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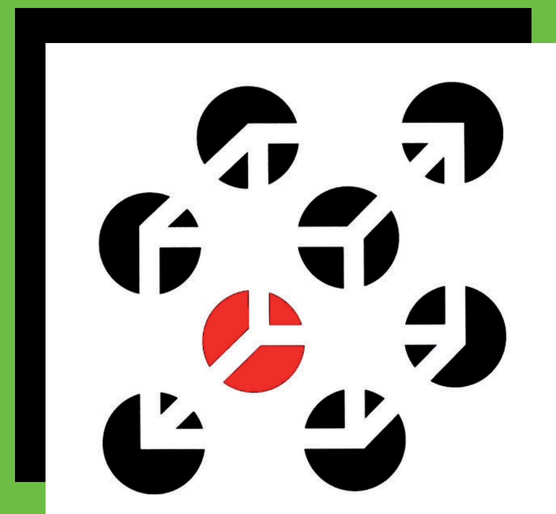

Xerxes D. Arsiwalla

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University of Amsterdam

Supersymmetric Black Holes as Probes of Quantum Gravity

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Publications

This thesis is based on the following publications:

- ✔ X. D. Arsiwalla and E. P. Verlinde, "A Black Hole Levitron", *accepted in* Phys. Rev. D, April (2010) arXiv:0902.0002 [hep-th].
- ✔ X. D. Arsiwalla, "Entropy Functions with 5D Chern-Simons terms", JHEP **0909**, 059 (2009) arXiv:0807.2246 [hep-th].
- ✔ X. D. Arsiwalla, "More Rings to rule them all : Fragmentation, 4D/5D and Split-Spectral Flows", JHEP **0802**, 066 (2008) arXiv:0709.0308 [hep-th].
- ✔ X. D. Arsiwalla, R. Boels, M. Marino and A. Sinkovics, "Phase Transitions in q-deformed 2d Yang-Mills theory and Topological Strings", Phys. Rev. D **73**, 026005 (2006) arXiv:0509002 [hep-th].

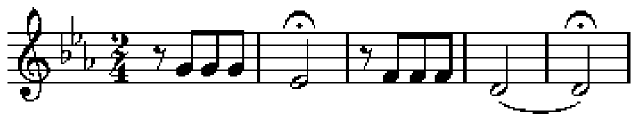
Other publications by the author:

- ☕ X. D. Arsiwalla, J. de Boer, K. Papadodimas and E. P. Verlinde, *in preparation* "Degenerate Stars and Gravitational Collapse in AdS/CFT".
- ✔ X. D. Arsiwalla, "Quantum Gravity: A Conflict of Paradigms and Emergent Unification", Blind - Magazine for Interdisciplinary Studies, Vol **22**, (2009).
- ✔ X. D. Arsiwalla, H. Donners, J. Meijer and M. Schraagen, "Predicting EEG based Brain Activity for Rapid Image Categorization", Proceedings of the Summer School on Consciousness and the Brain, Cognitive Science Center Amsterdam, (2009).

Contents

Chapter 1

Introduction



Opening motif of Beethoven's 5^{th}

Gravity has been one of the earliest known forces of nature, formulated as an inverse square law by Newton circa 1687. Yet to this day, it remains one of the most mysterious. It was only with the advent of Einstein's general theory of relativity in 1915, that a concrete mathematical foundation was laid for classical gravity. About the same time, in the early nineteen hundreds came the quantum revolution, lead by the likes of Schroedinger, Heisenberg and Dirac. Now quantum mechanics is a theory of microscopic particles and their interactions. These are objects typically characterized as small in size and light in weight. Classical gravity, on the other hand, is a theory of macroscopic bodies, which are typically large in size and heavy in weight. So the question is, how should one describe the physics of objects, which are small in size, yet huge in mass? Black holes are classic examples in this category and the rest of this thesis shall be devoted to that cause. To describe the these, a quantum description of gravity becomes pertinent. It was Einstein's dream to find a unified description of gravity that reconciled the classical and quantum paradigms. A promising candidate in this direction emerges in the form of string theory. In addition to unifying Einstein's theory to quantum mechanics, string theory seeks to go even further and unify all the forces of nature in such a way that they can be understood through a common set of fundamental principles.

Now how does one put together the pieces of this jig-saw? Let us start by laying out the fundamental ingredients of our universe. The physical universe is comprised of matter, radiation and their interactions. The fundamental building blocks of matter are particles called fermions. The building blocks of radiation are bosons. The fundamental interactions are electromagnetic, strong nuclear force, weak nuclear force and gravity. Before the advent of quantum mechanics and Einstein's special relativity, much of physics was based on Newtonian dynamics. Quantum theory shook the very foundations of the Newtonian paradigm and presented before us a whole new world which behaves very differently at microscopic scales, yet at large distances aggregates to classical laws. Moreover quantum mechanics and special relativity easily gelled together to give rise to what we now call relativistic quantum field theories. However general relativity as a classical theory of gravity remained evasive to any such "quantization".

Let us briefly see how fundamental interactions can be described in the language of quantum field theories. From this perspective a force between two fermions is mediated via the exchange of a specific boson. And these processes lend themselves to some of the most precise perturbative computations known. The predictions of field theory for each of these 3 forces (sans gravity) confer amazingly with experiment. Put together, this is what we call the standard model of particle physics. On the other hand, in general relativity, gravity is a property of space-time. From this point of view, space-time is a dynamic rather than static, whose geometry is responsible for the gravitational attraction between massive bodies. The presence of matter has the effect of distorting the 'shape' of the space-time around it. However, the standard model does not seem to incorporate gravity or the dynamism of space-time. Apparently there is no natural way to extend quantum field theories to include gravity. In field theory, an interaction is mediated via an exchange boson. The carriers of gravity are spin-2 bosons called gravitons. However, gravitons cannot be found in the spectrum of any conventional quantum field theory. A full description of quantum gravity should reconcile these two notions of force, one as an exchange of gravitons at microscopic scales and the other as a manifestation of space-time geometry at macroscopic scales. String theory is one such attempt to answer these questions.

The fundamental ingredients of string theory are not particles, but one dimensional objects called strings. These come in 2 types : open and closed. Analogous to the chords of a musical instrument, a string of a given length and fixed tension has a discrete range of vibrating frequencies, thus characterizing its energy spectrum. The idea now is that each vibrating mode represents a particle of nature. Low energy vibrations correspond to light particles, high energy modes to massive particles. It is indeed remarkable that the spectrum of these vibrations includes matter, radiation and gravity all in one package. Moreover strings interact with each other: closed strings intersect each other at a point, where they open up to form another closed string. Similarly open strings interact with other open strings by gluing at one of their ends once again resulting in an open string. Now this is a consistent interacting perturbative quantum field theory, not of particles but of strings. Subsequently non-perturbative techniques were also developed and higher dimensional objects called D-branes were included into the machinery. And as we shall soon see, the latter will play a very important role in the duality relating gravity to a gauge theory. The world is then fundamentally comprised of such quantum strings and branes; and the particles we observe around us are simply manifestations of their vibrations. Mathematical consistency requires that the theory be defined in 10 space-time dimensions, six of which are compactified on an internal manifold. In this thesis, we shall often encounter string theory compactified on a Calabi-Yau space, with D-branes wrapping internal cycles and thus resulting in black hole solutions in 4 dimensions. For practical purposes, we shall be interested in the low-energy effective theory in the bulk space-time, which turns out to be supergravity in 4 or 5D. Thus we now have a rigorous mathematical framework to compute observables involving graviton exchange such as correlation functions, scattering amplitudes, etc.

Before seeing how black holes enter the picture, let us briefly discuss the interplay between length and energy scales. This is crucial for understanding when stringy effects will be of relevance and also for subsequent unification of forces. A string length is typically of the order of 10^{-33} cm, called the Planck length. In physics, the scale of length is inversely proportional of that of energy, meaning that shorter distance interactions occur at higher energies and vice-versa. By the same logic, energies of dynamical processes that directly involve string interactions are of the order of cataclysmic explosions such as the big bang itself. In contrast, the shortest distance scales that present day technology can probe lie in giga electron volts, corresponding to sub-nuclear processes; the mass of the top quark is 174 GeV; and the LHC, when fully functional is anticipated to achieve energies of up to 7000 GeV. Now the energies at which stringy interactions can be probed are around 10^{19} GeV. This is called the Planck Scale. Unfortunately this far far beyond the reach of current laboratory technology. But these are precisely the scales relevant for processes that occurred during the early history of the universe. Moreover, this is the scale at which a fully quantum description of gravity becomes relevant. The reason is simply because the strength of the fundamental forces is not the same at every energy scale. It in fact varies as we probe physical processes at different energies. This running of coupling constants with energy is what eventually enables the unification of the fundamental interactions at the Planck scale. String theory offers a fully quantum description of Planck scale physics with the string coupling as the only free parameter in the theory and all other interactions described in terms of this parameter.

Having motivated why a quantum description of gravity is necessary for probing Planck scale physics, we now turn our attention to black holes. These are precisely the objects, whose underlying microscopics take us to the Planck scale, and that is how string theory enters the picture. Black holes thus serve as the test-beds of any theory of quantum gravity. Then in 1972 Bekenstein discovered that black holes are much more than mere voids in space-time, bound by event horizons: rather they behave like thermodynamic objects that carry a temperature and entropy! Putting these ideas on a firmer footing, Hawking later showed that black holes aren't really black when treated (semi-)quantum mechanically; they emit thermal radiation, later called Hawking radiation. In a sharp twist of ideas, black holes could now shrink and evaporate! Such an underlying thermodynamic association comes with its fair share of implications. Now a thermodynamic system can be can be formulated in terms of a statistical ensemble of an underlying structure that constitutes the microscopic degrees of freedom of the system. For instance, the temperature of a gas is a measure of the average kinetic energy of its molecules. However temperature is not a notion that can be assigned to an individual molecule of a gas; it is a meaningful concept only for the gas as a whole. Thus its origin lies in microscopic degrees of freedom of the gas as a whole. Similarly, the entropy of a thermodynamic system is precisely a measure of its underlying microstates.

The immediate question ensuing from this chain of ideas is what then are the microstates of a black hole and how do they interact? Since general relativity breaks down beyond the horizon, it is not suited to answer this question. One now needs to go beyond Hawking's approximate calculation and requires a full-fledged theory of quantum gravity, which probes physics at the smallest of length scales. And this is where string theory sheds some light into the picture. At large distances, string theory adequately reproduces Einstein's classical gravity, but at short distances it significantly modifies the latter - indicating that space-time geometry itself is not fundamental, but emerges as a macroscopic average. In string theory, a black hole is then described as a bound state of D-branes with stringy excitations. Hawking radiation is then the process of emission of closed strings from this ensemble. Within this one can now perform bulk computations via a low-energy effective analysis to compute say the leading-order result of several black holes. However, in general, such strongly gravitating systems will carry higher order curvature corrections, that can be difficult to compute using only low-energy effective techniques. This is where holography enters as a powerful new tool. 't Hooft's holographic principle is a statement about quantum gravity relating the degrees of freedom of a bulk gravitating system to those encoded in a holographically dual boundary quantum field theory without gravity. In string theory, this bulk/boundary correspondence manifests itself as Maldacena's gauge/gravity duality - also known as the AdS/CFT correspondence. What makes this correspondence useful is its realization as a strong/weak coupling duality - meaning that a strongly coupled gravitating system in the bulk maps to a weakly coupled gauge theory on the boundary and vice-versa. This is a remarkable feature of the correspondence that now permits us to perform strongly coupled and hence non-perturbative bulk computations simply via a perturbative analysis on a dual gauge theory living on the boundary. Indeed a lot of current research on black holes in string theory, including the work presented in this thesis, is focussed along this direction.

The starting point of the research in this thesis has been a recent conjecture in string theory due to Ooguri, Strominger and Vafa (OSV) relating a type of string theory in six dimensions to a supersymmetric black hole in four dimensions. The former known as topological string theory is defined on a six dimensional Calabi-Yau target space endowed with D-branes wrapped along homology cycles. In the bulk this corresponds to a four dimensional black hole, whose microstates can be accounted for by counting supersymmetric (known as BPS in this context) states on the world-volume theory of the bound state of branes. Topological string theory has deep mathematical connections to the field of algebraic geometry and in that context is concerned with the counting of algebraic invariants, known as Gromov-Witten invariants. However topological string theory is mostly understood only in the perturbative limit of a small string coupling constant. The reason why the OSV conjecture has managed to captivate so much attention is that it offers a rare glimpse into a non-perturbative definition of topological strings. The catch of course lies in the fact that in order to capture non-perturbative features in this theory, one has to know its equivalent in terms of corresponding black hole states, and as things turn out, neither is the latter fully understood. A modest approach might then be to look for specific limits of topological string theory by probing corresponding states of the associated black hole; and even this turns out to be rather difficult. This is the point at which it is useful to invoke the AdS/CFT duality. This opens out a new angle for making progress on the above-mentioned issues. In its manifestation as a strong/weak coupling duality, the strongly coupled regime of gravity in the bulk corresponds to the weakly coupled sector of the holographically dual gauge theory (without gravity) and vice-versa. This way computing observables of a quantum field theory on the boundary not only helps probe black holes in the bulk, but coupled with the OSV correspondence, it reveals hitherto unknown sectors in the spectrum of topological string theory. Now consider the scenario in which the brane system we are investigating, possesses a gauge theory which is exactly solvable. In rare cases when this does happen, one can carry out non-perturbative analysis and get a handle of the corresponding non-perturbative features in both, the bulk gravitational system as well as the associated string theory. The D0-D2-D4 BPS black hole that we have extensively investigated in this thesis, precisely allows for such a possibility, with the gauge theory being localized to a q-deformed version of 2D Yang-Mills on a Riemann surface with gauge group $U(N)$.

Putting these links together is part of an extensive on-going research program within the string theory community. The work presented in this thesis will focus on the gauge/gravity side of these connections. Within this backdrop, some of the research questions we pose in this thesis concern precision black hole entropy counting in 4D or 5D; observable charge shifts for these gravitational systems; the role of multi-center configurations as fragments of a single black hole geometry; and how one may probe the phase space of the holographically dual Yang-Mills theory. In this thesis we investigate these questions from several angles, incorporating new related developments such as the discovery of black rings in five dimensions; the 4D/5D connection relating black holes in four dimensions to those in five dimensions, and subsequently a multi-center extension of this connection along with the inclusion of extended black objects; the formulation of an entropy function technique that is well suited for computations involving higher order corrections due to the remarkable feature that within this formalism, all equations of motion straightforwardly reduce to algebraic equations; gauge theories dual to multi-center black hole configurations, necessary for a holographic understanding of microstates. A lot of the pieces of this jig-saw in fact compliment each other and thus a parallel rather than serial approach towards investigating these questions indeed leads to an integration of ideas and emergence of new insights. Nevertheless, the underlying theme behind all of this work shall still be the gravity/gauge duality connecting the macroscopic to the microscopic. In order to modestly achieve some of the above objectives, a large part of this work shall be devoted towards developing methodology and interpreting underlying mechanisms.

We begin our investigations in chapter 3 with macroscopic gravity calculations and further build up on the entropy function formalism of Sen. Our goal in this chapter is to develop an entropy function formalism for any extremal 5D black object, whose action contains what are called Chern-Simons terms. This is because Sen's original formulation was not incorporated to include such terms in the action owing to problems with manifest gauge invariance under large gauge transformations. We shall solve the problem and show that our 5D entropy function works for both black holes as well as black rings. With this 5D technology, it is now possible to correctly identify the physical charges in 5D black holes/rings. These are conserved Page charges, which are shifted relative to their 4D counterparts due to large gauge transformations originating from Chern-Simons terms. Here we shall interpret these charge shifts as what are known as spectral flow shifts and have also shown how spectral flow can be incorporated into the 5D entropy formalism, which at the same time remains gauge invariant and has an explicit dependence only on physical charges. Moreover, our 5D analysis enables us to fix a mismatch that arose in the electric charges of Goldstein and Jena's prior calculation. The utility of these techniques is that they now allow a thorough precision entropy counting in 5D with higher curvature corrections.

Then in chapter 4, we turned our attention to the 4D/5D conjecture. The question we investigate in this chapter is how should the 4D/5D connection work for these multi-center configurations? More specifically, we explicitly set-up a 5D construction of AdS-fragmentation, whereby a single black ring splits-up into a multi-black ring configuration. Furthermore it is shown that these fragmented rings are equivalent to a direct 5D lift of 4D multi-center black holes. In this way the 5D duals of these baby universes turn out to be a configuration of non-concentric multi-black rings. Once again we are faced with Chern-Simons induced charge shifts, but now for multi-center 5D systems. For single center configurations, the tools developed in chapter 3 gave us a geometric interpretation of these shifts as spectral flow. Even in the case of multi-center systems, we can show the manifestation of 4D/5D charge shifts as spectral flow, but now using insights from AdS fragmentation. Using an independent supergravity analysis, we also confirm that all conserved charges in 5D are once again Page charges, as expected. As an application of these methods, we then reproduce the total angular momentum of concentric black rings, originally due to Gauntlett and Gutowski. Finally, through this analysis we provide a geometric description of this system of multiple black rings, using the idea of split-spectral flows, wherein a given black ring's observables are influenced by fluxes generated in a background of neighboring rings. As a possible future research direction one may incorporate these split-flows into an entropy function so as to compute sub-leading degeneracies to multi-center systems as well.

Moving further, in chapter 5, we investigate a continuum limit of multi-center black hole configurations. We find solutions to integrability equations for large n centers, thus showing that such a limit indeed exists. We then construct a continuum distribution of black holes and performed a multipole expansion to find smeared black hole geometries with multipole moments. Using these solutions, one can now construct geometries with test black holes in multipole background fields, and that too along with the back-reaction. A very interesting application of precisely this is the black hole levitron. This entails spatially stabilizing a four dimensional black hole in background electric/magnetic fields. A stationary stable solution for this phenomenon is analytically found via the continuum multi-center limit developed in this thesis. Our levitron consists of a black hole levitating in stable equilibrium over a magnetic dipole base. We then go on to discuss how this construction strikes a resemblance to a mechanical Levitron.

Finally in chapter 6, we move on to microscopics. We investigate topological strings over a Calabi-Yau background of a Riemann surface endowed with two line bundles. The surface in this case is an S^2. Over this non-compact background, we seek to test the validity of the OSV conjecture and in the process discover a remarkable phase transition of the theory. We analyze this transition and comment on its implications for black hole physics. Here we investigate the dual gauge theory of the aforementioned D0-D2-D4 black hole, which turns out to localize to a quantum deformation of 2D Yang-Mills theory with gauge group $U(N)$, where N represents the magnetic D4 brane charge. In this rare case, the microscopic theory turns out to be fully solvable and hence lends itself as an interesting tool for non-perturbative analysis. For our analysis, the Yang-Mills gauge theory is most effectively studied using an equivalent matrix model in the large N limit, which in this case was derived from Chern-Simons theory. In this work we discover that an analogous phase transition occurring in two dimensional QCD on a sphere is replicated in its q-deformed cousin for specific values of deformation parameter. The phase diagram of the model is determined and we show that the theory exhibits a phase transition only for small values of the deformation parameter, whereas for large values of the deformation parameter the phase transition is absent. We explicitly see how this transition is triggered by instanton effects. Finally, we presented the solution of the model in the strongly coupled phase. Our analysis suggests that, on certain backgrounds, non-perturbative topological string theory has a new phase transitions at small radius. From the point of view of gauge theory, it suggests a mechanism to smooth out such phase transitions. One implication of our result is that for certain backgrounds, the usual geometric description of topological strings does not hold in the small area phase of the gauge theory and this has bearing on the validity of OSV itself in that regime. A likely scenario suggested by this work is that sub-leading contributions to the gauge theory partition sum are associated to AdS-fragmentation of black holes.

Chapter 2

A Brief History of Black Holes in String Theory

Life is complex - it has both real and imaginary parts

- Anonymous

2.1 From Information to Thermodynamics

From an empirical perspective, a classical black hole may be defined as a region of space, causally disconnected from its surroundings, such that no signal can convey information about its state to the outside world. The emphasis on the notion of information in this context was first put forth by Wheeler in [1]. Structurally, black holes are believed to be remnants of gravitational collapse, often formed in the aftermath of giant supernovae explosions. Not all stars, however, end up as black holes, only those with initial mass about twice the mass of the sun or greater. The ones below this critical bound either end up as brown dwarfs, white dwarfs, neutron stars or the hypothesized quark/strange matter stars [2], [3]. Gravitational collapse is thus responsible for formation of exotic states of matter that define these remnants. White dwarfs essentially constitute a degenerate Fermi gas of electrons, when the mass of the parent star is below a critical limit - known as the Chandrashekar limit. Beyond that limit, neutron stars are formed when atoms are crushed into each other overcoming the electron degeneracy pressure such that the atomic electrons are squeezed into the nucleus to combine with protons to form a degenerate Fermi gas of neutrons. In this case, the critical mass limit is the Tolman-Oppenheimer-Volkov limit. Going further, squeezing beyond the neutron degeneracy pressure one supposedly arrives at the regime of the quark-gluon plasma. Note that the more massive the collapse, the shorter are the distance scales corresponding to the state of matter constituting that remnant. The most massive collapses result in black holes, and string scale physics is believed to provide a microscopic description of these remnants. If so, one may be tempted to ask whether a black hole can be thought of as a "gas" of string and brane excitations.

Now, unlike stars and other astrophysical objects, black holes as such do not reveal their initial chemical composition. No matter what type of object collapsed to form a black hole, in 4 dimensional general relativity, all stationary, charged and rotating black hole solutions form a single 3-parameter family of Kerr-Neumann solutions. Consequently, all observables of such a system only depend on its mass, angular momentum and electric charge. This is the so-called "no-hair principle". So what this mean in terms of information ? The lack of knowledge of initial composition leading to the no-hair theorem coupled with the lack of correlation of any signal from the inside of a black hole to an observer outside implies that a black hole represents a large amount of missing information. Possibly the maximum there can be in that region of space. From an information theoretic setting, a way to quantify the information of a system is using the measure of entropy, which is defined as the average number of bits needed for storage or communication of information, pertaining to a random variable X. It relates to the uncertainty encountered in this variable and is expressed as

$$S(X) = -\sum_{x \in \mathcal{M}} p(x) \; log \; p(x) \tag{2.1.1}$$

where $p(x)$ is the probability of X for a given bit x and $\mathcal{M}$ is the set of all bits. But what does this mean for a black hole? First let us contrast this to a star such as the sun. Albeit, the entropy of a star is usually much less than that of a black hole. In this case though, the entropy has a clear physical interpretation in terms of the underlying microstates of the relativistic gas. Thermodynamic variables can then be computed in the hydrodynamic limit. On the other hand, if we believe that a black hole is basically a singularity (shielded or otherwise) of space-time, then implicitly it lacks any composition, which makes it inconceivable to think of it in terms of statistical microstates. Yet, from the discussion above, we see that the missing information associated to a black hole is far more than any stellar object. Approaching this problem from one end, one may ask whether these bits of information carry any physical relevance. From the other end, one may ask if the resolution of this singularity in a quantum theory of gravity can provide a quantitative understanding of $S(X)$ in terms of "appropriate" microstates. The first approach is what will lead us to gravitational thermodynamics, while the second will take us to statistical mechanics in a quantum field theory. And the bridge between the two lies at the heart of a deeper holographic duality of quantum gravity. It still remains a hard problem to understand what these black hole microstates are. Presumably these are relevant up to the Planck scale if we believe that nothing collapses beyond the stage of a black hole (within a finite volume of space). For instance, microstates which describe the sun or a neutron star certainly can not be the right description of states of the ensemble after one crosses the critical limit of the corresponding degeneracy pressure beyond which black hole formation occurs.

Above, we motivated a statistical mechanic framework for accounting the information bits of a black hole. Now we may ask how one might see the underlying thermodynamics from such an ensemble? Thermodynamic observables of the black hole, if they do exist, should be easier to study as they would not require all the microscopic knowledge of the ensemble, but would be related only to macroscopic variables of the black hole solution - in this case the mass, charge and angular momentum. Though we have not said much about the role of gravity in this discussion so far, only having considered a system with maximum missing information content, the link with gravity will enter the discussion via the holographic bound - in the sense that gravity sets the bound on the maximum amount of information that can be stored in a given region of space. Just from the information perspective, we have motivated the possibility of an underlying statistical description. If in addition, there is an exists emergent thermodynamics from this ensemble, what is the gravitational interpretation of that ? In a sense, the Einstein equations inherently "know" about that. This is most easily seen by comparing the law of black hole mechanics to the first law of thermodynamics as follows : Consider an electrically charges Reissner-Nordstrom black hole with charge Q. This is a solution to Einstein-Maxwell gravity with action

$$\mathcal{S} = \frac{1}{16\pi G}\int d^4x\sqrt{|g|}(R - F_{\mu\nu}F^{\mu\nu}) \tag{2.1.2}$$

The metric $g_{\mu\nu}$ is then given by

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \quad (2.1.3)$$

Setting $g_{tt} = 0$ determines the radial position of the horizon. In this case, there are two horizons located at r_+ and r_- respectively. The topology of both horizons is a 2-sphere. Now consider the area of the outer horizon. This is given by

$$A = 4\pi \left(M + \sqrt{M^2 - Q^2}\right)^2 \quad (2.1.4)$$

which is only valid when $M^2 \geq Q^2$ is satisfied. For later reference, let us also mention how M and Q relate to the actual mass and charge m respectively q. Upon restoring constants and dimensions, we get $M \equiv \frac{Gm}{c^2}$ and $Q \equiv \frac{\sqrt{G}q}{c^2}$.

Now taking differentials on both sides of eq.(2.1.4) gives the law of black hole mechanics

$$d\left(mc^2\right) = \frac{\kappa}{8\pi} dA + \Phi dQ \quad (2.1.5)$$

where

$$\kappa = \frac{4\pi c^4 \sqrt{M^2 - Q^2}}{GA} \quad (2.1.6)$$

is the surface gravity and

$$\Phi = \frac{q}{r_+} \quad (2.1.7)$$

is the electric potential on the hole's horizon. Comparing the above to the first law of thermodynamics yields

$$U \leftrightarrow mc^2 \quad (2.1.8)$$

$$TdS \leftrightarrow \frac{\kappa}{8\pi} dA \quad (2.1.9)$$

$$VdQ \leftrightarrow \Phi dQ \quad (2.1.10)$$

This observation together with the classical area theorem [4], [5], [6] - that black holes do not shrink and the area of the horizon cannot decrease under any circumstances - lead Bekenstein to associate the area of a black hole's horizon to an entropy. In general, this takes the form

$$S_{BH} = f(A) \quad (2.1.11)$$

$$\text{and} \quad T_{BH} = \frac{\kappa}{8\pi f'(A)} \quad (2.1.12)$$

To determine the function $f(A)$ we follow Bekenstein's gedanken experiment [7], [8]. Assume a generic power law function $f(A) = A^\gamma$, where γ can be an arbitrary positive power. Negative powers are excluded, since from eq.(2.1.11), it would imply that as the mass of a black hole increases in any physical process, its entropy decreases - thus violating thermodynamics. Coming back to the experiment, let us now drop some matter adiabatically into a stationary Schwarzschild black hole with entropy S_{BH} and mass M. The entropy content of the external matter being denoted by S_{matt} and its mass μ. As matter falls into the black hole, the latter's horizon area must increase. However, by the second law of thermodynamics, the growth in the black hole's entropy must compensate for the loss in S_{matt}, which is devoured by the black hole. That is,

$$\Delta S_{BH} \geq S_{matt} \tag{2.1.13}$$

must be satisfied. Now from eq.(2.1.4), we may determine the increase in the black hole's area as

$$\Delta A = 16\pi \left(\mu^2 + 2M\mu\right) \tag{2.1.14}$$

Taking differentials on both sides of eq.(2.1.11) and inserting the expressions determined above, we now have to satisfy the following inequality

$$\Delta S_{BH} \;=\; \gamma A^{\gamma-1} \cdot 16\pi \left(\mu^2 + 2M\mu\right) \;\geq\; S_{matt} \tag{2.1.15}$$

Now for $\gamma > 1$, we can always choose a small enough black hole such that the above inequality will be violated. This forces upon us a linear dependence of the function $f(A)$. The classical entropy can thus be expressed upto a multiplicative constant as follows

$$S_{BH} = \eta \frac{Ac^3}{G\hbar} \tag{2.1.16}$$

The constant η was later determined to be $1/4$ in Hawking's semi-classical calculation of quantum fields in a curved background [9]. Furthermore, this result lends T_{BH} the interpretation of a physical temperature associated to thermal radiation emitted by the black hole - the so-called Hawking radiation. Certainly, in light of this, the classical area law lends itself to a natural generalization - the generalized second law of black hole thermodynamics, which can be expressed as

$$\Delta S_{outside} + \Delta S_{BH} \geq 0 \tag{2.1.17}$$

stating that the total sum of ordinary entropy $S_{outside}$ outside the black hole and the black hole entropy never decreases and typically increases as a consequence of generic transformations of the system (black hole + environment).

These are the two most important laws. For completeness let us also state the zeroth and third law of black hole thermodynamics. The former is analogous to the zeroth law of thermodynamics, which claims that the temperature of a system in thermal equilibrium is constant everywhere in that system. For black holes this translates to the surface gravity being constant everywhere over the horizon of a stationary black hole.

In thermodynamics itself, the status of the third law is somewhat ambiguous. In its stronger version it goes as the Nernst-Simon law, which says that the entropy of a system at absolute zero temperature either vanishes or becomes independent of intensive thermodynamic parameters. But many condensed matter systems are known to violate this and so do extremal black holes (due to a non-vanishing horizon area with zero surface gravity). Hence this is not taken as a law. Instead in its weaker form, the third law states that it is impossible for a system to reach absolute zero temperature in any physical process in a finite amount of time. In this version the analog holds for black holes as well. A stationary black hole with Hawking temperature T_{BH} cannot by any physical process transform or decay to an extremal black hole.

To summarize this section, we see starting from an information theoretic perspective, the underpinnings of a thermodynamic connection to gravity, which in turn emerges from a "hidden" microscopic description of quantum gravity. Here string theory enters the picture as a candidate description of quantum gravity. In string theory, a black hole is described as a bound state of strings and branes. Gravity lives in the bulk, microstates live in the Hilbert space of the holographically dual gauge theory. In the special case of BPS black holes, these states are protected under deformation of the gravitational coupling, by supersymmetry. For these black holes, the microscopics and macroscopics yield satisfactory agreement and we shall encounter these systems later in this thesis. However, in the case of generic black holes, many of these questions still remain open.

2.2 BPS Black Holes

As a precursor to black hole solutions of four dimensional supergravity, we consider extremal Reissner-Nordstrom black holes with electric charge $Q \equiv \sqrt{G}q/c^2$ (as above). These are solutions to Einstein-Maxwell gravity, as we have seen above. In addition, they satisfy the extremality condition $M = |Q|$. Using this in eq.(2.1.4) for the area and inserting the resulting expression in eq.(2.1.16) gives the entropy of an extremal Reissner-Nordstrom black hole as

$$S_{RN} = \frac{\pi|q|^2}{c\hbar} \tag{2.2.18}$$

Note that this result now only depends on the black hole's charge and is completely independent of the gravitational constant G or any other moduli. This turns out to be an extremely useful property of extremal black holes as one can now tune the couplings to a regime of the theory that lends itself to say perturbative computations without changing the number of black hole microstates. More precisely, since $G \sim g_s^2 l_s^8$ (with l_s as the string length), we shall see within the context of the gauge/gravity duality that while the value of the string coupling g_s is tuned up in the gravity regime, it is more convenient to count black hole microstates in the gauge theory for a smaller value of g_s. The independence of the entropy on G is in some sense the reason why the gauge/gravity duality works.

In the near-horizon limit, the metric for the extremal Reissner-Nordstrom black hole can be derived from eq.(2.1.3) to be

$$ds^2 = -\frac{r^2}{|Q|^2}dt^2 + \frac{|Q|^2}{r^2}dr^2 + |Q|^2(d\theta^2 + sin^2\theta d\phi^2) \tag{2.2.19}$$

which refers to an $AdS_2 \times S^2$ geometry.

Now let us see how extremal black holes emerge in string theory/supergravity. In string theory/supergravity compactified to four dimensions, there also exist such black hole solutions with $AdS_2 \times S^2$ geometry, where the above-mentioned extremality condition generalizes to the BPS condition $M = |Z|$ for central charge Z. These are called BPS black holes. The low energy effective description of Type II A/B string theory is Type II A/B supergravity in ten dimensions. Compactifying this on a six dimensional Calabi-Yau gives $\mathcal{N} = 2$ supergravity in 4D. This theory has an $SU(2)_R$

R-symmetry and the massless fields fall into the gravity multiplet, vector multiplets or hypermultiplets.

For black hole entropy only the vector multiplets will concern us. Besides the gauge field $F^A_{\mu\nu}$, a vector multiplet contains a complex scalar field X^A. The number of (dynamical) vector multiplets is denoted by h_V. The scalars represent the moduli of the theory. Supersymmetry requires this moduli space to be a special Kahler manifold, with Kahler potential $\mathcal{K}$. The kinetic terms will be determined from the holomorphic prepotential $F(X)$ of the theory, which is determined from the Calabi-Yau geometry and can be computed from string theory. The scalars fields X^A together with F_A are projective coordinates on the vector multiplet moduli space. This gives rise to the "special geometry" of $\mathcal{N} = 2$ supergravity. Besides the h_V vector multiplets, the theory contains an auxiliary vector multiplet, also called the gravity multiplet, whose gauge field is the graviphoton. The expectation value of the scalar in the gravity multiplet is fixed in terms of the scalars in the dynamical multiplets. The same holds for the graviphoton and the gauge fields. The index A then runs from 0 to h_V.

We can then write the four dimensional bosonic two-derivative supergravity action as follows (with $c = \hbar = 1$)

$$S_{sugra} = \frac{1}{16\pi G}\int d^4x\sqrt{|g|}\,\Big(R + 2\mathcal{G}_{AB}(X,\bar{X})\partial^\mu X^A\partial_\mu\bar{X}^B + \frac{i}{4}\mathcal{N}_{AB}F^{+A}_{\mu\nu}F^{+B\mu\nu} - \frac{i}{4}\bar{\mathcal{N}}_{AB}F^{-A}_{\mu\nu}F^{-B\mu\nu}\Big) \quad (2.2.20)$$

where $F^{\pm A} = \frac{1}{2}\left(F^A \pm i * F^A\right)$ are the self-dual and anti-self-dual parts of the gauge field F^A. The scalar fields are constrained by

$$N_{AB}X^A\bar{X}^B = -1 \quad (2.2.21)$$

with the metric on the kinetic terms given by

$$\mathcal{G}_{AB}(X,\bar{X}) = N_{AB} + N_{AC}X^C N_{BD}\bar{X}^D \quad (2.2.22)$$

where N_{AB} and $\mathcal{N}_{AB}$ are given by

$$N_{AB} = 2ImF_{AB} \quad (2.2.23)$$

$$\mathcal{N}_{AB} = \bar{F}_{AB} + \frac{iN_{AC}X^C N_{BD}X^D}{N_{IJ}X^I X^J} \quad (2.2.24)$$

where we have

$$F_A = \partial_A F(X) \tag{2.2.25}$$

$$F_{AB} = \partial_A \partial_B F(X) \tag{2.2.26}$$

with $F(X)$ as the holomorphic prepotential of the theory. In terms of these, the Kahler potential of the vector multiplet moduli space can then be written as

$$e^{-\mathcal{K}(X,\bar{X})} = i\left(\bar{X}^A F_A - X^A \bar{F}_A\right) \tag{2.2.27}$$

The gauge fields are sourced by electric and magnetic charges q_A respectively p^A as follows

$$p^A = \frac{1}{2\pi} Re \int_{S^2} F^{+A} \tag{2.2.28}$$

$$q_A = \frac{1}{2\pi} Re \int_{S^2} \mathcal{N}_{AB} F^{+B} \tag{2.2.29}$$

The central charge function associated to this theory takes the form

$$Z = e^{\mathcal{K}/2}\left(X^A q_A - F_A p^A\right) \tag{2.2.30}$$

Black hole solutions to 4D $\mathcal{N}=2$ supergravity are parametrized by the ADM mass M and graviphoton charge of the theory. The latter is exactly the central charge Z given above. Black hole solutions exist when $M \geq |Z|$, where the equality refers to the BPS bound that characterizes a stable extremal black hole in supergravity. The BPS case is what shall concern us in what follows. This solution preserves half of the original 8 supersymmetries. This holds as long as there exists a covariantly constant spinor, which is obtained via setting the fermionic variations to vanish.

The metric for a 4D $\mathcal{N}=2$ BPS black hole is then given by

$$ds^2 = -e^{2U(r)}dt^2 + e^{-2U(r)}\left(dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)\right) \tag{2.2.31}$$

where

$$e^{-U(r)} = 1 + \frac{|Z|}{r} \tag{2.2.32}$$

In the near-horizon limit, this exactly reproduces the extremal Reissner-Nordstrom metric of eq.(2.2.19) with the replacement $Q \to Z$, thus giving back an $AdS_2 \times S^2$ near-horizon geometry. The entropy, obtained via the area law, then takes the form

$$S_{BH}(p,q) = \pi |Z|^2|_{horizon} \tag{2.2.33}$$

where the function Z is evaluated at the horizon. Once again the entropy will be independent of any moduli or coupling and will only be a function of the charges measured at infinity. In the next subsection we shall see how such macroscopic entropy computations can be performed.

In this sense, supersymmetric black holes are good analogs of extremal Einstein-Maxwell black holes and also serve as powerful test-beds for ideas of quantum gravity emerging in string theory.

2.2.1 The Entropy Function Formalism

For actions such as in eq.(2.2.20) and also others which include higher derivative corrections, there are several ways to compute the macroscopic black hole entropy. Two methods that will concern us in this thesis are the attractor mechanism (about which we will have more to say later) and the Sen entropy function formalism [29]. Moreover, since the latter of these will play a more crucial role in the research developed in this thesis, let us lay out its framework at this point. The entropy function method works for any black hole having $SO(2,1) \times SO(d-1)$ near-horizon isometry in arbitrary space-time dimension d. In recent works, it was also shown that the said isometry, ensures extremality.

Though the entropy function formalism is essentially a reformulation of Wald's formalism, for computational purposes it is far less tedious than the former, in the sense that the equations of motion elegantly reduce to algebraic equations. This is because this method only concerns itself with the near-horizon isometries and does not take into account whether the full global solution exists or not. Nevertheless it has served as a useful tool for higher derivative theories with local Lagrangian densities. It can be applied to non-SUSY extremal black holes and to higher dimensional black objects as well. Another reason why the entropy function formalism is a more reliable method than techniques emanating from topological string methods is that unlike the latter, which only takes into account holomorphic contributions to the prepotential, the former also works with non-holomorphic terms in the Lagrangian.

