of a random permutation (properly centered and re-scaled) converges, as the size of the permutation grows to infinity, to the famous Tracy-Widom distribution, which also models the behaviour of the largest eigenvalue of a random Gaussian Hermitian matrix. To prove this, they used the well known RSK correspondence to express the relevant probability as a Toeplitz determinant, by means of an identity of Gessel involving Schur polynomials. The analysis was concluded using the Riemann-Hilbert methodology, that exploits the connection of Toeplitz determinants with orthogonal polynomials on the unit circle.

Developments like this showcase the full extent of random matrix theory, where the appearance of techniques from diverse branches of mathematics is rather the norm than an exception. While the tools employed in these studies may sometimes be technical, they often reveal fundamental information about such universal objects as random and structured matrices.

One of the aims of our work is to investigate further the relationship between random matrices, Toeplitz and Hankel determinants and symmetric functions.

The main focus of this thesis is the study of minors of Toeplitz, Hankel and Toeplitz±Hankel matrices. The aforementioned relationship extends naturally to this setting, as these minors can also be expressed as random matrix models, and in terms of symmetric functions. In particular, their matrix model expression includes the insertion of Schur polynomials in the integrand.

We draw inspiration from the work of Bump and Diaconis [50] concerning Toeplitz minors. In particular, they expressed these as unitary matrix models, and proved a generalization of Szegő's theorem, describing their asymptotic behaviour. It turns out that, as long as the generating function is sufficiently smooth, Toeplitz minors behave asymptotically as the corresponding Toeplitz determinant times a combinatorial factor, independent of the size of the minor.

We will generalize these observations, make some new ones, and use the results of our investigations to study related mathematical structures and physical theories.

1.2 Background

One motivation to study the minors of Toeplitz and Toeplitz±Hankel matrices, besides their own mathematical interest, arises from the fact that these can be expressed as the "twisted" integrals

$$\int_{G(N)} \chi^{\lambda}_{G(N)}(U^{-1})\chi^{\mu}_{G(N)}(U)f(U)dU, \tag{1.1}$$

where dU denotes Haar measure on one of the classical Lie groups

$$G(N) = U(N), Sp(2N), O(2N), O(2N+1),$$

and the $\chi^{\mu}_{G(N)}(U)$ are the characters associated to the irreducible representations of these groups. Minors of Toeplitz matrices have appeared explicitly in the literature before, often in relation with symmetric functions and Schur polynomials; articles devoted to their study include [141, 186, 146, 64, 65, 8, 153]. However, besides the already mentioned work of Bump and Diaconis

[50], none of them exploits the equivalent formulation of Toeplitz minors as matrix models given by (1.1). These integrals represent a logical step forward in the study of matrix models other than the more classical ensembles. Let us explain this in more detail.

First, recall the fact that integrals of the type (1.1), without the insertion of irreducible characters, can be computed as determinants of Toeplitz and Toeplitz±Hankel matrices. Due to the ubiquity of these matrices and the amount of applications in numerous branches of mathematics and physics[4], many of their properties have been investigated and much is known about their determinants. In particular, their asymptotic behaviour is now well understood, thanks to the work of Szegő [181], Johansson [127] and Deift, Its and Krasovsky [67], among many other authors. While the properties and formulas concerning these determinants are often algebraic in nature, whenever the size of the matrix is finite (see for instance [62, 37], among many others), the study of their asymptotic features relies heavily on analytical tools. Moreover, qualitative differences in their behaviour arise depending on the analyticity of the function f in (1.1). For instance, the results of [127] make use of fine probabilistic estimations, and the work [67] features an impressively thorough application of the Riemann-Hilbert methodology.

On the other hand, if the function f is chosen to be the identity in (1.1), then the integral simplifies drastically. Due to the orthonormality of the characters, it vanishes unless the partitions indexing the two characters coincide, in which case it evalues to 1. This fundamental fact regarding characters associated to irreducible representations has been used extensively in the study of random matrices, along the lines of the pioneering work of Diaconis and Shahshahani [76]. As a consequence, the computation of correlations of algebraic functions on random matrices over the classical groups can be reduced to sums over the trivial correlations of their irreducible characters. In particular, the results in [50] are based on this fact, among other works[5], including [74, 49, 89]. This purely algebraic procedure is usually known as character expansion.

The presence of both an arbitrary integrable function and character insertions in these integrals, as in (1.1), leads naturally to the combination of both analytic and algebraic tools in their study. This entails both a challenge and an opportunity, as a richer approach to more complicated matrix models becomes available if the structure and properties of these integrals are well understood.

In addition to the already mentioned works [76, 50], we are also inspired by the studies of Baik and Rains [18] and Bump and Gamburd [49], who perform a systematic analysis of integrals over the classical groups with the aid of representation theoretical tools, in a spirit we adopt and generalize in the present thesis. We find further motivation in the works of Borodin and Okounkov [37] and Tracy and Widom [185], which discuss the interplay between the symmetric function formulation of Toeplitz determinants and their realization as determinants generated by actual functions, a central topic in this thesis. We also adopt partly the philosophy of Luque, Vivo and coauthors [145, 51] in their study of related matrix models, using symmetric function expansions to provide a computational alternative to the analysis of matrix integrals.

We find worth mentioning the work of Ishikawa and collaborators, who have also studied minors and minor summation formulas in relation to Schur polynomials and symmetric functions in a series of articles [118]-[123], also finding applications in random matrix theory (see for

[4]See for instance the references in section 2 of [198] for a sample of these, and [41] for an introductory monograph on Toeplitz and Hankel matrices.

[5]This feature is also rooted in the approach of Weingarten calculus to group integrals developed in [56].

instance [187]). Another topic intimately related to our work, which we feel deserves deeper inspection, is the use of symmetric functions in the study of orthogonal polynomials. See [142, 111, 35, 34] for some examples of this approach, among others.

Explicit expressions for the averages of Schur polynomials have been computed over several random matrix ensembles, even if their realization as Toeplitz or Hankel minors was not explicitly identified. These include averages over the Gaussian Unitary Ensemble [73], the Jacobi Unitary Ensemble [130], the Stieltjes-Wigert ensemble [78] and the real [177], complex and quaternionic [92] Ginibre ensembles.

1.3 Motivations

Further reasons to study the minors of Toeplitz, Hankel and Toeplitz±Hankel matrices include the following.

- **Generalized matrix models.** Schur polynomials form a basis in the ring of symmetric functions and, unlike other distinguished basis in this ring, they do as a vector space. Therefore, assuming it is possible to characterize the average of a Schur polynomial over a given ensemble, one can then in principle compute general insertions in the model by expressing these as sums over such averages. The same holds true for the irreducible characters of the symplectic and orthogonal groups and the spaces of class functions over these groups, which in particular are also symmetric[6].

  Although the above reasoning holds from a theoretical point of view, sometimes this approach is not satisfactory at the practical level. Often matrix models of interest, which can be identified with insertions in well understood ensembles, cannot be computed explicitly. This may be because the insertion itself is complicated, because it poses structural challenges, or simply because there is not an exact formula available for such integral. Even in these cases, expanding the insertions in terms of symmetric functions may be a useful tool, as the natural grading associated to Schur polynomials (by means of the weight of the associated partition) often provides a simple way to identify the higher order contributions to the sum on some of the parameters associated to the model, see [84] for instance. Such expansions also offer the possibility to perform a computer assisted analysis of the models, providing efficient implementations whenever the size of the ensemble is small.

- **Symmetric functions and combinatorics.** The techniques of random matrix theory can also be employed to investigate symmetric functions and to obtain explicit results concerning these objects. This is due to the deeper relationship between matrices chosen at random from the groups $G(N)$ with respect to Haar measure and the universal characters associated to them in the ring of symmetric functions, a connection rooted in the common framework of representation theory.

  In addition to this, and in a more direct fashion, several quantities of combinatorial interest have expressions in terms of Toeplitz or Toeplitz±Hankel determinants and minors [157]. For instance, the number of standard and semi-standard Young tableaux (respectively

[6]One might of course forget about the group theoretical origin of the ensembles considered and investigate non-symmetric insertions. Also in this case, usually the symmetry of the density of Haar measure on the groups $G(N)$ allows an easy expression of such integrals in terms of the symmetrized insertions.

symplectic and orthogonal tableaux) of some shape is given by a specialization of the Schur polynomial (respectively symplectic and orthogonal Schur polynomial) indexed by the corresponding partition [158]. A richer example is given by the already reviewed longest increasing subsequence problem, where the exponential specialization in the ring of symmetric functions plays a central role. Further examples include nonintersecting random walks [110, 4] and Brownian motions [109], and enumeration of plane partitions and rhombus tilings [54], among others.

Toeplitz minors also play a central role in the rich Schur process [163], which is a source of applications to the study of probabilistic properties of random partitions and other combinatorial objects.

- **Gauge theory.** Averages of Schur polynomials over random matrix ensembles also appear in contemporary physical theories. In gauge theories with a matrix model description, these correspond to non-local observables such as Wilson loops. The approach of symmetric functions at the structural level is still underdeveloped in this context (see [156, 12] nevertheless), but it is of particular interest. This is due to the fact that symmetric functions provide a unified tool to study theories with any symmetry group $G(N)$, while the classical techniques are usually best suited for the unitary setting. Particularly relevant for us is the case of Chern-Simons theory on S^3, for which both the partition function and Wilson loops are known and have been studied in detail for the unitary theory [148, 78], while only the partition function in the large N regime has been computed for the symplectic and orthogonal theories [176].

  It is worth mentioning that the determinants of Toeplitz±Hankel matrices have many applications in statistical mechanics problems and describe several physical properties of a number of strongly correlated systems [33], starting with their appearance in the Ising model [68]. In such applications, the Toeplitz±Hankel case corresponds to open boundary conditions, whereas Toeplitz determinants correspond to periodic boundary conditions [60]. Although the study of minors is less developed, they appear in the same context as the determinants, allowing the treatment of more general interaction patterns [32, 174].

- **Orthogonal polynomials.** Minors of Toeplitz and Hankel matrices also appear in the well known connection between the determinants of these matrices and the theory of orthogonal polynomials. Indeed, many quantities of interest for the orthogonal polynomials with respect to a given weight function in the unit circle or the real line can be expressed in terms of minors of the Toeplitz or Hankel matrix generated by this function, including the coefficients of the polynomials themselves, the coefficients in the three-term recurrence relations or the Christoffel-Darboux kernel, for instance.

- **Relations between unitary, orthogonal and symplectic matrix ensembles.** Matrix integrals over the unitary group have attracted much more attention in the literature than their symplectic and orthogonal counterparts. Despite some recent advances on the topic, where some classical results known for Toeplitz matrices have been generalized to the Toeplitz±Hankel case (see for instance [127, 67, 82, 90, 30]), much fewer works are concerned with the relation *between* ensembles with different symmetries, as in [91, 54]. Finding examples of such connections is a topic of interest, particularly for the reviewed applications in gauge theory and combinatorics, as they provide relations between objects with different underlying symmetries.

- **Representation theory.** While we have not adopted this perspective in the present work, we feel that the study of integrals of the type (1.1) may be valuable from the point of view of representation theory. Indeed, these integrals may be understood as deformed inner products in the space of class functions on the groups $G(N)$. This provides a natural generalization of a basic tool in character theory, which we believe is worth investigating. Furthermore, the equivalent expressions of the integrals in terms of symmetric functions may also yield information concerning the irreducible representations of the groups $G(N)$.

  Another topic which could benefit from this connection is the representation theory of the infinite symmetric group and the infinite dimensional versions of the classical groups $G(N)$. Relevant objects for these groups, such as extreme characters, can be approximated by their finite dimensional analogues, see for instance [38, 59]. It is then natural to wonder if the well developed study of the asymptotic behaviour of random matrix ensembles can be exploited in this context, by means of the equivalent expressions of the matrix models (1.1) in terms of symmetric functions. It is also worth noting that this kind of advances have numerous applications in related topics [108, 48].

1.4 Plan for the thesis

We outline now the structure of the remainder of the thesis. We adopt here a general point of view and refer the reader to the short summaries included at the beginning of each chapter for more detailed descriptions of their contents.

Chapter 2 is concerned with the study of Toeplitz minors, which correspond to the unitary case of the integral (1.1), and serves as a demonstration of the approach we later adopt for the rest of the groups $G(N)$. In particular, we introduce a key notion, which is that of matrices generated by formal expressions in terms of symmetric functions. Exploiting this idea at the structural level, rather than using symmetric functions as a tool in the study of matrix models, we obtain a deeper understanding of the objects involved in this connection.

Our main application involves semi-banded Toeplitz matrices, which are especially suited to this approach. This, in addition to the richer structure present in the unitary case, allows a fruitful investigation of some properties of skew Schur polynomials.

In chapter 3 we develop the analogous formalism for the case of Toeplitz±Hankel matrices, which correspond to the symplectic and orthogonal groups. A second key concept is displayed here, which is the fact that the use of symmetric functions allows a unified approach in the study of the matrix integrals (1.1) for any of the groups $G(N)$. This allows investigation of analogous features of four families of objects, those associated to each of the groups $G(N)$, in the same conceptual framework.

Once the results from chapter 2 have been established, their analogues in this setting follow using similar reasonings. We therefore turn to examining relations between matrix models with different underlying symmetries. We have found this to be an attractive but underdeveloped topic. A third key idea results of particular interest here, which is the fact that the equivalence between matrix models, symmetric functions and minors of structured matrices is also profitable at a practical level, and not just a piece of isolated theory. Indeed, once a feature concerning one of the objects in this connection has been established, usually using the properties intrinsic to that object, it may be translated into a statement regarding the rest of the items in the connection, which may not have been obvious in their separate contexts.

We then concentrate into the study of the exactly solvable ensemble corresponding to Jacobi's third theta function, which models Chern-Simons theory on the three-sphere. This serves as an opportunity to exploit the results and showcase the philosophy of the thesis. We use the various techniques developed and equivalent formulations of the model to analyze it in various regimes. We also approach fermion quantum models with matrix degrees of freedom by reducing their study to the Chern-Simons model, demonstrating how the ideas and features developed in the thesis might be of use at a practical level.

Lastly, we address the case of Hankel minors in chapter 4. We explore their relation with the theory of orthogonal polynomials, a topic we believe deserves further investigation. We focus on a particular example: the insertion of a characteristic polynomial in the Laguerre Unitary Ensemble. After providing tools for the study of the model in the finite regime, we pose and solve the associated Riemann-Hilbert problem to obtain its large N behaviour. While analogous problems have been considered in the literature from this perspective, we choose to employ this approach due to the prominence and reach of the Riemann-Hilbert methodology in modern random matrix theory, and in particular in some of the topics covered in this thesis.

We have chosen to prioritize clarity in the exposition and attempt to deliver a fluent presentation of our results. For ease of reading, let us briefly remark the main original contributions of our work. All the numbered theorems and corollaries in the text are new to the best of our knowledge, and comparisons with the existing literature are provided where appropriate. In addition to these, we obtain novel expressions for Toeplitz minors (2.30),(2.32), inverses of Toeplitz matrices (2.39), and the biorthogonal polynomials with respect to a given function on the unit circle (2.61) in terms of symmetric functions. We also provide explicit expressions of some specializations of certain skew Schur polynomials (as well as their asymptotic behaviour) (2.43),(2.44),(2.48),(2.52),(2.64),(2.65),(2.67),(2.68) and of the biorthogonal polynomials with respect to truncated theta functions (2.63),(2.66), which we have not been able to find in the literature. Moreover, we compute the Hopf links of Chern-Simons theory on S^3 with $G(N)$ symmetry (3.62)-(3.65).

Chapter 2

Toeplitz minors and specializations of skew Schur polynomials

Chapter summary

We introduce the formalism of Toeplitz minors and explain their relation with unitary integrals. After reviewing some results on symmetric functions, we express such minors in terms of skew Schur polynomials, and obtain their asymptotic behaviour in terms of the associated Toeplitz determinants. We then characterize a class of Toeplitz minors for which an exact asymptotic expression can be obtained, and a class of Toeplitz minors that can be realized as the specialization of a single skew Schur polynomial. We compute the inverses of several Toeplitz matrices, using Chebyshev polynomials, the Duduchava-Roch formula and the kernel associated to two sets of biorthogonal polynomials on the unit circle. Comparing our formulas on Toeplitz minors with these inverses, we obtain explicit evaluations of a Selberg-Morris integral and for specializations of certain skew Schur polynomials. Finally, we use Laplace expansion on a Toeplitz determinant to obtain a set of relations satisfied by skew Schur poynomials[7].

2.1 Preliminaries

2.1.1 Toeplitz minors

Let $f(e^{i\theta}) = \sum_{k\in\mathbb{Z}} d_k e^{ik\theta}$ be an integrable function on the unit circle. The Toeplitz matrix generated by f is the matrix

$$T(f) = (d_{j-k})_{j,k\geq 1} = \begin{pmatrix} d_0 & d_{-1} & d_{-2} & \\ d_1 & d_0 & d_{-1} & \ddots \\ d_2 & d_1 & d_0 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}.$$

[7]The content of this chapter is based on the preprint [105]. Some results that can be found here but not in [105] include corollary 2, the computations involving the pentadiagonal Toeplitz matrix in section 2.3.1, identity (2.69) and theorems 2 and 3.

That is, $T(f)$ is an infinite matrix, constant along its diagonals, which entries are the Fourier coefficients of the function f, given by

$$d_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\theta} f(e^{i\theta})d\theta. \tag{2.1}$$

We denote by $T_N(f)$ its principal submatrix of order N, and its determinant by

$$D_N(f) = \det T_N(f).$$

We record the statement and a proof of the classical Andréief's identity, a central result in random matrix theory, as it will play a significant role in the following.

Lemma (Andréief, [9])**.** *Let $g_1,\ldots,g_N$ and $h_1,\ldots,h_N$ be integrable functions on a measure space (X,σ). Then,*

$$\frac{1}{N!}\int_{X^N} \det\left(g_j(z_k)\right)_{j,k=1}^N \det\left(h_j(z_k)\right)_{j,k=1}^N \prod_{k=1}^N d\sigma(z_k) = \det\left(\int_X g_j(z)h_k(z)d\sigma(z)\right)_{j,k=1}^N. \tag{2.2}$$

Proof. Expanding the second determinant in the left hand side above we see that

$$\begin{aligned}
&\int_{X^N} \det\left(g_j(z_k)\right)_{j,k=1}^N \det\left(h_j(z_k)\right)_{j,k=1}^N \prod_{k=1}^N d\sigma(z_k)\\
&= \sum_{\pi\in S_N}\operatorname{sgn}(\pi)\int_{X^N}\det\left(g_j(z_k)\right)_{j,k=1}^N \prod_{k=1}^N h_{\pi(k)}(z_k)\prod_{k=1}^N d\sigma(z_k)\\
&= \sum_{\pi\in S_N}\operatorname{sgn}(\pi)\int_{X^N}\det\begin{pmatrix} f_1(z_1)g_{\pi(1)}(z_1) & f_1(z_2)g_{\pi(2)}(z_2) & \ldots & f_1(z_N)g_{\pi(N)}(x_N)\\ f_2(z_1)g_{\pi(1)}(z_1) & f_2(z_2)g_{\pi(2)}(z_2) & \ldots & f_2(z_N)g_{\pi(N)}(x_N)\\ \vdots & \vdots & & \vdots\\ f_N(z_1)g_{\pi(1)}(z_1) & f_N(z_2)g_{\pi(2)}(z_2) & \ldots & f_N(z_N)g_{\pi(N)}(x_N)\end{pmatrix}\prod_{k=1}^N d\sigma(z_k)\\
&= \sum_{\pi\in S_N}\operatorname{sgn}(\pi)\det\left(\int_X f_j(z)g_{\pi(k)}(z)d\sigma(z)\right)_{j,k=1}^N = N!\det\left(\int_X g_j(z)h_k(z)d\sigma(z)\right)_{j,k=1}^N,
\end{aligned}$$

which is precisely the desired conclusion. □

Choosing as measure $d\sigma(\theta) = \frac{1}{2\pi}f(e^{i\theta})d\theta$ on $[0,2\pi)$ in this identity, where $d\theta$ is the usual Lebesgue measure on this interval, and setting $g_j(z) = h_j(z^{-1}) = z^{N-j}$ for $j = 1,\ldots,N$, where $z = e^{i\theta}$, one obtains the following integral representation for the Toeplitz determinant of size N

$$D_N(f) = \frac{1}{N!}\frac{1}{(2\pi)^N}\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{1\le j<k\le N}|e^{i\theta_j}-e^{i\theta_k}|^2\prod_{j=1}^N f(e^{i\theta_j})d\theta_j. \tag{2.3}$$

Note that the first product in the integral above is the square modulus of the usual Vandermonde determinant on the points $e^{i\theta_j}$. This is known as Heine, or Heine-Szegő identity.

Toeplitz determinants can also be expressed as integrals over the group of unitary matrices $U(N)$, a fact of particular interest from the point of view of random matrix theory. Given a function f on the unit circle, we define, for any $U\in U(N)$, the function

$$f(U) = \prod_{j=1}^N f(e^{i\theta_j}), \tag{2.4}$$

where the $e^{i\theta_j}$ are the eigenvalues of U. Using Weyl's integral formula [195, 103], one can reduce integrals over $U(N)$ with respect to Haar measure to integrals over the subset of diagonal matrices, as these form a maximal torus in the group, which coincide precisely with the right hand side of (2.3). This leads to the expression

$$D_N(f) = \int_{U(N)} f(U)dU,$$

where dU denotes the normalized Haar measure on $U(N)$. That is, given a function f on the unit circle, the Toeplitz determinant of size N generated by this function coincides with the integral over the group of unitary matrices of size N of the function $f(U)$. In particular, the study of the matrix model (2.3) and its large N behaviour for different choices of f can be utilized to investigate the statistical properties of random unitary matrices, see for instance [75].

A Toeplitz minor is a minor of a Toeplitz matrix, obtained by striking a finite number of rows and columns from a Toeplitz matrix of finite size. Any particular striking can be encoded in a pair of integer partitions λ and μ (see section 2.1.2 for more details and some basic facts on partitions), and thus one can see that any Toeplitz minor can be realized as the determinant of a matrix of the form

$$T_N^{\lambda,\mu}(f) = (d_{j-\lambda_j-k+\mu_k})_{j,k=1}^N. \tag{2.5}$$

We denote the minor itself by

$$D_N^{\lambda,\mu}(f) = \det T_N^{\lambda,\mu}(f).$$

Choosing λ and μ to be empty partitions above we recover a Toeplitz determinant of size N. Setting $h_j(z) = z^{N-j+\lambda_j}$ and $g_k(z) = z^{-(N-k+\mu_k)}$ in Andreiéf's identity and using (2.10), we see that Toeplitz minors also have an integral representation [50]

$$D_N^{\lambda,\mu}(f) = \int_{U(N)} s_\lambda(U^{-1}) s_\mu(U) f(U) dU = \tag{2.6}$$

$$\frac{1}{N!}\frac{1}{(2\pi)^N}\int_0^{2\pi}\cdots\int_0^{2\pi} s_\lambda(e^{-i\theta_1},...,e^{-i\theta_N}) s_\mu(e^{i\theta_1},...,e^{i\theta_N}) \prod_{j=1}^N f(e^{i\theta_j}) \prod_{1\le j<k\le N} |e^{i\theta_j}-e^{i\theta_k}|^2 d\theta_1...d\theta_N,$$

where s_λ, s_μ are Schur polynomials[8]. We see that symmetric functions are present already in the elementary procedure of choosing a minor from a Toeplitz matrix. Let us review some basic facts about such functions before delving into this relationship.

2.1.2 Symmetric functions

We recall some basic results involving symmetric functions that can be found in [147, 178], for example. We denote $z = e^{i\theta}$ in the following, and treat z as a formal variable. A partition $\lambda = (\lambda_1, \dots, \lambda_l)$ is a finite and non-increasing sequence of positive integers. The number of nonzero entries is called the length of the partition and is denoted by $l(\lambda)$, and the sum $|\lambda| = \lambda_1 + \cdots + \lambda_{l(\lambda)}$ is called the weight of the partition. The entry λ_j is understood to be zero whenever the index j is greater than the length of the partition. The notation (a^b) stands for

[8] We abuse notation here; we assume it is clear when the expression $f(U)$ should be read as $\prod_j f(e^{i\theta_j})$ (i.e. when f is a function on the unit circle) and when it should be read as $f(e^{i\theta_1}, \dots, e^{i\theta_N})$ (i.e. when f is a symmetric function in several variables). See the next section for definitions and basic facts concerning Schur polynomials.

the partition with exactly b nonzero entries, all equal to a. A partition can be represented as a Young diagram, by placing λ_j left-justified boxes in the j-th row of the diagram. The conjugate partition λ' is then obtained as the partition which diagram has as rows the columns of the diagram of λ (see figure 2.1.2 for an example).

Figure 2.1: The partition $(3, 2, 2)$ and its conjugate $(3, 3, 1)$.

Lemma (See 1.7 in [147], for instance)**.** *Let λ be a partition satisfying $l(\lambda) \leq N$ and $\lambda_1 \leq K$ (that is, such that its Young diagram is contained in the rectangular shape (K^N)). Then, the $N + K$ numbers*

$$\{K + j - \lambda_j\}_{j=1}^{N} \cup \{\lambda'_j + K + 1 - j\}_{j=1}^{K} \tag{2.7}$$

are a permutation of $\{1, 2, \ldots, N + K\}$.

Some inspection shows that increasing sequences of N integers are in correspondence with arrays of the form $(j-\lambda_j)_{j=1}^{N}$, where λ is a partition of length not greater than N. Therefore, any particular choice of rows and columns from a Toeplitz matrix to form a minor can be encoded in a pair of partitions, by means of equation (2.5). The following procedure describes how to obtain this minor from the underlying Toeplitz matrix $T(f)$. We assume in the following that the length of the partitions λ and μ is less than or equal to N, the size of the minor under consideration.

- Strike the first $|\lambda_1-\mu_1|$ columns or rows of $T_{N+\max\{\lambda_1,\mu_1\}}(f)$, depending on whether $\lambda_1-\mu_1$ is greater or smaller than zero, respectively.
- Keep the first row of the matrix, and strike the next $\lambda_1 - \lambda_2$ rows. Keep the next row, and strike the next $\lambda_2 - \lambda_3$ rows. Continue until striking $\lambda_{l(\lambda)} - \lambda_{l(\lambda)+1} = \lambda_{l(\lambda)}$ rows.
- Repeat the previous step on the columns of the matrix with μ in place of λ. The resulting matrix is precisely $T_N^{\lambda,\mu}(f)$, as defined in (2.5).

Let $x = (x_1, x_2, ...)$ be a set of variables. Let us identify several distinguished families of generators of the ring of symmetric functions in the variables x, which will be useful in the following. The power-sum symmetric polynomials p_k are given by $p_k(x) = x_1^k + x_2^k + \ldots$ for every $k \geq 1$, and $p_0(x) = 1$. The elementary symmetric polynomials $e_k(x)$ and the complete homogeneous polynomials $h_k(x)$ are

$$h_k(x) = \sum_{i_1 \leq \cdots \leq i_k} x_{i_1} \ldots x_{i_k}, \qquad e_k(x) = \sum_{i_1 < \cdots < i_k} x_{i_1} \ldots x_{i_k}. \tag{2.8}$$

We also set $p_k(x) = h_k(x) = e_k(x) = 0$ for negative k, and we set empty sums to 1. These families of functions are related by the identities

$$\begin{aligned} H(x; z) &= \sum_{k=0}^{\infty} h_k(x) z^k = \exp\left(\sum_{k=1}^{\infty} \frac{p_k(x)}{k} z^k\right) = \prod_{j=1}^{\infty} \frac{1}{1 - x_j z}, \\ E(x; z) &= \sum_{k=0}^{\infty} e_k(x) z^k = \exp\left(\sum_{k=1}^{\infty} (-1)^{k+1} \frac{p_k(x)}{k} z^k\right) = \prod_{j=1}^{\infty} (1 + x_j z). \end{aligned} \tag{2.9}$$

The families $(h_k(x))$ and $(e_k(x))$, where $k \geq 0$, consist of algebraically independent functions. Moreover, each of the families form a complete set of generators of the ring of symmetric functions in x. Hence, we will see H and E as arbitrary functions on the unit circle depending on the parameters x, and we will use indistinctly their infinite product expression. Note that these two functions satisfy $H(x;z)E(x;-z) = 1$.

Another distinguished family of symmetric functions is that of Schur polynomials. These form a basis for the ring of symmetric functions, as a vector space, and are indexed by partitions. Among their several equivalent definitions, the classical Jacobi-Trudi identities express Schur polynomials as Toeplitz minors generated by the functions H and E

$$\begin{aligned} s_\mu(x) &= \det\left(h_{j-k+\mu_k}(x)\right)_{j,k=1}^N = D_N^{\varnothing,\mu}\left(H(x;z)\right), \\ s_{\mu'}(x) &= \det\left(e_{j-k+\mu_k}(x)\right)_{j,k=1}^N = D_N^{\varnothing,\mu}\left(E(x;z)\right), \end{aligned}$$

where μ verifies $l(\mu) \leq N$ (resp. $l(\mu') \leq N$) in the first (resp. second) identity, and $\varnothing$ denotes the empty partition. If the set of variables is finite, say $x = (x_1, \dots, x_N)$, one can also define the Schur polynomial indexed by μ as

$$s_\mu(x_1, \dots, x_N) = \frac{\det\left(x_j^{N-k+\mu_k}\right)_{j,k=1}^N}{\det\left(x_j^{N-k}\right)_{j,k=1}^N}, \tag{2.10}$$

where we set $s_\mu(x_1, \dots, x_N) = 0$ if $l(\mu) > N$. Note that the denominator in the above formula is actually the Vandermonde determinant on the variables x, while the determinant in the numerator is a minor of the Vandermonde matrix. In particular, the integral formula (2.6) can be deduced from this fact and Andréief's identity. Using L'Hôpital's rule in (2.10) one can deduce the identity

$$s_\mu(1^N) = \frac{1}{G(N+1)} \prod_{1\leq j<k\leq N} (\mu_j - \mu_k + k - j), \tag{2.11}$$

which holds for any $N \geq l(\mu)$, where G is the Barnes function.

Given two partitions λ and μ, the symmetric function $s_\lambda(x)s_\mu(x)$ can be expanded in the basis of Schur polynomials; we write this decomposition as

$$s_\lambda(x)s_\mu(x) = \sum_\nu c_{\lambda\mu}^\nu s_\nu(x). \tag{2.12}$$

The coefficients $c_{\lambda\mu}^\nu$ are known as Littlewood-Richardson coefficients. Skew Schur polynomials are defined by the expansion

$$s_{\mu/\lambda}(x) = \sum_\nu c_{\lambda\nu}^\mu s_\nu(x). \tag{2.13}$$

Skew Schur polynomials can also be expressed as Toeplitz minors generated by the functions H and E

$$s_{\mu/\lambda}(x) = D_N^{\lambda,\mu}(H(x;z)), \qquad s_{(\mu/\lambda)'}(x) = D_N^{\lambda,\mu}(E(x;z)), \tag{2.14}$$

where $l(\mu), l(\mu') \leq N$ respectively. A skew Schur polynomial vanishes if $\lambda \not\subseteq \mu$; this can be seen as a consequence of its Toeplitz minor representation and the fact that the Toeplitz matrices generated by H and E are triangular. A central result in the theory of symmetric functions is the Cauchy identity, and its dual form

$$\sum_\nu s_\nu(x)s_\nu(y) = \prod_{j=1}^\infty \prod_{k=1}^\infty \frac{1}{1 - x_j y_k}, \qquad \sum_\nu s_\nu(x)s_{\nu'}(y) = \prod_{j=1}^\infty \prod_{k=1}^\infty (1 + x_j y_k),$$

where $y = (y_1, y_2, \dots)$ is another set of variables and the sums run over all partitions ν.

Gessel [107] obtained the following expression for the Toeplitz determinant generated by the function $f(z) = H(y; z^{-1})H(x; z)$

$$D_N\left(\prod_{k=1}^{\infty}\frac{1}{1-y_k z^{-1}}\prod_{j=1}^{\infty}\frac{1}{1-x_j z}\right) = \sum_{l(\nu)\le N} s_\nu(y)s_\nu(x), \tag{2.15}$$

where the sum runs over all partitions ν of length $l(\nu) \le N$. If one of the sets of variables x or y is finite, say $y = (y_1, \dots, y_d)$, comparing the right hand side above with the sum in Cauchy identity and recalling that the Schur polynomial $s_\nu(y_1, ..., y_d)$ vanishes if $l(\nu) > d$ one obtains a well known identity of Baxter [27]

$$D_N\left(\prod_{k=1}^{d}\frac{1}{1-y_k z^{-1}}\prod_{j=1}^{\infty}\frac{1}{1-x_j z}\right) = \prod_{k=1}^{d}\prod_{j=1}^{\infty}\frac{1}{1-x_j y_k}, \tag{2.16}$$

valid when $N \ge d$. Note that the right hand side above is independent of N. An analogous identity follows if the factor $H(x; z)$ is replaced by $E(x; z)$, using the dual Cauchy identity instead. However, no such identity is available for Toeplitz determinants generated by functions of the type $E(y; z^{-1})E(x; z)$; this will be relevant later.

All the above identities should be regarded as formal identities in the ring of symmetric functions. In the following, we will sometimes specialize the variables x (or any set of generators in this ring) to obtain actual identities for particular functions, or, equivalently, for particular matrix models.

Given a partition λ satisfying $l(\lambda) \le N$ and $\lambda_1 \le K$ (that is, $\lambda \subset (K^N)$), we define a new partition by

$$L_{K,N}(\lambda) = (K - \lambda_N, \dots, K - \lambda_1) = (K^N) - \lambda^r, \tag{2.17}$$

where λ^r denotes the "reversed" array $(\lambda_N, \dots, \lambda_1)$. That is, $L_{K,N}(\lambda)$ is the partition that results from rotating 180º the complement of λ in the rectangular shape (K^N); see figure 2.1.2 for an example. We see that the following relation holds

$$L_{K,N}(\lambda) = \left(L_{N,K}(\lambda')\right)'.$$

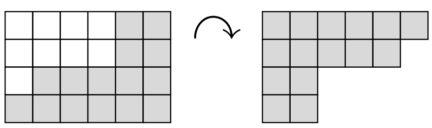


Figure 2.2: The partition $\lambda = (4, 4, 1)$ and the partition $L_{6,4}(\lambda) = (6, 5, 2, 2)$.

Lemma. *Let λ be a partition verifying $\lambda \subset (K^N)$ (that is, $l(\lambda) \le N$ and $\lambda_1 \le K$). We have*

$$s_\lambda(x_1^{-1}, \dots, x_N^{-1}) = s_{L_{K,N}(\lambda)}(x_1, \dots, x_N)\prod_{j=1}^{N} x_j^{-K}. \tag{2.18}$$

A proof follows from direct manipulations in (2.10), for instance. In sight of the integral representation (2.6), we obtain as a consequence the identity

$$D_N^{\lambda,\mu}(f) = D_N^{L_{K,N}(\mu),L_{K,N}(\lambda)}, \tag{2.19}$$

which holds for any K satisfying $\max(\lambda_1, \mu_1) \leq K$.

2.1.3 Asymptotic behaviour of Toeplitz determinants and minors generated by smooth functions.

We record now precise statements of the strong Szegő limit theorem and of its generalization to Toeplitz minors due to Bump and Diaconis.

Theorem (Szegő). *Let $f(e^{i\theta}) = \sum_{k\in\mathbb{Z}} d_k e^{ik\theta}$ be a function on the unit circle, and suppose it can be expressed as $f(e^{i\theta}) = \exp(\sum_{k\in\mathbb{Z}} c_k e^{ik\theta})$, where the coefficients c_k verify*

$$\sum_{k\in\mathbb{Z}} |c_k| < \infty, \qquad \sum_{k\in\mathbb{Z}} |k||c_k|^2 < \infty. \tag{2.20}$$

Let us assume, moreover, that $c_0 = 0$, without loss of generality. Then,

$$\lim_{N\to\infty} D_N(f) = \exp\left(\sum_{k=1}^{\infty} k c_k c_{-k}\right).$$

Note that, after dividing the Toeplitz determinant $D_N(f)$ by e^{Nc_0} (that is, multiplying the function f by a constant), one can always assume that the coefficient c_0 vanishes. We will therefore assume in the following that $c_0 = 0$ for all the functions involved, unless specified otherwise.

A function f satisfying the hypotheses of this theorem is continuous, nonzero, and has winding number zero [41]. Under these same conditions, the following theorem holds [50].

Theorem (Bump, Diaconis). *Let f verify the hypotheses in the previous theorem, and suppose λ and μ are partitions of weights n and m respectively. Then, as $N \to \infty$*

$$\lim_{N\to\infty} D_N^{\lambda,\mu}(f) = \left(\lim_{N\to\infty} D_N(f)\right) \sum_{\phi\vdash n} \sum_{\psi\vdash m} \chi_\phi^\lambda \, \chi_\psi^\mu \, z_\phi^{-1} z_\psi^{-1} \Delta(f,\phi,\psi), \tag{2.21}$$

where the sum runs over all the partitions ϕ of n and ψ of m, the terms z_ϕ, z_ψ are the orders of the centralizers of the equivalence classes of the symmetric groups S_n, S_m indexed by ϕ and ψ respectively, the functions χ^λ, χ^μ are the characters associated to the irreducible representations of S_n and S_m indexed by λ and μ respectively, and

$$\Delta(f,\phi,\psi) = \prod_{k=1}^{\infty} \begin{cases} k^{n_k} c_{-k}^{n_k-m_k} m_k! L_{m_k}^{(n_k-m_k)}(-k c_k c_{-k}), & \text{if } n_k \geq m_k \\ k^{m_k} c_k^{m_k-n_k} n_k! L_{n_k}^{(m_k-n_k)}(-k c_k c_{-k}), & \text{if } n_k \leq m_k \end{cases}.$$

Above, the coefficients n_k, m_k correspond to the partitions $\phi = (1^{n_1}2^{n_2}\dots)$ and $\psi = (1^{m_1}2^{m_2}\dots)$ in their frequency notation, and the $L_n^{(\alpha)}$ are Laguerre polynomials [181].

Note that the product in the factor $\Delta(f,\phi,\psi)$ is actually finite, since only a finite number of n_k's and m_k's are distinct from zero for each pair ϕ, ψ. We see that in the $N \to \infty$ limit the Toeplitz minor generated by a regular symbol factors as the corresponding Toeplitz determinant

λ	μ	$\lim_{N\to\infty} D_N^{\lambda,\mu}(f)/D_N(f)$	λ	μ	$\lim_{N\to\infty} D_N^{\lambda,\mu}(f)/D_N(f)$
$\varnothing$	$\square$	c_1	$\varnothing$	(2)	$\frac{1}{2}c_1^2 + c_2$
$\varnothing$	$(1,1)$	$\frac{1}{2}c_1^2 - c_2$	$\varnothing$	(3)	$\frac{1}{6}c_1^3 + c_1c_2 + c_3$
$\varnothing$	$(1,1,1)$	$\frac{1}{6}c_1^3 - c_1c_2 + c_3$	$\varnothing$	$(2,2)$	$\frac{1}{12}c_1^4 - c_1c_3 + c_2^2$

λ	μ	$\lim_{N\to\infty} D_N^{\lambda,\mu}(f)/D_N(f)$
$(1,1)$	$(1,1)$	$\frac{1}{4}c_{-1}^2c_1^2 + c_{-1}c_1 - \frac{1}{2}c_{-2}c_1^2 - \frac{1}{2}c_{-1}^2c_2 + c_{-2}c_2 + 1$
$\square$	(3)	$\frac{1}{6}c_{-1}c_1^3 + \frac{1}{2}c_1^2 + c_{-1}c_1c_2 + c_2 + c_{-1}c_3$

Table 2.1: Some values of the formula (2.21).

times a sum depending only on f and the partitions λ, μ (and not on N). The formula (2.21) can be implemented easily in a computer algebra system, leading to efficient evaluations for partitions of small weights. Table 2.1 shows some of these values for particular choices of λ and μ.

An equivalent expression for the sum in the right hand side of (2.21) was obtained by Tracy and Widom [186], and these were later compared by Dehaye in [65] in terms of symmetric functions. Further generalizations of this formula were given in [64, 146] by Dehaye and Lyons, respectively.

2.2 Toeplitz minors generated by symmetric functions

We turn to the computation of an equivalent formulation of the asymptotic formula (2.21). We start by proving a general result for the case of Toeplitz minors generated by formal power series, and then show how it implies an analogous result for minors generated by functions satisfying the hypotheses in Szegő's theorem.

Theorem 1. *Let x, y be some sets of variables, and consider the function*

$$f(z) = H(x; z)H(y; z^{-1}),$$

where H is given by (2.9)*. Then, for any two fixed partitions λ and μ we have*

$$\lim_{N\to\infty} D_N^{\lambda,\mu}(f) = \left(\lim_{N\to\infty} D_N(f)\right) \sum_{\nu} s_{\lambda/\nu}(y) s_{\mu/\nu}(x). \tag{2.22}$$

Note that we understand f as a formal Laurent power series whose coefficients are symmetric functions on x and y, and thus the convergence above is in the algebra of formal power series.

