Spinning particle: Is Newton-Wigner the only way?

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A rapidly spinning compact object couples to an ambient curved background via the so-called spin-curvature coupling. In expressing this, one has to deal with the ambiguity of the definition of the center of mass of the body. What is worse, in a Hamiltonian formalism, this choice corresponds to an unphysical "parasitic" degree of freedom in the dynamical system. A solution to this is to apply a Hamiltonian constraint on the system and to obtain a set of brackets where the center-of-mass degree of freedom is erased from the algebra. I report on my progress in this procedure in the case of the so-called Tulczyjew-Dixon (or "covariant") supplementary spin condition and in my effort to cover the resulting phase space with canonical coordinates.

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1. Introduction: Motion of spinning particle

Notation and convention: I use the $G = c = 1$ geometrized units and a $(-+++)$ metric signature. Greek indices $\mu, \nu, ...$ run from 0 to 3, and Einstein summation convention is assumed.

When the surface of a given body is rotating at relativistic speeds, it will couple to any curved space-time background by the so-called *spin-curvature coupling.* This effect is an important ingredient for the relativistic two-body problem and, in particular, for the modelling of gravitational waves from compact binaries.

To understand how the spin-curvature coupling arises, imagine a compact object such as a neutron star moving in an ambient curved space-time background where, in order to treat the problem step by step, we will for now understand the compact object exclusively as a "test" object. When the variability length, or the curvature scale of the background is much longer than the size of the body, it is meaningful to construct a set of Riemann normal coordinates X^μ with respect to the background geometry centred on the body. The coordinate acceleration of individual elements of the body caused by the background gravitational field then is $\sim \Gamma^\mu{}_{\nu\kappa} U^\nu U^\kappa$, where $\Gamma^\mu{}_{\nu\kappa}$ are Christoffel symbols of the background and U^ν the four-velocity of the body elements, both expressed in the local frame.

Let us now assume that the body is approximately rigidly rotating with respect to the time-foliation by the Riemann normal coordinates $X^0 = \text{const.}$. We can then formally write $U^i = (\vec{\omega} \times \vec{X})^i$ $(i = 1, 2, 3)$ where $\vec{\omega}$ is some angular velocity vector. Then let us further assume that we are comparing relative accelerations on the constant-time slice $X^0 = 0$ in the Riemann normal frame so that we obtain

$$\delta a^\mu \sim \Gamma^\mu{}_{\nu\kappa} U^\nu U^\kappa \sim R^\mu{}_{0j0} X^j + R^\mu{}_{0ji} X^j (\vec{\omega} \times \vec{X})^i + \dots , \tag{1}$$

where $R_{\mu\nu\kappa\lambda}$ is the Riemann tensor along the wordline of the particle and the three dots denote higher-order terms in the size of the body. Now by averaging over $\vec{X}$, we see that we are able to choose a centre of coordinates such that the first, dipole-type term vanishes. However, the second term cannot be reduced by any coordinate shift. If we then take any system of particles, be it dust or a body held together by some force, we always see that the centre of mass of this system is subject to a residual *spin-curvature force.*

1.1. *Mathisson-Papapetrou-Dixon equations*

Procedures such as the one sketched above were carried out in various degrees of generality by Mathisson,[1] Papapetrou[2] and Dixon[3] to derive the equations of motion of extended bodies in curved space-time

$$\frac{\mathrm{D}P_\mu}{\mathrm{d}\tau} = -\frac{1}{2} R_{\mu\nu\kappa\lambda} \dot{x}^\nu S^{\kappa\lambda} + \dots , \tag{2}$$

$$\frac{\mathrm{D}S^{\mu\nu}}{\mathrm{d}\tau} = 2P^{[\mu} \dot{x}^{\nu]} + \dots , \tag{3}$$

where $x^\nu(\tau)$ is a referential position within the body, P_μ its momentum, and $S^{\kappa\lambda}$ has the general meaning of an angular momentum or "gravitomagnetic dipole" about its own center of mass. The parameter chosen here is proper time τ, but the equations can be parametrized by any other parameter. It should be noted that even though P_μ has the meaning of the overall linear momentum of the system, it may not be strictly true that the word-line is chosen so that $\dot{x}^\nu \propto P^\nu$.[4]

The three dots in the equation above denote higher-order terms corresponding to the interactions of the higher-order multipoles with the background curvature. The ability to discard higher-order terms is possible only for compact objects, for many astrophysical bodies the spin-curvature term is vastly sub-dominant. However, one can also easily show that even for compact objects, the spin-curvature coupling is suppressed to the 1.5 post-Newtonian order on the level of the equations of motion of binaries, and to linear order in the mass ratio. Thus, there is a limit to the applicability of the discussion of the strictly "test-particle" motion and at some point cross-reaction with other "non-test" effects cannot be ignored any more.