However one of the shortcomings of this formalism in its its original form, was that it was only applicable to reparametrization invariant and gauge invariant Lagrangians. The means we have a problem when considering theories of gravity with Chern-Simons type of terms. Black rings become a prominent example of this class. In this thesis, we will present a resolution to this problem.

Let us first demonstrate how this formalism works for the simple case of an extremal Reissner-Nordstrom black hole. The entropy of such an object is computed by extremising the Sen function defined as follows

$$\mathcal{E}(v_1, v_2, q) = 2\pi \left(e\frac{\partial \mathcal{F}}{\partial e} - \mathcal{F} \right) \tag{2.2.34}$$

The above is a Legendre transform of the reduced action defined as

$$\mathcal{F}(v_1, v_2, q) = \int d\theta d\phi \sqrt{-g}\mathcal{L} \tag{2.2.35}$$

corresponding to an $AdS_2 \times S^2$ near-horizon geometry parametrized through the metric

$$ds^2 = v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + v_2 (d\theta^2 + sin^2\theta d\phi^2) \tag{2.2.36}$$

$\mathcal{L}$ denotes the Lagrangian density. The near-horizon geometry is parametrised by a constant electric field e and constant scalar moduli X. v_1 and v_2 respectively denote the radius of AdS_2 and S^2, which shall be determined through their equations of motion. The electric charge q is conjugate to the field e and is determined via

$$q = \frac{\partial \mathcal{F}}{\partial e} \tag{2.2.37}$$

The remaining equations of motion are given by extremising the entropy function

$$\frac{\partial \mathcal{E}}{\partial v_1} = \frac{\partial \mathcal{E}}{\partial v_2} = 0 \tag{2.2.38}$$

Solving these algebraic equations and eliminating v_1, v_2 and e in the entropy function gives the black hole entropy

$$S_{BH} = \pi q^2 \tag{2.2.39}$$

In later chapters, we shall exploit the power of this formalism for extended black objects such as rings and develop the necessary systematics in the case when the associated actions carry Chern-Simons terms.

2.3 Holography

2.3.1 A Principle of Quantum Gravity

The idea of holography has been a powerful tool in quantum gravity research. Stated in its most general form it can be expressed as follows :

Physical processes in the bulk space-time of a d-dimensional theory are reflected in processes occurring in a different $d-1$ dimensional theory living on the boundary of that space-time.

The motivation underlying this proposal, due to Gerard 't Hooft [10] as a quantum gravitational principle arose from insights in black hole physics; namely Bekenstein's entropic or holographic bound [11]. The latter can be construed via the following gedanken experiment proposed by Susskind in [12]. Consider a neutral non-rotating spherical body, which fits entirely in a region of space bounded by area $\mathcal{A}$. Let $\mathcal{S}$ denote the entropy of this object. Now allow this mass to collapse, forming a black hole, in this case of the Schwarzschild type. Clearly the black hole's horizon area $\mathcal{A}_{\mathcal{BH}} \leq \mathcal{A}$. But, by the second law of black hole thermodynamics, the black hole entropy must satisfy $\mathcal{S}_{\mathcal{BH}} \geq \mathcal{S}$. Therefore (with $c = \hbar = 1$), we have

$$\mathcal{S} \leq \mathcal{S}_{\mathcal{BH}} = \frac{\mathcal{A}_{\mathcal{BH}}}{4G} \leq \frac{\mathcal{A}}{4G} \tag{2.3.40}$$

Now in conventional QFT lore, the degrees of freedom scale as the volume V of a given region and not the area. So why does the boundary capture information of physics in the bulk ? The answer is that gravity imposes a cut-off on the number of states that a system can occupy within a given volume. As in Bekenstein's gedanken experiment discussed above, if all the quantum states within a given volume were occupied by throwing in more and more matter (so as to match the apparent QFT measure of entropy), it would soon result in black hole whose horizon exceeds the volume V.

Thus the entropy in a region of space is bound by its area and a black hole within that entire region carries the maximum possible energy, saturating the bound. Owing to the connection between entropy and information, as discussed above, the holographic bound suggests that information of a system in the bulk is somehow bound by what would be a natural measure of information on the boundary. This led 't Hooft to go a step further and and propose the holographic principle as a principle of any theory including gravity. Note that while the holographic bound is only applicable to an isolated system confined to a finite region, the holographic principle is a statement for the entire universe within which a system is contained. The latter thus has to be appropriately regularized, as is implemented via the UV/IR regulator in AdS/CFT.

That the degrees of freedom of a gravitational system scale as the area of its boundary (when it exists) rather than its volume is certainly intriguing considering that one would never arrive at such a premise in standard quantum field theories. But then again, field theoretic quantization procedures for gravity lead to non-renormalizable theories. What is remarkable though, about this proposal, is that observables of a gravitating system in the bulk are fully encoded in a theory on the boundary, which itself has no gravity at all and may well be a standard quantum field theory. In an earlier section, we have seen how gravitational quantities in the bulk carry a thermodynamic interpretation, which in turn is associated to an underlying statistical description of microstates. If, via the holographic principle, the boundary theory indeed captures the full quantum description of dynamics in the bulk, it should also be able to provide a microscopic calculation of quantities such as entropy, temperature and free energy of the gravitational system. As we shall soon see, the real utility of this prescription will emerge from the fact that it facilitates microscopic calculations of observables on the boundary, that are strongly coupled and therefore unfeasible to perform in the bulk.

2.3.2 The Maldacena Conjecture

So how does one realize this bulk/boundary duality? Within the context of string theory, the AdS/CFT correspondence serves as a realization of the holographic principle, in the form of a gauge/gravity duality [13], [14] (see also [15] for an excellent review). In its original form the AdS/CFT correspondence was conjectured as a duality between $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in 3+1 dimensions with gauge group $U(N)$ and Type IIB superstring theory on an $AdS_5 \times S^5$ background. On the string theory side, the parameter N denotes the 5-form RR flux through the S^5. Consequently, the radius of S^5 is given by $R^4 = 4\pi g_s N \alpha'^2$. This is equal to the radius of AdS_5. Furthermore, the string coupling g_s is related to the Yang-Mills coupling g_{YM} by $g_s = g_{YM}^2$. The radial direction in the bulk plays the role of an energy scale in the field theory, such that going to the boundary of AdS corresponds to going into the UV regime of the field theory. As per the equivalence goes, the claim is that observables, states and correlation functions of the two theories are equivalent to one another.

More precisely, we have

$$\left\langle e^{\int d^4x \phi_0(\vec{x})\mathcal{O}(\vec{x})} \right\rangle_{CFT} = \mathcal{Z}_{AdS}\left[\phi(\vec{x},z)|_{z=0} = \phi_0(\vec{x})\right] \tag{2.3.41}$$

where $\vec{x}$ are coordinates on the boundary and z denotes the radial variable. The left hand side of the above relation is the generating function of correlation functions in the field theory. The right hand side is the full partition function of string theory with the boundary condition that the field ϕ attains the value ϕ_0 on the boundary of AdS. Armed with this, we can calculate correlation functions of the operator $\mathcal{O}$ by taking functional derivatives with respect to ϕ_0 and then setting ϕ_0 to zero, since the latter is an arbitrary function.

An analogous relation is valid for other fields too. The above equation was written for massless fields. With massive fields involved, the only subtlety is that the $z = 0$ limit has to be taken with an appropriate regulator. The gauge theory above, $\mathcal{N} = 4$ SYM, lives on the world-volume of a parallel stack of N D3-branes placed in 10 dimensional flat space. In the near-horizon limit, the metric of these branes reduces to that of $AdS_5 \times S^5$, giving a string theory on a curved background. In its original form, this statement was made for the low-energy effective lagrangian, wherein only massless string states and their excitations contribute. This is achieved by sending $\alpha' \to 0$ and consequently the string length $l_s \to 0$, whilst keeping energies and dimensionless parameters fixed. This is known as the decoupling limit, wherein gravity becomes free in the bulk and decouples from the brane theory. A striking feature of the AdS/CFT correspondence is that the conjectured equivalence is a strong-weak duality. When the AdS radius of curvature is small compared to the string length

$$\frac{R^4}{l_s^4} << 1 \qquad \Rightarrow \qquad g_{YM}^2 N << 1 \tag{2.3.42}$$

wherein the gauge theory lies in the perturbative regime (g_s is taken smaller than 1), but gravity is strongly coupled. On the other hand, when the supergravity approximation is valid, we have that

$$\frac{R^4}{l_s^4} >> 1 \qquad \Rightarrow \qquad g_{YM}^2 N >> 1 \tag{2.3.43}$$

which refers to a strongly coupled field theory. The supergravity approximation is particularly useful for many applications of the correspondence. In this regime the string theory partition function takes the form $e^{-I_{sugra}}$, where I_{sugra} is the supergravity action evaluated on-shell on $AdS_5 \times S^5$. On the gauge theory side, this corresponds to taking both large N and large 't Hooft coupling $g_{YM}^2 N$. This yields

$$W_{gauge}[\phi_0] \approx \text{extremum } I_{sugra}[\phi(\vec{x}, z)|_{z=0} = \phi_0(\vec{x})] \tag{2.3.44}$$

where W_{gauge} is the generator of connected Green's functions in the gauge theory.

Though the discussion above was specified to the particular context of AdS_5, the conjecture has subsequently been proposed for arbitrary dimension d as an AdS_d/CFT_{d-1} correspondence.

2.3.3 AdS/CFT for Other Compactifications

The realization of AdS/CFT as a holographic principle of quantum gravity, being a powerful tool for probing microscopics of a gravitational system, has found application in a wide variety of problems, some of the most prominent being in the area of black hole physics. The fact that the near-horizon geometry of a black hole contains an AdS factor, facilitates the application of the correspondence, allowing for a counting of microscopic degrees of freedom via the dual gauge theory, which resides on the world-volume of D-branes wrapping compact cycles.

As an application of AdS/CFT to other black hole systems and for different compactifications, let us demonstrate the D1-D5-P system, which realises an extremal black hole in five dimensions from Type IIB string theory compactified on a five dimensional manifold $M_4 \times S^1$, where M_4 can either be $K3$ or T^4. The microscopic entropy of this black hole was first computed by Strominger and Vafa in [16] and this was shown to be in agreement with the macroscopic gravity calculation. The gravity set-up is as follows : a set of D5-branes with charge Q_5 wrap along $M_4 \times S^1$ and D1-branes carrying charge

Q_1 wrap along the same S^1. There are N units of momentum excitations along the S^1. This configuration preserves 1/8 of the supersymmetries and gives a 5D black hole in the non-compact directions at the location of the branes. In the decoupling limit, the solution in the string frame is given by

$$ds^2 = \frac{1}{\sqrt{f_1 f_5}}\left(-dt^2 + dz^2 + \frac{N}{r^2}(dz - dt)^2\right) + \sqrt{f_1 f_5}dx^i dx^i + \sqrt{\frac{f_1}{f_5}}ds^2(M_4) \quad (2.3.45)$$

Here $f_i = 1+Q_i/r^2$, z denotes the coordinate along the S^1, x^i denote the transverse non-compact directions and $ds^2(M_4)$ is the metric on the compact space. The supergravity description of this system is valid in the large charge limit. From the metric, the leading order entropy can be obtained as the classical area law given by

$$S_{5D} = \frac{A_5}{4G_5} = 2\pi\sqrt{Q_1 Q_5 N} \quad (2.3.46)$$

The dual CFT to this gravity is the world-volume theory living on the D1-D5 system. Since the volume of M_4 is taken to be of string scale, the low energy effective theory is a 1+1 dimensional CFT. More precisely, this is a deformation of the $\mathcal{N} = (4,4)$ sigma model on the orbifold $M_4^{Q_1Q_5}/S_n^{Q_1Q_5}$, which is a symmetric product of the compact manifold modded by the permutation group. In [16] this microscopic entropy counting was performed in the perturbative regime of the orbifold CFT. For an extremal black hole, microstates pertain to the BPS sector of the CFT, and these can be counted by a supersymmetrically protected index such as the helicity supertrace. Since this index remains invariant under deformation of continuous parameters in the theory, counting of BPS states in the CFT can be related to the supergravity result. At leading order, this was shown to yield an exact match [16]. Much of this discussion is valid for four dimensional black holes as well. In the next subsection, we shall discuss another tool for probing dual CFT's of four dimensional black holes with Calabi-Yau compactifications.

2.3.4 The OSV Conjecture

In this thesis, we shall mostly be interested in $\mathcal{N} = 2$ supersymmetric black holes obtained by Calabi-Yau compactification. As a recurring theme in string theory, descriptions of many gravitational systems are encoded in holographically dual gauge theories. In this regard, a recent conjecture by Ooguri, Strominger and Vafa (OSV)

[129] relating black holes to topological strings and how that relates to a brane-world theory which reduces to a deformed version of 2D Yang Mills [94], will concern part of the work we present here [17].

Let us briefly lay out the framework of this machinery. In its original form, the conjectured equality goes as follows

$$\mathcal{Z}_{BH} = |\mathcal{Z}_{top}|^2 \tag{2.3.47}$$

where the left-hand side denotes a black hole partition sum defined as follows

$$\mathcal{Z}_{BH}(p,\phi) \equiv e^{\mathcal{F}(p,\phi)} = \sum_{\{q_A\}} \Omega_p(\{q_A\}) e^{\pi \phi^A q_A} \tag{2.3.48}$$

where the weight $\Omega_p(\{q_A\})$ is a measure of black hole microstates and $\mathcal{F}$ is the free energy associated to the black hole ensemble. The latter is typically an inverse Legendre transform of the entropy. Using the $\mathcal{N}=2$ expression for entropy in eq.(2.2.33) with eq.(2.2.30) and the black hole attractor equations

$$Re[CX^A] = p^A \qquad Re[CF_A] = q_A \tag{2.3.49}$$

relates the free energy $\mathcal{F}$ to the holomorphic prepotential F as follows

$$\mathcal{F} = -\pi Im\ F \tag{2.3.50}$$

where C is a complex constant. The black hole partition sum above is a mixed ensemble that sums over all D2 and D0 branes (taken to be electrically charged) with fixed chemical potentials thereby treating them canonically while keeping D4 and D6 branes (magnetically charged) fixed, thus treating the latter micro-canonically. The string coupling g_s as well as the Kahler modulus t_s attain a specific functional dependence on the magnetic charges and electric chemical potentials of the black hole ensemble. An ingredient implicitly used in the construction of this conjecture is the $\mathcal{N}=2$ attractor mechanism of [18], [19], [20] (see [25] for an excellent review). Solving the resulting quantum corrected attractor equations, the authors of [22] obtain the R^2-corrected quantum black hole entropy S_{BH}, which was shown in [129] to simply be the Legendre transform of the holomorphic part of the topological string free energy F_{top} including genus g corrections :

$$S_{BH}(q,p) = \mathcal{F}(p,\phi) - \phi^A q_A \tag{2.3.51}$$

where the electric charges are conjugate variables to the chemical potential

$$q_A = \frac{\partial \mathcal{F}(p,\phi)}{\partial \phi^A} \tag{2.3.52}$$

and the topological string free energy $F_{top}(t)$ with complex Kahler modulus t enters via

$$\mathcal{F}(p,\phi) = F_{top}(t) + \overline{F_{top}(t)} \tag{2.3.53}$$

meaning just the holomorphic part. Topological strings basically count holomorphic maps from string world-sheets to a target Calabi-Yau space. Genus g coefficients F_g in the perturbative expansion of the topological free energy

$$F_{top} = \sum_{g=0}^{\infty} g_{top}^{2g-2} F_g \tag{2.3.54}$$

precisely compute the scattering amplitude of 2 gravitons and $2g-2$ graviphotons in the physical string theory. These processes manifest as higher order corrections to the low energy effective action. Subsequently in the 4D $\mathcal{N}=2$ supergravity action these higher curvature corrections are encoded by extending the holomorphic prepotential to also be a function of the chiral multiplet W^2, which derives itself from the Weyl multiplet. The supergravity prepotential can then be expanded as

$$F(CX^A, W^2) = \sum_{g=0}^{\infty} F_g(CX^A) W^{2g} \tag{2.3.55}$$

where $F_0(CX^A)$ denotes the tree level prepotential and the constant C may be determined as a normalization factor. Indeed the expansion coefficients in the two expansions in eq.(2.3.54) and eq.(2.3.55) are related. The equivalence between the free energies was demonstrated in [129] (building up on earlier work in [21], [22], [23], [24]) as follows

$$F(CX^A, 256) = -\frac{2i}{\pi} F_{top}(t^A, g_{top}) \tag{2.3.56}$$

where the following identifications are made

$$CX^A = p^A + i\frac{\phi^A}{\pi} \tag{2.3.57}$$

$$t^A = \frac{X^A}{X^0} \tag{2.3.58}$$

$$g_{top} = \pm\frac{4\pi i}{CX^0} \tag{2.3.59}$$

with C^2W^2 fixed to a constant value of 256. Hence eq.(2.3.56) together with the above identifications, when inserted in eq.(2.3.50) gives eq.(2.3.53) and that leads to the conjecture in eq.(2.3.47) including quantum corrections.

The immediate question that one may now ask is how can this conjecture be tested for any given compact or non-compact Calabi-Yau geometry ? and how effective is that as a tool for computing higher derivative corrections to black hole entropy if we knew the corresponding worldsheet instanton corrections to the topological string prepotential ? Even though, for a lot of interesting cases the OSV statement itself falls short of these aims, it nevertheless motivated developments that enabled a more computable approach to black holes and black rings including single and multi-center solutions in four and higher dimensions; as well as the question of precision entropy counting for these objects.

In the aftermath of the OSV result, developments in [132], [94] whilst attempting to verify the validity of the conjecture, found interesting non-perturbative gauge theories, which serve as gravitational duals for specific Calabi-Yau geometries. The case in point here is 2D q-deformed Yang-Mills theory, which is not only dual to a bound state of D0-D2-D4 BPS black holes, but its chiral sector is also touted to capture non-perturbative dynamics of topological strings on non-compact Calabi-Yau backgrounds, constituting a Riemann surface endowed with two line bundles, with $-p$ respectively $p + 2g - 2$ as the degrees of the line bundles and g as the genus of the Riemann surface over which the bundles are endowed. In some sense, this can be thought of as zooming onto a local section of an otherwise compact geometry. The gauge theory is localized on the world-volume of the branes. For the most part here, we shall be interested inthe case $g = 0$, where the surface is a sphere. 2D Yang-Mills on an S^2 can also be studied using the equivalent matrix model technology, in this case the Chern-Simons or Stieltjes-Wigert matrix model as it is known. In this thesis we describe the work in [17] where the large N Douglaas-Kazakov phase transition of 2D QCD on a sphere [109] is replicated in its q-deformed cousin for specific values of deformation parameter $p > 2$. Moreover, like in the original Douglaas-Kazakov theory, on the q-deformed case as well, the transition is trigerred via instanton effects. Geometrically p relates to degrees of the Calabi-Yau line bundles. One implication of this result is that for backgrounds with

$p > 2$, the usual geometric description of topological strings does not hold in the small area phase of the gauge theory and this has bearing on the validity of OSV itself in that regime. Moreover, knowledge of a dual non-perturbative gauge theory, whenever possible, potentially facilitates instanton weight computations as a tool to extract black hole degeneracies. On the other hand a precise gravitational interpretation of this transition is not fully understood. A possible gravitational interpretation of this phase transition is that it signals the onset of topology change, much like the baby universe scenario in [107] which result from AdS-fragmentation of black holes. We describe the above results in detail in chapter 6 of this thesis.

2.4 Higher Dimensions & Multiple Centers

2.4.1 Black Rings & Chern-Simons Charge Shifts

Unlike 4D, where uniqueness theorems prohibit black hole solutions with topologies other than spherical (when non-rotating) or oblate (rotating Kerr), in 5D a toroidal black hole solution was recently discovered [31]. These supersymmetric black rings have $S^2 \times S^1$ spatial horizons. Moreover, these rings also carry a dipole charge, which adds "hair" to the ring. For our purposes, we shall discuss BPS black rings. These are characterized by an $AdS_3 \times S^2$ near-horizon geometry. Hence, interest in this object was also generated due to the prospect of having a 2D CFT as the microscopic dual of AdS_3 gravity. In fact in [54] this CFT was claimed to be the same as that of the MSW theory [53] for black strings in 5D. However, before testing the correspondence between gravity and the field theory, it is important to have a rigourous understanding of the physically relevant quantities on both sides of the correspondence. In this respect, there have been some subtleties concerning the treatment of charges in the bulk theory. Part of the research in this thesis attempts to clarify these issues [26], [65] (discussed in chapters 3 and 4).

In any computation of macroscopic observables such as the entropy or conserved currents, it is necessary to identify the physically relevant charges and express observables only in terms of those, if one wants to make a meaningful comparison of bulk to boundary observables. In most theories, this is straightforward; exceptions being actions which include Chern-Simons terms. Since these terms do not leave the action invariant under large gauge transformations, they affect the definition of charges in the theory. The right prescription is to express observables via what are called Page charges (the different notions of charge in Chern-Simons type of theories has been elegantly described in [59]).

In the case of black rings, the details of the above implementation shall be demonstrated in chapter 3 first within the context of the entropy function for a single black ring and then in chapter 4 within the context of AdS fragmentation for multi-ring geometries. We then verify that this implementation yields correct observables, by comparing expressions to the literature wherever possible.

With regards to the first of the above implementations, a crucial step was to develop an explicit 5D entropy function formalism that works for both 5D extremal black holes and black rings. The problem with Sen's original formulation in [29] was that it was not suited to include terms in the action that are not manifestly gauge invariant, such as Chern-Simons terms. Hence prior computations involving 5D black objects, relied on an ad hoc recipe of reducing the action to 4D and adding a total derivative term by hand to restore gauge invariance. The trouble with this make-shift approach is that it does not correctly identify physical 5D observables. This refers to conserved charges in 5D which are shifted relative to their 4D counterparts due to large gauge transformations originating from Chern-Simons terms. This feature is also referred to as spectral flow (the phrase being coined due to an analogous shift in Virasoro generators of its dual CFT). We solve the problem by showing how spectral flow can be incorporated into a 5D entropy formalism, which at the same time remains gauge invariant and has an explicit dependence only on physical charges.

2.4.2 4D/5D Connection & Multi-Center Geometries

A closely related issue to the above is the 4D/5D connection [42]. Stating it rather generally, from an OSV perspective, it can be expressed as [43]

$$\mathcal{Z}_{4D}^{BH} = \mathcal{Z}_{5D}^{BH} = |\mathcal{Z}_{top}|^2 \tag{2.4.60}$$

Of course, this way of writing it is highly oversimplified. More so since such black hole partition sums, not only include single center states, but also multi-center configurations. A more refined form of the conjecture is to match a specific gravity solution in 5D to the corresponding one in 4D (which may in general be comprised of a different number of centers than the 5D solution to which it is being associated) and compare how observables relate. For our purposes, an interesting configuration is the supersymmetric black ring, whose 4D counterpart is given by a particular 4D 2-center solution. From the full string theory point of view, such a 4D/5D map is reminiscent of the M-theory/Type IIA correspondence. Now from the discussion we had above, let us recall that unlike in 4D, the 5D action is not explicitly gauge invariant. This immediately creates a puzzle over how we should match the 4D charges, angular momentum and entropy to the corresponding quantities in 5D, where physical charges are in fact not gauge invariant. The correct dictionary has to take into account these Chern-Simons induced charge shifts in 5D. In this thesis in chapters 3 and 4, we shall provide a resolution of this puzzle for the single-center as well as the multi-center geometry and furthermore provide an interpretation of these charge shifts as spectral flow in the gravity theory.

For multi-center geometries, we shall set-up an explicit 5D construction of AdS-fragmentation and show that the 5D duals of the baby universes in [107] turn out to be a configuration of non-concentric multi-black rings in Taub-NUT space. Here too, we encounter Chern-Simons induced charge shifts. After presenting how the 4D/5D multi-center charges transform we confirm that all conserved charges here are Page charges. Finally a geometric description is given to this system of rings using the idea of split-spectral flows, wherein a given black ring's observables are influenced by fluxes generated in a background of neighboring rings.

As an aside, let us also remark on a spin-off that resulted from our investigation of Denef's multi-center geometries. It is known that a sub-set of these going by the name of scaling solutions [27] are known to play a role in the problem of black hole microstates. However even for the simplest configurations with more than two centers, solving integrability constraints to determine the full metric becomes a highly formidable task. Interestingly enough, we find that in the limit of large N number of centers, the integrability constraints are solvable [28]. We can then construct a continuum distribution of black holes and obtain the metric. Upon this continuum system we perform a multipole expansion to find smeared black hole geometries with multipole moments. As an interesting application of these methods, we then construct a black hole levitron in chapter 5. Presumably, all this carries over to 5D as well.

Chapter 3

5D Entropy Functions with Chern-Simons Terms

Hell, there are no rules here– we're trying to accomplish something

- Thomas Edison

In this chapter we begin our investigations concerning BPS black holes by starting with macroscopic entropy calculations. The development of precision macroscopic techniques are necessary if one is to later compare results to holographically dual microscopic theories, or for that matter even for validating other manifestations of gauge/gravity conjectures such as the OSV conjecture. While in four dimensions, for most cases of interest, such macroscopic entropy computations are fairly straightforward (provided all the relevant higher order terms in the action have been satisfactorily determined); the five dimensional set-up however has proven to be more subtle. This is mainly due to the inclusion of Chern-Simons terms in the action and associated charge shifts in 5D as compared to 4D. In the current chapter of this thesis, we tackle these issues and in the process develop a 5D entropy function technology that builds over Sen's entropy formalism [29].

The entropy function formalism of Sen [29], [30] allows for a very systematic approach to computing black hole entropy in D dimensions with $AdS_2 \times S^{D-2}$ near-horizon geometry, especially including higher derivative corrections. Subsequently this formalism has also found application to other extremal black objects such as black rings and even black holes with reduced near-horizon isometry groups [39], [40]. However, in odd dimensions, the presence of Chern-Simons terms in the supergravity action no longer leaves the latter invariant under large gauge transformations; whereas Sen's original construction was formulated for gauge as well as reparametrization invariant actions. To overcome this hurdle, it was proposed in [41] to perform a dimensional reduction in order to bring the Lagrangian density into a gauge invariant form and then apply the entropy function method. Therefore whilst computing the black ring entropy function, the authors of [39] first perform a dimensional reduction of the 5D supergravity Lagrangian into a gauge invariant 4D Lagrangian, upon which the standard entropy function method can then be applied.

In this work we revisit the black ring and 5D static black hole entropy functions. Instead of taking recourse to a dimensional reduction, we propose that a meaningful 5D computation of the entropy function with Chern-Simons terms is possible[1]. While performing such a 5D analysis, a key issue which requires careful consideration is how we should treat charges in 5D and their corresponding spectral flows. For the benefit of our esteemed reader, let us recall that these are also the same questions that have been at the center of much debate [37], [38], [55], [84], [56] with regards to the 4D/5D conjecture for black holes and black rings [42], [43]. It is not surprising that those subtleties also come into play when trying to perform an intrinsic 5D analysis of the entropy function formalism. And that happens because the introduction of Chern-Simons terms brings in three different notions of charge : Brane-source charge, Maxwell charge and Page charge [59]. Which one is more relevant depends very much on the details of the geometric configuration one is interested in. Then expressing the entropy function in terms of the correct 5D charges will turn out to be the crucial step towards resolving its apparent lack of gauge invariance. We do this explicitly first for the black ring and then for the black hole.

In case of the black ring, even though we find that the reduced action is no longer invariant under large gauge transformations, it nevertheless turns out that the entropy function itself does remains gauge invariant. Furthermore we show that this invariance is no coincidence, but stems from an underlying spectral flow symmetry of the theory, which leaves the entropy function invariant under spectral flow transformations. In

[1] In this paper we only consider gauge-type Chern-Simons terms. Presumably our considerations are valid for gravitational or mixed gauge-gravitational Chern-Simons terms as well.

order to achieve this, we have to first demonstrate how the relevant spectral flow relations emerge within the 5D computation whilst solving the equations of motion in the presence of Chern-Simons terms. Through this we shall also be able to identify the 4D/5D dictionary, using which the 4D-reduced computation of Goldstein and Jena [39] can be recovered - except for one subtle issue on which our 5D computation differs from their 4D computation for reasons that will become clear in the calculations that follow.

In this context it is worth pointing out to the work of [45] on AdS_5 black holes in gauged supergravity where it was also suggested that Chern-Simons terms would somehow facilitate charge shifts of the form $q_I \to q_I + c_I$. However these authors propose a modified Sen's formalism with shifted charges directly implemented and the c_I being undetermined shift parameters. Then in [46] this issue was pushed further (see also [47] for work in a related context), where they propose a new entropy function for rotating 5D black holes in order to extract asymptotic charges from near horizon data. However the above attempts do not work for black ring type geometries. The philosophy we adopt in this work is that it is not necessary to modify Sen's formalism by imposing charge redefinitions ad hoc, but rather a consistent 5D evaluation of Sen's functional is possible and from which these charge shifts can be seen to emerge in a natural way. We will see that this is indeed the case and such charge shifts carry a natural interpretation as spectral flow shifts in 5D. This way we are able to uniquely determine the shift parameters and unlike previous attempts our procedure works simultaneously for both AdS_2 as well as AdS_3 near horizon factors.

After having treated the black ring, we proceed to check gauge invariance of the 5D black hole entropy function. Here again we see that a 5D calculation shows some interesting differences when compared to the 4D calculation of [39]. This will have something to do with the x^μ-dependence of the moduli a^I (which are ψ-components of the 5D gauge fields A^I). In the calculation of [39], the x^μ-dependence of a^I are retained throughout dimensional reduction of Chern-Simons terms to 4D and only then are they set to be constants. Apparently this is what seems to create a seemingly incorrect shift in electric charges when comparing their result for the black hole entropy to that of [48]. Here we claim that the way out is not to assume such a coordinate dependence (

which would even be incompatible with the isometries of the 5D near-horizon geometry) in a 5D calculation. In addition to finding an agreement with the result of [48], our claim also leads to the correct 5D electric charges which are seen to perfectly tally with recent results of [61], who perform an explicit near-horizon analysis pertaining to 5D supergravity.

The outline of this paper is as follows - In section 2 we compute the black ring entropy function without dimensional reduction. The 5D charges turn out to be Page charges, which exhibit spectral flow behaviour. The entropy function however is shown to be spectral flow invariant. Section 3 concerns gauge invariance of the 5D black hole entropy function. For both black objects, we compare the 5D charges computed here via the 5D entropy formalism to those computed in the supergravity analysis of [61]. In section 4 we clarify the subtleties in charges arising between explicit 4D and 5D applications of the entropy function. Then in section 5 we provide an interpretation for the $e^0 \leftrightarrow p^0$ switch within the entropy formalism as corresponding to a black hole $\leftrightarrow$ black ring interpolation in supergravity. Finally in section 6 we conclude with some discussions.

3.1 The Black Ring Entropy Function & Spectral Flow

Let us now perform a 5D computation of the black ring entropy function and derive the associated spectral flow relations from the equations of motion therein.

Consider the action of 5D minimal ungauged two-derivative supergravity theory coupled to $N-1$ abelian vector multiplets. Writing only the bosonic fields, we have

$$\mathcal{S}_5 = \frac{1}{16\pi G_5}\int R*1 - G_{IJ}dX^I\wedge *dX^J - \frac{1}{2}G_{IJ}F^I\wedge *F^J - C_{IJK}A^I\wedge F^J\wedge F^K \qquad (3.1.1)$$

where X^I are massless scalars parameterizing the five dimensional "very special geometry". These scalars define the compactification volume $\mathcal{V}$ via the relation

$$C_{IJK}X^I X^J X^K = \mathcal{V} \qquad (3.1.2)$$

The couplings G_{IJ} are functions of the scalar moduli and are defined as

$$G_{IJ} = -\frac{1}{2}\frac{\partial}{\partial X^I}\frac{\partial}{\partial X^J}\, ln\,\mathcal{V}\,\Bigg|_{\mathcal{V}=1} \tag{3.1.3}$$

The indices I, J, K run from 1 to N while C_{IJK} is a completely symmetric tensor and $F^I = dA^I$ are N $U(1)$ gauge fields.

Now let us consider the effect of large gauge transformations to the action in eq.(3.1.1). These transformations can be parametrised as

$$A^I \longrightarrow A^I + \Lambda^I \tag{3.1.4}$$

where Λ^I are one-forms whose components we shall shortly specify. Clearly the Chern-Simons term in the action is not invariant under large gauge transformations[2]. In fact large gauge transformations introduce integral shifts of the action that pick up a phase in the path integral. In this section, we revisit the black ring entropy function and show that instead of the 4D approach followed by [39], one can also perform an alternate well-defined 5D calculation. Consequently, we need to directly tackle the problematic Chern-Simons terms above; which we do so by invoking spectral flow shifts.

To begin with, the 5D geometry is expressed via a Kaluza-Klein ansatz for an $AdS_2 \times S^2 \times S^1$ topology (metric in eq.(3.1.8) below). Eventually of course, when one extremises the entropy function, the S^1 fibres over the AdS_2 (see [39]) precisely recovering the known near-horizon $AdS_3 \times S^2$ metric ([31], [32], [34], [35]) of a supersymmetric black ring. Also the 5D gauge potential A^I is expressed in terms of the aforementioned Kaluza-Klein decomposition as follows

$$A^I = A^I_\mu dx^\mu + a^I\left(d\psi + A^0_\mu dx^\mu\right) \tag{3.1.5}$$

where ψ parametrises the S^1 circle with a periodicity of 4π; the A^0_μ are off-diagonal entries in the 5D Kaluza-Klein metric (which we shall write down shortly); the

[2]Small gauge transformations pose no problems in this case. This is because the extra gauge terms in the action can be expressed as an integral of a total derivative which is then evaluated as a surface term at infinity, where the gauge parameters asymptotically vanish. However, with large gauge transformations this is not so. The latter are not obtained as continuous transformations from the identity element and hence cannot be expressed as exact forms that could be partially integrated and evaluated as surface terms.

scalars a^I, which are ψ-components of the 5D gauge potential A^I, are interpreted as axions in 4D; while A^I_μ would just be the usual gauge potential in the four non-compact dimensions. Typically a large gauge transformation applied to an on-shell gauge potential can be implemented by choosing $\Lambda^I = k^I d\psi$ where k^I are integral constants. Note however that A^I in eq.(3.1.5) is not yet on-shell since we have still to insert the values of a^I, A^I_μ and A^0_μ after solving their respective equations of motion. We therefore write down a more general ansatz for the gauge parameter given by $\Lambda^I = k^I \left(d\psi + A^0_\mu dx^\mu\right)$. This can be implemented in eq.(3.1.5) via a simple shift

$$a^I \longrightarrow a^I + k^I \tag{3.1.6}$$

where the k^I are again integral constants. A few comments are in order here. Though eq.(3.1.6) still represents a shift in the ψ-component of A^I, this quantity (a^I) also enters as a factor in the other x^μ-components making it natural to allow shifts of $K^I A^0_\mu$ in those respective components. Also it turns out, as will be clear in what follows, that eq.(3.1.6) in fact denotes the most general shift that correctly generates the full 5D spectral flow of charges. Moreover this choice of Λ^I will also leave the components of the on-shell field strength F^I independent of k^I once we solve the equation of motion for a^I and insert it into dA^I. These will be consistency checks of eq.(3.1.6) that we shall verify along the way.

The reduced action (terminology not to be confused with dimensionally reduced action) is now defined by integrating the 5D lagrangian density over $S^2 \times S^1$ - the spatial horizon of the black ring, spanned by θ, ϕ and ψ

$$\mathcal{F}_5^{br} = \frac{1}{16\pi G_5}\int_\Sigma d\theta d\phi d\psi \sqrt{-g_5}\mathcal{L}_5 \tag{3.1.7}$$

Our task then is to evaluate $\mathcal{F}_5^{br}$ in the background of the Kaluza-Klein metric for an $AdS_2 \times S^2 \times S^1$ near-horizon topology

$$ds^2 = \omega^{-1}\left[v_1\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + v_2\left(d\theta^2 + sin^2\theta d\phi^2\right)\right] + \omega^2\left(d\psi + A^0_\mu dx^\mu\right)^2 \tag{3.1.8}$$

with A^I_μ and A^0_μ specified by

$$A^I_\mu dx^\mu = e^I r dt + p^I cos\theta d\phi \qquad A^0_\mu dx^\mu = e^0 r dt \tag{3.1.9}$$

Here we take ω, v_1, v_2, X^I, a^I, e^I, e^0 to be constants in the near-horizon geometry. These will eventually be fixed as functions of the black ring charges upon extremisation. ω is the radius of the Kaluza-Klein circle; v_1, v_2 denote the AdS_2 and S^2 radii respectively; p^I are magnetic charges and e^I denote the corresponding electric fields in 4D (we shall soon write down the electric fields in 5D as well). e^0 is dual to the magnetic field associated to a p^0 charge (or D6-brane charge). However for rings, it is well known that the p^0 charge is absent in the immediate vicinity of the horizon. In 4D, e^0 too is treated as an electric field; however in 5D it will turn out to be associated to the angular momentum of the black ring along the S^1 direction.

Putting all this together, and computing the 5D reduced action gives

$$\begin{aligned}
\mathcal{F}_5^{br}(v_1, v_2, \omega, X^I, a^I, e^I, e^0, p^I) &= \left(\frac{2\pi}{G_5}\right)\Bigg[v_1 - v_2 + \frac{v_2\,\omega^3\,(e^0)^2}{4\,v_1} \\
- \frac{v_1}{v_2}\,\omega\,\frac{G_{IJ}}{2}\,p^I p^J &+ \frac{v_2}{v_1}\,\omega\,\frac{G_{IJ}}{2}\,(e^I + \tilde{a}^I e^0)\,(e^J + \tilde{a}^J e^0)\Bigg] \\
&+ \left(\frac{24\pi}{G_5}\right) C_{IJK}\left[\,(e^I + \tilde{a}^I e^0)\,p^J\,\tilde{a}^K\,\right]
\end{aligned} \tag{3.1.10}$$

We get the three terms in the first line of eq.(3.1.10) by computing the five dimensional Ricci scalar; the second line comes from the 5D Yang-Mills term in the action; and the last line is obtained from the Chern-Simons term. It is important to note that this result here differs from that of [39] on two counts[3] - Firstly we have shifts in the moduli $a^I \to \tilde{a}^I \equiv (a^I + k^I)$, which essentially encode large gauge transformations in 5D and consequently leave $\mathcal{F}_5^{br}$ with a gauge ambiguity, which is manifest through the explicit k^I dependence. In a 4D-reduced calculation these shifts do not appear. The second point on which $\mathcal{F}_5^{br}$ differs from its dimensionally reduced version $\mathcal{F}_4^{br}$ is a factor of $\frac{1}{2}$ in one of the two Chern-Simons contributions to the reduced action. This can be seen in the last line of eq.(3.7) in ref. [39] (note that their p^0 has to be set to zero when considering black rings). In a 5D calculation, the reduced action $\mathcal{F}_5^{br}$ does not contain this factor. In section 4 we shall see that this difference of factors arises because of the way the moduli a^I have been treated in a 5D calculation as opposed to how they were dealt with in the 4D case. This point will also turn out to be crucial in determining the correct 5D charges and in the end we shall justify our results by comparing with the analysis in [61].

