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Integrable equations associated with the finite-temperature deformation of the discrete Bessel point process

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Abstract
We study the finite-temperature deformation of the discrete Bessel point process. We show that its largest particle distribution satisfies a reduction of the 2D Toda equation, as well as a discrete version of the integro-differential Painlevé II equation of Amir–Corwin–Quastel, and we compute initial conditions for the Poissonization parameter equal to 0. As proved by Betea and Bouttier, in a suitable continuum limit the last particle distribution converges to that of the finite-temperature Airy point process. We show that the reduction of the 2D Toda equation reduces to the Korteweg–de Vries equation, as well as the discrete integro-differential Painlevé II equation reduces to its continuous version. Our approach is based on the discrete analogue of Its–Izergin–Korepin–Slavnov theory of integrable operators developed by Borodin and Deift.

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1 | INTRODUCTION AND RESULTS

1.1 | Finite-temperature discrete Bessel point process and the 2D Toda equation

In this paper we study the *finite-temperature discrete Bessel point process*, which is the determinantal point process on $\mathbb{Z}' := \mathbb{Z} + \frac{1}{2}$ with correlation kernel

$$K_\sigma^{\mathrm{Be}}(a,b) = \sum_{l\in\mathbb{Z}'} \sigma(l)\mathrm{J}_{a+l}(2L)\mathrm{J}_{b+l}(2L), \qquad a,b\in\mathbb{Z}', \tag{1.1}$$

where $L > 0$ is a parameter, $\mathrm{J}_k(\cdot)$ is the Bessel function of first kind of order k, and $\sigma : \mathbb{Z}' \to [0,1]$ is a function such that $\sigma \in \ell^1(\mathbb{Z}' \cap (-\infty, 0))$. The fact that the kernel (1.1) actually induces a determinantal point process on $\mathbb{Z}'$ and the role of the decay conditions on σ at $-\infty$ will be clarified in Section 2.

The specialization $\sigma = 1_{\mathbb{Z}'_+}$ of (1.1), where $\mathbb{Z}'_+ := \mathbb{Z}' \cap (0, +\infty)$, yields the standard discrete Bessel point process [11, 23], namely, the determinantal point process with correlation kernel

$$\begin{aligned} K^{\mathrm{Be}}(a,b) &= \sum_{l\in\mathbb{Z}'_+} \mathrm{J}_{a+l}(2L)\mathrm{J}_{b+l}(2L) \\ &= L\,\frac{\mathrm{J}_{a-\frac{1}{2}}(2L)\mathrm{J}_{b+\frac{1}{2}}(2L) - \mathrm{J}_{a+\frac{1}{2}}(2L)\mathrm{J}_{b-\frac{1}{2}}(2L)}{a-b}, \quad a,b\in\mathbb{Z}'. \end{aligned} \tag{1.2}$$

(The last equality easily follows from a property of the Bessel functions and will be proved for the reader's convenience in Lemma 2.2.) The discrete Bessel point process has the following combinatorial interpretation. Let $\mathbb{Y}$ be the set of integer partitions (or, equivalently, Young diagrams). Namely, elements $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathbb{Y}$ are half-infinite sequences of non-negative integers λ_i, for $i \geqslant 1$, satisfying $\lambda_i \geqslant \lambda_{i+1}$ and with finitely many non-zero λ_i's. In particular, for $\lambda \in \mathbb{Y}$, the *weight* $|\lambda| := \sum_{i\geqslant 1} \lambda_i$ is a finite number. The *Poissonized Plancherel measure* $\mathbb{P}_{\mathrm{Plan}}$ is the probability measure on $\mathbb{Y}$, depending on a parameter $L > 0$, defined by

$$\mathbb{P}_{\mathrm{Plan}}(\{\lambda\}) := \mathrm{e}^{-L^2} L^{2|\lambda|} \left(\frac{\dim\lambda}{|\lambda|!}\right)^2, \qquad \lambda\in\mathbb{Y}. \tag{1.3}$$

Here, $\dim\lambda$ is the dimension of the irreducible representation of the symmetric group $S_{|\lambda|}$ corresponding to λ, or, equivalently, $\dim\lambda$ is the number of standard Young tableaux of shape λ. If we associate to each $\lambda \in \mathbb{Y}$ a subset of $\mathbb{Z}'$ through the map $\lambda \mapsto \{\lambda_i - i + \frac{1}{2}\}_{i\geqslant 1}$, it was proven in [11, 29] that the push-forward of $\mathbb{P}_{\mathrm{Plan}}$ is the determinantal point process on $\mathbb{Z}'$ whose correlation kernel is precisely (1.2).

The kernel (1.1) has a similar interpretation when

$$\sigma(l) = (1 + u^l)^{-1}, \qquad l \in \mathbb{Z}', \tag{1.4}$$

for a parameter $u \in [0,1)$. Namely, introduce a probability measure $\mathbb{P}_{\mathsf{cPlan}}$ on $\mathbb{Y}$ (*cylindric Plancherel distribution* [8]), depending on parameters $L > 0$ and $u \in [0,1)$, by

$$\begin{aligned} \mathbb{P}_{\mathsf{cPlan}}(\{\lambda\}) &:= \frac{1}{Z(u,L)} \sum_{\mu \subset \lambda} u^{|\mu|} \left(\frac{\big(L(1-u)\big)^{|\lambda|-|\mu|} \dim(\lambda/\mu)}{(|\lambda|-|\mu|)!} \right)^2, \quad \lambda \in \mathbb{Y}, \\ Z(u,L) &:= \frac{\mathrm{e}^{L^2(1-u)}}{\prod_{n\geqslant 1}(1-u^n)}, \end{aligned} \tag{1.5}$$

where the sum runs over partitions $\mu \in \mathbb{Y}$ such that $\mu_i \leqslant \lambda_i$ for all $i \geqslant 1$, and $\dim(\lambda/\mu)$ is the number of standard Young tableaux of shape λ/μ. Consider also the probability measure $\mathbb{P}_{\mathsf{C}}$ on $\mathbb{Z}$ defined by

$$\mathbb{P}_{\mathsf{C}}(\{c\}) = \frac{u^{c^2/2}}{\sum_{n\in\mathbb{Z}} u^{n^2/2}}, \qquad c \in \mathbb{Z}. \tag{1.6}$$

It is proven in [5, 8] that, under the map $(\lambda, C) \mapsto \{\lambda_i - i + 1/2 + C\}_{i\geqslant 1}$, the push-forward of $\mathbb{P}_{\mathsf{cPlan}} \otimes \mathbb{P}_{\mathsf{C}}$ is the determinantal point process on $\mathbb{Z}'$ whose correlation kernel is (1.1) with σ as in (1.4).

Going back to the kernel (1.1) for general σ, we shall see in Lemma 2.4 that the induced determinantal point process has almost surely a largest particle a_{max}. We shall study its cumulative distribution function

$$Q_\sigma(L,s) := \mathbb{P}(a_{\mathsf{max}} \leqslant s), \qquad s \in \mathbb{Z}'. \tag{1.7}$$

By the general theory of determinantal point processes [9, 24, 31], this distribution can be expressed as

$$Q_\sigma(L,s) = \det(1 - \mathcal{P}_s \mathcal{K}_\sigma^{\mathsf{Be}} \mathcal{P}_s), \qquad s \in \mathbb{Z}'. \tag{1.8}$$

Here, $\mathcal{K}_\sigma^{\mathsf{Be}}$ is the operator on $\ell^2(\mathbb{Z}')$ induced[†] by the kernel (1.1), and $\mathcal{P}_s$ is the orthogonal projector onto $\ell^2(\{s+1, s+2, \ldots\})$, namely, $\mathcal{P}_s$ is induced by the kernel $P_s(a,b) = 1_{a>s}\,\delta(a,b)$, for $s \in \mathbb{Z}'$. The determinant in (1.8) is a Fredholm determinant, as the operator $\mathcal{P}_s \mathcal{K}_\sigma^{\mathsf{Be}} \mathcal{P}_s$ is trace class on $\ell^2(\mathbb{Z}')$ for all $s \in \mathbb{Z}'$ (Lemma 2.4).

It is also worth noting that $Q_\sigma(L,s)$ can be equivalently described as the following expectation with respect to the Poissonized Plancherel measure (1.3) (see Lemma 3.1):

$$Q_\sigma(L,s) = \mathbb{E}_{\mathsf{Plan}}\left[\prod_{i=1}^{+\infty} \big(1 - \sigma(\lambda_i - i - s)\big) \right]. \tag{1.9}$$

Finally, let us remark that $0 \leqslant Q_\sigma(L,s) \leqslant 1$ is a non-decreasing function of $s \in \mathbb{Z}'$ such that $Q_\sigma(L,s) \to 1$ as $s \to +\infty$. In particular, there exists $s_0 \in \mathbb{Z}' \cup \{-\infty\}$ (depending on σ) such that $Q_\sigma(L,s) = 0$ if $s < s_0$ and $Q_\sigma(L,s) > 0$ otherwise. In particular, since for any $\lambda \in \mathbb{Y}$ the set $\{\lambda_i - i + \frac{1}{2}\}_{i\geqslant 1}$ has largest particle $a_{\mathsf{max}} = \lambda_1 - \frac{1}{2} \geqslant -\frac{1}{2}$, we deduce by the discussion above of the

[†] Throughout this paper, we agree that a kernel $X : \mathbb{Z}' \times \mathbb{Z}' \to \mathbb{C}$ induces an operator $\mathcal{X}$ on $\ell^2(\mathbb{Z}')$ by $(\mathcal{X}\psi)(a) = \sum_{b\in\mathbb{Z}'} X(a,b)\psi(b)$, for $\psi \in \ell^2(\mathbb{Z}')$ and $a \in \mathbb{Z}'$.

Poissonized Plancherel measure that, when $\sigma = \mathbf{1}_{\mathbb{Z}'_+}$, we have $s_0 = -1/2$. On the other hand, when $\sigma(l) = (1+u^l)^{-1}$ as in (1.4), corresponding to the cylindric Plancherel measure, we have $s_0 = -\infty$, because

$$Q_\sigma(L,s) = \mathbb{P}(a_{\mathsf{max}} \leqslant s) \geqslant \mathbb{P}(a_{\mathsf{max}} = s) \geqslant \mathbb{P}_{\mathsf{cPlan}}\big(\{\emptyset\}\big)\mathbb{P}_{\mathsf{C}}(\{s+\tfrac{1}{2}\}) > 0, \qquad \text{for all } s \in \mathbb{Z}'. \tag{1.10}$$

Our first result is the following.

Theorem I. *For all $s \in \mathbb{Z}'$ such that $Q_\sigma(L,s) > 0$, we have*

$$\frac{\partial^2}{\partial L^2}\log Q_\sigma(L,s) + \frac{1}{L}\frac{\partial}{\partial L}\log Q_\sigma(L,s) + 4 = 4\,\frac{Q_\sigma(L,s+1)Q_\sigma(L,s-1)}{Q_\sigma(L,s)^2}. \tag{1.11}$$

The proof is given in Section 4.

The Equation (1.11) is, essentially, a reduction of the *2D Toda equation*. Indeed, it implies that

$$\tau_s(\theta_+,\theta_-) := \mathrm{e}^{\theta_+\theta_-}\, Q_\sigma(\sqrt{\theta_+\theta_-}, s) \tag{1.12}$$

is a 2D Toda tau function, that is, $\tau_s(\theta_+,\theta_-)$ satisfies the bilinear form of the 2D Toda equation [20, 34]

$$\frac{\partial^2}{\partial\theta_+\partial\theta_-}\log\tau_s(\theta_+,\theta_-) = \frac{\tau_{s+1}(\theta_+,\theta_-)\tau_{s-1}(\theta_+,\theta_-)}{\tau_s(\theta_+,\theta_-)^2}. \tag{1.13}$$

Equation (1.11), or rather the corresponding equation for the variables $\{\mathrm{e}^{L^2}Q_\sigma(L,s)\}_{s\in\mathbb{Z}'}$, is also known as *cylindrical Toda equation*. Another class of solutions of (1.11) written in terms of Fredholm determinants is studied in [33, 35]. More recently, using a Fredholm determinant representation, Matetski, Quastel, and Remenik proved that multi-point distributions associated to the polynuclear growth model with arbitrary initial data satisfy the *non-commutative Toda equation* [28].

It is appropriate to remark that, in the case $\sigma = \mathbf{1}_X$, with X a subset of $\mathbb{Z}'$ bounded below,† the connection to the 2D Toda equation is not new. Indeed, in this case, our result follows from [29, Theorem 3], which relates more generally the Schur measure on partitions with the Toda hierarchy. A particular case studied in even more detail is the one in which $X = \mathbb{Z}'_+$. In this situation, by the combinatorial interpretation of the discrete Bessel point process explained above, we have $Q_\sigma(L,s) = 0$ for $s \leqslant -\frac{3}{2}$ and $Q_\sigma(L,s) > 0$ for $s \geqslant -\frac{1}{2}$. Moreover, by the Borodin–Okounkov–Geronimo–Case formula [10, 19], the Fredholm determinant $Q_\sigma(L,s)$, for $s \in \mathbb{Z}'_+$, is related to a Toeplitz determinant of size $\lceil s\rceil = s + \frac{1}{2}$ as

$$Q_\sigma(L,s) = \mathrm{e}^{-L^2}\det\big[\mathrm{I}_{i-j}(2L)\big]_{i,j=1,\dots,\lceil s\rceil}, \qquad \mathrm{I}_k(2L) = \operatorname*{res}_{u=0}\mathrm{e}^{L(u+u^{-1})}u^{-k-1}\mathrm{d}u. \tag{1.14}$$

† If X is not bounded below, by (1.9) we have $Q_\sigma(L,s) = 0$ identically.

Once this connection with Toeplitz determinants is established, the 2D Toda equation can be obtained in several different ways, essentially exploiting the relation with orthogonal polynomials on the unit circle, as for instance in [1, 4, 21].

Therefore, Theorem I states that the connection of the discrete Bessel kernel to the 2D Toda equation extends to the deformation (1.1) of the kernel. We complement this result by computing small L asymptotics for $Q_\sigma(L,s)$.

Theorem II. *For any* $s \in \mathbb{Z}'$, *let* $Q^0_\sigma(s) := \prod_{i=1}^{+\infty}\big(1-\sigma(-i-s)\big)$. *For all* $s \in \mathbb{Z}'$ *such that* $Q^0_\sigma(s) > 0$, *there exists* $L_* = L_*(s) > 0$ *such that* $Q_\sigma(L,s) > 0$ *for* $0 \leqslant L < L_*$, *and*

$$\log Q_\sigma(L,s) = \log Q^0_\sigma(s) - \frac{\sigma(-s)-\sigma(-s-1)}{1-\sigma(-1-s)}L^2 + \mathrm{O}(L^4), \qquad L \to 0. \tag{1.15}$$

We note that when $Q^0_\sigma(s) > 0$, the denominator in the term of order L^2 of (1.15) does not vanish. The proof is given in Section 5.

1.2 | Continuum limit to the Korteweg–de Vries equation

The finite-temperature discrete Bessel kernels (1.1) have continuum limits to the finite-temperature Airy kernels [5]. These are kernels of the form

$$K^{\mathrm{Ai}}_\varsigma(\xi,\eta;t) = \int_{\mathbb{R}} \varsigma(t^{-2/3}r)\mathrm{Ai}(\xi+r)\mathrm{Ai}(\eta+r)\mathrm{d}r, \qquad \xi,\eta \in \mathbb{R}, \tag{1.16}$$

with Ai and Ai$'$ the Airy function and its derivative, respectively, $t > 0$ a positive real parameter, and $\varsigma : \mathbb{R} \to [0,1]$ a function which is smooth and satisfies $\varsigma(r) \in L^1\big((-\infty,0), \sqrt{|r|}\mathrm{d}r\big)$. In [5], the authors proved this limit for σ as in (1.4), but their result extends easily to more general functions, as long as $\sigma = \sigma_\epsilon$ depends on an additional parameter ϵ in such a way that $\sigma_\epsilon(\zeta/\epsilon) \to \varsigma(\zeta)$ for some function ς as $\epsilon \to 0$. More precisely, when σ is given by (1.4), one has to identify the parameter ϵ with $1-u$. Then, we have the convergence

$$\sigma\left(\frac{r}{t^{2/3}(1-u)}\right) \to \varsigma\left(\frac{r}{t^{2/3}}\right) = \frac{1}{1+\mathrm{e}^{-rt^{-2/3}}}, \qquad \text{as } u \to 1^-, \tag{1.17}$$

which is the scaling limit used in [5] to study the edge behavior of the cylindrical Plancherel measure.