Proof. First, we note that the limit $\lim_{N\to\infty} D_N(f)$ in the right hand side of (2.22) is well defined as a formal expression, since by the identities of Gessel and Cauchy we have

$$\lim_{N\to\infty} D_N(f) = \lim_{N\to\infty} \sum_{l(\nu)\le N} s_\nu(x)s_\nu(y) = \prod_{j=1}^{\infty}\prod_{k=1}^{\infty} \frac{1}{1 - x_j y_k}. \tag{2.23}$$

If R, S are two strictly increasing sequences of natural numbers, we denote by $\det_{R,S} M$ the minor of the matrix M obtained by taking the rows and columns of M indexed by R and S, respectively. Now, we start from identity (2.19), for notational convenience, and obtain

$$D_N^{\lambda,\mu}(f) = D_N^{L_{K,N}(\mu),L_{K,N}(\lambda)} = \det_{R,S} T(f),$$

where the sequences R, S are given by $R = (r_j)_{j=1}^N = (j + \mu_{N+1-j})_{j=1}^N$ and $S = (s_k)_{k=1}^N = (k + \lambda_{N+1-k})_{k=1}^N$. Since the Toeplitz matrices generated by each of the factors of f verify $T(f(z)) = T(H(y; z^{-1}))T(H(x; z))$, the use of Cauchy-Binet formula gives

$$\det_{R,S} T(f(z)) = \sum_T \det_{R,T} T(H(y; z^{-1})) \det_{T,S} T(H(x; z)), \tag{2.24}$$

where the summation is over all the strictly increasing sequences $T = (t_1, \ldots, t_N)$ of length N of positive integers[9]. There is a correspondence between such sequences and partitions ν of length $l(\nu) \leq N$, given by $\nu_{N+1-j} = t_j - j$, for $j = 1, ..., N$. Thus, for each T we have

$$\det_{T,S} T(H(x; z)) = \det(h_{t_j - s_k}(x))_{j,k=1}^N = \det(h_{j+\nu_{N+1-j}-k-\lambda_{N+1-k})}(x))_{j,k=1}^N.$$

Reversing the order of its rows and columns, we see that the last determinant above is $D_N^{\lambda;\nu}(H(x; z))$. According to (2.14) this is precisely the skew Schur polynomial $s_{\nu/\lambda}(x)$, and an analogous derivation yields $\det_{R,T} T(H(y; z^{-1})) = s_{\nu/\mu}(y)$. We thus obtain[10]

$$D_N^{\lambda,\mu}(f) = \sum_{l(\nu)\leq N} s_{\nu/\mu}(y) s_{\nu/\lambda}(x). \tag{2.25}$$

Combining this with the following identity between Schur and skew Schur polynomials (see e.g. Ex. I.5.26 in [147])

$$\sum_\nu s_{\nu/\mu}(y) s_{\nu/\lambda}(x) = \sum_\kappa s_\kappa(y) s_\kappa(x) \sum_\nu s_{\lambda/\nu}(y) s_{\mu/\nu}(x), \tag{2.26}$$

where the sums run over all partitions, we arrive at the desired conclusion, upon identification of the first sum in the right hand side above with the large N limit of the Toeplitz determinant generated by f. □

An analogous reasoning shows that identity (2.22) holds also for functions of the form

$$f(z) = E(x; z)E(y; z^{-1}),$$

after taking the conjugate of all the partitions indexing the skew Schur polynomials in the right hand side of (2.22).

Let us emphasize that the theorem is to be understood as an identity among symmetric functions. However, as usual in this context, one can specialize an algebraically independent family of symmetric functions to any given sequence of, say, real or complex numbers, and extend (2.22) to an identity involving more general Toeplitz matrices, as long as the formal manipulations are justified after this specialization (see [178, 185, 18] for examples of this).

[9] We are actually using an infinite dimensional generalization of the Cauchy-Binet formula, as the one that appears in [185]. This is allowed since the sum in the right hand side is well defined as a formal expression.

[10] This type of formula also appears in the transition weights of the Schur process [163].

Let us consider, for instance, a function f that satisfies the regularity conditions in Szegő's theorem. That is, assume $f(e^{i\theta}) = \exp\left(\sum_k c_k e^{ik\theta}\right)$, where the coefficients c_k satisfy the decay conditions (2.20). Then, assuming that $c_0 = 0$ without loss of generality, we can write $f(e^{i\theta}) = f^+(e^{i\theta})f^-(e^{i\theta})$, where

$$f^+(e^{i\theta}) = \exp\left(\sum_{k>0} c_k e^{ik\theta}\right) = 1 + \sum_{k\geq 1} d_k^+ e^{ik\theta}, \qquad f^-(e^{i\theta}) = \exp\left(\sum_{k<0} c_k e^{ik\theta}\right) = 1 + \sum_{k\geq 1} d_k^- e^{-ik\theta}. \tag{2.27}$$

Now, recall that the complete homogeneous symmetric polynomials are a complete set of algebraically independent generators of the ring of symmetric functions. Thus, we can consider the specializations

$$h_k(x) \mapsto d_k^+, \qquad h_k(y) \mapsto d_k^- \qquad (k \geq 0),$$

on theorem 1 to recover the function f from the formal power series $H(x;z)H(y;z^{-1})$. Note also that the specialization of the skew Schur polynomials in the theorem can be defined in terms of the Fourier coefficients d_k^+, d_k^- by means of the Jacobi-Trudi identities, so that the right hand side in (2.22) is well defined (the sum is actually finite for any fixed pair of partitions λ and μ). Therefore, we can rephrase theorem 1 as follows.

Corollary 1. *Let* $f(e^{i\theta}) = \exp\left(\sum_k c_k e^{ik\theta}\right)$*, where the* c_k *satisfy the conditions* (2.20)*, and assume moreover that* $c_0 = 0$*, without loss of generality. Define* f^+ *and* f^- *as in* (2.27)*, and assume that these functions are square integrable. Then,*

$$\lim_{N\to\infty} D_N^{\lambda,\mu}(f) = \exp\left(\sum_{k=1}^{\infty} k c_k c_{-k}\right) \sum_{\nu} s_{\lambda/\nu}(d^-) s_{\mu/\nu}(d^+), \tag{2.28}$$

where the convergence is now the usual convergence in $\mathbb{C}$*, and we have denoted by* $s_{\lambda/\nu}(d^\pm)$ *the determinants*

$$s_{\lambda/\nu}(d^\pm) = \det\left(d^\pm_{j-\nu_j-k+\lambda_k}\right)_{j,k=1}^{\max(l(\lambda),l(\nu))}, \tag{2.29}$$

in terms of the Fourier coefficients of $f^\pm$*.*

Similar examples where an algebraic result concerning Toeplitz determinants generated by formal Laurent series is seen to be equivalent to an analytic one for functions satisfying the hypothesis in Szegő's theorem[11] can be found in [185, 37], for instance. As in theorem (1), an analogous result holds if one considers the specializations $e_k(x) \mapsto d_k^+$ and $e_k(y) \mapsto d_k^-$ instead, transposing the partitions in (2.28).

We have assumed in the above discussion that f verifies the hypotheses in Szegő's theorem. This was necessary in order for the limit $\lim_{N\to\infty} D_N(f)$ to be finite, so that the formal manipulations in theorem 1 are justified. Numerical experiments suggest however that theorem 1 holds for more general functions for which this limit is not finite, such as functions with Fisher-Hartwig singularities. It follows from a result of Lyons, see theorem 3.1 in [146], that this is indeed true for the case of Toeplitz matrices generated by positive valued functions (as is the case, for instance, of pure Fisher-Hartwig singularities with zeros or poles, see section

[11] We have assumed in addition that the functions $f^\pm$ are square integrable, so that the use of the infinite dimensional generalization of the Cauchy-Binet formula [185] is still valid.

2.3.2). However, we have been unable to extend this result to the most general case of arbitrary functions with Fisher-Hartwig singularities.

We conclude this section showing that exact formulas are available whenever the function f can be obtained as a specialization with a finite number of nonzero variables. There are two possibilities:

- Case 1: There is a factor of the type H specialized to a finite set of variables. Suppose that f is of the form $f(z) = H(y_1, \ldots, y_d; z^{-1})H(x; z)$. Then, in the same fashion as in Baxter's identity (2.16), the corresponding Toeplitz minor (2.25) stabilizes and we obtain the formula

$$D_N^{\lambda,\mu}\left(\prod_{k=1}^{d}\frac{1}{1-y_k z^{-1}}\prod_{j=1}^{\infty}\frac{1}{1-x_j z}\right) = \prod_{k=1}^{d}\prod_{j=1}^{\infty}\frac{1}{1-x_j y_k}\sum_{\nu} s_{\lambda/\nu}(y)s_{\mu/\nu}(x), \tag{2.30}$$

  which holds for every $N \geq d$. An analogous result holds for symbols of the type $f(z) = H(y_1, \ldots, y_d; z^{-1})E(x; z)$.

- Case 2: There is a factor of the type E specialized to a finite set of variables. We assume, without loss of generality, that f is of the form $f(z) = E(y_1, \ldots, y_d; z^{-1})E(x; z)$. As mentioned above, no N-independent formula is available for these symbols. However, the Fourier coefficients of this function are (2.9)

$$f(z) = \prod_{j=1}^{d}(1 + y_j z^{-1})\prod_{j=1}^{\infty}(1 + x_j z) = \sum_{k=-d}^{\infty}\left(\prod_{j=1}^{d} y_j\right) e_{d+k}(y_1^{-1}, \ldots, y_d^{-1}, x)z^k, \tag{2.31}$$

  and therefore it follows from (2.14) that

$$D_N^{\lambda,\mu}(E(y_1, \ldots, y_d; z^{-1})E(x; z)) = \left(\prod_{k=1}^{d} y_k^N\right) s_{((d^N)+\mu/\lambda)'}(y_1^{-1}, \ldots, y_d^{-1}, x). \tag{2.32}$$

  We see that in this case the Toeplitz minor can be expressed essentially as the specialization of a single skew Schur polynomial[12], indexed by shapes of the type depicted in figure 2.2. This fact will have several consequences, and we will use the function (2.31) as a running example in the following. The case $\lambda = \mu = \varnothing$ of (2.32) was first obtained in [58], see also theorem 7 and the subsequent discussion. An analogous identity has also been obtained [8] for the case $f(z) = E(y_1, \ldots, y_d; z^{-1})$. Comparing with the analogous of equation (2.25) for this symbol we see that (2.32) coincides with

$$\sum_{\nu \subset (N^d)} s_{\nu/\mu'}(y_1, ..., y_d)s_{\nu/\lambda'}(x),$$

  where the (finite) sum runs over all partitions ν satisfying $l(\nu) \leq N$ and $\nu_1 \leq d$.

[12] Of course, in sight of identities (2.14), any Toeplitz minor can be expressed as the specialization of a single skew Schur polynomial, with an adequate specialization and partitions λ and μ. The main feature of identity (2.32) is that the only dependance on N of the skew Schur polynomial is via the rectangular shape (d^N).

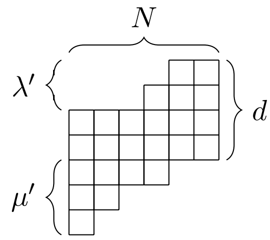


Figure 2.3: The shape $((d^N)+\mu/\lambda)'$, for $N=6, d=4$, $\mu=(3,2,1,1)$ and $\lambda=(2,2,2,1)$.

Corollary 2. *Let λ be a partition of length $l(\lambda)\leq d$, and let μ be any partition. The following identities hold*

$$\lim_{N\to\infty} s_{(N^d)/\lambda}(x) = \left(\lim_{N\to\infty} s_{(N^d)}(x)\right) s_\lambda(x_1^{-1},\dots,x_d^{-1}),$$
$$\lim_{N\to\infty} s_{(\underbrace{N,\dots,N}_{d},\mu_1,\dots,\mu_{l(\mu)})}(x) = \left(\lim_{N\to\infty} s_{(N^d)}(x)\right) s_\mu(x_{d+1},x_{d+2},\dots).$$

Note that due to the well known fact that the skew Schur polynomial indexed by a partition coincides with that indexed by the partition rotated 180$^{\text{o}}$ [168], the polynomial in the left hand side of the first identity in the corollary coincides with $s_{L_{N,d}(\lambda)}(x)$.

Proof. First of all, observe that due to the condition on λ and the fact that μ is a fixed partition that does not depend on N, the skew Schur and Schur polynomials in the left hand sides above are well defined for large enough N.

Now, let $x=(x_1,x_2,\dots)$ be a set of variables, and consider the function

$$f(z) = E(x_1^{-1},\dots,x_d^{-1};z^{-1})E(x_{d+1},x_{d+2},\dots;z).$$

The first and second identities result then from combining the case $\mu=\varnothing$ and $\lambda=\varnothing$ respectively of theorem 1 to the Toeplitz determinants and minors generated by f, in sight of their equivalent representation as Schur and skew Schur polynomials (2.32). □

An analogous result is available for the general case of theorem 1, where both of the partitions λ and μ are nonempty in (2.32).

2.3 Inverses of Toeplitz matrices and skew Schur polynomials

In the remaining of this chapter, we adapt classical results from linear algebra to the case of Toeplitz matrices, and exploit the formulation in terms of symmetric functions to obtain some new results and explicit evaluations of the objects under study.

The usual formula for the inversion of a matrix in terms of its cofactors reads as follows for the case of Toeplitz matrices

$$\left(T_N^{-1}(f)\right)_{j,k} = (-1)^{j+k}\frac{D_{N-1}^{(1^{k-1}),(1^{j-1})}(f)}{D_N(f)}. \tag{2.33}$$

Hence, whenever the inverse of a Toeplitz matrix is known explicitly, formula (2.33) gives explicit evaluations of the formulas appearing in section 2.2. For instance, if the function f is of the form $f(z) = H(y_1, ..., y_d; z^{-1})H(x; z)$, then it follows from (2.30) that for $N-1 \geq d$ we have

$$T_N^{-1}(H(y_1, ..., y_d; z^{-1})H(x; z)) = \begin{pmatrix} 1 & -e_1(y) & e_2(y) & \dots \\ -e_1(x) & 1 + e_1(x)e_1(y) & -(e_1(y) + e_1(x)e_2(y)) & \dots \\ e_2(x) & -(e_1(x) + e_2(x)e_1(y)) & 1 + e_1(x)e_1(y) + e_2(x)e_2(y) & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

We focus on functions of the form $f(z) = E(y_1, ..., y_d; z^{-1})E(x; z)$, and exploit the fact that the Toeplitz minor in the right hand side of (2.33) has several expressions: in terms of the inverse of the corresponding Toeplitz matrix

$$D_N^{(1^k),(1^j)}(f) = (-1)^{j+k} D_{N+1}(f)(T_{N+1}^{-1}(f))_{j+1,k+1}, \tag{2.34}$$

as a specialization of a skew Schur polynomial (2.32)

$$D_N^{(1^k),(1^j)}(f) = s_{(\underbrace{N, ..., N}_{d}, j)/(k)}(y_1^{-1}, \dots, y_d^{-1}, x) \prod_{r=1}^{d} y_r^N, \tag{2.35}$$

and as the multiple integral

$$D_N^{(1^k),(1^j)}(f) = \tag{2.36}$$

$$\frac{1}{N!}\frac{1}{(2\pi)^N}\int_0^{2\pi} \cdots \int_0^{2\pi} e_k(e^{-i\theta_1}, ..., e^{-i\theta_N})e_j(e^{i\theta_1}, ..., e^{i\theta_N}) \prod_{j=1}^{N} f(e^{i\theta_j}) \prod_{1\leq j<k\leq N} |e^{i\theta_j} - e^{i\theta_k}|^2 d\theta_1 ... d\theta_N,$$

where e_j, e_k are elementary symmetric polynomials (2.9) (we assume in the three last identities that $N \geq 1$ and $0 \leq j, k \leq N$). Moreover, theorem 1 gives the asymptotic behaviour

$$\lim_{N\to\infty} D_N^{(1^k),(1^j)}(f) = \left(\lim_{N\to\infty} D_N(f)\right) \sum_{r=0}^{\min(j,k)} h_{k-r}(y)h_{j-r}(x). \tag{2.37}$$

Note that the partitions indexing the sum in (2.22) are now conjugated[13].

Comparing (2.34) and (2.35) we obtain

$$T_N^{-1}(E(y_1, \dots, y_d; z^{-1})E(x; z)) = \tag{2.39}$$

$$\frac{1}{D_N(f)} \times \begin{pmatrix} s_{(5,5,5)}(y^{-1},x) & -s_{(5,5,5)/(1)}(y^{-1},x) & s_{(5,5,5)/(2)}(y^{-1},x) & \dots & \pm s_{(5,5,5)/(5)}(y^{-1},x) \\ -s_{(5,5,5,1)}(y^{-1},x) & s_{(5,5,5,1)/(1)}(y^{-1},x) & -s_{(5,5,5,1)/(2)}(y^{-1},x) & \dots & \mp s_{(5,5,5,1)/(5)}(y^{-1},x) \\ \vdots & \vdots & \vdots & & \vdots \\ \pm s_{(5,5,5,5)}(y^{-1},x) & \mp s_{(5,5,5,5)/(1)}(y^{-1},x) & \pm s_{(5,5,5,5)/(2)}(y^{-1},x) & \dots & s_{(5,5,5,5)/(5)}(y^{-1},x) \end{pmatrix},$$

---

[13] Direct comparison between formulas (2.33) and (2.37) yields the identity

$$\lim_{N\to\infty} (T_N^{-1}(f))_{j+1,k+1} = (-1)^{j+k} \sum_{r=0}^{\min(j,k)} h_{k-r}(y)h_{j-r}(x). \tag{2.38}$$

This follows also from the fact that the Toeplitz matrix generated by f satisfies $T_N(f) = T_{N\times\infty}(E(y; z^{-1}))T_{\infty\times N}(E(x; z))$. Therefore, as $N \to \infty$, we have $T^{-1}(f) = T^{-1}(E(x; z))T^{-1}(E(y; z^{-1}))$, and the $(j+1, k+1)$-th entry of this matrix is precisely the right hand side of (2.38).

where the diagram indexing the Schur polynomial in the first entry of the matrix is indexed by the partition $(N-1)^d$, and we remove a box from its first row or add a box to the last (empty) row as we move to the right or downwards along the entries of the matrix, respectively. The signs of the last row and column should be read as $\pm = (-1)^{N+1}$ and $\mp = (-1)^N$, and the notation (y^{-1}, x) stands for the specialization $(y_1^{-1}, \ldots, y_d^{-1}, x)$.

Let us remark that, as shown by these examples, the symmetric function approach may uncover hidden structure behind the Toeplitz determinants and minors generated by a given function. This allows investigation of some properties of Toeplitz matrices, providing new results (see for instance [76, 18, 74]) and new proofs of already known ones (as in [50, 49], for example). For example, Day's well known formula on Toeplitz determinants [62] can be deduced from basic properties of symmetric functions and the Toeplitz determinant and minor formulation, as shown in [49], as is also the case with the classical formulas of Baxter and Schmidt [28].

In the following, we recall some known explicit inverses of Toeplitz matrices and compute another two in order to obtain evaluations for the Toeplitz minor (2.34). Comparing these with equations (2.35) and (2.36) we will obtain explicit formulas for the corresponding skew Schur polynomials and multiple integrals, as well as their asymptotics. We assume in the following invertibility of all the matrices involved.

2.3.1 Tridiagonal and pentadiagonal Toeplitz matrices

A simple example is given by the Toeplitz matrix generated by the function $f(z) = E(y;z^{-1})E(x;z)$, where x and y are single (nonzero) variables

$$T_N(E(y;z^{-1})E(x;z)) = \begin{pmatrix} 1+xy & y & \\ x & 1+xy & \ddots \\ & \ddots & \ddots \end{pmatrix}. \tag{2.40}$$

The inverse of a tridiagonal Toeplitz matrix has an expression in terms of Chebyshev polynomials of the second kind [181]. These are defined by the recurrence relation

$$\begin{cases} U_{j+1}(z) = 2zU_j(z) - U_{j-1}(z) & (j \geq 1), \\ U_0(z) = 1, \quad U_1(z) = 2z. \end{cases}$$

The determinant of the matrix (2.40) is then given by [88]

$$D_N(E(y;z^{-1})E(x;z)) = \frac{(xy)^{N+1}-1}{xy-1} = (xy)^{N/2}U_N(c) \qquad \left(c = \frac{1+xy}{2\sqrt{xy}}\right), \tag{2.41}$$

and its inverse by

$$(T_N^{-1}(E(y;z^{-1})E(x;z)))_{j,k} = \begin{cases} (-1)^{j+k} \dfrac{y^{k-j}}{(xy)^{(k-j+1)/2}} \dfrac{U_{j-1}(c)U_{N-k}(c)}{U_N(c)} & (j \leq k), \\ (-1)^{j+k} \dfrac{x^{j-k}}{(xy)^{(j-k+1)/2}} \dfrac{U_{k-1}(c)U_{N-j}(c)}{U_N(c)} & (j > k). \end{cases} \tag{2.42}$$

Inserting these expressions in equation (2.35) we obtain the following expression for an arbitrary skew Schur polynomial indexed by a shape of at most two rows and specialized to two variables

$$\begin{aligned} s_{(N,j)/(k)}(x,y^{-1}) &= (xy^{-1})^{(N+j-k)/2} U_{\min(j,k)}(c) U_{N-\max(j,k)}(c) = \\ &= \frac{1}{x^k y^{N+j-k}} \sum_{r=0}^{\min(j,k)} (xy)^r \sum_{r=\max(j,k)}^{N} (xy)^r, \end{aligned} \tag{2.43}$$

for $j,k = 0,...,N$ and $N \geq 1$. Taking $k=0$ above we recover the known expression for a Schur polynomial specialized to two variables in terms of a Chebyshev polynomial [129]. We emphasize that the above formula also coincides with the integral (2.36), with $f(z) = (1+xz)(1+yz^{-1})$. We also obtain from formula (2.37) that

$$\lim_{N\to\infty} s_{(N,j)/(k)}(x,y^{-1})y^N = x^j y^k \frac{(xy)^{-\min(j,k)-1} - 1}{(xy)^{-1} - 1}, \tag{2.44}$$

where the convergence is in the ring of symmetric functions or the usual convergence in $\mathbb{C}$, if $|x|, |y| < 1$.

We can also use this to study the case of a pentadiagonal Toeplitz matrix. Let x_1, x_2, y_1, y_2 be some variables, and consider the function

$$f(z) = (1+x_1 z)(1+x_2 z)(1+y_1 z^{-1})(1+y_2 z^{-1}). \tag{2.45}$$

As proposed in [194], the inverse of this matrix can be computed by means of the Sherman – Morrison – Woodbury formula (SMW formula in the following), as follows. If we denote

$$f_1(z) = (1+x_1 z)(1+y_1 z^{-1}), \qquad f_2(z) = (1+x_2 z)(1+y_2 z^{-1}),$$

the Toeplitz matrices generated by these functions verify

$$T_N(f_1)T_N(f_2) = T_N(f) - \begin{pmatrix} x_1 y_2 & & \\ & & \\ & & x_2 y_1 \end{pmatrix}_{N\times N} \tag{2.46}$$

The SMW formula reads

$$(C + XY^t)^{-1} = C^{-1} - C^{-1}X(I + Y^t C^{-1} X)^{-1} Y^t C^{-1}, \tag{2.47}$$

where C is an $N \times N$ invertible matrix and X, Y are $N \times M$ matrices with $M \leq N$. Noting that the last matrix in (2.46) verifies

$$\begin{pmatrix} x_1 y_2 & & \\ & & \\ & & x_2 y_1 \end{pmatrix} = XY^t = \begin{pmatrix} x_1 y_2 & \\ & \\ & 1 \end{pmatrix}_{N\times 2} \begin{pmatrix} 1 & \\ & \\ & x_2 y_1 \end{pmatrix}^t_{N\times 2},$$

we see that the inverse $T_N^{-1}(f)$ can be expressed in terms of the matrix

$$C^{-1} = (a_{j,k}) = T_N^{-1}(f_2) T_N^{-1}(f_1)$$

Let us consider the symmetric case $x_1 = y_1$ and $x_2 = y_2$ in (2.45), for simplicity. Using (2.42) we find that, whenever $j \leq k$

$$a_{j,k} = \frac{1}{x_1 x_2} \frac{(-1)^{j+k}}{U_N(c_1)U_N(c_2)} \Bigg[\sum_{l=1}^{j} U_{l-1}(c_1)U_{l-1}(c_2)U_{N-k}(c_1)U_{N-j}(c_2) + \sum_{l=j+1}^{k} U_{l-1}(c_1)U_{j-1}(c_2)U_{N-k}(c_1)U_{N-l}(c_2) + \sum_{l=k+1}^{N} U_{k-1}(c_1)U_{j-1}(c_2)U_{N-l}(c_1)U_{N-l}(c_2) \Bigg],$$

where $c_j = (1 + x_j^2)/2x_j$ and the U_k are Chebyshev polynomials of the second kind. Similarly, if $j > k$ we have

$$a_{j,k} = \frac{1}{x_1 x_2} \frac{(-1)^{j+k}}{U_N(c_1)U_N(c_2)} \Bigg[\sum_{l=1}^{k} U_{l-1}(c_1)U_{l-1}(c_2)U_{N-k}(c_1)U_{N-j}(c_2) + \sum_{l=k+1}^{j} U_{k-1}(c_1)U_{l-1}(c_2)U_{N-l}(c_1)U_{N-j}(c_2) + \sum_{l=j+1}^{N} U_{k-1}(c_1)U_{j-1}(c_2)U_{N-l}(c_1)U_{N-l}(c_2) \Bigg].$$

Using (2.47) we then obtain the following expression for the (j,k)-th entry of the matrix $T_N^{-1}(f)$, where f is given by (2.45)

$$a_{j,k} - \frac{1}{D} \Bigg[x_1 x_2 a_{j1} a_{1k} + x_1 x_2 a_{jN} a_{Nk} + x_1^2 x_2^2 \Big(a_{NN} a_{j1} a_{1k} - a_{N1} a_{jN} a_{1k} - a_{1N} a_{j1} a_{Nk} + a_{11} a_{jN} a_{Nk} \Big) \Bigg], \tag{2.48}$$

where $a_{j,k}$ is given by the above expressions and

$$D = 1 + x_1 x_2 (a_{11} + a_{NN}) + x_1^2 x_2^2 (a_{11} a_{NN} - a_{1N} a_{N1}).$$

This can be combined with the known formulas for the determinant of the Toeplitz matrix generated by f (see for instance [81]) to obtain an evaluation of the Toeplitz minor (2.34), which coincides with the skew Schur polynomial

$$s_{(N,N,j)/(k)}(x_1, x_2, x_1^{-1}, x_2^{-1}) x_1^N x_2^N$$

and the matrix model

$$\frac{1}{N!} \frac{1}{(2\pi)^N} \int_{[0,2\pi]^N} |V(e^{i\theta})|^2 e_k(e^{-i\theta}) e_j(e^{i\theta}) \prod_{j=1}^{N} (1 + x_1 e^{i\theta_j})(1 + x_2 e^{i\theta_j})(1 + x_1 e^{-i\theta_j})(1 + x_2 e^{-i\theta_j}) d\theta_j.$$

Analogous expressions can be obtained whenever $x_1 \neq y_1$ and/or $x_2 \neq y_2$, following the same reasoning as above.

2.3.2 The pure Fisher-Hartwig singularity

The asymptotics of Toeplitz determinants generated by symbols that do not verify the regularity conditions in Szegő's theorem have been long studied. In the seminal work [85], Fisher and Hartwig conjectured the asymptotic behaviour of Toeplitz determinants generated by a class of (integrable) functions that violate these conditions. The functions in this class are products of a function which is smooth, in the sense of Szegő's theorem, and a finite number of so-called pure Fisher-Hartwig singularities. Their conjecture was later refined in [22] and [26], and only recently a complete description of the asymptotics of these determinants was achieved by Deift, Its and Krasovsky [67]. See [68] for a detailed historical account of the subject.

A pure Fisher-Hartwig singularity is a function of the form [41]

$$|1-e^{i\theta}|^{2\alpha}e^{i\beta(\theta-\pi)} \qquad (0<\theta<2\pi), \tag{2.49}$$

where the parameters α,β satisfy $\mathrm{Re}(\alpha)>-1/2$ and $\beta\in\mathbb{C}$. The factor $|1-e^{i\theta}|^{2\alpha}$ may have a zero, a pole, or an oscillatory singularity at the point $z=1$, while the factor $e^{i\beta(\theta-\pi)}$ has a jump if β is not an integer. Thus, depending on the different values of the parameters α and β, the symbol above may violate the regularity conditions in Szegő's theorem. It will be more convenient to work with the equivalent factorization [41]

$$(1-e^{i\theta})^{\gamma}(1-e^{-i\theta})^{\delta}.$$

This function coincides with (2.49) if $\gamma=\alpha+\beta$ and $\delta=\alpha-\beta$; we will assume in the following that the parameters γ and δ are positive integers. We can then express this function as the specialization

$$f(z)=\varphi_{\gamma,\delta}(z)=E(\underbrace{1,\ldots,1}_{\delta};z^{-1})E(\underbrace{1,\ldots,1}_{\gamma};z). \tag{2.50}$$

Functions with general Fisher-Hartwig singularities are obtained as the product of a function verifying the regularity conditions in Szegő's theorem times a finite number of translated pure singularities of the form $\varphi_{\gamma_r,\delta_r}(e^{i(\theta-\theta_r)})$. Each of these factors has a singularity with parameters γ_r,δ_r at the point $e^{i\theta_r}$.

The inverse of the Toeplitz matrix generated by the pure FH singularity can be computed by means of the Duduchava-Roch formula [79, 169, 40]

$$T((1-z)^{\gamma})M_{\gamma+\delta}T((1-z^{-1})^{\delta})=\frac{\Gamma(\gamma+1)\Gamma(\delta+1)}{\Gamma(\gamma+\delta+1)}M_{\delta}T(\varphi_{\gamma,\delta})M_{\gamma},$$

where M_a is the diagonal matrix with entries $(M_a)_{k,k}=\binom{a+k-1}{k-1}$, for $k\geq 1$. Böttcher and Silbermann [42] used this formula to give an explicit expression for the determinant of the Toeplitz matrix generated by the pure FH singularity

$$D_N\left(\varphi_{\gamma,\delta}\right)=G(N+1)\frac{G(\gamma+\delta+N+1)}{G(\gamma+\delta+1)}\frac{G(\gamma+1)}{G(\gamma+N+1)}\frac{G(\delta+1)}{G(\delta+N+1)}, \tag{2.51}$$

where G is the Barnes function [21]. Also the inverse of the corresponding Toeplitz matrix can be computed explicitly by means of this formula [40]

$$(T_N^{-1}(\varphi_{\gamma,\delta}))_{j,k}=(-1)^{j+k}\frac{\Gamma(\gamma+j)\Gamma(\delta+k)}{\Gamma(j)\Gamma(k)}\sum_{r=\max(j,k)}^{N}\frac{\Gamma(r)}{\Gamma(\gamma+\delta+r)}\binom{\gamma+r-k-1}{r-k}\binom{\delta+r-j-1}{r-j}.$$

Inserting these expressions in equation (2.35) we obtain

$$s_{(\underbrace{N,\ldots,N}_{d},j)/(k)}(1^M) = G(N+2)\frac{G(M+N+2)}{G(M+1)}\frac{G(M-d+1)}{G(M-d+N+2)}\frac{G(d+1)}{G(d+N+2)}\times \tag{2.52}$$
$$\frac{\Gamma(M-d+j+1)}{\Gamma(j+1)}\frac{\Gamma(d+k+1)}{\Gamma(k+1)}\sum_{r=\max(j,k)}^{N}\frac{\Gamma(r+1)}{\Gamma(M+r+1)}\binom{M-d+r-k-1}{r-k}\binom{d+r-j-1}{r-j},$$

for $j,k \leq N$ and $M > d$ (or $M \geq d$, if $j=0$). The above formula recovers known evaluations whenever $k=0$ and thus the function in the left hand side above is a Schur polynomial (these can be computed by means of the hook-content formula [178], for instance). Explicit expressions for such specialization of skew Schur polynomials indexed by partitions of certain shapes have been obtained recently in [158], and coincide with the above formula when the shapes are the same.

Using expression (2.49), we see that the integral form of a Toeplitz minor generated by the pure Fisher-Hartwig generality

$$D_N^{\lambda,\mu}(\varphi_{\gamma,\delta}) = s_{((\delta^N)+\mu/\lambda)'}(1^{\gamma+\delta}) = \tag{2.53}$$
$$\frac{1}{N!}\frac{1}{(2\pi)^N}\int_0^{2\pi}\ldots\int_0^{2\pi} s_\lambda(e^{-i\theta})s_\mu(e^{i\theta})\prod_{j=1}^{N} e^{\frac{1}{2}i\theta_j(\gamma-\delta)}|1+e^{i\theta_j}|^{\gamma+\delta}\prod_{1\leq j<k\leq N}|e^{i\theta_j}-e^{i\theta_k}|^2 d\theta_1\ldots d\theta_N,$$

is the $\beta=2$ case of the unit circle version of Selberg's integral known as Morris integral, with the insertion of two Schur polynomials. Its representation as a Toeplitz minor allows a direct computation for the case of a single polynomial.

Lemma. *Let μ be a partition of length $l(\mu) \leq N$. We have*

$$D_N^{\varnothing,\mu}(\varphi_{\gamma,\delta}) = D_N(\varphi_{\gamma,\delta})s_\mu(1^N)\prod_{k=1}^{l(\mu)}\frac{\Gamma(\gamma+k)}{\Gamma(\gamma+k-\mu_k)}\frac{\Gamma(\delta+N-k+1)}{\Gamma(\delta+N-k+\mu_k+1)}. \tag{2.54}$$

Proof. We follow the second of the two proofs given in [44] for the Toeplitz determinant $D_N(\varphi_{\gamma,\delta})$. We include this computation to showcase how the Toeplitz minor structure can be exploited to obtain evaluations of the more complicated objects considered (i.e. multiple integrals, skew Schur polynomials), rather than for its conceptual insight.

The Fourier coefficients of $\varphi_{\gamma,\delta}$ are [41]

$$d_k = \frac{\Gamma(\gamma+\delta+1)}{\Gamma(\gamma-k+1)\Gamma(\delta+k+1)}.$$

After extracting the factors

$$\prod_{j=1}^{N}\frac{\Gamma(\gamma+\delta+1)}{\Gamma(\gamma-\mu_N+N-j+1)}, \qquad \prod_{k=1}^{N}\frac{1}{\Gamma(\delta+\mu_k+N-k+1)},$$

coming from the rows and columns of $D_N^{\varnothing,\mu}(\varphi_{\gamma,\delta})$ respectively, we obtain the determinant

$$\begin{vmatrix} \frac{\Gamma(\gamma-\mu_N+N)}{\Gamma(\gamma-\mu_1+1)}\frac{\Gamma(\delta+\mu_1+N)}{\Gamma(\delta+\mu_1+1)} & \frac{\Gamma(\gamma-\mu_N+N)}{\Gamma(\gamma-\mu_2+2)}\frac{\Gamma(\delta+\mu_2+N-1)}{\Gamma(\delta+\mu_2)} & \cdots & \frac{\Gamma(\delta+\mu_N+1)}{\Gamma(\delta+\mu_N-N+2)} \\ \frac{\Gamma(\gamma-\mu_N+N-1)}{\Gamma(\gamma-\mu_1)}\frac{\Gamma(\delta+\mu_1+N)}{\Gamma(\delta+\mu_1+2)} & \frac{\Gamma(\gamma-\mu_N+N-1)}{\Gamma(\gamma-\mu_2+1)}\frac{\Gamma(\delta+\mu_2+N-1)}{\Gamma(\delta+\mu_2+1)} & \cdots & \frac{\Gamma(\delta+\mu_N+1)}{\Gamma(\delta+\mu_N-N+3)} \\ \vdots & \vdots & & \vdots \\ \frac{\Gamma(\gamma-\mu_N+1)}{\Gamma(\gamma-\mu_1-N+2)} & \frac{\Gamma(\gamma-\mu_N+1)}{\Gamma(\gamma-\mu_1-N+3)} & \cdots & 1 \end{vmatrix}. \tag{2.55}$$

Subtracting $(\delta+\mu_N-N+1+j)$ times the $(j+1)$-th row from the j-th row, for $j=1,...,N-1$, we can make the last column vanish except for the 1 at the bottom, thus obtaining a determinant of order $N-1$. After extracting the factor

$$\prod_{k=1}^{N-1}(\gamma+\delta+1)(\mu_k-\mu_N+N-k)$$

from the columns of the matrix, and the factor

$$\prod_{j=1}^{N-1}\frac{\Gamma(\gamma-\mu_N+j)}{\Gamma(\gamma-\mu_{N-1}+j)}$$

from its rows, we obtain a determinant with the same structure as (2.55), but with the following changes: N is replaced by $N-1$, δ is replaced by $\delta+1$ and μ is replaced by the partition $(\mu_1,\ldots,\mu_{N-1})$, that results from discarding the last part of μ. Making use of this recursive structure and identity (2.11) one arrives at the desired expression. □

This recovers a known formula [94] for the evaluation of the case $\lambda=\varnothing$ of the integral (2.53), although its expression as the specialization of a skew Schur polynomial appears to be new. Substituting $M-d$ by γ and d by δ, formula (2.52) gives an explicit evaluation of this integral valid for general values[14] of γ and δ whenever the Schur polynomials reduce to elementary symmetric polynomials $s_\lambda=e_k$, $s_\mu=e_j$.

2.3.3 Principal specializations

In order to study the principal specialization $x_j=q^{j-1}$ in the above formulas, we recall the well known method of Borodin for obtaining the inverse of the moment matrix of a biorthogonal ensemble. We follow the presentation in [36], where details and proofs can be found. The starting point is a random matrix ensemble of the form

$$\int\cdots\int \det\left(\xi_j(z_k)\right)_{j,k=1}^N \det\left(\eta_j(z_k)\right)_{j,k=1}^N \prod_{j=1}^N f(z_j)dz_j$$

(up to a constant), for a weight function f supported on some domain and two families of functions (ξ_j) and (η_j). If one is able to find two new families (ζ_j) and (ψ_j) that biorthogonalize[15] the former with respect to the weight f, that is

$$\begin{gathered}\zeta_j\in\operatorname{Span}\{\xi_1,\ldots,\xi_j\},\qquad \psi_j\in\operatorname{Span}\{\eta_1,\ldots,\eta_j\},\\ \int\zeta_j(z)\psi_k(z)f(z)dz=\delta_{j,k},\end{gathered}\tag{2.56}$$

then the matrix of coefficients of the kernel

$$K_N(z,\omega)=\sum_{r=1}^N\zeta_r(z)\psi_r(\omega)=\sum_{j,k=1}^N c_{j,k}\xi_j(z)\eta_k(\omega)\tag{2.57}$$

[14] We have only proved the validity of the formula for integer values of γ and δ. However, by Carlson's theorem, the formula holds for any positive γ and δ.

[15] Note that we are actually considering biorthonormal functions; we stick to the original terminology of [36] here and below and speak of biorthogonal functions in the following.

satisfies

$$\left[(c_{j,k})_{j,k=1}^N\right]^{-1} = \left(\int \xi_k(z)\eta_j(z)f(z)dz\right)_{j,k=1}^N. \tag{2.58}$$

If the ensemble is an orthogonal polynomial ensemble, then the moment matrix on the right hand side above is a Hankel matrix, the functions ξ_j and η_j are the monomials z^{j-1}, and we have that $\zeta_j = \psi_j = p_j$, the orthogonal polynomials with respect to the weight function f, that is supported on the real line. The case where the moment matrix on the right hand side above is the Toeplitz matrix generated by a function f supported on the unit circle corresponds to the biorthogonal ensemble with functions $\xi_j(z) = z^{-(j-1)}, \eta_j(z) = z^{j-1}$. Thus, the biorthogonality condition (2.56) amounts to finding two families of polynomials p_j and q_j such that

$$\frac{1}{2\pi}\int_0^{2\pi} p_j(e^{-i\theta})q_k(e^{i\theta})f(e^{i\theta})d\theta = \delta_{j,k}. \tag{2.59}$$

Let us remark that only when the Toeplitz matrix is Hermitian (that is, when the function f is real valued), these polynomials verify $p_j(e^{-i\theta}) = \overline{q_j(e^{i\theta})}$, the q_j are the orthogonal polynomials with respect to f, and the kernel above is the usual Christoffel-Darboux kernel (see [27, 131] for more details). In general, one needs to consider a biorthogonal ensemble as above. Nevertheless, one can compute the polynomials (p_j) and (q_j) in a similar fashion to the orthogonal case.