[3] Our G_{IJ} equals $2f_{IJ}$ in the notation of [39].

Now, the 5D entropy function is defined as the Legendre transform of $\mathcal{F}_5^{br}$ with respect to electric charges Q_I^{br}, Q_0^{br}

$$\mathcal{E}_5^{br} \;=\; 2\pi \left[\, Q_0^{br}\, e^0 \;+\; Q_I^{br}\, e^I \;-\; \mathcal{F}_5^{br}(v_1,\, v_2,\, \omega,\, X^I,\, \tilde{a}^I,\, e^I,\, e^0) \,\right] \tag{3.1.11}$$

where Q_I^{br} and Q_0^{br} are canonically conjugate to e^I and e^0 respectively[4]

$$Q_I^{br} = \frac{\partial \mathcal{F}_5^{br}}{\partial e^I} \qquad\qquad Q_0^{br} = \frac{\partial \mathcal{F}_5^{br}}{\partial e^0} \tag{3.1.12}$$

As we shall soon see, Q_I^{br}, Q_0^{br} are 5D Page charges and are physical observables of the black ring. These charges will differ from the 4D electric charges q_I respectively q_0 computed in [39].

Obtaining the entropy of a black ring then entails extremisation of the entropy function $\mathcal{E}_5^{br}$ with respect to its moduli variables

$$\frac{\partial \mathcal{E}_5^{br}}{\partial a^I} = \frac{\partial \mathcal{E}_5^{br}}{\partial v_1} = \frac{\partial \mathcal{E}_5^{br}}{\partial v_2} = \frac{\partial \mathcal{E}_5^{br}}{\partial \omega} = \frac{\partial \mathcal{E}_5^{br}}{\partial X^I} = 0 \tag{3.1.13}$$

But before that let us see how the gauge ambiguity in the reduced action $\mathcal{F}_5^{br}$, and consequently in the entropy function $\mathcal{E}_5^{br}$, can be resolved. For that purpose we will need to know exactly how the Chern-Simons terms in $\mathcal{F}_5^{br}$ affect physical charges Q_I^{br} and Q_0^{br}. It turns out that they induce spectral flow shifts in these charges. And we want to know how these shifts can be manifestly derived within the framework of the entropy function formalism itself. Consequently we shall see how $\mathcal{E}_5^{br}$ remains invariant under these shifts.

We begin evaluating eq.(3.1.12) for Q_I^{br} and Q_0^{br} by making use of $\mathcal{F}_5^{br}$ from eq.(3.1.10). To avoid cluttering of notation let us normalise the $\frac{4\pi}{G_5}$ factors in front of the charges to 1. Later in the final result we shall restore these constants. We then get the following expressions

$$Q_I^{br} = \left(\frac{v_2}{v_1}\right) \omega \frac{G_{IJ}}{2} \left(e^J + e^0 \tilde{a}^J\right) + 6 C_{IJK} \tilde{a}^J p^K \tag{3.1.14}$$

and

$$Q_0^{br} = \left(\frac{v_2}{v_1}\right) \left(\frac{1}{4}\omega^3 e^0 + \omega \frac{G_{IJ}}{2} \tilde{a}^I \left(e^J + e^0 \tilde{a}^J\right)\right) + 6 C_{IJK} \tilde{a}^I \tilde{a}^J p^K \tag{3.1.15}$$

[4]Formally the Q_I^{br} can be expressed as conjugates to $(e^I + \tilde{a}^I e^0)$. However since the Jacobian between the four and five dimensional electric variables (e^I respectively $e^I + \tilde{a}^I e^0$) is one, we end up with the first expression in eq.(3.1.12).

That these are in fact the correct 5D charges for a black ring can be checked by comparing these expressions to the 5D Page charges recently computed in the supergravity analysis of [61], who showed that the near-horizon region of a black ring also encodes full information of its charges measured at asymptotic infinity. The results of [61] yield

$$Q_I^{Page} = \frac{1}{16\pi^2}\int_\Sigma *F_I + 6C_{IJK}A^J \wedge F^K \quad (3.1.16)$$

$$Q_0^{Page} = -\frac{1}{16\pi^2}\int_\Sigma *d\xi + *(\xi\cdot A^I)F_I + 6C_{IJK}(\xi\cdot A^I)A^J \wedge F^K \quad (3.1.17)$$

where Σ is a 3-cycle over the spatial horizon. For the black ring Σ specialises to $S^2 \times S^1$. ξ denotes the axial Killing vector with respect to the ψ-direction, while $(\xi \cdot A^I)$ is an inner product between a vector field and a one-form. The Killing field ξ generates isometries along the ψ-direction; leading to a conserved charge, which is simply the angular momentum. In fact, the right-hand side of eq.(3.1.17) is just the Noether charge of Wald. Page charges are in fact not gauge invariant (due to an explicit A^I-dependence in these expressions), even though they are conserved and localised [59]. Now in order to strike a comparison between these charges of [61] and those computed here using the 5D entropy formalism, we will need to explicitly integrate the right-hand sides of eqs.(3.1.16) and (3.1.17). Since these are simply local integrations, it is sufficient to make use of only near-horizon data of the gauge fields and metric from eqs.(3.1.5) and (3.1.8). Computing the non-vanishing components of the 5D field strength gives $F^I_{rt} = e^I + \tilde{a}^I e^0$ and $F^I_{\theta\phi} = -p^I sin\theta$. In the near-horizon terminology, the axial vector ξ^i is found to be A^i_0, with non-vanishing components $A^t_0 = \frac{\omega^3 e^0}{v_1 r}$ and $A^\psi_0 = -1$. Using this we can determine F_0, which is just $d\xi$; and by $d\xi$ we mean $\partial_i \xi_j \, dx^i \wedge dx^j$. Note also that in the $\xi \cdot A^I$ term, it makes sense to only consider the projection of the Killing field on the physical (on-shell sector) gauge fields. Putting together all these quantities and inserting them into eqs.(3.1.16) and (3.1.17) exactly reproduces eqs.(3.1.14) and (3.1.15). Hence we see that Q_I^{br} and Q_0^{br} obtained from the entropy function indeed represent the correct five-dimensional supergravity Page charges Q_I^{Page} and Q_0^{Page} respectively.

Now in the entropy function formalism the 5D field A^I in eq.(3.1.5) depends on three different moduli e^I, e^0 and a^I. Extremising $\mathcal{E}_5^{br}$ with respect to these moduli and plugging the extremum values of these moduli back into eq.(3.1.5) basically determines the near-horizon gauge fields of the black ring. A^I can then be expressed purely in terms of electric and magnetic charges. For our purposes, these three extremisation conditions will fully determine the physical charges that source these gauge fields A^I. Hence eqs.(3.1.14) and (3.1.15) require further input from

$$\frac{\partial \mathcal{E}_5^{br}}{\partial a^I} = 0 \qquad \Longrightarrow \qquad F^I_{rt} = 0 \tag{3.1.18}$$

and this exactly corresponds to $\int_\Sigma *F_I = 0$ computed in [61] by explicit near-horizon integration. Physically, eq.(3.1.18) signifies a vanishing electric flux in the near-horizon geometry, which is simply what one would expect in the absence of a compact 3-cycle when the topology is that of $AdS_3 \times S^2$. Moreover the above result also tells us that the only non-vanishing on-shell components of the field strength (in this case the $F^I_{\theta\phi}$) are all indeed gauge invariant.

We are now ready to identify the black ring spectral flow shifts that emerge from within the structure of the entropy function formalism itself. Separating the k^I dependence in Q^{br}_I and Q^{br}_0 yields

$$Q^{br}_I = q_I + 6C_{IJK}k^J p^K \tag{3.1.19}$$

and

$$Q^{br}_0 = q_0 + 2k^I q_I + 6C_{IJK}k^I k^J p^K \tag{3.1.20}$$

where q_I and q_0 are read-off from eqs.(3.1.14) respectively (3.1.15) after replacing $\tilde{a}^I$ by a^I; and they can indeed be identified as the four dimensional (gauge invariant as well) electric charges that appeared in the calculation of [39]. In 5D however, Q^{br}_I and Q^{br}_0 are the correct physical observables [38], [61], [65].

Let us now determine what the conserved quantities, under spectral flow shifts of Q^{br}_I and Q^{br}_0 look like. It is easy to see that $\widehat{Q}_0$ defined by

$$\widehat{Q}_0 \;\equiv\; Q^{br}_0 - C^{IJ} Q^{br}_I \, Q^{br}_J \tag{3.1.21}$$

is left invariant under spectral flow transformations described in eqs.(3.1.19) and (3.1.20) in the following sense

$$\widehat{Q}_0 \left(Q^{br}_0 , \; Q^{br}_I\right) \;=\; \widehat{Q}_0 \left(q_0 , \; q_I\right) \tag{3.1.22}$$

where $C^{IJ} \equiv [C_{IJ}]^{-1}$ and $C_{IJ} \equiv 6C_{IJK}p^K$. Consequently the quantity $\widehat{Q}_0$ is completely independent of the shift parameters k^I and this fact will play an important role in maintaining invariance of the 5D black ring entropy function.

Putting together all the above ingredients into eq.(3.1.11) gives us the entropy function in terms of 5D variables

$$\mathcal{E}_5^{br} = \frac{4\pi^2}{G_5}\left\{v_2 - v_1 + \frac{v_1}{v_2}\left[\omega\frac{G_{IJ}}{2}p^I p^J + 4\omega^{-3}\left(\widehat{Q}_0\right)^2\right]\right\} \qquad (3.1.23)$$

The first term in the square brackets in $\mathcal{E}_5^{br}$ comes from the magnetic flux, while the second term is related to the effective momentum of D0-particles[5]. This brings us to the main result of this section that $\mathcal{E}_5^{br}$ is indeed invariant under spectral flow transformations, once the moduli of the gauge field A^I have been determined. Here we have obtained $\mathcal{E}_5^{br}$ in eq.(3.1.23) from a 5D calculation, and this agrees with the structural form of the dimensionally reduced $\mathcal{E}_4^{br}$ of [39] because of spectral flow invariance[6]. Note however that while the form of the expression in eq.(3.1.23) is the same as that obtained in the 4D calculation of [39], their $\widehat{Q}_0$ differs from ours in eq.(3.1.21) obtained above by a half in the last term. In section 4 we shall see that this is because of a slight discrepancy that enters the charges defined in [39]. Nevertheless the final 4D and 5D entropies reconcile despite the fact that $\mathcal{F}_5^{br}$ and $\mathcal{F}_4^{br}$ differ due to explicit gauge transformation parameters and also that the observable 5D charges are Page charges whereas the 4D ones are Maxwell [61], [65]. This illustrates the point that for a 5D action which includes Chern-Simons terms, there is another way besides a dimensional reduction to 4D; a direct 5D calculation will also give the correct result once the right 5D variables have been implemented into the calculation. Note that $\mathcal{E}_5^{br}$ is not yet an entropy and here what we see is that even when $\mathcal{E}_5^{br}$ is not at its stationary point, it is still gauge invariant. Hence we get

$$\mathcal{E}_5^{br}\left(Q_0^{br}, Q_I^{br}, p^I, v_1, v_2, \omega, X^I\right) = \mathcal{E}_5^{br}\left(q_0, q_I, p^I, v_1, v_2, \omega, X^I\right) \qquad (3.1.24)$$

upon inserting eqs.(3.1.19) and (3.1.20) into eq.(3.1.23). The left-hand side is what one gets from an explicit 5D calculation, whereas the right-hand side is what results from a dimensionally reduced computation.

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5 These are precisely the left-movers of the dual $(0, 4)$ SCFT [53].

6 The 4D/5D lift for black rings is in fact a special case of spectral flow transformations when the value of k^I is set to p^I [65].

A 5D calculation is necessary to illustrate the inherent spectral flow associated to a black ring geometry. The physical interpretation of spectral flow for black rings has been discussed in [65]. The 4D/5D transformations themselves are in fact a special case of spectral flow transformations. And that is actually the reason why application of the entropy function formalism to black rings should work well either in 4D or 5D (even though we think that an explicit 5D computation expresses charge/geometric data more naturally).

For the sake of completeness, let us also extremise with respect to the remaining moduli, as in eq.(3.1.13); and show that the resulting black ring entropy obtained from our 5D calculation indeed gives the right answer. Solving for v_1, v_2, ω gives

$$v_1 \;=\; v_2 \;=\; \omega \frac{G_{IJ}}{2} p^I p^J + 4\omega^{-3} (\widehat{Q}_0)^2 \tag{3.1.25}$$

and

$$\omega^4 \;=\; \frac{12\,(\widehat{Q}_0)^2}{\frac{G_{IJ}}{2}\,p^I p^J} \tag{3.1.26}$$

and upon using these values of v_1, v_2, ω back into $\mathcal{E}_5^{br}$ yields[7]

$$\mathcal{E}_5^{br} = \frac{8\pi^2}{G_5} \sqrt{\left(\frac{2\ G_{IJ}}{3} p^I p^J\right)^{\frac{3}{2}} \widehat{Q}_0} \tag{3.1.27}$$

Of course the couplings G_{IJ}, which are functions of the yet-to-be-extremised scalar moduli X^I, will depend on geometric data of the specific compactification space. For our purposes we leave it with the general expression in eq.(3.1.27).

3.2 The 5D Black Hole Entropy Function

We now repeat our calculation for the 5D black hole. The near-horizon metric ansatz is again taken to be $AdS_2 \times S^2 \times S^1$. However this time round it turns out that the S^1

[7] In this result the charge of the Kaluza-Klein monopole p^0 is taken to be unity. This corresponds to a black ring in Taub-NUT (or flat space whenever the Taub-NUT radius goes to infinity). The case $p^0 > 1$ corresponds to taking an orbifold of the Taub-NUT and that in turn leads to a near-horizon factor of $AdS_3/\mathbb{Z}_{p^0}$ for the black ring. In the notation of [39] this charge has been denoted as $\tilde{p}^0$ and like in that work its effect can be included by modding the S^1 circle by $\mathbb{Z}_{p^0}$.

fibres over the S^2, eventually leading to an $AdS_2 \times S^3$ geometry near the horizon [39]. It has been proven in [62], [63], [64] that even in the rotating case, the near-horizon isometry of an extremal black hole contains an $SO(2,1)$ symmetry. Moreover, that the entropy function formalism can also be applied to such rotating black holes having AdS_2 isometry was shown in [60]. Such a black hole in 5D carries a Kaluza-Klein monopole charge p^0, which comes from uplifting a D6-brane in Type II A theory to M-theory and the black hole sits at the origin of the KK monopole[8]. Even though this geometric configuration is different from that of a black ring, it is still reasonable to implement the Kaluza-Klein metric ansatz of eq.(3.1.8) provided the off-diagonal components A^0_μ are suitably modified for the black hole case. We consider the same type of black hole as in [39], so that the results of our analysis can be compared to theirs. Hence A^0_μ is taken as

$$A^0_\mu dx^\mu = p^0 \, cos\, \theta \, d\phi \tag{3.2.28}$$

where p^0 denotes the Kaluza-Klein monopole charge. Note also that the quantity e^0 is absent for these black holes, which corresponds to an absence of Kaluza-Klein momentum J_0^{KK}. Here $J_0^{KK} = 0$ is only to be thought of as vanishing of the intrinsic angular momentum (resulting from the absence of D0-charge in the brane bound state). In [39] it was claimed that this black hole is static. However there is a slight subtlety to that. The effective angular momentum is in fact non-vanishing. As a quick check one can easily compute the integral in eq.(3.1.17) and we see that the second term in the integrand carries a non-vanishing contribution. Nevertheless it will turn out that this effective contribution does not enter the entropy formula (and this last point was presumably the reason that this black hole was viewed as a static system in [39]). On the other hand a black hole of the BMPV type [51], is a true rotating black hole with an angular momentum that enters the entropy formula. Such a black hole would be obtained had we started with a bound state of spinning M2's in Taub-NUT (or a D0-D2-D6 bound state in Type II A). Instead what we have here is a black hole more of the type discussed in [52]. It can be conceived as a bound state of non-rotating M2's sitting at the tip of a Taub-NUT-flux geometry (D2-D4-D6 in II A), where

[8] Note that when $p^0 > 1$, the S^1 circle is modded by $\mathbb{Z}_{p^0}$ consequently giving an $AdS_2 \times S^3/\mathbb{Z}_{p^0}$ near-horizon geometry. This shall be appropriately implemented in what follows.

the intrinsic angular velocity of the horizon vanishes, leaving only the flux induced component of the angular momentum which affects the geometry but not the entropy formula - in some sense like a static black hole in a flux background.

Within this set-up we now compute $\mathcal{F}_5^{bh}$ to get

$$\begin{aligned}\mathcal{F}_5^{bh}(v_1,\, v_2,\, \omega,\, X^I,\, a^I,\, e^I,\, p^I,\, p^0) &= \left(\frac{2\pi}{G_5}\right)\Bigg[v_1 - v_2 - \frac{v_1\,\omega^3\,(p^0)^2}{4\,v_2} \\ + \frac{v_2}{v_1}\,\omega\,\frac{G_{IJ}}{2}\,e^I\,e^J &- \frac{v_1}{v_2}\,\omega\,\frac{G_{IJ}}{2}\,(p^I+\tilde{a}^I p^0)\,(p^J+\tilde{a}^J p^0)\Bigg] \\ &+ \left(\frac{24\pi}{G_5}\right) C_{IJK}\left[\,(p^I+\tilde{a}^I p^0)\,e^J\,\tilde{a}^K\,\right] \end{aligned} \quad (3.2.29)$$

which differs from eq.(3.1.10) with the replacement $p^I \longrightarrow p^I + \tilde{a}^I p^0$ and a $(p^0)^2$ term in the 5D Ricci scalar that replaces the $(e^0)^2$ term in the black ring computation. Just as in the black ring analysis before, we once again find that $\mathcal{F}_5^{bh}$ computed here is not exactly going to be the same as $\mathcal{F}_4^{bh}$ in [39]. Firstly, in a 5D approach the gauge parameters k^I show up and secondly, the relative factors in front of the Chern-Simons contributions will differ from those in the 4D computation of [39] (refer to eq.(3.7) in ref. [39] after setting $e^0 = 0$ therein). Once again in $\mathcal{F}_5^{bh}$ this factor does not appear. In the next section we shall see in detail how this affects the definition of electric charges in 5D and thereby fix a small mismatch, with respect to the definition of 5D charges, in the result for the entropy obtained by [39] when compared to that of [48].

Having eq.(3.2.29) in hand, we are now in a position to write the 5D black hole charges from the analog of the definition in eq.(3.1.12)

$$Q_I^{bh} = \left(\frac{v_2}{v_1}\right)\omega\frac{G_{IJ}}{2}e^J + 6C_{IJK}\left(p^J + \tilde{a}^J p^0\right)\tilde{a}^K \quad (3.2.30)$$

Moreover using

$$\frac{\partial \mathcal{E}_5^{bh}}{\partial a^I} = 0 \qquad \Longrightarrow \qquad p^I + \tilde{a}^I p^0 = 0 \quad (3.2.31)$$

we can write eq.(3.2.30) as

$$Q_I^{bh} = \frac{1}{16\pi^2}\int_\Sigma *F_I \quad (3.2.32)$$

since $F_{rt}^I = e^I$ and $F_{\theta\phi}^I = -(p^I + \tilde{a}^I p^0)\,sin\theta$. Here Σ is now an S^3, the spatial horizon of the black hole. Eq.(3.2.31) is just the condition for vanishing of the effective magnetic flux

$$\int_{S^2} F_I \;=\; 0 \tag{3.2.33}$$

in other words suggesting the absence of a compact 2-cycle in this black hole geometry. Eq.(3.2.31) also confirms that all the non-vanishing on-shell components of the field strength are gauge invariant. Moreover for given magnetic charges p^I and p^0, the constraint $p^I + \tilde{a}^I p^0 = 0$ imposes a restriction on the value of k^I. Therefore for this black hole, we cannot set-up arbitrary spectral flow shifts for the charges.

In the terminology of [59], eq.(3.2.32) implies that Q_I^{bh} is not a Page but a Maxwell charge[9], which is gauge invariant and does not show spectral flow behaviour. Q_I^{bh} therefore represents the same physical observable in 5D as well as in 4D alike.

Under these considerations, the entropy function for this black hole takes the form

$$\mathcal{E}_5^{bh} = \frac{4\pi^2}{G_5}\left\{v_2 - v_1 + \frac{v_1}{v_2}\left[\frac{1}{4}\omega^3(p^0)^2 + \omega^{-1}2G^{IJ}Q_I^{bh}Q_J^{bh}\right]\right\} \tag{3.2.34}$$

where G^{IJ} is defined as the inverse of G_{IJ}. Once again we have obtained a gauge invariant entropy function from an explicit 5D calculation in terms of physical 5D variables. Now it is straightforward to extremise $\mathcal{E}_5^{bh}$ with respect to v_1, v_2 and ω to get

$$v_1 \;=\; v_2 \;=\; \frac{1}{4}\omega^3(p^0)^2 + \omega^{-1}2G^{IJ}Q_I^{bh}Q_J^{bh} \tag{3.2.35}$$

and

$$\omega^4 \;=\; \frac{8G^{IJ}Q_I^{bh}Q_J^{bh}}{3(p^0)^2} \tag{3.2.36}$$

Then eliminating v_1, v_2 and ω by way of substituting their values at the stationary point back into $\mathcal{E}_5^{bh}$ leaves us with

$$\mathcal{E}_5^{bh} = \frac{4\pi^2}{G_5}\sqrt{p^0\,\left(\frac{8\,G^{IJ}}{3}Q_I^{bh}Q_J^{bh}\right)^{\frac{3}{2}}} \tag{3.2.37}$$

which finally gives us the entropy of this black hole. The couplings G^{IJ} can be determined depending on the specific choice of compactification. Here Q_I^{bh} is the observable

[9] Additionally, in this case the Maxwell charge is localised within Σ and does not require integration over all space because the source term $F^J \wedge F^K$ in the 5D supergravity equation of motion : $d * F_I = -6C_{IJK}F^J \wedge F^K$, vanishes following eq.(3.2.31).

electric charge in 5D and since we have shown above that this charge does not exhibit any spectral flow behaviour, it exactly equals the number of M2-branes wrapping Calabi-Yau 2-cycles. Upon shrinking the M-theory circle and reducing to Type II A, the M2-branes directly descend to D2-branes. Then Q_I^{bh} is also the physical charge for a 4D black hole.

3.3 Charge Comparison: 4D & 5D Approaches

In this section we demonstrate how the charge mismatch, obtained in [39] when compared to that of [48], is fixed by our 5D approach. We then provide the necessary consistency checks. Firstly comparing eq.(3.2.37) above to the entropy obtained by [48] (whose computation is performed via a 5D attractor mechanism), indeed gives an exact agreement; thereby fixing the mismatch in the result of [39] where the charges Q_I^{bh} in the entropy formula were shifted by $3C_{IJK}p^Jp^K/p^0$ (refer eq.(3.41) in ref. [39] where in their notation $\hat{q}_I$ enters the entropy formula rather than q_I; then in eq.(3.65) in the same reference they compare $\hat{q}_I$ to the charge in [48] where the latter itself does not contain any shift terms). In our case, using eqs.(3.2.30) and (3.2.31) we see that the charges entering the entropy are $Q_I^{bh} = \left(\frac{v_2}{v_1}\right)\omega\frac{G_{IJ}}{2}e^J$ without any p^I dependence. The extra $3C_{IJK}p^Jp^K/p^0$ terms in [39] do not enter our Q_I^{bh} and consequently the match to [48] is exact.

Before we delve into reasons underlying this mismatch, let us at this stage perform a consistency check for our charges computed above. We want to see whether Q_I^{bh} compares to the charge integral obtained in the supergravity analysis of [61], which would serve as an independent verification. For that purpose consider eq.(3.1.16) with Σ taken to be an S^3. Since we know the near-horizon components of A^I and F^I, we insert these into eq.(3.1.16) and evaluate the integral. Because $F^I_{\theta\phi} = 0$, the $\int_\Sigma A^J \wedge F^K$ part of the integral vanishes and the $\int_\Sigma *F_I$ term precisely reproduces $\left(\frac{v_2}{v_1}\right)\omega\frac{G_{IJ}}{2}e^J$. That verifies that our expression for Q_I^{bh} in eqs.(3.2.30) and (3.2.31) is indeed the correct electric charge of the black hole.

One may now ask why the charges of [39] picked up those incorrect shifts ? Which may be rephrased by asking what went wrong with their Chern-Simons contributions to the 4D reduced action $\mathcal{F}_4^{bh}$? The Chern-Simons terms in $\mathcal{F}_4^{bh}$ were obtained from a four dimensional reduction of the 5D supergravity action (refer appendix A in [39]). This then gave rise to the above-mentioned factor of $\frac{1}{2}$ in $\mathcal{F}_4^{bh}$ (eq.(3.7) in [39]), which subsequently lead to an erroneous shift in their definition of charges. In our calculation the factor of $\frac{1}{2}$ did not appear in the 5D reduced action $\mathcal{F}_5^{bh}$ and that gave the correct electric charge, which matches [48] and confers with [61]. This subtle difference in a factor of $\frac{1}{2}$ between the reduced actions computed in [39] and that computed here seems to be related to how we treated the moduli a^I in our calculation, as opposed to how the same was handled in [39]. There they assume an x^μ-dependence for the moduli a^I, while performing a dimensional reduction of Chern-Simons terms. These a^I are set to constants only when one arrives at the four dimensional set-up. Subsequently the four dimensionally reduced Chern-Simons Lagrangian density (see (A.11) in ref. [39]) picks up a factor of half in front of the second term therein. This is how the $\frac{1}{2}$ enters the 4D reduced action $\mathcal{F}_4^{bh}$ and consequently the charges. On the other hand, in our 5D calculation, in the absence of any dimensional reduction there is no natural way to assume an x^μ-dependence for a^I (whilst already in the 5D near-horizon geometry) and then suddenly set them to constants at some other stage of the calculation. The 5D components of the field strength F_{rt}, $F_{\theta\phi}$ are constants in the near-horizon geometry and giving the fields an x^μ-dependence through a^I would tantamount to a deformation of the near-horizon geometry and possibly interfere with the AdS isometries which were crucial to the formulation. Therefore in our calculations we have set all 5D near-horizon moduli as constants (whose values are determined upon extremisation) throughout the calculation and this procedure seems to give the correct answers. It would be interesting to see if this four dimensional reduction can also be re-done keeping all the a^I constant and then check if that leads to the right charges. The focus of this paper however was to show that an explicit five dimensional entropy function calculation also works and provides us with consistent answers.

Finally let us remark that the much-discussed factor of $\frac{1}{2}$ in the 4D reduced actions affects the charge definitions of both the black ring as well as the black hole. However in case of the black hole the effect is far more drastic. Let us clarify this point. We start with the ring. Suppose that the factor of half had also appeared in our reduced action as a coefficient of the last term in eq.(3.1.10) (multiplying $C_{IJK}e^0\tilde{a}^I\tilde{a}^Jp^K$). That would only change the charge Q_0^{br} in eq.(3.1.15) by replacing the '6' in the last term ($C_{IJK}e^0\tilde{a}^I\tilde{a}^Jp^K$) with a '3'. The term itself does not vanish when comparing to Q_0^{Page} in eq.(3.1.17). Of course this changes the numerical coefficients in eqs.(3.1.20) and (3.1.21). But eq.(3.1.22) will still be satisfied for the modified equations and a new $\widehat{Q}_0$ (with a half instead of a one in the last term in eq.(3.1.21)) will finally enter the black ring entropy function in eq.(3.1.23). Hence the changes in this case only show up as different coefficients of existing terms. But in the case of the black hole the erroneous factor in the reduced action adds another term to the charge which we clearly know does not exist. We can see this as follows. Suppose the last term in eq.(3.2.29) (the one with a p^0) carried a half. This would carry forward as an extra numerical factor in the definition of Q_I^{bh} in eq.(3.2.30). However after using eq.(3.2.31) (the factor half does not affect this equation because the C_{IJK} terms do not enter $\mathcal{E}_5^{bh}$) in eq.(3.2.30), we are left with an extra $3C_{IJK}p^Jp^K/p^0$ term in the definition of Q_I^{bh}. And as mentioned above, this extra term neither confers to the Page charge integrals in [61] nor to the literature in [123]. Hence the changes are far more conspicuous in case of the black hole.

3.4 Black Hole - Black Ring Interpolation

Earlier in section 2 we saw how the near-horizon solution of a black ring can be expressed via various moduli parameters. Among these e^I and e^0 are conjugate to the electric charges and angular momentum respectively, while the magnetic flux p^I is a fixed quantity. On the other hand, the 5D black hole of section 3 only carried electric variables e^I and fixed magnetic variables p^I, p^0. From the perspective of the entropy function formalism, obtaining the metric of a black hole from that of a black ring can simply be achieved by switching off the e^0 contribution to the metric and turning on a p^0 one instead (and then extremising with respect to these new moduli). This assignment was first proposed in [39], where it appears as an ad hoc choice that reproduces the leading order entropies of the two black objects. In this section we want to provide a physical justification for this assignment of parameters. We will soon see that switching the terms $e^0 rdt \leftrightarrow p^0 cos\theta d\phi$ among each other in the near-horizon Kaluza-Klein metric will in fact be equivalent to changing the modulus l (here l is the three dimensional distance of the black ring from the origin of the Taub-NUT base space) from a specified finite quantity to a vanishing limit in the complete 5D supergravity solution. Gravitationally this means we are shrinking the 5D black ring to the origin of the base space to get a 5D black hole. In this sense, we argue that the $e^0 \leftrightarrow p^0$ switch is actually a black hole - black ring interpolation rather than some sort of black hole - black ring duality, that was suggestively speculated in [39]. Let us now examine this in more detail.

In section 2 we demonstrated that p^I, Q_I and Q_0 computed from a 5D entropy function analysis, are the correct physical observables of a black ring. Moreover a glance at the microscopic description of a black ring as a bound state of branes will in fact reveal that the observable charges are not exactly the brane charges [36], [37]. Microscopically a black ring can be described by a Calabi-Yau compactification of M-theory on a circle [54] with M2-M5 branes wrapping 2- respectively 4-cycles on the Calabi-Yau. The remaining one leg of the M5-brane wraps the M-theory circle thus giving a black string along this S^1 (as in the description of [53]). This string is stabilised by angular momentum modes running along the circle. The relation between brane charges and observable charges in fact takes the form [38]

$$\begin{aligned} q_I^{M2} &= Q_I - 6C_{IJK}\, p_{M5}^J\, p_{M5}^K \\ p_{M5}^I &= p^I \\ J_0^{KK} &= Q_0 - p_{M5}^I\, q_I^{M2} - 6C_{IJK}\, p_{M5}^I\, p_{M5}^J\, p_{M5}^K \end{aligned} \tag{3.4.38}$$

These shifts from the actual brane charges have been shown in [65] to be manifestations of spectral flow when $k^I = p^I$. In this way the above relations also serve as a 4D/5D map between the two-center system of a D0-D2-D4 black hole in 4D, placed in the vicinity of a D6-charge; and a black ring in 5D. Hence when the M-theory circle shrinks to zero size then the charge shifts due to spectral flow disappear and the brane charges J_0^{KK}, q_I^{M2}, p_{M5}^I (which now become D0, D2, D4 charges respectively in the Type II A description) coincide with the observable charges. Having stated the relations between physical and brane charges of the black ring, we can now incorporate these into supergravity solutions.

In order to study a supergravity construction that interpolates between 5D black holes and black rings in its different limits, we start by considering the most general 5D $\mathcal{N} = 1$ ungauged supergravity solution [49], [50] which is given by the following 5D metric and gauge fields

$$\begin{aligned} ds_5^2 &= - f^2 \, (\, dt \, + \, \Omega \,)^2 \, + \, f^{-1} \, ds^2(M_4) \\ F^I &= d \left[\, f \, X^I \, (\, dt \, + \, \Omega \,) \, \right] \, - \, \frac{2}{3} \, f \, X^I \, (\, d\Omega \, + \, \star d\Omega \,) \end{aligned} \tag{3.4.39}$$

where X^I are scalar fields in abelian vector multiplets. They satisfy the constraint equation $C_{IJK}X^I X^J X^K = 1$ and X_I are defined by the condition $X^I X_I = 1$. $ds^2(M_4)$ above refers to the Gibbons-Hawking metric of a 4D hyper-Kahler base space, which in our case is simply taken to be $ds^2(TN)$, the Taub-NUT metric (or $ds^2(\mathbb{R}^4)$ when considering a black ring in flat space) having KK-monopole charge. Let r, θ, ϕ, ψ denote coordinates on the 4D base space with (r, θ, ϕ) locally parameterising an $\mathbb{R}^3$ and ψ running along a compact S^1 with periodicity 4π. The Hodge dual $\star$ is taken with respect to the 4D base space. The function f and the one-form Ω can then be determined in terms of four harmonic functions $H_{TN}(x)$, $K^I(x)$, $L_I(x)$ and $M(x)$ (with $x \in \mathbb{R}^3$) in the following sense

$$\begin{aligned} f^{-1} \, X_I &= \frac{1}{4} \, {H_{TN}}^{-1} \, C_{IJK} K^J K^K \, + \, L_I \\ \Omega &= (-\frac{1}{8} \, {H_{TN}}^{-2} \, C_{IJK} K^I K^J K^K \, - \, {H_{TN}}^{-1} \, L_I \, K^I \, + \, M \,) \\ &\times \, (\, d\psi \, + \, cos \, \theta \, d\phi \,) \, + \, \widehat{\Omega} \end{aligned} \tag{3.4.40}$$

where $\widehat{\Omega}$ is defined by

$$\nabla \times \widehat{\Omega} \, = \, H_{TN} \, \nabla M \, - \, M \, \nabla H_{TN} \, + \, K^I \, \nabla L_I \, - \, L_I \, \nabla K^I \tag{3.4.41}$$

Operating the gradient on both sides of this equation yields integrability conditions

$$H_{TN} \, \nabla^2 M \, - \, M \, \nabla^2 H_{TN} \, + \, K^I \, \nabla^2 L_I \, - \, L_I \, \nabla^2 K^I \, = \, 0 \tag{3.4.42}$$

which are evaluated at each pole (charge center) in $\mathbb{R}^3$.

Within the above framework, a supergravity solution for any black object is now reduced to the task of specifying four harmonic functions. Let us first write these down for a black ring and then we shall see how to interpolate them to a black hole solution. For a black ring we have the following

$$\begin{aligned} H_{TN}(x) &= \frac{4}{R_{TN}^2} + \frac{p^0_{KK}}{|x|} & L_I(x) &= v_I + \frac{q_I^{M2}}{|x-l|} \\ K^I(x) &= \frac{p^I_{M5}}{|x-l|} & M(x) &= v_0 + \frac{J_0^{KK}}{|x-l|} \end{aligned} \tag{3.4.43}$$

Here p^0_{KK} is the charge of the Kaluza-Klein monopole in M-theory, which reduce to p^0_{KK} D6-branes in Type II A. The case $p^0_{KK} = 1$ corresponds to a Taub-NUT, otherwise the 4D hyper-Kahler base space is an orbifold of Taub-NUT, such that its geometry in the neighbourhood of the origin is of the type $\mathbb{C}^2/\mathbb{Z}_{p^0_{KK}}$. Let us clarify the remaining notation as well : R_{TN} denotes the asymptotic radius of the original Taub-NUT; $x \in \mathbb{R}^3$ and l is a modulus in $\mathbb{R}^3$ which denotes the distance between the plane containing the S^1 of the ring and the origin of base space. v_I is a constant determined at infinity and v_0 will soon get fixed via the integrability conditions. These harmonic functions have been specified via brane charges in the system. The bound states of branes wrapping Calabi-Yau cycles form BPS point particles in $\mathbb{R}^3$ and the poles in the above harmonic functions are attained precisely at the location of these BPS particles. The M2-M5-J^{KK} particle sits at $x = l$, while the KK monopole is located at $x = 0$. From a 4D point of view this is a 2-center black hole system, but in 5D it's just a black ring in a Taub-NUT orbifold [43].

Now let us evaluate eq.(3.4.42) for the above harmonics at each of the two poles. This yields the following two integrability conditions

$$v_0 = -\frac{J_0^{KK}}{|l|} \tag{3.4.44}$$

$$J_0^{KK} = v_I \, p^I_{M5} \left(\frac{p^0_{KK}}{|l|} + \frac{4}{R^2_{TN}} \right)^{-1} \tag{3.4.45}$$

Physically this implies that J_0^{KK}; which contributes part of the angular momentum along the ψ-direction of the ring; cannot be arbitrarily chosen, but is fixed for a given configuration. The above conditions can then be inserted back into eq.(3.4.43) and thereafter implementing the charge transformations in eq.(3.4.38) (which were obtained as spectral flow shifts from the supergravity action), essentially lays down the complete black ring solution. This compares to the standard solutions of [31], [32],

[34], [35], [38] when expressed in more convenient coordinates - but we will not require that here.

Now let us study the behavior of this black ring in the limit $l \to 0$. From [33] we already know that we should recover a 5D black hole in this limit. However the purpose of our presentation is to make a clear distinction between branes that constitute a black ring bound state from those that constitute a black hole bound state when the modulus l is driven to zero. Then we want to relate these brane charges to the spectral flow of those respective black objects in order to determine the physical charges.

Let us begin with eqs.(3.4.44) and (3.4.45). When $l \to 0$, they reduce to

$$J_0^{KK} = 0 \tag{3.4.46}$$

$$v_0 = -\frac{v_I \, p_{M5}^I}{p_{KK}^0} \tag{3.4.47}$$

and the harmonics in eq.(3.4.43) become

$$H_{TN}(x) = \frac{4}{R_{TN}^2} + \frac{p_{KK}^0}{|x|} \qquad L_I(x) = v_I + \frac{q_I^{M2}}{|x|}$$

$$K^I(x) = \frac{p_{M5}^I}{|x|} \qquad M(x) = -v_I \, p_{M5}^I \tag{3.4.48}$$

after having used eqs.(3.4.46) and (3.4.47) therein. What we have now is a BPS configuration in which there is not only a KK monopole at the origin of the Taub-NUT orbifold, but also the M5-M2 charge is now bound to this monopole. Moreover these bound states of branes have vanishing J_0^{KK} charge. This is a 5D black hole (or a D2-D4-D6 black hole from the point of view of a 4D reduction). Furthermore from the analysis in section 3 we saw that in the case of the 5D black hole, there are no spectral flow shifts. Therefore for this configuration, the brane charges p_{KK}^0, p_{M5}^I and q_I^{M2} respectively correspond to the following physical charges

$$p_{KK}^0 = p^0$$

$$p_{M5}^I = p^I$$

$$q_I^{M2} = Q_I \tag{3.4.49}$$

Now recalling the entropy function formalism, these charges are precisely associated to the following near-horizon variables : p^0, p^I, e^I. To sum up the contents of this section, we find that the physical interpretation of switching e^0 with p^0 in the entropy formalism's near-horizon ansatz corresponds to interpolating between limits of the modulus l on a Taub-NUT orbifold, which in supergravity yields an interpolation between black hole/black ring geometries. Moreover building this association to supergravity also serves the purpose of providing a justification for the specific choice of moduli in the Kaluza-Klein metric ansatz of [39], for each of the two geometries.