1.2. *The ambiguity of x^ν and $S^{\kappa\lambda}$*

As discussed at length in the literature the procedure of finding a center of mass for a rotating body is ambiguous.[4,5] In particular, the result depends on the slicing (the frame) we choose for the worldtube of the body. This is an issue even in flat space-time, where one is free to transform to another center of mass and angular-momentum tensor by the shifts[6]

$$x'^{\mu} = x^{\mu} + \delta x^{\mu}\,, \quad S'^{\nu\kappa} = S^{\nu\kappa} + P^{\nu}\delta x^{\kappa} - P^{\kappa}\delta x^{\nu}\,, \tag{4}$$

where δx^{μ} is some shift vector. This worldline shifting has a natural covariant extension to curved space-time by using Synge's world functions, but the overall physical content of the ambiguity is the same in both cases.[7]

A way to fix this "gauge freedom" of the worldline and the angular-momentum tensor is to choose a so-called *supplementary spin condition* of the form $S^{\mu\nu}V_{\nu} = 0$ with V_{ν} some time-like vector. The vector V^{μ} then has the meaning of the frame in which the center of mass and angular momentum are measured. When chosen judiciously, the supplementary condition eliminates all the nonphysical degrees of freedom from the system and one is left with only the physical ones to be evolved. Sometimes the choice of V^{ν} is implicit; the vector V^{ν} refers to the quantities it is supposed to fix (such as $x^{\nu}, S^{\kappa\lambda}$). In such cases one often ends up in a system where the non-physical degrees of freedom are not entirely constrained.[4,5,8,9]

2. Hamiltonian formalism

There is a range of reasons to use the Hamiltonian formalism, for the spinning-particle motion, from efficient numerical methods,[9] through perturbation theory,[10] to the use of the results in the so-called Effective-one-body model for relativistic binaries.[7,11] Let us now briefly summarize the Hamiltonians introduced by Witzany *et al.* in a previous work.[9] Consider the three covariant Hamiltonians corresponding to different choice of the frame V^{ν}:

$$V^{\nu} \text{ parallel transported along } x^{\nu}(\tau)\,: \tag{5}$$

$$H_{\rm KS} = \frac{1}{2m} g^{\mu\nu} P_{\mu} P_{\nu}\,, \tag{6}$$

$$V^{\nu} \propto P^{\nu}\,: \tag{7}$$

$$H_{\rm TD} = \frac{1}{2\mathcal{M}} \left(g^{\mu\nu} - \frac{4 S^{\nu\gamma} R^{\mu}{}_{\gamma\kappa\lambda} S^{\kappa\lambda}}{4\mathcal{M}^2 + R_{\chi\eta\omega\xi} S^{\chi\eta} S^{\omega\eta}} \right) P_{\mu} P_{\nu}\,, \tag{8}$$

$$V^{\nu} = \dot{x}^{\nu}\,: \tag{9}$$

$$H_{\rm MP} = \frac{1}{2m} g^{\mu\nu} P_{\mu} P_{\nu}\,, \tag{10}$$

where $m = -\dot{x}^{\nu} P_{\nu}$, $\mathcal{M} = \sqrt{-P^{\nu} P_{\nu}}$ are mass parameters with slightly different meanings but all corresponding roughly to the mass of the spinning body.[9,12]