Lemma. *Suppose the determinants $D_N(f)$ are nonzero for every N. Then, the polynomials p_j and q_j in* (2.59) *are given by*

$$p_j(z) = \frac{1}{(D_j(f)D_{j+1}(f))^{1/2}}\begin{vmatrix} d_0 & d_{-1} & \dots & d_{-j} \\ d_1 & d_0 & \dots & d_{-(j-1)} \\ \vdots & \vdots & & \vdots \\ d_{j-1} & d_{j-2} & & d_{-1} \\ 1 & z & \dots & z^j \end{vmatrix},$$

$$q_j(z) = \frac{1}{(D_j(f)D_{j+1}(f))^{1/2}}\begin{vmatrix} d_0 & d_{-1} & \dots & d_{-(j-1)} & 1 \\ d_1 & d_0 & \dots & d_{-(j-2)} & z \\ \vdots & \vdots & & \vdots & \vdots \\ d_j & d_{j-1} & \dots & d_1 & z^j \end{vmatrix}.$$

Proof. The condition on the determinants implies the existence of the polynomials themselves (see proposition 2.9 in [36], for instance), and they are uniquely determined up to multiplicative constants. Hence, it suffices to verify the biorthogonality condition (2.59). We denote

$$p_j(z) = \sum_{r=0}^{j} a_r^{(j)} z^r, \qquad q_k(z) = \sum_{r=0}^{k} b_r^{(k)} z^r. \tag{2.60}$$

Now, if $j \geq k$ in (2.59) we can rewrite this integral as the sum

$$\frac{1}{2\pi}\int_0^{2\pi} p_j(e^{-i\theta})q_k(e^{i\theta})f(e^{i\theta})d\theta = \frac{1}{(D_k(f)D_{k+1}(f)D_j(f)D_{j+1}(f))^{1/2}}\times$$

$$\sum_{r=0}^{k} b_r^{(k)}\begin{vmatrix} d_0 & d_{-1} & \dots & d_{-j} \\ d_1 & d_0 & \dots & d_{-(j-1)} \\ \vdots & \vdots & & \vdots \\ d_{j-1} & d_{j-2} & & d_{-1} \\ \frac{1}{2\pi}\int_0^{2\pi} e^{ir\theta}f(e^{i\theta})d\theta & \frac{1}{2\pi}\int_0^{2\pi} e^{i(r-1)\theta}f(e^{i\theta})d\theta & \dots & \frac{1}{2\pi}\int_0^{2\pi} e^{i(r-j)\theta}f(e^{i\theta})d\theta \end{vmatrix},$$

which vanishes if $j > k$ and equals 1 if $j = k$, since the last row in the above determinants is precisely $(d_r, d_{r-1}, \ldots, d_{r-j})$. Analogously, if $j < k$ in (2.59) the integral equals

$$\frac{1}{(D_k(f)D_{k+1}(f)D_j(f)D_{j+1}(f))^{1/2}} \sum_{r=0}^{j} a_r^{(j)} \begin{vmatrix} d_0 & d_{-1} & \ldots & d_{-(k-1)} & \frac{1}{2\pi}\int_0^{2\pi} e^{-ir\theta} f(e^{i\theta})d\theta \\ d_1 & d_0 & \ldots & d_{-(k-2)} & \frac{1}{2\pi}\int_0^{2\pi} e^{-i(r-1)\theta} f(e^{i\theta})d\theta \\ \vdots & \vdots & & \vdots & \vdots \\ d_k & d_{k-1} & \ldots & d_1 & \frac{1}{2\pi}\int_0^{2\pi} e^{-i(r-k)\theta} f(e^{i\theta})d\theta \end{vmatrix},$$

and again all the determinants in the sum vanish. □

In sight of such determinantal expressions, we see that the coefficients of the biorthogonal polynomials can be expressed essentially as Toeplitz minors. From this remark we also obtain the following equivalent integral formulas for the polynomials (known as Heine's identities)

$$\begin{aligned} &(D_N(f)D_{N+1}(f))^{1/2} p_N(z) = \\ &\quad \frac{1}{N!}\frac{1}{(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{1\le j<k\le N} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^{N} (z - e^{i\theta_j}) f(e^{i\theta_j}) d\theta_j = \\ &\quad \sum_{k=0}^{N} (-1)^k z^{N-k} \frac{1}{N!}\frac{1}{(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} e_k(e^{i\theta_1}, \ldots, e^{i\theta_N}) \prod_{1\le j<k\le N} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^{N} f(e^{i\theta_j}) d\theta_j, \end{aligned}$$

as well as

$$\begin{aligned} &(D_N(f)D_{N+1}(f))^{1/2} q_N(z) = \\ &\quad \frac{1}{N!}\frac{1}{(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{1\le j<k\le N} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^{N} (z - e^{-i\theta_j}) f(e^{i\theta_j}) d\theta_j = \\ &\quad \sum_{k=0}^{N} (-1)^k z^{N-k} \frac{1}{N!}\frac{1}{(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} e_k(e^{-i\theta_1}, \ldots, e^{-i\theta_N}) \prod_{1\le j<k\le N} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^{N} f(e^{i\theta_j}) d\theta_j. \end{aligned}$$

In particular, if the function in the integrals above is of the form (2.31), it follows from (2.32) that the coefficients of the biorthogonal polynomials can be expressed as skew Schur polynomials. For instance, we have

$$\begin{aligned} (D_N(f)D_{N+1}(f))^{1/2} p_N(z) = \prod_{j=1}^{d} y_j^N \Big(& s_{(N^d)}(y_1^{-1}, \ldots, y_d^{-1}, x) z^N - s_{(N^d,1)}(y_1^{-1}, \ldots, y_d^{-1}, x) z^{N-1} + \\ & \cdots + (-1)^{N-1} s_{(N^d,N-1)}(y_1^{-1}, \ldots, y_d^{-1}, x) z + (-1)^N s_{(N^{d+1})}(y_1^{-1}, \ldots, y_d^{-1}, x) \Big), \end{aligned} \tag{2.61}$$

where the first shape in the sum above is (N^d) and the last one is (N^{d+1}), and we add a box to the last row of the diagram as the degree of the monomial z^j decreases. An analogous expression holds for the polynomials q_j. This fact can be combined with the asymptotic expression (2.37) to obtain the asymptotic behaviour of the coefficients of the biorthogonal polynomials associated to a given function, as long as it verifies the hypotheses in Szegő's theorem, for example.

These expressions serve also as further motivation to study the Toeplitz minors (2.34), as both the coefficients of orthogonal polynomials and their asymptotic behaviour are topics of interest.

Let us also emphasize the close relation between the orthogonal or biorthogonal polynomials associated to a given function and the inverse of the moment matrix defined by this same function [131]. We have already reviewed how this inverse coincides with the matrix of coefficients of the Christoffel-Darboux kernel (2.58), which is built precisely from the biorthogonal polynomials. We see now from the above formulas that also the polynomials themselves can be read off the first row and column of the inverse of the moment matrix.

We now use (2.58) to study the finite and infinite principal specializations of the skew Schur polynomials indexed by the shapes considered earlier. We assume in the following that q is a new (real) variable verifying $|q| < 1$. We will denote by Γ_q and G_q the q-Gamma and q-Barnes functions [162], that in particular verify

$$\Gamma_q(k+1) = \frac{\prod_{j=1}^{k}(1-q^j)}{(1-q)^k} = \frac{(q;q)_k}{(1-q)^k}, \qquad G_q(k+1) = \prod_{j=1}^{k-1}\Gamma_q(j+1), \tag{2.62}$$

whenever k is a natural number (we assume that an empty product takes the value 1). The q-binomial coefficient is then given by

$$\begin{bmatrix}\omega\\ z\end{bmatrix}_q = \frac{\Gamma_q(\omega+1)}{\Gamma_q(z+1)\Gamma_q(\omega-z+1)} \qquad (\mathrm{Re}(\omega) \geq \mathrm{Re}(z) > 0)\,.$$

These functions coincide with their classical counterparts in the $q \to 1$ limit, that is

$$\lim_{q\to 1}\Gamma_q(z) = \Gamma(z), \qquad \lim_{q\to 1} G_q(z) = G(z), \qquad \lim_{q\to 1}\begin{bmatrix}\omega\\ z\end{bmatrix}_q = \binom{\omega}{z},$$

for all the ω and z such that the right hand sides above make sense. We consider the following specialization [86]

$$f(z) = \Theta_{\gamma,\delta}(z) = E(1,q,\ldots,q^{\delta-1};z^{-1})E(q,q^2,\ldots,q^{\gamma};z) = \sum_{k=-\delta}^{\gamma}\begin{bmatrix}\delta+\gamma\\ \delta+k\end{bmatrix}_q q^{k(k+1)/2}z^k,$$

for some positive integers γ and δ. The Toeplitz determinant generated by this function equals

$$D_N(\Theta_{\gamma,\delta}) = G_q(N+1)\frac{G_q(\delta+\gamma+N+1)}{G_q(\delta+\gamma+1)}\frac{G_q(\delta+1)}{G_q(\delta+N+1)}\frac{G_q(\gamma+1)}{G_q(\gamma+N+1)},$$

and the biorthogonal polynomials p_j, q_j are given by

$$\begin{aligned} p_j(z) &= \left(\frac{(q;q)_{\delta+j}(q;q)_{\gamma+j}}{(q;q)_j(q;q)_{\delta+\gamma+j}}\right)^{1/2}\sum_{r=0}^{j}(-1)^{j+r}\begin{bmatrix}j\\ r\end{bmatrix}_q\frac{(q;q)_{\gamma+r}}{(q;q)_{\gamma+j}}\frac{(q;q)_{\delta+j-r-1}}{(q;q)_{\delta-1}}z^r,\\ q_j(z) &= \left(\frac{(q;q)_{\delta+j}(q;q)_{\gamma+j}}{(q;q)_j(q;q)_{\delta+\gamma+j}}\right)^{1/2}\sum_{r=0}^{j}(-1)^{j+r}\begin{bmatrix}j\\ r\end{bmatrix}_q\frac{(q;q)_{\gamma+j-r-1}}{(q;q)_{\gamma-1}}\frac{(q;q)_{\delta+r}}{(q;q)_{\delta+j}}q^r z^r, \end{aligned} \tag{2.63}$$

where $(q;q)_k$ is as defined in (2.62). The last three identities can be proved directly from their determinantal expressions. We do not include the computations here but point to the second method of proof in [44], followed also in the derivation of (2.54), that can be generalized to the present setting. Recalling the notation (2.60), we have that the kernel (2.57) is then given by

$$K_{N+1}(z,\omega) = \sum_{r=0}^{N}p_r(z)q_r(\omega^{-1}) = \sum_{j,k=0}^{N}\left(\sum_{r=\max(j,k)}^{N}a_j^{(r)}b_k^{(r)}\right)z^j\omega^{-k} =$$

$$\sum_{j,k=0}^{N}\left(\sum_{r=\max j,k}^{N}(-1)^{j+k}q^{j}\frac{\Gamma_q(\delta+j+1)\Gamma_q(\gamma+k+1)\Gamma_q(r+1)}{\Gamma_q(j+1)\Gamma_q(k+1)\Gamma_q(\delta+\gamma+r+1)}\begin{bmatrix}\gamma+r-k-1\\ r-k\end{bmatrix}_q\begin{bmatrix}\delta+r-j-1\\ r-j\end{bmatrix}_q\right)z^{j}\omega^{-k}.$$

Moreover, the coefficient of $z^j\omega^{-k}$ in the above sum is the $(j+1,k+1)$-th entry of the inverse of the matrix $T_{N+1}(\Theta_{\gamma,\delta})$. Inserting this into expression (2.35) we obtain

$$s_{(\underbrace{N,\dots,N}_{d},j)/(k)}(1,q,\dots,q^{M-1})= \tag{2.64}$$
$$q^{dj-(d-1)k+d(d-1)N/2}G_q(N+2)\frac{G_q(M+N+2)}{G_q(M+1)}\frac{G_q(M-d+1)}{G_q(M-d+N+2)}\frac{G_q(d+1)}{G_q(d+N+2)}\times$$
$$\sum_{r=\max(j,k)}^{N}\frac{\Gamma_q(M-d+j+1)\Gamma_q(d+k+1)\Gamma_q(r+1)}{\Gamma_q(j+1)\Gamma_q(k+1)\Gamma_q(M+r+1)}\begin{bmatrix}M-d+r-k-1\\ r-k\end{bmatrix}_q\begin{bmatrix}d+r-j-1\\ r-j\end{bmatrix}_q,$$

for $j,k\leq N$ and $M>d$ (or $M\geq d$, if $j=0$). As expected, this expression coincides with (2.52) in the $q\to 1$ limit. Also, as above, the formula recovers known expressions whenever $k=0$ (and thus we have a Schur polynomial, comparing again with the hook-content formula [178], for instance). Finally, it follows from (2.37) and the Cauchy identity that

$$\lim_{N\to\infty}s_{(\underbrace{N,\dots,N}_{d},j)/(k)}(1,q,\dots,q^{M-1})q^{-Nd(d-1)/2}= \tag{2.65}$$
$$\frac{q^{dj-(d-1)k}}{(1-q)^{d(M-d)}}\frac{G_q(d+1)G_q(M-d+1)}{G_q(M+1)}\sum_{r=0}^{\min(j,k)}q^{-r}\begin{bmatrix}M-d+j-r-1\\ j-r\end{bmatrix}_q\begin{bmatrix}d+k-r-1\\ k-r\end{bmatrix}_q.$$

Note that the inversion of a Toeplitz matrix by means of the kernel (2.57) is a general procedure that can be used to obtain explicit evaluations of other specializations of the skew Schur polynomials of the shapes considered above, as long as the biorthogonal polynomials (2.59) are available. In particular, the results in subsection 2.3.2 for the pure Fisher-Hartwig singularity can be obtained in such a way. The biorthogonal polynomials can be obtained[16] as the $q\to 1$ limit of the polynomials (2.63), leading to the same formula (2.52).

Finally, taking into account that only one set of variables in the specialization of f needs to be finite in equations (2.34)-(2.37), we can study the principal specialization of the above skew Schur polynomials with an infinite number of variables. To do so, we consider the function

$$f(z)=\Theta_\delta(z)=E(1,q^{-1},\dots,q^{-(\delta-1)};z^{-1})E(q^{\delta},q^{\delta+1},\dots;z)=\sum_{k=-\delta}^{\infty}\frac{q^{k\delta+k(k-1)/2}}{(q;q)_{\delta+k}}z^{k},$$

for some positive integer δ. The corresponding Toeplitz determinant is given by

$$D_N(\Theta_\delta)=\frac{1}{(1-q)^{\delta N}}\frac{G_q(\delta+1)G_q(N+1)}{G_q(\delta+N+1)},$$

and the biorthogonal polynomials on the unit circle with respect to the function Θ_δ are given

[16] In the Hermitian case $\gamma=\delta$, where the polynomials are a single family of orthogonal polynomials, one recovers the family $S_n^a(z)$ introduced in [11] after substituting q by $q^{1/2}$, z by $q^{-1/2}z$ and a by q^{γ}.

by

$$p_j(z) = \left(\frac{(q;q)_{\delta+j}}{(q;q)_j}\right)^{1/2} \sum_{r=0}^{j} (-1)^{j+r} \begin{bmatrix} j \\ r \end{bmatrix}_q \frac{(q;q)_{\delta+j-r-1}}{(q;q)_{\delta-1}} q^{-(\delta-1)(j-r)} z^r,$$
$$q_j(z) = \left(\frac{1}{(q;q)_j (q;q)_{\delta+j}}\right)^{1/2} \sum_{r=0}^{j} (-1)^{j+r} \begin{bmatrix} j \\ r \end{bmatrix}_q (q;q)_{\delta+r} q^{\delta(j-r)} z^r. \tag{2.66}$$

Again, these expressions can be verified from their determinantal and minor formulas. The kernel in this case is then

$$K_{N+1}(z,\omega) = \sum_{j,k=0}^{N} \left(\sum_{r=\max j,k}^{N} (-1)^{j+k} q^{r+(\delta-1)j-\delta k} \frac{(q;q)_{\delta+k}}{(q;q)_j} \begin{bmatrix} r \\ r-k \end{bmatrix}_q \begin{bmatrix} \delta+r-j-1 \\ r-j \end{bmatrix}_q \right) z^j \omega^{-k}.$$

Inserting this in equation (2.35) we arrive at

$$s_{(\underbrace{N,\dots,N}_{d},j)/(k)}(1,q,\dots) = \tag{2.67}$$
$$\frac{q^{(d-1)j-dk+d(d-1)N/2}}{(1-q)^{d(N+1)}} \frac{G_q(N+2)G_q(d+1)}{G_q(d+N+2)} \frac{(q;q)_{d+k}}{(q;q)_j} \sum_{r=\max(j,k)}^{N} q^r \begin{bmatrix} r \\ r-k \end{bmatrix}_q \begin{bmatrix} d+r-j-1 \\ r-j \end{bmatrix}_q.$$

Once again, this identity coincides with the one given by the hook-content formula for $k=0$. It follows from (2.37) and the Cauchy identity that

$$\lim_{N\to\infty} s_{(\underbrace{N,\dots,N}_{d},j)/(k)}(1,q,\dots) q^{-Nd(d-1)/2} = \tag{2.68}$$
$$q^{dj-(d-1)k} \frac{(1-q)^{d(d-1)/2} G_q(d+1)}{(q;q)_\infty^d} \sum_{r=0}^{\min(j,k)} q^{-r} \frac{1}{(q;q)_{j-r}} \begin{bmatrix} d+k-r-1 \\ k-r \end{bmatrix}_q,$$

where $(q;q)_\infty = \prod_{k=1}^{\infty}(1-q^k)$ denotes the Euler function.

We have focused throughout this section on the simplest example of a single row and single column Toeplitz minor $D_N^{(1^j),(1^k)}(f)$, which can be expressed essentially as an element of the corresponding Toeplitz inverse. As we have seen, several nontrivial results follow already from this representation. However, more complicated minors can be expressed in terms of the inverse of the associated Toeplitz matrix, thus allowing generalizations of the formulas presented here. At the level of matrix integrals, this means obtaining explicit expressions for integrals with arbitrary Schur polynomials on the integrand (not only elementary symmetric polynomials). At the level of specializations of skew Schur polynomials, this means allowing λ and μ to be general partitions in formula (2.32), so that more general shapes can be added to or skewed from the rectangle (N^d) in (2.35) (as opposed to the single rows (j) and (k)). We need to introduce some notation before showing this; given a matrix A of size $N \times N$, we write

$$[A]\begin{pmatrix} j_1 & \dots & j_K \\ k_1 & \dots & k_K \end{pmatrix}$$

to denote the minor of A formed by the intersection of rows $j_1 < \dots < j_K$ and columns $k_1 < \dots < k_K$. Then, given two partitions λ and μ, we have

$$D_N^{\lambda,\mu}(f) = (-1)^{|\lambda|+|\mu|} D_{N+K}(f) \left[T_{N+K}^{-1}(f)\right] \begin{pmatrix} 1+\mu'_K & 2+\mu'_{K-1} & \dots & K+\mu'_1 \\ 1+\lambda'_K & 2+\lambda'_{K-1} & \dots & K+\lambda'_1 \end{pmatrix}, \tag{2.69}$$

where $K = \max\{\lambda_1, \mu_1\}$ As noted above, the case $K = 1$ corresponds to the one row and one column case considered in this section. For greater values of K the formula expresses arbitrary minors $D_N^{\lambda,\mu}(f)$ in terms of a Toeplitz determinant and a minor of the inverse of the corresponding Toeplitz matrix[17].

2.4 Laplace expansion of Toeplitz determinants and skew Schur polynomials

The single row and single column minors of a matrix also play a role in the procedure of computing its determinant by means of Laplace expansion. For the case of Toeplitz matrices, this reads

$$D_N(f) = \sum_{j=1}^{N} (-1)^{j+k} d_{j-k} D_{N-1}^{(1^{j-1})(1^{k-1})}(f) = \sum_{k=1}^{N} (-1)^{j+k} d_{j-k} D_{N-1}^{(1^{j-1})(1^{k-1})}(f) \tag{2.70}$$

where the k-th column, for $k \in \{1, \ldots, N\}$ (resp. j-th row, for $j \in \{1, \ldots, N\}$) is fixed in the first (resp. second) identity. Once again, we choose f to be of the form $f(z) = E(y_1, \ldots, y_d; z^{-1})E(x; d)$. Substituting the explicit expressions for the coefficients d_k (2.31) and for the minors (2.35) in this identity we obtain the following result, after a relabeling of the variables.

Theorem 2. *We have*

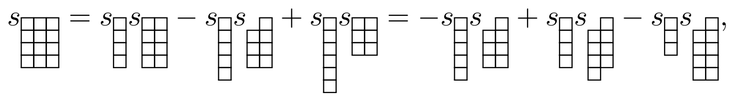
$$\begin{aligned} s_{(N^d)}(x) &= \sum_{j=1}^{N} (-1)^{j+k} e_{d+j-k}(x) s_{(d^{N-1})+(1^{k-1})/(1^{j-1})}(x) \\ &= \sum_{k=1}^{N} (-1)^{j+k} e_{d+j-k}(x) s_{(d^{N-1})+(1^{k-1})/(1^{j-1})}(x), \end{aligned} \tag{2.71}$$

where $k \in \{1, \ldots, N\}$ (resp. j-th row, for $j \in \{1, \ldots, N\}$) is fixed in the first (resp. second) identity.

Some particular cases of the above identities are already known relations between symmetric functions. For instance, choosing $j = 1$ and $d = 1$ in the second identity, one obtains the well known relation between the elementary and complete homogeneous symmetric polynomials

$$\sum_{j=0}^{N} (-1)^j e_j(x) h_{N-j}(x) = 0.$$

However, note that for every N, equation (2.71) actually contains $2N$ different expansions of the Schur polynomial $s_{(N^d)}$. For instance, choosing $N = 3$ and $d = 4$ in the theorem, and fixing $k = 1$ in the first identity and $j = 2$ in the second identity we obtain



[17]The proof of identity (2.69) is essentially a translation of Jacobi's identity on the minors of a matrix [191, 43] to the case of Toeplitz matrices, in combination with (2.7).

which can be easily checked using the Jacobi-Trudi dual identity and the Pieri formulas.

As was the case with the formula for the inverse of a matrix, Laplace's expansion formula can be generalized to an identity involving more general minors than the single row and single column case, as well as their complementary minors (see [191], for instance). For the Toeplitz case this reads

$$\begin{aligned} D_{N+\mu_1}(f) &= \sum_{\lambda\subset(\mu_1^N)} (-1)^{|\lambda|+|\mu|} D_N^{\lambda,\mu}(f) D_{\mu_1}^{L_{N,\mu_1}(\lambda'),L_{N,\mu_1}(\mu')}(f) \\ &= \sum_{\mu\subset(\lambda_1^N)} (-1)^{|\lambda|+|\mu|} D_N^{\lambda,\mu}(f) D_{\lambda_1}^{L_{N,\lambda_1}(\lambda'),L_{N,\lambda_1}(\mu')}(f). \end{aligned} \tag{2.72}$$

where μ (resp. λ) is a fixed partition of length smaller than or equal to N that determines the columns (resp. rows) with respect which we perform Laplace expansion in the first (resp. second) identity. Indeed, the only nontrivial step when translating the general formula to the Toeplitz case is showing that the complementary minor to that indexed by partitions λ and μ is the one indexed by the partitions $L_{K,N}(\lambda)$ and $L_{K,N}(\mu)$, where $K = \max\{\lambda_1, \mu_1\}$ (recall the definition (2.17)). Some inspection shows that this can be deduced from (2.7), for instance. Writing the equivalent expression of the minors in (2.72) in terms of skew Schur polynomials (2.32) as above, we arrive at the following result.

Theorem 3. *Let d be a positive integer. We have*

$$\begin{aligned} s_{(N+\mu_1)^d}(x) &= \sum_{\lambda\subset(\mu_1^N)} (-1)^{|\lambda|+|\mu|} s_{(((d^N)+\mu)/\lambda)'}(x) s_{\left(((d^{\mu_1})+L_{N,\mu_1}(\mu'))/L_{N,\mu_1}(\lambda')\right)'}(x) \\ s_{(N+\lambda_1)^d}(x) &= \sum_{\mu\subset(\lambda_1^N)} (-1)^{|\lambda|+|\mu|} s_{(((d^N)+\mu)/\lambda)'}(x) s_{\left(((d^{\lambda_1})+L_{N,\lambda_1}(\mu'))/L_{N,\lambda_1}(\lambda')\right)'}(x), \end{aligned}$$

where μ (resp. λ) is fixed in the first (resp. second) identity.

Chapter 3

Matrix models for classical groups and Toeplitz±Hankel minors with applications to Chern-Simons theory and fermionic models

Chapter summary

We study matrix integration over the classical Lie groups $G(N) = U(N), Sp(2N), O(2N)$ and $O(2N+1)$, using symmetric function theory and the equivalent formulation in terms of determinants and minors of Toeplitz±Hankel matrices, allowing the insertion of irreducible characters in the integrands ("twisted" integrals). After reviewing some facts from the theory of symmetric functions, we establish a number of relations between such integrals, including

1. Factorizations of unitary integrals as products and sums of products of symplectic and orthogonal integrals,
2. The expression of a class of models as the specialization of a single character associated to the corresponding symmetry group,
3. Expansions of symplectic and orthogonal integrals as weighted sums of twisted unitary integrals, or, equivalently, expansions of Toeplitz±Hankel determinants as weighted sums of Toeplitz minors,
4. Gessel type identities, expressing the $G(N)$ integrals under study as Schur function series, including the twisted case,
5. The asymptotic behaviour of the averages of irreducible characters over the aforementioned matrix models.

We then turn to an exactly solvable model, associated to Jacobi's third theta function. This allows us to compute both at finite and large N the partition functions, Wilson loops and Hopf links of Chern-Simons theory on S^3 with symmetry group $G(N)$, and we show that these models are Giambelli compatible. In this context, the general relations found before translate to identities between observables of the theories with different symmetry groups. Finally, we use character expansions and the asymptotic behaviour of the associated determinants to evaluate

averages in random matrix ensembles of Chern-Simons type, describing the spectra of solvable fermionic models with matrix degrees of freedom[18].

3.1 Preliminaries

3.1.1 Toeplitz±Hankel minors

Consider the groups of symplectic matrices of order $2N$, denoted by $Sp(2N)$, and of orthogonal matrices of orders $2N$ and $2N+1$, denoted by $O(2N)$ and $O(2N+1)$ respectively. We will also write $G(N)$ to refer to any of the groups $U(N), Sp(2N), O(2N)$ or $O(2N+1)$. In particular, the parameter N stands for the number of nontrivial eigenvalues of the matrices belonging to each of the groups, which are complex numbers of modulus 1. Given a square integrable function on the unit circle f, we define

$$f(U) = \prod_{k=1}^{N} f(e^{i\theta_k}) f(e^{-i\theta_k}), \tag{3.1}$$

for any matrix U belonging to one of the groups $G(N)$, where the $e^{i\theta_k}$ are the nontrivial eigenvalues of U. Note the difference with the definition (2.4) in the previous chapter; considering (3.1) instead amounts to considering symmetric Toeplitz matrices in the results of chapter 2 or, equivalently, functions that satisfy $f(z) = f(z^{-1})$. We will use definition (3.1) throughout the remainder of this chapter. Using Weyl's integral formula [195, 57], one can see that the integral of a function of the form (3.1) over one of the groups $G(N)$ with respect to Haar measure can be expressed as

$$\int_{G(N)} f(U)dU = C_{G(N)} \frac{1}{N!} \int_{[0,2\pi]^N} \det(M_{G(N)}(e^{-i\theta})) \det(M_{G(N)}(e^{i\theta})) \prod_{k=1}^{N} f(e^{i\theta_k}) f(e^{-i\theta_k}) \frac{d\theta_k}{2\pi}, \tag{3.2}$$

where dU denotes Haar measure, the constants $C_{G(N)}$ are

$$C_{U(N)} = 1, \qquad C_{Sp(2N)} = \frac{1}{2^N} = C_{O(2N+1)}, \qquad C_{O(2N)} = \frac{1}{2^{N+1}}$$

and $M_{G(N)}(e^{i\theta})$ is the matrix appearing in Weyl's denominator formula for the root system associated to each of the groups $G(N)$. In the unitary case, this is the Vandermonde matrix, while for the rest of the groups we have [137]

$$\det M_{U(N)}(z) = \det \left(z_j^{N-k} \right)_{j,k=1}^{N} = \prod_{1 \le j < k \le N} (z_j - z_k), \tag{3.3}$$

$$\det M_{Sp(2N)}(z) = \det \left(z_j^{N-k+1} - z_j^{-(N-k+1)} \right)_{j,k=1}^{N} = \prod_{1 \le j < k \le N} (z_j - z_k)(1 - z_j z_k) \prod_{j=1}^{N} (z_j^2 - 1) z_j^{-N}, \tag{3.4}$$

$$\det M_{O(2N)}(z) = \det \left(z_j^{N-k+\frac{1}{2}} - z_j^{-(N-k+\frac{1}{2})} \right)_{j,k=1}^{N} = 2 \prod_{1 \le j < k \le N} (z_j - z_k)(1 - z_j z_k) \prod_{j=1}^{N} z_j^{-N+1}, \tag{3.5}$$

[18] The content of this chapter is based on the preprint [106]. Some results displayed here and not in [106] include theorem 7 and corollaries 5 and 6.

$$\det M_{O(2N+1)}(z) = \det\left(z_j^{N-k} + z_j^{-(N-k)}\right)_{j,k=1}^{N} = \prod_{1\le j<k\le N}(z_j - z_k)(1 - z_j z_k)\prod_{j=1}^{N}(z_j - 1)z_j^{-N+1/2}, \tag{3.6}$$

where we denote $z_j = e^{i\theta_j}$. Choosing $\sigma(\theta) = \frac{1}{2\pi} f(e^{i\theta}) f(e^{-i\theta}) d\theta$ on $[0, 2\pi)$ as measure and suitable functions g_j and h_j for each of the groups $G(N)$ in Andréief's identity (2.2) we obtain from (3.2) the determinantal expressions

$$\int_{U(N)} f(U)dU = \det\left(d_{j-k}\right)_{j,k=1}^{N}, \tag{3.7}$$

$$\int_{Sp(2N)} f(U)dU = \det\left(d_{j-k} - d_{j+k}\right)_{j,k=1}^{N}, \tag{3.8}$$

$$\int_{O(2N)} f(U)dU = \frac{1}{2}\det\left(d_{j-k} + d_{j+k-2}\right)_{j,k=1}^{N}, \tag{3.9}$$

$$\int_{O(2N+1)} f(U)dU = \det\left(d_{j-k} - d_{j+k-1}\right)_{j,k=1}^{N}, \tag{3.10}$$

where d_k denotes the Fourier coefficient

$$d_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\theta} f(e^{i\theta}) f(e^{-i\theta}) d\theta \tag{3.11}$$

for each $k \in \mathbb{Z}$. Note that this definition also differs from the one given in the previous chapter (2.1). As remarked above, we now have $d_k = d_{-k}$ for all k. Recall that a matrix which (j,k)-th coefficient depends only on $j+k$ is called a Hankel matrix, and is constant along its anti-diagonals. Expressions for group integrals as determinants of Toeplitz±Hankel matrices have been obtained previously, see for instance [18]. Besides their own intrinsic interest, matrix integrals over the groups $G(N)$ enjoy connections with combinatorics [18], number theory [133] and integrable systems [3], among many other topics.

Note that while $\int_{G(N)} f(U)dU = \int_{G(N)} f(-U)dU$ for $G(N) = U(N), Sp(2N), O(2N)$ (as follows from the above determinantal expressions, for instance), we have

$$\int_{O(2N+1)} f(-U)dU = \det\left(d_{j-k} + d_{j+k-1}\right)_{j,k=1}^{N}. \tag{3.12}$$

It turns out that the minors of the above Toeplitz±Hankel matrices (which will be referred to as Toeplitz±Hankel minors in the following) also have equivalent integral representations, as in the unitary case. Indeed, one can also express the characters associated to the irreducible representations of the groups $G(N)$ as the quotient of a minor of the matrix $M_{G(N)}$, indexed by a partition λ, and the determinant of the matrix itself. See (3.13)-(3.16) for such expressions. Hence, the insertion of one or two characters of the group $G(N)$ in the integrand in (3.2) cancels one or two of the determinants. Therefore, denoting by $\chi^{\lambda}_{G(N)}$ the character of the group $G(N)$ indexed by the partition λ, we obtain the following result from Andreiéf's identity (2.2).

Theorem 4. *Let N be a positive integer, and let λ and μ be two partitions of lengths $l(\lambda), l(\mu) \le N$. Consider the "reversed" arrays*

$$\lambda^r = (\lambda_{N-j+1})_j = (\lambda_N, \lambda_{N-1}, \ldots, \lambda_2, \lambda_1), \qquad \mu^r = (\mu_{N-j+1})_j = (\mu_N, \ldots, \mu_1).$$

We then have

$$\int_{U(N)} \chi^{\lambda}_{U(N)}(U^{-1})\chi^{\mu}_{U(N)}(U)f(U)dU = \det\left(d_{j-\lambda_j-k+\mu_k}\right)_{j,k=1}^{N} = \det\left(d_{j+\lambda^r_j-k-\mu^r_k}\right)_{j,k=1}^{N},$$
$$\int_{Sp(2N)} \chi^{\lambda}_{Sp(2N)}(U)\chi^{\mu}_{Sp(N)}(U)f(U)dU = \det\left(d_{j+\lambda^r_j-k-\mu^r_k} - d_{j+\lambda^r_j+k+\mu^r_k}\right)_{j,k=1}^{N},$$
$$\int_{O(2N)} \chi^{\lambda}_{O(2N)}(U)\chi^{\mu}_{O(2N)}(U)f(U)dU = \frac{1}{2}\det\left(d_{j+\lambda^r_j-k-\mu^r_k} + d_{j+\lambda^r_j+k+\mu^r_k-2}\right)_{j,k=1}^{N},$$
$$\int_{O(2N+1)} \chi^{\lambda}_{O(2N+1)}(U)\chi^{\mu}_{O(2N+1)}(U)f(U)dU = \det\left(d_{j+\lambda^r_j-k-\mu^r_k} - d_{j+\lambda^r_j+k+\mu^r_k-1}\right)_{j,k=1}^{N},$$

where the d_k are given by (3.11)*.*

We have used above the fact that $\chi^{\lambda}_{G(N)}(U) = \chi^{\lambda}_{G(N)}(U^{-1})$ for $G(N) = Sp(2N), O(2N), O(2N+1)$. The resulting determinants are now minors of the Toeplitz and Toeplitz±Hankel matrices appearing in the right hand sides of formulas (3.7)-(3.10), obtained by striking some of their rows and columns. The procedure to obtain these minors from the corresponding partitions is the same as the one described in section 2.1.2 for the case of Toeplitz minors, with the exception that the order of rows and columns should now be inverted from first to last. Let us record here this procedure, for convenience.

- Start with one of the Toeplitz or Toeplitz±Hankel matrices in the right hand sides of (3.7)-(3.10), of size $N + \max\{\lambda_1, \mu_1\}$. Strike the last $|\lambda_1 - \mu_1|$ columns or rows of the matrix, depending on whether $\lambda_1 - \mu_1$ is greater or smaller than zero, respectively.

- Keep the last row of the resulting matrix, and strike the $\lambda_1 - \lambda_2$ next-to-last rows. Keep the next row, and strike the next $\lambda_2 - \lambda_3$ rows. Continue until striking $\lambda_{l(\lambda)} - \lambda_{l(\lambda)+1} = \lambda_{l(\lambda)}$ rows.

- Repeat the previous step on the columns of the matrix with μ in place of λ. The resulting matrix is precisely the minor indexed by the partitions λ and μ, as defined in theorem 4.

In particular, the striking of rows and columns performed on the underlying matrix only depends on the partitions λ and μ, and is the same for any of the matrices (3.7)-(3.10). Note also that in the Toeplitz case, the above procedure coincides with the one described in section 2.1.2, as the matrices are now symmetric.

3.1.2 Characters of $G(N)$ and symmetric functions

We summarize below some basic facts about the characters of the classical groups and their relation to symmetric functions. See [147, 102, 135] for more details.

Recall that Schur polynomials, which correspond to the irreducible characters of the unitary group $U(N)$, can be defined as the quotient of a minor of the Vandermonde matrix (indexed by a partition λ) over the determinant of the matrix itself (2.10). The irreducible characters associated to the other groups $G(N)$ can be defined analogously, replacing the Vandermonde matrix by the corresponding matrix $M_{G(N)}$, recall (3.3)-(3.6). More precisely, let λ be a partition of length $l(\lambda) \le N$; we have[19]

$$\chi^{\lambda}_{U(N)}(U) = \frac{\det M^{\lambda}_{U(N)}(z)}{\det M_{U(N)}(z)} = \frac{\det\left(z_j^{N-k+\lambda_k}\right)_{j,k=1}^{N}}{\det\left(z_j^{N-k}\right)_{j,k=1}^{N}}, \tag{3.13}$$

$$\chi^{\lambda}_{Sp(2N)}(U) = \frac{\det M^{\lambda}_{Sp(2N)}(z)}{\det M_{Sp(2N)}(z)} = \frac{\det\left(z_j^{N-k+\lambda_k+1} - z_j^{-(N-k+\lambda_k+1)}\right)_{j,k=1}^{N}}{\det\left(z_j^{N-k+1} - z_j^{-(N-k+1)}\right)_{j,k=1}^{N}}, \tag{3.14}$$

$$\chi^{\lambda}_{O(2N)}(U) = \frac{\det M^{\lambda}_{O(2N)}(z)}{\det M_{O(2N)}(z)} = \frac{\det\left(z_j^{N-k+\lambda_k} + z_j^{-(N-k+\lambda_k)}\right)_{j,k=1}^{N}}{\det\left(z_j^{N-k} + z_j^{-(N-k)}\right)_{j,k=1}^{N}}, \tag{3.15}$$

$$\chi^{\lambda}_{O(2N+1)}(U) = \frac{\det M^{\lambda}_{O(2N+1)}(z)}{\det M_{O(2N+1)}(z)} = \frac{\det\left(z_j^{N-k+\lambda_k+\frac{1}{2}} - z_j^{-(N-k+\lambda_k+\frac{1}{2})}\right)_{j,k=1}^{N}}{\det\left(z_j^{N-k+\frac{1}{2}} - z_j^{-(N-k+\frac{1}{2})}\right)_{j,k=1}^{N}}, \tag{3.16}$$

Of course, this is nothing but Weyl's character formula, specialized to each of the groups $G(N)$.