These types of kernels (and related Fredholm determinants) attracted a great deal of interest in the last 15 years. They first appeared in the field of random matrices [25], in the theory of the Kardar–Parisi–Zhang equation [2], and in relation with one-dimensional systems of fermions at finite temperature [17]. Riemann–Hilbert (RH) techniques for the study of related Fredholm determinants have been developed and used in [12–16]. In particular, Fredholm determinants on $L^2(\mathbb{R})$

of the form

$$F_\varsigma(x,t) = \det\left(1 - \mathbf{1}_{(-xt^{-1/3},+\infty)}\mathcal{K}_\varsigma^{\mathrm{Ai}}\mathbf{1}_{(-xt^{-1/3},+\infty)}\right) \tag{1.18}$$

have been shown to satisfy† [15]

$$\frac{\partial^2}{\partial t\partial x}\log F_\varsigma(x,t) + \frac{x}{t}\frac{\partial^2}{\partial x^2}\log F_\varsigma(x,t) + \left(\frac{\partial^2}{\partial x^2}\log F_\varsigma(x,t)\right)^2 + \frac{1}{6}\frac{\partial^4}{\partial x^4}\log F_\varsigma(x,t) = 0, \tag{1.19}$$

that is, the function

$$U_\varsigma(x,t) := \frac{\partial^2}{\partial x^2}\log F_\varsigma(x,t) + \frac{x}{2t} \tag{1.20}$$

satisfies the Korteweg–de Vries equation

$$\frac{\partial}{\partial t}U_\varsigma(x,t) + 2U_\varsigma(x,t)\frac{\partial}{\partial x}U_\varsigma(x,t) + \frac{1}{6}\frac{\partial^3}{\partial x^3}U_\varsigma(x,t) = 0. \tag{1.21}$$

It is instructive to look at how Equation (1.19) (closely related to the bilinear form of the Korteweg–de Vries equation) emerges in such continuum limit from Equation (1.11) (which is in turn related to the 2D Toda equation). Let the variables L, s be given in terms of variables x, t and of an additional parameter $\epsilon > 0$ as

$$s(x,t;\epsilon) = \frac{2}{\epsilon^3 t^2} - \frac{x}{\epsilon t}, \quad L(x,t;\epsilon) = \frac{1}{\epsilon^3 t^2}. \tag{1.22}$$

Under this transformation, we have

$$\frac{\partial}{\partial L} = \frac{\partial x}{\partial L}\frac{\partial}{\partial x} + \frac{\partial t}{\partial L}\frac{\partial}{\partial t} = -\frac{1}{2}\epsilon t\left((\epsilon^2 xt - 4)\frac{\partial}{\partial x} + \epsilon^2 t^2\frac{\partial}{\partial t}\right). \tag{1.23}$$

Moreover, let us introduce

$$F(x,t;\epsilon) := Q_\sigma(L(x,t;\epsilon), s(x,t;\epsilon)). \tag{1.24}$$

As shown in [5], $F(x,t;\epsilon)$ converges, as $\epsilon \to 0$, to $F_\varsigma(x,t)$, and we shall now explain how the equation for Q_σ of Theorem I reduces to (1.19). Expanding at $\epsilon = 0$ as

$$\log F(x,t;\epsilon) = f_0(x,t) + \epsilon f_1(x,t) + \epsilon^2 f_2(x,t) + \mathrm{O}(\epsilon^3), \tag{1.25}$$

the left-hand side of (1.11) is

$$\begin{aligned}\left(\frac{\partial^2}{\partial L^2} + \frac{1}{L}\frac{\partial}{\partial L}\right)\log F(x,t;\epsilon) + 4 &= 4 + 4\epsilon^2 t^2\frac{\partial^2}{\partial x^2}f_0(x,t) + 4\epsilon^3 t^2\frac{\partial^2}{\partial x^2}f_1(x,t)\\ &-2\epsilon^4 t^4\left(\frac{\partial^2}{\partial t\partial x}f_0(x,t) + \frac{x}{t}\frac{\partial^2}{\partial x^2}f_0(x,t) - \frac{2}{t^2}\frac{\partial^2}{\partial x^2}f_2(x,t)\right) + \mathrm{O}(\epsilon^5)\end{aligned} \tag{1.26}$$

† Only Equation (1.21) appears explicitly in [15, Theorem 1.3]. However, (1.19) can be obtained by substituting $U_\varsigma(x,t) = \partial_x^2 \log F(x,t) + x/(2t)$ in (1.21) and integrating once in x thanks to the asymptotics proved in [15, Section 5].

and, similarly, the right-hand side of (1.11) is

$$4\frac{F(x-\epsilon t,t;\epsilon)F(x+\epsilon t,t;\epsilon)}{F(x,t;\epsilon)^2} = 4+4\epsilon^2t^2\frac{\partial^2}{\partial x^2}f_0(x,t)+4\epsilon^3t^2\frac{\partial^2}{\partial x^2}f_1(x,t)$$
$$+\epsilon^4t^4\left(2\left(\frac{\partial^2}{\partial x^2}f_0(x,t)\right)^2+\frac{1}{3}\frac{\partial^4}{\partial x^4}f_0(x,t)+\frac{4}{t^2}\frac{\partial^2}{\partial x^2}f_2(x,t)\right)+\mathrm{O}(\epsilon^5). \qquad (1.27)$$

Terms of order up to ϵ^3 match identically, while at order ϵ^4 we obtain precisely (1.19) (whose relation to the Korteweg–de Vries equation has been explained above) for the function $F_\varsigma(x,t) = \exp\big(f_0(x,t)\big)$.

Remark 1.1. After submission, we learned that this scaling limit of the cylindrical Toda equation to the cylindrical KdV equation had already appeared in [27].

1.3 | A discrete version of the integro-differential Painlevé II equation

For the Korteweg–de Vries solutions $U_\varsigma(x,t)$ associated with Fredholm determinants (1.18) of the finite-temperature Airy kernel (1.16) there is an identity between the potential and the wave function.† Namely, provided exponential decay of ς at $-\infty$, it is shown in [15] that the solution to the boundary value problem

$$\frac{\partial^2}{\partial x^2}\psi(\zeta;x,t)=(\zeta-2U_\varsigma(x,t))\psi(\zeta;x,t), \qquad \psi(\zeta;x,t)\sim t^{1/6}\mathrm{Ai}(t^{2/3}\zeta-xt^{-1/3}),\ \ x\to-\infty, \qquad (1.28)$$

satisfies

$$U_\varsigma(x,t)=\frac{x}{2t}-\frac{1}{t}\int_{\mathbb{R}}\psi(\eta;x,t)^2\varsigma'(\eta)\mathrm{d}\eta. \qquad (1.29)$$

Plugging (1.29) into (1.28) one obtains the so-called *integro-differential Painlevé II equation* of Amir, Corwin, and Quastel [2]

$$\frac{\partial^2}{\partial x^2}\psi(\zeta;x,t)=\left(\zeta-\frac{x}{t}+\frac{2}{t}\int_{\mathbb{R}}\psi(\eta;x,t)^2\varsigma'(\eta)\mathrm{d}\eta\right)\psi(\zeta;x,t), \qquad (1.30)$$

whose solution (subject to the boundary value condition in (1.28)) characterizes the distribution F_ς, since, by (1.20) and (1.29),

$$\frac{\partial^2}{\partial x^2}\log F_\varsigma(x,t)=-\frac{1}{t}\int_{\mathbb{R}}\psi(\eta;x,t)^2\varsigma'(\eta)\mathrm{d}\eta. \qquad (1.31)$$

It is worth recalling that in the limit $\varsigma\to\mathbf{1}_{(0,+\infty)}$, the kernel (1.16) reduces to the classical Airy kernel, the integro-differential Painlevé II equation (1.30) reduces to the standard Painlevé II equa-

† We thank Percy Deift for pointing out that such relation is the analogue of the *Trace Formula* of [18] for potentials $U_\varsigma(x,t)$ which, unlike the classical setting of op. cit., do not vanish as $x\to\pm\infty$ but rather behave as $x/(2t)$.

tion, and its solution selected by the boundary behavior in (1.28) is the Hastings–McLeod solution (in agreement with the celebrated result by Tracy and Widom [32]).

The next result is an analogous property for the finite-temperature discrete Bessel kernels.

Theorem III. *Let $L > 0$ and $s_0 := \min\{s \in \mathbb{Z}' \,:\, Q_\sigma(L,s) > 0\} \in \mathbb{Z}' \cup \{-\infty\}$. For all $s \in \mathbb{Z}'$ with $s \geqslant s_0$, we introduce*

$$\mathfrak{a}(L,s) := \frac{\sqrt{Q_\sigma(L,s+1)Q_\sigma(L,s-1)}}{Q_\sigma(L,s)}, \qquad \mathfrak{b}(L,s+1) := \frac{\partial}{\partial L}\log\frac{Q_\sigma(L,s+1)}{Q_\sigma(L,s)}. \tag{1.32}$$

Then, for all $s \in \mathbb{Z}'$, $s \geqslant s_0$,

$$\mathfrak{a}^{-1}(L,s) - \mathfrak{a}(L,s) = \frac{1}{L}\sum_{l\in\mathbb{Z}'}\big(\sigma(l+1)-\sigma(l)\big)\varphi(l+1;L,s-1)\varphi(l;L,s), \tag{1.33}$$

$$\mathfrak{b}(L,s+1) = \frac{2}{L}\sum_{l\in\mathbb{Z}'}\big(\sigma(l+1)-\sigma(l)\big)\varphi(l+1;L,s)\varphi(l;L,s), \tag{1.34}$$

where $\varphi(l;L,s)$ are defined for $l \in \mathbb{Z}'$ and for $s \in \mathbb{Z}'$ with $s \geqslant s_0 - 1$ and satisfy the recursion

$$\mathfrak{a}(L,s+1)\varphi(l;L,s+1) + \mathfrak{a}(L,s)\varphi(l;L,s-1) = \left(\frac{l+s+1}{L} + \frac{\mathfrak{b}(L,s+1)}{2}\right)\varphi(l;L,s). \tag{1.35}$$

Moreover, for all $l \in \mathbb{Z}'$ we have

$$\varphi(l;L,s) \sim \sqrt{L}\,\mathrm{J}_{l+s+1}(2L), \qquad s \to +\infty. \tag{1.36}$$

The proof is given in Section 6 and is based on a Lax pair argument. In particular, when $\sigma = \mathbf{1}_{\mathbb{Z}'_+}$ we obtain a Lax pair which, although different from the one used by Borodin [7], can be equivalently used to prove the connection to the discrete Painlevé II equation established in op. cit. (and independently proved by other methods in [1, 3]); see Section 6.1 for more details.

It is worth observing that in the scaling limit (1.22) as $\epsilon \to 0$, the equations of Theorem III formally reduce to above mentioned equations for Fredholm determinants of the finite-temperature Airy kernel. More precisely, with the notations of (1.22), (1.24), and (1.25), we have, as $\epsilon \to 0$,

$$\mathfrak{a}\big(L(x,t;\epsilon),s(x,t;\epsilon)\big) = 1 + \frac{1}{2}\epsilon^2 t^2\frac{\partial^2}{\partial x^2}f_0(x,t) + \mathrm{O}(\epsilon^3), \tag{1.37}$$

$$\mathfrak{a}^{-1}\big(L(x,t;\epsilon),s(x,t;\epsilon)\big) - \mathfrak{a}\big(L(x,t;\epsilon),s(x,t;\epsilon)\big) \sim -\epsilon^2 t^2\frac{\partial^2}{\partial x^2}f_0(x,t), \tag{1.38}$$

$$\mathfrak{b}\big(L(x,t;\epsilon),s(x,t;\epsilon)+1\big) \sim -2\epsilon^2 t^2\frac{\partial^2}{\partial x^2}f_0(x,t). \tag{1.39}$$

Introducing ψ and ς by the $\epsilon \to 0$ expansions

$$(\epsilon t)^{1/2}\varphi(\zeta/\epsilon;L(x,t;\epsilon),s(x,t;\epsilon)) = \psi(\zeta;x,t) + \mathrm{O}(\epsilon), \qquad \sigma(\zeta/\epsilon) = \varsigma(\zeta) + \mathrm{O}(\epsilon), \tag{1.40}$$

we also have (by approximating a Riemann–Stieltjes sum with the corresponding integral)

$$\frac{1}{L}\sum_{l\in\mathbb{Z}'}\big(\sigma(l+1)-\sigma(l)\big)\varphi(l+1;L,s-1)\varphi(l;L,s)\Big|_{L=L(x,t;\epsilon),\ s=s(x,t;\epsilon)} \sim \epsilon^2 t\int_{\mathbb{R}}\varsigma'(\eta)\psi(\eta;x,t)^2\mathrm{d}\eta, \tag{1.41}$$

$$\frac{2}{L}\sum_{l\in\mathbb{Z}'}\big(\sigma(l+1)-\sigma(l)\big)\varphi(l+1;L,s)\varphi(l;L,s)\Big|_{L=L(x,t;\epsilon),\ s=s(x,t;\epsilon)} \sim 2\epsilon^2 t\int_{\mathbb{R}}\varsigma'(\eta)\psi(\eta;x,t)^2\mathrm{d}\eta. \tag{1.42}$$

By (1.33) we have equality of (1.38) and (1.41), and looking at the leading order terms gives (1.31). Similarly, by (1.34) we have equality of (1.39) and (1.42), and looking at the leading order terms gives again (1.31). Moreover, using (1.37) and (1.39), Equation (1.35) reduces to (1.28). Finally, also the asymptotic relation in (1.36) for φ formally matches with the one for ψ in (1.28) using [11, Lemma 4.4]

$$L^{1/3}\mathrm{J}_{2L+\xi L^{1/3}}(2L) \sim \mathrm{Ai}(\xi), \qquad L\to+\infty. \tag{1.43}$$

1.4 | Organization of the rest of the paper

In Section 2 we gather some properties of the discrete Bessel point process and its finite-temperature deformation. In Section 3 we prove a *discrete RH* characterization of $Q_\sigma(L,s)$, following a general strategy developed by Borodin and Deift [6] which parallels the theory of integrable operators of Its–Izergin–Korepin–Slavnov [22] in a discrete setting. Next, we prove Theorem I, II, and III in Sections 4, 5, and 6, respectively. We briefly discuss the connection of our approach to the results of Borodin [7] relative to the special case $\sigma = \mathbf{1}_{\mathbb{Z}'_+}$ in Section 6.1. An elementary technical lemma which is helpful in the discussion of discrete RH problems is deferred to the Appendix.

2 | PRELIMINARIES ON THE DISCRETE BESSEL KERNEL

The Bessel functions satisfy [30, Equation (10.6.1)]

$$L\big(\mathrm{J}_{k+1}(2L)+\mathrm{J}_{k-1}(2L)\big) = k\mathrm{J}_k(2L), \qquad k\in\mathbb{C}, \tag{2.1}$$

and [30, Equation (10.4.1)]

$$\mathrm{J}_{-k}(2L) = (-1)^k\mathrm{J}_k(2L), \qquad k\in\mathbb{Z}. \tag{2.2}$$

Lemma 2.1. *As $k\to+\infty$, we have*

$$\mathrm{J}_k(2L) \sim \frac{1}{\sqrt{2\pi k}}\left(\frac{\mathrm{e}L}{k}\right)^k, \qquad \frac{\partial}{\partial\varkappa}\mathrm{J}_\varkappa(2L)\Big|_{\varkappa=k} \sim \frac{\log(L/k)}{\sqrt{2\pi k}}\left(\frac{\mathrm{e}L}{k}\right)^k, \tag{2.3}$$

and, as $k \to +\infty$ through integer values, we have

$$\mathrm{J}_{-k}(2L) \sim \frac{(-1)^k}{\sqrt{2\pi k}}\left(\frac{\mathrm{e}L}{k}\right)^k, \qquad \frac{\partial}{\partial \varkappa}\mathrm{J}_\varkappa(2L)\bigg|_{\varkappa=-k} \sim (-1)^{k+1}\sqrt{\frac{2\pi}{k}}\left(\frac{k}{\mathrm{e}L}\right)^k. \tag{2.4}$$

Proof. For real $k > -1/2$, we can represent the Bessel function by the Poisson integral [30, Equation (10.9.4)]

$$\mathrm{J}_k(2L) = \frac{L^k}{\sqrt{\pi}\,\Gamma(k+\frac{1}{2})}\int_0^\pi \cos(2L\cos\theta)\mathrm{e}^{2k\log(\sin\theta)}\mathrm{d}\theta. \tag{2.5}$$

Since $\theta \mapsto \log(\sin\theta)$ has a unique non-degenerate maximum at $\theta = \pi/2$ for $\theta \in (0,\pi)$, it suffices to use Laplace's method to obtain the large k asymptotics of the integral. Combining with Stirling's asymptotics, we obtain the first relation in (2.3). The first relation in (2.4) then follows from (2.2).

