Quantum corrections to the Bianchi II transition under local rotational invariance

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A quantum state in a Bianchi II model is studied as it approaches the cosmological singularity, by means of the evolution of its moments. Classically this system presents a transition between two Bianchi I models. This phenomenon is described by a very specific and well-known transition law, which is derived based on the conservation of certain physical quantities. In the quantum theory fluctuations, as well as higher-order quantum moments, of the different variables arise. Consequently, these constants of motion are modified and hence also the transition rule. We focus on the so-called locally rotationally symmetric and vacuum case, as a first step towards a more complete study. Indeed, the future goal of this research line is to generalize this analysis to the Bianchi IX spacetime, which can be seen as a succession of Bianchi II models. Ultimately, these results will shed light on the role played by quantum effects in the BKL conjecture.

Keywords: Bianchi II; local rotational symmetry (LRS); quantum Bianchi spacetime; Bianchi transition.

1. Introduction

Close to a spacelike singularity, it is conjectured that the classical dynamics of any universe follows a chaotic behavior, according to Belinski-Khalatnikov-Lifshitz (BKL).[1] In this scenario, the dynamics of each point is decoupled from the rest and can be described by a Bianchi IX spacetime, which can be understood as a succession of Bianchi II models.[1] This model is much easier to analyze than the full Bianchi IX dynamics, and therefore provides a good first step to study the full BKL scenario. More precisely, as it was described in Misner's seminal work,[2] in the Bianchi II model the system undergoes a single transition between two Bianchi I models. However, close to the singularity quantum effects are expected to become relevant, and thus, in the present work we will analyze how they modify this classical transition.

2. Classical Model

A Bianchi II model is a type of spatially homogeneous but anisotropic four-dimensional spacetime. If we follow the usual 3+1 decomposition in order to describe the model canonically, the vacuum Hamiltonian constraint of this system is found to be

$$\mathcal{C} = \frac{1}{2} e^{-3\alpha} \left(-p_\alpha^2 + p_-^2 + p_+^2\right) + e^{\alpha - 8\beta_+} = 0, \tag{1}$$

where α describes the spatial volume and $\beta_\pm$ are the shape parameters that encode the spatial anisotropy of the spacetime. These are the so-called Misner variables and p_α and $p_\pm$ are their corresponding conjugate momenta. If α is chosen as the internal time, the physical Hamiltonian of the system takes the subsequent form:

$$H := \; -p_\alpha = \left(p_+^2 + p_-^2 + 2e^{4\alpha - 8\beta_+}\right)^{1/2}. \tag{2}$$

This is the most general vacuum description but, in order to further simplify the problem, we will remove one of the shape parameters by imposing $\beta_- = 0 = p_-$, which is known as the *locally rotationally symmetric* (LRS) case. Consequently, the Hamiltonian (2) is reduced to:

$$H = \left(p^2 + 2e^{4\alpha - 8\beta}\right)^{1/2}, \tag{3}$$

where, for compactness, $\beta := \beta_+$ and $p := p_+$ have been defined.

This model presents a singularity, even in the particular LRS case, precisely when $\alpha \to -\infty$. Hence, we will be interested in studying the evolution towards this limit. For this purpose, the first step is to obtain the equations of motion:

$$\dot{\beta} = \frac{p}{H}, \tag{4}$$

$$\dot{p} = \frac{8}{H} e^{4\alpha - 8\beta}, \tag{5}$$

where the dot represents the derivative with respect to the internal time α. From (4)-(5) we note that, when the exponential term $e^{4\alpha-8\beta}$ is negligible, the equations of motion can easily be solved. Under such an assumption, p would be a constant of motion and, in this sense, the system would follow a free dynamics, whereas the shape-parameter β would be a linear function of α:

$$\beta = \mathrm{sign}(p)\alpha + c, \tag{6}$$

with an integration constant c. During this period the system is equivalent to a LRS vacuum Bianchi I –or Kasner– solution. This Kasner regime is completely characterized by the values of the constants of motion p and c. As illustrated in Fig. 1 and Fig. 2, if we begin the evolution towards the singularity in one of these regimes, with $p > 0$ and at large values of α, the shape parameter β follows the linear behavior as given in (6) until the exponential term $e^{4\alpha-8\beta}$ ceases to be negligible. At that point, a transition happens and the system enters in another Kasner regime.