The characters $\chi^{\lambda}_{G(N)}$ can be lifted to the so called "universal characters" in the ring of symmetric functions in countably many variables [135]. In this fashion, the lifting of the characters of $U(N)$, $Sp(2N)$, $O(2N)$ and $O(2N+1)$ gives rise to the Schur s_λ, symplectic Schur sp_λ, even orthogonal Schur o^{even}_λ and odd orthogonal Schur o^{odd}_λ functions, respectively. When the length of the partition λ is less than or equal to the number of nontrivial eigenvalues of a matrix U, these functions coincide with the irreducible characters of the corresponding group, after specializing the corresponding variables back to the nontrivial eigenvalues z_j of U. For instance, we have $\chi^{\lambda}_{Sp(2N)}(U) = sp_\lambda(z_1, \dots, z_N)$ for any partition satisfying $l(\lambda) \le N$. We emphasize that while this condition is necessary in order for the characters $\chi^{\lambda}_{G(N)}(U)$ to be defined, the corresponding symmetric functions need not satisfy such restriction, and are defined for more general partitions. Indeed, given a (possibly infinite) set of variables $x = (x_1, x_2, \dots)$, one can define the Schur, symplectic Schur, and even/odd orthogonal Schur functions by means of the Jacobi-Trudi identities

$$s_\lambda(x) = \det\left(h_{j-k+\lambda_k}(x)\right)_{j,k=1}^{l(\lambda)} = \det\left(e_{j-k+\lambda'_k}(x)\right)_{j,k=1}^{\lambda_1}, \tag{3.17}$$

$$sp_\lambda(x) = \frac{1}{2}\det\left(h_{\lambda_j-j+k}(x,x^{-1}) + h_{\lambda_j-j-k+2}(x,x^{-1})\right)_{j,k=1}^{l(\lambda)} \tag{3.18}$$

$$= \det\left(e_{\lambda'_j-j+k}(x,x^{-1}) - e_{\lambda'_j-j-k}(x,x^{-1})\right)_{j,k=1}^{\lambda_1} \tag{3.19}$$

$$o^{even}_\lambda(x) = \det\left(h_{\lambda_j-j+k}(x,x^{-1}) - h_{\lambda_j-j-k}(x,x^{-1})\right)_{j,k=1}^{l(\lambda)} \tag{3.20}$$

$$= \frac{1}{2}\det\left(e_{\lambda'_j-j+k}(x,x^{-1}) + e_{\lambda'_j-j-k+2}(x,x^{-1})\right)_{j,k=1}^{\lambda_1}, \tag{3.21}$$

$$o^{odd}_\lambda(x) = \det\left(h_{\lambda_j-j+k}(x,x^{-1},1) - h_{\lambda_j-j-k}(x,x^{-1},1)\right)_{j,k=1}^{l(\lambda)} \tag{3.22}$$

[19]Recall that the character (3.15) does not correspond to an irreducible representation of $O(2N)$ if $\lambda_N \neq 0$. This fact is not relevant for our purposes so we ignore it in the following and work with the algebraic expression (3.15); minor modifications to the subsequent reasoning allow a treatment of the general case.

$$=\frac{1}{2}\det\Big(e_{\lambda'_j-j+k}(x,x^{-1},1)+e_{\lambda'_j-j-k+2}(x,x^{-1},1)\Big)_{j,k=1}^{\lambda_1}, \tag{3.23}$$

where the h_k and the e_k are the complete homogeneous and elementary symmetric polynomials respectively (2.8). These functions satisfy the Cauchy identities

$$\sum_\nu s_\nu(x)s_\nu(y)=\prod_{i,j=1}^{\infty}\frac{1}{1-x_iy_j}, \tag{3.24}$$

$$\sum_\nu sp_\nu(x)s_\nu(y)=\prod_{i<j}(1-y_iy_j)\prod_{i,j=1}^{\infty}\frac{1}{1-x_iy_j}\frac{1}{1-x_i^{-1}y_j}, \tag{3.25}$$

$$\sum_\nu o_\nu^{even}(x)s_\nu(y)=\prod_{i\le j}(1-y_iy_j)\prod_{i,j=1}^{\infty}\frac{1}{1-x_iy_j}\frac{1}{1-x_i^{-1}y_j}, \tag{3.26}$$

$$\sum_\nu o_\nu^{odd}(x)s_\nu(y)=\prod_{i\le j}(1-y_iy_j)\prod_{i,j=1}^{\infty}\frac{1}{1-x_iy_j}\frac{1}{1-x_i^{-1}y_j}\prod_{j=1}^{\infty}\frac{1}{1-y_j}, \tag{3.27}$$

and dual Cauchy identities

$$\sum_\nu s_\nu(x)s_{\nu'}(y)=\prod_{i,j=1}^{\infty}(1+x_iy_j), \tag{3.28}$$

$$\sum_\nu sp_\nu(x)s_{\nu'}(y)=\prod_{i\le j}(1-y_iy_j)\prod_{i,j=1}^{\infty}(1+x_iy_j)(1+x_i^{-1}y_j), \tag{3.29}$$

$$\sum_\nu o_\nu^{even}(x)s_{\nu'}(y)=\prod_{i<j}(1-y_iy_j)\prod_{i,j=1}^{\infty}(1+x_iy_j)(1+x_i^{-1}y_j). \tag{3.30}$$

$$\sum_\nu o_\nu^{odd}(x)s_{\nu'}(y)=\prod_{i<j}(1-y_iy_j)\prod_{i,j=1}^{\infty}(1+x_iy_j)(1+x_i^{-1}y_j)\prod_{j=1}^{\infty}(1+y_j). \tag{3.31}$$

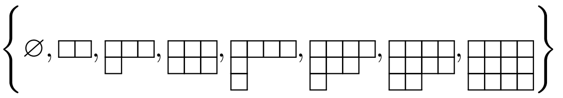
Since the groups $Sp(2N), O(2N), O(2N+1)$ can be embedded on the unitary group $U(2N)$ or $U(2N+1)$, the irreducible characters on each of these groups can be expressed in terms of the others, after applying the specialization homomorphisms $(z_1,\dots,z_{2N})\mapsto(z_1,\dots,z_N,z_1^{-1},\dots,z_N^{-1})$ for $Sp(2N)$, $O(2N)$ or $(z_1,\dots,z_{2N+1})\mapsto(z_1,\dots,z_N,z_1^{-1},\dots,z_N^{-1},1)$ for $O(2N+1)$. When seen as symmetric functions, they have the following expansions [135]

$$s_\lambda(x,x^{-1})=\sum_\alpha\sum_{\beta'\,even}c^\lambda_{\alpha\beta}sp_\alpha(x), \tag{3.32}$$

$$s_\lambda(x,x^{-1})=\sum_\alpha\sum_{\beta\,even}c^\lambda_{\alpha\beta}o_\alpha^{even}(x), \tag{3.33}$$

$$s_\lambda(x,x^{-1},1)=\sum_\alpha\sum_{\beta\,even}c^\lambda_{\alpha\beta}o_\alpha^{odd}(x), \tag{3.34}$$

where $c^\lambda_{\alpha\beta}$ are Littlewood-Richardson coefficients (2.12), and we say that a partition is even if it has only even parts. Reciprocally, Schur polynomials evaluated at a set of variables and their inverses can be expressed in terms of symplectic and orthogonal characters. To state this relation precisely, we first recall the Frobenius notation for partitions. We write $\lambda=(a_1,\dots,a_p|b_1,\dots,b_p)$, for some positive integers $a_1>\cdots>a_p$ and $b_1>\cdots>b_p$, if there are p boxes on the main diagonal of the Young diagram of λ, with the k-th box having a_k boxes immediately to the right and b_k boxes immediately below. Given a partition λ, we denote by $p(\lambda)$ the number of boxes on the main diagonal of its diagram. We can then introduce the sets $R(N), S(N)$ and $T(N)$ of partitions of shapes $(a_1+1,\ldots,a_p+1|a_1,\ldots,a_p)$, $(a_1,\ldots,a_p|a_1,\ldots,a_p)$ and $(a_1-1,\ldots,a_p-1|a_1,\ldots,a_p)$ respectively in Frobenius notation, with $a_1 \leq N-1$. For instance, the set $R(3)$ consists of the partitions



(3.35)

the set $S(3)$ is the set of self-conjugate partitions of length at most 3 and the set $T(3)$ is obtained as the set of partitions conjugated to those of $R(2)$. Note that there are exactly 2^N partitions in each of the sets $R(N)$ and $S(N)$, and 2^{N-1} in the set $T(N)$, all of them of length less than or equal to N. We can now state Littlewood's classical identities [135]

$$\begin{aligned} sp_\lambda(x) &= \sum_\alpha \sum_{\beta\in T(N)} (-1)^{|\beta|/2} c^\lambda_{\alpha\beta} s_\alpha(x,x^{-1}) = \sum_{\beta\in T(N)} (-1)^{|\beta|/2} s_{\lambda/\beta}(x,x^{-1}), \\ o^{even}_\lambda(x) &= \sum_\alpha \sum_{\beta\in R(N)} (-1)^{|\beta|/2} c^\lambda_{\alpha\beta} s_\alpha(x,x^{-1}) = \sum_{\beta\in R(N)} (-1)^{|\beta|/2} s_{\lambda/\beta}(x,x^{-1}), \\ o^{odd}_\lambda(x) &= \sum_\alpha \sum_{\beta\in R(N)} (-1)^{|\beta|/2} c^\lambda_{\alpha\beta} s_\alpha(x,x^{-1},1) = \sum_{\beta\in R(N)} (-1)^{|\beta|/2} s_{\lambda/\beta}(x,x^{-1},1). \end{aligned} \tag{3.36}$$

A distinctive feature of symmetric functions is that different Young diagrams may actually determine the same skew Schur polynomial. For instance, we have already used the fact that the skew Schur polynomial indexed by any given skew diagram coincides with the skew Schur polynomial indexed by a 180º rotation of the very same diagram in the previous chapter, but many other conditions under which this holds are known [168]. An example of different Young diagrams determining the same symplectic or orthogonal characters (and thus, symplectic or orthogonal Schur functions) is given in proposition 2.4.1 of [135]. Let us provide another example of this, which will be useful in the following. We will use the easily checked property

$$e_k(x_1,\ldots,x_N,x_1^{-1},\ldots,x_N^{-1}) = e_{2N-k}(x_1,\ldots,x_N,x_1^{-1},\ldots,x_N^{-1}). \tag{3.37}$$

Theorem 5. *Let λ be a partition. We have*

$$sp_\lambda(x_1,\ldots,x_N) = (-1)^{\lambda_1(\lambda_1+1)/2} sp_{L_{\lambda_1,2N+\lambda_1+1}(\lambda)}(x_1,\ldots,x_N), \tag{3.38}$$

$$o^{even}_\lambda(x_1,\ldots,x_N) = (-1)^{\lambda_1(\lambda_1-1)/2} o^{even}_{L_{\lambda_1,2N+\lambda_1-1}(\lambda)}(x_1,\ldots,x_N), \tag{3.39}$$

$$o^{odd}_\lambda(x_1,\ldots,x_N) = (-1)^{\lambda_1(\lambda_1-1)/2} o^{odd}_{L_{\lambda_1,2N+\lambda_1}(\lambda)}(x_1,\ldots,x_N), \tag{3.40}$$

where $L_{K,N}(\lambda)$ is as defined in (2.17).

That is, the characters of $Sp(2N)$, $O(2N)$ and $O(2N+1)$ indexed by a partition λ coincide (up to a sign) with the characters indexed by the partition obtained from rotating 180º the complement of λ in the rectangular diagrams $(\lambda_1^{2N+\lambda_1+1})$, $(\lambda_1^{2N+\lambda_1-1})$ and $(\lambda_1^{2N+\lambda_1})$ respectively.

Proof. It is instructive to start with two simple examples. For the case of partitions with a single column, it follows from (3.19) and (3.37) that

$$sp_{(1^k)}(x) = e_k(x,x^{-1}) - e_{k-2}(x,x^{-1}) = e_{2N-k}(x,x^{-1}) - e_{2N+2-k}(x,x^{-1}) = -sp_{(1^{2N+2-k})}(x). \tag{3.41}$$

For the case of partitions with two columns, we obtain from (3.19) and (3.41) that

$$sp_{(2^k1^j)}(x) = \begin{vmatrix} e_{j+k}-e_{j+k-2} & e_{j+k+1}-e_{j+k-3} \\ e_{k-1}-e_{k-3} & e_k-e_{k-4} \end{vmatrix} = \begin{vmatrix} sp_{(1^{j+k})} & sp_{(1^{j+k+1})}+sp_{(1^{j+k-1})} \\ sp_{(1^{k-1})} & sp_{(1^k)}+sp_{(1^{k-2})} \end{vmatrix} =$$

$$\begin{vmatrix} -sp_{(1^{2N+2-j-k})} & -sp_{(1^{2N+1-j-k})}-sp_{(1^{2N+3-j-k})} \\ -sp_{(1^{2N+3-k})} & -sp_{(1^{2N+2-k})}-sp_{(1^{2N+4-k})} \end{vmatrix} =$$

$$-\begin{vmatrix} sp_{(1^{2N+3-k})} & sp_{(1^{2N+4-k})}+sp_{(1^{2N+2-k})} \\ sp_{(1^{2N+2-j-k})} & sp_{(1^{2N+3-j-k})}+sp_{(1^{2N+1-j-k})} \end{vmatrix} = -sp_{(2^{2N+3-j-k}1^j)}(x),$$

where the fourth identity above results from exchanging the first and second rows of the determinant (we have omitted the dependence on x in the determinants for ease of notation).

The proof for the general case is a straightforward generalization of the above reasoning. Let now $\lambda = (1^{a_1}2^{a_2}\dots M^{a_M})$ be a general partition, written in frequency notation. That is, λ is the partition with exactly a_M parts equal to M, a_{M-1} parts equal to $M-1$, and so on. Then, we have

$$\lambda' = (a_M + a_{M-1} + \cdots + a_1, a_M + a_{M-1} + \cdots + a_2, \dots, a_M + a_{M-1}, a_M),$$

using the standard notation for partitions. Let us denote the j-th entry of λ' by b_j, for ease of notation. It follows from the Jacobi-Trudi identity (3.19) and (3.37) that $(-1)^M sp_\lambda$ can be expressed as (we omit again the dependence on x)

$$(-1)^M \begin{vmatrix} e_{b_1}-e_{b_1-2} & e_{b_1+1}-e_{b_1-3} & \cdots & e_{b_1+M-1}-e_{b_1-M-1} \\ e_{b_2-1}-e_{b_2-3} & e_{b_2}-e_{b_2-4} & \cdots & e_{b_2+M-2}-e_{b_2-M-2} \\ \vdots & \vdots & & \vdots \\ e_{b_{M-1}-M+2}-e_{b_{M-1}-M} & e_{b_{M-1}-M+3}-e_{b_{M-1}-M-1} & \cdots & e_{b_{M-1}+1}-e_{b_{M-1}-2M+1} \\ e_{b_M-M+1}-e_{b_M-M-1} & e_{b_M-M+2}-e_{b_M-M-2} & \cdots & e_{b_M}-e_{b_M-2M} \end{vmatrix} =$$

$$(-1)^M \begin{vmatrix} sp_{(1^{b_1})} & sp_{(1^{b_1+1})}+sp_{(1^{b_1-1})} & \cdots & sp_{(1^{b_1+M-1})}+\cdots+sp_{(1^{b_1-M+1})} \\ sp_{(1^{b_2-1})} & sp_{(1^{b_2})}+sp_{(1^{b_2-2})} & \cdots & sp_{(1^{b_2+M-2})}+\cdots+sp_{(1^{b_2-M})} \\ \vdots & \vdots & & \vdots \\ sp_{(1^{b_{M-1}-M+2})} & sp_{(1^{b_{M-1}-M+3})}+sp_{(1^{b_{M-1}-M+1})} & \cdots & sp_{(1^{b_{M-1}+1})}+\cdots+sp_{(1^{b_{M-1}-2M+3})} \\ sp_{(1^{b_M-M+1})} & sp_{(1^{b_M-M+2})}+sp_{(1^{b_M-M})} & \cdots & sp_{(1^{b_M})}+\cdots+sp_{(1^{b_M-2M+2})} \end{vmatrix} =$$

$$\begin{vmatrix} sp_{(1^{2N+2-b_1})} & sp_{(1^{2N+1-b_1})}+sp_{(1^{2N+3-b_1})} & \cdots & sp_{(1^{2N+3-M-b_1})}+\cdots+sp_{(1^{2N+1+M-b_1})} \\ sp_{(1^{2N+3-b_2})} & sp_{(1^{2N+2-b_2})}+sp_{(1^{2N+4-b_2})} & \cdots & sp_{(1^{2N+4-M-b_2})}+\cdots+sp_{(1^{2N+2+M-b_2})} \\ \vdots & \vdots & & \vdots \\ sp_{(1^{2N+M-b_{M-1}})} & sp_{(1^{2N-1+M-b_{M-1}})}+sp_{(1^{2N+1+M-b_{M-1}})} & \cdots & sp_{(1^{2N+1-b_{M-1}})}+\cdots+sp_{(1^{2N-1+2M-b_{M-1}})} \\ sp_{(1^{2N+1+M-b_M})} & sp_{(1^{2N+M-b_M})}+sp_{(1^{2N+2+M-b_M})} & \cdots & sp_{(1^{2N+2-b_M})}+\cdots+sp_{(1^{2N+2M-b_M})} \end{vmatrix}.$$

Reversing the order of the rows of the last determinant above, we see that it corresponds to another symplectic Schur function indexed by some partition μ, satisfying

$$\begin{aligned} \mu'_1 &= 2N + M + 1 - b_M = 2N + M + 1 - a_M, \\ \mu'_2 &= 2N + M + 1 - b_{M-1} = 2N + M + 1 - a_M - a_{M-1}, \\ &\vdots \end{aligned}$$

$$\mu'_M = 2N + M + 1 - b_1 = 2N + M + 1 - a_M - a_M - a_{M-1} - \cdots - a_1,$$

which corresponds precisely to $L_{\lambda_1, 2N+\lambda_1+1}(\lambda)$, thus yielding the desired conclusion. The proof of identities (3.39) and (3.40) follows analogously from the corresponding Jacobi-Trudi identities (3.21) and (3.23). □

3.1.3 Large N limit of Toeplitz and Toeplitz±Hankel determinants

We record now a generalization of Szegő's theorem to symplectic and orthogonal integrals (equivalently, determinants of Toeplitz±Hankel matrices) due to Johansson [127].

Theorem (Johansson)**.** *Let f be a function in the unit circle, and assume that it can be expressed as $f(e^{i\theta}) = \exp(\sum_{k=1}^{\infty} c_k e^{ik\theta})$, with $\sum_k |c_k| < \infty$ and $\sum_k k|c_k|^2 < \infty$, and define $f(U)$ by formula (3.1). We have*

$$\lim_{N\to\infty} \int_{Sp(2N)} f(U)dU = \exp\left(\frac{1}{2}\sum_{k=1}^{\infty} kc_k^2 - \sum_{k=1}^{\infty} c_{2k}\right), \tag{3.42}$$

$$\lim_{N\to\infty} \int_{O(2N)} f(U)dU = \exp\left(\frac{1}{2}\sum_{k=1}^{\infty} kc_k^2 + \sum_{k=1}^{\infty} c_{2k}\right), \tag{3.43}$$

$$\lim_{N\to\infty} \int_{O(2N+1)} f(U)dU = \exp\left(\frac{1}{2}\sum_{k=1}^{\infty} kc_k^2 - \sum_{k=1}^{\infty} c_{2k-1}\right). \tag{3.44}$$

We have stated the theorem for slightly different integrals that those appearing in [127]. The result, as stated here, follows after using the mapping $\cos\theta_j \mapsto x_j$ in the integrals[20] (3.2) and using the general version of Johansson's result. This allows to express the integrals in terms of the orthogonal polynomials with respect to a modified weight on $[-1, 1]$, which relation with the orthogonal polynomials with respect to the original weight is well known [181] (see also [18]). This result has been rederived in several different contexts, see for instance [23, 24, 67, 25].

The asymptotic behaviour of Toeplitz±Hankel determinants generated by functions with Fisher-Hartwig singularities has also attracted interest over the years [68]. For our purposes, we will only need to consider determinants generated by functions with a single Fisher-Hartwig singularity. This fact, together with the definition (3.1) allows us to consider only particular examples of the very general results known for this kind of asymptotics. Starting with the Toeplitz case, what follows is a particular case of a theorem of Widom [196] adapted for this setting. See [80, 67] for more general results on the topic.

Theorem (Widom)**.** *Let f be given by*

$$f(e^{i\theta}) = e^{V(e^{i\theta})}(1 - e^{i(\theta-\theta_0)})^{\alpha}, \tag{3.45}$$

where $Re(\alpha) > -1/2$, $0 < \theta_0 < 2\pi$, and the potential $V(e^{i\theta}) = \sum_{k=1}^{\infty} c_k e^{ik\theta}$ satisfies $\sum_k |c_k| < \infty$ and $\sum_k k|c_k|^2 < \infty$, as in Szegő's theorem. Define $f(U)$ by (3.1) for any $U \in U(N)$. Then, as $N \to \infty$, we have

$$\int_{U(N)} f(U)dU = \exp\left(\sum_{k=1}^{\infty} kc_k^2\right) N^{\alpha^2} e^{-2\alpha V(e^{i\theta_0})} \frac{G^2(\alpha+1)}{G(2\alpha+1)}(1 + o(1)), \tag{3.46}$$

where G is Barnes' G function

[20] The relation is more apparent working directly with the trigonometric expression of Haar measure on $G(N)$, see for instance equations (3.3)-(3.5) in [60].

We also quote a particular case of a theorem of Deift, Its and Krasovsky [67] for Toeplitz±Hankel determinants generated by functions with a single singularity at the point $z=-1$, which will be enough for our purposes. See [67] for more general results.

Theorem (Deift, Its, Krasovsky)**.** *Let f be given by* (3.45)*, with $\theta_0=\pi$, and define $f(U)$ by* (3.1) *for any $U\in Sp(2N), O(2N), O(2N+1)$. Then, as $N\to\infty$, we have*

$$\int_{G(N)} f(U)dU = \left(\int_{G(N)} e^{V(U)}dU\right) e^{-\alpha V(-1)} N^{\alpha^2/2+\alpha t} 2^{-\alpha^2/2-\alpha(s+t-1/2)} \frac{\pi^{\alpha/2}G(t+1)}{G(\alpha+t+1)}(1+o(1)), \tag{3.47}$$

where s and t depend on the group $G(N)$ and are given by

$$Sp(2N): s=t=\frac{1}{2}, \qquad O(2N): s=t=-\frac{1}{2}, \qquad O(2N+1): s=-t=\frac{1}{2}.$$

Note that the asymptotic behaviours of the integrals $\int_{G(N)} e^{V(U)}dU$ in equations (3.46) and (3.47) are given by Szegő's and Johansson's theorems, respectively.

3.2 Relations between Toeplitz±Hankel determinants and minors

We now turn to some computations exploiting the determinant and minor expressions for the group integrals introduced in section 3.1.1, as well as their symmetric function formulation.

3.2.1 Factorizations and group integrals as rectangular characters

Theorem 6. *We have*

$$\begin{aligned}
\int_{U(2N-1)} f(U)dU &= \int_{Sp(2N-2)} f(U)dU \int_{O(2N)} f(U)dU \\
&= \frac{1}{2}\int_{O(2N-1)} f(U)dU \int_{O(2N+1)} f(-U)dU + \frac{1}{2}\int_{O(2N+1)} f(U) \int_{O(2N-1)} f(-U)dU, \\
\int_{U(2N)} f(U)dU &= \int_{O(2N+1)} f(U)dU \int_{O(2N+1)} f(-U)dU \\
&= \frac{1}{2}\int_{Sp(2N)} f(U)dU \int_{O(2N)} f(U)dU + \frac{1}{2}\int_{Sp(2N-2)} f(U) \int_{O(2N+2)} f(U)dU.
\end{aligned}$$

Proof. The theorem follows immediately after expressing the above integrals as the Toeplitz and Toeplitz±Hankel determinants (3.7)-(3.10),(3.12) and noticing that these determinants satisfy the corresponding identities, see e.g. [191]. □

Theorem 7. *Let $x=(x_1,\dots,x_K)$ be some variables, and let λ be a partition satisfying $l(\lambda)\le N$ and $\lambda_1\le K$. We have*

$$\int_{Sp(2N)} \chi^{\lambda}_{Sp(2N)}(M)E(x_1,\dots,x_K;M)dM = \left(\prod_{j=1}^{K} x_j^N\right) sp_{L_{N,K}(\lambda')}(x_1,\dots,x_K) \tag{3.48}$$

$$\int_{O(2N)} \chi^{\lambda}_{O(2N)}(M)E(x_1,\ldots,x_K;M)dM = \left(\prod_{j=1}^{K} x_j^N\right) o^{even}_{L_{N,K}(\lambda')}(x_1,\ldots,x_K) \tag{3.49}$$

$$\int_{O(2N+1)} \chi^{\lambda}_{O(2N+1)}(M)E(x_1,\ldots,x_K;M)dM = (-1)^{|\lambda|+KN}\left(\prod_{j=1}^{K} x_j^N\right) o^{odd}_{L_{N,K}(\lambda')}(-x_1,\ldots,-x_K), \tag{3.50}$$

where $L_{N,K}(\lambda')$ *is the partition given by* (2.17).

Proof. Let us proceed with the symplectic case. We start from the case $\mu=\varnothing$ of the symplectic integral in theorem 4, which in sight of the Fourier coefficients of the function $E(x_1,\ldots,x_K;z)$ (2.31) equals

$$\int_{Sp(2N)} \chi^{\lambda}_{Sp(2N)}(M)E(x_1,\ldots,x_K;M)dM$$
$$= \det\left(\prod_{j=1}^{K} x_j\left(e_{K+j+\lambda^r_j-k}(x,x^{-1})-e_{K+j+\lambda^r_j+k}(x,x^{-1})\right)\right)_{j,k=1}^{N},$$

where we have denoted $x^{-1}=(x_1^{-1},\ldots,x_K^{-1})$. Using (3.37) we see that this determinant can also be expressed as

$$\det\left(\prod_{j=1}^{K} x_j\left(e_{K-\lambda_{N+1-j}-j+k}(x,x^{-1})-e_{K-\lambda_{N+1-j}-j-k}(x,x^{-1})\right)\right)_{j,k=1}^{N},$$

which, due to the Jacobi-Trudi identity (3.19), coincides with the right hand side of (3.48).

Identity (3.49) follows analogously. Let us turn however, to identity (3.50), as it requires some more computation. As in the symplectic case, using the Jacobi-Trudi identity (3.23), the fact that $e_k(x,1)=e_k(x)+e_{k-1}(x)$, and identity (3.37) we obtain

$$\left(\prod_{j=1}^{K} x_j^N\right) o^{odd}_{L_{N,K}(\lambda')}(-x)$$
$$=\frac{1}{2}\det\left(\prod_{j=1}^{K} x_j\left(e_{K-\lambda^r_j-j+k}(-x,-x^{-1},1)+e_{K-\lambda^r_j-j-k+2}(-x,-x^{-1},1)\right)\right)_{j,k=1}^{N}$$
$$=\frac{1}{2}\det\left(\prod_{j=1}^{N} x_j\left(e_{K-\lambda^r_j-j+k}(-x,-x^{-1})+e_{K-\lambda^r_j-j+k-1}(-x,-x^{-1})\right.\right.$$
$$\left.\left.+e_{K-\lambda^r_j-j-k+2}(-x,-x^{-1})+e_{K-\lambda^r_j-j-k+1}(-x,-x^{-1})\right)\right)_{j,k=1}^{N}$$
$$=\frac{1}{2}\det\left(\prod_{j=1}^{N} x_j\left(e_{K+j+\lambda^r_j-k}(-x,-x^{-1})+e_{K+j+\lambda^r_j-k+1}(-x,-x^{-1})\right.\right.$$
$$\left.\left.+e_{K+j+\lambda^r_j+k-2}(-x,-x^{-1})+e_{K+j+\lambda^r_j+k-1}(-x,-x^{-1})\right)\right)_{j,k=1}^{N}.$$

Adding $(-1)^{j+k}$ times the k-th column of the last matrix above, for each $k=1,...,j-1$, to the j-th column, for each $j=2,...,N$, we obtain

$$\left(\prod_{j=1}^{K} x_j^N\right) o^{odd}_{L_{N,K}(\lambda')}(-x) = \det\left(\prod_{j=1}^{K} x_j \left(e_{K+j+\lambda_j^r-k}(-x,-x^{-1}) + e_{K+\lambda_j^r+j+k-1}(-x,-x^{-1})\right)\right)_{j,k=1}^{N}.$$

Using the case $\mu=\varnothing$ of the odd orthogonal integral of theorem 4 and extracting the minus sign from the elementary symmetric polynomials in the last determinant above we arrive at (3.50). □

In particular, theorem 7 implies that the determinants of the corresponding Toeplitz±Hankel matrices in the left hand sides of the theorem can be expressed as the specialization of a single character associated to the irreducible representation of the corresponding group, indexed by a rectangular partition. This was first observed in [58] and has been generalized to integrals over other ensembles, see for instance [150, 151]. Combining this fact with theorem 6 we obtain the following result.

Corollary 3. *The following relations hold between the symmetric functions associated to the characters of the groups $G(N)$*

$$\begin{aligned}
&s_{((2N-1)^K)}(x_1,\ldots,x_K,x_1^{-1},\ldots,x_K^{-1}) = sp_{((N-1)^K)}(x_1,\ldots,x_K)o^{even}_{(N^K)}(x_1,\ldots,x_K)\\
&= \frac{(-1)^{NK}}{2} o^{odd}_{((N-1)^K)}(x_1,\ldots,x_K)o^{odd}_{(N^K)}(-x_1,\ldots,-x_K)\\
&+ \frac{(-1)^{NK}}{2} o^{odd}_{(N^K)}(x_1,\ldots,x_K)o^{odd}_{((N-1)^K)}(-x_1,\ldots,-x_K),\\
&s_{((2N)^K)}(x_1,\ldots,x_K,x_1^{-1},\ldots,x_K^{-1}) = (-1)^{NK} o^{odd}_{(N^K)}(x_1,\ldots,x_K)o^{odd}_{(N^K)}(-x_1,\ldots,-x_K)\\
&= \frac{1}{2} sp_{(N^K)}(x_1,\ldots,x_K)o^{even}_{(N^K)}(x_1,\ldots,x_K) + \frac{1}{2} sp_{((N-1)^K)}(x_1,\ldots,x_K)o^{even}_{((N+1)^K)}(x_1,\ldots,x_K).
\end{aligned}$$

The first and third identities in the corollary appeared before in [54]. There exist also identities expressing the sum of two Schur polynomials indexed by partitions of rectangular shapes in terms of orthogonal and symplectic Schur functions, as well as some other generalizations of these identities, see [54, 13, 14], but the second and fourth identities in the corollary are new to our knowledge.

3.2.2 Expansions in terms of Toeplitz minors

Theorem 8. *The integrals* (3.2) *verify*

$$\begin{aligned}
\int_{Sp(2N)} f(U)dU &= \frac{1}{2^N} \sum_{\rho_1,\rho_2\in R(N)} (-1)^{(|\rho_1|+|\rho_2|)/2} \int_{U(N)} \chi^{\rho_1}_{U(N)}(U^{-1})\chi^{\rho_2}_{U(N)}(U)f(U)dU,\\
\int_{O(2N)} f(U)dU &= \frac{1}{2^{N-1}} \sum_{\tau_1,\tau_2\in T(N)} (-1)^{(|\tau_1|+|\tau_2|)/2} \int_{U(N)} \chi^{\tau_1}_{U(N)}(U^{-1})\chi^{\tau_2}_{U(N)}(U)f(U)dU,\\
\int_{O(2N+1)} f(U)dU &= \frac{1}{2^N} \sum_{\sigma_1,\sigma_2\in S(N)} (-1)^{(|\sigma_1|+|\sigma_2|+p(\sigma_1)+p(\sigma_2))/2} \int_{U(N)} \chi^{\sigma_1}_{U(N)}(U^{-1})\chi^{\sigma_2}_{U(N)}(U)f(U)dU.
\end{aligned}$$

Proof. The main idea in the proof is that the determinants $\det M_{G(N)}$, whenever $G(N)$ is one of the groups $Sp(2N), O(2N)$ or $O(2N+1)$, contain as a factor the determinant $\det M_{U(N)}$, as can be seen in formulas (3.3)-(3.6). Hence, as a consequence of the definition (3.1), one can see the integrals over the groups $G(N)$ as integrals over $U(N)$ with an additional term in the integrand. Moreover, these additional terms can be expressed as Schur functions series as follows [147]

$$\frac{\det M_{Sp(2N)}(z)}{\det M_{U(N)}(z)} = \prod_{j=1}^{N} z_j^{-N} \prod_{j<k}(1-z_j z_k) \prod_{j=1}^{N}(1-z_j^2) = \prod_{j=1}^{N} z_j^{-N} \sum_{\rho\in R(N)} (-1)^{|\rho|/2} s_\rho(z_1,\ldots,z_N),$$

$$\frac{\det M_{O(2N)}(z)}{\det M_{U(N)}(z)} = 2\prod_{j=1}^{N} z_j^{-N+1} \prod_{j<k}(1-z_j z_k) \prod_{j=1}^{N} = 2\prod_{j=1}^{N} z_j^{-N+1} \sum_{\tau\in T(N)} (-1)^{|\tau|/2} s_\tau(z_1,\ldots,z_N),$$

$$\frac{\det M_{O(2N+1)}(z)}{\det M_{U(N)}(z)} = \prod_{j=1}^{N} z_j^{-N+1/2} \prod_{j<k}(1-z_j z_k) \prod_{j=1}^{N}(1-z_j)$$
$$= \prod_{j=1}^{N} z_j^{-N+1/2} \sum_{\sigma\in S(N)} (-1)^{(|\sigma|+p(\sigma))/2} s_\sigma(z_1,\ldots,z_N),$$

where $R(N), S(N)$ and $T(N)$ are defined in (3.35). Substituting these formulas into (3.2), for each of the groups $G(N) = Sp(2N), O(2N), O(2N+1)$, one obtains the desired result. □

Thus, we see that the integral of a function over one of the groups $G(N)$ can be expressed as a certain sum of integrals of the same function over $U(N)$ with Schur polynomials on the integrand. Note that the integrals in the right hand sides above are symmetric upon exchange of the partitions indexing the Schur polynomials. Since there are exactly 2^N partitions in the sets $R(N)$ and $S(N)$, and 2^{N-1} in $T(N)$, this[21] implies that there are at most 2^{2N-1} different terms in each of the sums.

According to identities (3.7)-(3.10), the integrals and twisted integrals over the groups $G(N)$ can be expressed as determinants and minors, respectively, of certain Toeplitz±Hankel matrices. Therefore, theorem 8 translates to the following result involving only the aforementioned matrices.

Corollary 4. *Let f be a function on the unit circle which Fourier coefficients verify $d_k = d_{-k}$. Given two partitions λ and μ, we denote the Toeplitz minor generated by f and indexed by λ and μ by*

$$D_N^{\lambda,\mu}(f) = \det\left(d_{j-\lambda_j-k+\mu_k}\right)_{j,k=1}^{N},$$

as in [50]. We have

$$\det\left(d_{j-k}-d_{j+k}\right)_{j,k=1}^{N} = \frac{1}{2^N}\sum_{\rho_1,\rho_2\in R(N)} (-1)^{(|\rho_1|+|\rho_2|)/2} D_N^{\rho_1,\rho_2}(f),$$

$$\det\left(d_{j-k}+d_{j+k-2}\right)_{j,k=1}^{N} = \frac{1}{2^{N-2}}\sum_{\tau_1,\tau_2\in T(N)} (-1)^{(|\tau_1|+|\tau_2|)/2} D_N^{\tau_1\tau_2}(f),$$

$$\det\left(d_{j-k}-d_{j+k-1}\right)_{j,k=1}^{N} = \frac{1}{2^N}\sum_{\sigma_1,\sigma_2\in S(N)} (-1)^{(|\sigma_1|+|\sigma_2|+p(\sigma_1)+p(\sigma_2))/2} D_N^{\sigma_1\sigma_2}(f).$$

[21] Together with further symmetries of the integral; for instance, $\int_{U(N)} s_{(a^N)}(U^{-1}) s_{(a^N)}(U) f(U) dU = \int_{U(N)} f(U) dU$ for every $a > 0$.

The minors appearing in the right hand sides above fit in the Toeplitz matrix generated by f of order $2N+1$, $2N$ and $2N-1$, respectively, and the sums have 2^{2N-1} different terms, as in theorem 8.

For example, taking $N=2$ in the first identity above we obtain the expansion

$$2\begin{vmatrix} d_0-d_2 & d_1-d_3 \\ d_1-d_3 & d_0-d_4 \end{vmatrix} = \begin{vmatrix} d_0 & d_1 \\ d_1 & d_0 \end{vmatrix} - \begin{vmatrix} d_2 & d_1 \\ d_3 & d_0 \end{vmatrix} + \begin{vmatrix} d_3 & d_0 \\ d_4 & d_1 \end{vmatrix} - \begin{vmatrix} d_1 & d_2 \\ d_4 & d_1 \end{vmatrix}$$
$$+ \begin{vmatrix} d_1 & d_0 \\ d_4 & d_3 \end{vmatrix} - \begin{vmatrix} d_0 & d_1 \\ d_3 & d_2 \end{vmatrix} + \begin{vmatrix} d_0 & d_3 \\ d_3 & d_0 \end{vmatrix} - \begin{vmatrix} d_3 & d_2 \\ d_4 & d_3 \end{vmatrix},$$

where all the determinants in the right hand side above are minors of the Toeplitz matrix $(d_{j-k})_{j,k=1}^5$. Analogous computations lead to expansions of minors of Toeplitz$\pm$Hankel matrices as sums of minors of Toeplitz matrices (equivalently, expansions of twisted integrals over $Sp(2N)$, $O(2N)$ or $O(2N+1)$ in terms of twisted integrals over $U(N)$), weighted with Littlewood-Richardson coefficients. However, the resulting expressions are rather cumbersome and we do not pursue this road further.

Setting $f(z) = E(x_1,\dots,x_K;z)$ in theorem 8 and making use of theorem 7 we also obtain the following result.

Corollary 5. *The characters of $G(N)$ indexed by rectangular shapes can be expanded in terms of skew Schur polynomials as follows*

$$sp_{(N^K)}(x_1,\dots,x_K) = \frac{1}{2^N}\sum_{\rho_1,\rho_2\in R(N)}(-1)^{(|\rho_1|+|\rho_2|)/2}s_{((K^N)+\rho_2/\rho_1)'}(x_1,\dots,x_K,x_1^{-1},\dots,x_K^{-1}),$$
$$o^{even}_{(N^K)}(x_1,\dots,x_K) = \frac{1}{2^{N-1}}\sum_{\tau_1,\tau_2\in T(N)}(-1)^{(|\tau_1|+|\tau_2|)/2}s_{((K^N)+\tau_2/\tau_1)'}(x_1,\dots,x_K,x_1^{-1},\dots,x_K^{-1}),$$
$$o^{odd}_{(N^K)}(x_1,\dots,x_K) =$$
$$\frac{1}{2^N}\sum_{\sigma_1,\sigma_2\in S(N)}(-1)^{(3|\sigma_1|+3|\sigma_2|+p(\sigma_1)+p(\sigma_2))/2}s_{((K^N)+\sigma_2/\sigma_1)'}(x_1,\dots,x_K,x_1^{-1},\dots,x_K^{-1}),$$

Note that these expressions are different from the classical identities (3.36).

3.2.3 Gessel-type identities

Another possibility for expressing integrals and twisted integrals over the classical groups in terms of symmetric functions is available in the form of Schur function series, as the classical identity of Gessel for Toeplitz determinants (2.15). Let us denote by $\mathfrak{s}^{\nu}_{G(N)}(x)$ the Schur, symplectic Schur or even/odd orthogonal Schur symmetric function indexed by the partition ν for $G(N) = U(N), Sp(2N), O(2N), O(2N+1)$ respectively, for this theorem only.

Theorem 9. *Let $x=(x_1,x_2,\dots)$ be a set of variables. Recall the definition*

$$H(x;z) = \prod_{j=1}^{\infty}\frac{1}{1-x_jz}.$$

The following Schur function series expansions hold

$$\int_{G(N)} H(x;U)dU = \sum_{l(\nu)\le N} s_\nu(x)\mathfrak{s}^{\nu}_{G(N)}(x), \tag{3.51}$$

$$\int_{G(N)} \chi^{\mu}_{G(N)}(U)H(x;U)dU = \sum_{l(\nu)\leq N} s_{\nu/\mu}(x)\mathfrak{s}^{\nu}_{G(N)}(x), \tag{3.52}$$

$$\int_{G(N)} \chi^{\lambda}_{G(N)}(U^{-1})\chi^{\mu}_{G(N)}(U)H(x;U)dU = \begin{cases} \sum\limits_{l(\nu)\leq N} s_{\nu/\lambda}(x)s_{\nu/\mu}(x), & G(N)=U(N), \\ \sum\limits_{l(\nu)\leq N}\sum\limits_{\kappa} b^{\kappa}_{\lambda\mu} s_{\nu/\kappa}(x)\mathfrak{s}^{\nu}_{G(N)}(x), & \textit{rest of } G(N), \end{cases} \tag{3.53}$$

where the coefficients $b^{\kappa}_{\lambda\mu}$ *can be expressed in terms of Littlewood-Richardson coefficients* $c^{\lambda}_{\sigma\tau}$ (2.12) *by the following formula*

$$b^{\kappa}_{\lambda\mu} = \sum_{\sigma,\rho,\tau} c^{\lambda}_{\sigma\tau}c^{\mu}_{\rho\tau}c^{\kappa}_{\sigma\rho}.$$

The same expansions hold for the function $E(x;z) = \prod_{j=1}^{\infty}(1+x_j z)$ (2.9)*, after transposing the partitions indexing all the symmetric functions in the above identities.*

We remark the fact that the choice of functions above is without loss of generality. Indeed, recall that the Fourier coefficients of the functions $H(x;z)$ and $E(x;z)$ are the complete homogeneous symmetric functions $h_k(x)$ and the elementary symmetric functions $e_k(x)$ respectively (2.8). Both of these families are sets of algebraically independent generators in the ring of symmetric functions, and thus one can specialize them to any given values to recover any function with arbitrary Fourier coefficients from $H(x;z)$ or $E(x;z)$, as discussed after theorem 1.

A similar proof of identity (3.51) for $G(N) = Sp(2N), O(2N)$ can be found in [30]. See also [121, 19, 20] for earlier related results. Different Schur and symmetric function series for some of these integrals can also be found in [18, 143].

Proof. The expansion (3.51) for $G(N) = U(N)$ is the aforementioned result of Gessel [107], which extends easily to the other groups. We sketch the proof for convenience of the reader. Denote the Toeplitz matrix of order N generated by a function f by $T_N(f)$. It is well known that if two functions a, b satisfy

$$a(z) = \sum_{k\leq 0} a_k z^k, \qquad b(z) = \sum_{k\geq 0} b_k z^k, \tag{3.54}$$

where $z = e^{i\theta}$, then the Toeplitz matrix generated by the function ab satisfies $T_N(ab) = T_N(a)T_N(b)$. It follows from the Cauchy-Binet formula that $\det T_N(ab)$ is then a sum over minors of the Toeplitz matrices of sizes $N\times\infty$ and $\infty\times N$ generated by a and b, respectively. The proof is completed upon noting that if $a(z^{-1}) = b(z) = H(x;z)$ then by the Jacobi-Trudi identity (3.17) the minors appearing in the sum are precisely the Schur polynomials appearing in (3.51), since the Fourier coefficients of the function $H(x;z)$ are the complete homogeneous symmetric polynomials $h_k(x)$. The proof for the other groups is analogous: now the factorization

$$TH_N(ab) = T_N(a)TH_N(b)$$

holds for each of the Toeplitz±Hankel matrices $TH_N(b)$ appearing in (3.8)-(3.10) and functions a, b satisfying (3.54). The result then follows from the Jacobi-Trudi identities (3.18)-(3.22) (although some additional computations are needed in the odd orthogonal case, as in corollary 3).

Identities (3.52), and (3.53) for $U(N)$, follow analogously from the generalization of Jacobi-Trudi formula for skew Schur polynomials. Identity (3.53) for the rest of the groups follows from (3.52) and the fact that the characters $\chi^{\lambda}_{G(N)}$ follow the multiplication rule [144]

$$\chi^{\lambda}_{G(N)}(U)\chi^{\mu}_{G(N)}(U) = \sum_{\nu} b^{\nu}_{\lambda\mu}\chi^{\nu}_{G(N)}(U) \tag{3.55}$$

for $G(N) = Sp(2N), O(2N)$ and $O(2N+1)$ (recall that $\chi^{\lambda}_{G(N)}(U) = \chi^{\lambda}_{G(N)}(U^{-1})$ for such groups).

The corresponding identities involving the function E follow analogously, using the dual Jacobi-Trudi identities instead (or, equivalently, using the involution $h_k \mapsto e_k$) in (3.51)-(3.53)). □

Observe that if we replace the left hand sides of (3.51) and (3.52) by their expression as a single character of $G(N)$, given by (2.32) and theorem 7, the above theorem gives Schur function series expansions for such characters, which are different to those obtained in corollary 5. Yet another expansion for symplectic and even and odd orthogonal functions indexed by rectangular shapes can be obtained by Laplace expansion of the corresponding Toeplitz±Hankel determinants, as done in theorem 2 for the Toeplitz case.

3.2.4 Large N limit

We will be interested in the following in computing the $N \to \infty$ limit of integrals of the type $\int_{G(N)} f(U)dU$. This can be achieved by means of the strong Szegő limit theorem and its generalization to the rest of the groups $G(N)$ due to Johansson (3.42)-(3.44), or equivalently, by means of theorem 9 and the Cauchy identities (3.24)-(3.27) (see section 3.3.1 below for such explicit computations). It turns out that the twisted integrals share a common asymptotic behavior.

Theorem 10. *Let λ and μ be two partitions. We have*

$$\lim_{N\to\infty} \frac{\int_{G(N)} \chi^{\lambda}_{G(N)}(U^{-1})\chi^{\mu}_{G(N)}(U)H(x;U)dU}{\int_{G(N)} H(x;U)dU} = \sum_{\nu} s_{\lambda/\nu}(x)s_{\mu/\nu}(x), \tag{3.56}$$

for any of the groups $G(N) = U(N), Sp(2N), O(2N), O(2N+1)$.

Note that if there is only one character in the integrand above the right hand side simplifies to a single Schur polynomial. As before, the theorem also holds for the function $E(x;e^{i\theta}) = \prod_{j=1}^{\infty}(1 + x_j e^{i\theta})$, after transposing the partitions indexing the skew Schur polynomials above.