3.5 Conclusions and Discussion

The inclusion of Chern-Simons terms in the entropy function formalism has rather been a bit of a puzzle due to its apparent lack of gauge invariance under large gauge transformations. This being because Sen's original derivation [29] was based on the premise of gauge and reparametrisation invariant lagrangian densities. The dimensional reduction approach was proposed [41] in order to rectify this. In view of the proposed 4D/5D connection [42], [43], that such a recipe works might not come as a total surprise though. However even in those developments several contentious subtleties stood out as regards the correct physical notion of charge in 4D and 5D [37], [38], [55], [84], [56]. In this note we have argued that there is no fundamental obstruction to a well-defined 5D treatment of entropy functions with Chern-Simons terms, provided one implements the correct physical 5D charges into the calculations. In general these 5D charges differ from those used in the dimensionally reduced approach due to spectral flow shifts. However to fully specify a charge, one needs to obtain the equation of motion of the corresponding gauge field which is sourced by that charge. Within the setting of the entropy formalism, these gauge fields are determined via moduli e^I, e^0 and a^I. Therefore upon extremising $\mathcal{F}_5$ with respect to these moduli one can determine the electric charges. On the other hand the magnetic charges are pre-fixed from the beginning. Our calculations demonstrate that once the physical 5D charges are made manifest in the entropy function, it immediately falls into a 5D gauge invariant expression, even without requiring to fix all the remaining moduli v_1, v_2, ω, X^I. In other words we do not need to modify Sen's formalism, but only correctly identify the physical 5D charges and perform computations manifestly in terms of these charges. Moreover because of the fact that gauge fields and consequently charges of 5D geometries with different near-horizon topologies will in general be quite different, we find that one cannot construct a universal entropy function that describes any 5D geometry in the presence of Chern-Simons terms and which is also gauge invariant. In reference [39], they do manage to write down a unified entropy function, however that can only be expressed in terms of off-shell charges and it is in fact not invariant under spectral flow transformations. Therefore in order to check 5D gauge invariance, we had to treat the $AdS_2 \times S^2 \times S^1$ black ring topology and the $AdS_2 \times S^3$ black hole topology separately.

As is well-known, Chern-Simons terms in odd dimensions induce spectral flow shifts in the supergravity action, which also reflect in the defining notion of charges in these theories [59]. In our analysis for the black ring, we have seen that these spectral flow equations also arise in a natural way out of Sen's formalism in 5D. Consequently the 5D electric charges were no longer gauge invariant and neither was the reduced action $\mathcal{F}_5^{br}$. Nonetheless the entropy function $\mathcal{E}_5^{br}$ itself turned out to remain invariant under gauge/spectral flow transformations if it is expressed as a function of the correct physical charges. We have also verified that the electric charges computed here from Sen's approach are identical to the Page charges expected from 5D supergravity : our charges calculations for the black ring give a precise match with the charge integrals recently computed by [61] on the basis of near-horizon data.

On the other hand, whilst computing for the 5D black hole we found that the electric charges turned out not to be Page but simply 5D Maxwell charges with no spectral flow shifts. This was because a vanishing magnetic flux in an $AdS_2 \times S^3$ geometry suppresses all spectral flow shifts. As a consequence, the 5D charges of this black hole exactly match those of its 4D counterpart upon compactification of the fifth dimension. This corroborates with the 4D/5D lift of [42]. Within this set-up, gauge invariance of the entropy function thereon follows in a straightforward manner. Then extremising $\mathcal{E}_5^{bh}$ to compute the black hole entropy indeed gave us an exact match with the result of [48], where the latter was obtained via an attractor mechanism calculation. This resolves the slight discrepancy in the result of [39] where their entropy did not quite match [48] : because their electric charges did not agree with those of [48]. Besides the comparison to [48], we have also provided additional evidence to support the claim that Q_I^{bh} computed here are the correct charges to work with by showing that they also match exactly with the charges of [61], which were obtained from a 5D supergravity approach. The discrepancy in the charges of [39] arise whilst dimensionally reducing the Chern-Simons terms to 4D : namely, they assume an x^μ-dependence for the moduli a^I; and only set the a^I to constants in the final step. Consequently this introduces terms in $\mathcal{L}_{CS}^{4D}$, which incorrectly shift their electric charges, thereby causing a mismatch with the entropy of [48]. However from the point of view of a manifestly 5D calculation, there was no natural way to assume such an x^μ-dependence (whilst already in the 5D near-horizon geometry) and then abruptly deem them constants later in the calculation. The 5D components of the field strength F_{rt}, $F_{\theta\phi}$ are constants in the near-horizon geometry and giving the fields an x^μ-dependence through a^I would seem to come in conflict with the isometries of the near-horizon geometry. Moreover from the result of [61] given in eq.(3.1.16), the $\int_\Sigma 6C_{IJK}A^J \wedge F^K$ term vanishes for this black hole in the absence of an effective magnetic flux ($p^I + \tilde{a}^I p^0$). It is only the $\int_\Sigma *F_I$ term that contributes to the charge. Inserting the expression for the near-horizon field strength into the integral of eq.(3.1.16), exactly reproduces our expression for Q_I^{bh}. The extra terms in the charges of [39] would simply not agree with the integral of [61]. This seems to suggest that assuming an x^μ-dependence on any of the moduli in the near-horizon geometry and then setting them to constants after dimensional reduction might be suspect. Within the entropy formalism, the isometries of the geometry are crucial to the analysis and all physical quantities ought to obey these. This imposes restrictions on the moduli, which works well when the latter are deemed constants in this geometry at any stage of the analysis.

A related line of interest which we have investigated in this chapter concerns black ring $\leftrightarrow$ black hole interpolation in the context of Sen's formalism. The idea behind such an interpolation between geometries has been familiar since the work of [33], where it was shown using black ring solutions from [31], [32]. For what we had in mind here, it was more convenient to reformulate this interpolation using the most general 5D $\mathcal{N} = 1$ ungauged supergravity solution of [49], [50] and varying the Taub-NUT modulus l from a specified point to a vanishing limit. This way the structure of harmonic functions and brane wrappings associated to the two geometries is more readily manifest. The supergravity solution of course captures the global structure of the geometry, whereas the entropy formalism is only a near-horizon analysis. Therefore in principle it is not possible to construct a full-fledged interpolation of solutions using the latter. However we have still managed to show within the Sen formalism that upon interchanging off-diagonal entries in the Kaluza-Klein metric bearing e^0 terms with those bearing p^0 ones, yields algebraic data that can be compared to the limiting supergravity solutions in such a way that parameters in the Kaluza-Klein metric can be specifically associated to brane wrappings in the supergravity solution for both the black ring and black hole. In retrospect, this also lends some physical intuition to the ad hoc assignment of variables made in the black hole/black ring metric ansatz proposed in [39]. Our original motivation in studying this $e^0 \leftrightarrow p^0$ exchange was in the hope of finding some sort of black ring/black hole duality loosely speculated by [39]. However within the context of our analysis, the $e^0 \leftrightarrow p^0$ exchange seems to relate more with the idea of a geometric interpolation rather than any string or gravitational duality. There is though an interesting work by [66] which might be more in the direction of seeking such a string duality between 5D black holes and black rings. In that work, the authors propose a duality between microstate degeneracies of a D0-D2-D4 system with those of a D0-D2-D6 system on the same Calabi-Yau via a Fourier-Mukai transform. From a 5D perspective, this would lift to a black hole/black string duality. From our discussion in section 4, we have seen that the M-theory lift of a D2-D4-D6 system gives a 5D black hole, whereas a D0-D2-D4 system in the vicinity of a D6 charge, lifts to a black ring. It would therefore be quite interesting to see if a microscopic duality along the lines of [66] can also be constructed for this black hole/black ring system.

Let us now briefly summarize our results with an outlook of what is to follow in subsequent chapters. We took-off by considering Sen's entropy function analysis in this chapter, for the case of 5D supergravity actions containing Chern-Simons terms. The key result of this work has been to develop an explicit 5D entropy function formalism that works for both 5D extremal black holes and black rings. The issue with Sen's original formulation [29] was that it was not suited to include terms in the action that are not manifestly gauge invariant, such as Chern-Simons terms. Hence prior computations involving 5D black objects, relied on an ad hoc recipe of reducing the action to 4D and adding a total derivative term by hand to restore gauge invariance. The trouble with this make-shift approach is that it does not correctly identify physical 5D observables. This refers to conserved charges in 5D which are shifted relative to their 4D counterparts due to large gauge transformations originating from Chern-Simons terms. This feature is also referred to as spectral flow (the phrase being coined due to an analogous shift in Virasoro generators of its dual CFT). Here we have solved the problem by showing how spectral flow can be incorporated into a 5D entropy formalism, which at the same time remains gauge invariant and has an explicit dependence only on physical charges. In particular, we have performed explicit calculations for the black ring and 5D black hole. In the black ring analysis, we found Chern-Simons induced spectral flow shifts emerging in a natural way out of the entropy function formalism. The entropy function nevertheless was seen to remain gauge invariant and the resulting electric charges were identified as Page charges. For the black hole too, 5D gauge invariance was confirmed. Our 5D analysis enabled us to fix a mismatch that arose in the electric charges of Goldstein and Jena's 4D-reduced calculation. Additionally, we have also provided an interpretation for the $e^0 \leftrightarrow p^0$ exchange in the entropy function as being associated to an interpolation between black hole and black ring geometries in Taub-NUT.

One of the reasons the entropy function is a more powerful tool than OSV is because the latter only takes into account holomorphic contributions to the prepotential. The entropy function method can be applied to non-SUSY extremal black holes and also to higher dimensional black objects. A future goal would be to extend this formulation to multi-center as well as non-extremal black holes if possible, and that promises interesting applications.

As further outlook in this program, one could consider using the 5D formulation developed here for the purpose of computing higher derivative corrections. More specifically, the case in point would be for geometries having an AdS_3 component in its near-horizon region. Using anomaly cancellation arguments, Kraus and Larsen [67] have demonstrated that the only supersymmetric higher derivative terms of the 5D low energy effective action that contribute to Wald's entropy in AdS_3, result from one-loop worldsheet instanton contributions, that is mixed gauge-gravitational Chern-Simons terms and its supersymmetric completion. Previous attempts at computing R^2 corrections to black ring entropy have failed to match exact microscopic results because the explicit supersymmetric completion of the above mixed Chern-Simons term was not known until recently in the work of Hanaki, Ohashi and Tachikawa [68], who make use of a 5D off-shell superconformal formalism. Since the 5D entropy function formulation presented in this thesis is well-suited to deal with Chern-Simons type corrections, a possible computation using the above technology would be to obtain one-loop corrected charge shifts for black rings as well as the exact higher derivative entropy for black rings from a macroscopic calculation. For 5D AdS_2 black holes the problem gets much harder as there are no anomaly cancellations and therefore all higher-loop effects have to be considered. Part of that has been attempted for spinning black holes using a supergravity analysis in [134].

Chapter 4

The 4D/5D Map and Multi-Center Geometries

Everything happens to everybody sooner or later if there's time enough

- George Bernard Shaw

In the previous chapter we have investigated Chern-Simons induced charge shifts for a black ring/black hole in 5D. These effects are certainly crucial for constructing a precise mapping between black objects in four and five dimensions. In this chapter, we want to construct a 4D/5D map for multi-center geometries, which in our case are taken to be multiple non-concentric black rings in Taub-NUT. These rings themselves are seen to emerge from AdS fragmentation of a single black ring. This picture is the 5D version of 4D fragmentation into baby universes, which in turn are related to finite N contributions to the OSV conjecture, thus bringing together different pieces of the puzzle for the case of multi-center geometries.

Starting from recent work in [42] and [43], a considerable interest has been generated in understanding 5D BPS degeneracies by constructing dualities to the better understood 4D sector [38], [56], [54], [84]. Matter of fact, this 4D/5D relation was put forth by [42] as a 5D version of the OSV conjecture [129]

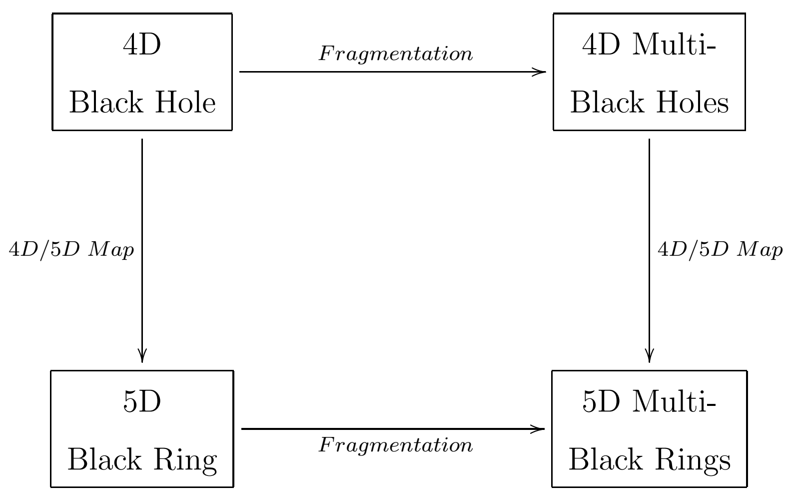
$$Z_{BH}^{5D} = Z_{BH}^{4D} = |Z_{top}|^2 \tag{4.0.1}$$

Evidence for this proposal was sought for by matching the entropy of the 5D BMPV [51] black hole in Taub-NUT space, to the entropy of a 4D Calabi-Yau black hole while making use of the M-theory $\leftrightarrow$ Type II A correspondence. Moreover, since Z_{BH}^{4D} counts degeneracies of single as well as multi-center black holes, it was pointed out by [43] that Z^{5D}_{BH} must also account for equivalent multi-black objects in 5D, assuming eq.(4.0.1) holds. While a single-center BPS black hole in 4D just lifts to a 5D BMPV black hole; in [43] a rather interesting result was demonstrated : a 4D two-center charge configuration consisting of a D6 charged point-particle at the origin (of $\mathbb{R}^3$) and a D4-D2-D0 charge at a distance $|\vec{L}|$ from it, will in fact lift to a supersymmetric black ring in 5D Taub-NUT space. $|\vec{L}|$ now becomes a modulus on Taub-NUT denoting the distance of the ring from the origin.

On a rather different footing, yet another offshoot of the OSV bandwagon was the work of [107], conceiving baby universes as finite (but still relatively large) N non-perturbative corrections to the OSV conjecture. These corrections go like e^{-N} and are realised as instanton effects in the holographically dual gauge theory. In turn, the holomorphic sector of the gauge theory is dual to the topological string partition sum Z_{top}. The gravitational realisation of these corrections were proposed as 4D multi-center black hole configurations, which can be generated via the mechanism of AdS fragmentation [73], [74] of a single black hole at $x_0 \in \mathbb{R}^3$ into multiple black holes at $\{x_i \in \mathbb{R}^3\}$. These multi-AdS throats are associated to a gravitational instanton action which describes the amplitude for tunneling, in Euclidean time, of a single black hole to multi-black holes. Based on that, [107] forward the idea of a third quantized Hilbert space of baby universes.

One of the motivations driving this note was to reconcile the two aforementioned streams of thought. We try and address some questions regarding the fragmentation of black rings in 5D. Analogous to the 4D case, where we saw how to split D4-D2-D0 charges, here we start with a black ring in Taub-NUT, since this is the pertinent 5D lift of a D4-D2-D0 black hole placed at a distance $|\vec{L}|$ from a single D6 charge (the sole D6 here does not participate in fragmentation). We then set up a fragmentation ansatz for this single ring and see that it splits up into non-concentric multiple black rings (in general). This construction is subject to charge splitting constraints, which as we shall soon see will turn out to be more subtle in the 5D case that they were in 4D due to the presence of cross-terms between multiple centers that must now be carefully tendered.

On the other hand, one might fairly well ask whether the fragmented multi-rings constructed in this manner could as well have been obtained from a direct 5D lift of the 4D multi-center solution. The answer turns out to be in the affirmative; and to do so we shall first require to construct the 4D/5D dictionary for multi-center charges. Compared to the 4D/5D map of [42] for a single black object, the analogous one for multi-centers will turn out a bit more involved again due to the relentless cross-terms. Nevertheless with such a map in hand, transforming amongst 4D/5D multi-center charges, we verify that our fragmented harmonic functions are indeed direct 5D lifts of 4D multi-center solutions. This enables us to confirm commutativity of the following box diagram.



(4.0.2)

As had already been hinted by in [107] in context to the 4D set-up; eq.(4.0.2) seems to predicate the suggestion in 5D, that fragmentation might be thought of as a possible recipe for generating classes of multi-center configurations once given corresponding single-center ones. Of course the multi-rings that we generate in this note by these methods, are by no means any new solutions which had previously been unheard of. For that matter, we point to some of the extensive literature, where several classes of 5D multi-center solutions have been worked out : [36], [77], [78], [79], [80], [81]. The focus in this note is based more in the spirit of the box diagram in eq.(4.0.2) and studying the details therein.

Whilst meandering amidst this impending scheme of things, we are duly confronted with issues concerning the physically meaningful definition of charges in 4D and 5D. We begin with an apprehension of the single black hole/ black ring duality by matching 4D two-center harmonics to 5D black ring harmonics. Such a comparison invokes symplectic charge transformations going from 4D to 5D. Additionally these 4D/5D transformations also make way for an alternative derivation of black ring angular momenta. A clear notion of single-center 4D/5D mapping, now equips us to move on to study the interpretation of 5D multi-center charges. First we procure the 5D charge splitting equations via implementation of the 4D charge splitting equations as well as the single-center 4D/5D lift. The 5D equations so obtained definitely carry the baggage of cross-terms, due to the fact that the 4D/5D transformations are non-linear in the dipole fields. Moreover we shall see that it now becomes relevant to identify which of these charges is of Maxwell type and which of Page type. This discussion picks up from [61] and continues further for the case of fragmented charges. In fact we shall see that in 5D the charges $Q_{A i(5D)}$ which actually engage in fragmentation are Page charges. These are really the physical multi-ring charges and not the charges $\widetilde{Q}_{A_{i\,(5D)}}$ in terms of which the multi-black ring metric is usually expressed. We also write down an explicit expression transforming between these two types of charges. In due course the multi-center 4D/5D dictionary falls in place.

As an application of charge fragmentation methods described here, we derive the total angular momentum of a system of non-concentric multi-black rings by simply starting from the angular momentum of a single black ring and making use of 5D charge splitting equations. As a check for our answer, we reduce to the special case of concentric black rings in order to compare the our result with the well-known expression of Gauntlett and Gutowski [82], [83]; and yes, their result is correctly reproduced !

The alluring calls for a geometric interpretation of these fragmented rings underscore the final act. In a multi-ring background, individual rings receive multiple spectral flow shifts due to fluxes emanating from split-charge centers; thus coining the notion of 'split-spectral flows'. Each ring may be thought of as sourcing a Dirac string generated due to its magnetic flux. In a Taub-NUT base, these rings are stacked in order of increasing radius. Hence, say the i^{th}-ring; in addition to its own Dirac string; also encircles Dirac strings sourced by each of the $(i-1)$ rings of smaller radius in the Taub-NUT base. And going around Dirac strings is by no means a free ride. It costs large gauge transformations, which can have long-term consequences if Chern-Simons terms are involved as well. This is how spectral flows arise. Therefore the case of our

i^{th}-ring multi-timing that many Dirac strings will face a horde of spectral flow shifts to its initial brane charges. This will completely account for the physical split-charges of fragmented rings. Moreover, adding up all the split-spectral flows of all of our wandering fragmented rings correctly gives back the spectral flow of an unfragmented single black ring system, as it should. This sheds light on a geometrical view of the origin of multi-ring Page charges and their cross terms. In fact such split-spectral flows divide the geometry into patches with locally defined gauge field potentials, such that adjacent patches are related up to gauge transformations.

The organization of this chapter is as follows : Section 2 provides a lay-out of the 4D multi-center black hole technology and comments on its physical interpretation as baby universes. Section 3 handles harmonics, charges and angular momenta of a single black ring in Taub-NUT from a 4D/5D map. Section 4 is where 5D fragmentation takes shape. We set-up conditions for black ring fragmentation and provide an interpretation for multi-center 5D charges. This follows by writing down a multi-center 4D/5D charge dictionary and also deriving the angular momenta of (non-)concentric multi-black rings. Section 5 seeks to unfold a geometric perspective on the above via the notion of split-spectral flows. Alas, we must wind up..... that's why there's section 6, concluding and throwing pointers at further directions.

4.1 A Glance at 4D Black Hole Fragmentation

In this section we briefly sketch the set-up of 4D black hole fragmentation and its interpretation of baby universes following the approach of [107]. The conceptual basis behind the idea of baby universes lies in the phenomenon of AdS fragmentation [73], [74], which was proposed as an instanton process wherein a single black hole, seen as an excitation in one vacuum configuration, tunnels to a multi-black hole state appearing as an excitation in another vacuum. The two vacua lie in the asymptotic limits of a "Euclidean time" co-ordinate, which is defined by an entropy functional $S(x)$. From the Euclidean metric (obtained after a Wick rotation) the $AdS_2 \times S^2$ geometry is seen to flow to a product geometry of $\otimes_{i=1}^{n} AdS_2^i \times S_i^2$ (to leading approximation).

As an explicit representation of multi-black hole configurations, the authors of [107] make use of the well-known multi-center solutions of $\mathcal{N} = 2$ supergravity from [69], [70], [71], [72]. The idea behind the fragmentation procedure is that the black hole harmonic functions interpolate between the single-center harmonics; at asymptotic infinity $x \to \infty$; and the multi-center harmonics; which are achieved upon approaching the near-horizon limit. In fact, near the i^{th}-horizon when $x \to x_i$, the i^{th}-black hole dominates the solution. Therefore given a single-center solution and implementing the above idea, one can set up an ansatz for harmonic functions of fragmented black holes. Additionally charge conservation constrains the distribution of charges at fragmented centers. In [107] it was shown that the supergravity configuration of [69], [70], [71], [72] can indeed be realised in this way via AdS fragmentation of a single black hole. For the sake of setting up notation as well as later reference, let us flash a quick glance at how this works.

Consider the harmonic functions of a single black hole in 4D with magnetic charges p^I and electric charges q_I placed at the spatial origin in $\mathbb{R}^3$

$$U^I(x) = \frac{p^I}{|x|} + u^I \qquad\qquad V_I(x) = \frac{q_I}{|x|} + v_I \tag{4.1.3}$$

here $I = 0, 1, ...$ denotes vector multiplet indices; $x \in \mathbb{R}^3$; and u^I, v_I are constants determined at infinity. In these co-ordinates the pole at $x = 0$ is the location of the horizon which has the topology of a two-sphere S^2. Another ingredient we will require is the entropy functional $S(x) \equiv S\left[U^I(x), V_I(x)\right]$. This is a specific polynomial function of the harmonics and only at the horizon does it attain the value of the entropy. Elsewhere $S(x)$ freely flows between its asymptotic limits. This flow in $S(x)$ will induce the harmonic functions $U^I(x)$, $V_I(x)$ to interpolate between single-center and multi-center solutions. At asymptotic infinity with $x \to \infty$, a single black hole geometry with charges p^I, q_I placed at the origin and harmonics given by eq.(4.1.3) leads to $S(x) \;\to\; c$ (a finite number). When these harmonics are inserted in the metric we get the well-known topology of $AdS_2 \times S^2$ and $S(x)$ enters this near-horizon Bertotti-Robinson metric as the square of the AdS_2 radius. The idea of AdS fragmentation now proposes treating $S(x)$ as a Euclidean time direction. Then the $S(x) \;\to\; \infty$ asymptote serves as another vacuum into which there exists a finite probability amplitude for a single black hole system to tunnel into a system of multi-black holes. The most general solution for harmonic functions, which interpolate between these asymptotic vacua, looks like

$$U^I(x) = \sum_{i=1}^{n} \frac{p_i^I}{|x - x_i|} + u^I \qquad V_I(x) = \sum_{i=1}^{n} \frac{q_{Ii}}{|x - x_i|} + v_I \tag{4.1.4}$$

where $U^I(x)$, $V_I(x)$ now describe a multi-black hole system with n horizons located at centers $\{x_i\}$. Charge splitting is subject to the following constraints

$$\sum_{i=1}^{n} p_i^I = p^I \qquad \text{and} \qquad \sum_{i=1}^{n} q_{Ii} = q_I \tag{4.1.5}$$

To fully specify the solution additional integrability conditions are also required

$$\left(p_i^I V_I(x) - q_{Ii} U^I(x)\right)\Big|_{x=x_i} = 0 \tag{4.1.6}$$

which have to be evaluated at each horizon. Now we can see how the above harmonic functions interpolate between single and multi-center geometries as follows : at asymptotic infinity $x \to \infty$, the harmonics in eq.(4.1.4) reduce to eq.(4.1.3) (by using the constraints in eq.(4.1.5)) and $S(x) \to c$; whereas at each $x \to x_i$, only the i^{th}-summands in eq.(4.1.4) dominate, describing multiple black holes located at $\{x_i\}$ respectively and consequently giving $S(x) \to \infty$. Hence flowing $S(x)$ from c to ∞ describes an AdS geometry fragmenting into a multi-AdS geometry. Eqs.(4.1.4), (4.1.5) and (4.1.6) were originally derived as part of the multi-center $\mathcal{N} = 2$ supergravity solution of [69]. In [107] this solution has been interpreted as remnants of an AdS fragmentation process.

4.2 Black Ring from 4D/5D Duality

In this section we demonstrate how charges, harmonics and angular momenta of a black ring can be determined purely in terms of a 4D/5D duality. While the charges and harmonics are straightforward to get; an explicit expression for ring angular momenta obtained from 4D/5D lifting will serve to compliment the usual supergravity derivations discussed in the literature.

We start by considering the following two-center system in 4D :

$$\begin{aligned} U^0(x) &= \frac{1}{|x|} + u^0 & V_A(x) &= \frac{q_{A(4D)}}{|x - x_0|} + v_A \\ U^A(x) &= \frac{p^A_{(4D)}}{|x - x_0|} + u^A & V_0(x) &= \frac{q_{0(4D)}}{|x - x_0|} + v_0 \end{aligned} \tag{4.2.7}$$

which consists of a single D6 charge ($p^0_{(4D)} = 1$) at the origin $x = 0$ (in $\mathbb{R}^3$); and $p^A_{(4D)}$, $q_{A(4D)}$ and $q_{0(4D)}$ respectively D4, D2, D0 charges, which form a 4D black hole at $x = x_0$. In [43], the 4D metric of this system is decompactified to yield a 5D black ring in Taub-NUT. Instead of doing that, here we go for a more direct comparison; namely, showing that the 4D harmonics above will be identical to 5D black ring harmonics once they are expressed via 5D charges. For this we will require the 4D/5D charge transformations

$$p^A_{(4D)} = p^A_{(5D)} \tag{4.2.8}$$

$$q_{A(4D)} = \left(Q_{A(5D)} - 3D_{ABC}p^B_{(5D)}p^C_{(5D)}\right) \tag{4.2.9}$$

where $Q_{A(5D)}$ and $p^A_{(5D)}$ respectively will turn out to be black ring electric and magnetic charges. We shall soon comment on their interpretation. An additional ingredient required to specify eq.(4.2.7) are the integrability conditions, which yield

$$v_0 = -\frac{q_{0(4D)}}{L} \tag{4.2.10}$$

$$q_{0(4D)} = v_A p^A_{(4D)} \left(\frac{1}{L} + \frac{4}{R^2_{TN}}\right)^{-1} \tag{4.2.11}$$

here L denotes the radial distance $|x_0|$. The presence of a D6-brane leads to a geometric transition when lifting to M-theory, giving a Taub-NUT space in the uncompactified directions. Therefore $U^0(x)$ becomes a harmonic function in Taub-NUT with $u^0 = \frac{4}{R^2_{TN}}$ (with R_{TN} as the asymptotic radius of Taub-NUT). u^A remains arbitrary and can be set to zero. Putting all this together, the 4D harmonics above can indeed be compared to the known Taub-NUT-black ring harmonics in the literature [38][1] (see also [33])

$$H_{TN}(x) = \frac{4}{R^2_{TN}} + \frac{1}{|x|} \qquad L_A(x) = v_A + \frac{Q_{A(5D)} - 3D_{ABC}p^B{}_{(5D)}p^C{}_{(5D)}}{|x - x_0|}$$

$$K^A(x) = \frac{p^A{}_{(5D)}}{|x - x_0|} \qquad M(x) = \frac{J_{tube}}{L} + \frac{-J_{tube}}{|x - x_0|} \tag{4.2.12}$$

where $J_{tube} \equiv -q_{0(4D)}$, which is determined from eq.(4.2.11), is indeed the intrinsic (not total) angular momentum of the ring along the S^1 circle and is the M-theory lift of the D0-charge. Thus the harmonic functions of the 4D two-center system under consideration are exactly equivalent to those of a 5D black ring in Taub-NUT[2].

[1] Compared to [38] we have scaled the p^A_{5D} charge by a factor of (-1).

[2] A black ring in $\mathbb{R}^4$ (see [31], [32], [34], [35]) can be extracted as a special case of eq.(4.2.12) by

Note that these functions in eq.(4.2.12) (along with integrability conditions) completely specify the black ring solution. For the sake of completeness, let us quickly demonstrate how this comes about. Consider the most general 5D $\mathcal{N} = 1$ ungauged supergravity solution [49], [50] which is given by the following 5D metric and gauge fields

$$\begin{aligned} ds_5^2 &= - f^2 \, (\, dt \, + \, \omega \,)^2 \, + \, f^{-1} \, ds^2(M_4) \\ F^A &= d \left[\, f \, X^A \, (\, dt \, + \, \omega \,) \, \right] \, - \, \frac{2}{3} \, f \, X^A \, (\, d\omega \, + \, \star d\omega \,) \end{aligned} \tag{4.2.13}$$

where X^A are scalar fields in abelian vector multiplets; they satisfy the constraint equation $D_{ABC} X^A X^B X^C = 1$ and X_A are defined by the condition $X^A X_A = 1$. $ds^2(M_4)$ in the equation above refers to the Gibbons-Hawking metric of a 4D hyper-Kahler base space, which in our case is simply taken to be $ds^2(TN)$, the Taub-NUT metric (or $ds^2(\mathbb{R}^4)$ when considering a black ring in flat space). Let r, θ, ϕ, ψ denote coordinates on the 4D base space with (r, θ, ϕ) locally parameterising an $\mathbb{R}^3$ and ψ running along a compact S^1 with periodicity 4π. The Hodge dual $\star$ is taken with respect to the 4D base space. The function f and one-form ω are fully nailed down in terms of four yet-to-be-specified harmonic functions as follows

$$\begin{aligned} f^{-1} \, X_A &= \frac{1}{4} \, {H_{TN}}^{-1} \, D_{ABC} K^B K^C \, + \, L_A \\ \omega &= (\, -\frac{1}{8} \, {H_{TN}}^{-2} \, D_{ABC} K^A K^B K^C \, - \, \frac{3}{4} {H_{TN}}^{-1} \, K^A \, L_A \, + \, M \,) \\ &\times \, (\, d\psi \, + \, cos \, \theta \, d\phi \,) \, + \, \widehat{\omega} \end{aligned} \tag{4.2.14}$$

The notation used in this equation is intentionally suggestive. Furthermore, $\widehat{\omega}$ is defined by

$$\nabla \times \widehat{\omega} \, = \, H_{TN} \, \nabla M \, - \, M \, \nabla H_{TN} \, + \, \frac{3}{4} \left(L_A \, \nabla K^A \, - \, K^A \, \nabla L_A \right) \tag{4.2.15}$$

Now inserting the explicit form of the harmonic functions of eq.(4.2.12) into eqs.(4.2.13), (4.2.14) and (4.2.15) simply reproduces the complete black ring solution of [38] in Taub-NUT (or [33] in $\mathbb{R}^4$). Moreover, operating the gradient on both sides of eq.(4.2.15) and evaluating at the poles, exactly recovers the integrability conditions of eq.(4.1.6),

taking the limit $R_{TN} \to \infty$. The conventions of [31], [32] differ from [38] by rescaling of charges; in this note we continue using the latter.

which are subsequently solved to get eqs.(4.2.10) and (4.2.11). This prescription goes through for multi-rings as well. Inserting appropriate multi-ring harmonics into the same 5D supergravity metric given above, one can recover the multi-black ring solution [82], [83]. In this sense, the harmonics and integrability conditions can be said to be sufficiently representative of the solutions of single as well as multi-black rings. For what follows here, we shall adopt this stance as well. Therefore the focus in this note shall not be on solving supergravity equations themselves, but rather on obtaining quantities such as multi-ring harmonics, charges and angular momenta from ring fragmentation and spectral flows.

Now coming back to the 4D/5D transformations, a comment on eqs.(4.2.8) and (4.2.9) is due. These equations were derived in [42] by considering symplectic shifts in electric charges due the presence of a magnetic flux such that the degeneracy of microstates remains invariant. Subsequently this leads to matching of leading order entropies for 4D and 5D black holes. Also, the authors of [38], [37], [76] further clarify these transformations when interpolating from a 4D black hole to a 5D black ring. While $q_{A(4D)}$ is the observable in 4D, from the 5D perspective it is $Q_{A(5D)}$ which is the observed charge. Let us point out to yet another interpretation of these transformations coming from spectral flow shifts (as in [57]) associated to the 5D Chern-Simons term. In a later section, we pursue this last observation further.

Much like the above-mentioned D2 charges, there also occurs a shift for D0 charges (again due to [42])

$$q_{0(5D)} = q_{0(4D)} - (p^A_{(4D)} q_{A(4D)} + D_{ABC} p^A_{(4D)} p^B_{(4D)} p^C_{(4D)}) \tag{4.2.16}$$

Starting from this relation we now obtain an independent identification of the total black ring angular momenta. Simply using eqs.(4.2.8), (4.2.9), (4.2.10) and (4.2.11) into eq.(4.2.16) yields

$$q_{0(5D)} = v_A p^A_{(5D)} \left(\frac{1}{L} + \frac{4}{R^2_{TN}} \right)^{-1} - p^A_{(5D)} (Q_{A(5D)} - 2D_{ABC} p^B_{(5D)} p^C_{(5D)}) \tag{4.2.17}$$

Now let us denote $q_{0(5D)} \equiv -\frac{G}{3\pi} J_\psi$, where G is the 5D Newton's constant. Then J_ψ exactly compares to the total angular momentum of the ring along the S^1 circle as given in [38] (or [31] on reducing to $\mathbb{R}^4$). The first term of J_ψ is the intrinsic angular momentum arising via the presence of D0 charges along the S^1 circle (ψ-direction); the second component describes the angular momentum induced in the presence of a magnetic flux. In addition there is yet another angular momentum characterising the ring; one associated to the ϕ-circle along the S^2, perpendicular to the ψ-circle. In the absence of D0 charges along the ϕ-circle, with only flux going through it, the angular momentum contribution (denote as J_ϕ) is solely flux-induced, thus giving

$$J_\phi = J_\psi - \frac{3\pi}{G} J_{tube} \tag{4.2.18}$$

Thus far we conclude that explicit application of the 4D/5D correspondence correctly identifies the charge prescription, harmonic functions as well as angular momenta of a black ring. Proceeding this way the leading order black ring entropy too can be obtained, as well as its one-loop correction. Since the references [37], [76], [43], [54] do justice to the former and [56] to the latter, we shall have no more to say on that. Equipped with these tools, we shall next test their application for the case of multi-center black holes/rings.

4.3 Black Ring Fragmentation & Charge Splitting in 5D

As seen in section 2, 4D charge fragmentation is given by simple linear relations in terms of fragmented charges. For D4, D2, D0 branes respectively, we denote these splittings as

$$\sum_{i=1}^{n} p_i^A{}_{(4D)} = p^A{}_{(4D)} \qquad \sum_{i=1}^{n} q_{A i(4D)} = q_{A(4D)} \qquad \sum_{i=1}^{n} q_{0 i(4D)} = q_{0(4D)} \tag{4.3.19}$$

Let us note that in 4D these are also the physically observed charges. We would now like to construct the analog of these equations in 5D. In that case, as we shall soon see, the charge fragmentation equations are not only non-linear (in the dipole charges) but also involve cross-term contributions arising from multiple charge centers.

4.3.1 Charges & Harmonic Functions of Fragmenting Black Rings

Owing to the trivial 4D/5D relation for magnetic charges ${p^A}_{(5D)}$ (as in eq.(4.2.8)), their splitting into 5D components is straightforward.

$$ {p^A}_{(5D)} = \sum_{i=1}^{n} {p_i^A}_{(5D)} \tag{4.3.20} $$

The more interesting case is that of the electric charge $Q_{A(5D)}$ of a single black ring. Since this charge differs from the corresponding 4D charge $q_{A(4D)}$ by large gauge transformations induced via the Chern-Simons term in the 5D action, therefore the 5D splitting for $Q_{A(5D)}$ will turn out to be more involved. Analogous to the 4D case, let us define the fragmentation of this charge to be

$$ Q_{A(5D)} \equiv \sum_{i=1}^{n} Q_{Ai(5D)} \tag{4.3.21} $$

where we now have to determine $Q_{Ai(5D)}$ and then provide it with a physical interpretation. To do this, we substitute the conditions given in eq.(4.3.19) into eqs.(4.2.8) and (4.2.9). Upon further rearranging we get

$$ Q_{A(5D)} = \left\{ \sum_{i=1}^{n} q_{Ai(4D)} + \sum_{i=1}^{n}\sum_{j=1}^{n} 3D_{ABC} {p_i^B}_{(4D)} {p_j^C}_{(4D)} \right\} \tag{4.3.22} $$

$$ = \sum_{i=1}^{n} \left\{ \left(\widetilde{Q}_{A_i\,(5D)} - 3D_{ABC} {p_i^B}_{(5D)} {p_i^C}_{(5D)} \right) + \sum_{j=1}^{n} 3D_{ABC} {p_i^B}_{(5D)} {p_j^C}_{(5D)} \right\} \tag{4.3.23} $$

where the last line has been converted to 5D quantities with the intent of extracting 5D charge fragments. $\widetilde{Q}_{A_i\,(5D)}$ is introduced as a new 5D variable defined by the following 4D/5D transformation

$$ \widetilde{Q}_{A_i\,(5D)} = q_{Ai(4D)} + 3D_{ABC} {p_i^B}_{(4D)} {p_i^C}_{(4D)} \tag{4.3.24} $$

Notice that the right-hand side of eq.(4.3.23) has been expressed in a way that facilitates comparison to the literature. $\widetilde{Q}_{A_i\,(5D)}$ is actually a 5D charge associated to the i^{th} black ring and is the one that appears in the usual 5D multi-ring supergravity solutions (for instance see [82], [83]). In this way, eq.(4.3.23) is simply the ADM mass[3] of [82],

[3] Even though [82], [83] only refer to concentric rings, the above comparison is still meaningful because effects due to non-concentricity only start showing up for quantities involving the position vector $\vec{L}$, such as angular momentum, entropy, etc.