The equations of motion are obtained with the use of the Poisson bracket

$$\{x^\mu, P_\nu\} = \delta^\mu_\nu\,,\ \{x^\mu, S^{\kappa\lambda}\} = 0\,,\ \{x^\mu, x^\nu\} = 0\,, \tag{11}$$

$$\{P_\mu, P_\nu\} = -\frac{1}{2} R_{\mu\nu\kappa\lambda} S^{\kappa\lambda}\,, \tag{12}$$

$$\{S^{\mu\nu}, P_\kappa\} = -\Gamma^\mu{}_{\lambda\kappa} S^{\lambda\nu} - \Gamma^\nu{}_{\lambda\kappa} S^{\mu\lambda}\,, \tag{13}$$

$$\{S^{\mu\nu}, S^{\kappa\lambda}\} = g^{\mu\kappa} S^{\nu\lambda} - g^{\mu\lambda} S^{\nu\kappa} + g^{\nu\lambda} S^{\mu\kappa} - g^{\nu\kappa} S^{\mu\lambda}\,, \tag{14}$$

There is a set of canonical coordinates that reduce the Poisson brackets to the simple canonical form, these were also discussed by Witzany *et al.*[9]

2.1. *Hunting down degrees of freedom*

Amongst the choices for the supplementary conditions discussed above, only the so-called Tulczyjew-Dixon or "covariant" supplementary spin condition $V^\nu \propto P^\nu$ constrains the gauge freedom fully. This can be seen, in particular, by counting the degrees of freedom in phase space within the Hamiltonian formalism. A covariant formalism for a spinning particle parametrized by proper time τ should then not have more than 4 orbital degrees of freedom (corresponding to the 4 space-time coordinates), and a *single* degree of freedom corresponding to the spin sector (recall that each degree of freedom corresponds to a pair of canonically conjugate coordinates). If a larger number degrees of freedom crops up in the system of evolution equations, there are necessarily nonphysical or gauge-type degrees of freedom left in the system.

This being said, the issue with the Hamiltonian $H_{\rm TD}$ presented in equation (8) is the following. The supplementary spin condition $S^{\mu\nu}P_\nu = 0$ is only an *integral of motion* of $H_{\rm TD}$ when used along with the brackets (11)-(14). This means that in a numerical evolution one finds initial data such that $S^{\mu\nu}P_\nu = 0$ initially and then evolves them – the system consequently conserves the supplementary condition up to numerical noise.

Nevertheless, what one *cannot* do is to assume $S^{\mu\nu}P_\nu = 0$ as an identity to reduce the number of degrees of freedom that are evolved numerically while respecting the geometric structure of the problem.[9] That, is the system still keeps an additional gauge degree of freedom in the numerical evolution, even though it stays zero up to numerical error. Is there perhaps a way to obtain just the minimal Hamiltonian system without this redundant degree of freedom?

Efforts of this type have actually been carried out perturbatively with the use of the so-called Newton-Wigner supplementary spin condition $V^\mu \propto P^\mu/\mathcal{M} + \xi^\mu(x^\nu)$, where $\xi^\mu(x^\nu)$ is a fixed time-like vector field.[7,8] As such, this supplementary spin condition is necessarily non-covariant and refers to an additional fixed structure on the background. In other words, the vector $\xi^\mu(x^\nu)$ defines a privileged observer frame and the dynamics are in some sense defined with respect to this privileged frame. This impression is even more strengthened by the fact that the constraint procedure of the algebra (11)-(14) by the Newton-Wigner condition seems to be necessarily coupled to a constraint procedure on the time parameter. To rephrase this, the Newton-Wigner condition forces one's hand into an explicit 3+1 formalism – and covariance is lost.

2.2. *Dirac constraint procedure*

Let me now sketch the Dirac constraint procedure for the Tulczyjew-Dixon constraint $S^{\mu\nu}P_\nu = 0$ applied to the Poisson bracket (11)-(14). This should serve as a covariant counter-example to the procedures using the Newton-Wigner condition.

It was already shown in the paper by Witzany *et al.*[9] that one can add the Tulczyjew-Dixon Lagrangian-constraint term to the "minimal" Hamiltonian (6), solve for the Lagrange multiplier, and the resulting Hamiltonian gives the correct Mathisson-Papapetrou-Dixon equations for the motion of the body under the Tulczyjew-Dixon supplementary spin condition. The resulting Hamiltonian is actually $H_{\rm TD}$ presented in eq. (8). That is, we already *know* that applying the constraint $S^{\mu\nu}P_\nu = 0$ as a Lagrangian constraint to the system will lead to the correct dynamics, one only needs to figure out the correct *Dirac bracket* corresponding to this constraint.