Proof. If $G(N) = U(N)$, the result is the content of theorem 1. A proof for the rest of the groups $G(N)$ goes as follows. Start by considering a single character in the integral (3.56). Then, using the Cauchy identity (3.24) and the restriction rules (3.32)-(3.34) we obtain

$$\int_{G(N)} \chi^{\mu}_{G(N)}(U)H(x;U)dU = \sum_{l(\nu)\le N}\sum_{\alpha}\widetilde{\sum_{\beta}} c^{\nu}_{\alpha\beta}s_{\nu}(x)\int_{G(N)} \chi^{\mu}_{G(N)}(U)\chi^{\alpha}_{G(N)}(U)dU,$$

where $\widetilde{\sum}$ denotes that the sum on β runs over all even partitions for $G(N) = O(2N), O(2N+1)$, and over all partitions whose conjugate is even, for $G(N) = Sp(2N)$ (we say that a partition is even if it has only even parts), and the sum on α runs over all partitions. Taking $N \to \infty$ in the above expression and using the orthogonality of the characters with respect to Haar measure we obtain

$$\lim_{N\to\infty} \int_{G(N)} \chi^{\mu}_{G(N)}(U) H(x;U) dU = s_\mu(x) \sum_{\beta}^{\sim} s_\beta(x). \tag{3.57}$$

This gives the desired result upon noting that the sum on the right hand side is precisely the $N \to \infty$ limit of the integral $\int_{G(N)} H(x;U)dU$. The result for the integral (3.56) twisted by two characters then follows from (3.57) and the multiplication rules (2.12) and (3.55). □

In particular, we see that the $N \to \infty$ limit of the average is independent of the particular group $G(N)$ considered. This was noted in [64] for a single character, which automatically implies the same for two characters for $G(N) = Sp(2N), O(2N), O(2N+1)$. Indeed, since $\chi^{\lambda}_{G(N)}(U^{-1}) = \chi^{\lambda}_{G(N)}(U)$ for these groups, one can expand the product of two characters in the integrand using the multiplication rule (3.55), use theorem 10 on the resulting averages and then use (3.55) again to recover (3.56). However, this is not immediate for $G(N) = U(N)$, as the characters in the integrand are not evaluated at the same variables.

We remark that the convergence above is in the ring of symmetric functions, as in theorem 1. Likewise, specializing the variables x (or any family of generators in the ring of symmetric functions on these variables) to a particular function such that the limit $\lim_{N\to\infty} \int_{G(N)} f(U)dU$ is finite we obtain the asymptotic behaviour in (3.56) for the corresponding specialization of the right hand side, where the skew Schur polynomials are replaced by their specializations (2.29).

We also have an analogous result to corollary 2 for symplectic and orthogonal Schur functions.

Corollary 6. *Let λ be a partition with $l(\lambda) \leq K$. We have*

$$\lim_{N\to\infty} sp_{L_{N,K}(\lambda)}(x_1,\dots,x_K) = \left(\lim_{N\to\infty} sp_{(N^K)}(x_1,\dots x_K)\right) s_\lambda(x_1,\dots,x_K),$$
$$\lim_{N\to\infty} o^{even}_{L_{N,K}(\lambda)}(x_1,\dots,x_K) = \left(\lim_{N\to\infty} o^{even}_{(N^K)}(x_1,\dots x_K)\right) s_\lambda(x_1,\dots,x_K),$$
$$\lim_{N\to\infty} o^{odd}_{L_{N,K}(\lambda)}(x_1,\dots,x_K) = (-1)^{|\lambda|}\left(\lim_{N\to\infty} o^{odd}_{(N^K)}(x_1,\dots x_K)\right) s_\lambda(x_1,\dots,x_K).$$

The proof follows after combining the case $\mu = \varnothing$ of theorem 10 with theorem 7, as in the proof of corollary 2. Alternatively, it can be seen as a consequence of corollary 2 and the fact that the highest degree term in the Schur polynomial expansion of symplectic and orthogonal Schur functions is precisely the Schur polynomial indexed by the same partition, as can be seen from (3.36), for instance.

3.3 An exactly solvable model: Jacobi's third theta function

We particularize the previous results to the case of a completely solvable model, for both finite and large N. The objects under study appear in several contexts, such as $G(N)$ Chern-Simons theory on S^3, the skein of the annulus [160] and Fourier and sine/cosine transforms [104].

Let q be a parameter satisfying $|q| < 1$, and consider Jacobi's third theta function

$$\sum_{k\in\mathbb{Z}} q^{k^2/2} z^k = (q;q)_\infty \prod_{j=1}^{\infty} (1 + q^{j-1/2} z)(1 + q^{j-1/2} z^{-1}), \tag{3.58}$$

where $(q;q)_\infty = \prod_{j=1}^{\infty}(1-q^j)$. We then define $f(U)$ for $U \in G(N)$ as in (3.1), with

$$f(z) = \Theta(z) = E(q^{1/2}, q^{3/2}, \ldots; z^{-1}), \tag{3.59}$$

where E is given by (2.9). For this choice of function, the integral

$$Z_{G(N)} = (q;q)_\infty^N \int_{G(N)} \Theta(U)dU \tag{3.60}$$

recovers the partition function of Chern-Simons theory on S^3 with symmetry group $G(N)$, and the coefficients in the corresponding Toeplitz and Toeplitz±Hankel matrices are $d_k = q^{k^2/2}$, according to (3.58). After a matrix model description was obtained for Chern-Simons theory on manifolds such as S^3 or lens spaces [148], the solvability of the theory has been well known, and a number of equivalent representations have been obtained [182, 170]. Moreover, the averages

$$\langle W_\mu \rangle_{G(N)} = \frac{1}{Z_{G(N)}} \int_{G(N)} \chi^{\mu}_{G(N)}(U)\Theta(U)dU$$

and

$$\langle W_{\lambda\mu} \rangle_{G(N)} = \frac{1}{Z_{G(N)}} \int_{G(N)} \chi^{\lambda}_{G(N)}(U^{-1})\chi^{\mu}_{G(N)}(U)\Theta(U)dU,$$

where $l(\lambda), l(\mu) \leq N$, are, respectively, the Wilson loop and Hopf link of the theory. As we will see below, the formalism of Toeplitz and Toeplitz±Hankel determinants and minors provides an elementary mean for computing these objects. Moreover, as will be clear throughout the rest of the chapter, the symmetric function structure behind these models allows a unified approach in their study, since properties or explicit results for the different groups $G(N)$ will follow from completely analogous reasonings. This is particularly useful in sight of the lack of results concerning the partition function and observables of the symplectic and orthogonal theories [176].

3.3.1 Partition functions of $G(N)$ Chern-Simons theory on S^3

Unitary group

We start by reviewing the simplest and best known case. We obtain from the determinant expression (3.7)

$$Z_{U(N)} = \det\left(q^{(j-k)^2/2}\right)_{j,k=1}^N = q^{\sum_{j=1}^N j^2} \det\left(q^{-jk}\right)_{j,k=1}^N = \prod_{j<k}(1-q^{k-j}) = \prod_{j=1}^{N-1}(1-q^j)^{N-j},$$

where the third identity follows from the fact that the second determinant above is essentially the determinant of the matrix $M_{U(N)}(z)$ (3.3), with $z_j = q^{j-1}$.

The large N limit of this expression is given by Szegő's theorem, which shows that as $N \to \infty$

$$Z_{U(N)} \sim \exp\left(-N\sum_{k=1}^{\infty}\frac{1}{k}\frac{q^k}{1-q^k} + \sum_{k=1}^{\infty}\frac{1}{k}\frac{q^k}{(1-q^k)^2}\right).$$

The same formula can be obtained using Cauchy's identity (3.24) in formula (3.51), as explained in [185].

Symplectic group

We can proceed analogously for the rest of the groups. The determinants will now be specializations of the corresponding matrix $M_{G(N)}(z)$ with $z_j = q^j$, which can be computed explicitly by means of the formulas (3.3)-(3.6). For the symplectic group we obtain

$$Z_{Sp(2N)} = \det\left(q^{(j-k)^2/2} - q^{(j+k)^2/2}\right)_{j,k=1}^{N} = q^{\sum_{j=1}^{N} j^2} \det(q^{-jk} - q^{jk})_{j,k=1}^{N}$$
$$= \prod_{j=1}^{N-j}(1-q^j)^{N-j} \prod_{j=3}^{N}(1-q^j)^{[\frac{j-1}{2}]} \prod_{j=N+1}^{2N-1}(1-q^j)^{[\frac{2N+1-j}{2}]} \prod_{j=1}^{N}(1-q^{2j}) = \prod_{j=1}^{2N}(1-q^j)^{\epsilon(j)},$$

where

$$\epsilon(j) = \begin{cases} N - \frac{j}{2} - \frac{1}{2}, & j \text{ odd } 1 \le j \le N, \\ N - \frac{j}{2}, & j \text{ even, } 1 \le j \le N, \\ N - \frac{j}{2} + \frac{1}{2}, & j \text{ odd, } N+1 \le j \le 2N, \\ N - \frac{j}{2} + 1, & j \text{ even, } N+1 \le j \le 2N. \end{cases}$$

As with the unitary model, this result is exact and holds for every N, and coincides with the expression obtained in [176] for the large N regime. We see that the partition function of the symplectic model is obtained as the product of the partition function of the unitary model and extra factors.

For the large N limit, we obtain from Johansson's generalization of Szegő's theorem (3.42) that as $N \to \infty$

$$Z_{Sp(2N)} \sim \exp\left(-N\sum_{k=1}^{\infty}\frac{1}{k}\frac{q^k}{1-q^k} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}\frac{q^k}{(1-q^k)^2} + \sum_{k=1}^{\infty}\frac{1}{2k}\frac{q^k}{1-q^{2k}}\right).$$

Again, the same result is obtained using Cauchy's identity for symplectic characters (3.25) in equation (3.51). Notice that in the large N limit, the partition function for the $Sp(2N)$ model is a factor of the partition function of the $U(N)$ model, while precisely the opposite occurred at finite N.

Orthogonal groups

Proceeding analogously, we see that by identity (3.6)

$$Z_{O(2N)} = \frac{1}{2}\det\left(q^{(j-k)^2/2} + q^{(j+k-2)^2/2}\right)_{j,k=1}^{N}$$
$$= \prod_{j=1}^{N-1}(1-q^j)^{N-j} \prod_{j=1}^{N-1}(1-q^j)^{[\frac{j+1}{2}]} \prod_{j=N}^{2N-3}(1-q^j)^{[\frac{2N-j-1}{2}]} = \prod_{j=1}^{2N-3}(1-q^j)^{\epsilon(j)},$$

where

$$\epsilon(j) = \begin{cases} N - \frac{j}{2} + \frac{1}{2}, & j \text{ odd, } 1 \le j \le N-1, \\ N - \frac{j}{2}, & j \text{ even, } 1 \le N-1, \\ N - \frac{j}{2} - \frac{1}{2}, & j \text{ odd, } N \le j \le 2N-3, \\ N - \frac{j}{2} - 1, & j \text{ even, } N \le j \le 2N-3, \end{cases}$$

in agreement with [176]. Again, the partition function contains as a factor the partition function of the unitary model. For $O(2N+1)$ we have

$$\begin{aligned} Z_{O(2N+1)} &= \det\left(q^{(j-k)^2/2} - q^{(j+k-1)^2/2}\right)_{j,k=1}^{N} \\ &= \prod_{j=1}^{N-1}(1-q^j)^{N-j}\prod_{j=2}^{N}(1-q^j)^{[\frac{j}{2}]}\prod_{j=N+1}^{2N-2}(1-q^j)^{[\frac{2N-j}{2}]}\prod_{j=1}^{N}(1-q^{j-1/2}) \\ &= \prod_{j=1}^{2N-2}(1-q^j)^{\epsilon(j)}\prod_{j=1}^{N}(1-q^{j-1/2}), \end{aligned}$$

where

$$\epsilon(j) = \begin{cases} N - \frac{j}{2} - \frac{1}{2}, & j \text{ odd}, 1 \le j \le 2N-2, \\ N - \frac{j}{2}, & j \text{ even}, 1 \le j \le 2N-2, \end{cases}$$

in agreement with [176]. We see once again that the partition function can be seen as the partition function of the unitary model times an extra factor. In this case, also factors with half-integer exponents $(1-q^{j/2})$ are present.

Let us also record here the value of the closely related integral (3.12) for this choice of function, for completeness. We have

$$(q;q)_\infty^N \int_{O(2N+1)} \Theta(-U)dU = \prod_{j=1}^{2N-3}(1-q^j)^{\epsilon(j)}\prod_{j=1}^{N}(1+q^{j-1/2}) = Z_{O(2N+1)}\prod_{j=1}^{N}\frac{(1+q^{j-1/2})}{(1-q^{j-1/2})},$$

where $\epsilon(j)$ is as in $Z_{O(2N+1)}$.

For the large-N limit, we obtain from Johansson's theorem (3.43),(3.44) that as $N \to \infty$,

$$\begin{aligned} Z_{O(2N)} &\sim \exp\left(-N\sum_{k=1}^{\infty}\frac{1}{k}\frac{q^k}{1-q^k} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}\frac{q^k}{(1-q^k)^2} - \sum_{k=1}^{\infty}\frac{1}{2k}\frac{q^k}{1-q^{2k}}\right), \\ Z_{O(2N+1)} &\sim \exp\left(-N\sum_{k=1}^{\infty}\frac{1}{k}\frac{q^k}{1-q^k} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}\frac{q^k}{(1-q^k)^2} - \sum_{k=1}^{\infty}\frac{1}{2k-1}\frac{q^{k-1/2}}{1-q^{2k-1}}\right). \end{aligned}$$

One can verify directly from the expressions obtained that in the large N limit we recover the partition function of $U(N)$ as the product of the partition functions of $Sp(2N)$ and $O(2N)$, consistently with corollary 6.

3.3.2 Gross-Witten-Wadia model

The factorization properties obtained in theorem 6 hold for any choice of function, and thus are applicable to gauge theories with other matrix model descriptions, such as the Gross-Witten-Wadia model [112, 193]. This is of particular interest in sight of the renewed interest in this topic [164, 5, 6, 125].

The partition function of this model with $G(N)$ symmetry is given by

$$Z_{G(N)}^{GWW}(\beta) = \int_{G(N)} f_{GWW}(U)dU,$$

where

$$f_{GWW}(z) = e^{-\beta\left(z+z^{-1}\right)}.$$

Its large N behaviour follows from the strong Szegő limit theorem and its generalization to the rest of the groups $G(N)$, for instance

$$\lim_{N\to\infty} Z^{GWW}_{U(N)}(\beta) = e^{\beta^2}.$$

A similar analysis as in the Chern-Simons case can be performed for this choice of function. Moreover, since all but two of the Fourier coefficients of the potential in the function f_{GWW} vanish, often simpler relationships follow from the results in the previous sections. For instance, we see from theorem 6 that

$$Z^{GWW}_{U(2N)}(\beta) = Z^{GWW}_{O(2N+1)}(\beta)\, Z^{GWW}_{O(2N+1)}(-\beta)\,,$$

and, using also the asymptotic expressions (3.42) and (3.43), we find that

$$Z^{GWW}_{U(2N-1)}(\beta), Z^{GWW}_{U(2N)}(\beta) \sim Z^{GWW}_{Sp(2N)}(\beta) Z^{GWW}_{O(2N)}(\beta) = (Z^{GWW}_{O(2N)}(\beta))^2 = (Z^{GWW}_{Sp(2N)}(\beta))^2,$$

as $N \to \infty$. This relationship also has a XX spin chain interpretation [192], but is however modified in the usual double scaling limit [161, 95]. At any rate, it seems that large N results for the unitary Gross-Witten-Wadia model can be translated to the $O(2N)$ and $Sp(2N)$ models. It would also be interesting to exploit the factorizations in theorem 6 in other contexts, taking into account the known connections of the group integrals $Z^{GWW}_{G(N)}$ with Painlevé equations [96, 99, 125] or increasing subsequence problems [18].

3.4 Wilson loops and Hopf links of $G(N)$ Chern-Simons theory on S^3

We now turn to computing Wilson loops and Hopf links of Chern-Simons theory on S^3 with symmetry group $G(N)$, for each of the classical groups. Let us fix two partitions λ and μ of lengths $l(\lambda), l(\mu) \leq N$ throughout the rest of the section.

3.4.1 Unitary group

The insertion of a Schur polynomial on the unitary model gives

$$\begin{aligned}(q;q)^N_\infty \int_{U(N)} s_\mu(U)\Theta(U)dU &= \det(q^{(j-k-\mu^r_k)^2/2})^N_{j,k=1}\\ &= q^{\sum_{j=1}^N\left(\mu_j^2/2+(N-j+1)\mu_j+j^2\right)} \det\left(q^{-j(k+\mu^r_k)}\right)^N_{j,k=1}.\end{aligned}$$

We see that the determinant in the right hand side above is now essentially the minor $M^\mu_{U(N)}(z)$ in (3.13) after setting $z_j = q^{-j}$. This yields

$$\langle W_\mu\rangle_{U(N)} = q^{\sum_{j=1}^N \mu_j(\mu_j/2-j+1)} s_\mu(1,q,\ldots,q^{N-1}), \tag{3.61}$$

which, up to a prefactor of a power of q, recovers the original result in [78]. We recall that the above specialization of the Schur polynomial is a polynomial on q with positive and integer coefficients [147].

Inserting two Schur polynomials in the integral we obtain

$$(q;q)_\infty^N \int_{U(N)} s_\lambda(U^{-1}) s_\mu(U)\Theta(U)dU = \det(q^{(j+\lambda_j^r-k-\mu_k^r)^2/2})_{j,k=1}^N$$
$$= q^{\sum_{j=1}^N \left(\lambda_j^2/2+\mu_j^2/2+(N-j)(\lambda_j+\mu_j)+(j-1)^2\right)} \det(q^{-(N-j+\lambda_j)(N-k+\mu_k)})_{j,k=1}^N.$$

The determinant is now a minor of $M_{U(N)}^{\lambda}(z)$, obtained by striking some of its rows. That is, a minor obtained by striking rows *and* columns of the Vandermonde matrix $M_{U(N)}(1,q,\dots,q^{N-1})$, as noted in [160]. One can express this in terms of Schur polynomials by setting $z_j = q^{N-j+\mu_j}$ in this matrix, which yields

$$\langle W_{\lambda\mu}\rangle_{U(N)} = q^{\sum_{j=1}^N \left(\lambda_j^2/2+\mu_j^2/2-(j-1)(\lambda_j+\mu_j)\right)} s_\mu(1,q,\dots,q^{N-1}) s_\lambda(q^{-\mu_1}, q^{1-\mu_2},\dots,q^{N-1-\mu_N}).$$

The above expression can also be written in terms of the quadratic Casimir element of $U(N)$, which we denote by $C_2^{U(N)}(\lambda) = \sum_j \lambda_j(\lambda_j + N - 2j + 1)$, as follows

$$q^{\left(-(N-1)(|\lambda|+|\mu|)+C_2^{U(N)}(\lambda)+C_2^{U(N)}(\mu)\right)/2} s_\mu(1,q,\dots,q^{N-1}) s_\lambda(q^{-\mu_1}, q^{1-\mu_2},\dots,q^{N-1-\mu_N}). \quad (3.62)$$

Further interest in the minors of the Vandermonde matrix $M_{U(N)}(1,q,\dots,q^{N-1})$ and the rest of the matrices $M_{G(N)}$ arises from their relation with Chebotarëv's theorem[22] and the recent related advances in the topic [104].

We also see that a phenomenon already present when computing the partition functions takes place when computing averages of Schur polynomials. For the theta function, integrating the determinant $\det M_{G(N)}(z)$ in (3.2) amounts essentially to computing the determinant of the matrix $M_{G(N)}(z)$ itself, after a certain specialization of the variables z. We also see that the average of one or two Schur polynomials is expressed precisely as the corresponding Schur polynomials, after some specialization to the same number of nonzero variables as the size of the model.

This property has been noted in [156, 159] for models of Hermitian Gaussian matrices. It is argued in [159] that "the main feature of Gaussian matrix measures is that they preserve Schur functions". Indeed, we shall see that the same property holds when changing the symmetry of the ensemble from unitary to symplectic or orthogonal, by simply replacing Schur polynomials by symplectic or orthogonal Schur functions.

3.4.2 Symplectic group

Performing analogous computations to the unitary case, we see that

$$(q;q)_\infty^N \int_{Sp(2N)} \overline{sp_\lambda(U)} sp_\mu(U)\Theta(U)dU = \det\left(q^{(j+\lambda_j^r-k-\mu_k^r)^2/2} - q^{(j+\lambda_j^r+k+\mu_k^r)^2/2}\right)_{j,k=1}^N$$
$$= q^{\sum_{j=1}^N \left(\lambda_j^2/2+\mu_j^2/2+(N-j+1)(\lambda_j+\mu_j)+j^2\right)} \det\left(q^{-(j+\lambda_j^r)(k+\mu_k^r)} - q^{(j+\lambda_j^r)(k+\mu_k^r)}\right)_{j,k=1}^N,$$

which leads to

$$\langle W_{\lambda\mu}\rangle_{Sp(2N)} = q^{\left(N(|\lambda|+|\mu|)+C_2^{Sp(2N)}(\lambda)+C_2^{Sp(2N)}(\mu)\right)/2} sp_\mu(q,q^2,\dots,q^N) sp_\lambda(q^{1+\mu_N},\dots,q^{N+\mu_1}), \quad (3.63)$$

[22]The matrix $M_{U(N)}(1,q,\dots,q^{N-1})$, for q a p-th root of unity, is the matrix associated to the discrete Fourier transform (DFT), and Chebotarëv's classical theorem [179] states that every minor of this matrix is nonzero if p is prime. An analogue of this theorem for the matrices of the discrete sine and cosine transforms, which correspond to $M_{Sp(2N)}(q,\dots,q^N)$ and $M_{O(2N)}(1,\dots,q^{N-1})$ respectively, has been proved recently [104].

where we have identified $C_2^{Sp(2N)}(\lambda) = \sum_j \lambda_j(\lambda_j + N - 2j + 2)$, the quadratic Casimir element of $Sp(2N)$. As before, the second identity in (3.63) follows from the fact that integrating the function Θ we recover a (row and column-wise) minor of the matrix $M_{Sp(2N)}(z)$ itself, specialized to $z_j = q^j$. We note that λ and μ are interchangeable in the above formula, and also that setting one of the partitions to be empty we obtain a formula for the average of a single character $\langle W_\mu \rangle_{Sp(2N)}$.

3.4.3 Orthogonal groups

For the orthogonal models we have

$$(q;q)_\infty^N \int_{O(2N)} o_\lambda^{even}(U) o_\mu^{even}(U)\Theta(U)dU = \frac{1}{2}\det\left(q^{(j+\lambda_j^r-k-\mu_k^r)^2/2} + q^{(j+\lambda_j^r+k+\mu_k^r-2)^2/2}\right)_{j,k=1}^N$$
$$= \frac{1}{2} q^{\sum_{j=1}^N \left(\lambda_j^2/2+\mu_j^2/2+(N-j)(\lambda_j+\mu_j)+(j-1)^2\right)} \det\left(q^{-(N-j+\lambda_j)(N-k+\mu_k)} + q^{(N-j+\lambda_j)(N-k+\mu_k)}\right)_{j,k=1}^N,$$

which can be rewritten as

$$\langle W_{\lambda\mu}\rangle_{O(2N)} = q^{\left(N(|\lambda|+|\mu|)+C_2^{O(2N)}(\lambda)+C_2^{O(2N)}(\mu)\right)/2} o_\mu^{even}(1,q,\dots,q^{N-1}) o_\lambda^{even}(q^{\mu_N},\dots,q^{N-1+\mu_1}), \tag{3.64}$$

where $C_2^{O(2N)}(\lambda) = \sum_{j=1}^N \lambda_j(\lambda_j + N - 2j)$ is the quadratic Casimir of $O(2N)$. As before, setting one partition to be empty we obtain a formula for the Wilson loop $\langle W_\mu\rangle_{O(2N)}$. For the odd orthogonal group $O(2N+1)$ we obtain

$$(q;q)_\infty^N \int_{O(2N+1)} o_\lambda^{odd}(U) o_\mu^{odd}(U)\Theta(U)dU = \det\left(q^{(j+\lambda_j^r-k-\mu_k^r)^2/2} - q^{(j+\lambda_j^r+k+\mu_k^r-1)^2/2}\right)_{j,k=1}^N$$
$$= q^{\sum_{j=1}^N \left(\lambda_j^2/2+\mu_j^2/2+(N-j+1/2)(\lambda_j+\mu_j)+(j-1/2)^2\right)}$$
$$\times \det\left(q^{-(N-j+\lambda_j+1/2)(N-k+\mu_k+1/2)} - q^{(N-j+\lambda_j+1/2)(N-k+\mu_k+1/2)}\right)_{j,k=1}^N,$$

which yields

$$\begin{aligned}\langle W_{\lambda\mu}\rangle_{O(2N+1)} =& q^{\left((N+1/2)(|\lambda|+|\mu|)+C_2^{O(2N+1)}(\lambda)+C_2^{O(2N+1)}(\mu)\right)/2} \\ &\times o_\mu^{odd}(q^{1/2},q^{3/2},\dots,q^{N-1/2}) o_\lambda^{odd}(q^{1/2+\mu_N},q^{3/2+\mu_{N-1}},\dots,q^{N-1/2+\mu_1}),\end{aligned} \tag{3.65}$$

with $C_2^{O(2N+1)}(\lambda) = \sum_{j=1}^N \lambda_j(\lambda_j + N - 2j + 1/2)$ the quadratic Casimir of $O(2N+1)$.

3.4.4 Giambelli compatible processes

The classical Giambelli identity expresses a Schur polynomial indexed by a general partition λ as the determinant of a matrix whose entries are Schur polynomials indexed only by "hook-shaped" partitions. More precisely

$$s_{(a_1,\dots,a_p|b_1,\dots,b_p)}(x) = \det\left(s_{(a_j|b_k)}(x)\right)_{j,k=1}^p,$$

where we have used the Frobenius notations for the partitions in the above identity (see the beginning of section 3.2.2). In [39], the notion of "Giambelli compatible" processes was introduced to refer to probability measures on point configurations that preserve the Giambelli identity above, in the sense that

$$\langle s_{(a_1,\dots,a_p|b_1,\dots,b_p)}\rangle = \det\left(\langle s_{(a_j|b_k)}\rangle\right)_{j,k=1}^{p},$$

where the bracket notation $\langle s_\lambda\rangle$ denotes the average of the Schur polynomial λ with respect the corresponding probability measure. Since then, several matrix models and gauge theories have been proved to be Giambelli compatible, including biorthogonal ensembles [183], ABJM theory [113], and supersymmetric Chern-Simons theory [83, 152].

Using the formulas obtained in the previous sections, one can easily prove that the matrix models corresponding to the theta function (3.59) with $G(N)$ symmetry are Giambelli compatible in a slightly generalized sense. Indeed, we have seen that the average of a character over these ensembles can be evaluated as the precise same character, with a certain specialization, times a prefactor in the parameter q (equations (3.61),(3.63),(3.64),(3.65)). This fact, together with the Giambelli identity for the characters of the groups $G(N)$ [2, 101]

$$\chi_{G(N)}^{(a_1,\dots,a_p|b_1,\dots,b_p)}(U) = \det\left(\chi_{G(N)}^{(a_j|b_k)}(U)\right)_{j,k=1}^{p},$$

and some straightforward computations to take care of the prefactors in q, let us obtain the following conclusion.

Theorem 11. *The Wilson loops of Chern-Simons theory on S^3 verify the following Giambelli identity*

$$\langle W_{(a_1,\dots,a_p|b_1,\dots.b_p)}\rangle_{G(N)} = \det\left(\langle W_{(a_j|b_k)}\rangle_{G(N)}\right)_{j,k=1}^{N}.$$

That is, the Giambelli identity is preserved, after replacing the Schur polynomials in both sides of the identity with the corresponding character $\chi_{G(N)}^{\lambda}$. For $G(N)=U(N)$ this is a known result, as we are considering an orthogonal polynomial ensemble (which were proven to be Giambelli compatible in [39]). However, for the rest of the groups $G(N)$ this provides an example of an ensemble with non unitary symmetry that is Giambelli compatible.

3.4.5 Large N limit and Hopf link expansions

The expansions found in theorem 8 have particular consequences when considering the Chern-Simons model. Considering the function Θ in this theorem and taking into account the results in section 3.3.1, we see that at finite N the partition functions of $Sp(2N), O(2N)$ and $O(2N+1)$ Chern-Simons theories can be expressed as sums of unnormalized Hopf links of the unitary theory. On the other hand, theorem 10 implies that

$$\lim_{N\to\infty}\langle W_{\lambda\mu}\rangle_{G(N)} = \sum_{\nu} s_{(\lambda/\nu)'}(q^{1/2},q^{3/2},\dots)s_{(\mu/\nu)'}(q^{1/2},q^{3/2},\dots) \tag{3.66}$$

for each of the groups[23] $G(N)$. Note that if there is only one character in the average the above formula simplifies to

$$\lim_{N\to\infty}\langle W_{\mu}\rangle_{G(N)} = s_{\mu'}(q^{1/2},q^{3/2},\dots). \tag{3.67}$$

[23]The partitions in (3.56) appear now conjugated, since the function is Θ is expressed as a specialization of $E(x;e^{i\theta})$.

Putting these two facts together we arrive at the following expansions

$$\frac{Z_{Sp(2N)}}{Z_{U(N)}} \sim \frac{1}{2^N} \sum_{\rho_1,\rho_2 \in R(\infty)} (-1)^{(|\rho_1|+|\rho_2|)/2} \langle W_{\rho_1\rho_2}\rangle_{G(N)},$$
$$\frac{Z_{O(2N)}}{Z_{U(N)}} \sim \frac{1}{2^{N-1}} \sum_{\tau_1,\tau_2 \in T(\infty)} (-1)^{(|\tau_1|+|\tau_2|)/2} \langle W_{\tau_1\tau_2}\rangle_{G(N)}$$
$$\frac{Z_{O(2N+1)}}{Z_{U(N)}} \sim \frac{1}{2^N} \sum_{\sigma_1,\sigma_2 \in S(\infty)} (-1)^{(|\sigma_1|+|\sigma_2|+p(\sigma_1)+p(\sigma_2))/2} \langle W_{\sigma_1\sigma_2}\rangle_{G(N)}$$

as $N \to \infty$, where the sets $R(\infty), S(\infty)$ and $T(\infty)$ are defined as the sets $R(N), S(N)$ and $T(N)$ respectively (see theorem 8) without the restriction $\alpha_1 \leq N-1$. That is, at large N the partition functions of the symplectic or orthogonal theories can be expressed as that of the unitary theory with an infinite number of corrections, which correspond to Wilson loops and Hopf links, indexed by partitions of increasing complexity[24] (and which are the same in this limit for each of the groups $G(N)$). Previous examples of partition functions of Chern-Simons theory expressed as sums of averages of characters can be found in [116, 117, 45, 149].

3.5 Fermion quantum models with matrix degrees of freedom

Some interest has arised recently in the study of fermionic quantum mechanical models with matrix degrees of freedom [10, 184, 134]. These models appear as specific instances of tensor quantum mechanical models [134] and have distinctive spectrums of harmonic oscillator type, but with exponentially degenerated energy levels, which suggests connections with other solvable models and to integrability.

These spectra can be computed analytically, see for instance [184, 61], based on the matrix model description obtained in [10], and also [134], where their identification of the Hamiltonian with quartic interactions in terms of Casimirs was used. We compute here averages of insertions of characteristic polynomial type in the $G(N)$ Chern-Simons matrix model. This is in analogy with the model in [10], which described $U(N) \times U(L)$ fermion models in terms of the average of the L-th moment of a determinant insertion in $U(N)$ Chern-Simons matrix models. One motivation for this is that more complex models than the one in [10, 184], with symmetries such as $SO(N) \times SO(L)$, are given in [134] with qualitatively the same spectra, after numerically diagonalizing the Hamiltonian.

The models we study correspond to the average of the function

$$\Theta^{(L,m)}(e^{i\theta}) = \left(2\cos\frac{\theta + im}{2}\right)^L \Theta(e^{i\theta})$$

over the groups $G(N)$, where L is a positive integer and m is a real parameter. In sight of (3.1) and the identity $2\cos\frac{\theta}{2} = |1+e^{i\theta}|$, we see that for U belonging to any of the groups $G(N)$ we have

$$\Theta^{(L,m)}(U) = \Theta(U) e^{Lm} \prod_{j=1}^{N} (1+e^{-m}e^{i\theta_j})^L (1+e^{-m}e^{-i\theta_j})^L, \tag{3.68}$$

[24] Note that the empty partition belongs to each of the sets $R(\infty), S(\infty)$ and $T(\infty)$, and thus the first term in the sums is always a 1.

where the $e^{i\theta_j}$ are the nontrivial eigenvalues of U. We will denote this average by

$$Z_{G(N)}^{(L,m)} = \frac{1}{Z_{G(N)}} \int_{G(N)} \Theta^{(L,m)}(U)dU,$$

where $Z_{G(N)}$ is as defined in (3.60). Taking the limit $m \to 0$ of the unitary model $Z_{U(N)}^{(L,m)}$ we recover the compactly supported analogue of the model considered[25] in [184]. In the unitary case, this corresponds to the average of a characteristic polynomial over the Chern-Simons model. Averages of characteristic polynomials over the classical groups have attracted interest over the years, in particular for their applications in number theory, since the appearance of the seminal works [132, 133], and in the study of many physical systems, see for instance [47] and references therein.

3.5.1 Unitary group

Using the dual Cauchy identity (3.28) twice to expand the product in (3.68) and identity (3.62) we obtain

$$\begin{aligned} Z_{U(N)}^{(L,m)} &= e^{Lm} \sum_{\lambda,\mu} s_{\lambda'}(\underbrace{e^{-m},\ldots,e^{-m}}_{L}) s_{\mu'}(\underbrace{e^{-m},\ldots,e^{-m}}_{L}) \langle W_{\lambda\mu}\rangle_{U(N)} \\ &= e^{Lm} \sum_{\lambda,\mu} e^{-m(|\lambda|+|\mu|)} s_{\lambda'}(1^L) s_{\mu'}(1^L) q^{(C_2^{U(N)}(\lambda)+C_2^{U(N)}(\mu))/2} \\ &\quad \times s_\mu(1,q^{-1},\ldots,q^{-(N-1)}) s_\lambda(q^{-\mu_N},q^{-(\mu_{N-1}+1)},\ldots,q^{-(\mu_1+N-1)}), \end{aligned} \tag{3.69}$$

where 1^L denotes the specialization $x_1 = \cdots = x_L = 1$. Recall that an explicit formula for $s_\mu(1^L)$ is available (2.11). Now, since $s_\nu(x_1,\ldots,x_N) = 0$ if $l(\nu) > N$, we see that the above sum is actually over all partitions λ,μ contained in the rectangular diagram[26] (L^N). Several nontrivial features of the model can be deduced from this fact.

First of all, we see that $Z_{U(N)}^{(L,m)}$ is a polynomial on $q^{1/2}$ and e^{-m}. The high number of terms in this polynomial compared to its relatively low degree on q implies the high number of degeneracies in the spectrum mentioned above. Figure 3.1 shows some examples where this phenomenon is apparent. Secondly, using the dual Cauchy identity again we see that in the limit $q \to 1$ we have

$$\lim_{q\to 1} Z_{U(N)}^{(L,m)} = e^{Lm}(1+e^{-m})^{2NL}.$$

Up to the prefactor e^{Lm}, this shows the duality between the parameters (N,L) in this limit [184]. Finally, the expression (3.69) allows direct computation of the model for low values of N and L and implementation in a computer algebra system. For instance, for $L = 1$ we have

$$\langle \Theta^{(L=1,m)}\rangle_{U(N)} = e^m \sum_{r,s=0}^{N} e^{-m(r+s)} q^{s-s^2/2+r/2} \begin{bmatrix} N \\ r \end{bmatrix}_q e_s(q^{-1},1,q,\ldots,q^{r-2},q^r,q^{r+1},\ldots,q^{N-1}),$$

where e_k denotes the k-th elementary symmetric polynomial (2.8).

[25] This model is also related with the Ewens measure on the symmetric group, see [165] for instance.
[26] See [155] for recent results on asymptotics on the number of such partitions as L and N grow to infinity.

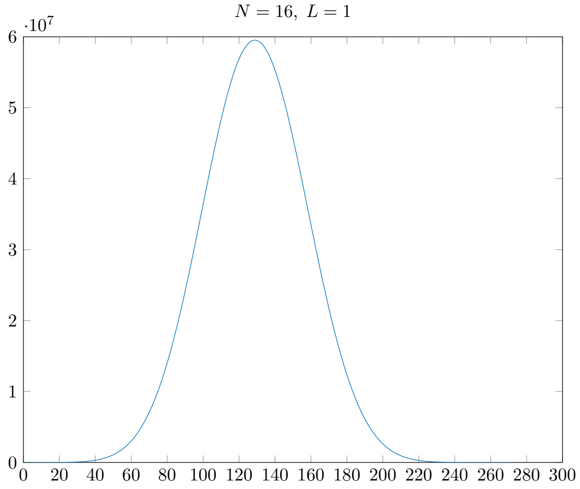


Figure 3.1: For each n in the x axis, the y axis shows the coefficient of the monomial $q^{n/2}$ in $Z_{U(16)}^{(L=1,m=0)}$ (left) and $Z_{U(6)}^{(L=2,m=0)}$ (right).

Large N limit

The large N limit of the model can be computed by two different means, depending on the value of m. If m is nonzero, it follows from (3.69) and the identity (3.66) that

$$\begin{aligned}\lim_{N\to\infty} Z_{U(N)}^{(L,m)} &= e^{Lm}\sum_{\lambda,\mu} s_{\lambda'}(\underbrace{e^{-m},\dots,e^{-m}}_{L})s_{\mu'}(\underbrace{e^{-m},\dots,e^{-m}}_{L}) \\ &\qquad\times\sum_{\nu} s_{(\lambda/\nu)'}(q^{1/2},q^{3/2},\dots)s_{(\mu/\nu)'}(q^{1/2},q^{3/2},\dots) \\ &= e^{Lm}(1-e^{-2m})^{-L^2}\prod_{k=1}^{\infty}\frac{1}{(1-e^{-m}q^{k-1/2})^{2L}},\end{aligned}$$

where the second identity above follows from standard manipulations of Schur and skew Schur polynomials[27].

The above expression is no longer valid in the massless case, $m = 0$. Nevertheless, the large N limit of the model can still be computed, using the fact that $Z_{U(N)}^{(L,m)}$ can be seen as the determinant of the Toeplitz matrix generated by the function $\Theta^{(L,m)}$ (recall identity (3.7)). For $m = 0$, this function does not verify the hypotheses in Szegő's theorem, but it can be written as the product of a function that does verify these hypotheses (the function Θ, as in section 3.3.1) and a Fisher-Hartwig singularity, recall sections 2.3.2 and 3.1.3.

According to (2.49), we see that the function $\Theta^{(L,m=0)}$ corresponds to the product of the smooth function Θ (in the sense of Szegő's theorem) and a single singularity at the point $z = -1$, with parameters $\alpha = L$ and $\beta = 0$. This implies that as $N \to \infty$ we have (3.46)

$$Z_{G(N)}^{(L,m=0)} = N^{L^2}\frac{G(L+1)^2}{G(2L+1)}\prod_{k=1}^{\infty}\frac{1}{(1-q^{k-1/2})^{2L}}(1+o(1)), \tag{3.70}$$

[27]More precisely, we have used the expansion $s_{\lambda/\nu} = \sum_\alpha c_{\nu\alpha}^{\lambda} s_\alpha$, the multiplication rule $\sum_\lambda c_{\nu\alpha}^{\lambda} s_\lambda = s_\nu s_\alpha$ and the Cauchy identity (3.24), where the $c_{\nu\alpha}^{\lambda}$ are Littlewood-Richardson coefficients.

Model	$N=4$	$N=6$	$N=8$	Value of q
$U(N)$	1.0018	1.0005	1.0003	$q=0.1$
$Sp(2N)$	0.9559	0.9692	0.9768	$q=0.25$
$O(2N)$	0.9726	0.9970	0.9997	$q=0.33$
$O(2N+1)$	0.8616	0.9631	0.9906	$q=0.5$

Table 3.1: The table shows the quotient between the numerical value of the spectrums $Z_{G(N)}^{(L=1,m=0)}$, computed directly by means of the formulas (3.69),(3.73),(3.75),(3.76), and the predicted value given by formulas (3.70),(3.74),(3.77). The high rate of convergence is apparent already at low values of N. The rightmost column shows the value of q at which the spectrum is computed.

where G is Barnes' G function. Using its well known asymptotic expansion[28] we see that as $L \to \infty$ the free energy of the model satisfies

$$\lim_{L\to\infty} \log Z_{U(N\to\infty)}^{(L,m)} \sim L^2 \log\left(\frac{N}{L}\right) - L^2\left(2\log 2 - 3/2\right) - \frac{\log L}{12} - 2L\log\left(\sqrt{q},q\right)_\infty ,$$

where we have written the last term as a $q-$Pochhammer symbol[29]. We have considered the large L limit after the large N limit; this is non-rigorous but standard in estimating free energies in the regime where one defines a Veneziano parameter[30] $\zeta = L/N$ and the double scaling is $\zeta = cte$ for $N \to \infty$ and $L \to \infty$. As we see, the leading term of the free energy vanishes for $\zeta = 1$, and changes sign with $\zeta \to 1/\zeta$ otherwise.

Table 3.5.1 shows some numerical tests of the accuracy of formula (3.70) (as well as the analogous formulas for the rest of the models, see the following subsections) for several values of q and N.

Let us emphasize that both the symmetric function approach and the Toeplitz determinant realization of the matrix model prove to be useful for computing its large N limit. Indeed, in the massive case, the character expansion is immediate and gives a manageable expression of the model, while the massless case is also readily handled with the aid of a particular example of Fisher-Hartwig asymptotics.

3.5.2 Symplectic group

We can proceed analogously for the rest of the groups $G(N)$. The expression resulting from the character expansion is actually simpler in this case, although some extra care needs to be taken before integrating. Let us start with the symplectic group. First, we use the dual Cauchy

[28]For any z in a sector not containing the negative real axis it holds that

$$\log G(z+1) = \frac{1}{12} - \log A + \frac{z}{2}\log 2\pi + \left(\frac{z^2}{2} - \frac{1}{12}\right)\log z - \frac{3z^2}{4} + \sum_{k=1}^{N} \frac{\mathrm{B}_{2k+2}}{4k(k+1)z^{2k}} + \mathcal{O}\left(\frac{1}{z^{2N+2}}\right), \qquad (3.71)$$

where A is the Glaisher–Kinkelin constant and the B_k are the Bernouilli numbers.