Next, by (2.5), for real $k > -1/2$,

$$\begin{aligned}\frac{\partial}{\partial k}\mathrm{J}_k(2L) &= \frac{\partial}{\partial k}\left(\log\frac{L^k}{\sqrt{\pi}\,\Gamma(k+\frac{1}{2})}\right)\mathrm{J}_k(2L)\\ &\quad+\frac{L^k}{\sqrt{\pi}\,\Gamma(k+\frac{1}{2})}\int_0^\pi 2\log(\sin\theta)\cos(2L\cos\theta)\mathrm{e}^{2k\log(\sin\theta)}\mathrm{d}\theta.\end{aligned} \tag{2.6}$$

Using the asymptotics for the digamma function Γ'/Γ, as well as the already established first relation in (2.3) for the first term, and again Laplace's method for the second term, we obtain (after some computations) the second relation in (2.3). Finally, for the last relation we use that, for $k \in \mathbb{Z}$, we have [30, 10.2.4],

$$(-1)^k\frac{\partial}{\partial\varkappa}\mathrm{J}_\varkappa(2L)\bigg|_{\varkappa=-k} = \pi\mathrm{Y}_k(2L) - \frac{\partial}{\partial\varkappa}\mathrm{J}_\varkappa(2L)\bigg|_{\varkappa=k}, \tag{2.7}$$

where $\mathrm{Y}_k(\cdot)$ is the Bessel function of second kind of order k, and it suffices to use the second relation in (2.3) along with the asymptotics $\mathrm{Y}_k(2L) \sim -\sqrt{2/(\pi k)}(\mathrm{e}L/k)^{-k}$ as $k \to +\infty$ [30, Equation (10.19.2)]. □

Let us recall the discrete Bessel kernel $K^{\mathrm{Be}}(a,b) = \sum_{l\in\mathbb{Z}'_+}\mathrm{J}_{a+l}(2L)\mathrm{J}_{b+l}(2L)$, as in (1.2). It is worth observing that only Bessel functions of integer order appear in this expression.

Lemma 2.2. *We have*

$$K^{\mathrm{Be}}(a,b) = L\,\frac{\mathrm{J}_{a-\frac{1}{2}}(2L)\mathrm{J}_{b+\frac{1}{2}}(2L) - \mathrm{J}_{a+\frac{1}{2}}(2L)\mathrm{J}_{b-\frac{1}{2}}(2L)}{a-b}, \qquad a,b\in\mathbb{Z}',\ a\neq b, \tag{2.8}$$

$$K^{\mathrm{Be}}(a,a) = L\left(\mathrm{J}_{a+\frac{1}{2}}(2L)\frac{\partial\mathrm{J}_{a-\frac{1}{2}}(2L)}{\partial a} - \mathrm{J}_{a-\frac{1}{2}}(2L)\frac{\partial\mathrm{J}_{a+\frac{1}{2}}(2L)}{\partial a}\right), \qquad a\in\mathbb{Z}'. \tag{2.9}$$

In particular, $K^{\mathrm{Be}}(a,a)$ is a decreasing function of $a \in \mathbb{Z}'$ satisfying

$$K^{\mathrm{Be}}(a,a) \to 1, \qquad a\to-\infty,\ a\in\mathbb{Z}'. \tag{2.10}$$

Proof. Fix $M \in \mathbb{Z}'_+$. Using (2.1) we compute, for any real $a \neq b$, omitting the argument $2L$ of the Bessel functions,

$$\begin{aligned}(a-b)\sum_{l\in\mathbb{Z}'_+\cap[\frac{1}{2},M]} \mathrm{J}_{a+l}\mathrm{J}_{b+l} &= \sum_{l\in\mathbb{Z}'_+\cap[\frac{1}{2},M]} \big((a+l)\mathrm{J}_{a+l}\mathrm{J}_{b+l} - (b+l)\mathrm{J}_{a+l}\mathrm{J}_{b+l}\big)\\ &= L\sum_{l\in\mathbb{Z}'_+\cap[\frac{1}{2},M]} \big(\mathrm{J}_{a+l-1}\mathrm{J}_{b+l} + \mathrm{J}_{a+l+1}\mathrm{J}_{b+l} - \mathrm{J}_{a+l}\mathrm{J}_{b+l-1} - \mathrm{J}_{a+l}\mathrm{J}_{b+l+1}\big)\\ &= L\big(\mathrm{J}_{a-\frac{1}{2}}\mathrm{J}_{b+\frac{1}{2}} + \mathrm{J}_{a+M+1}\mathrm{J}_{b+M} - \mathrm{J}_{a+\frac{1}{2}}\mathrm{J}_{b-\frac{1}{2}} - \mathrm{J}_{a+M}\mathrm{J}_{b+M+1}\big), \end{aligned} \tag{2.11}$$

where in the last step we telescope the sum. Sending $M \to +\infty$ and using the first asymptotics in (2.3) and (2.4), we obtain (2.8). Sending instead $a \to b$ first and then sending $M \to +\infty$ we obtain (2.9). Finally, it suffices to insert (2.4) in (2.9) to obtain (2.10). □

Lemma 2.3. *For all $a, b \in \mathbb{Z}'$ we have $\sum_{l\in\mathbb{Z}'} \mathrm{J}_{a+l}(2L)\mathrm{J}_{b+l}(2L) = \delta_{a,b}$.*

Proof. Let $M, N \in \mathbb{Z}'$ with $N < 0 < M$. Using a similar argument as in (2.11), we obtain, for real $a \neq b$,

$$(a-b)\sum_{l\in\mathbb{Z}'\cap[N,M]} \mathrm{J}_{a+l}\mathrm{J}_{b+l} = L\big(\mathrm{J}_{a+N-1}\mathrm{J}_{b+N} - \mathrm{J}_{a+N}\mathrm{J}_{b+N-1} + \mathrm{J}_{a+M+1}\mathrm{J}_{b+M} - \mathrm{J}_{a+M}\mathrm{J}_{b+M+1}\big), \tag{2.12}$$

and so sending $M \to +\infty, N \to -\infty$ and using (2.3) we obtain the thesis for $a \neq b$. Sending instead $a \to b$ first, and then sending $M \to +\infty, N \to -\infty$ using (2.4) and (2.10) we obtain the thesis for $a = b$. □

Lemma 2.4. *We have $0 \leqslant \mathcal{K}^{\mathrm{Be}}_\sigma \leqslant 1$. Moreover, if $\sigma \in \ell^1(\mathbb{Z}' \cap (-\infty, 0))$, the operator $\mathcal{P}_s\mathcal{K}^{\mathrm{Be}}_\sigma\mathcal{P}_s$ is trace class, where $\mathcal{P}_s$ is the orthogonal projector onto $\ell^2(\{s+1, s+2, \dots\})$, for all $s \in \mathbb{Z}'$.*

Proof. It follows from Lemma 2.3 that the operator $\mathcal{J}$ induced by the kernel $\mathrm{J}_{a+b}(2L)$ is an unitary involution of $\ell^2(\mathbb{Z}')$, that is, $\mathcal{J} = \mathcal{J}^\dagger = \mathcal{J}^{-1}$. By a slight abuse of notation, denote with σ the operator of multiplication by σ, that is, the operator on $\ell^2(\mathbb{Z}')$ induced by the kernel $\sigma(a)\delta(a, b)$. Then, by definition, $\mathcal{K}^{\mathrm{Be}}_\sigma = \mathcal{J}\sigma\mathcal{J}$. Let $\langle\cdot,\cdot\rangle$ be the scalar product on $\ell^2(\mathbb{Z}')$: since $0 \leqslant \sigma \leqslant 1$, we have

$$\langle(\mathcal{K}^{\mathrm{Be}}_\sigma)^2\psi, \psi\rangle = \langle\sigma^2\mathcal{J}\psi, \mathcal{J}\psi\rangle \leqslant \langle\sigma\mathcal{J}\psi, \mathcal{J}\psi\rangle = \langle\mathcal{K}^{\mathrm{Be}}_\sigma\psi, \psi\rangle, \qquad \text{for all } \psi \in \ell^2(\mathbb{Z}'). \tag{2.13}$$

Therefore, $\mathcal{K}^{\mathrm{Be}}_\sigma \geqslant (\mathcal{K}^{\mathrm{Be}}_\sigma)^2 \geqslant 0$, which also implies $1 - \mathcal{K}^{\mathrm{Be}}_\sigma \geqslant (1 - \mathcal{K}^{\mathrm{Be}}_\sigma)^2 \geqslant 0$.

For the second statement, observe that $\mathcal{P}_s\mathcal{K}^{\mathrm{Be}}_\sigma\mathcal{P}_s = \mathcal{H}_s\mathcal{H}_s^\dagger$ where $\mathcal{H}_s$ is induced by kernel

$$H_s(a, b) = \mathbf{1}_{a>s}\mathrm{J}_{a+b}(2L)\sqrt{\sigma}(b), \qquad a, b \in \mathbb{Z}'. \tag{2.14}$$

For a fixed $s \in \mathbb{Z}'$, the operator $\mathcal{H}_s$ is Hilbert–Schmidt on $\ell^2(\mathbb{Z}')$ if and only if

$$\sum_{a,b\in\mathbb{Z}'} |H_s(a, b)|^2 = \sum_{a\in\mathbb{Z}'_+}\sum_{b\in\mathbb{Z}'} \mathbf{1}_{a>s}\mathrm{J}_{a+b}(2L)^2\sigma(b) = L\sum_{l\in\mathbb{Z}'} \sigma(l - s - \tfrac{1}{2})K^{\mathrm{Be}}(l, l) < +\infty. \tag{2.15}$$

The convergence of the latter series at $l \to +\infty$ follows from (2.9) and the first asymptotic relations in (2.3) and (2.4), along with $0 \leqslant \sigma \leqslant 1$. The convergence at $l \to -\infty$ follows instead by (2.9) and the second asymptotic relations in (2.3) and (2.4), along with the summability assumption on σ. □

It follows from this lemma and the Macchi–Soshnikov criterion [31, Theorem 3] that there exists a unique determinantal point process on $\mathbb{Z}'$ whose correlation kernel is K^{Be}_{σ}. It also follows from the general theory of determinantal point processes, for example, from [31, Theorem 4], that this process has almost surely a largest particle $a_{\max}$, whose distribution is given by the Fredholm determinant as in (1.8).

3 | DISCRETE RIEMANN–HILBERT CHARACTERIZATION OF Q_σ

Let us introduce the operator $\mathcal{M}_s$ on $\ell^2(\mathbb{Z}')$, for $s \in \mathbb{Z}'$, induced by the kernel

$$M_s(a,b) = \sqrt{\sigma}\big(a-s-\tfrac{1}{2}\big)K^{\mathrm{Be}}(a,b)\sqrt{\sigma}\big(b-s-\tfrac{1}{2}\big), \qquad a,b\in\mathbb{Z}', \tag{3.1}$$

where K^{Be} is as in (1.2). The operator $\mathcal{M}_s$ is of *discrete integrable form* [6, 7], namely, using (2.8) the off-diagonal entries of the kernel can be expressed as

$$M_s(a,b) = \frac{\mathbf{f}^\top(a)\mathbf{g}(b)}{a-b}, \qquad a,b,\in\mathbb{Z}',\ a\neq b, \tag{3.2}$$

where

$$\mathbf{f}(a) := \sqrt{\sigma}(a-s-\tfrac{1}{2})\begin{pmatrix} \mathrm{J}_{a-\frac{1}{2}}(2L) \\ L\mathrm{J}_{a+\frac{1}{2}}(2L)\end{pmatrix}, \qquad \mathbf{g}(b) := \sqrt{\sigma}(b-s-\tfrac{1}{2})\begin{pmatrix} L\mathrm{J}_{b+\frac{1}{2}}(2L) \\ -\mathrm{J}_{b-\frac{1}{2}}(2L)\end{pmatrix}. \tag{3.3}$$

Using (2.9) we can express the diagonal entries as

$$\begin{aligned} M_s(a,a) &= \sigma(a-s-\tfrac{1}{2})K^{\mathrm{Be}}(a,a) \\ &= L\,\sigma(a-s-\tfrac{1}{2})\left(\mathrm{J}_{a+\frac{1}{2}}(2L)\frac{\partial \mathrm{J}_{a-\frac{1}{2}}(2L)}{\partial a} - \mathrm{J}_{a-\frac{1}{2}}(2L)\frac{\partial \mathrm{J}_{a+\frac{1}{2}}(2L)}{\partial a}\right). \end{aligned} \tag{3.4}$$

Lemma 3.1.

(i) *The operator $\mathcal{M}_s$ is trace class and we have*

$$Q_\sigma(L,s) = \det(1-\mathcal{M}_s). \tag{3.5}$$

(ii) *The identity (1.9) holds true.*

(iii *For all $s\in\mathbb{Z}'$ such that $Q_\sigma(L,s)>0$, we have*

$$\frac{Q_\sigma(L,s-1)}{Q_\sigma(L,s)} - 1 = \operatorname{tr}\big((1-\mathcal{M}_s)^{-1}\mathcal{N}_s\big), \tag{3.6}$$

where $\mathcal{N}_s$ is the rank one operator on $\ell^2(\mathbb{Z}')$ induced by the kernel

$$N_s(a,b) = \sqrt{\sigma}(a-s-\tfrac{1}{2})\,\mathrm{J}_{a-\frac{1}{2}}(2L)\,\sqrt{\sigma}(b-s-\tfrac{1}{2})\,\mathrm{J}_{b-\frac{1}{2}}(2L). \tag{3.7}$$

Proof.

(i) We have

$$Q_\sigma(L,s) = \det(1 - \mathcal{P}_s \mathcal{K}_\sigma^{\mathrm{Be}} \mathcal{P}_s) = \det(1 - \mathcal{H}_s \mathcal{H}_s^\dagger) = \det(1 - \mathcal{H}_s^\dagger \mathcal{H}_s), \tag{3.8}$$

where $\mathcal{H}_s$ is induced by the kernel (2.14). Let $\mathcal{T}$ be the shift operator on $\ell^2(\mathbb{Z}')$, induced by the kernel $T(a,b) = \delta_{a,b+1}$. It is straightforward to verify that $\mathcal{H}_s^\dagger \mathcal{H}_s = \mathcal{T}^{s+\frac{1}{2}} \mathcal{M}_s \mathcal{T}^{-s-\frac{1}{2}}$. This identity implies that $\mathcal{M}_s$ is trace class, and, by combining it with (3.8), we obtain (3.5).

(ii) We have, by the previous point,

$$Q_\sigma(L,s) = \det(1 - \sigma(\cdot - s - \tfrac{1}{2})\mathcal{K}^{\mathrm{Be}}), \tag{3.9}$$

where $\sigma(\cdot - s - \frac{1}{2})$ denotes the multiplication operator induced by the kernel $\sigma(a - s - \frac{1}{2})\delta_{a,b}$. Then (1.9) follows from a general property of determinantal point processes (for example, see [9, Equation (11.2.4)]).

(iii) Let $\mathcal{S}_s$ be the operator on $\ell^2(\mathbb{Z}')$ induced by the kernel $S_s(a,b) = \sqrt{\sigma}(a - s - \frac{1}{2})\delta_{a,b}$. Then, $\mathcal{M}_s = \mathcal{S}_s \mathcal{K}^{\mathrm{Be}} \mathcal{S}_s$ where $\mathcal{K}^{\mathrm{Be}}$ is the operator induced by the discrete Bessel kernel K^{Be}, defined in (1.2). Recalling the shift operator $\mathcal{T}$, induced by the kernel $T(a,b) = \delta_{a,b+1}$, we observe that $\mathcal{S}_{s-1} = \mathcal{T}^{-1}\mathcal{S}_s\mathcal{T}$ so that

$$Q_\sigma(L,s-1) = \det\left(1 - \mathcal{S}_s \mathcal{T} \mathcal{K}^{\mathrm{Be}} \mathcal{T}^{-1} \mathcal{S}_s\right) = \det\left(1 - \mathcal{M}_s + \mathcal{S}_s\left(\mathcal{K}^{\mathrm{Be}} - \mathcal{T}\mathcal{K}^{\mathrm{Be}}\mathcal{T}^{-1}\right)\mathcal{S}_s\right). \tag{3.10}$$

From (1.2), we note that $\mathcal{N}_s := \mathcal{S}_s(\mathcal{K}^{\mathrm{Be}} - \mathcal{T}\mathcal{K}^{\mathrm{Be}}\mathcal{T}^{-1})\mathcal{S}_s$ is the rank one operator induced by the kernel (3.7). As long as $Q_\sigma(L,s) \neq 0$, we have

$$\begin{aligned} Q_\sigma(L,s-1) &= \det(1 - \mathcal{M}_s + \mathcal{N}_s) = \det(1 - \mathcal{M}_s)\det\left(1 + (1 - \mathcal{M}_s)^{-1}\mathcal{N}_s\right) \\ &= Q_\sigma(L,s)\left(1 + \operatorname{tr}\left((1 - \mathcal{M}_s)^{-1}\mathcal{N}_s\right)\right), \end{aligned} \tag{3.11}$$

using a standard formula for the determinant of a rank one perturbation of the identity.

□

The next key step is to apply the discrete version of Its–Izergin–Korepin–Slavnov procedure [22], as developed for instance by Borodin [7]. This approach provides us with an effective way of computing the resolvent operator $\mathcal{R}_s := (1 - \mathcal{M}_s)^{-1} - 1$ that proves useful to investigate (3.6). Indeed, the main result of this theory (Theorem 3.4 below, following from general results of Borodin) is that the resolvent operator $\mathcal{R}_s$ is also induced by a kernel of integrable form expressed through a meromorphic 2×2 matrix-valued function $Y(\cdot)$ (parametrically depending on σ, s, L as well) which is uniquely characterized by the following RH conditions.