[83]. Note that because these references dwell in conventions different from ours, the following rescaling of charges must be used : $p_i^A{}_{(5D)} \longrightarrow \sqrt{2} p_i^A{}_{(5D)}$. Also they use C_{ABC} as the intersection number, which relates to the one used here via $C_{ABC} = 6D_{ABC}$.

Despite the above comparison, let us remark that in our case eq.(4.3.23) is obtained as a result of 5D fragmentation. Therefore it is clear that summing all the $\widetilde{Q}_{A_i(5D)}$'s over all i would not conserve $Q_{A(5D)}$. The charges that are actually involved in 5D fragmentation are clearly the $Q_{Ai(5D)}$'s and not $\widetilde{Q}_{A_i(5D)}$'s. So the question arises, which of these two is the correct physical observable ? In order to answer this, we shall take a closer look at the interpretation of each of these charges via their integral representations. It will turn out that it is in fact the $Q_{Ai(5D)}$'s that are the physically observable quantities and not the $\widetilde{Q}_{A_i(5D)}$'s. The subtlety between $Q_{Ai(5D)}$ and $\widetilde{Q}_{A_i(5D)}$ arises precisely due to the presence of cross-terms relating different charge centers. The consequences of these cross-terms will also be evident in other quantities such as multi-ring angular momenta. For later reference, let us note down the relation between the two charges

$$Q_{Ai(5D)} = \widetilde{Q}_{A_i(5D)} + \sum_{j=1}^{i-1} 3D_{ABC} \left(p_i^B{}_{(5D)} p_j^C{}_{(5D)} + p_j^B{}_{(5D)} p_i^C{}_{(5D)} \right) \qquad (4.3.25)$$

Whilst plucking this expression from eq.(4.3.23) one has also to keep in mind that $Q_{Ai(5D)}$ should be independent of how the cycles B and C have been labelled. Therefore the resulting expression for $Q_{Ai(5D)}$ has to be symmetrised as done above.

Now let us try to understand the various 5D charges discussed above in the form of integrals over near-horizon patches. In [61] it was shown that in terms of purely near-horizon fields (and not requiring data from the complete solution) of a single black ring, the charge $Q_{A(5D)}$ can be understood as a Page charge rather than a Maxwell charge (see also [59] for a clear exposition on the different notions of charge)

$$Q_{A(5D)} = \int_\Sigma \left(\star a_{AB} F^B + 3D_{ABC} A^B \wedge F^C \right) \qquad (4.3.26)$$

where the range of integration, denoted by Σ, is a spatial 3-cycle in the vicinity of the black ring horizon. The a_{AB}, which is a function of the scalar moduli, serve the usual purpose of lowering vector multiplet indices. A^B denote near-horizon $U(1)$ gauge fields around the black ring. A Page charge is conserved, localized and quantized, but not gauge invariant. The near-horizon integral on the right-hand side of eq.(4.3.26) implicitly represents a Page charge. In [61], they explicitly compute this integral and show that it indeed results in the black ring charge $Q_{A(5D)}$.

Adapting the results of [61] to the present context of fragmented rings, we now argue that the $Q_{Ai(5D)}$'s are also Page charges. This is consistent with the role of eq.(4.3.21) as a charge conservation equation. Then $Q_{Ai(5D)}$ should also have an expression as a localized charge resulting from a near-horizon integral

$$Q_{Ai(5D)} \stackrel{?}{=} \int_{\Sigma_i} \left(\star a_{AB} F_i^B + 3D_{ABC} A_i^B \wedge F_i^C\right) \tag{4.3.27}$$

for A_i^B as $U(1)$ gauge fields locally defined in the neighborhood of the i^{th}-ring horizon. Σ_i denotes a 3-cycle enclosing the i^{th}-horizon and $F_i^B = dA_i^B$. So the question then is : does this integral in eq.(4.3.27) work out to give $Q_{Ai(5D)}$? Upon inserting the following expression for the gauge field :

$$\begin{aligned} A_i^B = \; - \; & \left[\left(D^{BC}\left(\widetilde{Q}_{A_i\,(5D)} - 3D_{CDE}p_i^D{}_{(5D)}p_i^E{}_{(5D)}\right) + 2\sum_{j=1}^{n} p_j^B{}_{(5D)}\right) d\psi \right. \\ & \left. + \left(p_i^B{}_{(5D)}(1+x) + 2\sum_{j=i+1}^{n} p_j^B{}_{(5D)}\right) d\chi\right] \end{aligned} \tag{4.3.28}$$

into the integral in eq.(4.3.27), the authors of [61] indeed do obtain the expression[4] we had in eq.(4.3.25). In eq.(4.3.28), the variables ψ, χ and x are the usual ring-coordinates (notation follows from [31]). $(\psi + \chi/2)$ and x parametrise the S^2, while $(\psi - \chi/2)$ runs along the S^1 near the horizon of the i^{th}-black ring. The gauge fields A_i^B are locally defined patch-wise. Gluing of adjacent patches is achieved via gauge transformations. In eq.(4.3.28), $i = 1$ refers to the innermost ring (smallest radius) and the radial parameter monotonically increases with increasing i. The expression for A_i^B used in [61] was extracted from the supergravity solution of [82], [83] for concentric black rings. The same is reliable for non-concentric rings too, since restrictions to concentricity mainly become relevant when evaluating integrability conditions (and those bear consequences for multi-ring angular momenta).

From the expression for A_i^B above, we see that the gauge field around the i^{th}-ring also feels the back-reaction due to dipole fields from neighboring rings. It is precisely

[4] In [61] the computation was first done for the special case of only one vector field, and then it was generalized to n $U(1)$ fields by simply carrying through the same calculation with vector indices.

this dipole field back-reaction that leads to cross-terms in the computation of $Q_{Ai(5D)}$. In our case we tried to derive these terms from the construction of 5D fragmentation. It is gratifying to note that they exactly compare with those coming from the integral of [61]. As we shall see that fragmentation of a single ring indeed does reproduce the correct multi-ring charges.

Now turning our attention to $\widetilde{Q}_{A_i\,(5D)}$, let us see why this is in fact not a physical charge. From the definition of $\widetilde{Q}_{A_i\,(5D)}$ in eq.(4.3.24), its 4D/5D transformation is identical to that of a single black ring system with electric charge $\widetilde{Q}_{A_i\,(5D)}$ and magnetic charge $p_i^A{}_{(5D)}$. This is in stark contrast to the analogous transformation of $Q_{Ai(5D)}$ (which can be read-off from eq.(4.3.22)). Unlike $Q_{Ai(5D)}$, we see that $\widetilde{Q}_{A_i\,(5D)}$ clearly does not sense the background reaction due to neighboring rings. Hence such a charge cannot be given a global physical meaning in a multi-ring geometry. Its presence is at best only a local approximation. Therefore its integral representation is trivially identical to eq.(4.3.26) after all charges (which enter into the explicit expressions for the gauge potentials) have been replaced by those at the i^{th}-center.

Eqs.(4.3.22) and (4.3.24) essentially describe the multi-center 4D/5D dictionary for electric charges. As expected the physical multi-center Page charge $Q_{Ai(5D)}$ transforms in a more complicated way than $Q_{A(5D)}$ (eq.(4.2.9)), due to the multi-black ring background. On the other hand, the charges $\widetilde{Q}_{A_i\,(5D)}$, though unphysical, retain manifest symplectic invariance of the original single-center solution. Each of the $\left(p_i^A{}_{(5D)}\,,\,\widetilde{Q}_{A_i\,(5D)}\right)$ manifestly transform as a symplectic pair. This underlying property often makes it convenient to express multi-black ring solutions in terms of these charges (as has been usual practice in the literature).

Having explicitly constructed the 5D charge fragmentation equations for magnetic and electric charges $\left(p^A{}_{(5D)}\,,\,Q_{A(5D)}\right)$ along with the relevant multi-center 4D/5D transformations, we are now equipped to derive two of the multi-ring harmonic functions $\left(K^A(x), L_A(x)\right)_{multi}$ from the single-ring harmonics $\left(K^A(x), L_A(x)\right)_{single}$ by merely implementing the fragmentation recipe of section 2. As in eqs.(4.1.4) and (4.1.5) we have

$$\begin{aligned} L_A(x)\Big|_{single} &= v_A + \frac{Q_{A(5D)} - 3D_{ABC}p^B{}_{(5D)}p^C{}_{(5D)}}{|x-x_0|} \\ &\longrightarrow v_A + \sum_{i=1}^{n} \frac{Q^{\bullet}_{Ai(5D)} - 3D_{ABC}p^{\bullet B}_{i\ (5D)}p^{\bullet C}_{i\ (5D)}}{|x-x_i|} = L_A(x)\Big|_{multi} \end{aligned} \quad (4.3.29)$$

which is subject to the constraint

$$Q_{A(5D)} - 3D_{ABC}p^B{}_{(5D)}p^C{}_{(5D)} = \sum_{i=1}^{n}\left(Q^{\bullet}_{Ai(5D)} - 3D_{ABC}p^{\bullet B}_{i\ (5D)}p^{\bullet C}_{i\ (5D)}\right) \quad (4.3.30)$$

Eqs.(4.3.29) and (4.3.30) constitute a natural 5D fragmentation ansatz with newly-defined charges $Q^{\bullet}_{Ai(5D)}$ and $p^{\bullet A}_{i\ (5D)}$ such that at $x \to \infty$ one recovers $L_A(x)\Big|_{single}$ while at $x \to x_i$ the solution (at leading approximation) appears like a single black ring at the i^{th} location. Now the constraint in eq.(4.3.30) above is identical in form to the charge splitting eq.(4.3.23), which suggests the identification

$$Q^{\bullet}_{Ai(5D)} \equiv \widetilde{Q}_{A_{i\,(5D)}} \qquad p^{\bullet A}_{i\ (5D)} \equiv p^A_{i\ (5D)} \quad (4.3.31)$$

From this we also see how the charges $\widetilde{Q}_{A_{i\,(5D)}}$ enter into the 5D harmonics and subsequently into the metric. Of course the above harmonic function could also have been written in terms of $Q_{Ai(5D)}$, but then the expressions would only get a little messy as we proceed.

Another remark that we can make at this stage is that eq.(4.3.29) (along with the conditions in eqs.(4.3.30) and (4.3.31)) could also have been obtained via a different route; namely, by direct use of the multi-ring 4D/5D transformation (eq.(4.3.24)) into eqs.(4.1.4) and (4.1.5). This is consistent with the commutativity of the diagram in eq.(4.0.2), which suggests that fragmenting a single black ring into multiple black rings reproduces the same configuration as that obtained by a direct 5D lift of the appropriate 4D multi-center black holes.

Dealing with the harmonic function $K^A(x)$ for magnetic charges is now straightforward :

$$\begin{aligned} K^A(x)\Big|_{single} &= \frac{p^A{}_{(5D)}}{|x-x_0|} \\ &\longrightarrow \sum_{i=1}^{n}\frac{p^A_{i\ (5D)}}{|x-x_i|} = K^A(x)\Big|_{multi} \end{aligned} \quad (4.3.32)$$

which is again subject to

$$p^A{}_{(5D)} = \sum_{i=1}^{n} p_i^A{}_{(5D)} \tag{4.3.33}$$

As per the other two black ring harmonic functions $H_{TN}(x)$ and $M(x)$: the former remains unchanged under fragmentation as our brane configuration includes only a single D6 charge (which lifts to a Kaluza-Klein monopole in 5D); while fragmentation of the latter comes up in the following sub-section.

4.3.2 Angular Momenta from Black Ring Fragmentation

We are now ready to derive the expressions for angular momenta of a multi-black ring system from 5D fragmentation techniques. Our starting point is eq.(4.2.17) : the angular momentum of a single black ring along the ψ-direction

$$J_\psi = \frac{3\pi}{G} J_{tube} + \frac{3\pi}{G} p^A_{(5D)} (Q_{A(5D)} - 2D_{ABC} p^B_{(5D)} p^C_{(5D)}) \tag{4.3.34}$$

Inserting the 5D charge splitting eqs.(4.3.20) and (4.3.23) into the above we readily obtain

$$\begin{aligned} J_\psi \;=\; \frac{3\pi}{G} \sum_{i=1}^{n} J^i_{tube} \;\;+\;\; \frac{3\pi}{G} \Bigg[& \sum_{i,j=1}^{n} p_i^A{}_{(5D)} \left(\widetilde{Q}_{A_j\,(5D)} - 3D_{ABC} p_j^B{}_{(5D)} p_j^C{}_{(5D)} \right) \\ & + \sum_{i,j,k=1}^{n} D_{ABC} p_i^A{}_{(5D)} p_j^B{}_{(5D)} p_k^C{}_{(5D)} \Bigg] \end{aligned} \tag{4.3.35}$$

where the quantities J^i_{tube} have yet to be determined from integrability conditions. As a special case of our result in eq.(4.3.35), we shall be able to reproduce the expression for angular momentum of concentric black rings which was first derived by Gauntlett and Gutowski in [82], [83] in the context of 5D supergravity. In order to obtain J^i_{tube}, we will first have to determine the multi-ring harmonic function $M(x)$, from where J^i_{tube} can be extracted. Therefore, fragmenting the function $M(x)$ yields

$$\begin{aligned} M(x)\Big|_{single} \;&=\; v_0 + \frac{-J_{tube}}{|x - x_0|} \\ &\longrightarrow\; v_0 + \sum_{i=1}^{n} \frac{-J^i_{tube}}{|x - x_i|} = M(x)\Big|_{multi} \end{aligned} \tag{4.3.36}$$

subject to

$$J_{tube} = \sum_{i=1}^{n} J^i_{tube} \tag{4.3.37}$$

Additionally, the multi-ring harmonics $(H_{TN}(x), K^A(x), L_A(x), M(x))_{multi}$ above also have to satisfy integrability conditions as in eq.(4.1.6). These are to be evaluated at each horizon. Starting with $x = 0$, we get

$$v_0 = \sum_{i=1}^{n} \frac{J^i_{tube}}{L_i} \tag{4.3.38}$$

This determines the constant v_0 in terms of J^i_{tube} (which we still have to fix in terms of electric and magnetic charges) and L_i (which is the radial distance in $\mathbb{R}^3$ of the i^{th} pole from the origin). However, as discussed earlier, v_0 is a constant predetermined at infinity and should not be affected by the process of fragmentation. As a consistency check we shall see in what follows that eq.(4.3.38) is indeed identical to eq.(4.2.10) obtained earlier in section 3. Before that we will require to compute the remaining n conditions at the horizons $\{x_i\}$. This yields

$$\begin{aligned} - J^i_{tube} \;=\; & \left(\frac{4}{R^2_{TN}} + \frac{1}{L_i}\right)^{-1} \left(p^A_{i\,(5D)} v_A + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{p^A_{i\,(5D)} \left(\widetilde{Q}_{A_{j\,(5D)}} - 3D_{ABC} p^B_{j\,(5D)} p^C_{j\,(5D)}\right)}{\sqrt{L_i^2 - 2L_iL_j cos\theta_{ij} + L_j^2}} \right. \\ & \left. - \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\left(\widetilde{Q}_{A_{i\,(5D)}} - 3D_{ABC} p^B_{i\,(5D)} p^C_{i\,(5D)}\right) p^A_{j\,(5D)}}{\sqrt{L_i^2 - 2L_iL_j cos\theta_{ij} + L_j^2}} \right) \end{aligned} \tag{4.3.39}$$

where θ_{ij} is the angle between $\vec{L_i}$, $\vec{L_j} \in \mathbb{R}^3$. Now rearranging eq.(4.3.39) for $\frac{J^i_{tube}}{L_i}$ and then inserting back into eq.(4.3.38) produces

$$v_0 = -\frac{4\,J_{tube}}{R^2_{TN}} - v_A p^A{}_{(5D)} \tag{4.3.40}$$

after also using eqs.(4.3.20) and (4.3.37). Indeed eq.(4.3.40) is precisely the value of v_0 obtained earlier by inserting eq.(4.2.11) into eq.(4.2.10).

Now with eqs.(4.3.39) and (4.3.40) the function $M(x)\Big|_{multi}$ is fully specified. Thus simply from 5D black ring fragmentation we were able to construct all of the multi-black ring harmonic functions. Moreover inserting eq.(4.3.39) for J^i_{tube} into eq.(4.3.35) results in the complete expression for the total multi-black ring angular momentum in the ψ-direction : J_ψ. Also the angular momentum in the ϕ-direction : J_ϕ, can then be read-off from J_ψ since

$$J_\phi = J_\psi - \frac{3\pi}{G}\sum_{i=1}^{n} J^i_{tube} \tag{4.3.41}$$

still continues to hold.

An additional comment on eq.(4.3.39) is due. Let us take a closer look at the last two terms on the right-hand side of this equation. As long as the multi-center charges are constrained to remain mutually non-local, then $\vec{L_i} \neq \vec{L_j}$ will hold and that avoids any potential singularity in eq.(4.3.39). Hence the sum of the two numerators (within the summation symbols) is allowed to assume any non-zero value. From the 4D point of view, this is precisely the condition for the dual 4D charges $(p^A_{i\,(4D)}, q_{Ai(4D)})$ to be non-parallel (on the charge lattice). This was the interesting new feature in the multi-center solution of [69], [70], [71], [72]. On the other hand, if the condition $\vec{L_i} \neq \vec{L_j}$ were to be relaxed; then we would be required to impose

$$p^A_{i\,(5D)}\left(\widetilde{Q}_{A_{j\,(5D)}} - 3D_{ABC}p^B_{j\,(5D)}p^C_{j\,(5D)}\right) - \left(\widetilde{Q}_{A_{i\,(5D)}} - 3D_{ABC}p^B_{i\,(5D)}p^C_{i\,(5D)}\right)p^A_{j\,(5D)} = 0 \tag{4.3.42}$$

for all $i \neq j$, thereby eliminating the last two terms in eq.(4.3.39). The corresponding 4D charge vectors $(p^A_{i\,(4D)}, q_{Ai(4D)})$ are now parallel-aligned on the charge lattice[5]. The reason we made the above comment is because the construction in [82], [83] does restrict to eq.(4.3.42) and hence we too will need to make use of it whenever comparing to their results. For all other purposes, our results continue to hold for non-parallel charges in general.

Eq.(4.3.35) along with eq.(4.3.39) gave us the most general result for the angular momentum (along the ψ-coordinate) of non-concentric multi-black rings. We would

[5] Note that being parallel on the charge lattice should not be confused with co-linearity of the poles in $\mathbb{R}^3$. Even for parallel charges the multi-center poles are still free to remain non-colinear. From a 4D/5D perspective, non-colinear D4-D2-D0 poles in 4D lift to non-concentric rings in 5D.

now like to reduce our result to the case of concentric rings so as to compare it with the well-known answer of [82], [83], which was derived using 5D supergravity techniques of [49] and [50]. First we set all angles θ_{ij} between the poles to zero. The co-linear alignment of poles in $\mathbb{R}^3$ translates to concentric rings in 5D. In order to eliminate Dirac-Misner strings, [82], [83] choose to impose eq.(4.3.42), which can be interpreted as a restriction to parallel 4D charge vectors[6]. From our discussion above, we see that it is still possible to continue with non-parallel charges by trading-off mutual locality of charges. Nevertheless to make contact with [82], [83]; we use eq.(4.3.42) in eq.(4.3.39) with all angles $\theta_{ij} = 0$ and thus arrive at the desired result upon plugging everything back into eq.(4.3.35). To facilitate a direct comparison, let us also connect with the notation of [82], [83]; which is achieved via simple charge redefinitions. Firstly we note that the 4D/5D transformations - eqs.(4.2.9) and (4.2.16) - match their counterparts in [82], [83] after the following two redefinitions : $q_{0(5D)} \longrightarrow (q_{0(5D)} + p^A_{(4D)} q_{A(4D)})/2$ and $p^A_{(4D)} \longrightarrow \sqrt{2} p^A_{(4D)}$. We have already seen how the latter conformed to 5D split-charges and played a role in matching eq.(4.3.23) to the above literature. Now coming to the multi-ring angular momentum in eq.(4.3.35), it can be seen after some algebra that the first of the above two redefinitions simply gives a factor of 2 to the last term of eq.(4.3.35). Then making use of the second redefinition in the form $p^A_{i\,(5D)} \longrightarrow \sqrt{2} p^A_{i\,(5D)}$ produces

$$J_\psi = -\frac{6\sqrt{2}\pi}{G} \sum_{i=1}^{n} L_i p^A_{i\,(5D)} v_A + \frac{\sqrt{2}\pi}{G} \left[3 \sum_{i,j=1}^{n} p^A_{i\,(5D)} \left(\widetilde{Q}_{A_{j\,(5D)}} - C_{ABC} p^B_{j\,(5D)} p^C_{j\,(5D)} \right) + 2 \sum_{i,j,k=1}^{n} C_{ABC} p^A_{i\,(5D)} p^B_{j\,(5D)} p^C_{k\,(5D)} \right] \quad (4.3.43)$$

which exactly agrees (upto an overall factor which we leave to one's taste) with [82], [83] as the total angular momentum of concentric black rings in $\mathbb{R}^4$.

Finally let us remark that writing the 5D charge $q_{0(5D)}$ in the form

$$q_{0(5D)} = \sum_{i=1}^{n} q_{0i(5D)} \quad (4.3.44)$$

[6] In fact this is not the most general way to eliminate Dirac-Misner strings and admittedly ends up making the solution of [82], [83] highly restrictive. In general it suffices to impose the integrability conditions as we have done in this note. The difference with [82], [83] is that those authors impose eq.(4.3.39) in a very special way.

its fragments can be easily read-off from eq.(4.3.35) above. Just as was the case earlier with the $Q_{Ai(5D)}$ charge, we see again that the multi-ring 4D/5D transformations for $q_{0i(5D)}$ are more complicated due to the presence of cross-terms which must be carefully taken into account while performing a 4D/5D lift. In the next section, we proceed to discuss the physical origin of these cross-terms and their geometric interpretation.

4.4 Geometric Interpretation using Split-Spectral Flows

In this section we try to provide a geometric understanding of multi-black rings, based on successive application of spectral flow transformations. Such split-spectral flows now assume relevance in the presence of multiple $AdS_3 \times S^2$ horizons. This generalizes the spectral flow discussions of [57], [58] to a multi-center setting.

Let us first consider a single black ring, whose near-horizon geometry is $AdS_3 \times S^2$. This will be seen to fit exactly within the considerations of [57], [58]. In this background geometry, the 5D supergravity action contains a Chern-Simons term

$$S_{CS} = \int_{AdS_3 \times S^2} D_{ABC} A^A \wedge F^B \wedge F^C \tag{4.4.45}$$

which is not invariant under large gauge transformations. $F^A = dA^A$ is the usual two-form $U(1)$ magnetic flux passing through the S^2. The electric charge is obtained by varying the 5D action with respect to the field strength F^A. Due to the presence of the above-mentioned Chern-Simons term, the electric charge so obtained also varies under large gauge transformations

$$q_A = \int_{S^2 \times S^1} \left(\star F_A \; + \; 3 D_{ABC} A^B \wedge F^C \right) \tag{4.4.46}$$

Since the 5D supergravity action can be obtained from a Calabi-Yau compactification of M-theory, the electric charge q_A is the M2-brane charge from a 11-dimensional perspective (or D2 charge in Type II A) and the magnetic charge p^A defined as

$$p^A = \int_{S^2} F^A \tag{4.4.47}$$

is the M5-brane charge[7] (or D4 in 10 dimensions).

It can be seen by inspection that the second term in the integrand in eq.(4.4.46) will decay rapidly when evaluated over a homologous 3-surface sufficiently distant from the horizon, leaving only the first term to contribute. However, prior to integration, let us consider the effect of a large gauge transformation of A^A, of the type

$$A^A \;\longrightarrow\; A^A \,+\, k^A\, d\,(\psi/2\pi) \tag{4.4.48}$$

with k^A an integer and $0 \leq \psi \leq 2\pi$ a coordinate running along the S^1. This leaves us with A^A-independent terms that do not vanish at infinity, thereby producing shifts in the electric charge q_A of the type

$$q_A \;\longrightarrow\; q_A \,+\, 3D_{ABC}k^B p^C \tag{4.4.49}$$

This charge is clearly not gauge invariant and the physical explanation shall soon follow. For now, let us note that this equation compares to the 4D/5D charge transformation that we encountered earlier in eq.(4.2.9), since it is the lack of gauge invariance of the 5D Chern-Simons term in the action that is responsible for inducing shifts in the original gauge invariant 4D charges.

Similarly the M-theory angular momentum q_0 (or D0 charge in Type II A) is again not a gauge invariant quantity and we now proceed to derive its charge shifts, obtained via gauge transforming an integral representation of angular momentum. For a 5D supergravity action (to be thought of as a semi-classical reduction of M-theory in our context), such an integral can be extracted from appropriate contributions to the gauge field energy-momentum tensor. For the aforesaid 5D action, this has been derived in [61] making use of Wald's method [85]

$$q_0 = -\int_{S^2\times S^1}\left(\star\, d\xi \,+\, \star(\xi\cdot A^A)\,F_A \,+\, D_{ABC}\,(\xi\cdot A^A)\,A^B\wedge F^C\right) \tag{4.4.50}$$

Here ξ denotes the axial Killing vector with respect to the ψ-direction, while $(\xi \cdot A^A)$ is an interior product between a vector field and a one-form. The Killing field ξ generates isometries along the ψ-direction; leading to a conserved charge, which is the angular

[7] Strictly speaking, this definition remains valid so long as the NUT charge (the KK monopole at the origin) is not encompassed by the S^2.

momentum. In fact, the right-hand side of eq.(4.4.50) is simply the Noether charge of Wald. Asymptotically, the A^A dependent terms in the integrand (in eq.(4.4.50)) drop off and the integral reduces to Komar's formula for the angular momentum. However, large gauge transforming with eq.(4.4.48) yields precisely two more asymptotically non-vanishing remnants. Recognizing the asymptotic form of eq.(4.4.46) and eq.(4.4.47) leads to the following charge shifts in angular momentum

$$q_0 \longrightarrow q_0 - k^A q_A - D_{ABC} k^A k^B p^C \tag{4.4.51}$$

This again can be compared to the 4D/5D transformation in eq.(4.2.16).

Now eqs.(4.4.49), (4.4.51) are in fact the spectral flow transformations in question. The name spectral flow arises due to the fact that in the dual $(0,4)$ SCFT these transformations correspond to automorphisms of the conformal algebra. Moreover spectral flow is a symmetry of the theory as it leaves the generalized elliptic genus of the CFT invariant. Note that such flows are characteristic of an odd dimensional theory. For a 4D black hole with $AdS_2 \times S^2$ horizon, the supergravity action is gauge invariant. Therefore the electric charge equals the actual number of D2 branes wrapped on Calabi-Yau 2-cycles; while the D0 charge counts the physical D0 branes. Because of this we can also interpret eqs.(4.4.49), (4.4.51) as a 5D lift of 4D charges.

The gauge transformation in eq.(4.4.48) is picked up upon going around (perpendicular to the ψ-direction) the ring with a probe particle; which has been given the interpretation of M5-anti-M5 branes being pair-produced, going around the ring in opposite directions and mutually annihilating (see fig. 1 in [57]). More precisely, this can be visualized as follows. The spatial near-horizon geometry of a bound state of M5-M2 branes (with angular momentum) is a product of Euclidean AdS_2 and S^2 (refer to fig. 4.1 (a) below). On the AdS_2 disc, the black ring is depicted as a circle along the ψ-direction. The radial coordinate on the disc is the same as the Taub-NUT radial direction. Now consider the pair-production of k^A M5-anti-M5 pairs. These wrap 4-cycles on the Calabi-Yau, while the fifth direction goes around the equator of an S^2. This S^2 is a point on the AdS_2 disc, located on the inside of the circle representing the black ring. The M5-anti-M5 rings along the S^2 equator move apart in opposite directions towards the poles, where they self-annihilate leaving behind a Dirac surface on the S^2. Since the location of the Dirac surface is unphysical, it can be moved away to spatial infinity. Upon crossing the ring, it causes a shift of gauge potential by an amount $k^A\, d\,(\psi/2\pi)$. Thus the presence of a magnetic flux k^A shifts the gauge potential A^A and consequently the charges q_A, q_0. For the case of the single ring described above, this flux is the dipole flux passing through the ring and is generated by its own M5 charges. Hence $k^A = p^A$ here, which leads to eqs.(4.4.49), (4.4.51).

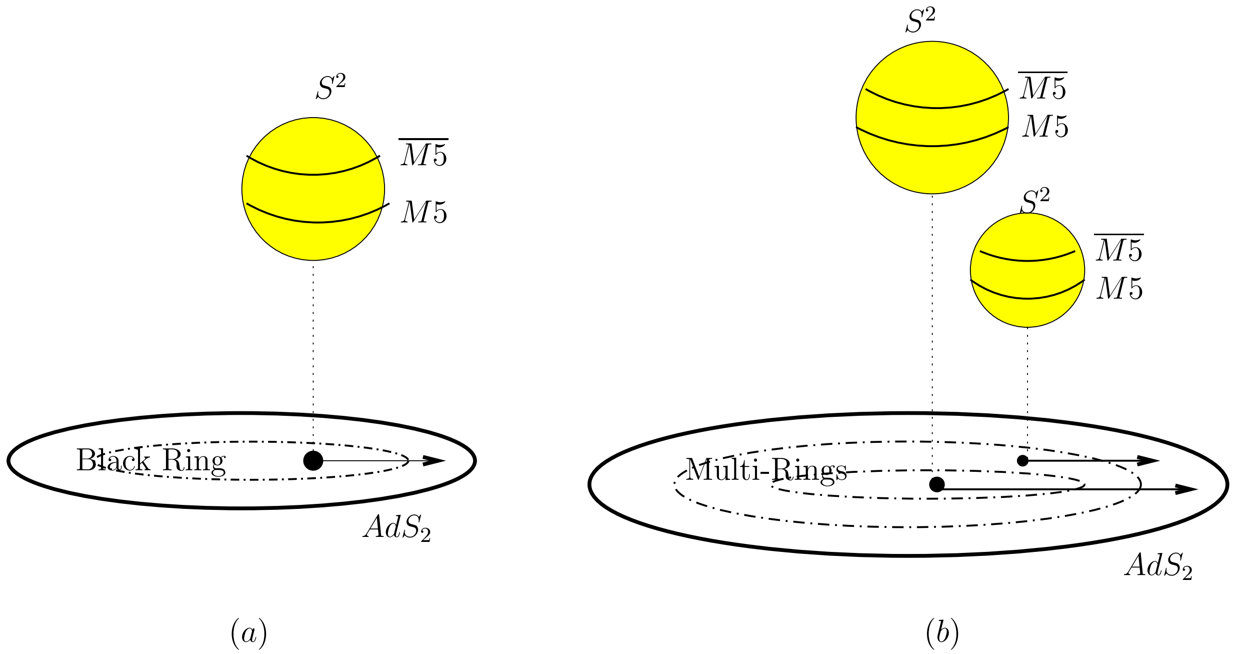


Figure 4.1: Visualising the spectral flow for black rings : (a) Nucleation of an M5-anti-M5 pair around a single black ring leading to a large gauge transformation. (b) The same idea now extended to a multi-black ring background leads to multiple gauge transformations in a geometrically ordered way.

4.4.1 Electric Charges and Split-Spectral Flows

Now extending the above discussion, we shall systematically derive multi-black ring electric charges and angular momenta as a split-spectral flow argument. We begin with electric charges. Let us label the n rings with an index i, in increasing order of radius. The innermost ring is labeled $i = 1$. Its brane charges are p_1^A, q_{A_1}, q_{0_1}. Here p_1^A exhibits a dipole behavior, generating a magnetic flux $k^A = p_1^A$. This in turn shifts q_{A_1} by spectral flow as in eq.(4.4.49). Indeed this innermost ring behaves just like the single ring case encountered in the previous discussion above. Moving onto the next ring, this has brane charges p_2^A, q_{A_2}, q_{0_2}. As depicted in fig. 4.2 below, the total flux passing through this ring is not only that generated by its own charge p_2^A, but also that emanating from the inner ring. These distinct fluxes give rise to the following spectral flows :

$$q_{A_2} \longrightarrow q_{A_2} + 3\, D_{ABC}\; p_\delta^B \;\; p_\gamma^C \longrightarrow \begin{cases} \delta = 2,\ \gamma = 2 \quad \text{with } \ k^B = p_2^B \\ \delta = 1,\ \gamma = 2 \quad \text{with } \ k^B = p_1^B \\ \delta = 2,\ \gamma = 1 \quad \text{with } \ k^C = p_1^C \end{cases} \tag{4.4.52}$$

where the last transformation occurs due to the fact that the flux has also to be symmetrised with respect to the cycles. The physical electric charge of this ring is then obtained by adding up all these shifts to the original brane charge.

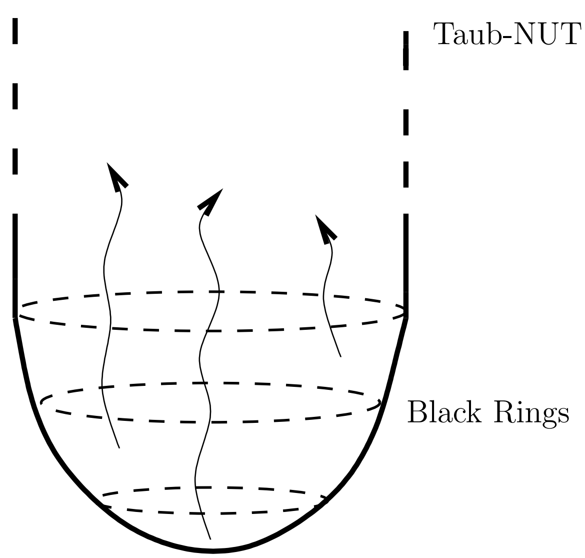


Figure 4.2: A Taub-NUT perspective of the influence of magnetic flux generated by individual black rings upon neighbouring black rings.

From the point of view of fig. 4.1 (b), multi-rings are depicted as n-circles on the disc, one inside the other. Nucleation of an M5-anti-M5 pair now occurs in the vicinity of each of the n rings, creating n Dirac surfaces. Upon moving these surfaces to infinity, the i^{th}-ring is crossed by i Dirac surfaces each with flux p_j^A, giving in total a flux $k_{tot}^A = \sum_{j=1}^{i} p_j^A$ passing through this ring. This is the origin of multiple spectral flows for a multi-ring system.

We can now directly write down the result for the i^{th}-ring with all the spectral flows put together : those resulting from the intrinsic (due to ring's own magnetic charge) flux as well as those from background (generated by those rings which are encircled by the i^{th} one) flux, we get

$$q_{A_i} \longrightarrow q_{A_i} + 3D_{ABC} p_i^B p_i^C + 3D_{ABC} \sum_{j=1}^{i-1} (p_i^B p_j^C + p_j^B p_i^C) \tag{4.4.53}$$

Much like the analogy in electrostatics, the fluxes due to rings which encircle the i^{th}-ring from the outside, do not affect it. With respect to fig. 4.1 (b), each ring acts as a source, emanating flux; while the sink is at infinity. Hence only those rings placed to the exterior of the source ring will lie in its flux field. Eq.(4.4.53) gives us the physical charge of the i^{th}-ring from a spectral flow analysis. This can be compared to eq.(4.3.25), where the same quantity emerged from a fragmentation analysis.

Furthermore upon adding up the split-spectral flow shifts of all of the n rings leads to the total spectral flow shift of the full multi-ring configuration

$$\begin{aligned} Q_A^{total} &\equiv \sum_{i=1}^{n} q_{A_i} + 3D_{ABC} \sum_{i=1}^{n} p_i^B p_i^C + 3D_{ABC} \sum_{i=1}^{n} \sum_{j=1}^{i-1} (p_i^B p_j^C + p_j^B p_i^C) \\ &= \sum_{i=1}^{n} q_{A_i} + 3D_{ABC} \sum_{i,j=1}^{n} p_i^B p_j^C \end{aligned} \tag{4.4.54}$$

where in the last step, the identity

$$\sum_{i=1}^{n} \sum_{j=1}^{i-1} (A_{ij} + A_{ji}) \quad = \quad \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} \; - \; \sum_{i=1}^{n} A_{ii} \tag{4.4.55}$$

has been used. Indeed Q_A^{total} exactly equates to $Q_{A(5D)}$ in eq.(4.3.22), which is simply the electric charge of a single black-ring system. Therefore, adding up all the spectral flow shifts as well as the total brane charge gets us back to the geometry of a single black ring. In this sense the spectral flow transforms of a multi-ring system are really split-spectral flows of a single ring system.

4.4.2 Angular Momenta and Split-Spectral Flows

The multi-ring angular momentum can now be obtained in a similar fashion. Once again consider the i^{th}-ring with brane charges p_i^A, q_{A_i}, q_{0_i}. The relevant angular momentum spectral flows for this ring are

$$q_{0_i} \longrightarrow q_{0_i} - p_i^A \sum_{j=1}^{i} q_{A_j} - D_{ABC} p_i^A p_i^B p_i^C \tag{4.4.56}$$

$$q_{0_i} \longrightarrow q_{0_i} - \sum_{j=1}^{i-1} p_j^A q_{A_i} - D_{ABC} \sum_{j=1}^{i-1} p_j^{\big(A} \sum_{k=1}^{i-1} p_k^B \; p_i^{C\big)} \tag{4.4.57}$$

In the above flow equations, firstly we have the intrinsic magnetic flux $k_i^A = p_i^A$, generated by M5 charges on the i^{th}-ring itself. This flux interacts with M2 charges as well as M5 charges (carried on other Calabi-Yau cycles), both on the i^{th}-ring. Then there is the background magnetic flux $k^A_{back} = \sum_{j=1}^{i-1} p_j^A$ because this ring is placed in the background fields generated by the $i-1$ rings to its interior. Now a new addition to the above is a background electric flux $\sum_{j=1}^{i-1} q_{A_j}$, which also interacts with electric charges on the i^{th}-ring. That explains the second term on the right-hand side of eq.(4.4.56). And eq.(4.4.57) then accounts for interactions of the magnetic background with the i^{th}-brane charges in the usual way. The last term there has to be symmetrised and therefore the brackets in superscripts denote a sum over all symmetric permutations of cycles. Then adding up all these contributions will result in the angular momentum of the i^{th}-ring.