[29]This type of piece also appears in the free energy of some $4d$ supersymmetric gauge theories [172].

[30]In analogy with localization, L could be interpreted as number of flavours, but with hypermultiplets describing fermionic matter, and hence in the numerator in the matrix model. For example, in [31] we see this type of insertions in the context of matrix quantum mechanics.

identity (3.29) to expand the product in (3.68), obtaining

$$Z^{(L,m)}_{Sp(2N)} = e^{Lm}(1-e^{-2m})^{-L(L+1)/2} \sum_{\mu} e^{-|\mu|m} s_{\mu'}(1^L) \int_{Sp(2N)} sp_{\mu}(U)\Theta(U)dU. \tag{3.72}$$

Since $sp_{\mu}(x_1,\ldots,x_N)=0$ if $l(\mu)-\mu_1-1>2N$ (as can be seen from (3.18), for instance), we see that the sum above actually runs over all partitions contained in the rectangular diagram (L^{2N+L+1}), and therefore is finite. However, we can only use formula (3.63) and substitute the integral in (3.72) by the Wilson loop $\langle W_{\mu}\rangle_{Sp(2N)}$ for those partitions satisfying $l(\mu)\leq N$. One can bypass this constraint in the following way. It is proven in [135] (see proposition 2.4.1) that any $sp_{\mu}(U)$ (seen as a symmetric function, specialized to the nontrivial eigenvalues of U) indexed by a partition of length $l(\mu)>N$ either vanishes or coincides with an irreducible character $\chi^{\lambda}_{Sp(2N)}(U)$, with $l(\lambda)\leq N$, up to a sign. One can then substitute those $sp_{\mu}(U)$ in (3.72) by the corresponding $\chi^{\lambda}_{Sp(2N)}(U)$, use formula (3.63) to write the integrals as the Wilson loops $\langle W_{\lambda}\rangle_{Sp(2N)}$, and then undo the change to recover the $\langle W_{\mu}\rangle_{Sp(2N)}$ indexed by the original partition μ (recall that these coincide themselves with a symplectic Schur function, up to a prefactor). This yields the formula

$$Z^{(L,m)}_{Sp(2N)} = e^{Lm}(1-e^{-2m})^{-L(L+1)/2} \sum_{\mu} e^{-|\mu|m} s_{\mu'}(1^L)\langle W_{\mu}\rangle_{Sp(2N)}, \tag{3.73}$$

where the sum runs over all partitions contained in the rectangular shape (L^{2N+L+1}). An analogous analysis to the unitary case can be performed now. In particular, in the $q\to 1$ limit we obtain

$$\lim_{q\to 1} Z^{(L,m)}_{Sp(2N)} = e^{Lm}(1+e^{-m})^{2NL}$$

using the dual Cauchy identity (3.29). Thus, not only does the (N,L) duality hold for the symplectic group, up to the prefactor e^{Lm}, but the model is actually the same as the unitary one in the $q\to 1$ limit.

Also as in the unitary case, the above sum gives rise to a highly degenerated spectrum. See figure 3.2 for an example; explicit instances for lower values of N and L can also be computed easily. For instance, using the fact that $sp_{(1^k)}(x_1,\ldots,x_N) = -sp_{(1^{2N+2-k})}(x_1,\ldots,x_N)$ (which follows from (3.18)), we obtain for $L=1$ the expression

$$\begin{aligned} Z^{(L=1,m)}_{Sp(2N)} &= e^{m}(1-e^{-2m})^{-1} \sum_{k=0}^{2N+2} e^{-km} q^{Nk+k-k^2/2} sp_{(1^k)}(q,\ldots,q^N) \\ &= e^{m}(1-e^{-2m})^{-1} \sum_{k=0}^{N} e^{-km}(1-e^{-(N-k+1)2m}) q^{Nk+k-k^2/2} sp_{(1^k)}(q,\ldots,q^N) \\ &= e^{m} \sum_{k=0}^{N} e^{-km}(1+e^{-2m}+e^{-4m}+\cdots+e^{-(N+k)2m}) q^{Nk+k-k^2/2} sp_{(1^k)}(q,\ldots,q^N). \end{aligned}$$

We see that the prefactor $(1-e^{-2m})^{-1}$ cancels due to the mentioned coincidence among symplectic characters indexed by single row partitions. The prefactor also cancels for greater values of L, due to the identity (3.38). In particular, this shows that the model is well defined in the massless limit $m\to 0$, which was not immediate from (3.73).

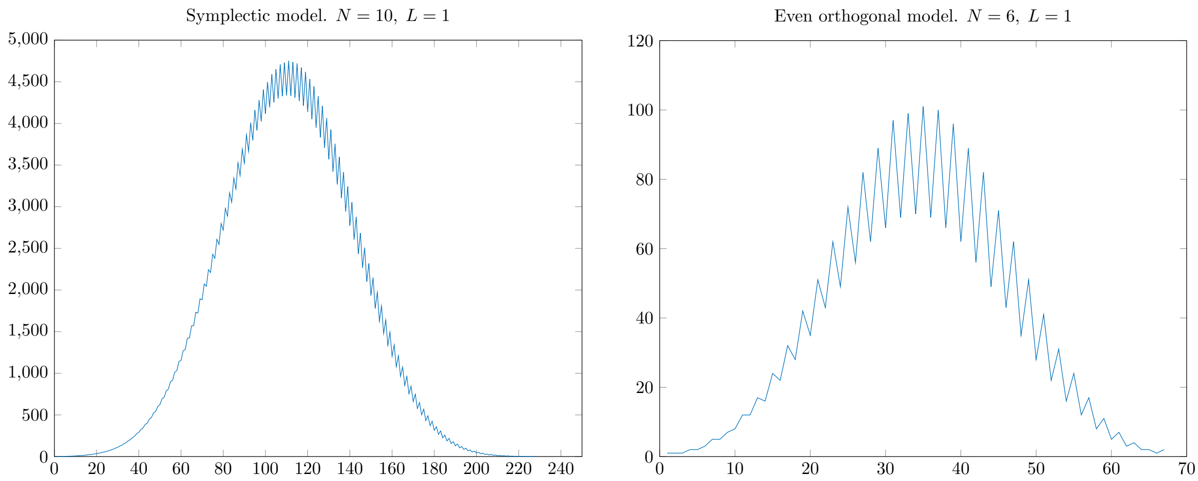


Figure 3.2: For each n in the x axis, the y axis shows the coefficient of the monomial $q^{n/2}$ in $Z^{(L=1,m=0)}_{Sp(20)}$ (left) and $Z^{(L=1,m=0)}_{O(12)}$ (right).

Large N limit

Using identity (3.67) and the dual Cauchy identity (3.24) we see that if $m \neq 0$ we have

$$\lim_{N\to\infty} Z^{(L,m)}_{Sp(2N)} = e^{Lm}(1-e^{-2m})^{-L(L+1)/2}\prod_{k=1}^{\infty}\frac{1}{(1-e^{-m}q^{k-1/2})^L}.$$

For the massless case, we can proceed as in the unitary model, and use known results on the asymptotics of Toeplitz±Hankel determinants generated by functions with Fisher-Hartwig singularities. It follows from (3.47) that for a single singularity at -1 with parameters $\alpha = L$ and $\beta = 0$ we have

$$Z^{(L,m=0)}_{Sp(2N)} = \left(\frac{N}{2}\right)^{L(L+1)/2}\frac{\pi^{L/2}G(3/2)}{G(3/2+L)}\prod_{k=1}^{\infty}\frac{1}{(1-q^{k-1/2})^L}(1+o(1)) \tag{3.74}$$

as $N\to\infty$. Table 3.5.1 shows some numerical tests of the accuracy of this formula.

3.5.3 Orthogonal groups

A similar reasoning applies to the orthogonal groups. For the even orthogonal group, it follows from (3.30) that

$$Z^{(L,m)}_{O(2N)} = e^{Lm}(1-e^{-2m})^{-L(L-1)/2}\sum_{\mu} e^{-|\mu|m}s_{\mu'}(1^L)\langle W_\mu\rangle_{O(2N)}. \tag{3.75}$$

The even orthogonal characters verify $o^{even}_\mu(x_1,\ldots,x_N) = 0$ if $l(\mu)-\mu_1+1 > 2N$, and thus the sum above is now over all the partitions μ contained in the rectangle (L^{2N+L-1}) (a similar reasoning to the symplectic case holds, and in the end one can replace every even orthogonal Schur function $o^{even}_\mu(U)$ in the sum by the corresponding Wilson loop $\langle W_\mu\rangle_{O(2N)}$). See figure 3.2 for an example of this spectrum. A direct computation shows also that for $L = 1$ the sum simplifies to

$$Z^{(L=1,m)}_{O(2N)} = e^m\sum_{k=0}^{2N} e^{-km}q^{Nk-k^2/2}o_{(1^k)}(1,q,\ldots,q^{N-1}) =$$

$$= e^{m}\sum_{k=0}^{N-1} e^{-km}(1+e^{-(N-k)2m})q^{Nk-k^2/2}o_{(1^k)}(1,q,\dots,q^{N-1}) + e^{-(N-1)m}q^{N^2/2}o_{(1^N)}(1,q,\dots,q^{N-1}).$$

As in the symplectic model, the prefactor $(1-e^{-2m})^{-L(L-1)/2}$ in (3.75) cancels for higher values of L, due to the identity (3.39).

For the odd orthogonal group we have

$$Z^{(L,m)}_{O(2N+1)} = e^{Lm}(1+e^{-m})^{-L}(1-e^{-2m})^{-L(L-1)/2}\sum_{\mu} e^{-|\mu|m}s_{\mu'}(1^L)\langle W_\mu\rangle_{O(2N+1)}, \qquad (3.76)$$

using (3.31). Since $o^{odd}_{\mu}(x_1,\dots,x_N)=0$ whenever $l(\mu)-\mu_1>2N$, we see that the sum runs now over all the partitions μ contained in the rectangular shape (L^{2N+L}). The $L=1$ model can be computed explicitely, yielding

$$Z^{(L=1,m)}_{O(2N+1)} = e^{m}(1+e^{-m})^{-1}\sum_{k=0}^{2N+1} e^{-km}q^{Nk+k/2-k^2/2}o^{odd}_{(1^k)}(q^{1/2},q^{3/2},\dots,q^{N-1/2}) =$$
$$= e^{m}(1+e^{-m})^{-1}\sum_{k=0}^{N} e^{-km}(1+e^{-(N-k+1/2)2m})q^{Nk+k/2-k^2/2}o^{odd}_{(1^k)}(q^{1/2},\dots,q^{N-1/2}).$$

As above, the prefactor $(1-e^{-2m})^{-L(L-1)/2}$ cancels for every L, this time because of the identity (3.40).

Using the dual Cauchy identities (3.30),(3.31) and identities (3.64) and (3.65) we see that also for the orthogonal models we have that

$$\lim_{q\to 1} Z^{(L,m)}_{O(2N)} = \lim_{q\to 1} Z^{(L,m)}_{O(2N+1)} = e^{Lm}(1+e^{-m})^{2NL},$$

preserving the (N,L) duality and coincidence of the models in this limit.

Large N limit

As in the symplectic model, using (3.67) and the Cauchy identity (3.24) we see that if $m\neq 0$ then we have

$$\lim_{N\to\infty} Z^{(L,m)}_{O(2N)} = e^{Lm}(1-e^{-2m})^{-L(L-1)/2}\prod_{k=1}^{\infty}\frac{1}{(1-e^{-m}q^{k-1/2})^L}$$

and

$$\lim_{N\to\infty} Z^{(L,m)}_{O(2N+1)} = e^{Lm}(1+e^{-m})^{-L}(1-e^{-2m})^{-L(L-1)/2}\prod_{k=1}^{\infty}\frac{1}{(1-e^{-m}q^{k-1/2})^L}.$$

If $m=0$ we can use again the known results on Fisher-Hartwig asymptotics reviewed in the appendix (3.47) to obtain that, as $N\to\infty$,

$$\begin{aligned} Z^{(L,m=0)}_{O(2N)} &= \left(\frac{N}{2}\right)^{L(L-1)/2}\frac{(4\pi)^{L/2}G(1/2)}{G(1/2+L)}\prod_{k=1}^{\infty}\frac{1}{(1-q^{k-1/2})^L}(1+o(1)),\\ Z^{(L,m=0)}_{O(2N+1)} &= \left(\frac{N}{2}\right)^{L(L-1)/2}\frac{(\pi/4)^{L/2}G(1/2)}{G(1/2+L)}\prod_{k=1}^{\infty}\frac{1}{(1-q^{k-1/2})^L}(1+o(1)).\end{aligned} \qquad (3.77)$$

Product formula	Schur function series
$\prod_j(1-x_j)^{-1}\prod_{j<k}(1-x_jx_k)^{-1}$	$\sum_\mu s_\mu$, over all partitions
$\prod_j(1-x_j^2)^{-1}\prod_{j<k}(1-x_jx_k)^{-1}$	$\sum_\mu s_\mu$, over all even partitions (all parts even)
$\prod_j(1-tx_j)^{-1}\prod_{j<k}(1-x_jx_k)^{-1}$	$\sum_\mu t^{c(\mu)} s_\mu$, over all partitions ($c(\lambda)$ = number of columns of odd length)
$\prod_j \frac{(1+tx_j)}{(1-x_j^2)}\prod_{j<k}(1-x_jx_k)^{-1}$	$\sum_\mu t^{r(\mu)} s_\mu$, over all partitions ($r(\lambda)$ = number of rows of odd length)
$\prod_{j<k}(1-x_jx_k)$	$\sum_\mu (-1)^{\|\mu\|/2} s_\mu(x_1,\ldots,x_n)$ over partitions $\mu=(\alpha_1-1,\ldots,\alpha_p-1\|\alpha_1,\ldots,\alpha_p)$ with $\alpha_1\le n-1$
$\prod_j(1-x_j^2)\prod_{j<k}(1-x_jx_k)$	$\sum_\mu (-1)^{\|\mu\|/2} s_\mu(x_1,\ldots,x_n)$ over partitions $\mu=(\alpha_1+1,\ldots,\alpha_p+1\|\alpha_1,\ldots,\alpha_p)$ with $\alpha_1\le n-1$
$\prod_j(1-x_j)\prod_{j<k}(1-x_jx_k)$	$\sum_\mu (-1)^{(\|\mu\|+p(\mu))/2} s_\mu(x_1,\ldots,x_n)$ over partitions $\mu=(\alpha_1,\ldots,\alpha_p\|\alpha_1,\ldots,\alpha_p)$ with $\alpha_1\le n-1$, where $p(\mu)=p$
$\frac{1-\prod_{j=1}^N x_jy_j}{1-t\prod_{j=1}^N x_jy_j}\prod_{j,k=1}^N(1-x_jy_k)^{-1}$	$\sum_\mu t^{\mu_N} s_\mu(x_1,\ldots,x_N)s_\mu(y_1,\ldots,y_N)$ over partitions of length $l(\mu)\le N$

Table 3.2: Some examples of Schur function series.

Let us make some remarks, to end this section, concerning possible generalizations of the above analysis. First of all, let us stress the fact that the explicit expressions found for the averages of Schur and symplectic and orthogonal Schur functions over the matrix model associated to the Θ function provide a useful tool in the study of more general ensembles. We have already given an example of this, by reducing the analysis of the $\Theta^{(L,m)}$ model to sums of Schur averages over the simpler Θ model, but more complicated insertions can be considered. Indeed, several Schur function series are known for closed factors that can be interpreted as functions on the eigenvalues of the matrices in $G(N)$, as we have done with the characteristic polynomial and the dual Cauchy identity. Table 3.2 shows a few examples among the numerous known cases, taken from [147, 119]. See [128, 122, 118, 123, 103, 175] for instance, for more examples and some generalizations.

Secondly, even if one is interested in insertions that are too complicated for such a character expansion to be useful in practice, or if these insertions pose analytical obstacles, one can still approximate the model to a given order of the parameters of the theory, by truncating the sums over characters up to a certain weight of the indexing partitions, see for instance [84]. This type of approximation becomes particularly interesting in combination with computer-assisted calculations of the models of interest. Indeed, the implementations of the corresponding expressions should be straightforward in any computer algebra system, as long as a closed expression for the average of a single Schur polynomial is available, and may provide a different tool for investigating the statistical properties of random matrix ensembles by looking only at the finite N models. This is particularly useful whenever the model is such that a large number of cancellations occur in the sum over averages of Schur polynomials.

Chapter 4

Hankel minors and the Laguerre Unitary Ensemble

Chapter summary

We introduce the formalism of Hankel minors, and establish some connections with the theory of orthogonal polynomials. In particular, we express the Christoffel-Darboux kernel associated to a set of orthogonal polynomials as a weighted sum over Chebyshev polynomials which coefficients are minors of the associated Hankel matrix. After providing a brief overview of the Riemann-Hilbert methodology, we turn our attention to the Laguerre Unitary Ensemble. We study the insertion of a characteristic polynomial in the corresponding matrix model, both in the finite N regime, by means of Schur polynomial expansions, and as the size of the model grows to infinity, solving the associated Riemann-Hilbert problem[31].

4.1 Preliminaries

4.1.1 Hankel minors

Let w be a function supported on the real line, with moments

$$w_k = \int_{\mathbb{R}} t^k w(t) dt < \infty,$$

for all $k \geq 0$. We denote the Hankel matrix of size N generated by this function by

$$H_N(w) = (w_{j+k-2})_{j,k=1}^N = \begin{pmatrix} w_0 & w_1 & w_2 & \dots & w_{N-1} \\ w_1 & w_2 & w_3 & \dots & w_N \\ w_2 & w_3 & w_4 & \dots & w_{N+1} \\ \vdots & \vdots & \vdots & & \vdots \\ w_{N-1} & w_N & w_{N+1} & \dots & w_{2N-2} \end{pmatrix} \tag{4.1}$$

[31]The contents of this chapter are based on joint work with Dr. Alfredo Deaño. We would like to express our gratitude to Alfredo for his hospitality during a visit to University of Kent on September 2018 and for his valuable help in the learning process of the Riemann-Hilbert methodology.

The use of Andréief's identity leads to the well known expression for the determinant of a Hankel determinant as a matrix model

$$\det H_N(w) = \det\left(w_{j+k-2}\right)_{j,k=1}^{N} = \frac{1}{N!}\int_{\mathbb{R}^{\mathbb{N}}} \prod_{j<k}(t_j-t_k)^2 \prod_{j=1}^{N} w(t_j)dt_j. \tag{4.2}$$

As in the Toeplitz and Toeplitz±Hankel case, one can also consider minors of Hankel matrices, obtained by removing some of their rows and columns. We will refer to these as Hankel minors. An analogous reasoning as in the Toeplitz case leads to their equivalent integral representation in terms of Schur polynomials. Indeed, given two partitions λ and μ and a positive integer N, we recall the definition of the "reversed" arrays

$$\lambda^r = (\lambda_j^r)_{j=1}^N = (\lambda_{N+1-j})_{j=1}^N, \qquad \mu^r = (\mu_j^r)_{j=1}^N = (\mu_{N+1-j})_{j=1}^N. \tag{4.3}$$

We then obtain from Andreiéf's identity that a Hankel minor can be expressed as the integral

$$\det H_N^{\lambda,\mu}(w) = \det\left(w_{j+\lambda_j^r+k+\mu_k^r-2}\right)_{j,k=1}^{N} = \frac{1}{N!}\int_{\mathbb{R}^N} s_\lambda(t)s_\mu(t)\prod_{j<k}(t_j-t_k)^2\prod_{j=1}^{N} w(t_j)dt_j, \tag{4.4}$$

where the $s_\lambda(t)$ are Schur polynomials evaluated at the variables of integration $t_1,\dots,t_N$, and the first identity above serves as a definition. We find a formally identical situation to the Toeplitz case, where the moments of the function w play the role of the Fourier coefficients of the function f. Moreover, the Hankel minor (4.4) can be obtained from the underlying Hankel matrix and the partitions λ and μ following the procedure described in section 3.1.1 for the case of Toeplitz±Hankel matrices.

4.1.2 Orthogonal polynomials

One of the standard approaches to the computation of matrix models is that of orthogonal polynomials. Let us review some well known facts and basic properties of these objects, which can be found in [181, 124] for instance.

We say that an infinite sequence of polynomials $(p_N)_{N\geq 0}$, where each p_N has degree exactly N, is orthonormal with respect to the weight w if

$$\int_{\mathbb{R}} p_j(t)p_k(t)w(t)dt = \delta_{jk} \tag{4.5}$$

for each $j,k\geq 0$. We denote the leading coefficient of the polynomial p_N by γ_N, and by

$$\pi_N(u) = \frac{1}{\gamma_N}p_N(u) \tag{4.6}$$

the monic polynomials of degree N associated to the sequence. The orthonormal polynomials verify a three term recurrence relation

$$\begin{aligned} up_0(u) &= b_0p_1(u) + a_0p_0(u), \\ up_j(u) &= b_jp_{j+1}(u) + a_jp_j(u) + b_{j-1}p_{j-1}(u), \qquad j\geq 1, \end{aligned} \tag{4.7}$$

with $p_0 \equiv 1$. As reviewed in section 2.3.3, these polynomials have both a determinantal and a matrix model expression, usually known as Heine's formula

$$\begin{aligned}\pi_N(u) &= \frac{1}{\det H_N(w)}\begin{vmatrix} w_0 & w_1 & \dots & w_{N-1} & 1 \\ w_1 & w_2 & \dots & w_N & u \\ \vdots & \vdots & & \vdots & \vdots \\ w_N & w_{N+1} & \dots & w_{2N-2} & u^N \end{vmatrix} \\ &= \frac{1}{\det H_N(w)}\frac{1}{N!}\int_{\mathbb{R}^N}\prod_{j<k}(t_j-t_k)^2\prod_{j=1}^N (u-t_j)w(t_j)dt_j.\end{aligned} \tag{4.8}$$

Note that since the moment matrix of the ensemble is Hankel, the families p_N and q_N of section 2.3.3 actually coincide, and we have a single family of orthogonal polynomials instead of two families of biorthogonal ones[32]. Also the Christoffel-Darboux kernel (2.57) can be expressed as a matrix model

$$\begin{aligned}K_N(u_1,u_2) &= \sum_{k=0}^{N-1} p_k(u_1)p_k(u_2) = \gamma_{N-1}^2\frac{\pi_N(u_1)\pi_{N-1}(u_2)-\pi_{N-1}(u_1)\pi_N(u_2)}{u_1-u_2} \\ &= \gamma_{N-1}^2\frac{1}{\det H_N(w)}\frac{1}{N!}\int_{\mathbb{R}^N}\prod_{j<k}(t_j-t_k)^2\prod_{j=1}^N (u_1-t_j)(u_2-t_j)w(t_j)dt_j,\end{aligned} \tag{4.9}$$

where the second identity above follows from the Christoffel-Darboux formula. Equations (4.8) and (4.9) can be seen as particular cases of the more general identity

$$\begin{aligned}&\det\begin{pmatrix} \pi_N(u_1) & \pi_{N+1}(u_1) & \dots & \pi_{N+m-1}(u_1) \\ \pi_N(u_2) & \pi_{N+1}(u_2) & \dots & \pi_{N+m-1}(u_2) \\ \vdots & \vdots & & \vdots \\ \pi_N(u_m) & \pi_{N+1}(u_m) & \dots & \pi_{N+m-1}(u_m) \end{pmatrix} \\ &= \prod_{j<k}(u_j-u_k)\frac{1}{\det H_N(w)}\frac{1}{N!}\int_{\mathbb{R}^N}\prod_{j<k}(t_j-t_k)^2\prod_{j=1}^N\prod_{k=1}^m (u_k-t_j)w(t_j)dt_j,\end{aligned} \tag{4.10}$$

due originally to Brézin and Hikami [46] (see also [188]). In its matrix model expression, this corresponds to the average of m characteristic polynomials over the ensemble with weight function w, evaluated at the points u_k. This identity has been generalized in several directions in subsequent works, see for instance [16, 47] and references therein. In particular, replacing the characteristic polynomials in (4.10) by their inverses amounts to replacing the polynomials in the corresponding row of the determinant in the left hand side of (4.10) by their Cauchy transforms. Among other properties, one interesting feature about the averages of characteristic polynomials and their ratios is universality. This concept is used in random matrix theory to describe the fact that eigenvalues of large random matrices (or other quantities depending on their eigenvalues) share a common statistical behaviour at the microscopical level, that depends only on the symmetry class of the ensemble, rather than on its particular characteristics. Universality has

[32] In general, the existence of the polynomials themselves is not guaranteed. As shown by (4.8), a sufficient condition for this is the nonvanishing of the Hankel determinants $\det H_N(w)$, for every $N \geq 1$. We will assume in the following that the weight function w is such that this condition holds. Another sufficient condition for this is the positive definiteness of the Hankel matrices $H_N(w)$.

been proved for averages of the type (4.10) and more general ones in the bulk [180] and at the edges of the spectrum [189, 29].

Some quantities of interest for the theory of orthogonal polynomials can be expressed in terms of Hankel determinants and minors. Besides the obvious remark that the coefficients of the polynomials are minors themselves, in sight of identity (4.8), this is also true for the coefficients in the three term recurrence relation (4.7). Indeed, comparing the terms in u^{N+1} and u^N in equation (4.7) and using (4.8), we find that

$$a_N = \langle e_1 \rangle_N - \langle e_1 \rangle_{N+1},$$
$$b_N = \frac{\gamma_N}{\gamma_{N+1}} = \frac{(\det H_N(w) \det H_{N+2}(w))^{1/2}}{\det H_{N+1}(w)},$$

where e_1 is the first elementary symmetric polynomial (2.8) and, given a symmetric function s, the notation in the right hand side of the first identity above stands for the average of s over the ensemble of size N

$$\langle s \rangle_N = \frac{1}{\det H_N(w)} \frac{1}{N!} \int_{\mathbb{R}^N} s(t_1, \ldots, t_N) \prod_{j<k} (t_j - t_k)^2 \prod_{j=1}^{N} w(t_j) dt_j. \tag{4.11}$$

The averages $\langle e_1 \rangle_N$ appearing above coincide with the quotients $\det H_N^{\varnothing,(1)}(w)/\det H_N(w)$ of the minor of the Hankel matrix of size $N+1$ obtained by removing its next to last row and its last column, and the Hankel determinant of size N.

More general quantities can also be studied in terms of Hankel minors. Indeed, using the dual Cauchy identity (3.28) and the formula (2.18) we obtain the Schur function expansion

$$\prod_{j=1}^{N} \prod_{k=1}^{m} (u_k - t_j) = \sum_{\nu \subset (N^m)} (-1)^{mN+|\nu|} s_\nu(u_1, \ldots, u_m) s_{L_{m,N}(\nu')}(t_1, \ldots, t_N), \tag{4.12}$$

where $L_{m,N}(\nu')$ is defined in (2.17). Therefore, substituting the product in (4.10) by this expression, we see that the average of a characteristic polynomial over a random matrix ensemble can be computed equivalently as a finite sum over averages of Schur polynomials. See [171] for a related result. This is particularly useful whenever the choice of weight function is such that an explicit expression for the minors of the underlying moment matrix is available, as in section 3.5. More general insertions can also be computed in a similar fashion, including ratios of characteristic polynomials, using the Cauchy identity (3.24) or both the Cauchy and dual Cauchy identity. These correspond to determinants of the form (4.10), involving also the Cauchy transform of the orthogonal polynomials π_N, as mentioned above, and give rise to infinite Schur function series.

In particular, we obtain the following consequence of the expansion (4.12). Recall the fact, derived in section 2.3, that a Schur polynomial indexed by a partition of length at most 2 can be expressed as (2.43)

$$s_{(\nu_1,\nu_2)}(u_1, u_2) = (u_1 u_2)^{|\nu|/2} U_{\nu_1-\nu_2} \left(\frac{1}{2} \left(\sqrt{\frac{u_1}{u_2}} + \sqrt{\frac{u_2}{u_1}} \right) \right),$$

where U_k denotes the k-th Chebyshev polynomial of the second kind. Thus, noting that the expansion for the Christoffel-Darboux kernel (4.9) involves only Schur polynomials indexed by partitions with at most two parts, we arrive to the following result.

Theorem 12. *Let u_1, u_2 be nonzero. The Christoffel-Darboux kernel* (4.9) *associated to a family of orthogonal polynomials can be expressed as the sum*

$$K_N(u_1, u_2) = \sum_{0 \le \nu_1 \le \nu_2 \le N} \langle s_{(2^{N-\nu_1}, 1^{\nu_1 - \nu_2})} \rangle_N (-\sqrt{u_1 u_2})^{\nu_1+\nu_2} U_{\nu_1-\nu_2}\left(\frac{1}{2}\left(\sqrt{\frac{u_1}{u_2}} + \sqrt{\frac{u_2}{u_1}}\right)\right),$$

where the bracket notation stands for the average (4.11) *over the corresponding random matrix ensemble.*

That is, the kernel of order N built from the polynomials orthogonal with respect a weight function w can be expressed as a sum over Chebyshev polynomials, where the coefficients in the sum are given by minors of the Hankel matrix generated by w. Some inspection following the procedure described in section 3.1.1 to obtain minors from the underlying partitions shows that all these minors are obtained by striking two columns from the Hankel matrix of size $N \times (N+2)$ generated by w. Examples of Schur function series involving Chebyshev polynomials of the second kind have appeared previously in the literature, see for instance [122].

4.1.3 Riemann-Hilbert methodology

We now outline the main ideas behind the Riemann-Hilbert approach to the study of the asymptotic behaviour of orthogonal polynomials. See for instance [66, 138], among others, for more detailed introductory expositions.

In general, a Riemann-Hilbert problem consists of finding an analytic function on the complex plane $\mathbb{C}$ minus a collection of oriented curves Σ, on which the boundary values of the function from both of their sides are given, usually together with some normalization condition. We follow the usual convention and define, for a given collection of oriented curves Σ,

$$Y_+(z) = \lim_{\substack{z' \to z \\ z' \in \text{left side of } \Sigma}} Y(z'), \qquad Y_-(z) = \lim_{\substack{z' \to z \\ z' \in \text{right side of } \Sigma}} Y(z'),$$

for any $z \in \Sigma$ and any function Y analytic in $\mathbb{C} \setminus \Sigma$. Such values are well defined except for endpoints of curves or points of intersection of curves, where one needs to impose extra conditions when addressing the Riemann-Hilbert problem.

The connection with orthogonal polynomials[33] is due to Fokas, Its and Kitaev [87], who noticed that these can be expressed in terms of a Riemann-Hilbert problem for a matrix-valued function. More precisely, given some weight function $w(x)$ supported on the real line, they considered a function $Y : \mathbb{C} \mapsto \mathbb{C}^{2x2}$ solving the following problem.

Riemann-Hilbert problem for Y

1. Y is analytic in $\mathbb{C} \setminus \text{supp } w$.
2. For $x \in \text{supp } w$, the matrix Y verifies the jump condition

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}.$$

[33] We describe here the case of orthogonal polynomials with respect to a weight function supported on the real line; analogous premises hold in more general settings. For instance, the case of functions supported on the unit circle, which corresponds to Toeplitz determinants, was formulated in [15].

3. As $z \to \infty$, we have

$$Y(z) = \left(I + \mathcal{O}(z^{-1})\right) z^{N\sigma_3}. \tag{4.13}$$

We denote in (3) above and in the following by $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ the third Pauli matrix. We also omit the dependance on N of the matrix Y in the notation, for ease of presentation.

The authors of [87] proved that, under suitable conditions on w, the solution to this problem is given by

$$Y(z) = \begin{pmatrix} \pi_N(z) & \mathcal{C}(\pi_N w)(z) \\ -2\pi i \gamma_{N-1}^2 \pi_{N-1}(z) & -2\pi i \gamma_{N-1}^2 \mathcal{C}(\pi_{N-1} w)(z) \end{pmatrix}, \tag{4.14}$$

where π_N is the N-th monic orthogonal polynomial with respect to the weight function w, the constant γ_N is the leading coefficient of the N-th orthonormal polynomial (4.6), and the operator

$$\mathcal{C}(f)(z) = \frac{1}{2\pi i} \int_0^\infty \frac{f(t)dt}{t - z}, \qquad z \in \mathbb{C} \setminus [0, \infty),$$

is the usual Cauchy transform. Furthermore, several quantities of interest related to the orthogonal polynomials can be expressed in terms of the entries of the matrix Y. For instance, we see from (4.14) that

$$\gamma_{N-1}^2 = -\frac{1}{2\pi i} \lim_{z\to\infty} \frac{Y_{21}(z)}{z^{N-1}}, \qquad \gamma_N^{-2} = -2\pi i \lim_{z\to\infty} Y_{12}(z) z^{N+1}, \qquad \eta_N = \lim_{z\to\infty} \frac{Y_{11}(z) - z^N}{z^{N-1}}, \tag{4.15}$$

where η_N is the subleading coefficient of the monic orthogonal polynomial π_N. Similar identities hold for the coefficients in the three term recurrence relation.

Hence, explicit expressions for the matrix Y and its asymptotic behaviour can be used in particular to describe the behaviour of the orthogonal polynomials with respect to the weight w. In order to obtain these expressions, one usually considers a series of transformations for the matrix Y, that allow to reduce the Riemann-Hilbert problem for this matrix to several problems that can be explicitly solved, following the strategy pioneered in [69]-[71] by Deift and collaborators. Each of these problems is defined in a different domain of the complex plane, in such a way that the union of these regions covers the whole plane and the matchings between the different domains are smooth. An outline of the transformations is as follows

$$Y \mapsto T \mapsto S \mapsto R, \tag{4.16}$$

where the last matrix R is asymptotically close to the identity as $N \to \infty$. Moreover, this matrix is constructed in terms of known explicit solutions to different Riemann-Hilbert problems, so called local and global parametrices. Reversing the series of transformations, this provides explicit expressions for the matrix Y and its asymptotic behaviour in the various domains of the plane.

We end this section posing three standard model Riemann-Hilbert problems and record their solutions, which will be used in section 4.3. We give more details about the purpose of the transformations (4.16) in the following section. Unless specified otherwise, we will consider the main branches of all the functions used in the following.

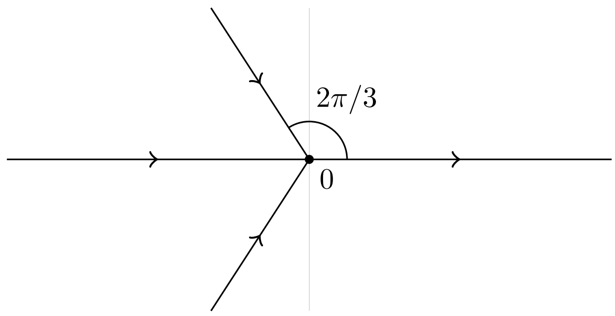


Figure 4.1: Jump contour for Φ_{Ai}.

Airy model Riemann-Hilbert problem

1. $\Psi_{\mathrm{Ai}} : \mathbb{C} \setminus \Sigma_{\mathrm{Ai}} \to \mathbb{C}^{2\times 2}$ is analytic, where Σ_{Ai} consists of the real axis and the two rays $e^{2\pi i/3}\mathbb{R}^+$ and $e^{-2\pi i/3}\mathbb{R}^+$, as shown in figure 4.1.
2. The matrix Ψ_{Ai} has the following jump relations

$$\Psi_{\mathrm{Ai}}(z)_+ = \Psi_{\mathrm{Ai}}(z)_- \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in \mathbb{R}^-, \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & z \in \mathbb{R}^+, \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & z \in \left(e^{2\pi i/3}\mathbb{R}^+ \cup e^{-2\pi i/3}\mathbb{R}^+\right), \end{cases}$$

3. As $z \to \infty$, $z \notin \Sigma_{\mathrm{Ai}}$, we have

$$\Psi_{\mathrm{Ai}}(z) = z^{-\frac{\sigma_3}{4}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \left(I + \sum_{k=1}^{\infty} \frac{\Psi_{\mathrm{Ai},k}}{z^{3k/2}} \right) e^{-\frac{2}{3}z^{3/2}\sigma_3}, \tag{4.17}$$

where $\Psi_{\mathrm{Ai},k}$ are constant matrices that can be computed explicitly.
4. As $z \to 0$, we have $\Psi_{\mathrm{Ai}}(z) = \mathcal{O}(1)$.

This problem was posed and solved in [71], where also explicit solutions of the constant matrices $\Psi_{\mathrm{Ai},k}$ can be found. Its solution is given by

$$\Psi_{\mathrm{Ai}}(z) = \sqrt{2\pi} e^{i\pi/6} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \begin{cases} \begin{pmatrix} \mathrm{Ai}(z) & \mathrm{Ai}(\omega^2 z) \\ \mathrm{Ai}'(z) & \omega^2 \mathrm{Ai}'(\omega^2 z) \end{pmatrix} e^{-\frac{i\pi}{6}\sigma_3}, & 0 < \arg z < \frac{2\pi}{3}, \\ \begin{pmatrix} \mathrm{Ai}(z) & \mathrm{Ai}(\omega^2 z) \\ \mathrm{Ai}'(z) & \omega^2 \mathrm{Ai}'(\omega^2 z) \end{pmatrix} e^{-\frac{i\pi}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, & \frac{2\pi}{3} < \arg z < \pi, \\ \begin{pmatrix} \mathrm{Ai}(z) & -\omega^2 \mathrm{Ai}(\omega z) \\ \mathrm{Ai}'(z) & -\mathrm{Ai}'(\omega z) \end{pmatrix} e^{-\frac{i\pi}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & -\pi < \arg z < -\frac{2\pi}{3}, \\ \begin{pmatrix} \mathrm{Ai}(z) & -\omega^2 \mathrm{Ai}(\omega z) \\ \mathrm{Ai}'(z) & -\mathrm{Ai}'(\omega z) \end{pmatrix} e^{-\frac{i\pi}{6}\sigma_3}, & -\frac{2\pi}{3} < \arg z < 0, \end{cases} \tag{4.18}$$

where $\omega = e^{\frac{2\pi i}{3}}$ and $\mathrm{Ai}(z)$ is the Airy function [1].

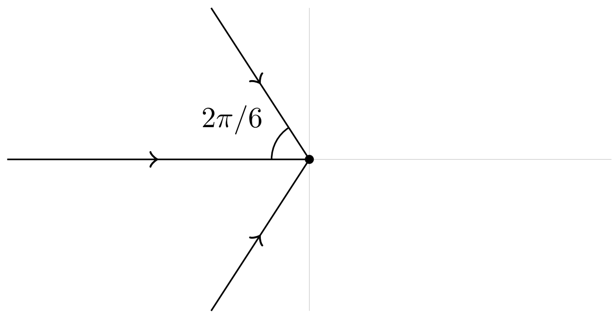


Figure 4.2: Jump contour for Φ_{Be}.

Bessel model Riemann-Hilbert problem

1. $\Psi_{\mathrm{Be}} : \mathbb{C} \setminus \Sigma_{\mathrm{Be}} \to \mathbb{C}^{2\times 2}$ is analytic, where Σ_{Be} consists of the negative real axis and the two rays $e^{2\pi i/3}\mathbb{R}^+$ and $e^{-2\pi i/3}\mathbb{R}^+$, as shown in figure 4.2
2. The matrix Ψ_{Be} has the following jump relations

$$\Psi_{\mathrm{Be}}(z)_+ = \Psi_{\mathrm{Be}}(z)_- \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in \mathbb{R}^-, \\ \begin{pmatrix} 1 & 0 \\ e^{i\pi\alpha} & 1 \end{pmatrix}, & z \in e^{2\pi i/3}\mathbb{R}^+, \\ \begin{pmatrix} 1 & 0 \\ e^{-i\pi\alpha} & 1 \end{pmatrix}, & z \in e^{-2\pi i/3}\mathbb{R}^+, \end{cases}$$

   where α is some complex number.
3. As $z \to \infty$, $z \notin \Sigma_{\mathrm{Be}}$, we have

$$\Psi_{\mathrm{Be}}(z) = (2\pi z^{1/2})^{-\frac{\sigma_3}{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \left(I + \sum_{k=1}^{\infty} \frac{\Psi_{\mathrm{Be},k}}{z^{k/2}} \right) e^{2z^{1/2}\sigma_3}, \tag{4.19}$$

   where $\Psi_{\mathrm{Be},k}$ are constant matrices that can be computed explicitly.
4. As $z \to 0$, we have

$$\Psi_{\mathrm{Be}}(z) = \begin{cases} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix} z^{\frac{\alpha}{2}\sigma_3}, & |\arg z| < \frac{2\pi}{3}, \\ \begin{pmatrix} \mathcal{O}(z^{-\frac{\alpha}{2}}) & \mathcal{O}(z^{-\frac{\alpha}{2}}) \\ \mathcal{O}(z^{-\frac{\alpha}{2}}) & \mathcal{O}(z^{-\frac{\alpha}{2}}) \end{pmatrix}, & \frac{2\pi}{3} < |\arg z| < \pi, \end{cases} \qquad \text{if Re } \alpha > 0,$$

$$\Psi_{\mathrm{Be}}(z) = \begin{cases} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log z) \\ \mathcal{O}(1) & \mathcal{O}(\log z) \end{pmatrix}, & |\arg z| < \frac{2\pi}{3}, \\ \begin{pmatrix} \mathcal{O}(\log z) & \mathcal{O}(\log z) \\ \mathcal{O}(\log z) & \mathcal{O}(\log z) \end{pmatrix}, & \frac{2\pi}{3} < |\arg z| < \pi, \end{cases} \qquad \text{if Re } \alpha = 0,$$

$$\Psi_{\mathrm{Be}}(z) = \begin{pmatrix} \mathcal{O}(z^{\frac{\alpha}{2}}) & \mathcal{O}(z^{\frac{\alpha}{2}}) \\ \mathcal{O}(z^{\frac{\alpha}{2}}) & \mathcal{O}(z^{\frac{\alpha}{2}}) \end{pmatrix}, \qquad \text{if Re } \alpha < 0.$$

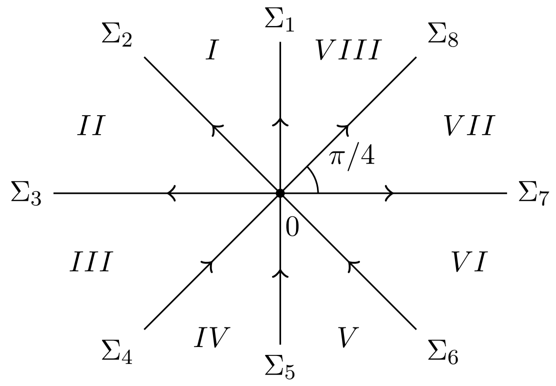


Figure 4.3: Jump contour for Φ_{HG}.