Discrete RH problem for Y

(a) *$Y(z)$ is a 2×2 matrix-valued meromorphic function of z with simple poles at $\mathbb{Z}'$ only.*

(b) *For all $a \in \mathbb{Z}'$, the function*

$$Y_a^{\mathrm{reg}}(z) := Y(z)\left(I - \frac{W_Y(a)}{z - a}\right) \tag{3.12}$$

has a removable singularity at $z = a$, where

$$W_Y(a) := \frac{\mathbf{f}(a)\mathbf{g}^\top(a)}{1 - M_s(a,a)}, \qquad a \in \mathbb{Z}'. \tag{3.13}$$

Here, $\mathbf{f}(a)$, $\mathbf{g}(a)$, and $M_s(a,a)$ are given explicitly in (3.3) and (3.4).

(c) *We have $\lim_{n\to+\infty} \sup_{|z|=n} |Y(z) - I| = 0$, where the limit is taken over integer values of n, I denotes the 2×2 identity matrix and $|\cdot|$ denotes any matrix norm.*

Before describing how Y allows us to express the resolvent operator $\mathcal{R}_s$, we make a few observations.

Remark 3.2.

(i) The usual formulation of condition (b) in the discrete RH problem is the slightly different but completely equivalent requirement that, for all $a \in \mathbb{Z}'$, the limit $\lim_{z\to a} Y(z)W_Y(a)$ exists and that

$$\lim_{z\to a} Y(z)W_Y(a) = \operatorname*{res}_{z=a} Y(z)\,\mathrm{d}z. \tag{3.14}$$

(ii) Since $0 \leqslant \sigma(a) \leqslant 1$ and $K^{\mathrm{Be}}(a,a) < 1$ for all $a \in \mathbb{Z}'$ (see Lemma 2.2), we get $1 - M_s(a,a) > 0$ for all $a \in \mathbb{Z}'$. In particular, (3.13) is well defined.

(iii) For any solution Y to the above discrete RH problem, we have $\det Y(z) = 1$ identically in z. Indeed, $\mathbf{f}^\top(a)\mathbf{g}(a) = 0$ implies $W_Y^2(a) = 0$, hence $\det Y(z) = \det Y_a^{\mathrm{reg}}(z)$ for all $a \in \mathbb{Z}'$ and so $\det Y(z)$ extends to an entire function of z. By condition (c) together with the maximum modulus theorem we conclude that $\det Y(z) = 1$ identically in z.

(iv) The solution Y to the above discrete RH problem is unique, if any exists. Indeed, for any two solutions $Y(z)$ and $\widetilde{Y}(z)$, the matrix $T(z) := \widetilde{Y}(z)Y^{-1}(z)$ has removable singularities at $\mathbb{Z}'$ by condition (b), because $T(z) = \widetilde{Y}_a^{\mathrm{reg}}(z)(Y_a^{\mathrm{reg}})^{-1}(z)$ for all $a \in \mathbb{Z}'$, hence $T(z)$ extends to an entire matrix function of z. By condition (c) together with the maximum modulus theorem, we infer that $T(z) = I$ identically in z.

(v) Condition (b) in the discrete RH problem for Y implies that $Y(z)$ has the following Laurent expansion near $z = a \in \mathbb{Z}'$:

$$Y(z) = C_Y(a)\left(\frac{W_Y(a)}{z-a} + I + Y_1(a)(z-a) + O\big((z-a)^2\big)\right), \tag{3.15}$$

where $C_Y(a)$ is an invertible matrix. In particular, although $Y(z)$ has a pole as $z \to a \in \mathbb{Z}'$, the limits $\lim_{z\to a} Y(z)\mathbf{f}(a)$ and $\lim_{z\to a} Y^{-\top}(z)\mathbf{g}(a)$ for $a \in \mathbb{Z}'$ exist and are finite. In the interest of lighter notations, we suppress the limit notation in such expressions, namely for $a \in \mathbb{Z}'$ we define

$$Y(a)\mathbf{f}(a) := \lim_{z\to a} Y(z)\mathbf{f}(a), \quad Y^{-\top}(a)\mathbf{g}(a) := \lim_{z\to a} Y^{-\top}(z)\mathbf{g}(a). \tag{3.16}$$

Similarly, for $a \in \mathbb{Z}'$ we also define

$$Y'(a)\mathbf{f}(a) := \lim_{z\to a} \frac{\mathrm{d}Y(z)}{\mathrm{d}z}\mathbf{f}(a) = C_Y(a)Y_1(a)\mathbf{f}(a). \tag{3.17}$$

Similarly, the inverse matrix Y^{-1} has the Laurent expansion

$$Y^{-1}(z) = \left(-\frac{W_Y(a)}{z-a} + I + \widetilde{Y}_1(a)(z-a) + O\big((z-a)^2\big)\right)\widetilde{C}_Y(a), \tag{3.18}$$

where $\widetilde{C}_Y(a)$ is an invertible matrix, which does not necessarily coincide with $C_Y^{-1}(a)$.

(vi) In what follows we shall need also the subleading terms in the expansion at $z \to \infty$:

$$Y(z) = I + \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} z^{-1} + O(z^{-2}), \tag{3.19}$$

for functions $\alpha = \alpha(L,s)$, $\beta = \beta(L,s)$ and $\gamma = \gamma(L,s)$. This matrix is traceless because $\det Y(z) = 1$ identically in z. Here, as in condition (c) of the discrete RH problem, $|z| \to +\infty$ through integer values.

Lemma 3.3. *Fix $a \in \mathbb{Z}'$. Let $C_Y(a)$ and $Y_1(a)$ be as in (3.15), and let $c_Y(a) := \det C_Y(a)$. We have*

$$\frac{\mathbf{g}^\top(a) Y_1(a)\mathbf{f}(a)}{1 - M_s(a,a)} = \frac{c_Y(a) - 1}{c_Y(a)} \tag{3.20}$$

and, for some $d_Y(a) \in \mathbb{C}$,

$$\widetilde{C}_Y(a) C_Y(a) = c_Y(a) I + d_Y(a) W_Y(a). \tag{3.21}$$

Proof. Since $\mathbf{f}(a), \mathbf{g}(a)$ are orthogonal and non-zero, the 2×2 matrix

$$U := \left(\frac{\mathbf{f}(a)}{|\mathbf{f}(a)|} \;\middle|\; \frac{\mathbf{g}(a)}{|\mathbf{g}(a)|} \right) \tag{3.22}$$

is an orthogonal matrix, $UU^\top = I$. Here, we denote $|\mathbf{v}| := \sqrt{\mathbf{v}^\top \mathbf{v}}$ for a column vector $\mathbf{v} \in \mathbb{C}^2$. Introducing

$$\varkappa := \frac{|\mathbf{f}(a)| \cdot |\mathbf{g}(a)|}{1 - M_{\sigma,s}(a,a)}, \tag{3.23}$$

we have

$$W_Y(a) = U \begin{pmatrix} 0 & \varkappa \\ 0 & 0 \end{pmatrix} U^\top. \tag{3.24}$$

Using that $\det Y(z) = 1$ identically in z (Remark 3.2) and (3.15),

$$\begin{aligned} \frac{1}{c_Y(a)} = \frac{\det Y(z)}{\det C_Y(a)} &= \det\left(\frac{1}{z-a} U \begin{pmatrix} 0 & \varkappa \\ 0 & 0 \end{pmatrix} U^\top + I + Y_1(a)(z-a) + O\big((z-a)^2\big)\right) \\ &= \det\left(\frac{1}{z-a} \begin{pmatrix} 0 & \varkappa \\ 0 & 0 \end{pmatrix} + I + U^\top Y_1(a) U (z-a) + O\big((z-a)^2\big)\right) \\ &= 1 - \varkappa\big(U^\top Y_1(a) U\big)_{2,1} + O(z-a) = 1 - \varkappa\big(U^\top Y_1(a) U\big)_{2,1}. \end{aligned} \tag{3.25}$$

Finally, using (3.22) and (3.23) we get

$$\varkappa\big(U^\top Y_1(a)U\big)_{2,1} = \frac{\mathbf{g}^\top(a)Y_1(a)\mathbf{f}(a)}{1-M_{\sigma,s}(a,a)} \tag{3.26}$$

and (3.20) follows.

By multiplying the Laurent expansion of Y^{-1}, given in (3.18), on the right by that of Y, given in (3.15), vanishing of terms of order $(z-a)^{-1}$ implies $W_Y(a)\widetilde{C}_Y(a)C_Y(a) = \widetilde{C}_Y(a)C_Y(a)W_Y(a)$. In turn, this means that $\widetilde{C}_Y(a)C_Y(a) = e_Y(a)I + d_Y(a)W_Y(a)$ for some constants $d_Y(a), e_Y(a)$. Next, the fact that the constant term is the identity gives

$$\widetilde{C}_Y(a)C_Y(a) + \widetilde{Y}_1(a)\widetilde{C}_Y(a)C_Y(a)W_Y(a) - W_Y(a)\widetilde{C}_Y(a)C_Y(a)Y_1 = I$$
$$\Rightarrow (e_Y(a)-1)I + \big(d_Y(a)I + e_Y(a)\widetilde{Y}_1(a)\big)W_Y(a) = e_Y(a)W_Y(a)Y_1(a). \tag{3.27}$$

Multiplying the last relation by $\mathbf{f}^\top(a)$ on the left and by $\mathbf{f}(a)$ on the right, and combining with (3.20), we obtain $e_Y(a) = c_Y(a)$, and so also (3.21) is proved. □

Using [7, Theorem 1.1], we immediately obtain the following result.

Theorem 3.4. *Let $s \in \mathbb{Z}'$ be such that $Q_\sigma(L,s) > 0$, so that $1-\mathcal{M}_s$ is invertible. Then, the discrete RH problem for Y has a unique solution and the resolvent operator $\mathcal{R}_s := (1-\mathcal{M}_s)^{-1} - 1$ is induced by the kernel*

$$R_s(a,b) = \frac{\widetilde{\mathbf{f}}^\top(a)Y^\top(a)Y^{-\top}(b)\widetilde{\mathbf{g}}(b)}{a-b}, \quad R_s(a,a) = \frac{M_s(a,a)}{1-M_s(a,a)} + \widetilde{\mathbf{g}}^\top(a)Y^{-1}(a)Y'(a)\widetilde{\mathbf{f}}(a), \tag{3.28}$$

for $a, b \in \mathbb{Z}'$, $a \neq b$, where

$$\widetilde{\mathbf{f}}(a) := \frac{\mathbf{f}(a)}{1-M_s(a,a)}, \qquad \widetilde{\mathbf{g}}(a) := \frac{\mathbf{g}(a)}{1-M_s(a,a)}. \tag{3.29}$$

Thanks to this result we can prove the following variational formulas for Q_σ.

Theorem 3.5. *For all $L > 0$ and all $s \in \mathbb{Z}'$ such that $Q_\sigma(L,s) > 0$, we have*

$$\frac{Q_\sigma(L,s-1)}{Q_\sigma(L,s)} - 1 = \beta(L,s), \qquad \frac{\partial}{\partial L}\log Q_\sigma(L,s) = -\frac{2\alpha(L,s)}{L}, \tag{3.30}$$

where $\alpha(L,s)$ and $\beta(L,s)$ are defined in (3.19).

Proof. We start with the first equation in (3.30). By (3.6) and $(1-\mathcal{M}_s)^{-1} = 1 + \mathcal{R}_s$, we have

$$\frac{Q_\sigma(L,s-1)}{Q_\sigma(L,s)} - 1 = \operatorname{tr}\big((1-\mathcal{M}_s)^{-1}\mathcal{N}_s\big) = \sum_{a,b\in\mathbb{Z}'} \mathrm{J}_{a-\frac{1}{2}}\mathrm{J}_{b-\frac{1}{2}}\sqrt{\widetilde{\sigma}(a)}\sqrt{\widetilde{\sigma}(b)}\big(\delta_{a,b} + R_s(a,b)\big), \tag{3.31}$$

where $\widetilde{\sigma}(a) := \sigma(a-s-\frac{1}{2})$ and $R_s(a,b)$ is explicitly given in (3.28), and, throughout this proof, we omit the argument $2L$ of the Bessel functions. We start by computing the part of the sum that comes from $a \neq b$; denoting $\Delta = \{(a,a) : a \in \mathbb{Z}'\}$, this is

$$\sum_{a,b\in\mathbb{Z}'\setminus\Delta} \frac{\mathrm{J}_{a-\frac{1}{2}}\sqrt{\widetilde{\sigma}}(a)}{1-M_s(a,a)} \frac{\mathbf{f}^\top(a)Y^\top(a)Y^{-\top}(b)\mathbf{g}(b)}{a-b} \frac{\mathrm{J}_{b-\frac{1}{2}}\sqrt{\widetilde{\sigma}}(b)}{1-M_s(b,b)} = \sum_{a,b\in\mathbb{Z}'\setminus\Delta} \operatorname*{res}_{z=a} \operatorname*{res}_{w=b} \frac{\boldsymbol{\varphi}^\top(z)\boldsymbol{\psi}(w)}{z-b} \mathrm{d}w\mathrm{d}z, \tag{3.32}$$

where we introduce the meromorphic vector functions

$$\boldsymbol{\varphi}(z) := Y(z)\begin{pmatrix} 0 \\ -1 \end{pmatrix}, \qquad \boldsymbol{\psi}(w) := Y^{-\top}(w)\begin{pmatrix} -1 \\ 0 \end{pmatrix}. \tag{3.33}$$

Indeed, condition (b) in the discrete RH problem for Y implies that

$$\operatorname*{res}_{z=a} Y(z)\mathrm{d}z = \frac{Y(a)\mathbf{f}(a)\mathbf{g}^\top(a)}{1-M_s(a,a)}, \tag{3.34}$$

yielding

$$\operatorname*{res}_{z=a} \boldsymbol{\varphi}(z)\mathrm{d}z = Y(a)\mathbf{f}(a)\frac{\mathrm{J}_{a-\frac{1}{2}}\sqrt{\widetilde{\sigma}}(a)}{1-M_s(a,a)}, \qquad \operatorname*{res}_{w=b} \boldsymbol{\psi}(w)\mathrm{d}w = Y^{-\top}(b)\mathbf{g}(b)\frac{\mathrm{J}_{b-\frac{1}{2}}\sqrt{\widetilde{\sigma}}(b)}{1-M_s(b,b)}. \tag{3.35}$$

Using condition (c) in the discrete RH problem for Y, we can represent $\boldsymbol{\psi}$ by its (infinite) partial fraction expansion (see Lemma A.1), namely

$$\boldsymbol{\psi}(z) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \sum_{b\in\mathbb{Z}'} \frac{\operatorname*{res}_{w=b} \boldsymbol{\psi}(w)\mathrm{d}w}{z-b}. \tag{3.36}$$

Hence we can rewrite (3.32) as

$$\begin{aligned} &\sum_{a\in\mathbb{Z}'} \operatorname*{res}_{z=a} \boldsymbol{\varphi}^\top(z)\left[\boldsymbol{\psi}(z) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{\operatorname*{res}_{w=a} \boldsymbol{\psi}(w)\mathrm{d}w}{z-a}\right]\mathrm{d}z \\ &= \sum_{a\in\mathbb{Z}'} \operatorname*{res}_{z=a} \boldsymbol{\varphi}^\top(z)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathrm{d}z - \sum_{a\in\mathbb{Z}'} \operatorname*{res}_{z=a}\operatorname*{res}_{w=a} \frac{\boldsymbol{\varphi}^\top(z)\boldsymbol{\psi}(w)}{z-a}\mathrm{d}w\mathrm{d}z, \end{aligned} \tag{3.37}$$

where we use that $\boldsymbol{\varphi}^\top(z)\boldsymbol{\psi}(z) = 0$. For the first term in (3.37) we appeal to Cauchy theorem to write the sum as a formal residue at $z = \infty$;

$$\sum_{a\in\mathbb{Z}'} \operatorname*{res}_{z=a} \boldsymbol{\varphi}^\top(z)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathrm{d}z = \lim_{n\to+\infty} \frac{1}{2\pi\mathrm{i}} \oint_{|z|=n} \boldsymbol{\varphi}^\top(z)\mathrm{d}z\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \beta(L,s), \tag{3.38}$$

where $\beta(L,s)$ is introduced in (3.19). Using the Laurent expansion (3.15) and (3.21), we compute the second part in (3.37) as

$$-\sum_{a\in\mathbb{Z}'} \operatorname*{res}_{z=a}\operatorname*{res}_{w=a} \frac{\boldsymbol{\varphi}^\top(z)\boldsymbol{\psi}(w)}{z-a}\mathrm{d}w\mathrm{d}z = -\sum_{a\in\mathbb{Z}'} \frac{c_Y(a)\mathrm{J}^2_{a-\frac{1}{2}}\widetilde{\sigma}(a)}{1-M_s(a,a)}, \tag{3.39}$$

where $c_Y(a) := \det C_Y(a)$. We now compute the terms in (3.31) coming from the diagonal $\Delta \subset \mathbb{Z}' \times \mathbb{Z}'$; this contribution is, using (3.28) and Lemma 3.3,

$$\begin{aligned}
&\sum_{a\in\mathbb{Z}'} \mathrm{J}^2_{a-\frac12}\,\widetilde{\sigma}(a)\left(1+\frac{M_s(a,a)}{1-M_s(a,a)}+\widetilde{\mathbf{g}}^\top(a)Y^{-1}(a)Y'(a)\widetilde{\mathbf{f}}(a)\right)\\
&\qquad=\sum_{a\in\mathbb{Z}'} \mathrm{J}^2_{a-\frac12}\,\widetilde{\sigma}(a)\left(\frac{1}{1-M_s(a,a)}+\widetilde{\mathbf{g}}^\top(a)\widetilde{C}_Y(a)C_Y(a)Y_1(a)\widetilde{\mathbf{f}}(a)\right)\\
&\qquad=\sum_{a\in\mathbb{Z}'} \mathrm{J}^2_{a-\frac12}\,\widetilde{\sigma}(a)\left(\frac{1}{1-M_s(a,a)}+\frac{c_Y(a)}{1-M_s(a,a)}\,\frac{\mathbf{g}^\top(a)Y_1(a)\mathbf{f}(a)}{1-M_s(a,a)}\right)\\
&\qquad=\sum_{a\in\mathbb{Z}'} \mathrm{J}^2_{a-\frac12}\,\widetilde{\sigma}(a)\left(\frac{1}{1-M_s(a,a)}+\frac{c_Y(a)-1}{1-M_s(a,a)}\right)\\
&\qquad=\sum_{a\in\mathbb{Z}'} \frac{c_Y(a)\mathrm{J}^2_{a-\frac12}\,\widetilde{\sigma}(a)}{1-M_s(a,a)}.
\end{aligned} \tag{3.40}$$

The proof of the first equation in (3.30) is obtained by combining (3.32), (3.37)–(3.40).