In order to study this transition, a useful procedure is to exploit the conserved quantities of the system. The mentioned constants of motions are found to be the following:

$$R_1 := 2H - p, \tag{7}$$

$$R_2 := e^{2(\alpha+\beta)}(H - p). \tag{8}$$

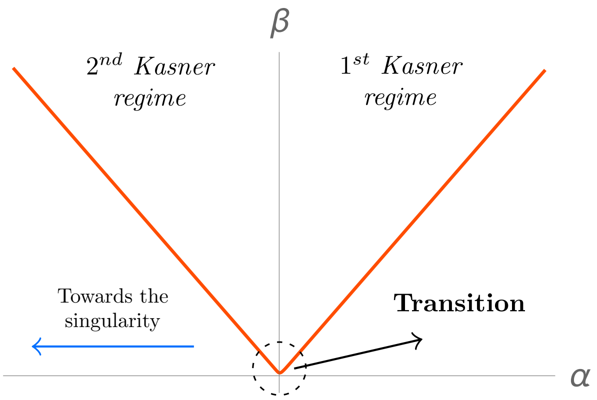

Fig. 1. Classical evolution of the variable β with respect to α, where $p > 0$ for large values of α and β.

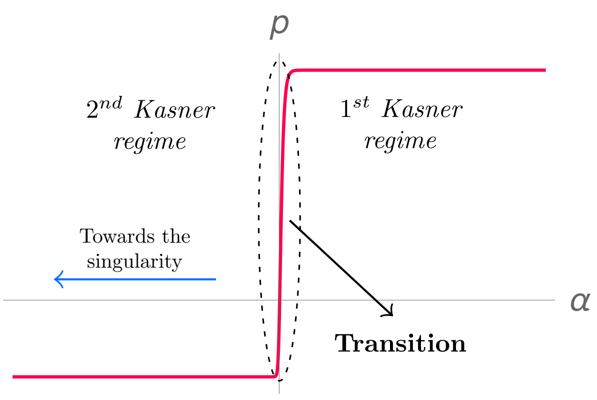

Fig. 2. Classical evolution of the variable p with respect to α, where $p > 0$ for large values of α and β.

If we evaluate these quantities in the initial and final states and equate them, we obtain a system of two equations, which relate the parameters of the first Kasner epoch $(\overline{c}, \overline{p})$ with those of the second one $(\widetilde{c}, \widetilde{p})$. Thus, after solving it we obtain the law that describes the transition between the two Kasner regimes, namely:

$$\widetilde{p} = -\frac{1}{3}\overline{p}, \tag{9}$$

$$\widetilde{c} = -3\overline{c} - \frac{1}{2}\ln\left(\frac{2}{3}\overline{p}^3\right). \tag{10}$$

3. Quantum Transition Rules

Once that we have provided a canonical description of the classical model, we are ready to develop the quantum analysis. The quantum dynamics is governed by the following Hamiltonian operator, which is defined from the classical one and chosen to be Weyl-ordered, that is, with a totally symmetric ordering of the basic operators:

$$\hat{H}(\hat{p},\hat{\beta}) = \left(\hat{p}^2 + 2e^{4\alpha-8\hat{\beta}}\right)^{1/2}_{\text{Weyl}}. \tag{11}$$

Then, instead of studying the evolution of the wave function itself, we follow a formalism developed for quantum cosmology and quantum mechanics in general, which is based on the moment decomposition of the wave function.[4] Thus, according to this formalism, the system is fully described by the expectation values of the basic variables,

$$\beta := \langle\hat{\beta}\rangle, \quad p = \langle\hat{p}\rangle, \tag{12}$$

as well as the following infinite set of quantum moments:

$$\Delta(\beta^i p^j) := \langle(\hat{\beta}-\beta)^i(\hat{p}-p)^j\rangle_{\text{Weyl}} \ \ (i,j\in\mathbb{N}). \tag{13}$$

The sum of the indices $i+j$ will be referred as the order of the corresponding moment. The evolution of these variables is ruled by a quantum effective Hamiltonian H_Q, that is defined as the expectation value of $\hat{H}$. By performing a Taylor expansion around the expectation values of $\hat{\beta}$ and $\hat{p}$, it can be written as the following infinite series:

$$\begin{aligned} H_Q &= \left\langle\hat{H}(\hat{p},\hat{\beta})\right\rangle \\ &= H(\beta,p) + \sum_{i+j=2}^{\infty}\frac{1}{i!j!}\frac{\partial^{i+j}H(\beta,p)}{\partial\beta^i\partial p^j}\Delta(\beta^i p^j), \end{aligned} \tag{14}$$

where $H(\beta,p)$ is the classical Hamiltonian (3). This effective Hamiltonian H_Q encodes the complete dynamical information of the system. Indeed, the equations of motion for the variables of the system –β, p and $\Delta(\beta^i p^j)$– are obtained by computing their Poisson brackets with H_Q:

$$\frac{d\beta}{d\alpha} = \{\beta, H_Q\} = \frac{\partial H_Q}{\partial p}, \tag{15}$$

$$\frac{dp}{d\alpha} = \{p, H_Q\} = -\frac{\partial H_Q}{\partial \beta}, \tag{16}$$