To get the total angular momentum of the multi-ring system we add up those of each of the rings

$$\begin{aligned} J^{total} &\equiv \sum_{i=1}^{n} q_{0_i} - \sum_{i=1}^{n}\sum_{j=1}^{i-1}(p_i^A q_{A_j} + p_j^A q_{A_i}) - \sum_{i=1}^{n} p_i^A q_{A_i} - D_{ABC}\sum_{i=1}^{n} p_i^A p_i^B p_i^C \\ &- D_{ABC}\sum_{i=1}^{n}\sum_{j=1}^{i-1} p_j^{\big(A} \sum_{k=1}^{i-1} p_k^B \; p_i^{C\big)} \\ &= \sum_{i=1}^{n} q_{0_i} - \sum_{i,j=1}^{n} p_i^A q_{A_j} - D_{ABC} \sum_{i,j,k=1}^{n} p_i^A p_j^B p_k^C \end{aligned} \tag{4.4.58}$$

Upon substituting q_{A_j} in the last equality above with $\widetilde{Q}_{A_{i\,(5D)}}$ via eq.(4.3.24), we see that eq.(4.4.58) indeed compares[8] to eq.(4.3.35) leading to $J^{total} = -\frac{G}{3\pi} J_\psi$. A split-spectral

[8] Of course spectral flow does not determine q_{0_i} as a function of L_i. That input still relies on the

flow analysis thus provides us with a physical understanding of where all the different multi-ring angular momentum contributions actually come from. In particular, it gives a clear description of how individual rings behave in the background of other rings.

Consequently a geometric picture of this multi-black ring configuration emerges from such split-spectral flow considerations. In fact what these split-flows are really doing is to break up the global multi-ring geometry into patches with locally defined gauge potentials; such that gauge fields in neighboring patches are related up to large gauge transformations. In fig. 4.1 (b) these patches can be identified as follows : first there's the innermost disc inside the first ring, defining a patch with gauge potential A_1^A; then there are the annular regions all around it, with gauge potentials A_2^A, A_3^A,......... respectively. This defines a chain of potentials spanning the entire geometry

$$A_1 \xrightarrow{\beta_1} A_2 \xrightarrow{\beta_2} A_3 \cdots\cdots\cdots\cdots \xrightarrow{\beta_{n-1}} A_n \xrightarrow{\beta_n} A_n + \beta_n \qquad (4.4.59)$$

(suppressed vector indices may be readily reinstated here) the β_i are large gauge transformations between A_i and A_{i+1}. In fact these local regions emerging here due to split-spectral flow considerations might provide a conceptual basis for the analysis of [61] where the authors compute localised charge integrals for black rings by dividing the geometry into local patches which are all glued together. The existence of such patches enable near-horizon integrals such as those in eqs.(4.3.26), (4.3.27) to capture all the data normally extracted from the full geometry.

4.5 Conclusions and Discussion

Two remarkable set of ideas pertaining to string theoretic descriptions of black holes, that have generated lots of excitement in the aftermath of the OSV conjecture are : (1) the 4D/5D connection between black holes/rings [42], [43]; and (2) multi-center black holes as non-perturbative corrections to the black hole partition function [107]. In this note we have sought for a modest attempt at combining these two, in the sense of the commutative box diagram of eq.(4.0.2).

We have approached the problem by setting-up an explicit 5D construction of black

integrability conditions.

ring fragmentation and thereafter also show that fragmented black rings are equivalent to a direct 5D lift of 4D multi-black holes. For the purposes of the latter, we determine the multi-center 4D/5D charge transformations as well. Related to these issues is the important issue of interpretation of charges in 5D, especially for our multi-center split charges. In [61] it was shown that the electric charge (and angular momentum) of a single black ring could be expressed purely in terms of near-horizon data as a Page charge. In our analysis we see that the 5D charges $Q_{Ai(5D)}$ which participate in fragmentation are in fact also Page charges (as opposed to being Maxwell charges) and in that sense these are the physical charges of the system. Whereas the multi-center charges $\widetilde{Q}_{A_{i\,(5D)}}$ that usually appear in the supergravity multi-ring metric are not physical charges. Even though the latter-mentioned charges can be algebraically related to the former ones, we find it nevertheless important to distinguish the physically relevant ones for the multi-ring configuration.

Another rather interesting application of the 5D fragmentation methods developed in this chapter is an alternative derivation of the angular momenta of concentric black rings. It is indeed gratifying to note that we are able to exactly reproduce the results of Gauntlett and Gutowski.

Lastly, we saw how the introduction of split-spectral flows lends a geometric perspective to shifts in brane charges of fragmented black rings by accounting how a Dirac string generated by a given ring influences other rings in such a multi-ring background. This serves as yet another derivation for the total angular momentum of a multi-ring system. Moreover summing up all the split-spectral flow shifted charges of all the fragmented rings exactly gives back the observed electric charge of a single black ring. The split-spectral flows basically divide the geometry into patches with locally defined gauge fields. The significance of these patches becomes relevant when computing near-horizon integrals.

From a broader perspective, one might contemplate over the role of fragmented configurations on the black hole/ring partition function. In [107], each fragmented configuration is viewed as a multi-AdS throat geometry; and further following [73], [74], each such geometry is associated to some saddle point of the partition function. In that sense Z_{BH} is presumed to sum over all possible geometries subject to charge conservation constraints. Fragmentation is thus a euclidean tunneling process from one minima to another. These leading order contributions therefore dominate the multi-AdS partition sum of [107]. However there ought to be further sub-leading corrections to each multi-center configuration that should be computable from any complete partition sum. At this stage, it would be very tempting to think that the black hole farey tail partition function of [75], [57], [58] might be precisely the object that captures the multi-center saddle points as well as its sub-leading corrections. Whether or not these multi-center geometries lend a physical description to the farey tail story remains to be seen.

Let us make a few more remarks with a view towards subsequently outlook for these results. Ever since investigations into non-perturbative corrections to the OSV conjecture began, one of the earliest applications of Denef et al's multi-center black hole solutions [69] was its realization as a gravity dual to finite N effects of a 2D q-deformed U(N) Yang-Mills gauge theory localized on the world-volume of branes wrapping a non-compact Calabi-Yau constituted by a Riemann surface endowed with two line bundles. These gravity duals have been interpreted as 4D baby universes [107], being viewed as end-products of AdS-fragmentation. The question investigated in this chapter was how does the 4D↔5D connection of [42] work for these multi-center configurations? More specifically, after having explicitly set-up a 5D construction of AdS-fragmentation, whereby a single black ring splits-up into a multi-black ring configuration, it was shown that these fragmented rings are equivalent to a direct 5D lift of 4D multi-center black holes. In this sense, the 5D duals of these baby universes are simply a configuration of non-concentric multi-black rings in Taub-NUT space. However Chern-Simons induced charge shifts once again appeared in this context. Therefore after having motivated the 4D/5D charge transformations for multi-center configurations, we have confirmed that all conserved charges are in fact Page charges arising due to 5D Chern-Simons terms and provide a geometric interpretation for this system of rings using the idea of split-spectral flows, wherein a given black ring's observables are influenced by fluxes generated in a background of neighboring rings. A future research direction is to incorporate these split-flows into an entropy function so as to compute sub-leading degeneracies to multi-center systems.

Chapter 5

Continuum Solutions & Black Hole Levitrons

Gravity cannot be held responsible for people falling in love;
nor levitrons, for those rising above it
- with apologies to Albert Einstein

Continuing our investigation into multi-center geometries, in this chapter of the thesis we look for limiting cases where one still has some analytic control on the solution. This becomes a relevant issue because whilst performing calculations involving multi-center geometries, it soon becomes apparent that even for the simplest configurations with more than two centers, solving integrability constraints to determine the full metric becomes highly formidable. Therefore in this chapter, as a curiosity we probe the other extreme, namely the continuum limit of multi-center black holes in 4D and look for solutions. It turns out that that regime is indeed amenable to analytic results. Furthermore as an interesting application of these solutions, we investigate the problem of spatially stabilizing four dimensional extremal black holes in background electric/magnetic fields. This construction of black holes levitating over external magnetic fields strikes a close resemblance to a mechanical Levitron.

Moreover in the light of on-going interest in questions concerning black hole production; it is interesting to consider how one could go about stabilizing such a black hole using external fields, thus leading to a black hole analog of a particle-trap or rather as we shall see that of a Levitron. However unlike the more familiar subatomic particle traps or even Millikan's famous oil drop experiment [86], the effects of general relativity give rise to interesting new features. We shall describe how this idea can in fact be materialized by writing down solutions for black holes levitating in electromagnetic as well as constant gravitational fields.

For our purposes in this chapter, we shall consider four dimensional extremal black holes as solutions to minimal $\mathcal{N} = 2$ SUGRA ([72], [71], [69]). Furthermore, let us confine these configurations to only include electric and magnetic charges q and p respectively. These extremal black holes are known to satisfy the BPS constraint. The most general metric ansatz consistent with supersymmetry can then be written as

$$\begin{aligned} ds^2 &= -\frac{\pi}{S(\vec{x})}(dt + \omega_i dx^i)^2 + \frac{S(\vec{x})}{\pi} dx^i dx^i \\ \text{with} \quad S(\vec{x})/\pi &= \mathcal{P}^2(\vec{x}) + \mathcal{Q}^2(\vec{x}) \\ \text{and} \quad \mathcal{A} &= 2\pi \mathcal{Q}(\vec{x}) \left(dt + \omega_i dx^i\right) + \Theta \end{aligned} \tag{5.0.1}$$

is the four dimensional gauge field. $\mathcal{P}(\vec{x})$, $\mathcal{Q}(\vec{x})$ are harmonic functions associated to charges p and q respectively. Θ is the Dirac part of the vector potential satisfying $d\Theta = *d\mathcal{P}(\vec{x})$ with the Hodge star $*$ defined on $\mathbb{R}^3$. For a single spherically symmetric black hole in vacuum, it holds that $\vec{\omega} = 0$. However for our considerations, we shall be looking for solutions when the black hole is placed in external electric and magnetic fields. There is now a non-zero Poynting vector corresponding to a rotating geometry. We first look for levitating solutions in constant background fields. It turns out these are inadequate for stabilization in all three directions. Then we look for more non-trivial backgrounds obtained using a continuum limit of Denef et al's [72], [71], [69] multi-center solutions and find that turning on dipole fields achieves the desired result.

5.1 Black Hole Levitation in Constant External Fields

Given the metric ansatz in eq.(5.0.1), we begin by looking for stationary solutions of a black hole placed in constant electric, magnetic and gravitational fields. In order to achieve this we have to specify explicit harmonic functions describing this configuration, then compute the off-diagonal elements $\vec{\omega}$ and solve the associated integrability equations. We claim that the desired harmonic functions describing this configuration are

$$\mathcal{P}(\vec{x}) = u + \frac{p}{|\vec{x} - \vec{l}|} + Bz \qquad \mathcal{Q}(\vec{x}) = v + \frac{q}{|\vec{x} - \vec{l}|} + Ez \tag{5.1.2}$$

where B and E are constant magnetic respectively electric fields oriented along the z-direction and z denotes the z-coordinate. $\vec{l}$ marks the position of the black hole's horizon, which we determine via integrability conditions. u, v are constants. In principle, we can absorb u and v via a shift in the z-coordinate. This point will be made clear when we solve for $\vec{l}$. The Bz and Ez in eq.(5.1.2) are linear terms that satisfy Laplace's equation and can be recognized as the usual electro/magneto-static potentials associated to constant fields. Note that extremality implies the above linear terms also source constant gravitational fields.

A nice way to motivate the expressions for $\mathcal{P}(\vec{x})$ and $\mathcal{Q}(\vec{x})$ is to extract them via a special limit of Denef et al's multi-center solutions [72], [71], [69]. More specifically, let us consider the two-center solution. This is a regular BPS solution of four dimensional $\mathcal{N} = 2$ supergravity. It is stationary but non-static and hence caries an intrinsic angular momentum. Moreover the black holes comprising this bound state possess mutually non-local charges. Let us denote the corresponding two charge vectors as $\mathbf{\Gamma} = (p, q)$ and $\tilde{\mathbf{\Gamma}} = (\tilde{p}, \tilde{q})$. The idea is now to carry the charge $\tilde{\mathbf{\Gamma}}$ all the way to infinity while scaling $(\tilde{p}, \tilde{q})$ and the radial coordinate of the charges in such a way that the magnitudes of the electric/magnetic fields themselves are held fixed. Applying this limit to the expressions for electro/magneto-static fields of point charges indeed leaves us with constant fields oriented opposite to the direction of the source charges $\tilde{\mathbf{\Gamma}}$. Without loss of generality, the z-axis can then be chosen to point in the direction of the sources. Integrating these fields along the line element, precisely yields the linear potential terms in eq.(5.1.2).

In fact we may also use this limiting two-center system to captures other features of our original configuration of a black hole in constant external fields. Following [72], [71], [69], we can determine the off-diagonal terms in the metric using

$$\nabla \times \vec{\omega} = \mathcal{P}(\vec{x}) \nabla \mathcal{Q}(\vec{x}) - \mathcal{Q}(\vec{x}) \nabla \mathcal{P}(\vec{x}) \tag{5.1.3}$$

Below we shall solve $\vec{\omega}$ for a class of non-static solutions. Furthermore operating a gradient on both sides of eq.(5.1.3) leads to the following integrability equation

$$\mathcal{P}(\vec{x}) \nabla^2 \mathcal{Q}(\vec{x}) - \mathcal{Q}(\vec{x}) \nabla^2 \mathcal{P}(\vec{x}) = 0 \tag{5.1.4}$$

which we evaluate at $\vec{x} = \vec{l}$ to get

$$l = \frac{qu - pv}{pE - qB} \tag{5.1.5}$$

This gives us the position of the black hole. Here $\vec{l} = (0, 0, l)$ can be chosen on grounds of symmetry. One can also perform a shift of coordinates, so as to place the black hole at the origin. This can be achieved by setting constants $u = v = 0$. Note however that $(pE - qB) \neq 0$ is required in order to preserve mutual non-locality.

Eq.(5.1.3) can be conveniently solved using spherical coordinates (r, θ, ϕ). And that leads to a system of coupled differential equations

$$\begin{aligned}
(\nabla \times \vec{\omega})_r &= \frac{2\ \cos\theta\,(pE - qB)}{r} \\
(\nabla \times \vec{\omega})_\theta &= -\frac{\sin\theta\,(pE - qB)}{r}
\end{aligned} \tag{5.1.6}$$

while $(\nabla \times \vec{\omega})_\phi = 0$ due to ϕ-independence on the right-hand side. Our objective is now to seek out a non-trivial solution which confers to the description of a black hole rotating in the presence of external electromagnetic fields. We find that there exists such a simple solution with azimuthal symmetry

$$\omega_\phi = \sin\theta\,(pE - qB) \tag{5.1.7}$$

while $\omega_r = \omega_\theta = 0$. For completeness let us also mention that the solution presented in eq.(5.1.7) is certainly not the most general. For instance, we also find that solutions with harmonic variations such as $\frac{\partial\omega_\theta}{\partial\phi} = \cos\phi$ also exist and very likely one may well find a more general class of these. But we shall not require that for our purposes.

The solution above allows us to levitate a black hole at a fixed height on the xy-plane owing to the balancing act between gravitational attraction and electro/magneto-static repulsion. However it is not stable in all three directions and can move about the surface of the plane. To localise the black hole in all three directions we need a more complicated background field where the black hole can be held at a local minimum of an effective potential.

5.2 Continuum Limit of Multi-Center Solutions

In this section we start looking for extremal stationary solutions to Einstein-Maxwell gravity that admit backgrounds with multipole electromagnetic fields. As before, we work with four dimensional gravity with just one gauge field. Generalizations to $n-1$ vector fields or inclusion of other charges such as D0 and/or D6 in Type II A are rather straightforward. Let us now see how taking a continuum limit of Denef et al's multi-center solutions yields the desired backgrounds. In order to write down harmonic functions for such a smeared distribution of black holes, we define density functions $\rho_e(\vec{x}')$, $\rho_m(\vec{x}')$ via

$$\int_V \rho_e(\vec{x}')d\tau' = Q \qquad \text{and} \qquad \int_V \rho_m(\vec{x}')d\tau' = P \tag{5.2.8}$$

where $d\tau'$ is a volume element within a compact support V, that covers the distribution. In the continuum limit, harmonic functions for multiple black holes take the form

$$\mathcal{Q}(\vec{x}) = v + \int_V \frac{\rho_e(\vec{x}')}{|\vec{x}-\vec{x}'|}d\tau' \qquad \mathcal{P}(\vec{x}) = u + \int_V \frac{\rho_m(\vec{x}')}{|\vec{x}-\vec{x}'|}d\tau' \tag{5.2.9}$$

To these harmonics one may also add linear terms Ez and Bz corresponding to constant fields, whenever required. From a computational point of view, the real utility of the above-mentioned smeared distributions shows up in their respective multipole expansions. Expressing this in the regime that $|\vec{x}| >> |\vec{x}'|$ holds, we have

$$\begin{aligned}
\mathcal{Q}(\vec{x}) &= v + \frac{Q}{|\vec{x}|} + \frac{x_i\Delta_e^i}{|\vec{x}|^3} + \frac{1}{2}\frac{x_i x_j T_e^{ij}}{|\vec{x}|^5} + \cdots\cdots \\
\mathcal{P}(\vec{x}) &= u + \frac{P}{|\vec{x}|} + \frac{x_i\Delta_m^i}{|\vec{x}|^3} + \frac{1}{2}\frac{x_i x_j T_m^{ij}}{|\vec{x}|^5} + \cdots\cdots
\end{aligned} \tag{5.2.10}$$

where Q, P are electric respectively magnetic monopole moments; $\mathbf{\Delta_e}$, $\mathbf{\Delta_m}$ are electric and magnetic dipole moment vectors; and $\mathbf{T_e}$, $\mathbf{T_m}$ are respectively electric and magnetic quadrupole moment tensors - all defined in the usual way. We employ boldface characters to denote vectors as well as tensors. The "$\cdots\cdots$" in eq.(5.2.10) denote terms with higher order moments. When $|\vec{x}| >> |\vec{x}'|$, the series is convergent and these functions can be used to describe supergravity solutions associated to any specific multi-moment source, provided all lower moments vanish for that distribution. As an illustrative example, we analyze the solution for a charge distribution with dipole order corrections.

First let us check that the functions in eq.(5.2.9) yield meaningful expressions for continuum black hole configurations. Evaluating eq.(5.1.4) for these harmonics gives

$$\rho_e(\vec{x})\mathcal{P}(\vec{x}) - \rho_m(\vec{x})\mathcal{Q}(\vec{x}) = 0 \tag{5.2.11}$$

Outside the support V, this expression vanishes identically; whereas points within the support region ought to satisfy

$$u\rho_e(\vec{x}) + \rho_e(\vec{x})\int_V \frac{\rho_m(\vec{x}')}{|\vec{x}-\vec{x}'|}d\tau' - v\rho_m(\vec{x}) - \rho_m(\vec{x})\int_V \frac{\rho_e(\vec{x}')}{|\vec{x}-\vec{x}'|}d\tau' = 0 \tag{5.2.12}$$

After performing the relevant integrals, the above expression can be evaluated for all points $\vec{x} \in V$, and that defines the locus of solutions for the black hole distribution. In following sections, we will solve this condition for specific distribution functions. At the moment though, as a consistency check, let us confirm that, analogous to any multi-center configuration, asymptotically the above continuum configurations also behave like a single-center black hole with total charge P and Q. This can be done by seeing how the constants u and v (which themselves are asymptotically defined) relate to the total monopole charges Q and P, and if this relation is the same as that obtained for a single-center black hole with the same monopole charges. In order to do this we simply integrate both sides of eq.(5.2.12) over all $\vec{x} \in V$. This yields

$$uQ - vP = 0 \tag{5.2.13}$$

which is precisely what one obtains for a single-center solution with charges Q and P; thereby confirming the asymptotic dependence of u and v for an arbitrary continuum configuration having fixed total (monopole) charges Q and P.

Having checked consistency of integrability conditions, we next compute the off-diagonal elements $\vec{\omega}$ in the metric via

$$\nabla \times \vec{\omega} = -\mathcal{P}(\vec{x})\mathbf{E}(\vec{x}) + \mathcal{Q}(\vec{x})\mathbf{B}(\vec{x}) \tag{5.2.14}$$

where $\mathbf{E}(\vec{x})$ and $\mathbf{B}(\vec{x})$ refer to exact electric and magnetic fields corresponding to distributions $\rho_e(\vec{x})$ and $\rho_m(\vec{x})$ respectively. In this sense the continuum limit described here is much simpler than a finite N many body black hole system for which integrability equations turn out to be quite hard to solve in full generality.

For our objectives, it will suffice to solve eq.(5.2.14) using its multipole expansion. As an illustration, we consider a smeared distribution where the monopole contributions to $\vec{\omega}$ get magnetic dipole corrections coming from $\mathbf{\Delta_m}$, which is aligned along the z-axis. In spherical coordinates, eq.(5.2.14) takes the form

$$\begin{aligned}(\nabla \times \vec{\omega})_r &= \frac{2\,v\,\Delta_m\,\cos\theta}{r^3} + \frac{Q\,\Delta_m\,\cos\theta}{r^4} \\ (\nabla \times \vec{\omega})_\theta &= \frac{v\,\Delta_m\,\sin\theta}{r^3} + \frac{Q\,\Delta_m\,\sin\theta}{r^4}\end{aligned} \tag{5.2.15}$$

while $(\nabla \times \vec{\omega})_\phi = 0$ due to symmetry in the ϕ-direction. Note that whilst writing down eq.(5.2.15), we make use of the integrability constraint eq.(5.2.13) (inserting it into eq.(5.2.14)). As before, we seek solutions characterised by azimuthal symmetry. The ensuing result is

$$\omega_\phi = \frac{v\Delta_m \sin\theta}{r^2} + \frac{Q\Delta_m \sin\theta}{2\,r^3} \tag{5.2.16}$$

and $\omega_r = \omega_\theta = 0$. At large distances away from the smeared sources, eq.(5.2.16) gives dipole corrections to leading order contributions in the metric. In fact these constitute sub-leading contributions to the geometry. It is these multipole corrections that distinguish a true one-centered black hole from a multi-center distribution of black holes, when viewed at asymptotic infinity. For a pure one-center solution, $\vec{\omega}$ identically vanishes. While for the multi-center case, it is non-trivial but quite difficult to compute for any given discrete configuration. The continuum limit, on the other hand, facilitates viable computations, at least order by order in a multipole series expansion.

5.3 Towards a Black Hole Levitron

We are now ready to combine results of the last two sections to construct stable levitating black hole solutions and realize a Levitron-like construction. We perturb the constant background fields of section 5.1 with a magnetic dipole field and over this perturbed background solve for a black hole held at a fixed height. The dipole fields are produced by the smeared distribution discussed in section 5.2. For simplicity we consider a black hole with only electric charge q (a dyonic generalization is also straightforward). This construction is captured by the following harmonics

$$\mathcal{Q}(\vec{x}) = v + \frac{q}{|\vec{x} - \vec{l}|} + Ez \quad \mathcal{P}(\vec{x}) = u + \frac{\Delta_m \cos\theta}{|\vec{x}|^2} + Bz \tag{5.3.17}$$

The dipole moment is aligned parallel to the z-axis and carries a magnitude Δ_m. While θ is a coordinate denoting the angle that the position vector $\vec{x}$ makes with the z-axis. Below we shall see, how solving integrability conditions for these harmonics constrains allowed solutions for $|\vec{l}|$ and θ, where a black hole with charge q is held stable in the vicinity of a continuum distribution with dipole charge Δ_m.

For the rest of the computation however, it will suffice to turn off the constant fields E and B. This is because a dipole background will turn out to be sufficient hold the black hole at a fixed height and keep it stable in all three directions. Superposing constant fields do not affect stability of the solution but ultimately we will need the constant fields for giving an interpretation of black hole levitation in a constant gravitational field (as would be the case if we were ever to trap a small black hole in a laboratory somewhere on Earth !).

Continuing with the calculation, the position of the black hole $\vec{l}$ is determined by evaluating eq.(5.1.4) at the location of the pole $\vec{x} = \vec{l}$ using harmonics in eq.(5.3.17) with $E = B = 0$. This gives

$$|\vec{l}| = \sqrt{\frac{-\Delta_m \cos\theta}{u}} \tag{5.3.18}$$

This gives us a locus of solutions $|\vec{l}|, \theta$ for the black hole configuration described in eq.(5.3.17) (with $E = B = 0$). Before discussing further reality constraints on these solutions, let us also evaluate the integrability equation at the other pole $\vec{x} = 0$. This then determines the constant v as

$$v = -\frac{q}{|\vec{l}|} \tag{5.3.19}$$

Note that physical solutions only exist l ($\equiv |\vec{l}|$) real and non-negative and this restricts the values that the angle θ can assume. For instance, let us first consider the case when $u > 0$. Then θ can attain values only from 0 to $\frac{\pi}{2}$ provided the dipole is directed along the negative z-axis, while the ϕ co-ordinate remains unconstrained. On the other hand, for a dipole pointing in the positive z-direction, the angle θ can only span the range

$\frac{\pi}{2}$ to π (as shown in fig. 5.1 below). In the other case, when $u < 0$, then the signs appropriately reverse, namely when the dipole is directed along the negative z-axis, then θ goes from $\frac{\pi}{2}$ to π; whereas with a dipole along the positive z-orientation, θ spans values from 0 to $\frac{\pi}{2}$. The solution space of the black hole is now confined to a restricted parameter space. More precisely these are circular orbits corresponding to given values of θ on an equipotential surface of a dipole field. And in turn each orbit refers to a solution with a specified radial distance l. We plot the solution space for physical values of (l, θ, ϕ) in fig. 5.1 below. The dipole surface in the figure represents locations where a single black hole with a point charge can be stabilized in the gravitational and magnetic field of a continuum black hole distribution centered around the origin and carrying a magnetic dipole moment.

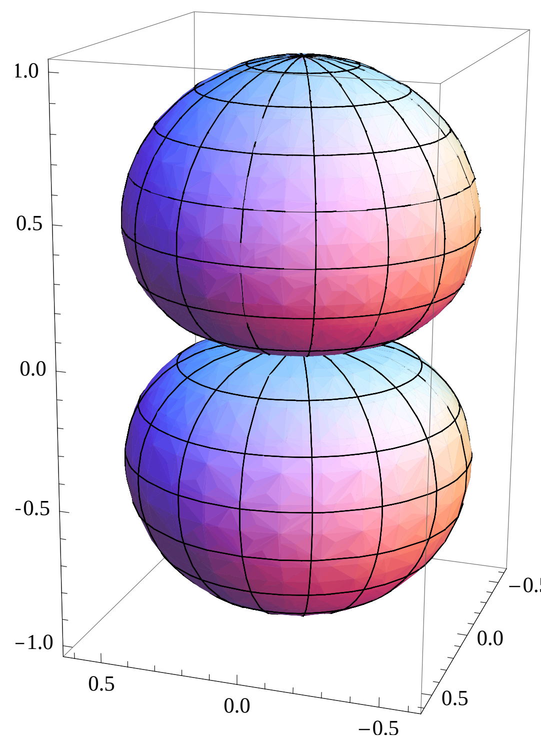


Figure 5.1: Here we make a 3D plot of eq.(5.3.18) for the solution space of $\vec{l}$ for positive as well as negative dipole orientations. Points on the upper globular surface correspond to (l, θ, ϕ) for $\Delta_m < 0$ and $u > 0$. Points on the lower globular surface correspond to those with $\Delta_m > 0$ when $u > 0$.

In fig. 5.1 above, we plot eq.(5.3.18). At $\theta = 0$ the black hole sits at a fixed height on the z-axis; at $\theta = \frac{\pi}{2}$ it falls into the origin; while the case $0 < \theta < \frac{\pi}{2}$ corresponds to the black hole being located anywhere on a circular orbit centered at height $l\cos\theta$ and having radius $l\sin\theta$. Solutions on the positive z-axis correspond to the case when $\Delta_m < 0$ (for $u > 0$), while those on the negative axis refer to $\Delta_m > 0$. For each value of θ in eq.(5.3.18) there exists a solution for $\vec{\omega}$. At $\theta = 0$ the solution space is just a single point and that is when the black hole achieves stability in all three directions at a fixed height on the z-axis.

For completeness we first compute $\vec{\omega}$ when the black hole is still sitting at the origin, that is when $\vec{l} = 0$. After that we shall determine the modification in $\vec{\omega}$ required to achieve stable levitation at a fixed height on the z-axis. In fact the solution at $\vec{l} = 0$. can simply be borrowed from our calculation in eq.(5.2.16) once we make the substitutions $Q \to q$ and $P \to 0$.

On the other hand, when the black hole is made to levitate at a fixed height l on the z-axis we have to solve the following system of equations

$$
\begin{aligned}
(\nabla \times \vec{\omega})_r &= -\frac{q\,u\,(r - l\,\cos\theta)}{(r^2 + l^2 - 2rl\,\cos\theta)^{\frac{3}{2}}} - \frac{2\,q\,\Delta_m\,\cos\theta}{l\,r^3} - \frac{q\,\Delta_m\,\cos\theta\,(r - l\,\cos\theta)}{r^2\,(r^2 + l^2 - 2rl\,\cos\theta)^{\frac{3}{2}}} \\
&\quad + \frac{2\,q\,\Delta_m\,\cos\theta}{r^3\,(r^2 + l^2 - 2\,r\,l\,\cos\theta)^{\frac{1}{2}}} \\
(\nabla \times \vec{\omega})_\theta &= -\frac{q\,u\,l\,\sin\theta}{(r^2 + l^2 - 2rl\,\cos\theta)^{\frac{3}{2}}} - \frac{q\,\Delta_m\,\sin\theta}{l\,r^3} - \frac{q\,l\,\Delta_m\,\sin\theta\,\cos\theta}{r^2\,(r^2 + l^2 - 2rl\,\cos\theta)^{\frac{3}{2}}} \\
&\quad + \frac{q\,\Delta_m\,\sin\theta}{r^3\,(r^2 + l^2 - 2\,r\,l\,\cos\theta)^{\frac{1}{2}}}
\end{aligned}
\tag{5.3.20}
$$

and again $(\nabla \times \vec{\omega})_\phi = 0$. Also $\vec{l} = (0, 0, l)$. This now becomes fairly more complicated compared to the non-levitating case. The modification in the metric reflects a modification to the geometry of the system. If we restrict to azimuthally symmetric cases, we find that eq.(5.3.20) has a solution only for small heights of levitation, that is when $l << r$. This can be understood in the following way. In this set-up the system consists of the black hole plus the source of the dipole field. Let us call the latter the base. The levitating we are looking for requires that the base be rigid against the gravitational pull of the black hole, that is the center of mass of the whole system be as close to the base as possible. For very large charges, corresponding to large values of l, a stable symmetric levitating solution does not seem to exist (we see this from numerical checks). In that case more complicated non-symmetric solutions may be sought for, but we would hardly call those levitating.

Narrowing down to our regime of interest, we expand around $l << r$ and solve eq.(5.3.20) order by order in l. Truncating up to second order terms we get

$$\begin{aligned}\omega_\phi &= -\frac{q\,u\,(1\,-\,\cos\theta)}{r\,\sin\theta} - \frac{q\,\Delta_m\,\sin\theta}{l\,r^2} + \frac{q\,\Delta_m\,\sin\theta}{2\,r^3} - \left\{\frac{q\,u\sin\theta}{r^2}\right\}\cdot l \\ &\quad + \left\{-\frac{3q\,u\cos\theta\,\sin\theta}{2\,r^3} - \frac{q\,\Delta_m\,(1\,+\,3\,\cos^2\theta)\sin\theta}{8\,r^5}\right\}\cdot l^2 + \mathcal{O}(l^3) \quad (5.3.21)\end{aligned}$$

while $\omega_r = \omega_\theta = 0$. This solution enables us to write down the full metric for a stationary system of a black hole levitating in equilibrium above a magnetic dipole field. Also this calculation easily extends to the case of a dyonic black hole.

5.3.1 Comparison to a Levitron

We now compare the levitation of black holes discussed above with that of a *Levitron*[89]. The latter is a spin stabilized magnetic levitation device first invented by Roy Harrigan[88]. It basically consists of a permanent base magnet above which a spinning top with a magnetic dipole moment levitates mid-air and is stable in all three directions. This gives rise to an apparent paradox due to Earnshaw's theorem [90] which states that no stationary configuration composed of electric/magnetic charges and masses can be held in stable equilibrium purely by static forces. And the reason for this is simply that all static potentials satisfy Laplace's equation whose solutions only exhibit saddles at critical points : there are neither any maxima nor minima. It was Sir Michael Berry's [91] (see also [92]) remarkable insight invoking adiabatic averaging that helped resolve the apparent paradox. He showed that a slow precession mode (when averaged over the fast rotation mode) was responsible for creating an effective stationary potential with a stable minimum. This is the same principle used in neutron traps as well as other particles carrying magnetic dipole moment.

A natural question which arises is whether our black hole construction also mimics the physics of the *Levitron* and how it overcomes Earnshaw's theorem. The latter it already seems to evade since it is based on Einstein's gravity rather than Newton's. However the gravitational interpretation of our Black Hole *Levitron's* balancing mechanism admittedly requires further investigation. Nevertheless a naive classical intuition can be obtained from the fact that a non-vanishing Poynting vector gives rise to a rotating black hole geometry and in turn a rotating electric distribution induces a magnetic field that repels the base magnet. It is the $\vec{\omega}$ in the metric that is responsible for inducing this balancing force. On the other hand the gauge theoretic interpretation of this multi-center balancing has been better understood in terms of Denef's quiver quantum mechanics [87] wherein the distance between centers is determined via an effective potential whose minima determine the stability loci $\vec{l}$.

5.4 Conclusions and Discussion

As we have seen from the discussion in earlier chapters, multi-center solutions are also interesting for the role they play in the problem of black hole microstate counting [27]. However even for the simplest configurations with more than two centers, solving integrability constraints to determine the full metric becomes a highly formidable task. Hence, in this chapter, as a curiosity, we asked ourselves the question whether analytic results could be obtained in some limiting cases of these geometries? And indeed we found that such a limit exists in the form of a large n number of centers. In this work we have constructed a continuum distribution of black holes and solved integrability conditions towards obtaining the metric. Upon this continuum system we have performed a multipole expansion to find smeared black hole geometries with multipole moments.

Furthermore, as an interesting application of these continuum solutions, we have constructed a levitating black hole solution. Our Black Hole *Levitron* stabilizes an extremal black hole at a fixed location in an electromagnetic field produced by a continuous distribution. Our work is built-up using Denef et al's multi-center solutions, which by themselves are stable, stationary BPS solutions with non-local charges. Our harmonic functions and integrability conditions can all be retrieved as special limits of the discrete multi-center case. Therefore our levitating solutions also describe stable, stationary configurations. This black hole construction very much resembles a mechanical *Levitron* and it would be interesting to investigate if Berry's mechanism can be proven to apply to this set-up as well. And finally it would be of practical relevance (in future!) to construct solutions for non-extremal Black Hole *Levitrons*!

Other interesting directions might be further investigation into other applications of the continuum limit of multi-center solutions. Compared to discrete-centered configurations, the smeared distribution lends itself to more viable computations. One may ask what role these distributions play in microstate counting of multiple-black hole geometries.

Chapter 6

Testing OSV on a 2D q-Yang-Mills Dual

Life is what happens, while you're busy making other plans

- John Lennon

Finally in this chapter, we arrive at the gauge theoretic side of the OSV conjecture. The world-volume theory on the D-brane ensemble, comprising the black hole bound state, localizes to a two dimensional q-deformed Yang-Mills theory on specific Calabi-Yau backgrounds to be described below. An exactly solvable dual gauge theory can serve as a useful tool for comparing with results in the bulk, thereby enabling a check of the gauge/gravity correspondence itself. Moreover in certain cases, nonperturbative completions of string theory too can be obtained by considering a holographic description in terms of a D-brane gauge theory. In this chapter we conduct a through investigation of this theory and its implications for string theory. In the process we discover a large N phase transition in the theory and also discuss its possible gravitational interpretations.

In the case of topological strings, the OSV [129] proposal for a non-perturbative completion was based on the connection to the black hole attractor mechanism. According to [129], the nonperturbative description of topological string theory on a Calabi-Yau background is encoded in a D-brane gauge theory living on some appropriate cycles of the manifold.

In [132, 94] this proposal was made more concrete by considering Calabi-Yau backgrounds of the form

$$L_1 \oplus L_2 \to \Sigma_g, \tag{6.0.1}$$

where Σ_g is a Riemann surface of genus g and L_1, L_2 are line bundles such that $\deg(L_1)+\deg(L_2) = 2g-2$. In this case, the relevant D-brane gauge theory reduces to a q-deformed version of two-dimensional Yang-Mills (YM) theory on the Riemann surface Σ_g. q-deformed 2d YM can be regarded as a one-parameter deformation of the standard 2d YM theory. As we will explain below, the deformation can be parametrized by a real, positive number p, in such a way that as $p \to \infty$ one recovers the standard YM theory. The q-deformed theory is exactly solvable and one can compute its partition function on any Riemann surface. This partition function has a strong coupling expansion as a sum over representations of the gauge group, which can be written, following [113], in terms of a product of a chiral and an antichiral sector. The perturbative topological string partition function, which was computed in [101] for this class of geometries, is given by a certain limit of this expansion in which the antichiral sector decouples. Once we have a nonperturbative description of the theory, it is natural to ask what new phenomena emerge in this description and what their implications are for string theory. For example, in [107] the fermionic description of 2d YM on the torus was used to study baby universes in string theory.

2d YM theory on the sphere exhibits an interesting phenomenon: as shown by Douglas and Kazakov [109], there is a large N, third order phase transition at a critical value of the area $A = \pi^2$ between a large area phase and a small area phase. From the point of view of the small area/weak coupling phase, the phase transition is triggered by instantons [112]. From the point of view of the large area/strong coupling phase and its string description in terms of branched coverings [111, 113], the transition is triggered by the entropy of branch-point singularities [130]. Due to the Douglas-Kazakov transition, the large area expansion of 2d YM theory on the sphere has a finite radius of convergence [130].

In this project we study the possibility of large N phase transitions in q-deformed 2d YM. Since as the deformation parameter p goes to infinity we recover the usual theory, it is natural to expect the transition to occur at large enough values of p. In fact, our result show that the transition persists for all $p > 2$, and we find a critical line smoothly connected to the Douglas-Kazakov transition of the standard 2d YM theory. We also show that for $p \leq 2$, in the regime of strong q-deformation, the phase transition does not occur. We also perform a detailed instanton analysis which shows that, as in the standard YM case studied in [112], the transition is triggered by instanton effects.

Most of the analysis here is done in the small area phase. In 2d YM theory this phase is described by a Gaussian matrix model. In the q-deformed case, this phase is essentially described by the Chern-Simons or Stieltjes-Wigert matrix model introduced in [125] and studied in [93, 130, 126][1]. This model, albeit complicated, is exactly solvable (in terms of, for example, orthogonal polynomials), and this is the underlying reason that we can make exact statements about the location of the critical line and the instanton contributions. The large area phase turns out to be more difficult to handle. In this paper we present some preliminary results and derive the equations that determine the full solution (including an explicit expression for the two-cut resolvent). We expect the phase transition of the q-deformed theory to be of third order for $p > 2$, since it is smoothly connected to the transition of Douglas and Kazakov, and indeed we give indirect evidence that this is so.