The proof of the second equation in (3.30) is similar. We have

$$\begin{aligned}
\frac{\partial}{\partial L}\log Q_\sigma(L,s) &= -\mathrm{tr}\left((1-\mathcal{M}_s)^{-1}\frac{\partial \mathcal{M}_s}{\partial L}\right)\\
&= -\sum_{a,b\in\mathbb{Z}'}\left(\mathrm{J}_{a-\frac12}\mathrm{J}_{b+\frac12}+\mathrm{J}_{a+\frac12}\mathrm{J}_{b-\frac12}\right)\sqrt{\widetilde{\sigma}}(a)\sqrt{\widetilde{\sigma}}(b)\big(\delta_{a,b}+R_s(a,b)\big)\\
&= -2\sum_{a,b\in\mathbb{Z}'}\mathrm{J}_{a+\frac12}\mathrm{J}_{b-\frac12}\sqrt{\widetilde{\sigma}}(a)\sqrt{\widetilde{\sigma}}(b)\big(\delta_{a,b}+R_s(a,b)\big),
\end{aligned} \tag{3.41}$$

where we use the identity $(1-\mathcal{M}_s)^{-1} = 1+\mathcal{R}_s$, the symmetry $R_s(b,a) = R_s(a,b)$, and we compute $\partial M_s/\partial L$ using

$$\partial_L \begin{pmatrix} \mathrm{J}_{a+\frac12}(2L) \\ L\mathrm{J}_{a-\frac12}(2L)\end{pmatrix} = \begin{pmatrix} \frac{a-\frac12}{L} & -\frac{2}{L} \\ 2L & -\frac{a-\frac12}{L}\end{pmatrix}\begin{pmatrix} \mathrm{J}_{a+\frac12}(2L) \\ L\mathrm{J}_{a-\frac12}(2L)\end{pmatrix}. \tag{3.42}$$

As before, we start by computing the part of the sum that comes from $a \neq b$; denoting $\Delta = \{(a,a) : a \in \mathbb{Z}'\}$, this contribution to (3.41) is

$$\begin{aligned}
&-2\sum_{a,b\in\mathbb{Z}'\setminus\Delta}\frac{\mathrm{J}_{a+\frac12}\sqrt{\widetilde{\sigma}}(a)}{1-M_s(a,a)}\,\frac{\mathbf{f}^\top(a)Y^\top(a)Y^{-\top}(b)\mathbf{g}(b)}{a-b}\,\frac{\mathrm{J}_{b-\frac12}\sqrt{\widetilde{\sigma}}(b)}{1-M_s(b,b)}\\
&\qquad= -2\sum_{a,b\in\mathbb{Z}'\setminus\Delta}\operatorname*{res}_{z=a}\operatorname*{res}_{w=b}\frac{\boldsymbol{\omega}^\top(z)\boldsymbol{\psi}(w)}{z-b}\mathrm{d}w\mathrm{d}z,
\end{aligned} \tag{3.43}$$

where we introduce the meromorphic vector functions $\boldsymbol{\psi}$, as in (3.33), and

$$\boldsymbol{\omega}(z) := Y(z)\begin{pmatrix}1/L\\0\end{pmatrix}, \qquad \operatorname*{res}_{z=a}\boldsymbol{\omega}(z)\mathrm{d}z = Y(a)\mathbf{f}(a)\frac{\mathrm{J}_{a+\frac{1}{2}}\sqrt{\widetilde{\sigma}}(a)}{1-M_s(a,a)}, \quad a\in\mathbb{Z}', \tag{3.44}$$

the last equality stemming from (3.34). Thanks to (3.36), we rewrite (3.43) as

$$\begin{aligned} &-2\sum_{a\in\mathbb{Z}'}\operatorname*{res}_{z=a}\boldsymbol{\omega}^{\top}(z)\left[\boldsymbol{\psi}(z)+\begin{pmatrix}1\\0\end{pmatrix}-\frac{\operatorname*{res}_{w=a}\boldsymbol{\psi}(w)\mathrm{d}w}{z-a}\right]\mathrm{d}z\\ &\quad=-2\sum_{a\in\mathbb{Z}'}\operatorname*{res}_{z=a}\boldsymbol{\omega}^{\top}(z)\begin{pmatrix}1\\0\end{pmatrix}\mathrm{d}z+2\sum_{a\in\mathbb{Z}'}\operatorname*{res}_{z=a}\operatorname*{res}_{w=a}\frac{\boldsymbol{\omega}^{\top}(z)\boldsymbol{\psi}(w)}{z-a}\mathrm{d}w\mathrm{d}z, \end{aligned}\tag{3.45}$$

where we use that $\boldsymbol{\omega}^{\top}(z)\boldsymbol{\psi}(z)$ is regular at $\mathbb{Z}'$. Again, the first term is a formal residue at $z=\infty$:

$$-2\sum_{a\in\mathbb{Z}'}\operatorname*{res}_{z=a}\boldsymbol{\omega}^{\top}(z)\begin{pmatrix}1\\0\end{pmatrix}\mathrm{d}z=-2\lim_{n\to+\infty}\frac{1}{2\pi\mathrm{i}}\oint_{|z|=n}\boldsymbol{\omega}^{\top}(z)\mathrm{d}z\begin{pmatrix}1\\0\end{pmatrix}=-\frac{2\alpha(L,s)}{L}, \tag{3.46}$$

where $\alpha(L,s)$ is introduced in (3.19). Using the Laurent expansion (3.15) and (3.21), we compute the second part in (3.45) as

$$2\sum_{a\in\mathbb{Z}'}\operatorname*{res}_{z=a}\operatorname*{res}_{w=a}\frac{\boldsymbol{\omega}^{\top}(z)\boldsymbol{\psi}(w)}{z-a}\mathrm{d}w\mathrm{d}z=2\sum_{a\in\mathbb{Z}'}\frac{c_Y(a)\mathrm{J}_{a-\frac{1}{2}}\mathrm{J}_{a+\frac{1}{2}}\widetilde{\sigma}(a)}{1-M_s(a,a)}, \tag{3.47}$$

where, as before, $c_Y(a) := \det C_Y(a)$. With a computation completely analogous to (3.40) we compute the terms in (3.41) coming from the diagonal $\Delta\subset\mathbb{Z}'\times\mathbb{Z}'$ as

$$\begin{aligned} &-2\sum_{a\in\mathbb{Z}'}\mathrm{J}_{a-\frac{1}{2}}\mathrm{J}_{a+\frac{1}{2}}\widetilde{\sigma}(a)\left(1+\frac{M_s(a,a)}{1-M_s(a,a)}+\widetilde{\mathbf{g}}^{\top}(a)Y^{-1}(a)Y'(a)\widetilde{\mathbf{f}}(a)\right)\\ &\quad=-2\sum_{a\in\mathbb{Z}'}\frac{c_Y(a)\mathrm{J}_{a-\frac{1}{2}}\mathrm{J}_{a+\frac{1}{2}}\widetilde{\sigma}(a)}{1-M_s(a,a)}. \end{aligned}\tag{3.48}$$

The proof of the second equation in (3.30) is complete by combining (3.43), (3.45)–(3.48). □

4 | PROOF OF THEOREM I

Throughout this section we shall assume that $s\in\mathbb{Z}'$ is such that $Q_\sigma(L,s)>0$. In particular (Theorem 3.4), the matrix $Y(z)$ introduced in the last section exists and is unique.

4.1 | Dressing

We proceed to a *dressing* of the discrete RH problem for Y, mimicking a common technique for continuous RH problems, see, for example, [26]. Introduce the following entire matrix function of z:

$$\Phi(z) := \begin{pmatrix} \mathrm{J}_{z-\frac{1}{2}}(2L) & \mathrm{i}\pi \mathrm{H}^{(1)}_{z-\frac{1}{2}}(2L) \\ L\mathrm{J}_{z+\frac{1}{2}}(2L) & \mathrm{i}\pi L \mathrm{H}^{(1)}_{z+\frac{1}{2}}(2L) \end{pmatrix}, \tag{4.1}$$

where $\mathrm{H}^{(1)}_k(2L)$ is the Hankel function of the first kind of order k and argument $2L$ [30]. The vectors $\mathbf{f}$ and $\mathbf{g}$ in (3.3) can be expressed as

$$\mathbf{f}(a) = \sqrt{\sigma}(a - s - \tfrac{1}{2})\Phi(a)\begin{pmatrix}1\\0\end{pmatrix}, \qquad \mathbf{g}(a) = \sqrt{\sigma}(a - s - \tfrac{1}{2})\Phi^{-\top}(a)\begin{pmatrix}0\\-1\end{pmatrix}. \tag{4.2}$$

Moreover, we have $\det \Phi(z) = 1$ identically in z [30, Equation (10.5.3)]; thus $\Phi^{-1}(z)$ is also entire in z. For later convenience, we also note that Φ satisfies

$$\Phi(z+1) = \frac{1}{L}\begin{pmatrix} 0 & 1 \\ -L^2 & z+\frac{1}{2}\end{pmatrix}\Phi(z), \qquad \frac{\partial}{\partial L}\Phi(z) = \frac{1}{L}\begin{pmatrix} z-\frac{1}{2} & -2 \\ 2L^2 & -z+\frac{1}{2}\end{pmatrix}\Phi(z), \tag{4.3}$$

as it follows from the identities [30, Equation (10.6.1)]

$$\mathrm{B}_{k+1}(2L) + \mathrm{B}_{k-1}(2L) = \frac{k}{L}\mathrm{B}_k(2L), \qquad \frac{\partial}{\partial L}\mathrm{B}_k(2L) = \mathrm{B}_{k-1}(2L) - \mathrm{B}_{k+1}(2L), \tag{4.4}$$

where $\mathrm{B}_k(\cdot)$ is either of the functions $\mathrm{J}_k(\cdot), \mathrm{H}^{(1)}_k(\cdot)$.

In the interest of clarity, let us momentarily restore the dependence $Y(z) = Y_\sigma(z; L, s)$ and $\Phi(z) = \Phi(z; L)$. We introduce the matrix $\Psi(z) = \Psi_\sigma(z; L, s)$ by

$$\Psi_\sigma(z; L, s) := Y_\sigma(z + s + \tfrac{1}{2}; L, s)\Phi(z + s + \tfrac{1}{2}; L). \tag{4.5}$$

As we shall now prove, $\Psi(z)$ is uniquely characterized by the following conditions.

Discrete RH problem for Ψ

(a) *$\Psi(z)$ is a 2×2 matrix-valued meromorphic function of z with simple poles at $\mathbb{Z}'$ only.*
(b) *For all $a \in \mathbb{Z}'$, the function*

$$\Psi^{\mathrm{reg}}_a(z) := \Psi(z)\left(I - \frac{W_\Psi(a)}{z-a}\right) \tag{4.6}$$

has a removable singularity at $z = a$, where

$$W_\Psi(a) := \begin{pmatrix} 0 & -\sigma(a) \\ 0 & 0 \end{pmatrix}, \qquad a \in \mathbb{Z}'. \tag{4.7}$$

(c) *We have* $\lim_{n\to+\infty} \sup_{|z|=n} |\Psi(z)\Phi^{-1}(z + s + \frac{1}{2}) - I| = 0$, *where the limit is taken over integer values of n, I denotes the identity 2×2 matrix and $|\cdot|$ denotes any matrix norm.*

Proof. The only condition that does not directly follow from the analogous conditions of the discrete RH problem for Y, thus deserving a proof, is (b). For, we need to show that with $W_\Psi(a)$ as given, for all $a \in \mathbb{Z}'$

$$\Psi_a^{\mathrm{reg}}(z) = \Psi(z)\left(I - \frac{W_\Psi(a)}{z-a}\right) \tag{4.8}$$

is regular at $z = a$. Using that $W_Y^2(a) = 0$ and the definition (4.5) of Ψ, this condition is equivalent to regularity at $z = a$ of

$$Y_{a+\widehat{s}}^{\mathrm{reg}}(z+\widehat{s})\left(I + \frac{W_Y(a+\widehat{s})}{z-a}\right)\Phi(z+\widehat{s})\left(I - \frac{W_\Psi(a)}{z-a}\right), \tag{4.9}$$

where we denote $\widehat{s} := s + \frac{1}{2} \in \mathbb{Z}$. Since $Y_{a+\widehat{s}}^{\mathrm{reg}}(z+\widehat{s})$ is regular at $z = a$, we only need to prove that

$$\left(I + \frac{W_Y(a+\widehat{s})}{z-a}\right)\Phi(z+\widehat{s})\left(I - \frac{W_\Psi(a)}{z-a}\right) \text{ is regular at } z = a. \tag{4.10}$$

To this end we consider the Laurent expansion at $z = a$ of the previous expression, which is

$$-\frac{W_Y(a+\widehat{s})\Phi(a+\widehat{s})W_\Psi(a)}{(z-a)^2}$$
$$+\frac{W_Y(a+\widehat{s})\Phi(a+\widehat{s}) - \Phi(a+\widehat{s})W_\Psi(a) - W_Y(a+\widehat{s})\Phi'(a+\widehat{s})W_\Psi(a)}{z-a} + O(1). \tag{4.11}$$

Vanishing of the coefficient of $(z-a)^{-1}$ implies

$$W_\Psi(a) = \Phi^{-1}(a+\widehat{s})W_Y(a+\widehat{s})\Phi(a+\widehat{s})$$
$$\times\left(I + \Phi^{-1}(a+\widehat{s})\Phi'(a+\widehat{s})\Phi^{-1}(a+\widehat{s})W_Y(a+\widehat{s})\Phi(a+\widehat{s})\right)^{-1}. \tag{4.12}$$

Since $W_Y^2 = 0$, this also implies that the coefficient of $(z-a)^{-2}$ vanishes and that the series is regular. It remains to show that (4.12) simplifies to (4.7). To this end we deduce from (4.2) that

$$W_Y(a+\widehat{s}) = \rho(a,s)\Phi(a+\widehat{s})\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\Phi^{-1}(a+\widehat{s}), \qquad \rho(a,s) := -\frac{\sigma(a)}{1 - M_s(a+\widehat{s}, a+\widehat{s})}, \tag{4.13}$$

such that

$$W_\Psi(a) = \rho(a,s)\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\left(I + \rho(a,s)\Phi^{-1}(a+\widehat{s})\Phi'(a+\widehat{s})\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right)^{-1}. \tag{4.14}$$

We now observe by a direct computation that

$$I + \rho(a,s)\Phi^{-1}(a+\widehat{s})\Phi'(a+\widehat{s})\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = \begin{pmatrix}1 & \star\\ 0 & \frac{1}{1-M_s(a+\widehat{s},a+\widehat{s})}\end{pmatrix}, \tag{4.15}$$

where $\star$ denotes a term whose explicit expression is inconsequential in this computation. The right-hand side of (4.15) is invertible and so we finally get

$$W_\Psi(a) = \rho(a,s)\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & -(1-M_s(a+\widehat{s},a+\widehat{s}))\star\\ 0 & 1-M_s(a+\widehat{s},a+\widehat{s})\end{pmatrix} = \begin{pmatrix}0 & -\sigma(a)\\ 0 & 0\end{pmatrix}, \tag{4.16}$$

as claimed in (4.7). □

4.2 | Lax pair

The main result achieved by the dressing procedure is that $W_\Psi(a)$ is independent of s, L. This enables us to obtain the following equations. It is convenient here to restore the full dependence $\Psi(z) = \Psi(z; L, s)$ (omitting anyway the dependence on σ to have lighter notations).