This problem was introduced and solved in [140]. Its solution is given explicitely by

$$\Psi_{\text{Be}}(z) = \begin{cases} \begin{pmatrix} I_\alpha(2z^{\frac{1}{2}}) & \frac{i}{\pi}K_\alpha(2z^{\frac{1}{2}}) \\ 2\pi i z^{\frac{1}{2}} I'_\alpha(2z^{\frac{1}{2}}) & -2z^{\frac{1}{2}}K'_\alpha(2z^{\frac{1}{2}}) \end{pmatrix}, & |\arg z| < \frac{2\pi}{3}, \\ \begin{pmatrix} \frac{1}{2}H_\alpha^{(1)}(2(-z)^{\frac{1}{2}}) & \frac{1}{2}H_\alpha^{(2)}(2(-z)^{\frac{1}{2}}) \\ \pi z^{\frac{1}{2}}\left(H_\alpha^{(1)}\right)'(2(-z)^{\frac{1}{2}}) & \pi z^{\frac{1}{2}}\left(H_\alpha^{(2)}\right)'(2(-z)^{\frac{1}{2}}) \end{pmatrix} e^{\frac{i\pi\alpha}{2}\sigma_3}, & \frac{2\pi}{3} < \arg z < \pi, \\ \begin{pmatrix} \frac{1}{2}H_\alpha^{(2)}(2(-z)^{\frac{1}{2}}) & -\frac{1}{2}H_\alpha^{(1)}(2(-z)^{\frac{1}{2}}) \\ -\pi z^{\frac{1}{2}}\left(H_\alpha^{(2)}\right)'(2(-z)^{\frac{1}{2}}) & \pi z^{\frac{1}{2}}\left(H_\alpha^{(1)}\right)'(2(-z)^{\frac{1}{2}}) \end{pmatrix} e^{-\frac{i\pi\alpha}{2}\sigma_3}, & -\pi < \arg z < -\frac{2\pi}{3}, \end{cases}$$

where $H_\alpha^{(1)}$ and $H_\alpha^{(2)}$ are the Hankel functions of the first and second kind, and I_α and K_α are the modified Bessel functions of the first and second kind.

Confluent hypergeometric model Riemann-Hilbert problem

1. $\Psi_{\text{HG}} : \mathbb{C} \setminus \Sigma_{\text{HG}} \to \mathbb{C}^{2\times 2}$ is analytic, where Σ_{HG} consists of the real and imaginary axis, as well as the two rays $e^{\pi i/4}\mathbb{R}$ and $e^{-\pi i/4}\mathbb{R}$, as shown in figure 4.3.
2. The matrix Ψ_{HG} verifies the jump relations

$$\Psi_{\text{HG}}(z)_+ = \Psi_{\text{HG}}(z)_- J_k, \qquad z \in \Sigma_j,$$

where the curves Σ_j are depicted in figure 4.3 and

$$J_1 = \begin{pmatrix} 0 & e^{-i\pi\beta} \\ -e^{i\pi\beta} & 0 \end{pmatrix}, \quad J_5 = \begin{pmatrix} 0 & e^{i\pi\beta} \\ -e^{-i\pi\beta} & 0 \end{pmatrix}, \quad J_3 = J_7 = \begin{pmatrix} e^{i\pi\alpha/2} & 0 \\ 0 & e^{-i\pi\alpha/2} \end{pmatrix},$$
$$J_2 = \begin{pmatrix} 1 & 0 \\ e^{-i\pi\alpha}e^{i\pi\beta} & 1 \end{pmatrix}, \quad J_4 = \begin{pmatrix} 1 & 0 \\ e^{i\pi\alpha}e^{-i\pi\beta} & 1 \end{pmatrix}, \quad J_6 = \begin{pmatrix} 1 & 0 \\ e^{-i\pi\alpha}e^{-i\pi\beta} & 1 \end{pmatrix}, \quad J_8 = \begin{pmatrix} 1 & 0 \\ e^{i\pi\alpha}e^{i\pi\beta} & 1 \end{pmatrix},$$

where α and β are complex parameters.
3. As $z \to \infty$, $z \notin \Sigma_{\text{HG}}$, we have

$$\Psi_{\text{HG}}(z) = \left(I + \sum_{k=1}^{\infty} \frac{\Psi_{\text{HG},k}}{z^k}\right) z^{-\beta\sigma_3} e^{-\frac{z}{2}\sigma_3} L^{-1}(z), \tag{4.20}$$

where $z^{-\beta}$ has a cut along $i\mathbb{R}^-$, so that $z^{-\beta} \in \mathbb{R}$ for $z \in \mathbb{R}^+$, $\Psi_{\mathrm{HG},k}$ are constant matrices that can be computed explicitly, and

$$L(z) = \begin{cases} e^{\frac{i\pi\alpha}{4}\sigma_3} e^{-i\pi\beta\sigma_3}, & \frac{\pi}{2} < \arg z < \pi, \\ e^{-\frac{i\pi\alpha}{4}\sigma_3} e^{-i\pi\beta\sigma_3}, & \pi < \arg z < \frac{3\pi}{2}, \\ e^{\frac{i\pi\alpha}{4}\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & -\frac{\pi}{2} < \arg z < 0, \\ e^{-\frac{i\pi\alpha}{4}\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & 0 < \arg z < \frac{\pi}{2}. \end{cases} \tag{4.21}$$

4. As $z \to 0$, we have

$$\Psi_{\mathrm{HG}}(z) = \begin{cases} \begin{pmatrix} \mathcal{O}(z^{\frac{\mathrm{Re}\,\alpha}{2}}) & \mathcal{O}(z^{-\frac{\mathrm{Re}\,\alpha}{2}}) \\ \mathcal{O}(z^{\frac{\mathrm{Re}\,\alpha}{2}}) & \mathcal{O}(z^{-\frac{\mathrm{Re}\,\alpha}{2}}) \end{pmatrix}, & z \in II \cup III \cup VI \cup VII, \\ \begin{pmatrix} \mathcal{O}(z^{-\frac{\mathrm{Re}\,\alpha}{2}}) & \mathcal{O}(z^{-\frac{\mathrm{Re}\,\alpha}{2}}) \\ \mathcal{O}(z^{-\frac{\mathrm{Re}\,\alpha}{2}}) & \mathcal{O}(z^{-\frac{\mathrm{Re}\,\alpha}{2}}) \end{pmatrix}, & z \in I \cup IV \cup V \cup VIII, \end{cases} \quad \text{if } \mathrm{Re}\,\alpha > 0,$$

$$\Psi_{\mathrm{HG}}(z) = \begin{cases} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log z) \\ \mathcal{O}(1) & \mathcal{O}(\log z) \end{pmatrix}, & z \in II \cup III \cup VI \cup VII \\ \begin{pmatrix} \mathcal{O}(\log z) & \mathcal{O}(\log z) \\ \mathcal{O}(\log z) & \mathcal{O}(\log z) \end{pmatrix}, & z \in I \cup IV \cup V \cup VIII, \end{cases} \quad \text{if } \mathrm{Re}\,\alpha = 0,$$

$$\Psi_{\mathrm{HG}}(z) = \begin{pmatrix} \mathcal{O}(z^{\frac{\mathrm{Re}\,\alpha}{2}}) & \mathcal{O}(z^{\frac{\mathrm{Re}\,\alpha}{2}}) \\ \mathcal{O}(z^{\frac{\mathrm{Re}\,\alpha}{2}}) & \mathcal{O}(z^{\frac{\mathrm{Re}\,\alpha}{2}}) \end{pmatrix}, \quad \text{if } \mathrm{Re}\,\alpha < 0,$$

where the regions I to $VIII$ are displayed in figure 4.3.

This problem was introduced and solved in [126] for the case $\alpha = 0$, and then in [100, 67] for the general case. Defining

$$\widehat{\Psi}_{\mathrm{HG}}(z) = \begin{pmatrix} \frac{\Gamma(1+\alpha/2-\beta)}{\Gamma(1+\alpha)} G(\alpha/2+\beta, \alpha; z) e^{-i\pi\alpha/2} & -\frac{\Gamma(1+\alpha/2-\beta)}{\Gamma(\alpha/2+\beta)} H(1+\alpha/2-\beta, \alpha; ze^{-i\pi}) \\ \frac{\Gamma(1+\alpha/2+\beta)}{\Gamma(1+\alpha)} G(1+\alpha/2+\beta, \alpha; z) e^{-i\pi\alpha/2} & H(\alpha/2-\beta, \alpha; ze^{-i\pi}) \end{pmatrix} e^{-\frac{i\pi\alpha}{4}\sigma_3},$$

where G and H are related to the Whittaker functions [1] by the following identities

$$G(a, \alpha; z) = \frac{M_{\kappa,\mu}(z)}{\sqrt{z}}, \qquad H(a, \alpha; z) = \frac{W_{\kappa,\mu}(z)}{\sqrt{z}} \qquad \left(\mu = \frac{\alpha}{2}, \qquad \kappa = \frac{1}{2} + \frac{\alpha}{2} - a\right), \tag{4.22}$$

we have that the solution of the confluent hypergeometric model Riemann-Hilbert problem is given by

$$\Psi_{\mathrm{HG}}(z) = \begin{cases} \widehat{\Psi}_{\mathrm{HG}}(z) J_2^{-1}, & z \in I, \\ \widehat{\Psi}_{\mathrm{HG}}(z), & z \in II, \\ \widehat{\Psi}_{\mathrm{HG}}(z) J_3, & z \in III, \\ \widehat{\Psi}_{\mathrm{HG}}(z) J_3 J_4^{-1}, & z \in IV, \\ \widehat{\Psi}_{\mathrm{HG}}(z) J_2^{-1} J_1^{-1} J_8^{-1} J_7^{-1} J_6, & z \in V, \\ \widehat{\Psi}_{\mathrm{HG}}(z) J_2^{-1} J_1^{-1} J_8^{-1} J_7^{-1}, & z \in VI, \\ \widehat{\Psi}_{\mathrm{HG}}(z) J_2^{-1} J_1^{-1} J_8^{-1}, & z \in VII, \\ \widehat{\Psi}_{\mathrm{HG}}(z) J_2^{-1} J_1^{-1}, & z \in VIII. \end{cases}$$

4.2 The Laguerre Unitary Ensemble

Consider the orthogonal polynomial ensemble with weight function

$$w(t) = t^{\alpha} e^{-t}, \qquad t \in [0, \infty),$$

where $\alpha > -1$ is a fixed parameter. The polynomials orthogonal with respect to this function are the classical Laguerre polynomials, which have been the subject of many studies and whose properties are well understood (see [181, 139, 63] for instance, among many others). In particular, the following features will be relevant for our purposes.

Lemma. *Let $L_N^{(\alpha)}$ be the classical Laguerre polynomials, given by*

$$L_N^{(\alpha)}(u) = \sum_{k=0}^{N} \binom{N+\alpha}{N-k} \frac{(-u)^k}{k!}, \tag{4.23}$$

for every $N \geq 0$. They verify the orthogonality relation

$$\int_0^{\infty} L_j^{(\alpha)}(t) L_k^{(\alpha)}(t) t^{\alpha} e^{-t} dt = \frac{\Gamma(\alpha + k + 1)}{\Gamma(k+1)} \delta_{jk}$$

and the second order differential equation

$$u(L_N^{(\alpha)}(u))'' + (\alpha + 1 - u)(L_N^{(\alpha)}(u))' + N L_N^{(\alpha)}(u) = 0. \tag{4.24}$$

Moreover, the largest zero z_N of the polynomial $L_N^{(\alpha)}$ verifies

$$z_N = 4N(1 + \mathcal{O}(N^{-1})), \text{ as } N \to \infty. \tag{4.25}$$

We will focus instead on a deformation of the Laguerre weight, given by

$$w_{u,m}(t) = (t-u)^{2m} t^{\alpha} e^{-t}, \tag{4.26}$$

where m is a positive integer[34] and $\alpha > 0$. In its matrix model expression, this corresponds to the insertion of the $2m$-th power of the characteristic polynomial of the ensemble evaluated at the point u. We denote

$$Z_{LUE(N)}^{u,m} = \det \left(\int_0^{\infty} t^{j+k-2} w_{u,m}(t) dt \right)_{j,k=1}^{N} = \frac{1}{N!} \int_0^{\infty} \cdots \int_0^{\infty} \prod_{j<k} (t_j - t_k)^2 \prod_{j=1}^{N} (t_j - u)^{2m} t_j^{\alpha} e^{-t_j} dt_j.$$

In section 4.3, we will study this model in the double scaling regime

$$N \to \infty, \qquad u \to \infty, \qquad \frac{u}{4N} = const \in (0, 1). \tag{4.27}$$

The double scaling in u means in particular that the rescaled parameter $u/4N$ lies in the bulk of the spectrum of the Laguerre Unitary Ensemble, which gives rise to a richer analysis of the model, as we will see in the following.

[34] We have chosen the power of the factor $(t-u)$ in the weight $w_{u,m}$ to be an even integer in order to avoid a more technical development of the problem. In the language of Fisher-Hartwig singularities for weight functions supported on the real line (see [52, 53]), the weight (4.26) corresponds to the product of a root type singularity with the weight of the Laguerre Unitary Ensemble, without any jump type singularities. The choice of α (instead of the usual, more general condition $\alpha > -1$) simplifies slightly the analysis as well, see in particular the proof of equation (4.46).

As reviewed in section 4.1.1, there is a number of equivalent interpretations for this object. For $m = 1$, it coincides with the diagonal of the Christoffel-Darboux kernel of degree N built from the monic Laguerre polynomials, in sight of identity (4.9). For general m, it can also be realized as the Wronskian of $2m$ consecutive polynomials

$$\frac{Z^{u,m}_{LUE(N)}}{Z_{LUE(N)}} = \frac{(-1)^{m(2m-1)}}{G(m+1)} \det \begin{pmatrix} \pi_N(u) & \pi_{N+1}(u) & \dots & \pi_{N+2m-1}(u) \\ \pi'_N(u) & \pi'_{N+1}(u) & \dots & \pi'_{N+2m-1}(u) \\ \vdots & \vdots & & \vdots \\ \pi^{(2m-1)}_N(u) & \pi^{(2m-1)}_{N+1}(u) & \dots & \pi^{(2m-1)}_{N+2m-1}(u) \end{pmatrix},$$

as shown by (4.10), where

$$\pi_N(u) = (-1)^N N! \sum_{k=0}^{N} (-1)^k \binom{N+\alpha}{N+k} \frac{u^k}{k!} \tag{4.28}$$

is the monic Laguerre polynomial of degree N, and the partition function of the Laguerre Unitary Ensemble is given by

$$Z_{LUE(N)} = \frac{1}{N!} \int_0^\infty \cdots \int_0^\infty \prod_{j<k} (t_j - t_k)^2 \prod_{j=1}^{N} t_j^\alpha e^{-t_j} dt_j = \frac{G(N+1)G(\alpha+N+1)}{G(\alpha+1)}. \tag{4.29}$$

Similar and more general insertions in the Laguerre Unitary Ensemble have been studied before, in particular showing their relation with the τ functions of the Painlevé III and V systems and the smallest and largest eigenvalues of this ensemble [96]-[98]. The case of general insertions of Fisher-Hartwig type in the bulk of the spectrum was addressed recently in [53], as well as the case of a single insertion approaching the soft edge of the spectrum [197] (see also [199] for a similar setting over the Jacobi Unitary Ensemble).

When addressing the large N analysis, it will be convenient to consider the re-scaled matrix model

$$\widehat{Z}^{v,m}_{LUE(N)} = \frac{1}{N!} \int_0^\infty \cdots \int_0^\infty \prod_{j<k} (t_j - t_k)^2 \prod_{j=1}^{N} (t_j - v)^{2m} t_j^\alpha e^{-4Nt_j} dt_j = (4N)^{-(N+2m+\alpha)N} Z^{u,m}_{LUE(N)}, \tag{4.30}$$

where $4Nv = u$, (so that, in particular, $v \in (0,1)$, recall (4.27)). We will denote the weight function of this model by

$$\widehat{w}_{v,m}(t) = (t-v)^{2m} t^\alpha e^{-4Nt}. \tag{4.31}$$

4.2.1 Equilibrium measure

A central object in the study of random matrix ensembles, and in particular in the Riemann-Hilbert methodology, is the equilibrium measure associated to the potential of the weight function of the ensemble. This measure, which we denote by $d\mu_V$, has several equivalent characterizations, see [66, 173] for instance. Its relevance from the point of view of orthogonal polynomials and random matrix theory is due to the fact that it coincides with the weak limit of the zero-counting measures of the monic orthogonal polynomials with respect to the weight $e^{-NV(t)}$. Therefore, it describes the distribution of the eigenvalues of the matrices belonging to the ensemble with weight function $e^{-NV(t)}$ as the size of the ensemble grows to infinity. It can also be defined as the unique minimizer of the functional

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \log|t-s|^{-1} d\mu(t)d\mu(s) + \int_{\mathbb{R}} V(t)d\mu(t)$$

among all Borel probability measures μ on $\mathbb{R}$. At the same time, it is uniquely determined by the following Euler-Lagrange variational conditions: there exists $\ell \in \mathbb{R}$ such that

$$\begin{aligned} &2\int \log|t-s|d\mu_V(s) - V(t) - \ell = 0, \qquad \text{for } t \in \operatorname{supp}\mu_V, \\ &2\int \log|t-s|d\mu_V(s) - V(t) - \ell < 0, \qquad \text{for } t \in \mathbb{R}\setminus \operatorname{supp}\mu_V. \end{aligned} \tag{4.32}$$

Lemma. *The equilibrium measure associated to the potential $V(t) = 4t$ is given by*

$$d\mu_V(t) = \frac{2}{\pi}\sqrt{\frac{1-t}{t}}dt, \qquad t \in (0,1). \tag{4.33}$$

Proof. We outline the proof in [166], which is itself based on the methods proposed in [173], where more details can be found. The monic orthogonal polynomials with respect to the re-scaled Laguerre weight $t^{\alpha}e^{-NV(t)}$ are

$$\widehat{\pi}_N(u) = \frac{1}{(4N)^N}\pi_N(4Nu),$$

where the π_N are the monic Laguerre polynomials, given by (4.28). It follows from (4.25) that if we denote the zeros of the polynomial $\widehat{\pi}_N(u)$ by $0 < x_{N,1} < \cdots < x_{N,N}$, we have that $\lim_{N\to\infty} x_{N,N} = 1$. We denote the normalized zero-counting measure associated to these polynomials by

$$d\mu_N(t) = \frac{1}{N}\sum_{k=1}^{N}\delta(t - x_{N,k})dt.$$

As mentioned above, these measures converge weakly to the sought equilibrium measure $d\mu_V$, so in particular we have

$$\frac{1}{N}\log\widehat{\pi}_N(u) = \int_0^{\infty}\log(u-t)d\mu_N(t) \to \int_0^1 \log(u-t)d\mu_V(t)$$

as $N\to\infty$. Therefore, after differentiating in both sides above we obtain

$$h_N(u) = \frac{1}{N}\frac{\widehat{\pi}'_N(u)}{\widehat{\pi}_N(u)} \to \int_0^1 \frac{d\mu_V(t)}{u-t} = h(u), \tag{4.34}$$

where the identities in the last equation serve as definitions. Now, it follows from (4.24) that the polynomials $\widehat{\pi}_N$ verify

$$u\widehat{\pi}_N(u)'' + (\alpha+1-4Nu)\widehat{\pi}_N(u)' + 4N^2\widehat{\pi}_N(u) = 0.$$

Substituting in (4.34) we see that the function h_N verifies itself a differential equation

$$uh'_N(u) + (\alpha+1-4Nu)h_N(u) + 4N + uNh_N^2(u) = 0.$$

Dividing by $4N$ in the above equation, letting $N \to \infty$ and comparing with the right hand side of (4.34) we find that the function $h(u)$ satisfies the following equation

$$\frac{u}{4}h^2(u) - uh(u) + 1 = 0,$$

from which we obtain

$$h(u) = 2 - 2\left(\frac{u-1}{u}\right)^{1/2}.$$

Using the Plemelj formula and comparing with the right hand side of (4.34) we arrive at the claimed expression for the equilibrium measure $d\mu_V$. □

Note that as a consequence of the re-scaling (4.30) and the behaviour of the zeros of the Laguerre polynomials, the equilibrium measure $d\mu_V$ is compactly supported. This is the main reason for introducing the rescaled model $\widehat{Z}^{v,m}_{LUE(N)}$, as it will simplify the subsequent analysis.

We also define some auxiliary functions, which properties will be key in the large N study of the model. First, we consider the g function

$$g(z) = \int_0^1 \log(z-t)d\mu_V(t). \tag{4.35}$$

Secondly, we extend the density of the measure μ_V to a meromorphic function on the complex plane $r(z) = \frac{2}{\pi}\left(\frac{z-1}{z}\right)^{1/2}$, with a branch cut on $[0,1]$. Lastly, we define

$$\xi(z) = 2\pi i \int_z^1 r(t)dt. \tag{4.36}$$

The following lemma follows from the definitions of the functions g and ξ, equation (4.32), and the jump properties of the logarithm and square root functions (see [190, 166] for instance, for more details).

Lemma. *The functions g and ξ verify the following properties.*

1. *The function g is analytic in $\mathbb{C}\setminus(-\infty,1]$, and satisfies*

$$g_+(x) - g_-(x) = \begin{cases} 2\pi i, & x<0, \\ 2\pi i \int_x^1 d\mu_V(t), & 0<x<1, \\ 0, & x>1, \end{cases} \tag{4.37}$$

as well as the variational conditions

$$\begin{aligned} g_+(x) + g_-(x) - V(x) - \ell = 0, &\qquad x \in supp\,\mu_V, \\ 2g(x) - V(x) - \ell < 0, &\qquad x \in \mathbb{R}\setminus supp\,\mu_V, \end{aligned} \tag{4.38}$$

2. *The function ξ is analytic in $\mathbb{C}\setminus(-\infty,1]$ and verifies*

$$\begin{aligned} \xi_+(x) = -\xi_-(x) = g_+(x) - g_-(x), &\qquad x \in (0,1), \\ \xi_+(x) - \xi_-(x) = 4\pi i, &\qquad x \in (-\infty,0), \end{aligned} \tag{4.39}$$

4.2.2 Schur polynomial expansions

Following the same strategy as in section 3.5, we can study the model $Z^{u,m}_{LUE(N)}$ in terms of averages of Schur polynomials over the Laguerre Unitary Ensemble. These can be computed using their Hankel minor representation, as follows.

Theorem 13. *Let λ be a partition of length $l(\lambda) \leq N$. We have*

$$\langle s_\lambda \rangle_{LUE(N)} = s_\lambda(1^N) \prod_{j=1}^{l(\lambda)} \frac{\Gamma(\alpha + N - j + \lambda_j + 1)}{\Gamma(\alpha + N - j + 1)}, \tag{4.40}$$

where the notation $\langle s \rangle_{LUE(N)}$ stands for the average of a symmetric function s over the Laguerre Unitary Ensemble.

Proof. According to (4.4), the insertion of the Schur polynomial indexed by a partition λ can be expressed as the Hankel minor

$$\det \begin{pmatrix} m_{\lambda_N} & m_{\lambda_N+1} & \cdots & m_{\lambda_N+N-1} \\ m_{\lambda_{N-1}+1} & m_{\lambda_{N-1}+2} & \cdots & m_{\lambda_{N-1}+N} \\ \vdots & \vdots & & \vdots \\ m_{\lambda_1+N-1} & m_{\lambda_1+N} & \cdots & m_{\lambda_1+2N-2} \end{pmatrix},$$

where

$$m_k = \int_0^\infty t^{k+\alpha} e^{-t} dt = \Gamma(\alpha + k + 1).$$

After extracting the factor $\Gamma(\alpha + \lambda_j + N - j + 1)$ from the j-th row of this determinant, for $j = 1, \ldots, N$, we are left with the determinant

$$\det \begin{pmatrix} 1 & \alpha + \lambda_N + 1 & (\alpha + \lambda_N + 1)(\alpha + \lambda_N + 2) & \cdots & (\alpha + \lambda_N + 1) \ldots (\alpha + \lambda_N + N - 1) \\ 1 & \alpha + \lambda_{N-1} + 2 & (\alpha + \lambda_{N-1} + 2)(\alpha + \lambda_{N-1} + 3) & \cdots & (\alpha + \lambda_{N-1} + 2) \ldots (\alpha + \lambda_N + N) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \alpha + \lambda_1 + N & (\alpha + \lambda_1 + N)(\alpha + \lambda_1 + N + 1) & \cdots & (\alpha + \lambda_1 + N) \ldots (\alpha + \lambda_1 + 2N - 2) \end{pmatrix}.$$

Performing elementary column operations, the above determinant can be reduced to a Vandermonde determinant on the points $\alpha + \lambda_j + N + 1 - j$, for $j = 1, \ldots, N$. Combining this fact with identity (2.11) we obtain

$$\frac{1}{N!} \int_0^\infty \cdots \int_0^\infty s_\lambda(t) \prod_{j<k} (t_j - t_k)^2 \prod_{j=1}^N t_j^\alpha e^{-t_j} dt_j = G(N+1) s_\lambda(1^N) \prod_{j=1}^N \Gamma(\alpha + \lambda_j + N - j + 1).$$

The proof is concluded after considering the quotient over the partition function of the Laguerre Unitary Ensemble $Z_{LUE(N)}$, which is given by (4.29). ☐

Note that a particular consequence of formula (4.40) is that the average $\langle s_\lambda \rangle_{LUE(N)}$ is a polynomial in α with integer coefficients, as well as a polynomial in N.

We can combine this result with equation (4.12) to expand the insertion of the characteristic polynomial in $Z^{u,m}_{LUE(N)}$ in terms of Schur polynomials. This leads to the formula

$$\frac{Z^{u,m}_{LUE(N)}}{Z_{LUE(N)}} = \sum_{\nu \subset ((2m)^N)} (-u)^{|\nu|} s_{L_{N,2m}(\nu')}(1^{2m}) \langle s_\nu \rangle_{LUE(N)}, \tag{4.41}$$

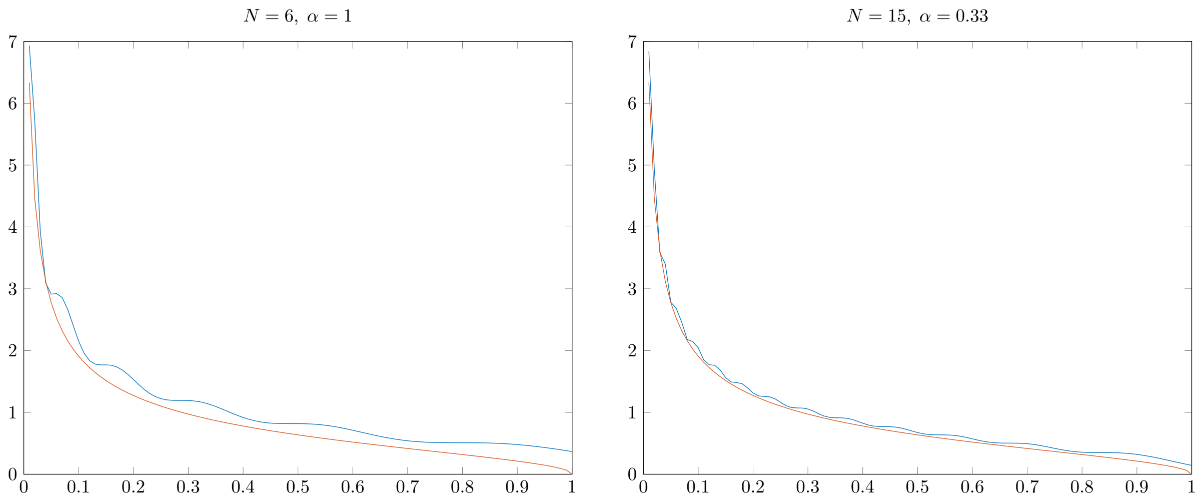


Figure 4.4: The re-scaled densities of states $\widetilde{\rho}_N$ of the Laguerre Unitary ensemble (4.43), for $N = 6$ and $N = 15$ and $\alpha = 1$ and $\alpha = 0.33$, respectively (in blue), and the density of the equilibrium measure $d\mu_V$ (4.33) (in red).

where $Z_{LUE(N)}$ is given by (4.29) and $L_{N,m}(\nu')$ is the partition defined in (2.17). The above formula can be implemented in a computer algebra system, providing quick evaluations of the model as long as its size N is not too big. As an example and consistency check, this formula can be used to computed the density of states of the Laguerre Unitary Ensemble. Recall that the normalized density of states is given by [154]

$$\rho_N(t) = \frac{1}{N} t^\alpha e^{-t} \sum_{k=0}^{N-1} p_k^2(t) = \frac{1}{N} t^\alpha e^{-t} \gamma_{N-1}^2 Z_{LUE(N)}^{u,m=1}, \tag{4.42}$$

where the p_k are the orthonormal Laguerre polynomials, the γ_N are defined in (4.6), and the second identity follows from equation (4.9). Integrating this function over a subset of the real line one recovers the normalized expected number of eigenvalues of the ensemble of size N to be found on this subset. In particular, the density of states converges as $N \to \infty$ to the density of the equilibrium measure $d\mu_V$, after the re-scaling

$$\widehat{\rho}_N(t) = 4N\rho_N(4Nt), \tag{4.43}$$

in order to make the limit function $\lim_{N\to\infty} \widehat{\rho}_N$ compactly supported as well. Figure 4.4 shows two instances of the re-scaled densities, computed by means of formulas (4.42) and (4.41), for several values of N and α, together with the density of the equilibrium measure $d\mu_V$. The convergence is apparent already at low values of N, whenever the size of the parameter α is not too big compared to N.

4.3 Large N analysis

We are now ready to solve the Riemann-Hilbert problem for the orthogonal polynomials with respect to the modified weight $\widehat{w}_{v,m}$, given by (4.31). To be precise, we consider the following restatement of the problem introduced in section 4.1.3: we seek $Y : \mathbb{C} \setminus [0, \infty) \to \mathbb{C}^{2\times 2}$ verifying the following conditions.

Riemann-Hilbert problem for Y

1. Y is analytic in $\mathbb{C} \setminus [0, \infty)$.
2. For $x \in (0, \infty)$, the matrix Y verifies the jump condition

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & (x-v)^{2m} x^\alpha e^{-4Nx} \\ 0 & 1 \end{pmatrix}$$

3. As $z \to \infty$, we have

$$Y(z) = \left(I + \mathcal{O}(z^{-1})\right) z^{N\sigma_3}. \tag{4.44}$$

4. As $z \to 0$, we have

$$Y(z) = \begin{cases} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}, & \alpha > 0, \\ \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log z) \\ \mathcal{O}(1) & \mathcal{O}(\log z) \end{pmatrix}, & \alpha = 0, \\ \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(z^\alpha) \\ \mathcal{O}(1) & \mathcal{O}(z^\alpha) \end{pmatrix}, & \alpha < 0. \end{cases}$$

This normalizing condition follows from equation (4.14), in sight of the particular form of the weight $\widehat{w}_{v,m}$, and is chosen to ensure uniqueness of the solution of the Riemann-Hilbert problem. Note that there is no need to impose any conditions at the point v, as the fact that m is a positive integer implies that $Y(z)$ is bounded around this point.

Before proceeding, let us remark that, besides the orthogonal polynomials with respect to the weight $\widehat{w}_{v,m}$, also the matrix model $\widehat{Z}^{v,m}_{LUE(N)}$ can be expressed in terms of the entries of the matrix Y. In order to show this, we need to introduce some notation for the subleading coefficient of the orthonormal polynomials with respect to $\widehat{w}_{v,m}$, say

$$p_N(u) = \gamma_N(u^N + \eta_N u^{N-1} + \dots). \tag{4.45}$$

Lemma. *The Hankel determinant $\widehat{Z}^{v,m}_{LUE(N)}$ satisfies*

$$\begin{aligned} \frac{d}{dm} \log \widehat{Z}^{v,m}_{LUE(N)} = &- (N + \alpha + 2m) \frac{d}{dm} \log(\gamma_{N-1}\gamma_N) + 4N \frac{d}{dm} \eta_N \\ &+ \alpha \left((Y^{-1} \frac{d}{dm} Y)_{11}(0) + Y_{11}(0) Y_{22}(0) \frac{d}{dm} \log(\gamma_{N-1}\gamma_N) \right) \\ &+ 2m \left((Y^{-1} \frac{d}{dm} Y)_{11}(v) + Y_{11}(v) Y_{22}(v) \frac{d}{dm} \log(\gamma_{N-1}\gamma_N) \right). \end{aligned} \tag{4.46}$$

Proof. The proof is analogous to the one[35] in [136]. Performing column operations in the Vandermonde determinants in (4.2) and using Andreiéf's identity one obtains the well known relation

$$\widehat{Z}^{v,m}_{LUE(N)} = \prod_{j=0}^{N-1} \gamma_j^{-2}, \tag{4.47}$$

[35] Note, however, that we do not need to consider the regularized integrals of [136], due to our assumptions on the parameters α and m.

where the γ_j are given by (4.45). Using the orthogonality relation (4.5) we obtain

$$\begin{aligned}\frac{d}{dm}\log \widehat{Z}^{v,m}_{LUE(N)} &= -2\sum_{j=0}^{N-1}\frac{\frac{d}{dm}\gamma_j}{\gamma_j} = -2\sum_{j=0}^{N-1}\int_0^\infty p_j(t)\left(\frac{d}{dm}p_j(t)\right)\widehat{w}_{v,m}(t)dt\\ &= -\frac{\gamma_{N-1}}{\gamma_N}\int_0^\infty \frac{d}{dm}\left(p_{N-1}(t)p_N'(t)-p_N(t)p_{N-1}'(t)\right)\widehat{w}_{v,m}(t)dt,\end{aligned}$$

where the last identity above follows from the Christoffel-Darboux formula. We use prime and dot notations for the derivatives with respect to t and m, respectively, for the remainder of the proof. Using the orthogonality condition (4.5), we see that the last integral above can be computed as follows

$$\begin{aligned}\frac{d}{dm}\log \widehat{Z}^{v,m}_{LUE(N)} &= -N\frac{\dot{\gamma_{N-1}}}{\gamma_{N-1}}+\frac{\gamma_{N-1}}{\gamma_N}\int_0^\infty\left(\dot{p}_N(t)p_{N-1}'(t)-p_N'(t)\dot{p}_{N-1}(t)\right)\widehat{w}_{v,m}(t)dx\\ &= -N\left(\frac{\dot{\gamma_{N-1}}}{\gamma_{N-1}}+\frac{\dot{\gamma_N}}{\gamma_N}\right)+\frac{\gamma_{N-1}}{\gamma_N}\int_0^\infty\left(p_N(t)\dot{p}_{N-1}(t)-\dot{p}_N(t)p_{N-1}(t)\right)\widehat{w}_{v,m}'(t)dt,\end{aligned}$$

where the last identity above follows from integration by parts. We see that the resulting integral can be split as the sum of three integrals. In each of these integrals the term between parentheses in the last integral above multiplies the factors

$$\alpha\frac{\widehat{w}_{v,m}(t)}{t},\qquad 2m\frac{\widehat{w}_{v,m}(t)}{t-v},\qquad -4N\widehat{w}_{v,m}(x).$$

Note that all these integrals are convergent, due to the assumptions on the parameters α and m. In order to compute the first one, we replace the term between parentheses by

$$p_N(t)\dot{p}_{N-1}(t)-p_N(t)\dot{p}_{N-1}(0)+p_N(t)\dot{p}_{N-1}(0)-\dot{p}_N(t)p_{N-1}(t)+\dot{p}_N(0)p_{N-1}(t)-\dot{p}_N(0)p_{N-1}(t),$$

without changing its value. Using the orthogonality properties of the polynomials p_N, we see that this integral evaluates to

$$\alpha\frac{\gamma_{N-1}}{\gamma_N}\left(-\frac{\dot{\gamma}_N}{\gamma_{N-1}}+2\pi i\dot{p}_{N-1}(0)\mathcal{C}(p_N\widehat{w}_{v,m})(0)-2\pi i\dot{p}_N(0)\mathcal{C}(p_{N-1}\widehat{w}_{v,m})(0)\right).$$

The second of the integrals can be computed following the same procedure, while the third one can be evaluated directly with aid of the orthogonality condition (4.5). Using (4.14) and the fact that $1=\det Y(z)=Y_{11}(z)Y_{22}(z)-Y_{12}(z)Y_{21}(z)$ we arrive at the desired conclusion. □

Due to its technical nature, providing an accessible and at the same time fully rigorous account of the Riemann-Hilbert methodology represents a task that lies outside the scope of this thesis. We choose to prioritize clarity in the following, and develop in detail the parts of the analysis that we believe to be more enlightening. We point to [190, 136, 166, 52, 53] for works concerned with the study of similar models, where more details can be found.

4.3.1 Transformations of the problem and global parametrix

We can now start with the series of transformations for the matrix Y described in section 4.1.3. For the first transformation, we recall the definition of the g function given in (4.35), as well as the variational conditions (4.32). With this, we define

$$T(z)=e^{-\frac{N\ell}{2}\sigma_3}Y(z)e^{-N(g(z)-\frac{\ell}{2})\sigma_3},$$

Using the definition of g and equations (4.37) and (4.38), we see that T solves the following problem.

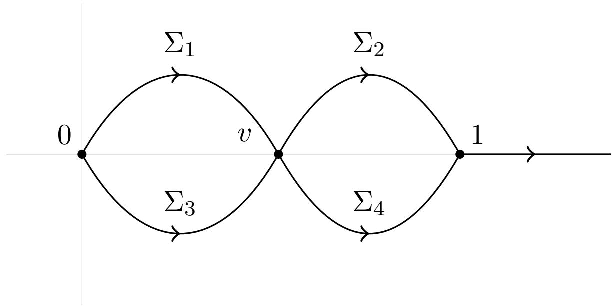


Figure 4.5: Lenses for the jumps of the matrix $S(z)$.

Riemann-Hilbert problem for T

1. T is analytic in $\mathbb{C} \setminus [0, \infty)$.
2. For $x \in (0, \infty)$ we have
$$T_+(x) = T_-(x) \begin{cases} \begin{pmatrix} e^{-N(g_+(x)-g_-(x))} & (x-v)^{2m}x^\alpha \\ 0 & e^{N(g_+(x)-g_-(x))} \end{pmatrix}, & 0 < x < 1, \\ \begin{pmatrix} 1 & (x-v)^{2m}x^\alpha e^{-N(-2g(x)+V(x)+\ell)} \\ 0 & 1 \end{pmatrix}, & 1 < x. \end{cases}$$
3. As $z \to \infty$, we have
$$T(z) = I + \mathcal{O}(z^{-1}).$$
4. As $z \to 0$, the matrix $T(z)$ has the same behaviour as $U(z)$.

The purpose of this transformation is to normalize the problem at infinity. Note that T is now asymptotically close to the identity matrix as $z \to \infty$, and more importantly, this is achieved without creating new singularities at other points of the plane. This is due to the fact that as $z \to \infty$
$$e^{Ng(z)} = z^N \left(1 - \frac{N}{4z} + \mathcal{O}(z^{-2})\right). \tag{4.48}$$

For the next transformation, known as nonlinear steepest descent [72], we choose four oriented contours on the complex plane joining the points $0, v$ and 1 of the form depicted in figure 4.5. By means of this transformation, we factorize the jump matrix for T into a product of three matrices, each of them having jumps on one of the chosen contours, or on the interval $[0, 1]$. The advantage of this factorization is that the jumps on the contours will be asymptotically close to the identity (outside some small neighbourhoods around the points 0 and 1), and the remaining problem on the interval $[0, 1]$ will have a solution that can be constructed explicitly. This process is known as opening lenses; we will call the contours, which are denoted by Σ_j in figure 4.5, the lips of the lenses.

We define
$$S(z) = T(z) \begin{cases} I, & z \text{ outside the lenses,} \\ \begin{pmatrix} 1 & 0 \\ -z^{-\alpha}(z-v)^{-2m}e^{-N\xi(z)} & 1 \end{pmatrix}, & z \text{ in the upper part of the lenses,} \\ \begin{pmatrix} 1 & 0 \\ z^{-\alpha}(z-v)^{-2m}e^{-N\xi(z)} & 1 \end{pmatrix}, & z \text{ in the lower part of the lenses.} \end{cases}$$

Note that we are not making a particular choice for the lips Σ_j. The precise contours depend on the local parametrices at the points $0, v$ and 1 and will be specified later, see section 4.3.2. We see that S solves the following problem.