$$\begin{aligned} \frac{d\Delta(\beta^i p^j)}{d\alpha} &= \{\Delta(\beta^i p^j), H_Q\} \\ &= \sum_{m,n=0}^{+\infty}\frac{1}{m!n!}\frac{\partial^{m+n}H(\beta,p)}{\partial\beta^m\partial p^n}\{\Delta(\beta^i p^j),\Delta(\beta^m p^n)\}, \end{aligned} \tag{17}$$

where the Poisson brackets between expectation values are defined in the usual way in terms of the commutator, that is, $\{\langle\hat{A}\rangle,\langle\hat{B}\rangle\} := -i\langle[\hat{A},\hat{B}]\rangle/\hbar$. Let us emphasize that this evolution of the variables is completely equivalent to that given by the Schrödinger picture in terms of the wave function.

We observe that the classical Hamiltonian H defined in (3) is not polynomial, which means that the series in (14) is indeed infinite and it is not truncated at certain order. Thus, the equations of motion (15)-(17) form an infinite system of coupled equations. Therefore, in order to study and compute the dynamics explicitly, a cut-off must be introduced. That is, we will consider all the moments of order higher than a certain value N to be vanishing:

$$\Delta(\beta^i p^j) \approx 0 \ \text{ for } \ i+j > N. \tag{18}$$

In fact, such a truncation corresponds to considering a semiclassical regime, where the quantum state is peaked on a classical trajectory and hence the high-order moments are negligible.

However, even if we apply a cut-off, instead of solving the coupled system of equations for any value of α, we will focus on the so-called Kasner regimes, as we have done in the classical analysis. During these periods, the equations of motion take the form

$$\frac{d\beta}{d\alpha} \approx \mathrm{sign}(p), \quad \frac{dp}{d\alpha} \approx 0, \quad \frac{d\Delta(\beta^i p^j)}{d\alpha} \approx 0, \tag{19}$$

which are immediate to solve:

$$\beta \approx \mathrm{sign}(p)\alpha + c, \tag{20}$$

$$p \approx const., \tag{21}$$

$$\Delta(\beta^i p^j) \approx const., \tag{22}$$

with an integration constant c. Comparing these results with the classical ones, we observe that the behavior of p and β in these states remains identical in the quantum analysis. Moreover, the moments hold a constant value. In summary, the parameters that characterize each quantum Kasner epoch will be, as in the classical case, the constants c and p, along with the infinite constant set of moments $\Delta(\beta^i p^j)$. Then, in order to analyze the transition between two of these quantum Kasner regimes, we will once again exploit the constants of motion. In this quantum scenario, the conserved quantities are found to be the expectation values $\langle \hat{R}_1 \rangle$ and $\langle \hat{R}_2 \rangle$, where $\hat{R}_1$ and $\hat{R}_2$ are the operator counterpart of the classical conserved quantities (7) and (8). In fact, any of the following combinations is also a constant of motion:

$$\langle \hat{R}_i^n \hat{R}_j^m \rangle = const., \ \text{ for } \ m, n \in \mathbb{N} \text{ and } i, j \in \{1, 2\}. \tag{23}$$

Nonetheless, as a technical subtlety, it must be mentioned that the choice of ordering of these operators is far from trivial: one must find the proper ordering so that they are conserved. Furthermore, at higher orders in moments, there may appear some terms that go with some explicit powers of $\hbar$. A more detailed discussion about these issues can be found in.[3]

Based on the classical results, we will consider that we have an initial Kasner regime characterized by the parameters $(\overline{c}, \overline{p}, \overline{\Delta(\beta^i p^j)})$ and that, as we approach the singularity, another such regime is reached described by the parameters $(\widetilde{c}, \widetilde{p}, \widetilde{\Delta(\beta^i p^j)})$. Hence, if we evaluate the constants of motion in these two states and equate them, we obtain a system of algebraic equations that relate the different parameters. After solving this system, we get the quantum transition law, which takes the following form:

$$\widetilde{p} = -\frac{1}{3}\overline{p}, \tag{24}$$

$$\widetilde{c} = -3\overline{c} - \frac{1}{2}\ln\left(\frac{2\overline{p}^2}{3}\right) + \sum_{n=2}^{N} \frac{(-1)^n}{n\overline{p}^n}\overline{\Delta(p^n)}, \tag{25}$$

$$\widetilde{\Delta(\beta^m p^n)} = \sum_{l=0}^{N-(m+n)} \sum_{k=0}^{m} \frac{a_{mnkl}}{\overline{p}^{m+l-k}} \overline{\Delta(\beta^k p^{m+n+l-k})} + \sum_{r=2}^{N-2} \sum_{k=0}^{m-1} \sum_{l=l_{min}}^{l_{max}} \frac{b_{mnklr}}{\overline{p}^{m+r+l-k}} \overline{\Delta(p^r)}\ \overline{\Delta(\beta^k p^{m+n+l-k})} + \sum_{k=1}^{\lfloor N/4 \rfloor} \frac{c_{mnk}}{\overline{p}^{2k-n}} \hbar^{2k}, \tag{26}$$