Proposition 4.1. *The matrix* $\Psi(z; L, s)$ *satisfies*

$$\Psi(z; L, s+1) = \widetilde{A}(z; L, s)\Psi(z; L, s), \qquad \frac{\partial}{\partial L}\Psi(z; L, s) = \widetilde{B}(z; L, s)\Psi(z; L, s), \tag{4.17}$$

where

$$\widetilde{A}(z; L, s) = \frac{1}{L}\begin{pmatrix} 0 & 1+\beta(L, s+1) \\ -L^2-\gamma(L, s) & z+s+1+\alpha(L, s)-\alpha(L, s+1) \end{pmatrix}, \tag{4.18}$$

$$\widetilde{B}(z; L, s) = \frac{1}{L}\begin{pmatrix} z+s & -2\big(1+\beta(L, s)\big) \\ 2\big(L^2+\gamma(L, s)\big) & -z-s \end{pmatrix}, \tag{4.19}$$

with $\alpha(L, s), \beta(L, s), \gamma(L, s)$ *as in (3.19).*

Proof. The fact that $W_\Psi(a)$ is independent of s allows us to write

$$\widetilde{A}(z; L, s) := \Psi(z; L, s+1)\Psi^{-1}(z; L, s) = \Psi_a^{\mathrm{reg}}(z; L, s+1)(\Psi_a^{\mathrm{reg}})^{-1}(z; L, s) \tag{4.20}$$

for all $a \in \mathbb{Z}'$. Hence $\widetilde{A}(z; L, s)$ has removable singularities at $z \in \mathbb{Z}'$ by condition (b) in the discrete RH problem for Ψ, and so is an entire function of z. Further, due to (4.5) we can write

$$\begin{aligned}\widetilde{A}(z; L, s) &= Y(z+s+\tfrac{3}{2}; L, s+1)\Phi(z+s+\tfrac{3}{2}; L)\Phi^{-1}(z+s+\tfrac{1}{2}; L)Y^{-1}(z+s+\tfrac{1}{2}; L, s) \\ &= Y(z+s+\tfrac{3}{2}; L, s+1)\begin{pmatrix} 0 & \frac{1}{L} \\ -L & \frac{z+s+1}{L} \end{pmatrix}Y^{-1}(z+s+\tfrac{1}{2}; L, s),\end{aligned} \tag{4.21}$$

where we use (4.3). This identity, together with condition (c) in the RH problem for Y, shows that $\widetilde{A}(z; L, s)$ grows linearly as $z \to \infty$; Liouville theorem then implies that $\widetilde{A}(z; L, s)$ is a linear function of z, explicitly obtained by the asymptotic relation (3.19) plugged in (4.21), which gives the claimed formula for $\widetilde{A}(z; L, s)$.

Similarly, $\widetilde{B}(z; L, s) := \big(\partial_L\Psi(z; L, s)\big)\Psi^{-1}(z; L, s)$ is an entire function of z because $\widetilde{B}(z; L, s) = \big(\partial_L\Psi_a^{\mathrm{reg}}(z; L, s)\big)(\Psi_a^{\mathrm{reg}})^{-1}(z; L, s)$ for all $a \in \mathbb{Z}'$ hence the singularities at $\mathbb{Z}'$ are removable. (Here we use again that $W_\psi(a)$ does not depend on L, s.) Moreover, using (4.5) and (4.3), we obtain

$$\begin{aligned}\widetilde{B}(z; L, s) = {}& \big(\partial_L Y(z+s+\tfrac{1}{2}; L, s)\big)Y^{-1}(z+s+\tfrac{1}{2}; L, s) + Y(z+s+\tfrac{1}{2}; L, s) \\ &\times \begin{pmatrix} \frac{z+s}{L} & -\frac{2}{L} \\ 2L & -\frac{z+s}{L} \end{pmatrix}Y^{-1}(z+s; L, s).\end{aligned} \tag{4.22}$$

Finally, condition (c) in the RH problem for Y shows that $\widetilde{B}(z; L, s)$ grows linearly as $z \to \infty$, and by Liouville theorem it coincides with the linear function of z explicitly obtained by the asymptotic relation (3.19) plugged in (4.22), and this gives the claimed formula for $\widetilde{B}(z; L, s)$. □

Remark 4.2. Since $\det\Psi(z) = 1$ identically in z, we must have $\det\widetilde{A}(z) = 1$ identically in z as well. Looking at (4.18), this implies the relation

$$\big(1 + \beta(L, s + 1)\big)\big(L^2 + \gamma(L, s)\big) = L^2. \tag{4.23}$$

Proof of Theorem I. By Proposition 4.1 and Equations (4.5) and (4.3), we have

$$\frac{\partial Y(z; L, s)}{\partial L} Y^{-1}(z; L, s) = \frac{1}{L} Y(z; L, s)\begin{pmatrix} z - \frac{1}{2} & -2 \\ 2L^2 & -z + \frac{1}{2}\end{pmatrix} Y^{-1}(z; L, s) + \widetilde{B}(z - s - \tfrac{1}{2}; L, s). \tag{4.24}$$

Consider the asymptotic expansion of this identity as $z \to \infty$. Looking at the entry (1,1) of the coefficient of z^{-1} we obtain, also using (4.23),

$$\frac{\partial}{\partial L}\alpha(L, s) = 2L\left(1 - \frac{1 + \beta(L, s)}{1 + \beta(L, s + 1)}\right). \tag{4.25}$$

The proof is completed using (3.30). □

5 | PROOF OF THEOREM II

The discrete RH problem for Y can be described equivalently as a linear equation on $\ell^2(\mathbb{Z}') \otimes \mathbb{C}^2$, as we now explain following the classical operator theory for continuous RH problems and the works of Borodin [6, 7].

By conditions (a) and (c) in the discrete RH problem for Y and Lemma A.1, we can write the solution in the form

$$Y(z) = I + \sum_{b\in\mathbb{Z}'} \frac{R_b}{z - b}. \tag{5.1}$$

The matrices R_a must satisfy, by condition (b) or, equivalently, (3.14),

$$R_a = W_Y(a) + \sum_{b\in\mathbb{Z}'\setminus\{a\}} \frac{R_b W_Y(a)}{a - b}, \qquad a \in \mathbb{Z}'. \tag{5.2}$$

By (3.13) we can write

$$W_Y(a) = \widehat{\mathbf{f}}(a)\widehat{\mathbf{g}}^\top(a), \qquad \widehat{\mathbf{f}}(a) := \sigma(a - s - \tfrac{1}{2})\begin{pmatrix} \mathrm{J}_{a-\frac{1}{2}}(2L) \\ L\mathrm{J}_{a+\frac{1}{2}}(2L)\end{pmatrix},$$

$$\widehat{\mathbf{g}}(a) := \frac{1}{1 - M_s(a, a)}\begin{pmatrix} L\mathrm{J}_{a+\frac{1}{2}}(2L) \\ -\mathrm{J}_{a-\frac{1}{2}}(2L)\end{pmatrix}, \tag{5.3}$$

and so Equation (5.2) implies that R_a is a rank one matrix of the form

$$R_a = \mathbf{r}(a)\widehat{\mathbf{g}}^\top(a) \tag{5.4}$$

for some $\mathbf{r}(a) \in \mathbb{C}^2$ (column vector). Since $\widehat{\mathbf{g}}(a) \neq 0$ for all $a \in \mathbb{Z}'$, (5.2) implies that

$$\mathbf{r}(a) = \widehat{\mathbf{f}}(a) + \sum_{b\in\mathbb{Z}'\setminus\{a\}} \frac{\mathbf{r}(b)\widehat{\mathbf{g}}^\top(b)\widehat{\mathbf{f}}(a)}{a-b}. \tag{5.5}$$

Introduce the operator $\mathsf{D} : \ell^2(\mathbb{Z}') \otimes \mathbb{C}^2 \to \ell^2(\mathbb{Z}') \otimes \mathbb{C}^2$ by

$$\mathsf{D} : \big(\mathbf{r}(a)\big)_{a\in\mathbb{Z}'} \mapsto \big((\mathsf{D}\mathbf{r})(a)\big)_{a\in\mathbb{Z}'}, \qquad (\mathsf{D}\mathbf{r})(a) := \sum_{b\in\mathbb{Z}'\setminus\{a\}} \frac{\mathbf{r}(b)\widehat{\mathbf{g}}^\top(b)\widehat{\mathbf{f}}(a)}{a-b}, \ \ a \in \mathbb{Z}'. \tag{5.6}$$

It is a well-defined operator on $\ell^2(\mathbb{Z}') \otimes \mathbb{C}^2$ by (2.3) and the fact that $M_s(b,b) = \sigma(b-s-\frac{1}{2})K^{\mathsf{Be}}(b,b)$ is at a bounded distance from 1 for all $b \in \mathbb{Z}'$ by the assumptions on σ.

If $1-\mathsf{D}$ is invertible, the discrete RH problem admits a solution, constructed via (5.1) and (5.4) with

$$\mathbf{r} := (1-\mathsf{D})^{-1}\widehat{\mathbf{f}}. \tag{5.7}$$

The convenience of this approach to the discrete RH problem is evident when the operator D is small. This is the case when $L \to 0$. For precision's sake, let us fix the norm on $\ell^2(\mathbb{Z}') \otimes \mathbb{C}^2$ to be the one induced by the standard norm on $\ell^2(\mathbb{Z}')$ and the Euclidean norm on $\mathbb{C}^2$.

Proposition 5.1. *Let $s \in \mathbb{Z}'$ be such that $Q^0_\sigma(s) := \prod_{i=1}^{+\infty}\big(1-\sigma(-i-s)\big) > 0$. There exists $L_* = L_*(s), c = c(s) > 0$ such that $\|\mathsf{D}\| < cL$ for $0 \leqslant L < L_*$, where $\|\mathsf{D}\|$ is the operator norm of D.*

Proof. For $k \geqslant 0$, we have the Taylor series $\mathrm{J}_k(2L) = L^k \sum_{j\geqslant 0} \frac{(-L^2)^j}{j!(k+j)!} = (-1)^k \mathrm{J}_k(2L)$ which implies

$$\begin{aligned}\mathrm{J}_k(2L) = \delta_{k,0} + L(\delta_{k,1}-\delta_{k,-1}) + L^2\left(\frac{\delta_{k,-2}+\delta_{k,2}}{2} - \delta_{k,0}\right) \\ + L^3\left(\frac{\delta_{k,3}-\delta_{k,-3}}{6} - \frac{\delta_{k,1}-\delta_{k,-1}}{2}\right) + \mathrm{O}(L^4),\end{aligned} \tag{5.8}$$

as $L \to 0$, with remainder uniform in $k \in \mathbb{Z}$ because $|\mathrm{J}_k(2L)| = |\mathrm{J}_{-k}(2L)| \leqslant L^k/k!$ for $k \geqslant 0$ integer [30, Equation (10.14.4)]. In particular, we have the following estimate for $L \to 0$, uniform in $a \in \mathbb{Z}'$,

$$K^{\mathsf{Be}}(a,a) = \sum_{l\in\mathbb{Z}'_+} \mathrm{J}_{a+l}(2L)^2 = \mathbf{1}_{a<0} + L^2(\delta_{a,\frac{1}{2}} - \delta_{a,-\frac{1}{2}}) + \mathrm{O}(L^4). \tag{5.9}$$

(The explicit term of order L^2 will be needed later.) Therefore, for all $a \in \mathbb{Z}'$, we have

$$1 - M_s(a,a) = 1 - \sigma(a-s-\tfrac{1}{2})K^{\mathsf{Be}}(a,a) = \begin{cases} 1 + \mathrm{O}(L^2), & a > 0, \\ 1 - \sigma(a-s-\frac{1}{2}) + \mathrm{O}(L^2), & a < 0, \end{cases} \tag{5.10}$$

with remainders uniform in $a \in \mathbb{Z}'$. As long as we assume $Q^0_\sigma(s) \neq 0$ we have $\sigma(a - s - \frac{1}{2}) \neq 1$ for all $a \in \mathbb{Z}'$ with $a < 0$, and so we can estimate, for L sufficiently small,

$$\frac{1}{1 - M_s(a, a)} \leqslant c_1(s), \tag{5.11}$$

for a constant $c_1(s)$ depending on s only. Finally, we can estimate the square of the Hilbert–Schmidt norm of D (which is an upper bound of the square of the operator norm of D) as

$$\sum_{a,b\in\mathbb{Z}',\, a\neq b} \left| \frac{\widehat{\mathbf{g}}^\top(b)\widehat{\mathbf{f}}(a)}{b-a} \right|^2 \leqslant L^2 c_1(s) \sum_{a,b\in\mathbb{Z}'} \left| \mathrm{J}_{b+\frac{1}{2}}(2L)\mathrm{J}_{a-\frac{1}{2}}(2L) - \mathrm{J}_{b-\frac{1}{2}}(2L)\mathrm{J}_{a+\frac{1}{2}}(2L) \right|^2 \leqslant 2L^2 \mathrm{e}^{2L} c_1(s) \tag{5.12}$$

(where we use again $|\mathrm{J}_k(2L)| = |\mathrm{J}_{-k}(2L)| \leqslant L^k/k!$ for $k \geqslant 0$ integer in the last step) and the proof is complete. □

Corollary 5.2. *Let $s \in \mathbb{Z}'$ be such that $Q^0_\sigma(s) := \prod_{i=1}^{+\infty}\big(1 - \sigma(-i-s)\big) > 0$. There exists $L_* = L_*(s)$ such that the discrete RH problem for Y is solvable for $0 \leqslant L \leqslant L_*$, and moreover we have*

$$Y(z; L, s) = Y^{[0]}(z; s) + LY^{[1]}(z; s) + L^2 Y^{[2]}(z; s) + L^3 Y^{[3]}(z; s) + \mathrm{O}(L^4), \qquad L \to 0, \tag{5.13}$$

where $Y^{[i]}(z; s)$ are 2×2 matrix-valued meromorphic functions of z independent of L.

Proof. By the discussion above, if the operator $1 - \mathsf{D}$ is invertible, the discrete RH problem for Y admits a solution. By Theorem 3.4, if $1 - \mathsf{D}$ is invertible, then $Q_\sigma(L, s) > 0$. It is then enough to use Proposition 5.1 as well as the formula

$$Y(z) = I + \sum_{b\in\mathbb{Z}'} \frac{\Big((1-\mathsf{D})^{-1}\widehat{\mathbf{f}}\Big)(b)\widehat{\mathbf{g}}^\top(b)}{z-b}, \tag{5.14}$$

stemming from (5.1), (5.4), and (5.7), along with the Neumann series $(1 - \mathsf{D})^{-1} = \sum_{k\geqslant 0} \mathsf{D}^k$. □

Proof of Theorem II. The proof follows from the above Corollary 5.2 and the following computations. In the limit $L \to 0$, the Poissonized Plancherel probability measure converges to a delta measure supported on the empty partition. From (1.9), we obtain

$$Q_\sigma(L, s)\big|_{L=0} = \prod_{i\geqslant 0}\big(1 - \sigma(-i-s)\big) =: Q^0_\sigma(s), \qquad s \in \mathbb{Z}'. \tag{5.15}$$

Next, by (5.9) and (5.8), we have that, denoting $\widetilde{\sigma}_s(a) := \sigma(a - s - \frac{1}{2})$,

$$W_Y(a) = \frac{\widetilde{\sigma}_s(a)}{1 - \widetilde{\sigma}_s(a)\big(\mathbf{1}_{a<0} + L^2(\delta_{a,\frac{1}{2}} - \delta_{a,-\frac{1}{2}}) + \mathrm{O}(L^4)\big)} \times \left[\begin{pmatrix} 0 & -\delta_{a,\frac{1}{2}} \\ 0 & 0 \end{pmatrix} + L^2 \begin{pmatrix} -\delta_{a,-\frac{1}{2}} + \delta_{a,\frac{1}{2}} & -\delta_{a,-\frac{1}{2}} + 2\delta_{a,\frac{1}{2}} - \delta_{a,\frac{3}{2}} \\ \delta_{a,-\frac{1}{2}} & \delta_{a,-\frac{1}{2}} - \delta_{a,\frac{1}{2}} \end{pmatrix} + \mathrm{O}(L^4)\right] \tag{5.16}$$

or, equivalently,

$$W_Y(a) = W_Y^{[0]}(a) + L^2 W_Y^{[1]}(a) + \mathrm{O}(L^4), \tag{5.17}$$

where

$$W_Y^{[0]}(a) = -\sigma(-s)\delta_{a,\frac{1}{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad W_Y^{[1]}(a) = \delta_{a,-\frac{1}{2}}V_{-\frac{1}{2}} + \delta_{a,\frac{1}{2}}V_{\frac{1}{2}} + \delta_{a,\frac{3}{2}}V_{\frac{3}{2}},$$

$$V_{-\frac{1}{2}} = \frac{\sigma(-s-1)}{1-\sigma(-s-1)}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \quad V_{\frac{1}{2}} = \sigma(-s)\begin{pmatrix} 1 & 2-\sigma(-s)^2 \\ 0 & -1 \end{pmatrix},$$

$$V_{\frac{3}{2}} = \sigma(-s+1)\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}. \tag{5.18}$$