As in the standard 2d YM, the existence of the phase transition in the q-deformed version indicates that the large area expansion has a finite radius of convergence. According to [130, 94], this theory provides a nonperturbative description of topological string theory on certain Calabi-Yau backgrounds. This suggests that the large area expansion breaks down in the full topological string theory, and there is a phase transition between a small area phase and a large area phase. From the gauge theory point of view, our analysis shows that when the q-deformation is strong enough, the model exhibits a single phase. This suggests that q-deformations give a mechanism to smooth out large N phase transitions.

The structure of this chapter is as follows: in section 3.1 we briefly review the Douglas-Kazakov transition in 2d YM theory. In section 3.2 we determine the phase diagram of the q-deformed theory and we find a line of critical points parametrized by p, for $p > 2$. In section 3.3 we adapt the analysis of [112] and study the phase transition of the q-deformed theory in terms of instantons in the weakly coupled phase. We find an explicit expression for the one-instanton suppression factor which indicates that,

[1] Connections between Chern-Simons theory and q-deformed 2d Yang-Mills theory have been made, from a different perspective, in [94] and [116].

indeed, the transition is triggered by instanton effects. In section 3.4 we analyze the large area phase, which can be encoded by standard techniques in a two-cut solution to an auxiliary matrix model. Finally, in section 3.5, we discuss the implications of our results for topological string theory and outline some problems opened by this investigation.

6.1 The Douglas-Kazakov Transition

2d YM theory is an exactly solvable model. In particular, the partition function of the $U(N)$ theory on the sphere is given by a sum over representations of $U(N)$ (see [105] and references therein)

$$Z = \sum_R \left(\dim R\right)^2 e^{-AC_2(R)/2N} e^{i\theta C_1(R)}, \tag{6.1.1}$$

where $\dim R$ is the dimension of the representation R, A is a real and positive parameter that can be identified with the area of the sphere, and $C_1(R)$, $C_2(R)$ are the first and second Casimir of R. We will represent R by a set of integers $\{l_1, l_2, \cdots, l_N\}$ satisfying the inequality

$$\infty \geq l_1 \geq l_2 \geq \cdots \geq l_N \geq -\infty. \tag{6.1.2}$$

In terms of these integers, the Casimirs have the expression

$$\begin{aligned} C_1(R) &= \sum_{i=1}^{N} l_i, \\ C_2(R) &= \sum_{i=1}^{N} l_i(l_i - 2i + N + 1). \end{aligned} \tag{6.1.3}$$

Although the above partition function looks rather simple, this theory turns out to have a very rich structure. In [111, 113] it was shown that at large area the partition function (6.1.1) admits a string representation in terms of branched coverings of Riemann surfaces (see [105] for an excellent review). Douglas and Kazakov found that the planar free energy on the sphere exhibits a third order phase transition at the critical value

$$A_* = \pi^2. \tag{6.1.4}$$

This large N transition is a continuum analogue of the Gross-Witten-Wadia phase transition for 2d YM theory on the lattice [114, 133]. Since in this paper we will be considering a generalization of the Douglas-Kazakov phase transition, we will briefly review how this transition is found. For the rest of this section we will set $\theta = 0$.

At large N it is natural to introduce a distribution of Young tableaux

$$n(x) = \frac{l_i}{N}, \quad x = \frac{i}{N}. \tag{6.1.5}$$

Defining the shifted distribution

$$h(x) = -n(x) + x - \frac{1}{2}, \tag{6.1.6}$$

one finds that the planar free energy is given by

$$F_0(A) = -S_G[h], \tag{6.1.7}$$

where the functional $S_G[h]$ reads

$$S_G[h] = -\int_0^1 dx \int_0^1 dy \log|h(x) - h(y)| + \frac{A}{2}\int_0^1 dx h(x)^2 - \frac{A}{24} - \frac{3}{2}. \tag{6.1.8}$$

Let us now introduce the density function

$$\rho(h) = \frac{dx}{dh}, \tag{6.1.9}$$

which is normalized to unity,

$$\int dh\, \rho(h) = 1. \tag{6.1.10}$$

One crucial observation of [109] is that, because of the inequality (6.1.2), this density has to satisfy

$$\rho(h) \leq 1 \tag{6.1.11}$$

for all h. We can now write (6.1.8) as

$$S_G[\rho] = -\int dh \int dh'\, \rho(h)\rho(h') \log|h - h'| + \frac{A}{2}\int dh \rho(h) h^2 - \frac{A}{24} - \frac{3}{2}. \tag{6.1.12}$$

This is (up to the ρ-independent terms) the saddle-point functional for a Gaussian matrix model with 't Hooft parameter $t = 1/A$. It then follows that the density $\rho(h)$ is given by Wigner's semicircle law,

$$\rho_G(\lambda, t) = \frac{1}{2\pi t}\sqrt{4t - \lambda^2}, \tag{6.1.13}$$

and we find

$$\rho(h) = \rho_G(h, 1/A). \tag{6.1.14}$$

However, it is clear that this solution can be valid only for $A \leq \pi^2$, since after this point the inequality (6.1.11) is violated. This indicates that there is a phase transition at the critical value (6.1.4).

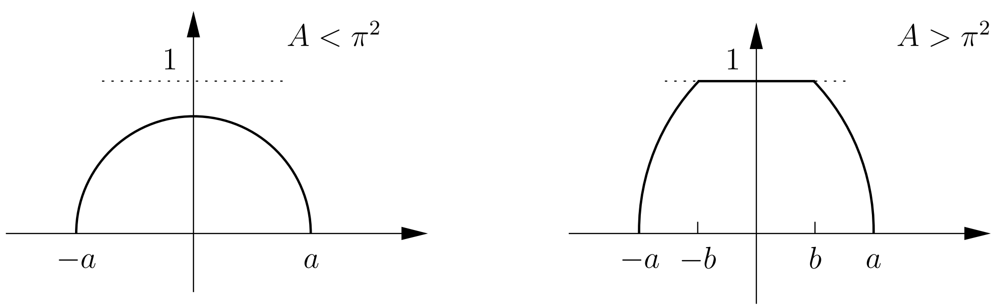


Figure 6.1: This figure shows the density $\rho(h)$ before and after the Douglas-Kazakov transition. The solution for $A \geq \pi^2$ can be interpreted as a two-cut solution of an auxiliary matrix model.

For $A \geq \pi^2$ the Gaussian solution is no longer valid, and Douglas and Kazakov argued that one could obtain a solution for the large area phase by considering a a density of eigenvalues of the form,

$$\rho(h) = \begin{cases} \tilde{\rho}(h), & -a \leq h \leq -b,\ b \leq h \leq a, \\ 1, & -b \leq h \leq b, \end{cases} \tag{6.1.15}$$

where $b < a$ are points in the real positive axis. From the point of view of the density $\rho(h)$, the Douglas-Kazakov transition can be represented as in Fig. 6.1: for $A < \pi^2$ the Gaussian density gives a good description, but as $A \geq \pi^2$ one finds a new density of the form (6.1.15). It is easy to see that finding $\tilde{\rho}(h)$ amounts to finding a two-cut solution for a modified matrix model with a logarithmic potential. The explicit solution to this problem was worked out in [109], and this allowed them to verify that the phase transition at $A = \pi^2$ is of third order. It was also verified that the large area solution agrees with the string expansion of [113].

The mechanism behind the Douglas-Kazakov phase transition was further elucidated in [127, 112, 104]. In particular, it was shown by Gross and Matytsin in [112] that the Douglas-Kazakov phase transition is driven by instantons. The small area phase is dominated by the perturbative vacuum, and instantons are suppressed with an $\exp(-N)$ factor. The one-instanton suppression factor at leading order in N was computed in [112] to be given by

$$\exp\left[-\frac{N}{A}\gamma_{\rm GM}(A)\right], \tag{6.1.16}$$

where

$$\gamma_{\rm GM}(A) = 2\pi\sqrt{\pi^2 - A} - A\log\left[\frac{(\pi + \sqrt{\pi^2 - A})^2}{A}\right]. \tag{6.1.17}$$

Since $\gamma_{\rm GM}(A = \pi^2) = 0$, as we reach the critical point instantons are not anymore suppressed and they trigger the phase transition, which is then a consequence of $\exp(-N)$ effects which are not visible in the $1/N$ expansion.

6.2 The Phase Diagram of q-Deformed 2D YM

The q-deformed two-dimensional Yang-Mills theory arises as a natural deformation of the usual model. This model has been considered in [102, 121] and more recently, in the context of topological string theory, in [94]. The partition function of the q-deformed theory on the sphere can be obtained by replacing the dimensions of representations in (6.1.1) by their quantum counterpart, in the sense of quantum group theory. The resulting partition function depends on the rank N of the gauge group, two real parameters, p, g_s, and an angle θ. It reads,

$$Z^q = \sum_R \left(\dim_q R\right)^2 q^{pC_2(R)/2} e^{i\theta C_1(R)}, \tag{6.2.1}$$

where the quantum dimension of R is given by

$$\dim_q R = \prod_{1\le i<j\le N} \frac{[l_i - l_j + j - i]}{[j-i]}, \tag{6.2.2}$$

and the q-numbers appearing here are defined as

$$[x] = q^{\frac{x}{2}} - q^{-\frac{x}{2}}, \quad q = e^{-g_s}. \tag{6.2.3}$$

The free energy of the model is defined as

$$F^q = \frac{1}{N^2}\log Z^q. \tag{6.2.4}$$

It is convenient to define the parameter A as

$$p\, g_s = \frac{A}{N}. \tag{6.2.5}$$

As we will see in a moment, A corresponds to the area of the sphere in (6.1.1). As in 2d YM, we will require A to be positive. Notice that the q-deformed theory is symmetric under $p, g_s \to -p, -g_s$. Therefore, we can restrict ourselves to the range of parameters $p > 0$, $g_s > 0$.

An important property of the q-deformed theory is that in a suitable double-scaling limit, one recovers ordinary 2D YM. This limit is defined as follows:

$$p \longrightarrow \infty, \qquad g_s \longrightarrow 0, \qquad A,\, N \ \text{fixed}. \tag{6.2.6}$$

As $g_s \to 0$ with N fixed, the quantum dimension becomes the classical dimension:

$$\dim_q R \longrightarrow \dim R, \tag{6.2.7}$$

and

$$q^{pC_2(R)/2} \longrightarrow \exp\left(-\frac{AC_2(R)}{2N}\right), \tag{6.2.8}$$

which is the standard weight factor for 2d YM. We then recover the partition function (6.1.1) for a sphere of area A. The q-deformed theory can then be regarded as a one-parameter deformation of 2d YM.

In this paper we will be interested in the large N dynamics of the deformed theory. It is useful to introduce the 't Hooft parameter, which is defined as

$$t = N g_s, \tag{6.2.9}$$

and we will consider the 't Hooft large N limit in which $N \to \infty$ and t and p are fixed. The planar free energy

$$F_0^q(t,p) = \lim_{N\to\infty} F^q \tag{6.2.10}$$

will then be a function of t and p. Notice that the limit (6.2.6) that gives ordinary Yang-Mills theory can be implemented order by order in the $1/N$ expansion by taking

$$p \longrightarrow \infty, \qquad t \longrightarrow 0, \qquad pt = A \ \text{fixed}. \tag{6.2.11}$$

In this way, we recover planar 2d YM on the sphere. We will check many of our results for the q-deformed theory, by verifying that in the limit (6.2.11) one recovers the known results in 2d YM.

In order to compute the planar free energy, we follow the steps outlined in the previous section for the undeformed theory and represent the planar free energy in terms of a functional of a distribution $h(x)$, which is defined as in (6.1.6). It is easy to see that in the large N limit the planar free energy derived from (6.2.1) is given by

$$F_0^q(t,p) = -S[h], \tag{6.2.12}$$

where the functional $S[h]$ reads

$$\begin{aligned} S[h] = &-\int_0^1 dx \int_0^1 dy \log|2\sinh\frac{t}{2}(h(x)-h(y))| + \frac{pt}{2}\int_0^1 dx h(x)^2 \\ &+ i\theta\int_0^1 dx h(x) - \frac{pt}{24} + \int_0^1 dx\int_0^1 dy \log|2\sinh\frac{t}{2}(x-y)|, \end{aligned} \tag{6.2.13}$$

and in (6.2.12) $S[h]$ is evaluated on the configuration $h(x)$ which minimizes the above functional. The last term in (6.2.13) comes from the denominator of the quantum dimension and it is given by

$$\int_0^1 dx \int_0^1 dy \log|2\sinh\frac{t}{2}(x-y)| = \frac{2}{t^2}F_0^{\rm CS}(t), \tag{6.2.14}$$

where

$$F_0^{\rm CS}(t) = \frac{t^3}{12} - \frac{\pi^2 t}{6} - {\rm Li}_3(e^{-t}) + \zeta(3). \tag{6.2.15}$$

This function is the planar free energy of Chern-Simons theory [110], and we recall that the polylogarithm of order n is given by

$${\rm Li}_n(x) = \sum_{k=1}^{\infty}\frac{x^k}{k^n}. \tag{6.2.16}$$

If we redefine

$$h(x) \to h(x) + \frac{i\theta}{tp}, \tag{6.2.17}$$

the functional (6.2.13) becomes

$$\begin{aligned} S[h] = &-\int_0^1 dx \int_0^1 dy \log|2\sinh\frac{t}{2}(h(x)-h(y))| + \frac{pt}{2}\int_0^1 dx h(x)^2 \\ &-\frac{pt}{24} + \frac{\theta^2}{2pt} + \frac{2}{t^2}F_0^{\rm CS}(t). \end{aligned} \tag{6.2.18}$$

Since the inclusion of θ only leads to an additive term in the planar free energy, we will set $\theta = 0$ from now on. After introducing a density function $\rho(h)$ as in (6.1.9), the ρ-dependent part of the effective action can be written (6.2.13) as

$$S[\rho] = -\int dh \int dh' \, \rho(h)\rho(h') \log|2\sinh \frac{t}{2}(h-h')| + \frac{pt}{2}\int dh \rho(h) h^2. \qquad (6.2.19)$$

As explained in the previous section, to see if there is a phase transition one first solves for the $\rho(h)$ that extremizes (6.2.19), assuming a one-cut structure. In order to compute $\rho(h)$, we have to solve the integral equation derived from (6.2.19),

$$ph = \mathrm{P}\int dh' \rho(h') \coth \frac{t}{2}(h-h'), \qquad (6.2.20)$$

where P denotes principal value. The density $\rho(h)$ is supported on a symmetric interval $(-a, a)$. A similar integral equation appears in the saddle-point analysis of the Chern-Simons matrix model on the three-sphere [125, 93, 126]. In fact, after the change of variables $\beta = th$, (6.2.19) becomes the planar functional for the Chern-Simons matrix model

$$Z_N = \int \prod_{i=1}^{N} \frac{d\beta_i}{2\pi} \prod_{i<j} \Big(2\sinh\frac{\beta_i - \beta_j}{2}\Big)^2 \exp\Big\{-\frac{N}{2\xi}\sum_{i=1}^{N}\beta_i^2\Big\}, \qquad (6.2.21)$$

with 't Hooft parameter $\xi = t/p$. This connection suggests an effective way of solving (6.2.20). As in [93, 131, 126], we change variables

$$\lambda = \mathrm{e}^{th + t/p}, \qquad (6.2.22)$$

and we introduce the density for the new variable λ,

$$\rho(\lambda) = \frac{dh}{d\lambda}\rho(h) = \frac{1}{t\lambda}\rho(h). \qquad (6.2.23)$$

The integral equation (6.2.20) becomes

$$\frac{1}{2}\frac{p}{t}\frac{\log\lambda}{\lambda} = \mathrm{P}\int d\lambda' \frac{\rho(\lambda')}{\lambda - \lambda'}. \qquad (6.2.24)$$

This is exactly the saddle-point equation for the Chern-Simons/Stieltjes-Wigert matrix model, and we can solve it in a variety of ways [93, 115, 126]. The direct computation performed in [126] is the most convenient one in view of the two-cut solution that we will introduce later, so let us briefly review it. As usual, we introduce a resolvent

$$\omega_0(\lambda) = \int d\lambda' \frac{\rho(\lambda')}{\lambda - \lambda'}, \qquad (6.2.25)$$

which due to the normalization (6.1.10) and the redefinition (6.2.23), satisfies the following asymptotic behaviour

$$\omega_0(\lambda) = \frac{1}{\lambda} + \mathcal{O}(\lambda^{-2}), \tag{6.2.26}$$

as $\lambda \to \infty$. The density $\rho(\lambda)$ is recovered from the resolvent $\omega_0(\lambda)$ through the standard equation

$$\rho(\lambda) = -\frac{1}{2\pi i}\big(\omega_0(\lambda + i\epsilon) - \omega_0(\lambda - i\epsilon)\big). \tag{6.2.27}$$

We are looking for a one-cut solution to the problem, therefore we assume that the density of eigenvalues is supported in the interval (a^-, a^+), where

$$a^{\pm} = e^{\pm ta + t/p}. \tag{6.2.28}$$

It is well known that $\omega_0(\lambda)$ can be computed as [128]

$$\omega_0(\lambda) = r(\lambda) \oint_{\mathcal{C}} \frac{dz}{2\pi i} \frac{g(z)}{(\lambda - z) r(z)}, \tag{6.2.29}$$

where $\mathcal{C}$ is a contour around the cut (a^-, a^+), and

$$g(\lambda) = \frac{p}{2t}\frac{\log \lambda}{\lambda}, \quad r(\lambda) = \sqrt{(\lambda - a^-)(\lambda - a^+)}. \tag{6.2.30}$$

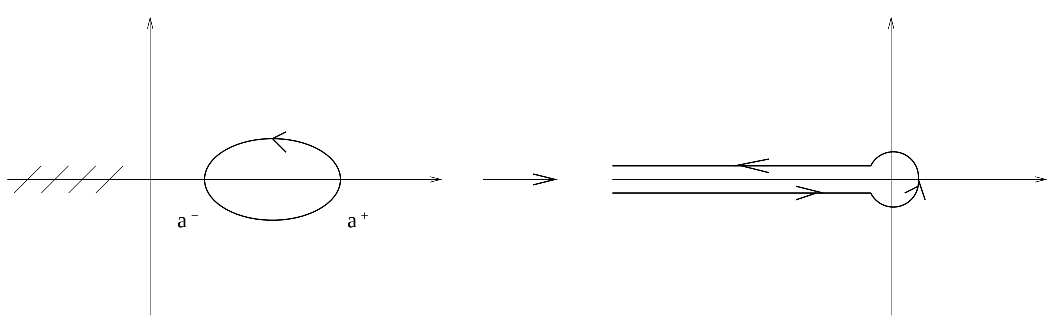


Figure 6.2: This figure shows the deformation of the contour needed to compute the resolvent in (6.2.29). We pick a residue at $z = p$, and we have to encircle the singularity at the origin as well as the branch cut of the logarithm, which on the left hand side is represented by the dashed lines.

The standard way to compute an integral like (6.2.29) is to deform the contour. Since the logarithm has a branch cut, we cannot push the contour to infinity. Instead, we deform the contour as indicated in Fig. 6.2. We pick a pole at $z = \lambda$, and then we surround the cut of the logarithm along the negative real axis and the singularity at $z = 0$ with a small circle C_ϵ of radius ϵ. A similar situation appears in, for example, [119]. The final formula for the resolvent is

$$\omega_0(\lambda) = \frac{p}{2t}\frac{\log\lambda}{\lambda} - \frac{p}{2t}r(\lambda)\lim_{\epsilon\to 0}\left\{-\int_{-\infty}^{-\epsilon}\frac{dz}{z(z-\lambda)r(z)} + \oint_{C_\epsilon}\frac{dz}{2\pi i}\frac{\log z}{z(z-\lambda)r(z)}\right\}. \quad (6.2.31)$$

The integrals in the second line have $\log\epsilon$ singularities as $\epsilon \to 0$, but they cancel each other, and after some computations one finds for the resolvent:

$$\begin{aligned}\omega_0(\lambda) =& -\frac{p}{2t\lambda}\log\left[\frac{(\sqrt{a^-}\sqrt{\lambda-a^+}-\sqrt{a^+}\sqrt{\lambda-a^-})^2}{(\sqrt{\lambda-a^-}-\sqrt{\lambda-a^+})^2\lambda^2}\right] \\ &+\frac{p}{2t\lambda}r(\lambda)\frac{1}{\sqrt{a^-a^+}}\log\left[\frac{4a^-a^+}{2\sqrt{a^-a^+}+a^-+a^+}\right].\end{aligned} \quad (6.2.32)$$

In order to satisfy the asymptotics (6.2.26) the second term must vanish, and the first one must go like $1/\lambda$. This implies

$$\begin{aligned}4a^-a^+ =& 2\sqrt{a^-a^+}+a^-+a^+, \\ \sqrt{a^-}+\sqrt{a^+} =& 2e^{t/p},\end{aligned} \quad (6.2.33)$$

and from here we obtain the positions of the endpoints of the cut a^-, a^+ as a function of t/p:

$$\begin{aligned}a^- =& 2e^{2t/p} - e^{t/p} - 2e^{\frac{3t}{2p}}\sqrt{e^{t/p}-1}, \\ a^+ =& 2e^{2t/p} - e^{t/p} + 2e^{\frac{3t}{2p}}\sqrt{e^{t/p}-1}.\end{aligned} \quad (6.2.34)$$

The final expression for the resolvent is then

$$\omega_0(\lambda) = -\frac{p}{t\lambda}\log\left[\frac{1+e^{-t/p}\lambda+\sqrt{(1+e^{-t/p}\lambda)^2-4\lambda}}{2\lambda}\right], \quad (6.2.35)$$

and from here we easily find the density of eigenvalues

$$\rho(\lambda) = \frac{p}{\pi t\lambda}\tan^{-1}\left[\frac{\sqrt{4\lambda-(1+e^{-t/p}\lambda)^2}}{1+e^{-t/p}\lambda}\right]. \quad (6.2.36)$$

We can now go back to the original variable h, to find

$$\rho(h) = \frac{p}{\pi}\tan^{-1}\left[\frac{\sqrt{e^{A/p^2}-\cosh^2(Ah/(2p))}}{\cosh(Ah/(2p))}\right], \quad (6.2.37)$$

which has its support on $(-a, a)$ with

$$a = \frac{2p}{A} \cosh^{-1}(e^{A/(2p^2)}). \tag{6.2.38}$$

As a test of this result, notice that in the double-scaling limit (6.2.11) one finds

$$\rho(h) = \rho_G(h, 1/A) + \mathcal{O}(1/p^2), \tag{6.2.39}$$

therefore the leading term is exactly the Wigner semi-circle distribution obtained by [109].

In order to assess the possibility of phase transitions, we have to verify the condition (6.1.11). Notice first that $|\tan^{-1}(x)| \leq \frac{\pi}{2}$, therefore

$$\rho(h) \leq p/2 \tag{6.2.40}$$

for all h. A first conclusion is that *there is no phase transition for* $p \leq 2$. For $p > 2$ there is indeed a phase transition which occurs when the value of A is such that the maximum of the distribution reaches the value 1. Since the maximum occurs at $h = 0$, we immediately find the following line of critical points:

$$A_*(p) = p^2 \log\left(1 + \tan^2\left(\frac{\pi}{p}\right)\right), \quad p > 2. \tag{6.2.41}$$

As $p \to \infty$,

$$A_*(p) \to \pi^2, \tag{6.2.42}$$

in agreement with the result of Douglas and Kazakov (6.1.4). Notice that $A_*(p)$ is a decreasing function of p for $p > 2$, and as $p \to 2^+$, the critical area increases to infinity. For a given p, the small area phase occurs for $A \leq A_*(p)$, and in this phase the planar free energy is well described by the distribution (6.2.37).

We then have the phase diagram represented in Fig. 6.3. The horizontal axis represents the parameter p, while the vertical axis represents A. The critical line, described by the function (6.2.41), has two asymptotes, represented by dashed lines: as $p \to \infty$ it approaches the horizontal dashed line $A = \pi^2$, which corresponds to the Douglas-Kazakov phase transition. As $p \to 2^+$ it approaches the vertical asymptote. For $p \in (0, 2]$ there is no phase transition. Notice that, if we parametrize the planar q-deformed theory in terms of p and A, the region $p \to \infty$ corresponds to a small

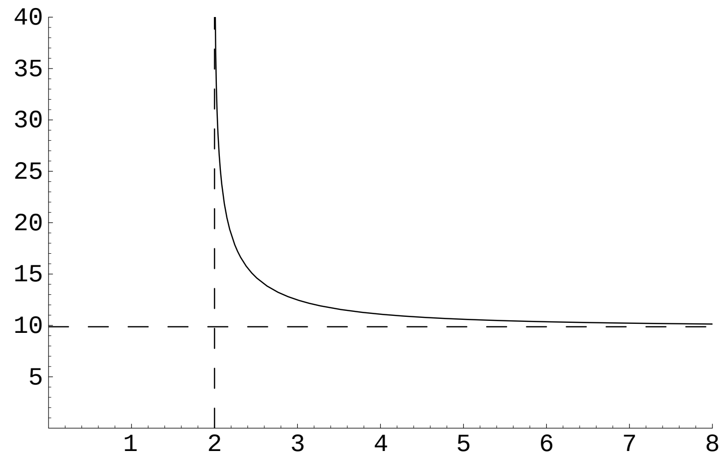


Figure 6.3: This figure represents the phase diagram of q-deformed 2d YM theory. The horizontal axis represents the parameter p, while the vertical axis represents A. The curve shown in the figure is the critical line (6.2.41), which separates the phases of small and large area. The horizontal dashed line, which is the asymptote of the curve as $p \to \infty$, represents the $A = \pi^2$ critical point of Douglas and Kazakov.

deformation, while the region $p < 2$ corresponds to a large deformation. We then see that, if we start with ordinary 2d YM and we turn on the deformation parameter $1/p$, the Douglas-Kazakov phase transition persists although the critical area increases. At $p = 2$ there is a "barrier" where the critical area becomes infinite. Therefore, when the deformation parameter is large enough, the large N phase transition is smoothed out.

To find the free energy in the small area phase, we have to compute the functional (6.2.19) evaluated on the density (6.2.37). Since this functional is closely related to the functional describing the planar Chern-Simons matrix model, we can borrow the results from [110, 126]. From [126] it follows that, at large N, the matrix integral (6.2.21) is given by

$$\exp\Big(N^2 F_0(\xi)\Big), \tag{6.2.43}$$

with

$$F_0(\xi) = \frac{1}{\xi^2} F_0^{\rm CS}(\xi) + \frac{\xi}{12}. \tag{6.2.44}$$

Since $\xi = t/p$ in our example, we finally obtain

$$F_0^q(t,p) = \frac{1}{t^2}\Big(p^2 F_0^{\rm CS}(t/p) - 2F_0^{\rm CS}(t)\Big) + \frac{t}{12p} + \frac{pt}{24}. \tag{6.2.45}$$

As a further check of this expression, notice that, after using the expansion,

$$\mathrm{Li}_3(e^{-t}) = \zeta(3) - \frac{\pi^2}{6}t + \Big(\frac{3}{4} - \frac{1}{2}\log\, t\Big)t^2 + \mathcal{O}(t^3), \tag{6.2.46}$$

one finds in the double-scaling limit (6.2.11)

$$F_0^q(t,p) \to \frac{A}{24} - \frac{1}{2}\log\, A + \frac{3}{4}, \tag{6.2.47}$$

which indeed is the free energy of the usual 2d YM theory in the small area phase.

6.3 Instanton Analysis

Since q-deformed 2d YM theory is a one-parameter deformation of the standard one, we expect the phase transition discovered in the previous section to be triggered by instantons as well. In this section we will verify this by computing the one-instanton suppression factor in the q-deformed case. This will also give an intuitive explanation of why the phase transition is absent for $p \leq 2$.

The starting point of the discussion is to write the partition function of the theory in a way that makes manifest the instanton content of the model. Since q-deformed 2d YM theory has the same action as standard 2d YM, but differs in the measure [94], we expect the partition function to be expressed in terms of a sum over instantons,

$$Z^q = \sum_{n_j} w(n_j) \exp\Big(-\frac{2\pi^2 N}{A}\sum_{j=1}^{N} n_j^2\Big), \tag{6.3.1}$$

where n_j, $j = 1, \cdots, N$, are the instanton numbers characterizing a classical solution [112], and $w(n_i)$ is the weight of such a configuration in the semiclassical expansion. In order to compute the weights $w(n_j)$, we follow the technique used by Minahan and Polychronakos [127] in standard 2d YM and perform a Poisson resummation of the original expression (6.2.1). This can be regarded as a duality transformation which takes us from the large A phase where the expansion (6.2.1) is valid, to the small area phase where the semiclassical expansion (6.3.1) is valid. The partition function can then be written as

$$Z^q = C\sum_{n_j} F_2(2\pi n_j), \tag{6.3.2}$$

where $F_2(x_j)$ is a Fourier transform with respect to the variables $p_j = l_j - j + 1/2$:

$$F_2(x_j) = \int \prod_j dp_j e^{-i\sum_j x_j p_j} \prod_{j<k} \Big(2\sinh\frac{t}{N}(p_j - p_k)\Big)^2 \exp\Big(-\frac{A}{2N}\sum_j p_j^2\Big), \tag{6.3.3}$$

and we are setting the θ angle to zero. This transform can be performed by first computing

$$F_1(x_j) = \int \prod_j dp_j e^{-i\sum_j x_j p_j} \prod_{j<k} \Big(2\sinh\frac{t}{N}(p_j - p_k)\Big) \exp\Big(-\frac{A}{2N}\sum_j p_j^2\Big), \tag{6.3.4}$$

and then doing a convolution. The integral (6.3.4) reduces to a Gaussian after using Weyl's denominator formula for a general Lie algebra,

$$\sum_{w\in\mathcal{W}} \epsilon(w) e^{w(\rho)\cdot u} = \prod_{\alpha>0} 2\sinh\frac{\alpha\cdot u}{2}, \tag{6.3.5}$$

where α are the positive roots, $w \in \mathcal{W}$ are the elements of the Weyl group, and $\epsilon(w)$ is the parity of w. We find, up to a multiplicative constant,

$$F_1(x_j) = \exp\Big(-\frac{N}{2A}\sum_{j=1}^N x_j^2\Big) \prod_{j<k} 2\sin\frac{t}{2A}(x_j - x_k), \tag{6.3.6}$$

and using convolution we finally obtain

$$\begin{aligned} F_2(x_j) = &\exp\Big(-\frac{N}{2A}\sum_{j=1}^N x_j^2\Big) \\ &\times \int \prod_{j=1}^N dy_j \prod_{j<k} \Big(4\sin\frac{t}{2A}(x_{jk} + y_{jk}) \sin\frac{t}{2A}(x_{jk} - y_{jk})\Big) \exp\Big(-\frac{N}{2A}\sum_{j=1}^N y_j^2\Big), \end{aligned} \tag{6.3.7}$$

where we introduced the notation $x_{jk} = x_j - x_k$. The instanton weight has then the expression

$$w(n_j) = \int \prod_{j=1}^N dy_j \prod_{j<k} \Big(4\sin\frac{t}{2A}(2\pi n_{jk} + y_{jk}) \sin\frac{t}{2A}(2\pi n_{jk} - y_{jk})\Big) \exp\Big(-\frac{N}{2A}\sum_{j=1}^N y_j^2\Big), \tag{6.3.8}$$

which is a q-deformed version of the result in [127] for standard 2d YM.

As it was pointed out in [112], a precise way to evaluate the importance of instanton contributions to the partition function is to compare the contribution of the one-instanton term in the semiclassical expansion (6.3.1) to the contribution of the perturbative vacuum. The relative weight of these contributions defines a function $\gamma(A,p)$ as follows

$$\exp\Big[-\frac{N}{A}\gamma(A,p)\Big] = \exp\Big(-N\frac{2\pi^2}{A}\Big)\frac{w_1}{w_0}, \qquad (6.3.9)$$

where the exponent in the right hand side involves the instanton action for $n_1 = 1, n_{i>1} = 0$, and we have denoted

$$\frac{w_1}{w_0} = \frac{w(1,0,\cdots,0)}{w(0,\cdots,0)}. \qquad (6.3.10)$$

We call the function in (6.3.9) the one-instanton suppression factor. Notice that, as long as $\gamma(A,p)$ is different from zero, instantons will be suppressed in the large N limit. The suppression is bigger the larger $\gamma(A,p)$ is. In the remaining of this section, we will compute $\gamma(A,p)$ in the small area phase of q-deformed 2d YM, and we will study its properties.

Let us first define the partition function

$$Z_N = \int \prod_{j=1}^{N} dy_j \prod_{j<k}\Big(2\sin\frac{t}{2A}(y_j - y_k)\Big)^2 \exp\Big(-\frac{N}{2A}\sum_{j=1}^{N} y_j^2\Big). \qquad (6.3.11)$$

This is very close to the partition function of the Chern-Simons matrix model, although it has a sin interaction between eigenvalues instead of a sinh interaction. We can then use the results of the previous section after changing

$$p \to -i\frac{p}{A}, \quad A \to \frac{1}{A}, \qquad (6.3.12)$$

and doing carefully the analytic continuation of p to the imaginary axis. Equivalently, we can change variables $y = -iA\beta/t$ in (6.3.11) to obtain the matrix model (6.2.21) with $\xi = -A/p^2$. One can then see from the formulae presented in the last section that the planar limit of (6.3.11) is controlled by the following density of eigenvalues,

$$\zeta(y) = \frac{p}{\pi A}\tanh^{-1}\Bigg[\frac{\sqrt{\cos^2(y/(2p)) - e^{-A/p^2}}}{\cos(y/(2p))}\Bigg], \qquad (6.3.13)$$

with endpoints located at

$$Y = 2p\cos^{-1}(e^{-A/(2p^2)}). \qquad (6.3.14)$$

As $p \to \infty$, one can easily check that $\zeta(y) \to \rho_G(y,A)$.

We can now evaluate (6.3.10). Notice first that $w_0 = Z_N$. On the other hand, as in [112], one has

$$w_1 = Z_{N-1}\int_{-\infty}^{\infty} dy_1 e^{-\frac{N}{2A}y_1^2}\Big\langle \prod_{j=2}^{N}\Big(4\sin\frac{1}{2p}\big(2\pi+(y_j-y_1)\big)\sin\frac{1}{2p}\big(2\pi-(y_j-y_1)\big)\Big)\Big\rangle_{N-1}, \tag{6.3.15}$$

where the correlator is computed in the model (6.3.11) with $N-1$ variables. Since we are interested in the large N behavior of the one-instanton suppression factor, we can compute the different integrals in the saddle-point approximation. This in particular means that we can set $y_1 = 0$ inside the correlator in (6.3.15). We find,

$$\frac{w_1}{w_0} = \left(\frac{2\pi A}{N}\right)^{1/2}\frac{Z_{N-1}}{Z_N}\exp\Big\{(N-1)\int dy\zeta(y)\log\Big(4\sin\frac{1}{2p}(2\pi+y)\sin\frac{1}{2p}(2\pi-y)\Big)\Big\}. \tag{6.3.16}$$

We have first to evaluate the quotient Z_{N-1}/Z_N in the large N limit. It is easy to see that, at leading order in N, this quotient is

$$\exp\Big\{-N(2F_0(\xi)+\xi F_0'(\xi))\Big\}, \tag{6.3.17}$$

where $F_0(\xi)$ is given in (6.2.44). Here, $\xi = -A/p^2$, and after an analytic continuation $\xi \to -\xi$ we find,

$$2F_0(\xi)+\xi F_0'(\xi) = \frac{p^2}{A}\Big(\mathrm{Li}_2(e^{-A/p^2}) - \frac{\pi^2}{6}\Big), \tag{6.3.18}$$

up to an overall sign $(-1)^N$ in Z_{N-1}/Z_N. Putting everything together, we obtain the following formula for the function $\gamma(A,p)$ defined in (6.3.9):

$$\begin{aligned}\gamma(A,p) =& 2\pi^2 + p^2\Big(\mathrm{Li}_2(e^{-A/p^2}) - \frac{\pi^2}{6}\Big)\\ &- A\int dy\zeta(y)\log\Big(4\sin\frac{1}{2p}(2\pi+y)\sin\frac{1}{2p}(2\pi-y)\Big).\end{aligned} \tag{6.3.19}$$

The integral in (6.3.19) can be evaluated analytically. Notice first that in any matrix model one has

$$F(v) \equiv \int d\lambda\rho(\lambda)\log\big(1-\lambda/v\big) = \int_{\infty}^{v} dv'\big(\omega_0(v')-1/v'\big). \tag{6.3.20}$$

This follows directly from the definition of the resolvent in (6.2.25). Taking into account the redefinition (6.3.12), we find that the integral in (6.3.19) is given by

$$2\mathrm{Re}\,F(e^{-A/p^2+2\pi i/p}), \tag{6.3.21}$$

where $F(v)$ is obtained as in (6.3.20), and the relevant resolvent is (6.2.35). After some work, and using standard identities for the dilogarithm, one finds the following expression:

$$\gamma(A,p) = 2\pi^2 - p^2\Big(\mathrm{Li}_2(e^{-A/p^2}) + \frac{\pi^2}{6}\Big) + 2p^2 \mathrm{Re}\,\mathcal{G}(f_+(p,A), f_-(p,A)), \qquad (6.3.22)$$

where

$$\begin{aligned} \mathcal{G}(x,y) &= \frac{1}{2}(\log\, x)^2 + \log\, x \log\,(1-y) + \mathrm{Li}_2(1-x) + \mathrm{Li}_2(y), \\ f_\pm(p,A) &= \exp\Big(\pm A/(2p^2) + i(\varphi - \pi/p)\Big), \\ \varphi &= \tan^{-1}\left(\frac{\sqrt{e^{-A/p^2} - \cos^2(\pi/p)}}{\cos(\pi/p)}\right). \end{aligned} \qquad (6.3.23)$$

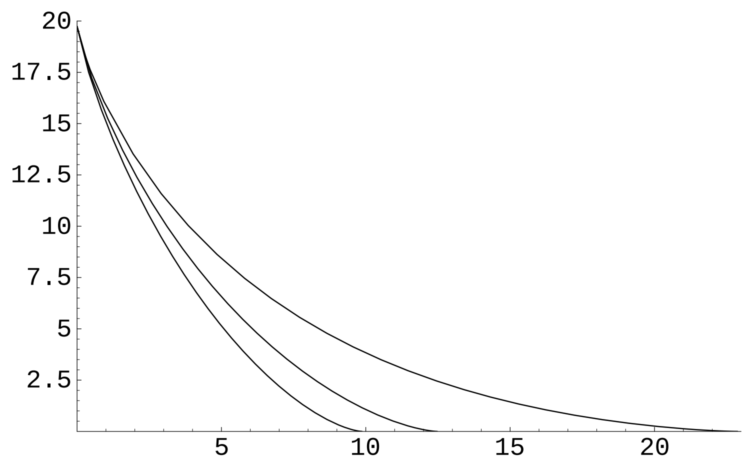


Figure 6.4: This figure shows the function $\gamma(A,p)$ appearing in the one-instanton suppression factor, plotted as a function of A, and for the values $p = 2.1, 3, \infty$, from top to bottom. For each p it is a decreasing function of the area and vanishes at the critical value $A_*(p)$.