where N denotes the order of the truncation. On the one hand, as can be seen from (24), the transition law for p remains identical as the classical one. However, we do obtain some quantum corrections for the transition law of β, as can be seen by comparing (10) with (25). Nevertheless, as expected, if all the moments are set to zero we recover the classical transition law. On the other hand, the transition law for the moments can be written in the compact form (26), with certain numerical coefficients a_{mnkl}, b_{mnklr} and c_{mnk}. Up to fifth order in moments, we have obtained the explicit expression of these coefficients and we refer the reader to[3] for the detailed expressions. If we analyze this formula in detail, we observe that it is linear in all of the moments except for some quadratic terms that contain pure moments of p. Moreover, regardless of the order of the truncation, there is a strong dependence on the index m of the shape parameter β of the corresponding moment. In particular, this index determines which initial moments will directly affect the value of a given final moment. For instance, the value of $\widetilde{\Delta(p^n)}$ (which corresponds to $m = 0$) only depends on the initial pure p-moments $\overline{\Delta(p^n)}$. Nevertheless, the value of $\widetilde{\Delta(\beta p^n)}$, that is $m = 1$, depends not only on pure p-moments, but also on $\overline{\Delta(\beta p^n)}$. In general, the value of the final moment $\widetilde{\Delta(\beta^m p^n)}$ will depend on the initial moments $\overline{\Delta(\beta^j p^i)}$ with $j \leq m$.

4. Quantum Dynamics

Once we have studied the asymptotic characteristics of this model in the Kasner regimes, we complete it by performing a numerical analysis to examine in detail its dynamical evolution. For this purpose, we have chosen an initial Gaussian state, namely

$$\overline{\Delta(\beta^{2m}p^{2n})} = 2^{-2(m+n)}\hbar^{2m}\sigma^{2(n-m)}\frac{(2m)!(2n)!}{m!n!}, \tag{27}$$

for $\forall m, n \in \mathbb{N}$, and vanishing otherwise. As expected, the numerical study shows that starting from high values of α and evolving towards lower ones, the system follows the Kasner dynamics until it undergoes a transition, which happens near $\alpha = 0$. This transition takes place in a very small period of time and then the system reaches its final equilibrium value –the final Kasner regime– very quickly.

Regarding the evolution of the expectation values p and β, we remark that they remain almost identical as in the classical case, that is, as depicted in Fig. 1 and Fig. 2. In fact, only in a very close zoom into the transition (near $\alpha = 0$) we can appreciate a slight modification in their evolution: quantum effects slightly increase the value of p and β during the transition.

Moreover, the evolution of the moments show that during the initial Kasner regime they hold a constant value, at certain point they begin to perform strong oscillations and then quickly relax to their final constant value, which characterizes the coherent state during the final Kasner regime. The precise dynamics of the moments are quite complicate but, if we look into the transition in more detail, some general features can be detected. For instance, we remark that the higher the index of p, the earlier the moments reach their final equilibrium value. Furthermore, we observe that the lower the order of a given moment, the sooner it starts to oscillate and with a larger amplitude, which is in agreement with the semiclassical hierarchy of moments we are assuming. More specifically, at each given order the pure fluctuations of p show the biggest amplitudes. In addition, moments with at least one odd index, *i.e.*, the initially vanishing ones, present more oscillations.

It is particularly interesting to observe that the moments that were vanishing in the initial Kasner epoch –due to the choice of the Gaussian state– are not vanishing in the final one, due to the mentioned oscillations. This is indeed an expected outcome according to the transition laws (26). However, it means that even when we choose an initial state where there are no correlations between β and p, the transition will generate correlations for the final state. In particular, all the initial vanishing moments are activated in a similar way: as they approach the transition they experience an excitation and start to grow exponentially, until they begin to oscillate.

5. Conclusions

In summary, we have described the quantum LRS Bianchi II model making use of a decomposition of the wave function into its infinite set of moments. For the classical case, we have obtained the explicit constants of motion for the full dynamics, and thus we have completely integrated the equations of motion. For the quantum case, making use of the conserved quantities (23), we have been able to obtain the exact quantum transition law for every variable (24)-(26) up to fifth-order in moments. Furthermore, by performing a numerical analysis of the model, we have studied the dynamical part of the evolution. In particular we have focused on the region where the transition takes place, and we have derived some general features of the behavior of the different variables. The follow-up of this work is to remove the LRS symmetry and to study the most general Bianchi II model. Then, the next step will be to extend the analysis to the full Bianchi IX model, and ultimately, to obtain the quantum corrections to the BKL transitions.

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