By Corollary 5.2, we can solve the discrete RH problem order by order in L, that is, we can plug the expansion (5.13) into the conditions of the discrete RH problem for $Y(z)$. Due to the parity of the series (5.17) it is easy to check that the terms $Y^{[1]}$ and $Y^{[3]}$ in (5.13) vanish. In particular, the leading term $Y^{[0]}(z)$ is characterized by the fact that it is analytic in $\mathbb{C} \setminus \{\frac{1}{2}\}$, with a simple pole at $1/2$, and satisfies

$$\operatorname*{res}_{z=1/2} Y^{[0]}(z)\mathrm{d}z = \lim_{z\to 1/2} Y^{[0]}(z)W_Y^{[0]}(\tfrac{1}{2}), \tag{5.19}$$

as well as $\sup_{|z|=n}|Y^{[0]}(z) - I| \to 0$ as $n \to +\infty$ through integer values. It follows that

$$Y^{[0]}(z) = I + \frac{W_Y^{[0]}(\frac{1}{2})}{z-\frac{1}{2}}. \tag{5.20}$$

Similarly, $Y^{[2]}(z)$ is characterized by the fact that it is analytic in $\mathbb{C} \setminus \{-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\}$, with simple poles at $\pm 1/2, 3/2$, and satisfies

$$\operatorname*{res}_{z=-1/2} Y^{[2]}(z)\mathrm{d}z = \lim_{z\to -1/2} Y^{[0]}(z)V_{-\frac{1}{2}}, \qquad \operatorname*{res}_{z=3/2} Y^{[2]}(z)\mathrm{d}z = \lim_{z\to 3/2} Y^{[0]}(z)V_{\frac{3}{2}}, \tag{5.21}$$

and

$$\operatorname*{res}_{z=1/2} Y^{[2]}(z)\mathrm{d}z = \lim_{z\to 1/2}\left(Y^{[0]}(z)V_{\frac{1}{2}} + Y^{[2]}(z)W_Y^{[0]}(\tfrac{1}{2})\right), \tag{5.22}$$

as well as $\sup_{|z|=n}|Y^{[2]}(z)| \to 0$ as $n \to +\infty$ through integer values. The solution is found in the form

$$Y^{[2]}(z) = \frac{N_{-\frac{1}{2}}}{z+\frac{1}{2}} + \frac{N_{\frac{1}{2}}}{z-\frac{1}{2}} + \frac{N_{\frac{3}{2}}}{z-\frac{3}{2}}. \tag{5.23}$$

The residues $N_{-\frac{1}{2}}$ and $N_{\frac{3}{2}}$ are found from (5.21) as

$$N_{-\frac{1}{2}} = \left(I - W_Y^{[0]}(\tfrac{1}{2})\right)V_{-\frac{1}{2}} = \frac{\sigma(-s-1)}{1-\sigma(-s-1)}\begin{pmatrix} \sigma(-s)-1 & \sigma(-s)-1 \\ 1 & 1 \end{pmatrix}, \tag{5.24}$$

$$N_{\frac{3}{2}} = \left(I + W_Y^{[0]}(\tfrac{1}{2})\right)V_{\frac{1}{2}} = \begin{pmatrix} 0 & -\sigma(-s+1) \\ 0 & 0 \end{pmatrix}. \tag{5.25}$$

For the residue $N_{\frac{1}{2}}$, we use (5.22) to get

$$N_{\frac{1}{2}} = \lim_{z\to 1/2} \frac{W_Y^{[0]}(\frac{1}{2})V_{\frac{1}{2}} + N_{\frac{1}{2}} W_Y^{[0]}(\frac{1}{2})}{z-\frac{1}{2}} + V_{\frac{1}{2}} + (N_{-\frac{1}{2}} - N_{\frac{3}{2}})W_Y^{[0]}(\tfrac{1}{2}). \tag{5.26}$$

Existence of the limit implies

$$W_Y^{[0]}(\tfrac{1}{2})V_{\frac{1}{2}} + N_{\frac{1}{2}} W_Y^{[0]} = 0 \tag{5.27}$$

which can be used to show that

$$N_{\frac{1}{2}} = \begin{pmatrix} \sigma(-s) & \star \\ 0 & \star \end{pmatrix}. \tag{5.28}$$

The remaining entries, denoted with $\star$, of $N_{\frac{1}{2}}$ can be found then by (5.26) but are not needed for the present argument†. Indeed, we have shown that

$$-\frac{L}{2}\frac{\partial}{\partial L}\log Q_\sigma(L,s) = \alpha(L,s) = L^2\left(N_{-\frac{1}{2}} + N_{\frac{1}{2}} + N_{\frac{3}{2}}\right)_{1,1} + \mathrm{O}(L^4), \qquad L \to 0_+, \tag{5.29}$$

where we use (3.30), hence the proof is completed by the explicit computation

$$\left(N_{-\frac{1}{2}} + N_{\frac{1}{2}} + N_{\frac{3}{2}}\right)_{1,1} = \frac{\sigma(-s) - \sigma(-s-1)}{1-\sigma(-1-s)}. \tag{5.30}$$

□

6 | PROOF OF THEOREM III

Throughout this section we assume $s \in \mathbb{Z}'$ is large enough such that $Q_\sigma(L, s-1) > 0$. Introduce, as in (1.32),

$$\mathfrak{a}(L,s) := \frac{\sqrt{Q_\sigma(L,s+1)Q_\sigma(L,s-1)}}{Q_\sigma(L,s)} = \sqrt{\frac{1+\beta(L,s)}{1+\beta(L,s+1)}}, \tag{6.1}$$

$$\mathfrak{b}(L,s) := \frac{\partial}{\partial L}\log\frac{Q_\sigma(L,s)}{Q_\sigma(L,s-1)} = -\frac{2}{L}\big(\alpha(L,s) - \alpha(L,s-1)\big), \tag{6.2}$$

where we use (3.30), and

$$\Theta(z;L,s) := \begin{pmatrix} \frac{L}{1+\beta(L,s)} & 0 \\ 0 & 1 \end{pmatrix} \Psi(z;L,s). \tag{6.3}$$

The following proposition is a consequence of (4.18) and (4.19) whose proof is a simple computation that we omit.

† It is however important to note that (5.26) is compatible with the structure (5.28) of $N_{\frac{1}{2}}$, so that we can really solve for these entries and thus fully determine $Y^{[2]}(z)$.

Proposition 6.1. *We have*

$$\Theta(z;L,s+1) = A(z;L,s)\Theta(z;L,s), \qquad \frac{\partial}{\partial L}\Theta(z;L,s) = B(z;L,s)\Theta(z;L,s) \tag{6.4}$$

where

$$A(z;L,s) = \begin{pmatrix} 0 & 1 \\ -\mathfrak{a}^2(L,s) & \frac{z+s+1}{L} + \frac{\mathfrak{b}(L,s+1)}{2} \end{pmatrix}, \quad B(z;L,s) = \begin{pmatrix} \frac{z+s+1}{L} + \mathfrak{b}(L,s) & -2 \\ 2\mathfrak{a}^2(L,s) & -\frac{z+s}{L} \end{pmatrix}. \tag{6.5}$$

In particular,

$$\Theta(z;L,s) = \begin{pmatrix} \chi(z;L,s-1) & \widetilde{\chi}(z;L,s-1) \\ \chi(z;L,s) & \widetilde{\chi}(z;L,s) \end{pmatrix}, \tag{6.6}$$

where $f(s) = \chi(z;L,s)$ or $f(s) = \widetilde{\chi}(z;L,s)$ are both solutions to

$$f(s+1) + \mathfrak{a}^2(L,s) f(s-1) = \left(\frac{z+s+1}{L} + \frac{\mathfrak{b}(L,s+1)}{2}\right) f(s). \tag{6.7}$$

Remark 6.2. It is worth noting that the compatibility of (6.4) is expressed by the identity

$$B(z;L,s+1)A(z;L,s) - A(z;L,s)B(z;L,s) = \frac{\partial}{\partial L}A(z;L,s). \tag{6.8}$$

Spelling out this equation gives the two relations

$$\frac{\partial \mathfrak{a}(L,s)}{\partial L} = \frac{\mathfrak{a}(L,s)}{2}\big(\mathfrak{b}(L,s+1) - \mathfrak{b}(L,s)\big), \qquad \frac{\partial \mathfrak{b}(L,s)}{\partial L} = 4\big(\mathfrak{a}^2(L,s) - \mathfrak{a}^2(L,s-1)\big) - \frac{\mathfrak{b}(L,s)}{L}. \tag{6.9}$$

The first identity is manifest already by comparing (6.1) and (6.2). Combining it with the second one immediately implies the 2D Toda equation

$$\frac{\partial^2}{\partial\theta_+\partial\theta_-} q_s(\theta_+,\theta_-) = \mathrm{e}^{q_{s+1}(\theta_+,\theta_-) - q_s(\theta_+,\theta_-)} - \mathrm{e}^{q_s(\theta_+,\theta_-) - q_{s-1}(\theta_+,\theta_-)} \tag{6.10}$$

for $q_s(\theta_+,\theta_-) := \log\big(Q_\sigma(\sqrt{\theta_+\theta_-}, s)/Q_\sigma(\sqrt{\theta_+\theta_-}, s-1)\big)$, which is a consequence of Theorem I.

The matrix $W_\Psi(a)$ is not constant in a, and so we shall now obtain a difference equation in z which, in general, has a meromorphic non-rational matrix coefficient. To this end, it is convenient to work with the gauge transformed matrix $\Theta(z;L,s)$ introduced in (6.3) and with the functions

$$\varphi(z;L,s) := \sqrt{\frac{1+\beta(L,s+1)}{L}}\,\chi(z;L,s). \tag{6.11}$$

In particular, from (6.7) we have

$$\mathfrak{a}(L,s+1)\varphi(z;L,s+1) + \mathfrak{a}(L,s)\varphi(z;L,s-1) = \left(\frac{z+s+1}{L} + \frac{\mathfrak{b}(L,s+1)}{2}\right)\varphi(z;L,s). \tag{6.12}$$

Proposition 6.3. *We have*

$$\Theta(z+1;L,s) = C(z;L,s)\Theta(z;L,s), \tag{6.13}$$

where

$$C(z) = \begin{pmatrix} 0 & 1 \\ -\mathfrak{a}^2(L,s) & \frac{z+s+1}{L} \end{pmatrix} + \sum_{l\in\mathbb{Z}'} \frac{\Delta\sigma(l)}{z-l} \begin{pmatrix} \mathfrak{a}(L,s)\varphi(l+1;L,s-1)\varphi(l;L,s) & -\varphi(l+1;L,s-1)\varphi(l;L,s-1) \\ \mathfrak{a}^2(L,s)\varphi(l+1;L,s)\varphi(l;L,s) & -\mathfrak{a}(L,s)\varphi(l+1;L,s)\varphi(l;L,s-1) \end{pmatrix}, \tag{6.14}$$

where $\Delta\sigma(l) := \sigma(l+1) - \sigma(l)$, $\mathfrak{a}(L,s)$ *is defined in (6.1), and* $\varphi(z;L,s)$ *is defined in (6.11).*

Proof. Define $C(z;L,s) := \Theta(z+1;L,s)\Theta^{-1}(z;L,s)$. Using (4.5), (6.3), and (4.3), we obtain

$$C(z;L,s) = \begin{pmatrix} \frac{L}{1+\beta(L,s)} & 0 \\ 0 & 1 \end{pmatrix} Y(z+s+\tfrac{3}{2};L,s) \begin{pmatrix} 0 & \frac{1}{L} \\ -L & \frac{z+s+1}{L} \end{pmatrix} Y^{-1}(z+s+\tfrac{1}{2};L,s) \begin{pmatrix} \frac{1+\beta(L,s)}{L} & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.15}$$

Using this relation, it is straightforward to see that

$$C_{-}(z;L,s) := C(z;L,s) - \begin{pmatrix} 0 & 1 \\ -\mathfrak{a}^2(L,s) & \frac{z+s+1}{L} \end{pmatrix} = \mathrm{O}(1/z) \tag{6.16}$$

as $z\to\infty$, in the sense of Lemma A.1, and applying this lemma we get

$$C_{-}(z;L,s) = \sum_{l\in\mathbb{Z}'} \frac{\operatorname{res}_{w=l} C_{-}(w)\mathrm{d}w}{z-l} = \sum_{l\in\mathbb{Z}'} \frac{\operatorname{res}_{w=l} C(w)\mathrm{d}w}{z-l}. \tag{6.17}$$

We are left with the task of computing these residues. For all $l\in\mathbb{Z}'$, using (6.3) we have

$$\begin{aligned} C(z) &= \Theta(z+1)\Theta^{-1}(z) \\ &= \Theta^{\mathrm{reg}}_{l+1}(z+1)\left(I + \frac{W_\Psi(l+1)}{z-l}\right)\left(I - \frac{W_\Psi(l)}{z-l}\right)(\Theta^{\mathrm{reg}}_{l})^{-1}(z) \\ &= \Theta^{\mathrm{reg}}_{l+1}(z+1)(\Theta^{\mathrm{reg}}_{l})^{-1}(z) + \frac{1}{z-l}\Theta^{\mathrm{reg}}_{l+1}(z+1)\big(W_\Psi(l+1) - W_\Psi(l)\big)(\Theta^{\mathrm{reg}}_{l})^{-1}(z), \end{aligned} \tag{6.18}$$

where

$$\Theta^{\mathrm{reg}}_{l}(z) := \begin{pmatrix} \frac{L}{1+\beta(L,s)} & 0 \\ 0 & 1 \end{pmatrix} \Psi^{\mathrm{reg}}_{l}(z) = \begin{pmatrix} \frac{L}{1+\beta(L,s)} & 0 \\ 0 & 1 \end{pmatrix} \Psi(z)\left(I - \frac{W_\Psi(l)}{z-l}\right), \tag{6.19}$$

which is regular at $z = l$. Hence, using (4.7), (6.6), and the fact that $\det \Theta(z; L, s) = \frac{L}{1+\beta(L,s)}$, we compute $\operatorname{res}_{w=l} C(w)\mathrm{d}w$ as

$$\Theta^{\mathrm{reg}}_{l+1}(l+1)\begin{pmatrix} 0 & \sigma(l)-\sigma(l+1) \\ 0 & 0 \end{pmatrix}\left(\Theta^{\mathrm{reg}}_{l}\right)^{-1}(l)$$

$$= \frac{1+\beta(L,s)}{L}\big(\sigma(l)-\sigma(l+1)\big)\begin{pmatrix} -\chi(l+1;L,s-1)\chi(l;L,s) & \chi(l+1;L,s-1)\chi(l;L,s-1) \\ -\chi(l+1;L,s)\chi(l;L,s) & \chi(l+1;L,s)\chi(l;L,s-1) \end{pmatrix} \tag{6.20}$$

and the proof is complete using the definition (6.11). □

Proof of Theorem III. We first prove (1.33) and (1.34). To this end, let C_{-1} be the coefficient of z^{-1} in the asymptotic series for $C(z; L, s)$ at $z = \infty$. On the one hand, using (6.14), we have

$$(C_{-1})_{1,1} = \mathfrak{a}(L,s)\sum_{l\in\mathbb{Z}'}\Delta\sigma(l)\varphi(l+1;L,s-1)\varphi(l;L,s). \tag{6.21}$$

On the other hand, using (6.15) and (3.19) instead,

$$(C_{-1})_{1,1} = L\left(1-\frac{1+\beta(L,s)}{1+\beta(L,s+1)}\right) = L\big(1-\mathfrak{a}^2(L,s)\big), \tag{6.22}$$

where we use (6.1) in the last equality. Hence, (6.21) and (6.22) are equal and we get (1.33). Similarly, using (6.14), we have

$$\operatorname{tr} C_{-1} = \mathfrak{a}(L,s)\sum_{l\in\mathbb{Z}'}\Delta\sigma(l)\big(\varphi(l+1;L,s-1)\varphi(l;L,s)-\varphi(l+1;L,s)\varphi(l;L,s-1)\big), \tag{6.23}$$

while using (6.15) and (3.19) we have $\operatorname{tr} C_{-1} = \alpha(L,s)/L$, and so

$$\alpha(L,s) = L\mathfrak{a}(L,s)\sum_{l\in\mathbb{Z}'}\Delta\sigma(l)\big(\varphi(l+1;L,s-1)\varphi(l;L,s)-\varphi(l+1;L,s)\varphi(l;L,s-1)\big). \tag{6.24}$$

Using this expression and (6.12) we finally simplify

$$\begin{aligned}
&\frac{\alpha(L,s+1)-\alpha(L,s)}{L}\\
&\quad= \sum_{l\in\mathbb{Z}'}\Delta\sigma(l)\Big\{\mathfrak{a}(L,s+1)\big[\varphi(l+1;L,s)\varphi(l;L,s+1)-\varphi(l+1;L,s+1)\varphi(l;L,s)\big]\\
&\qquad -\mathfrak{a}(L,s)\big[\varphi(l+1;L,s-1)\varphi(l;L,s)-\varphi(l+1;L,s)\varphi(l;L,s-1)\big]\Big\}\\
&\quad= \sum_{l\in\mathbb{Z}'}\Delta\sigma(l)\Big\{\varphi(l+1;L,s)\big[\mathfrak{a}(L,s+1)\varphi(l;L,s+1)+\mathfrak{a}(L,s)\varphi(l;L,s-1)\big]\\
&\qquad -\varphi(l;L,s)\big[\mathfrak{a}(L,s+1)\varphi(l+1;L,s+1)+\mathfrak{a}(L,s)\varphi(l+1;L,s-1)\big]\Big\}\\
&\quad= -\frac{1}{L}\sum_{l\in\mathbb{Z}'}\Delta\sigma(l)\varphi(l;L,s)\varphi(l+1;L,s).
\end{aligned} \tag{6.25}$$