In order to understand the properties of the instanton suppression factor, we have studied (analytically and numerically) the properties of (6.3.22) as a function of A and p for $p > 2$, $A \leq A_*(p)$. The main results of this analysis are the following:

1. As $p \to \infty$, the function $\gamma(A,p)$ becomes the function $\gamma_{\mathrm{GM}}(A)$ introduced in (6.1.17). This is a consistency check of the solution.

2. For any fixed $p > 2$, $\gamma(A,p)$ takes the value $2\pi^2$ at $A = 0$ and then it decreases

monotonically as the area is increased. At the critical area (6.2.41) one has

$$\gamma(A_*(p), p) = 0. \tag{6.3.24}$$

The vanishing of $\gamma(A, p)$ at the critical area can be proved analytically, since at $A = A_*(p)$,

$$f_\pm(p, A_*(p)) = \Big(\cos(\pi/p)\Big)^{\mp 1} e^{-i\pi/p}. \tag{6.3.25}$$

For arguments of this form (which are algebraic numbers) the dilogarithm satisfies nontrivial identities [120] that can be easily shown to lead to (6.3.24).

3. For $p < p'$, one has that $\gamma(A, p) > \gamma(A, p')$ in their common range $A \leq A_*(p')$.

These properties are illustrated in Fig. 6.4, which shows the function $\gamma(A, p)$ as a function of the area for the values $p = 2.1, 3, \infty$, from top to bottom. The above properties show that the one-instanton suppression factor in the small area phase decreases as the area grows, until it vanishes at $A_*(p)$. Therefore, at the line of critical points found in section 3, the instantons are not suppressed anymore and they become favorable configurations. This shows that the phase transition for the q-deformed theory is indeed triggered by instantons, and follows a mechanism similar to the one studied in [112]: for $A > A_*(p)$, the entropy of the instantons dominates over their Boltzmann weight. The above analysis also shows that, as p decreases, the instanton suppression factor becomes bigger and bigger, pushing the critical value of the area to ever larger values. This indicates that the smoothing out of the phase transition for $p \leq 2$ is due to the fact that the instantons are suppressed for all values of A and we only have one phase dominated by the perturbative vacuum $n_i = 0$.

6.4 The Two-Cut Solution

In this section we give some preliminary results about the large area phase of the theory. After the phase transition found in section 3, we expect a distribution $\rho(h)$ a la Douglas-Kazakov, with the shape shown in the r.h.s. of Fig. 6.1 and characterized by two points $\hat{a}, \hat{b}$. The distribution governing the large area distribution is then of the form

$$\rho(h) = \begin{cases} \tilde{\rho}(h), & -\hat{a} \le h \le -\hat{b},\ \hat{b} \le h \le \hat{a}, \\ 1, & -\hat{b} \le h \le \hat{b}. \end{cases} \tag{6.4.1}$$

After changing variables $\lambda = \exp(th + t/p)$ as in the previous section, the new density of eigenvalues $\tilde{\rho}(\lambda) = 1/(t\lambda)\tilde{\rho}(h)$ has support on the two intervals (a^-, b^-), (b^+, a^+), where

$$a^\pm = e^{t/p \pm t\hat{a}}, \quad b^\pm = e^{t/p \pm t\hat{b}}. \tag{6.4.2}$$

This density satisfies the following integral equation,

$$g(\lambda) \equiv \frac{p}{2t}\frac{\log\lambda}{\lambda} + \frac{1}{t\lambda}\log\frac{\lambda/b^+ - 1}{\lambda/b^- - 1} = \mathrm{P}\int \frac{\tilde{\rho}(\lambda')}{\lambda - \lambda'}d\lambda' \tag{6.4.3}$$

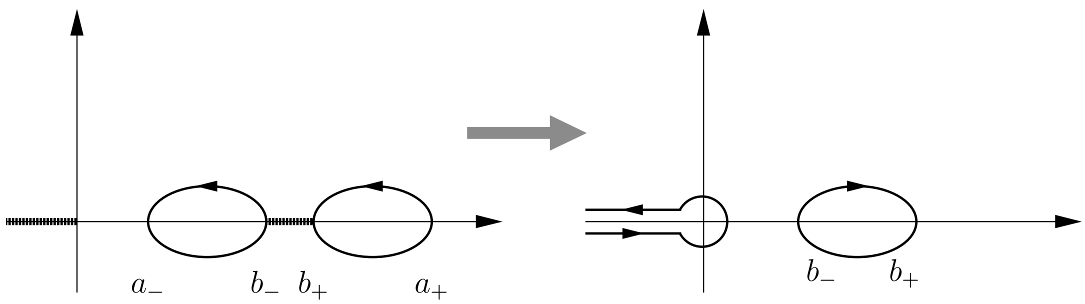


Figure 6.5: This figure shows the deformation of the contour needed to compute the resolvent in the two-cut solution. We have to encircle the singularity at the origin, and the two branch cuts denoted by thick lines on the left.

As in the one-cut case, we introduce a resolvent

$$\tilde{\omega}_0(\lambda) = \int \frac{\tilde{\rho}(\lambda')}{\lambda - \lambda'}d\lambda'. \tag{6.4.4}$$

This can be again computed by the contour integral (6.2.29), but now

$$r(z) = \sqrt{(z - a^-)(z - a^+)(z - b^-)(z - b^+)}. \tag{6.4.5}$$

and $\mathcal{C}$ is the union of the contours surrounding the cuts (a^-, b^-), (a^+, b^+). To perform the integral (6.2.29) we deform the contours in the way shown in Fig. 6.5: we now encircle the branch cut along $(-\infty, 0)$, coming from $\log\lambda$, and the branch cut of the integrand along (b^-, b^+). The answer for the resolvent is

$$\begin{aligned}\tilde{\omega}_0(\lambda) =& \frac{p}{2t}\frac{\log\lambda}{\lambda} + \frac{1}{t\lambda}\log\frac{\lambda/b^+ - 1}{\lambda/b^- - 1} \\ &- r(\lambda)\frac{p}{2t}\lim_{\epsilon\to 0}\left\{-\int_{-\infty}^{-\epsilon}\frac{dz}{z(z-\lambda)r(z)} + \oint_{C_\epsilon}\frac{dz}{2\pi i}\frac{\log z}{z(z-\lambda)r(z)}\right\} \\ &+ \frac{r(\lambda)}{t}\int_{b^-}^{b^+}\frac{dz}{z(z-\lambda)r(z)}.\end{aligned} \tag{6.4.6}$$

The above integrals can be expressed in terms of elliptic functions. We will change notation a^-, b^-, b^+, a^+ to d, c, b, a. Define

$$\begin{aligned}I(\lambda, u) &\equiv \int_u^d \frac{dz}{(z-\lambda)r(z)} \\ &= \frac{2}{(\lambda-c)(\lambda-d)\sqrt{(a-c)(b-d)}}\Big\{(c-d)\Pi(\phi, n, k) + (d-\lambda)F(\phi, k)\Big\},\end{aligned} \tag{6.4.7}$$

where $\Pi(\phi, n, k)$ and $F(\phi, k)$ are incomplete elliptic integrals of the third and the first kind, respectively, and

$$\sin^2\phi = \frac{(a-c)(d-u)}{(a-d)(c-u)}, \quad n = \frac{(a-d)(\lambda-c)}{(a-c)(\lambda-d)}, \quad k^2 = \frac{(b-c)(a-d)}{(a-c)(b-d)}. \tag{6.4.8}$$

In what follows it will be convenient to introduce the following angles ϕ_1, ϕ_2 and variables n_1 and n_2:

$$\begin{aligned}\sin^2\phi_1 =& \frac{a-c}{a-d}, \quad \sin^2\phi_2 = \frac{d}{c}\frac{a-c}{a-d}, \\ n_1 =& \frac{a-d}{a-c}, \quad n_2 = \frac{c}{d}\frac{a-d}{a-c}.\end{aligned} \tag{6.4.9}$$

In terms of these variables one finds,

$$\begin{aligned}I(\lambda, -\infty) =& \frac{2}{(\lambda-c)(\lambda-d)\sqrt{(a-c)(b-d)}}\Big\{(c-d)\Pi(\phi_1, n, k) + (d-\lambda)F(\phi_1, k)\Big\}, \\ I(\lambda, 0) =& \frac{2}{(\lambda-c)(\lambda-d)\sqrt{(a-c)(b-d)}}\Big\{(c-d)\Pi(\phi_2, n, k) + (d-\lambda)F(\phi_2, k)\Big\}, \\ I(0, -\infty) =& \frac{2}{cd\sqrt{(a-c)(b-d)}}\Big\{(c-d)\Pi(\phi_1, n_2, k) + dF(\phi_1, k)\Big\}.\end{aligned} \tag{6.4.10}$$

The first integral in the second line of (6.4.6) is given by

$$\frac{1}{\lambda}\Big(I(\lambda, -\infty) - I(\lambda, 0) - I(0, -\infty) + I(0, -\epsilon)\Big). \tag{6.4.11}$$

The second integral in the second line is simply a residue and it can be computed immediately:

$$-\frac{1}{\lambda}\frac{1}{\sqrt{abcd}}\log\epsilon. \tag{6.4.12}$$

We now compute $I(0,-\epsilon)$ at next-to-leading order in ϵ. This will have a logarithmic singularity which will cancel (6.4.12). In order to do that, we need the following identity [100]:

$$\Pi(\phi,n,k)=\delta(n)\left\{\frac{1}{2}\log\frac{\vartheta_1(v+\beta)}{\vartheta_1(v-\beta)}-\frac{\vartheta_4'(\beta)}{\vartheta_4(\beta)}v\right\}, \tag{6.4.13}$$

where

$$\delta(n)=\left(\frac{n}{(1-n)(k^2-n)}\right)^{\frac{1}{2}},\quad v=\frac{F(\phi,k)}{2K(k)},\quad \beta=\frac{F(\sin^{-1}(n^{-\frac{1}{2}}),k)}{2K(k)}, \tag{6.4.14}$$

and the τ parameter in the theta functions is given as usual by

$$q=e^{2\pi i\tau}=\exp(-\pi K'(k)/K(k)). \tag{6.4.15}$$

Notice that, when

$$\sin^2\phi=\frac{1}{n} \tag{6.4.16}$$

we have a logarithmic singularity in the elliptic integral $\Pi(\phi,n,k)$. This is immediately checked in the integral representation of the elliptic function[2]. We can now use (6.4.13) to extract the next-to-leading behavior. Since

$$\sin^2\phi=\frac{a-c}{a-d}\frac{d+\epsilon}{c+\epsilon},\quad n=n_2, \tag{6.4.17}$$

the leading behaviour of $\Pi(\phi,k,n_2)$ is given by

$$\begin{aligned}&\delta(n_2)\left(-\beta_2\frac{\vartheta_4'(\beta_2)}{\vartheta_4(\beta_2)}+\frac{1}{2}\log\frac{\vartheta_1(2\beta_2)}{\vartheta_1'(0)}-\frac{1}{2}\log\left(\frac{c-d}{4cd}\delta(n_2)\right)+\frac{1}{2}\log\,K(k)\right)\\&-\frac{\delta(n_2)}{2}\log\epsilon+\mathcal{O}(\epsilon),\end{aligned} \tag{6.4.18}$$

where β_2 is given by

$$\beta_2=\frac{F(\phi_2,k)}{2K(k)}. \tag{6.4.19}$$

This leads to the following expression

$$I(0,-\epsilon)=-\frac{1}{\sqrt{abcd}}\log\epsilon+I(0,0)+\mathcal{O}(\epsilon), \tag{6.4.20}$$

[2]Notice that in the conventions we are using the n in $\Pi(\phi,n,k)$ corresponds to $-n$ in the definition given in [100].

where

$$\begin{aligned} I(0,0) =& \frac{1}{\sqrt{abcd}}\Big(-2\beta_2\frac{\vartheta_4'(\beta_2)}{\vartheta_4(\beta_2)} + \log\frac{\vartheta_1(2\beta_2)}{\vartheta_1'(0)} + \log\Big(\frac{c-d}{4cd}\delta(n_2)\Big) - \log\, K(k)\Big) \\ &+ \frac{2}{c\sqrt{(a-c)(b-d)}}F(\phi_2,k). \end{aligned} \tag{6.4.21}$$

From the above result we see that the singularities as $\epsilon \to 0$ cancel, as wished.

We now consider the remaining integral. Define

$$\begin{aligned} J(\lambda) &\equiv \int_c^b \frac{dz}{(z-\lambda)r(z)} \\ &= \frac{2}{(\lambda-a)(\lambda-b)\sqrt{(a-c)(b-d)}}\Big\{(a-b)\Pi(m,k) + (b-\lambda)K(k)\Big\}, \end{aligned} \tag{6.4.22}$$

where

$$m = \frac{(b-c)(\lambda-a)}{(a-c)(\lambda-b)}. \tag{6.4.23}$$

We then have,

$$\int_c^b \frac{dz}{(z-\lambda)zr(z)} = \frac{1}{\lambda}\Big(J(\lambda) - J(0)\Big), \tag{6.4.24}$$

where $J(0)$ is given explicitly as

$$J(0) = \frac{2}{ab\sqrt{(a-c)(b-d)}}\Big\{(a-b)\Pi(m(0),k) + bK(k)\Big\}. \tag{6.4.25}$$

Putting everything together, we find the following expression for the resolvent:

$$\begin{aligned} \tilde{\omega}_0(\lambda) =& \frac{p}{2t}\frac{\log\lambda}{\lambda} + \frac{1}{t\lambda}\log\frac{\lambda/b-1}{\lambda/c-1} \\ &+ \frac{pr(\lambda)}{2t\lambda}\Big(I(\lambda,-\infty) - I(\lambda,0) - I(0,-\infty) + I(0,0)\Big) \\ &+ \frac{r(\lambda)}{t\lambda}\Big(J(\lambda) - J(0)\Big). \end{aligned} \tag{6.4.26}$$

As $\lambda \to \infty$, this is indeed a Laurent series in λ: using again (6.4.13) it is easy to see that $I(\lambda,-\infty)$ contains a term of the form $-\log(\lambda)/r(\lambda)$ that cancels against the first term in (6.4.26). In order to derive the conditions for the endpoints of the cut, we must impose the asymptotic behaviour

$$\tilde{\omega}_0(\lambda) = \frac{1-2\hat{b}}{\lambda} + \mathcal{O}(\lambda^{-2}). \tag{6.4.27}$$

We find three conditions. First of all, notice that there is a term of order λ coming from the integrals $I(0,0)$, $I(0,-\infty)$, and $J(0)$. Imposing the cancellation of this term, one obtains the condition

$$p(I(0,0) - I(0,-\infty)) - 2J(0) = 0. \tag{6.4.28}$$

The vanishing of the constant term leads to the condition

$$p\Big(F(\phi_2,k) - F(\phi_1,k)\Big) = 2K(k), \tag{6.4.29}$$

Finally, the fact that the $1/\lambda$ term has the coefficient $1-2\hat{b}$ leads to a third condition,

$$\begin{aligned}
&p\Bigg((a+b+d-c)(F(\phi_1,k) - F(\phi_2,k)) - 2(c-d)\Pi(\phi_2,n_1,k)\\
&+\sqrt{(a-c)(b-d)}\Big(-2\beta_1\frac{\vartheta_4'(\beta_1)}{\vartheta_4(\beta_1)} + \log\frac{\vartheta_1(2\beta_1)}{\vartheta_1'(0)} - \log\Big(\frac{c-d}{4}\delta(n_1)\Big) - \log\,K(k)\Big)\Bigg)\\
&+2(b+d+c-a)K(k) + 2(d-b)\Pi(m_\infty,k) = t,
\end{aligned} \tag{6.4.30}$$

where

$$\beta_2 = \frac{F(\phi_2,k)}{2K(k)}, \qquad m_\infty = \frac{b-c}{a-c}. \tag{6.4.31}$$

These conditions determine the endpoints $\hat{a}, \hat{b}$ as functions of the parameters t, p. We seem to have three conditions for two unknowns, but since we started with a symmetric problem and we just changed variables, one of the conditions is redundant. This is not easy to verify from the above expressions, but can be checked, for example, by doing a small t expansion of the equations, and assuming a power series ansatz for the endpoints:

$$\hat{a}(t,A) = \sum_{n=0}^{\infty} \hat{a}_n(A)\, t^n, \quad \hat{b}(t,A) = \sum_{n=0}^{\infty} \hat{b}_n(A)\, t^n. \tag{6.4.32}$$

The ansatz is justified by the fact that, as $t \to 0$ with A fixed, we must recover the standard YM result obtained in [109]. One can see that, at leading order in t, the three conditions above lead to the same equation, namely

$$\frac{\hat{a}_0 + \hat{b}_0}{2} A = 2K(k_0), \tag{6.4.33}$$

where

$$k_0^2 = \frac{4\hat{a}_0\hat{b}_0}{(\hat{a}_0 + \hat{b}_0)^2}. \tag{6.4.34}$$

Using standard properties of elliptic functions, one can easily check that this condition becomes

$$A = \frac{4}{\hat{a}_0} K(\hat{b}_0/\hat{a}_0), \tag{6.4.35}$$

which is precisely one of the equations found in [109]. Notice that, by making use of (6.4.28), we can simplify the expression of the resolvent to

$$\begin{aligned} \tilde{\omega}_0(\lambda) =& \frac{p}{2t}\frac{\log\lambda}{\lambda} + \frac{1}{t\lambda}\log\frac{\lambda/b-1}{\lambda/c-1} \\ &+ \frac{pr(\lambda)}{2t\lambda}\Big(I(\lambda,-\infty) - I(\lambda,0)\Big) + \frac{r(\lambda)}{t\lambda}J(\lambda). \end{aligned} \tag{6.4.36}$$

In principle, the above conditions for $\hat{a}, \hat{b}$, together with the explicit expression for the resolvent in (6.4.36), determine completely the solution for the large area phase. These conditions are rather intricate to be treated analytically, but one could study them numerically.

The most important question to address is the order of the phase transition for different values of p. This of course can be seen, as in [109], by computing the free energy in the large area phase that we have just analyzed. Since the line of critical points is smoothly connected to the Douglas-Kazakov transition, we should expect the transition in the q-deformed theory to be of third order for any $p > 2$. Indeed, one can find indirect evidence that this is the case by using an argument in [112] based on double-scaling limits. If we consider a theory with a large N n-th order phase transition at a critical area $A = A_*$ between phases I and II, the free energy has the following behaviour

$$F_0^{\mathrm{I}}(A) - F_0^{\mathrm{II}}(A) \sim (A_* - A)^n. \tag{6.4.37}$$

To define a double-scaling limit of such a theory, one should introduce a string coupling constant μ_s through

$$\mu_s^{-2} = N^2(A_* - A)^n. \tag{6.4.38}$$

The nonperturbative effects of such a theory are expected to be of the form $\exp(-1/\mu_s)$. But this means that the instanton effects in the original theory should have the behaviour $\exp(-N\gamma(A))$, with

$$\gamma(A) \sim (A_* - A)^{n/2}. \tag{6.4.39}$$

Indeed, in [112] it is found that the function (6.1.17) appearing in the instanton suppression factor has exactly the behaviour (6.4.39) with $n = 3$ near the Douglas-Kazakov transition point, as required for the existence of a double-scaling limit at a third order phase transition. According to this argument, the behaviour of the instanton suppression factor near the critical point can be indeed regarded as an indirect way to probe the order of the phase transition. We have checked numerically that the function $\gamma(A, p)$ that we found in (6.3.22) behaves indeed as

$$\gamma(A, p) \sim (A_*(p) - A)^{3/2} \tag{6.4.40}$$

near $A_*(p)$, for various values of $p > 2$. This is indeed consistent with the large N phase transition of the q-deformed theory being of third order for all $p > 2$.

We should also mention that, in [96], general criteria have been formulated to determine the order of a phase transition for a model based on a distribution of Young tableaux. These criteria only depend on the behavior of the density (6.2.37) in the small area phase. It can be easily seen that according to these criteria, the phase transition of the q-deformed theory is of third order for any $p > 2$.

6.5 Conclusions and Outlook

In this chapter we have shown that q-deformed 2d YM theory exhibits an interesting phase structure, with a Douglas-Kazakov phase transition smoothly connected to that of the standard YM theory, and a "barrier" at $p = 2$. One of the original motivations of this analysis was the appearance of the q-deformed theory as a nonperturbative completion of topological string theory on certain Calabi-Yau backgrounds. q-deformed 2d YM on the sphere has been proposed in [132, 94] as a nonperturbative, holographic description of topological strings on the local Calabi-Yau manifold

$$\mathcal{O}(-p) \oplus \mathcal{O}(p-2) \to \mathbf{P}^1, \tag{6.5.41}$$

where the integer number $p > 0$ corresponds to the parameter p appearing in (6.2.1). Explicit computations in [94] show that the perturbative partition function computed in [101] appears as a certain decoupling limit of the large area expansion of the q-deformed theory. However, the fact that this theory exhibits a phase transition suggests that, for geometries of the form (6.5.41) with $p > 2$, the large area expansion has a finite radius of convergence which, in terms of the 't Hooft parameter t, is given by $t_*(p) = A_*(p)/p$. As p becomes larger, the radius of convergence becomes smaller. Therefore, the conjecture of [129] suggests that for the geometries (6.5.41) with $p > 2$, there will be a phase transition at small radius in the full, nonperturbatively completed topological string theory. What are the possible interpretations of this phase transition in the topological string theory context? We will mention here three possibilities, although a better understanding of the implications of the phase transition of q-deformed YM to nonperturbative topological strings will require a more detailed treatment:

1. A first possibility is that the phase transition in the q-deformed theory indicates a topology change in the Calabi-Yau background. After all, the small and the large area phases are described by different master fields of the two-dimensional theory, corresponding to the one-cut and two-cut solutions discussed above, and it is known that in large N dualities the master field encodes the geometry of the target [108, 93]. This topology change might be also interpreted, as in [107], in terms of a process involving a splitting of baby universes.

2. A second possibility is that the small area phase does not have a geometric interpretation. One indication of that is the string description of standard 2d YM: the analysis of [111, 113] shows that the large area expansion has an interpretation in terms of branched coverings of the sphere. However, it has been argued that the existence of a large N phase transition suggests that this geometric picture does not hold for the small area phase [112]. In the same vein, it is likely that the small area phase of the q-deformed theory is not described appropriately by topological strings with a geometric target. This is in fact very reminiscent of the analysis of [98] (see also [99, 97]), where it was shown that the large N phase transition of the unitary matrix model corresponds, in AdS/CFT at finite temperature, to the point where the horizon of the small AdS black hole becomes comparable to the string scale. At this point, the supergravity/geometry picture breaks down. The situation we are considering here could be a topological string analogue of the large N transition of [98].

3. A more conservative possibility is that the conjecture of [129] does not fully apply

to the local geometries (6.5.41) when $p > 2$, or at least does not apply to the small area phase. The original conjecture was formulated for compact Calabi-Yau threefolds, and there may be subtleties when applying it to the noncompact case. It turns out that precisely for $p > 2$ there are obstructions for contracting the $\mathbf{P}^1$ inside (6.5.41) to a point [123, 118], and because of this reason one can expect these geometries not to arise as a decompactification limit of a compact Calabi-Yau. It is intriguing that the "barrier" $p = 2$ that we found in this paper is the same that occurs in the geometric setting.

In extracting the consequences of our analysis for the nonperturbative physics of topological strings, there is another point that should be mentioned. In our analysis we considered the saddle-point solution of the functional $S[h]$, and we found that this leads to a distribution where $\langle h \rangle = 0$ and the dependence on the θ angle is trivial. However, it has been argued in [117], by studying the instanton weight factors, that the presence of a nonzero θ changes the location of the critical line. This is an interesting possibility and deserves further study. Also, we have restricted ourselves to solutions with zero $U(1)$ charge. This is indeed the true vacuum of the theory [95], but one could also consider saddle-point solutions like those in [127]: one imposes the constraint $\langle h \rangle = Q$, where Q is the $U(1)$ charge, solves for the density, and then finally sums over all integer charges with a weight $\exp(iQ\theta)$. It may happen that, in order to compare our results with those of [94], one should use this prescription to include the $U(1)$ charges.

It is also worth pointing out that the instanton weight factors considered in section 4 are closely related to the degeneracies of BPS states analyzed in [94]. It is likely that the techniques of [112] that we used and extended to the q-deformed case in order to compute these weights lead to a useful technique to obtain the degeneracies.

From the point of view of the two-dimensional gauge theory, the results presented in this chapter indicate that, when the deformation parameter is sufficiently large, the large N phase transition is smoothed out already at the planar level. This is an interesting, new mechanism for smoothing out large N transitions which may have implications in other contexts (the other mechanism we are aware of to smooth out these transitions requires performing a double-scaling limit, as in [124, 98], and involves a resummation of the $1/N$ expansion).

There are also various open questions concerning the gauge theory aspects of our analysis. Of course, the two-cut solution that we presented in this work should be investigated in more detail. One could also investigate the phase structure and free energy of the chiral version of the q-deformed theory (in the 2d YM case, this has been done in [106, 122]). Since the chiral sector makes a more direct contact with the perturbative topological string amplitudes, this may help in understanding better the holographic description proposed in [132, 94]. It would be also very interesting to analyze the subleading $1/N$ corrections to the planar result in the small area phase. In [112] this was done for the standard YM case by using a discretized version of orthogonal polynomials, but it is not obvious how to generalize this to a discrete model with a sinh interaction. Such a generalization would make it also possible to define a double-scaled theory near the critical line of the q-deformed theory, as we briefly discussed in the last section. These are some of the open questions that would be interesting to consider for future research.

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Summary

Our greatest glory is not in never failing, but in rising up every time we do

- adapted from Confucius

Supersymmetric Black Holes as Probes of Quantum Gravity

The research presented in this thesis features a specific class of black holes arising in string theory that offer a rare window towards probing quantum gravity. The driving force behind much of this research in black hole physics lies in the idea of holography which is a duality between a gravitational theory in a bulk space-time and a quantum theory (without gravity) living on the boundary of that space-time. In string theory this duality is famously manifest as the AdS/CFT correspondence. Macroscopic observables refer to gravitational quantities in the bulk; whereas, microscopics corresponds to the theory living on a bound state of higher dimensional objects in string theory called branes. The black holes that we work with in this thesis are analogs of zero temperature black holes (also known as extremal) in Einstein's gravitational theory with electromagnetic charge; and are obtained through compactification of a type of closed string theory in ten dimensions, where the compactified six dimensions are endowed with 4, 2 and 0 dimensional (spatially) branes, thus giving a black hole solution in four non-compact dimensions. Lifting this set-up to eleven dimensions by opening a circular spatial dimension gives a five dimensional black string in M-theory, where the branes now lift to 5 & 2 dimensional membranes and respectively angular momentum along the M-theory circle. Subsequent fragmentation of this system leads to interesting multi-center configurations in 4 as well as 5 dimensions. Investigating this specific system and its various manifestations, sheds new insights into the quantum

field theory(gauge theory)/gravity paradigm.

An interesting recent development that initiated the research in this thesis is the Ooguri-Strominger-Vafa (OSV) conjecture, which is a correspondence relating a class of supersymmetric black holes to topological string theory - a lower dimensional string theory living on a six dimensional compactification space. This conjecture has raised several questions as well as opened up novel possibilities. Firstly, one had to verify that it indeed held true for a range of interesting gravitational systems. Of specific interest for this thesis was the D0-D2-D4 black hole solution from Type II A string theory compactified on a specific Calabi-Yau background. The dual gauge theory, in this case, turned out to be a quantum deformed version of two dimensional Yang-Mills on a closed surface. Since the latter lends itself to non-perturbative analysis, it opens up the interesting possibility to extract non-perturbative information from the gauge theory and thus determine corresponding corrections to the black hole system. Of course, as an independent check, it is still be useful to compute the black hole's entropy and observed charges with the inclusion of higher order corrections. To sum it up, both microscopic as well as macroscopic computations are necessary for furthering this research.

Besides the OSV conjecture, other developments in the direction of black holes with non-trivial topology and their respective entropy counting issues also began to gain momentum around the same time. At first sight, many of these apparently diverse developments appeared seemingly unrelated. As a case in point, we list those developments here: solutions for multi-center supersymmetric black holes; the discovery of black rings in five dimensions; the 4D/5D connection relating black holes in four dimensions to those in five dimensions, and subsequently a multi-center extension of this connection along with the inclusion of extended black objects; the formulation of an entropy function technique that is well suited for computations involving higher order corrections due to the remarkable feature that within this formalism, all equations of motion straightforwardly reduce to algebraic equations; quiver[3] gauge theories dual

[3]Literally speaking, a quiver is a case for holding arrows. In mathematics, a quiver is a directed graph, with loops and links between vertices. For our purposes here, the vertices represent gauge groups.

to multi-center black hole configurations, necessary for a holographic understanding of microstates. Our take in this work is that a lot of these themes in fact compliment each other and thus a parallel rather than serial approach to research in this field can in fact lead to integration of ideas and emergence of new insights therein. Nevertheless, the underlying theme behind all of this is still the gravity/gauge duality connecting the macroscopic to the microscopic. Therefore in order to modestly achieve some of these objectives, a large emphasis of the work in this thesis has been placed on developing methodology and interpreting underlying mechanisms.

Here we briefly summarize our results. In chapter 3, we began our investigations with macroscopic gravity calculations. We developed an entropy formalism suited for 5D black objects. This is then applied to both 5D black holes as well as black rings. In chapter 4, we turned our attention to the 4D/5D conjecture and carefully investigated subtle charge shifts that result in the process for black holes and black rings. These are issues that have stirred considerable debate in the literature. For single center configurations, the new tools developed in chapter 3 provide us with a geometric interpretation of the above shifts via spectral flow. We then moved on to understand this picture for multi-center geometries and interpret these results via the corresponding split-spectral flows. To do so, insights from AdS fragmentation were found to be extremely beneficial. In chapter 5, we investigated continuum multi-center black hole configurations, thus finding solutions to integrability equations for large n centers. We have subsequently used these solutions for generating interesting electro-magneto-gravitational backgrounds. As an interesting application we then discussed this in the context of a black hole levitron. On the microscopic side, we have studied the dual gauge theory of the aforementioned black hole, constructed as a bound state of D-branes. In this rare case, the microscopic theory turns out to be fully non-perturbative and thus lends itself as a very interesting tool for instanton analysis. In order to do this, we considered topological strings over a non-compact Calabi-Yau background, over which we sought to test the validity of the OSV conjecture and in the process discovered a remarkable phase transition in the theory. We analyzed this transition and commented on its implications for black hole physics.

Samenvatting

It is our choices, more than our abilities, that show who we truly are

- J. K. Rowling

Supersymmetrische Zwarte Gaten als Testgrond voor Kwantumzwaartekracht[4]

Het onderzoek dat in dit proefschrift gepresenteerd is, behandelt een bepaalde klasse van zwarte gaten in snaartheorie, die een zeldzame blik op kwamtumzwaartekracht bieden. De drijvende kracht achter een groot deel van dit onderzoek naar de natuurkunde van zwarte gaten is het holografisch principe. Dit is een dualiteit tussen een theorie in een bepaalde ruimtetijd waarin zwaartekracht een rol speelt aan de ene kant en een kwantumtheorie (zonder zwaartekracht) die op de rand van eerdergenoemde ruimtetijd leeft aan de andere kant. In snaartheorie manifesteert deze dualiteit zich als de AdS/CFT correspondentie. Macroscopische observabelen verwijzen naar grootheden in de zwaartekrachtstheorie, terwijl het label microscopisch gereserveerd is voor gebruik in de theorie die leeft op een gebonden toestand van hoger dimensionale objecten in snaartheorie, branen genaamd. De zwarte gaten waar we mee werken in dit proefschrift zijn analoog aan nul graden zwarte gaten (de zogenaamde extremale zwarte gaten) uit Einsteins zwaartekrachtstheorie met elektromagnetische lading. Bovendien worden ze verkregen door middel van een compactificatie van een bepaald type gesloten snaartheorie in tien dimensies, waarbij een aantal van de zes gecompactificeerde dimensies zijn gevuld met 4, 2 en 0 dimensionale (ruimtelijke) branen, hetgeen resulteert in een zwart gat oplossing in de vier niet compacte dimensies. Deze configuratie

[4] It's a pleasure to acknowledge Joost Hoogeveen for being so kind to provide me with the following Dutch translation of the summary.

kan verheven worden naar elf dimensies door een cirkelvormige ruimtelijke dimensie te openen. Op deze wijze ontstaat een vijf dimensionale zwarte snaar in M-theorie, waarbij de branen nu verheven zijn tot respectievelijk 5 & 2 dimensionale membranen en impulsmoment langs de M-theorie cirkel. Verdere fragmentatie van dit systeem leidt tot interessante “meervoudig middelpunt” configuraties in zowel 4 als 5 dimensies. Onderzoek van dit specifieke systeem en zijn verscheidene verschijningsvormen, leidt tot nieuwe inzichten in het kwantumveldentheorie (ijktheorie)/zwaartekracht paradigma.

Een interessante recente ontwikkeling, die de basis heeft gelegd voor het onderzoek in dit proefschrift, is het Ooguri-Strominger-Vafa (OSV) vermoeden. Dit is een correspondentie die een klasse van supersymmetrische zwarte gaten relateert aan topologische snaar theorie, i.e. een lager dimensionale snaartheorie die op een zes dimensionale compactificatieruimte leeft. Dit vermoeden heeft zowel verschillende vragen opgewekt als de weg vrijgemaakt voor nieuwe mogelijkheden. Ten eerste moest men verifiëren dat het vermoeden opging voor een scala aan interessante zwaartekrachtssystemen. Een belangrijke rol in dit proefschrift is weggelegd voor de D0-D2-D4 zwarte gat oplossing in type II A snaar theorie, die gecompactificeerd is op een specifieke Calabi-Yau achtergrond. De duale ijktheorie bleek, in dit geval, een kwantum gedeformeerde versie van twee dimensionale Yang-Mills theorie op een gesloten oppervlak te zijn. Aangezien deze theorie ook niet perturbatief geanalyseerd kan worden, opent dit de interessante mogelijkheid om niet perturbatieve informatie uit de ijktheorie te extraheren en vervolgens de corresponderende correcties van het zwarte gat systeem te bepalen. Als een onafhankelijke controle blijft het uiteraard nuttig om de entropie van het zwarte gat en de waargenomen ladingen direct uit te rekenen met inbegrip van hogere orde correcties. Samenvattend kan gezegd worden dat zowel microscopische als macroscopische berekeningen noodzakelijk zijn voor de vooruitgang van dit onderzoek.

In de tijd dat het OSV vermoeden werd geformuleerd, raakten ontwikkelingen op het gebied van zwarte gaten met niet triviale topologie en het hieraan gerelateerde probleem om hun entropie te bepalen ook in een stroomversnelling. Op het eerste gezicht leken deze ontwikkelingen ongerelateerd. Meer specifiek zijn deze ontwikkelingen: oplossingen voor “meervoudig middelpunt” supersymmetrische zwarte gaten; de ontdekking van zwarte ringen in vijf dimensies; de 4D/5D verbinding die zwarte gaten in vier dimensies relateert aan die in vijf dimensies en vervolgens een "meervoudig middelpunt" uitbreiding van deze verbinding alsook de generalisatie naar uitgebreide zwarte objecten; de formulering van een entropie functie techniek, die geschikt is voor berekeningen die hogere orde correcties meenemen vanwege de opmerkelijke eigenschap dat binnen dit formalisme alle bewegingsvergelijkingen reduceren tot algebraïsche vergelijkingen; quiver[5] ijktheoriën duaal aan "meervoudig middelpunt" zwart gat configuraties, noodzakelijk voor een holografisch begrip van microtoestanden. Onze perceptie van dit werk is dat veel van deze thema's elkaar aanvullen en dientengevolge kan een parallelle, in plaats van een seriële, benadering leiden tot integratie van ideeën en het verkrijgen van nieuwe inzichten. Desalniettemin is het onderliggende thema van dit alles nog steeds de zwaartekracht/ijk dualiteit die macroscopisch met microscopisch verbindt. Teneinde sommige van deze doeleinden op een bescheiden manier te bereiken, ligt de nadruk van dit proefschrift op het ontwikkelen van methodologie en het interpreteren van onderliggende mechanismes.

Tenslotte vatten we onze resultaten kort samen. In hoofdstuk 3 zijn we ons onderzoek begonnen met macroscopische zwaartekrachtsberekeningen. We hebben een entropie formalisme ontwikkeld dat geschikt is voor 5D zwarte objecten. Dit wordt vervolgens toegepast op zowel 5D zwarte gaten als zwarte ringen. In hoofdstuk 4 hebben we onze aandacht gericht op het 4D/5D vermoeden en hebben we verschuivingen in ladingen van zwarte gaten en zwarte ringen nauwkeurig onderzocht. Deze kwesties hebben een hoop stof doen opwaaien in de wetenschappelijke literatuur. Voor "enkelvoudig middelpunt" configuraties verschaffen de nieuwe technieken, ontwikkeld in hoofdstuk 2, ons een meetkundige interpretatie van bovengenoemde verschuivingen via spectrale stroom. Vervolgens hebben we de analoge stappen gezet voor "meervoudig middelpunt" configuraties en de resultaten geïnterpreteerd door middel van de relevante gespleten spectrale stromen. In deze interpretatie zijn inzichten vanuit AdS fragmentatie van grote waarde gebleken. In hoofdstuk 5 hebben we continuüm "meervoudig middelpunt" zwart gat configuraties onderzocht, hetgeen leidde tot oplossingen van integreerbaarheidsvergelijkingen voor grote n middelpunten. Vervolgens hebben

[5]Quiver literally translates as pijlkoker in Dutch. In mathematics, a quiver is a directed graph, with loops and links between vertices. For our purposes here, the vertices represent gauge groups.

we deze oplossingen gebruikt om interessante achtergronden te genereren, die zowel zwaartekrachts- als elektromagnetische velden bevatten. Als interessante toepassing hebben we dit behandeld in de context van een zwart gat levitron. Aan de microscopsiche kant hebben we de duale ijktheorie van het eerdergenoemde zwarte gat, dat een gebonden toestand van D-branen is, bestudeerd. In dit zeldzame geval blijkt de microscopische theorie volledig niet perturbatief te zijn en daarom is deze theorie zeer interessant in de analyse van instantonen. Als onderdeel hiervan beschouwen we in dit proefschrift topologische snaren op een niet compacte Calabi-Yau achtergrond en we hebben getracht de geldigheid van het OSV vermoeden te testen. Tijdens deze exercitie hebben we een opmerkelijke faseovergang ontdekt. We hebben deze overgang geanalyseerd en de hieruit volgende implicaties voor de natuurkunde van zwarte gaten becommentariëerd.

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Fortitude is a virtue of mind,
Gratitude, that of heart.
- Anonymous

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