Riemann-Hilbert problem for S

1. S in analytic in $\mathbb{C} \setminus ([0,\infty) \cup \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4)$.
2. The matrix S has the following jumps

$$S_+(z) = S_-(z) \begin{cases} \begin{pmatrix} 0 & x^{\alpha}(x-v)^{2m} \\ -x^{-\alpha}(x-v)^{-2m} & 0 \end{pmatrix}, & 0 < x < 1, \\ \begin{pmatrix} 1 & x^{\alpha}(x-v)^{2m} e^{-N(-2g(x)+V(x)+\ell)} \\ 0 & 1 \end{pmatrix}, & 1 < x, \\ \begin{pmatrix} 1 & 0 \\ z^{-\alpha}(z-v)^{-2m} e^{-N\xi(z)} & 1 \end{pmatrix}, & z \in \Sigma_j. \end{cases}$$

3. The function $S(z)$ has the same behaviour as $T(z)$ as $z \to \infty$.
4. The function $S(z)$ has the same behaviour as $T(z)$ as $z \to 0$ from outside the lenses. As $z \to 0$ from inside the lenses, we have

$$S(z) = \begin{cases} \begin{pmatrix} \mathcal{O}(z^{-\alpha}) & \mathcal{O}(1) \\ \mathcal{O}(z^{-\alpha}) & \mathcal{O}(1) \end{pmatrix}, & \alpha > 0, \\ \begin{pmatrix} \mathcal{O}(\log z) & \mathcal{O}(\log z) \\ \mathcal{O}(\log z) & \mathcal{O}(\log z) \end{pmatrix}, & \alpha = 0, \\ \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(z^{\alpha}) \\ \mathcal{O}(1) & \mathcal{O}(z^{\alpha}) \end{pmatrix}, & \alpha < 0. \end{cases}$$

Note that the product of the jump matrices of S on the upper contours, the interval $(0,1)$ and the lower contours recovers the jump matrix of T on $(0,1)$, as mentioned above.

As we will see below, the properties of the functions g and ξ imply that the jumps of S on the interval $(1,\infty)$ and on the contours Σ_j converge to the identity as $N \to \infty$. We can thus approximate the solution of the Riemann-Hilbert problem for S outside these disks by the solution to the following problem, usually called the global parametrix.

Riemann-Hilbert problem for $P^{(\infty)}$

1. $P^{(\infty)}$ is analytic in $\mathbb{C} \setminus [0,1]$.
2. For $x \in (0,1)$, we have

$$P_+^{(\infty)}(x) = P_-^{(\infty)}(x) \begin{pmatrix} 0 & (x-v)^{2m} x^{\alpha} \\ -(x-v)^{-2m} x^{-\alpha} & 0 \end{pmatrix}.$$

3. As $z \to \infty$, we have $P^{(\infty)}(z) = I + \mathcal{O}(z^{-1})$.

The standard procedure to build the solution to the Riemann-Hilbert problem for $P^{(\infty)}$ is to consider the Szegő function associated to the function $(x-v)^{2m}x^{\alpha}$ on the interval $[0,1]$. This is a function $D(z)$, analytic and non-zero on $\mathbb{C}\setminus[0,1]$, such that

$$D_{+}(x)D_{-}(x)=(x-v)^{2m}x^{\alpha}, \qquad \text{for } t\in(0,1), \tag{4.49}$$

and such that the limit $\lim_{z\to\infty}D(z)$ exists and is a positive real number.

Lemma. *The Szegő function associated to $(x-v)^{2m}x^{\alpha}$ on the interval $[0,1]$ is*

$$D(z)=\frac{(z-v)^{m}z^{\frac{\alpha}{2}}}{\varphi(z)^{(m+\frac{\alpha}{2})}} \tag{4.50}$$

where φ is a conformal map from $\mathbb{C}\setminus[0,1]$ onto the exterior of the unit circle, which is given by

$$\varphi(z)=2z-1+2(z(z-1))^{1/2}.$$

Proof. It follows from the definition of φ that this function takes negative values on the negative real axis. Therefore, we see that D is analytic (and non-zero) on $\mathbb{C}\setminus[0,1]$. The jump condition (4.49) follows from the fact that $\varphi_{+}(x)\varphi_{-}(x)=1$ on $(0,1)$. Lastly, we have

$$D_{\infty}=\lim_{z\to\infty}D(z)=4^{-\left(m+\frac{\alpha}{2}\right)}>0. \tag{4.51}$$

□

We can now provide an explicit expression for $P^{(\infty)}$. The Riemann-Hilbert problem on the interval $[0,1]$ with jump matrix $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ has a well known explicit solution [66], which is given by

$$M(z)=\frac{1}{2}\begin{pmatrix}\gamma(z)+\gamma(z)^{-1} & -i(\gamma(z)-\gamma(z)^{-1})\\ i(\gamma(z)-\gamma(z)^{-1}) & \gamma(z)+\gamma(z)^{-1}\end{pmatrix}, \qquad \text{where } \gamma(z)=\left(\frac{z-1}{z}\right)^{1/4}. \tag{4.52}$$

Therefore, it follows from the properties of the Szegő function that the unique solution of the Riemann-Hilbert problem for $P^{(\infty)}$ is given by

$$P^{(\infty)}(z)=D_{\infty}^{\sigma_3}M(z)D(z)^{-\sigma_3}. \tag{4.53}$$

However, we need to take into account the following consideration. Even if the power of the characteristic polynomial insertion in $\widehat{w}_{v,m}$ is only allowed to be an even integer, we do need an asymptotic expansion for Y that is valid for more general values of m. This is due to the fact that we need to integrate this function over a whole range of the parameter m in order to recover the model $\widehat{Z}^{v,m}_{LUE(N)}$, as shown by the differential identity (4.46). Allowing m to be a general parameter introduces also a jump type singularity in the weight function (see footnote 34), which would require a more involved analysis. We can bypass this obstacle by noting that the Szegő function associated to the function $|x-v|^{2m}x^{\alpha}$ on the interval $[0,1]$ is also given by (4.50), for any (positive) value of m. Since the model corresponding to this function coincides with $\widehat{Z}^{v,m}_{LUE(N)}$ for integer values of m, we can proceed with the Szegő funcion (4.50), thus avoiding the need to consider jump type singularities. Some inspection shows that the analysis for both of the models is identical for the moment; we will introduce and comment the required modifications in the following.

4.3.2 Local parametrices

As mentioned above, the matrix $P^{(\infty)}$ provides a good approximation to the solution of the Riemann-Hilbert problem for S, in the sense that the matrix $S(z)\left(P^{(\infty)}(z)\right)^{-1}$ converges to the identity as $N \to \infty$. This holds everywhere, except in a neighbourhood of the points $0, v$ and 1, due to the singularities of $P^{(\infty)}$ at these points, in sight of (4.53). We thus need to consider additional problems around each of these points, known as local parametrices. These will be built in terms of the model Riemann-Hilbert problems reviewed in section 4.1.3. It is at this point of the construction that we will fix the specific choice of contours Σ_j.

We start with the local parametrix at the point 1. We consider a disk $D(1,\delta_1)$ for some small fixed δ_1, which will be specified later. We consider the following problem in this disk.

Riemann-Hilbert problem for $P^{(1)}$

1. $P^{(1)}(z)$ is analytic in $D(1,\delta_1)\setminus((1-\delta_1,1+\delta_1)\cup\Sigma_2\cup\Sigma_4)$.
2. $P^{(1)}(z)$ has the same jumps inside $D(1,\delta_1)$ as $S(z)$.
3. Uniformly for $z\in\partial D(1,\delta_1)$ we have

$$P^{(1)}(z) = (I+\mathcal{O}(N^{-1}))P^{(\infty)}(z).$$

The strategy now is to transform this problem into one with constant jump matrices. We will then identify these jumps with those of a Riemann-Hilbert problem with a known solution. The composition of this solution with a conformal mapping gives a solution for the problem for $P^{(1)}$ on the disk $D(1,\delta_1)$, apart from a suitable modification to take account of the matching condition (3). We start by defining the matrix $\widehat{P}^{(1)}$ as follows

$$P^{(1)}(z) = \widehat{P}^{(1)}(z)e^{-\frac{N\xi(z)}{2}\sigma_3}(z-v)^{-m\sigma_3}z^{-\frac{\alpha}{2}\sigma_3}. \tag{4.54}$$

Using the properties of the function ξ (4.39), we find that $\widehat{P}^{(1)}$ solves the following problem.

Riemann-Hilbert problem for $\widehat{P}^{(1)}$

1. $\widehat{P}^{(1)}(z)$ is analytic in $D(1,\delta_1)\setminus((1-\delta_1,1+\delta_1)\cup\Sigma_2\cup\Sigma_4)$.
2. $\widehat{P}^{(1)}(z)$ verifies:

$$\widehat{P}^{(1)}(z)_+ = \widehat{P}^{(1)}(z)_- \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & 1-\delta_1 < z < 1, \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & 1 < z < 1+\delta_1, \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & z \in (\Sigma_2\cup\Sigma_4)\cap D(1,\delta_1). \end{cases} \tag{4.55}$$

3. Uniformly for $z\in\partial D(1,\delta_1)$ we have

$$\widehat{P}^{(1)}(z) = (I+\mathcal{O}(N^{-1}))P^{(\infty)}(z)z^{\frac{\alpha}{2}\sigma_3}(z-v)^{m\sigma_3}e^{\frac{N\xi(z)}{2}\sigma_3}. \tag{4.56}$$

We see that the jumps of the matrix $\widehat{P}^{(1)}$ coincide with those of the Airy model Riemann-Hilbert problem described in section 4.1.3. The idea now is to compose the solution of this problem, which has an explicit expression, with a locally conformal map from a neighbourhood of 1 onto a neighbourhood of 0, so that the contours of the jumps of this model problem are mapped to the disk $D(1,\delta_1)$. More precisely, we seek a solution for this problem of the form

$$\widehat{P}^{(1)}(z) = E_1(z)\Psi_{\mathrm{Ai}}(\zeta_1(z)), \tag{4.57}$$

where E_1 is a function analytic in a neighbourhood of 1 that will be specified later, Ψ_{Ai} is the solution of the Airy model problem, and ζ_1 is the aforementioned locally conformal map. In order to construct this map, we compare the asymptotic behaviour of Ψ_{Ai} as $z \to \infty$, given by (4.17), with the definition of $\widehat{P}^{(1)}$, and choose ζ_1 to compensate for the exponential factor in (4.54). That is, we set

$$\zeta_1(z) = \left(-\frac{3N}{4}\xi(z)\right)^{\frac{2}{3}}, \qquad z \in D(1,\delta_1) \setminus (1-\delta_1, 1]. \tag{4.58}$$

It follows from (4.36) that ζ_1 is indeed a locally conformal map from a neighbourhood of 1 onto a neighbourhood of 0, as desired. Therefore, we can now choose δ_1 small enough so that ζ_1 is conformal in the whole disk $D(1,\delta_1)$. We also set now the lips of the right lens Σ_2 and Σ_4 to be the preimages of the rays $e^{2\pi i/3}\mathbb{R}^+$ and $e^{2\pi i/3}\mathbb{R}^-$ under the map ζ_1 (more precisely, we set the parts of the lips that lie inside the disk $D(1,\delta_1)$ to be the preimages of the pieces of the rays that lie inside $\zeta_1(D(1,\delta_1))$). It follows from this construction that the matrix $\Psi_{\mathrm{Ai}}(\zeta_1(z))$ has the jumps specified in (4.55).

We still need to take care of the matching condition (4.56). To this end, note that we have not specified the choice of the analytic function E_1 introduced in (4.54). Another effect of composing the solution of the Airy problem with ζ_1 is that as $N \to \infty$ the asymptotic behaviour (4.17) is attained at the boundary of the disk $D(1,\delta_1)$. Comparing this behaviour with the matching (4.56), we see that the appropriate choice is

$$E_1(z) = P^{(\infty)}(z) z^{\frac{\alpha}{2}\sigma_3}(z-v)^{m\sigma_3}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\zeta_1(z)^{\sigma_3/4}.$$

As desired, this matrix is analytic in a neighbourhood of 1. Indeed, it follows from the jump condition for $P^{(\infty)}$ and the fact that $\zeta_1(x)_+^{1/4} = \zeta_1(x)_-^{1/4}e^{i\pi/2}$ for $x \in (1-\delta_1, 1)$ that E_1 has no jumps on $D(1,\delta_1)$. Moreover, the singularity of E_1 at the point 1 is at most of square-root type (recall the explicit construction of $P^{(\infty)}$, given by (4.53)), and therefore removable.

Hence, we arrive at the conclusion that the solution to the Riemann-Hilbert problem for $P^{(1)}$ is given by

$$P^{(1)}(z) = E_1(z)\Psi_{\mathrm{Ai}}(\zeta_1(z))e^{-\frac{N\xi(z)}{2}\sigma_3}(z-v)^{m\sigma_3}z^{-\frac{\alpha}{2}\sigma_3}.$$

Let us now construct a local parametrix in a neighbourhood of 0, following an analogous procedure to the one done for the parametrix around 1. We consider a small disk $D(0,\delta_0)$, for some δ_0 that will be fixed later.

Riemann-Hilbert problem for $P^{(0)}$

1. $P^{(0)}(z)$ is analytic in $D(0,\delta_0) \setminus ([0,\delta_0) \cup \Sigma_1 \cup \Sigma_3)$.
2. $P^{(0)}(z)$ has the same jumps inside $D(0,\delta_0)$ as $S(z)$.

3. Uniformly for $z \in \partial D(0,\delta_0)$ we have

$$P^{(0)}(z) = (I + \mathcal{O}(N^{-1}))P^{(\infty)}(z).$$

4. $P^{(0)}$ has the same behaviour at 0 as S.

As before, we can transform this problem into one with constant jump matrices by means of the following transformation

$$P^{(0)}(z) = \widehat{P}^{(0)}(z)e^{-\frac{N\xi(z)}{2}\sigma_3}(v-z)^{-m\sigma_3}(-z)^{-\frac{\alpha}{2}\sigma_3}. \tag{4.59}$$

Note that the exponential factor does not introduce new jumps on $(-\delta_0, 0)$, in sight of (4.39). It thus follows from the properties of the function ξ that $\widehat{P}^{(0)}$ solves the following Riemann-Hilbert problem.

Riemann-Hilbert problem for $\widehat{P}^{(0)}$

1. $\widehat{P}^{(0)}(z)$ is analytic in $D(0,\delta_0) \setminus ([0,\delta_0) \cup \Sigma_1 \cup \Sigma_3)$.
2. $\widehat{P}^{(0)}(z)$ has the following jumps

$$\widehat{P}^{(0)}(z)_+ = \widehat{P}^{(0)}(z)_- \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & 0 < z < \delta_0, \\ \begin{pmatrix} 1 & 0 \\ e^{-i\pi\alpha} & 1 \end{pmatrix}, & z \in \Sigma_1 \cap D(0,\delta_0), \\ \begin{pmatrix} 1 & 0 \\ e^{i\pi\alpha} & 1 \end{pmatrix}, & z \in \Sigma_3 \cap D(0,\delta_0). \end{cases}$$

3. Uniformly for $z \in \partial D(0,\delta_0)$ we have

$$\widehat{P}^{(0)}(z) = (I + \mathcal{O}(N^{-1}))P^{(\infty)}(z)(-z)^{\frac{\alpha}{2}\sigma_3}(v-z)^{m\sigma_3}e^{\frac{N\xi(z)}{2}\sigma_3}. \tag{4.60}$$

4. As $z \to 0$, the matrix $\widehat{P}^{(0)}$ has the following behaviour

$$\widehat{P}^{(0)}(z) = \begin{cases} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix} z^{\frac{\alpha}{2}\sigma_3}, & z \text{ outside the lens}, \\ \begin{pmatrix} \mathcal{O}(z^{-\frac{\alpha}{2}}) & \mathcal{O}(z^{-\frac{\alpha}{2}}) \\ \mathcal{O}(z^{-\frac{\alpha}{2}}) & \mathcal{O}(z^{-\frac{\alpha}{2}}) \end{pmatrix}, & z \text{ inside the lens}, \end{cases} \qquad \text{if } \operatorname{Re}\alpha > 0,$$

$$\widehat{P}^{(0)}(z) = \begin{cases} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log z) \\ \mathcal{O}(1) & \mathcal{O}(\log z) \end{pmatrix}, & z \text{ outside the lens}, \\ \begin{pmatrix} \mathcal{O}(\log z) & \mathcal{O}(\log z) \\ \mathcal{O}(\log z) & \mathcal{O}(\log z) \end{pmatrix}, & z \text{ inside the lens}, \end{cases} \qquad \text{if } \operatorname{Re}\alpha = 0,$$

$$\widehat{P}^{(0)}(z) = \begin{pmatrix} \mathcal{O}(z^{\frac{\alpha}{2}}) & \mathcal{O}(z^{\frac{\alpha}{2}}) \\ \mathcal{O}(z^{\frac{\alpha}{2}}) & \mathcal{O}(z^{\frac{\alpha}{2}}) \end{pmatrix}, \qquad \text{if } \operatorname{Re}\alpha < 0.$$

Note that both the jumps of the matrix $\widehat{P}^{(0)}$ and its behaviour at 0 coincide with those of the Bessel model Riemann-Hilbert problem. As before, we compose the explicit solution of this problem $\Psi_{\rm Be}$ with a locally conformal map that sends the jump contours of $\widehat{P}^{(0)}$ to those of $\Psi_{\rm Be}$. That is, we seek a solution of the form

$$\widehat{P}^{(0)}(z) = E_0(z)\Psi_{\rm Be}(\zeta_0(z)), \tag{4.61}$$

where ζ_0 denotes the desired conformal map and E_0 is an analytic prefactor, to be determined later. In sight of the exponential factor in (4.59) and the asymptotic behaviour (4.19), we see that a suitable choice for this mapping could be the function $\left(\frac{N}{4}\xi(z)\right)^2$. However, while being conformal, it follows from (4.36) that small neighbourhoods of 0 are mapped to neighbourhoods of 1 under the action of this map. We can nevertheless remedy this situation as follows. Consider the function

$$\widetilde{\xi}(z) = 2\pi i \int_z^0 r(s)ds, \tag{4.62}$$

where r is the analytic extension of the density of the equilibrium measure, see (4.36). It follows from the fact that $d\mu_V$ is a probability measure that $\xi(z) = \widetilde{\xi}(z) \pm 2\pi i$ on $\mathbb{C}\setminus(-\infty, 1]$. Therefore, we see that if we set

$$\zeta_0(z) = \left(\frac{N}{4}\widetilde{\xi}(z)\right)^2,$$

we obtain

$$e^{2\zeta_0^{1/2}(z)} = (-1)^N e^{\frac{N\widetilde{\xi}(z)}{2}}.$$

Now we have that the map ζ_0 is locally conformal, and maps neighbourhoods of 0 onto neighbourhoods of 0, as follows from (4.62). Moreover, the exponential factors in the asymptotic behaviour of the functions under interest are still compensated, up to the prefactor $(-1)^N$. However, this term can be included in the analytic function E_0, which has not been fixed yet. Indeed, it follows from the asymptotic behaviour (4.19) and (4.60) that choosing

$$E_0(z) = (-1)^N P^{(\infty)}(z)(-z)^{\frac{\alpha}{2}\sigma_3}(v-z)^{m\sigma_3}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}(2\pi\zeta_0^{1/2}(z))^{\frac{1}{2}\sigma_3}, \tag{4.63}$$

the matching condition is verified. As above, one can also check that the matrix E_0 is analytic in a neighbourhood of 0. We can also fix now the choice of δ_0 and of the lips Σ_1 and Σ_3: we choose them so that ζ_0 is conformal in the whole disk $D(0, \delta_0)$, and such that the parts of the lips that lie inside this disk coincide with the preimages of the rays $e^{2\pi i/3}\mathbb{R}^{\pm}$ under ζ_0. Some inspection confirms that the resulting matrix

$$P^{(0)}(z) = E_0(z)\Psi_{\rm Be}(\zeta_0(z))e^{-\frac{N\xi(z)}{2}\sigma_3}(v-z)^{-m\sigma_3}(-z)^{-\frac{\alpha}{2}\sigma_3} \tag{4.64}$$

solves indeed the Riemann-Hilbert problem for $P^{(0)}$.

Lastly, we construct the local parametrix at the point v. The procedure is analogous to the previous cases, and the solution will be expressed now in terms of the confluent hypergeometric model Riemann-Hilbert problem. We consider a small disk $D(v, \delta_v)$ for some δ_v to be determined later.

Riemann-Hilbert problem for $P^{(v)}$

1. $P^{(v)}(z)$ is analytic in $D(v, \delta_v)\setminus((v-\delta_v, v+\delta_v)\cup\Sigma_1\cup\Sigma_2\cup\Sigma_3\cup\Sigma_4)$.
2. $P^{(v)}(z)$ has the same jumps as $S(z)$ inside $D(v, \delta_v)$.

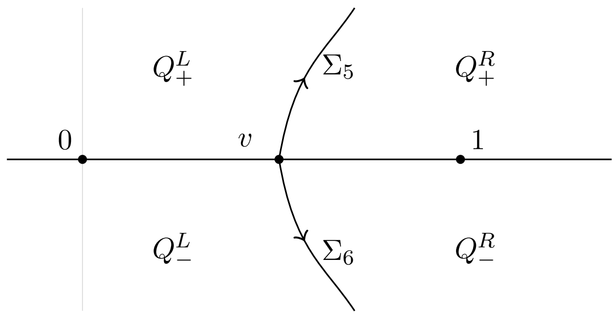


Figure 4.6: The contours and quadrants introduced in the definition of $W(z)$ (4.65).

3. Uniformly for $z \in \partial D(v, \delta_v)$ we have

$$P^{(v)}(z) = (I + \mathcal{O}(N^{-1}))P^{(\infty)}(z).$$

We follow the same approach as in the previous two cases. Nevertheless, a more subtle situation arises, as in the construction of the global parametrix. Recall that we are interested in obtaining an asymptotic expression valid for a whole range of the parameter m, due to the differential identity (4.46). When constructing the global parametrix, this situation was solved by noticing that the analysis for the model with weight function $|t-v|^{2m}t^{\alpha}e^{-4Nt}$ was identical to that of the model $\widehat{Z}^{v,m}_{LUE(N)}$. While this is still true for the local parametrices at 0 and 1, as can be seen from the explicit expressions of $P^{(0)}$ and $P^{(1)}$, the local parametrix at v needs to be constructed taking into account this modification. We use the same strategy as before, and think of the local parametrix as that associated to the model with weight function $|t-v|^{2m}t^{\alpha}e^{-4Nt}$ instead, where m need not be an integer anymore. Following [100, 52], we consider another contour on the complex plane, which will be fixed later, intersecting the real axis at the point v, and denote by Σ_5 and Σ_6 the parts of it that lie in the upper or lower half plane, respectively. We orient these contours away from the point v, and label the resulting four quadrants in the complex plane by $Q^L_\pm$ and $Q^R_\pm$, as depicted in figure 4.6. With this, we define the following extension of the function $(z-v)^m$ to the complex plane

$$W(z) = \begin{cases} (z-v)^m e^{-i\pi m}, & z \in Q^R_+, \\ (z-v)^m e^{i\pi m}, & z \in Q^R_-, \\ (v-z)^m e^{i\pi m}, & z \in Q^L_+, \\ (v-z)^m e^{-i\pi m}, & z \in Q^L_-, \end{cases} \tag{4.65}$$

where m is a general positive parameter.

We can now proceed as in the previous parametrices. First, we transform the problem into one with constant jump matrices by setting

$$P^{(v)}(z) = \widehat{P}^{(v)}(z)e^{-\frac{N\xi(z)}{2}\sigma_3}W(z)^{-\sigma_3}z^{-\frac{\alpha}{2}\sigma_3}. \tag{4.66}$$

We find that $\widehat{P}^{(v)}$ solves the following Riemann-Hilbert problem.

Riemann-Hilbert problem for $\widehat{P}^{(v)}$

1. $\widehat{P}^{(v)}(z)$ is analytic in $D(v,\delta_v) \setminus ((v-\delta_v, v+\delta_v) \cup \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4 \cup \Sigma_5 \cup \Sigma_6)$.

2. $\widehat{P}^{(v)}(z)$ has the following jumps

$$\widehat{P}^{(v)}(z)_+ = \widehat{P}^{(v)}(z)_- \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & v-\delta_v < z < v+\delta_v, \\ \begin{pmatrix} 1 & 0 \\ e^{2\pi i m} & 1 \end{pmatrix}, & z \in (\Sigma_1 \cup \Sigma_4) \cap D(v,\delta_v), \\ \begin{pmatrix} 1 & 0 \\ e^{-2\pi i m} & 1 \end{pmatrix}, & z \in (\Sigma_2 \cup \Sigma_3) \cap D(v,\delta_v), \\ \begin{pmatrix} e^{i\pi m} & 0 \\ 0 & e^{-i\pi m} \end{pmatrix}, & z \in (\Sigma_5 \cup \Sigma_6) \cap D(v,\delta_v), \end{cases}$$

3. Uniformly for $z \in \partial D(v,\delta_v)$ we have

$$\widehat{P}^{(v)}(z) = (I + \mathcal{O}(N^{-1}))P^{(\infty)}(z)z^{\frac{\alpha}{2}\sigma_3}W(z)^{\sigma_3}e^{\frac{N\xi(z)}{2}\sigma_3}. \tag{4.67}$$

We see that the jumps of $\widehat{P}^{(v)}$ coincide with those of the hypergeometric model Riemann-Hilbert problem with parameters $\alpha = 2m, \beta = 0$, after rotating the contours of this problem. Using (4.66), we see that the behaviour of the function $\widehat{P}^{(v)}$ also coincides with that of Ψ_{HG} at the point 0. We are therefore interested in a solution of the type

$$\widehat{P}^{(v)}(z) = E_v(z)\Psi_{\mathrm{HG}}(\zeta_v(z)),$$

where, as in the previous cases, E_v is an analytic prefactor to be determined, and ζ_v is a locally conformal map from a neighbourhood of v onto a neighbourhood of 0. Comparing (4.66) and the asymptotic behaviour (4.20), we see that a map of the form $\zeta_v = N\xi(z)$ would compensate the exponential factors in these equations. As for the local parametrix at 0, we define instead the function

$$\tilde{\tilde{\xi}}(z) = 2\pi i \int_z^v r(s)ds,$$

and choose as map

$$\zeta_v = N\tilde{\tilde{\xi}}(z),$$

which is indeed locally conformal from a neighbourhood of v onto a neighbourhood of 0. With this, we can set now the analytic prefactor[36]

$$E_v(z) = P^{(\infty)}(z)z^{\frac{\alpha}{2}\sigma_3}W(z)^{\sigma_3} \left\{ \begin{array}{ll} e^{i\pi\frac{m}{2}\sigma_3}, & z \in Q_+^R \\ e^{-i\pi\frac{m}{2}\sigma_3}, & z \in Q_+^L \\ e^{i\pi\frac{m}{2}\sigma_3}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in Q_-^L \\ e^{-i\pi\frac{m}{2}\sigma_3}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in Q_-^R \end{array} \right\} e^{i\pi N\int_1^v r_\mp(s)ds\,\sigma_3}, \tag{4.68}$$

[36]We choose $e^{i\pi N\int_1^v r_\mp(s)ds}$ in equation (4.68) for $\mathrm{Im}\, z > 0$ and $\mathrm{Im}\, z < 0$, respectively. This constant now plays the role of the factor $(-1)^N$ in the local parametrix at 0. It follows from the jump conditions of the functions $P^{(\infty)}$ and W that E_v is indeed analytic in a neighbourhood of v.

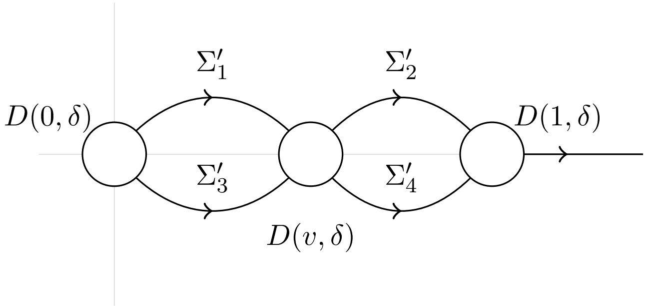


Figure 4.7: Final contour Σ_R.

as well as the lips Σ_j near the point v. Following the previous reasonings, we set the parts of them that lie inside the disk $D(v, \delta_v)$ to be the preimages of the rays $e^{\pm i\pi/4}\mathbb{R}^{\pm}, e^{\pm i\pi/2}\mathbb{R}^{+}$ under the map ζ_v. With this, we conclude that the matrix

$$P^{(v)}(z) = E_v(z)\Psi_{\text{HG}}(\zeta_v(z))e^{-\frac{N\xi(z)}{2}\sigma_3}W(z)^{-\sigma_3}z^{-\frac{\alpha}{2}\sigma_3} \tag{4.69}$$

solves the Riemann-Hilbert problem stated above. We recall that the parameters of the matrix Ψ_{HG} are set to $\alpha = 2m, \beta = 0$.

4.3.3 Final transformation and conclusion of the analysis

We can introduce the final transformation of the Riemann-Hilbert problem, which will allow us to obtain the sought asymptotic behaviour of the matrix Y. We consider the contour Σ_R, formed by the boundaries of three disks centered at the points $0, v$ and 1 of radius $\delta = \min\{\delta_0, \delta_v, \delta_1\}$, the interval $(1+\delta, \infty)$, and four curves Σ'_j, for $j = 1, \ldots, 4$. These curves are chosen as follows: they provide analytic continuations of the parts of the contours Σ_j that lie inside the disks, which have been fixed in the previous section. The resulting contour is depicted in figure 4.7. Now, we define

$$R(z) = S(z)\begin{cases} \left(P^{(1)}(z)\right)^{-1}, & \text{for } z \in D(1,\delta), \\ \left(P^{(v)}(z)\right)^{-1}, & \text{for } z \in D(v,\delta), \\ \left(P^{(0)}(z)\right)^{-1}, & \text{for } z \in D(0,\delta), \\ \left(P^{(\infty)}(z)\right)^{-1}, & \text{for } z \text{ elsewhere.} \end{cases} \tag{4.70}$$

Some inspection shows that the behaviour at the points $0, v$ and 1 of the local parametrices at each of these points implies in particular that the possible singularities of the matrix R at these points are removable. Therefore, we find that R solves the following problem.

Riemann-Hilbert problem for R.

1. $R(z)$ is analytic in $\mathbb{C} \setminus \Sigma_R$, where the contour Σ_R is shown in figure 4.7.

2. $R(z)$ has the following jumps

$$R(z)_+ = R(z)_- \begin{cases} P^{(1)}(z)\left(P^{(\infty)}\right)^{-1}, & z \in \partial D(1,\delta), \\ P^{(v)}(z)\left(P^{(\infty)}\right)^{-1}, & z \in \partial D(v,\delta), \\ P^{(0)}(z)\left(P^{(\infty)}\right)^{-1}, & z \in \partial D(0,\delta), \\ P^{(\infty)}(z)\begin{pmatrix} 1 & 0 \\ z^{-\alpha}(z-v)^{-m}e^{-N\xi(z)} & 1 \end{pmatrix}\left(P^{(\infty)}\right)^{-1}, & z \in \Sigma_j', \\ P^{(\infty)}(z)\begin{pmatrix} 1 & x^{\alpha}(x-v)^m e^{N(2g(x)-V(x)-\ell)} \\ 0 & 1 \end{pmatrix}\left(P^{(\infty)}\right)^{-1}, & 1 < x. \end{cases}$$

3. As $z \to \infty$, we have
$$R(z) = I + \mathcal{O}(z^{-1}).$$

According to the third condition in the Riemann-Hilbert problems for $P^{(0)}, P^{(v)}$ and $P^{(1)}$, the jumps of the matrix R on the boundary of the disks tend to the identity matrix as $N \to \infty$. Due to equations (4.39) and(4.37), we see that the function ξ is purely imaginary on the interval $(0,1)$, and its imaginary part strictly decreases from 2π to 0. Therefore, as a consequence of the Cauchy-Riemann equations, there exists a neighbourhood of the interval $(0,1)$ on which ξ has positive real part away from the real axis. Hence, after possibly replacing δ with a smaller radius for the disks above, so that the curves Σ_j' lie inside this neighbourhood, we conclude that the jumps of R converge to the identity as $N \to \infty$ on the parts of the lips of the lenses that lie above the real axis. Using (4.36), we see that the same conclusion holds for the parts of the lips that lie below the real axis. Moreover, it follows from (4.38) that the jump of R on the interval $(1,\infty)$ also tends to the identity as $N \to \infty$.

Thus, we see that the matrix R is asymptotically close to the identity as $N \to \infty$, as claimed above, in the sense that[37]
$$R(z) = I + \mathcal{O}(N^{-1}).$$

We can now reverse the series of transformations leading to R to recover an asymptotic approximation to the matrix Y. Inserting this into the differential identity (4.46), we arrive at the desired large N expression for the model $\widehat{Z}^{v,m}_{LUE(N)}$. This is precisely the content of the next theorem.

Theorem 14. *Let $\alpha > 0$, $v \in (0,1)$ and m be a positive integer. As $N \to \infty$, we have*

$$\frac{\widehat{Z}^{v,m}_{LUE(N)}}{\widehat{Z}^{v,0}_{LUE(N)}} = \left[\left(\frac{e^v}{2}\right)^{4Nm}\left(\frac{N}{e}\sqrt{\frac{1-v}{v}}\right)^{m^2}\frac{G(m+1)^2}{G(2m+1)}(4v)^{-m\alpha}\right](1+\mathcal{O}(N^{-1})), \tag{4.71}$$

where G is Barnes' G-function.

Proof. Recalling the differential identity (4.46), we see that we need approximations for the functions $(Y^{-1}\frac{d}{dm}Y)_{11}$ and $Y_{11}Y_{22}$ at the points 0 and v, as well as for the coefficients γ_N, γ_{N-1} and η_N.

[37] More detailed estimations on the norm of the matrix R on the contours of the Riemann-Hilbert problem can be obtained by means of contour integration; see [53] for instance, among many others.

We start with the coefficients γ_N, γ_{N-1} and η_N. We can obtain the asymptotic behaviour of these constants by means of equations (4.15) and the expression

$$Y(z) = e^{\frac{N\ell}{2}\sigma_3} R(z) P^{(\infty)}(z) e^{N(g(z)-\frac{\ell}{2})\sigma_3},$$

which holds for points z lying outside the disks $D(\varepsilon, \delta)$, for $\varepsilon \in \{0, v, 1\}$ and outside the lenses, as follows after reversing the transformations of the Riemann-Hilbert problem. Combining this remark with equation (4.48), the fact that $\ell = -2 - 4\log 2$ (see [197], for instance), and the asymptotic behaviour $D(z) = D_\infty(1 - \frac{vm}{z} + \mathcal{O}(z^{-2}))$, as $z \to \infty$, we find that as $N \to \infty$

$$\begin{aligned}
\gamma_{N-1}^2 &= \frac{1}{\pi} e^{2N} 2^{4N+4m+2\alpha-3}(1 + \mathcal{O}(N^{-1})),\\
\gamma_N^{-2} &= \pi e^{-2N} 2^{-(4N+4m+2\alpha+1)}(1 + \mathcal{O}(N^{-1})),\\
\eta_N &= -\frac{N}{4} + vm - \frac{1}{2} + \mathcal{O}(N^{-1}).
\end{aligned}$$

We next consider the behaviour of the functions $(Y^{-1}\frac{d}{dm}Y)_{11}$ and $Y_{11}Y_{22}$. We approach the points 0 and v taking points z on the disks $D(0, \delta)$ and $D(v, \delta)$ respectively and outside of the lenses (and also lying in the intersection of the preimage of the region II under the map ζ_v, depicted in figure 4.3, with the quadrant Q_+^R (4.65), in the case of v). Reversing the transformations of the Riemann-Hilbert problem for Y we see that for such points the matrix Y can be expressed as

$$Y(z) = e^{\frac{N\ell}{2}\sigma_3} R(z) P^{(\varepsilon)}(z) e^{N(g(z)-\frac{\ell}{2})\sigma_3}, \tag{4.72}$$

where $\varepsilon \in \{0, v\}$.

Let us start with the point v. Substituting the explicit expression of the local parametrix $P^{(v)}$, given in (4.69), and using the fact that $g_+(v) - \ell/2 - \xi_+(v)/2 = V(v)/2$ (which follows from (4.38) and (4.39)), we obtain

$$Y(v) = e^{\frac{N\ell}{2}\sigma_3}(I + \mathcal{O}(N^{-1})) E_v(v) \begin{pmatrix} \Psi_v & -\frac{1}{2}\Psi_v^{-1} \\ \Psi_v & \frac{1}{2}\Psi_v^{-1} \end{pmatrix} v^{-\frac{\alpha}{2}\sigma_3} e^{2Nv\sigma_3}, \tag{4.73}$$

where

$$\Psi_v = \frac{\Gamma(m+1)}{\Gamma(2m+1)} \left(4N\sqrt{\frac{1-v}{v}}\right)^m,$$

and

$$E_v(v) = D_\infty^{\sigma_3} M(v) e^{i\left((m+\frac{\alpha}{2})\arccos(2v-1) - \frac{m\pi}{2} + \pi N\int_1^v r_-(s)ds\right)\sigma_3},$$

where D_∞ and $M(z)$ are given by (4.51) and (4.52), respectively. We have used in the derivation of equation (4.73) above the asymptotic behaviour [1]

$$G(a, b; z) = z^{\frac{b}{2}}(1 + \mathcal{O}(z)), \qquad H(a, b; z) = \frac{\Gamma(b)}{\Gamma(a)} z^{-\frac{b}{2}} + \mathcal{O}(z^{1-\frac{1}{2}\mathrm{Re}\, b}) + \mathcal{O}(z^{\mathrm{Re}\, b}),$$

as $z \to 0$, for the functions G and H introduced in the explicit solution of the model hypergeometric Riemann-Hilbert problem (4.22), together with the approximation

$$\zeta_v(z) = -2\pi i N r(v)(z - v)(1 + \mathcal{O}(z - v))$$

as $z \to v$, where r is the extension (4.36) of the density of the equilibrium measure $d\mu_V$.

We now consider the behaviour of Y at 0. Following an analogous reasoning as before, we find that

$$Y(0) = e^{\frac{N\ell}{2}\sigma_3}(I + \mathcal{O}(N^{-1}))E_0(0) \begin{pmatrix} \Psi_0 & -\frac{1}{2\pi i\alpha}\Psi_0^{-1} \\ i\pi\alpha\Psi_0 & \frac{1}{2}\Psi_0^{-1} \end{pmatrix} v^{-m\sigma_3},$$

where

$$\Psi_0 = \frac{1}{\Gamma(\alpha+1)}(2N)^{\alpha}$$

and

$$E_0(0) = (-1)^N D_\infty^{\sigma_3} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i(2m+\alpha+1) \\ i & -2m-\alpha+1 \end{pmatrix} (4\pi N)^{\frac{\sigma_3}{2}},$$

after substituting the explicit expression for the local parametrix $P^{(0)}$, given by (4.64), in (4.72). We have used the approximations [1]

$$I_\alpha(z) = \frac{1}{\Gamma(\alpha+1)}\left(\frac{z}{2}\right)^{\alpha}(1+\mathcal{O}(z^2)), \qquad K_\alpha(z) = \frac{\Gamma(\alpha)}{2}\left(\frac{z}{2}\right)^{-\alpha} + \mathcal{O}(z^{1-\operatorname{Re}\alpha}) + \mathcal{O}(z^{\operatorname{Re}\alpha}),$$

as $z \to 0$, together with the fact that as $z \to 0$

$$\zeta_0^{1/2}(z) = -4Nz^{1/2}(1+\mathcal{O}(z)).$$

Substituting the obtained expressions in the differential identity (4.46) and performing some computations we arrive at

$$\begin{aligned} \frac{d}{dm}\log \widehat{Z}^{v,m}_{LUE(N)} &= 4N(v - \log 2) + 2m\log\left(4N\sqrt{\frac{1-v}{v}}\right) \\ &\quad - (2m+\alpha)\log 4 - \alpha\log v + 2m\frac{d}{dm}\log\left(\frac{\Gamma(m+1)}{\Gamma(2m+1)}\right) + \mathcal{O}(N^{-1}). \end{aligned}$$

Integrating this identity from $m = 0$ to an arbitrary integer and using the formula (see [53], for instance)

$$\int_0^z x\frac{d}{dx}\log\frac{\Gamma\left(\frac{x}{2}+1\right)}{\Gamma(x+1)}dx = -\frac{z^2}{4} + \log\frac{G\left(\frac{z}{2}+1\right)^2}{G(z+1)}$$

we arrive at the desired conclusion. □

We can use the result given in theorem 14 to obtain the large behaviour of the matrix model $Z^{u,m}_{LUE(N)}$ introduced in section 4.2. Indeed, combining equations (4.30) and (4.71) we find that as $N \to \infty$ and $u \to \infty$, with $u/4N = cte \in (0,1)$, we have

$$\frac{Z^{u,m}_{LUE(N)}}{Z_{LUE(N)}} = N^{2mN+m^2+m\alpha} e^{um-m^2} u^{-m\alpha} \left(\frac{4N-u}{u}\right)^{m^2/2} \frac{G(m+1)^2}{G(2m+1)}(1+\mathcal{O}(N^{-1})).$$

Note that the large N behaviour of the partition function of the Laguerre Unitary Ensemble $Z_{LUE(N)}$ can be obtained by means of equations (4.29) and (3.71).

Let us make some comments to end this section. We have focused only on the leading terms of the model $\widehat{Z}^{v,m}_{LUE(N)}$, but several generalizations are possible with some additional considerations. First of all, we note that the asymptotic behaviour of the orthogonal polynomials with respect to the weight $\widehat{w}_{v,m}$ in the various regions of the complex plane determined by the contour Σ_R is readily available from the asymptotic expressions for the matrix Y, in sight of equations (4.14) and (4.15). Moreover, as usual when solving Riemann-Hilbert problems, we observe that more terms in the asymptotic expression for Y can be obtained, with increased effort. This involves the analysis of the function R, besides more detailed approximations of the functions appearing in the explicit construction of Y used in the proof of theorem 14.

Finally, let us also remark that during the preparation of the current work, the article [53] appeared, which addresses much more general cases of insertions of Fisher-Hartwig singularities in the Laguerre and Jacobi Unitary Ensembles. Our results are consistent with those in [53], although the different choices of potentials and supports of the weights make the direct comparison slightly involved.

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