Next, (1.35) is exactly (6.12), and it remains only to show the asymptotic relation (1.36). To this end, we first observe that as $s \to +\infty$ we have $Q_\sigma(L,s) \to 1$ and so, by (3.30),

$$\beta(s) = \frac{Q_\sigma(L,s-1)}{Q_\sigma(L,s)} - 1 \to 0, \quad \text{as } s \to +\infty, \tag{6.26}$$

implying, by (6.11), (6.3), and (6.6), that

$$\varphi(z;L,s) \sim \frac{1}{\sqrt{L}}\chi(z;L,s), \qquad s \to +\infty. \tag{6.27}$$

It is therefore enough to show that for all $z \in \mathbb{Z}'$ we have

$$\chi(z;L,s) \sim L\mathrm{J}_{z+s+1}(2L), \qquad s \to +\infty. \tag{6.28}$$

To this end we first write, using (4.5), (6.3), (6.6), and (5.14),

$$\begin{aligned}\left|\frac{\chi(z;L,s)}{L\mathrm{J}_{z+s+1}(2L)} - 1\right| &= \left|\frac{1}{L\mathrm{J}_{z+s+1}(2L)}\left((Y(z+s+\tfrac{1}{2}) - I)\Phi(z+s+\tfrac{1}{2})\right)_{2,1}\right| \\ &\leqslant \sum_{b\in\mathbb{Z}'}\left|(0,1)\big((1-\mathsf{D})^{-1}\widehat{\mathbf{f}}\big)(b)\frac{\widehat{\mathbf{g}}^\top(b)\Phi(z+s+\frac{1}{2})}{(z+s+\frac{1}{2}-b)L\mathrm{J}_{z+s+1}(2L)}\begin{pmatrix}1\\0\end{pmatrix}\right|,\end{aligned} \tag{6.29}$$

where we recall that the operator D is defined in (5.6). We need to show that (6.29) vanishes as $s \to +\infty$. To this end, we first estimate it as follows

$$\left|\frac{\chi(z;L,s)}{L\mathrm{J}_{z+s+1}(2L)} - 1\right| \leqslant ce^L \sum_{b\in\mathbb{Z}'}\left|(0,1)\big((1-\mathsf{D})^{-1}\widehat{\mathbf{f}}\big)(b)\right| \tag{6.30}$$

because we claim that there exists $c > 0$ such that for s sufficiently large (depending on L, z only, not on b) we have, for all $b \in \mathbb{Z}'$,

$$\left|\frac{\widehat{\mathbf{g}}^\top(b)\Phi(z+s+\frac{1}{2})}{(z+s+\frac{1}{2}-b)L\mathrm{J}_{z+s+1}(2L)}\begin{pmatrix}1\\0\end{pmatrix}\right| \leqslant ce^L. \tag{6.31}$$

To prove this last assertion, we rewrite

$$\begin{aligned}&\frac{\widehat{\mathbf{g}}^\top(b)\Phi(z+s+\frac{1}{2})}{(z+s+\frac{1}{2}-b)L\mathrm{J}_{z+s+1}(2L)}\begin{pmatrix}1\\0\end{pmatrix} \\ &= \frac{K^{\mathrm{Be}}(z+s+\frac{1}{2},b)}{\big(1-M_s(b,b)\big)L\mathrm{J}_{z+s+1}(2L)} = \sum_{l\in\mathbb{Z}'_+}\frac{\mathrm{J}_{z+s+\frac{1}{2}+l}(2L)}{\mathrm{J}_{z+s+1}(2L)}\frac{\mathrm{J}_{b+l}(2L)}{1-M_s(b,b)}.\end{aligned} \tag{6.32}$$

Observe that

$$M_s(b,b) = \sigma(b-s-\tfrac{1}{2})K^{\mathrm{Be}}(b,b) \leqslant \begin{cases}\sup\limits_{l\in\mathbb{Z}',\, l<-s/2}\sigma(l), & \text{if } b < s/2,\\[2ex] K^{\mathrm{Be}}(\lfloor\frac{s+1}{2}\rfloor+\frac{1}{2},\lfloor\frac{s+1}{2}\rfloor+\frac{1}{2}), & \text{if } b > s/2,\end{cases} \tag{6.33}$$

which implies

$$\frac{1}{1-M_s(b,b)} = \mathrm{O}(1), \quad \text{as } s \to +\infty, \text{ uniformly in } b \in \mathbb{Z}'. \tag{6.34}$$

Next, for k real and sufficiently large, $\mathrm{J}_k(2L)$ is positive and monotonically decreasing in k, as it follows, for instance, by (2.3). Therefore, we can bound (6.32), provided s is sufficiently large,

$$\left| \frac{\widehat{\mathbf{g}}^\top(b)\Phi(z+s+\frac{1}{2})}{(z+s+\frac{1}{2}-b)L\mathrm{J}_{z+s+1}(2L)} \begin{pmatrix}1\\0\end{pmatrix} \right| \leqslant c \sum_{l\in\mathbb{Z}'_+} |\mathrm{J}_{b+l}(2L)| \leqslant c \sum_{k\in\mathbb{Z}} |\mathrm{J}_k(2L)| \leqslant 2c\mathrm{e}^L, \tag{6.35}$$

for some $c > 0$, where in the last step we use again the inequality $\mathrm{J}_{\pm k}(2L) \leqslant L^k/k!$ for all integers $k \geqslant 0$. (In (6.30) we rename $c \mapsto c/2$.) Next, we claim that

$$\|\mathsf{D}\mathbf{r}\|_{\ell^1(\mathbb{Z}')} \leqslant \frac{1}{2}\|\mathbf{r}\|_{\ell^1(\mathbb{Z}')} \tag{6.36}$$

provided s is sufficiently large. Postponing for a while the proof of this claim, let us show how to complete the estimate of (6.30): note that D commutes with multiplying on the left by the vector (0,1), and therefore so does $(1-\mathsf{D})^{-1}$, to write, using (6.36),

$$\begin{aligned}
\sum_{b\in\mathbb{Z}'} \left|(0,1)\big((1-\mathsf{D})^{-1}\widehat{\mathbf{f}}\big)(b)\right| &= \left\|(0,1)(1-\mathsf{D})^{-1}\widehat{\mathbf{f}}\right\|_{\ell^1(\mathbb{Z}')} \\
&\leqslant 2\left\|(0,1)\widehat{\mathbf{f}}\right\|_{\ell^1(\mathbb{Z}')}. \\
&= 2L \sum_{a\in\mathbb{Z}'} \sigma(a-s-\tfrac{1}{2})|\mathrm{J}_{a+\frac{1}{2}}(2L)| \\
&= 2L \sum_{\substack{a\in\mathbb{Z}',\\ a<s/2}} \sigma(a-s-\tfrac{1}{2})|\mathrm{J}_{a+\frac{1}{2}}(2L)| + 2L \sum_{\substack{a\in\mathbb{Z}',\\ a>s/2}} \sigma(a-s-\tfrac{1}{2})|\mathrm{J}_{a+\frac{1}{2}}(2L)| \\
&\leqslant \left(\sup_{l\in\mathbb{Z}',\, l<-\frac{s}{2}} \sigma(l) \right) \sum_{k\in\mathbb{Z}} |\mathrm{J}_k(2L)| + \sum_{\substack{a\in\mathbb{Z}',\\ a>s/2}} \frac{L^{a+\frac{1}{2}}}{(a+\frac{1}{2})!} = o(1),
\end{aligned} \tag{6.37}$$

as $s \to +\infty$. Finally, it remains to prove the claim (6.36). To this end, we have

$$\sum_{a,b\in\mathbb{Z}',\, a\neq b} \left| \frac{\mathbf{r}(b)\widehat{\mathbf{g}}^\top(b)\widehat{\mathbf{f}}(a)}{a-b} \right| \leqslant \sum_{b\in\mathbb{Z}'} |\mathbf{r}(b)| \sum_{a\in\mathbb{Z}'} |\widehat{\mathbf{g}}^\top(b)\widehat{\mathbf{f}}(a)|. \tag{6.38}$$

We can bound this quantity, using $|\mathrm{J}_{b\pm\frac{1}{2}}(2L)| \leqslant \mathrm{e}^L$ and $(1-M_s(b,b))^{-1} \leqslant c$ for s sufficiently large and for all $b \in \mathbb{Z}'$, as we proved in (6.34), as

$$(6.38) \leqslant cL\mathrm{e}^L \|\mathbf{r}(b)\|_{\ell^1(\mathbb{Z}')} \sum_{a\in\mathbb{Z}'} \sigma(a-s-\tfrac{1}{2})\Big(|\mathrm{J}_{a+\frac{1}{2}}(2L)| + |\mathrm{J}_{a-\frac{1}{2}}(2L)|\Big) \tag{6.39}$$

and we can bound the last sum over a exactly as in (6.37) by splitting it for $a < s/2$ and $a > s/2$, to obtain

$$\sum_{a\in\mathbb{Z}'} \sigma(a - s - \tfrac{1}{2})\Big(|\mathrm{J}_{a+\frac{1}{2}}(2L)| + |\mathrm{J}_{a-\frac{1}{2}}(2L)|\Big) = o(1), \qquad s \to +\infty, \tag{6.40}$$

such that, indeed, for s sufficiently large we have (6.36). □

6.1 | Connection with the discrete Painlevé II equation

Let us now consider, more specifically, the case $\sigma = \mathbf{1}_{\mathbb{Z}'_+}$, studied in depth by Borodin [7]. In this case, (6.14) reduces to

$$C(z) = \begin{pmatrix} 0 & 1 \\ -\mathfrak{a}^2(L,s) & \frac{z+s+1}{L} \end{pmatrix} + \frac{1}{z+\frac{1}{2}} \begin{pmatrix} \mathfrak{a}(L,s)\varphi_+(L,s-1)\varphi_-(L,s) & -\varphi_+(L,s-1)\varphi_-(L,s-1) \\ \mathfrak{a}^2(L,s)\varphi_+(L,s)\varphi_-(L,s) & -\mathfrak{a}(L,s)\varphi_+(L,s)\varphi_-(L,s-1) \end{pmatrix}, \tag{6.41}$$

where we denoted, for sake of brevity, $\varphi_\pm(L,s) = \varphi(\pm 1/2; L, s)$. In this case, the compatibility conditions between the Lax equations (6.4) and (6.13) greatly simplify, and we recover the well-known relations between the discrete Bessel kernel, the discrete Painlevé II and the modified Volterra equation (see [7] and also [1, 21]).

Let us start by noting that the identities (1.33) and (1.34) reduce to

$$\mathfrak{b}(L,s+1) = \frac{2}{L}\varphi_+(L,s)\varphi_-(L,s), \quad L\big(\mathfrak{a}^{-1}(L,s) - \mathfrak{a}(L,s)\big) = \varphi_+(L,s-1)\varphi_-(L,s). \tag{6.42}$$

Taking the ratio of these gives

$$\varphi_+(L,s) = \frac{\mathfrak{a}(L,s)\mathfrak{b}(L,s+1)}{2(1-\mathfrak{a}^2(s))}\varphi_+(L,s-1). \tag{6.43}$$

We then expand the determinant of $C(z;L,s)$, which we know to be equal to 1, around $z = \infty$. Using (6.43), the term of order z^{-1} yields

$$L\mathfrak{a}^2(L,s)(\mathfrak{b}(L,s+1) + \mathfrak{b}(L,s)) = (2s+1)(1-\mathfrak{a}^2(L,s)). \tag{6.44}$$

Next, let us consider the compatibility condition between the first equation in (6.4) and (6.13)

$$A(z+1;L,s)C(z;L,s) - C(z;L,s+1)A(z;L,z) = 0. \tag{6.45}$$

Inspecting the entry (2,2) of this condition, once written in terms of $\mathfrak{b}^2(L,s+1)$, $\mathfrak{a}(L,s+1)$ and $\mathfrak{a}(L,s)$ (using the equations obtained before), yields

$$\mathfrak{b}^2(L,s+1) = 4(1-\mathfrak{a}^2(L,s))(1-\mathfrak{a}^2(L,s+1)). \tag{6.46}$$

Equations (6.44) and (6.46) are the same (up to a change of variable) as [7, Equations (3.9), (3.10)] and lead to an expression of the Fredholm determinants $Q_{\mathbf{1}_{\mathbb{Z}'_+}}(L,s)$ in terms of a discrete recursion known as the *discrete Painlevé II equation*, Equation (6.47). For the reader's convenience, we explain here how to derive it, closely following [7]. Note, however, that the Lax pair used to obtain (6.44) and (6.46) is not the same as the one in op. cit.

Proposition 6.4 (cf. Borodin, [7]). *Let $v(L,s)$, for $s \in \mathbb{Z}'$ with $s \geqslant -\frac{1}{2}$, be the sequence of functions defined by the second-order recursion*

$$v(L,s+1)+v(L,s-1)=\frac{(s+\frac{1}{2})v(L,s)}{L(v^2(L,s)-1)} \tag{6.47}$$

with initial conditions $v(L,-\frac{1}{2})=1$, $v(L,\frac{1}{2})=-\mathrm{I}_1(2L)/\mathrm{I}_0(2L)$, where $\mathrm{I}_k(2L)$ is defined in (1.14). Then, for all $s\in\mathbb{Z}'$ satisfying $s\geqslant -\frac{1}{2}$,

$$\frac{Q_{\mathbf{1}_{\mathbb{Z}'_+}}(L,s+1)Q_{\mathbf{1}_{\mathbb{Z}'_+}}(L,s-1)}{Q^2_{\mathbf{1}_{\mathbb{Z}'_+}}(L,s)}=1-v^2(L,s). \tag{6.48}$$

Moreover, the functions $v(L,s)$ satisfy the modified Volterra equation

$$\frac{\partial}{\partial L}v(L,s)=\big(1-v^2(L,s)\big)\big(v(L,s+1)-v(L,s-1)\big). \tag{6.49}$$

Proof. We start by defining $v^2(L,-\frac{1}{2})=1$ (which satisfies (6.48)) and then recursively

$$v(L,s+1):=-\mathfrak{b}(L,s+1)v^{-1}(L,s). \tag{6.50}$$

Using (6.46), we have $v^2(L,s)=1-\mathfrak{a}^2(L,s)$ for all $s\geqslant -\frac{1}{2}$. We can now write (6.44) just in terms of the functions $v(L,s)$, and in this way we obtain (6.47). As for (6.48), it comes from the equality $v^2(L,s)=1-\mathfrak{a}^2(L,s)$ combined with (6.1). Finally, the initial condition for $v(L,1/2)$ can be deduced from the recursive definition $v(L,s+1)=-\mathfrak{b}(L,s)v^{-1}(L,s)$ combined with (6.2) and the fact that $Q_{\mathbf{1}_{\mathbb{Z}'_+}}(-1/2)=\mathrm{e}^{-L^2}$, $\quad Q_{\mathbf{1}_{\mathbb{Z}'_+}}(1/2)=\mathrm{e}^{-L^2}\mathrm{I}_0(2L)$, see (1.14). Finally, the modified Volterra equation (6.49) is merely a rewriting of (6.9) in terms of the functions $v(s,L)$. □

APPENDIX: PARTIAL FRACTION EXPANSION

Lemma A.1. *Let $f(\cdot)$ be a meromorphic function with simple poles at $\mathbb{Z}':=\mathbb{Z}+\frac{1}{2}$ such that*

$$\max_{|z|=n}|f(z)|\to 0,\qquad \textit{as } n\to+\infty \textit{ through integer values.} \tag{A.1}$$

Then, for all $z\in\mathbb{C}\setminus\mathbb{Z}'$ we have

$$f(z)=\sum_{a\in\mathbb{Z}'}\frac{\operatorname*{res}_{w=a} f(w)\mathrm{d}w}{z-a}. \tag{A.2}$$

Proof. Fix $z\in\mathbb{C}\setminus\mathbb{Z}'$: for all integers $n>|z|$, Cauchy theorem implies that

$$\frac{1}{2\pi \mathrm{i}}\oint_{|w|=n}\frac{f(w)}{z-w}\mathrm{d}w=\sum_{a\in\mathbb{Z}',|a|<n}\frac{\operatorname*{res}_{w=a} f(w)\mathrm{d}w}{z-a}-f(z). \tag{A.3}$$

As $n \to +\infty$, the left-hand side tends to 0 because

$$\left| \oint_{|w|=n} \frac{f(w)}{z-w} \mathrm{d}w \right| \leqslant \max_{|w|=n} |f(w)| \oint_{|w|=1} \frac{|\mathrm{d}w|}{|w - z/n|} \to 0, \tag{A.4}$$

as $n \to +\infty$ by assumption, and the proof is complete. □

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