Let me now briefly summarize the Dirac constraint procedure.[13,14] One starts with a set of n independent constraints $\Phi_a = 0$, $a = 1...n$ with a Poisson bracket $\{\Phi_a, \Phi_b\} = C_{ab}$, where one assumes the matrix C to be invertible, $\exists (C^{-1})^{bc}$: $C_{ab}(C^{-1})^{bc} = \delta^c_a$. Then we define the Dirac bracket $[,]$ for any phase-space functions A, B as

$$[A, B] = \{A, B\} - \{A, \Phi_a\}(C^{-1})^{ab}\{\Phi_b, B\}\,. \tag{15}$$

Now it is easy to see that this bracket fulfils the Jacobi identity and that $[\Phi_c, A] = 0$ for any A when evaluated on the $\Phi_a = 0$ hypersurface in phase space. As a result, the set of constraints $\Phi_a = 0$ can then be used as an *identity* on phase space in the sense that its use will commute with the Dirac bracket. In particular, it can be shown that the system with a given Hamiltonian and a Dirac bracket is equivalent to the system with the same Hamiltonian and a Lagrangian-constraint term.[13,14]

Now to the specific case of the constraint $S^{\mu\nu}P_\nu = 0$. Even though there are 4 components to this equation, not all of them are linearly independent. For instance, independently on whether the constrain is fulfilled or not, it holds that $P_\mu(S^{\mu\nu}P_\nu) = 0$, and $w_\mu(S^{\mu\nu}P_\nu) = 0$, where $w_\mu = \epsilon_{\mu\rho\sigma\chi}S^{\rho\sigma}P^\chi$ and $\epsilon_{\mu\rho\sigma\chi}$ is the Levi-Civita pseudo-tensor. As such, there are only *two* non-trivial constraints on the system hidden in the formula $S^{\mu\nu}P_\nu = 0$. Nevertheless, as already shown partially by Witzany *et al.*,[9] one can proceed in a covariant fashion by defining

$$C^{\mu\nu} = \{S^{\mu\kappa}P_\kappa, S^{\nu\lambda}P_\lambda\} = -\tilde{\mathcal{M}}^2 S^{\mu\nu}\,, \tag{16}$$

$$\tilde{\mathcal{M}}^2 \equiv -P_\lambda P^\lambda + \frac{1}{4}R_{\gamma\rho\sigma\chi}S^{\gamma\rho}S^{\sigma\chi}\,, \tag{17}$$

$$C^\dagger_{\kappa\lambda} = -\frac{1}{\tilde{\mathcal{M}}^2 S^2}S_{\kappa\lambda}\,, \tag{18}$$

where $S^2 \equiv S^{\mu\nu}S_{\mu\nu}/2$. The matrix $C^{\dagger}_{\kappa\lambda}$ is then a pseudo-inverse of $C^{\mu\nu}$ on the non-trivial sub-spaces of the constraint (specifically, $C^{\mu\nu}C^{\dagger}_{\nu\lambda}$ projects out the sub-spaces $\sim P_\mu, w_\mu$ discussed above). One can then covariantly define the Dirac bracket using the original Poisson bracket (11)-(14) as

$$[A, B] = \{A, B\} - \{A, S^{\mu\nu}P_\nu\}C^{\dagger}_{\mu\kappa}\{S^{\kappa\lambda}P_\lambda, B\}\,. \tag{19}$$

The resulting brackets for $P_\mu, x^\nu, S^{\kappa\lambda}$ are now straightforward to compute and they contain various orders of new spin and curvature terms as compared to the original brackets. (The full expressions and simplified bases will be reported on elsewhere.)

2.3. *The zeroth-order problem*

The bracket (19) is essentially what we have been looking for: only a minimal number of degrees of freedom is left in the phase space and one can happily evolve them in the Hamiltonian formalism defined by the Dirac bracket and the minimal Hamiltonian (6). In particular, it is advantageous to define the spin vector

$$s^\lambda \equiv \frac{1}{2\mathcal{M}}\epsilon^{\lambda\gamma\rho\sigma}S_{\gamma\rho}P_\sigma\,, \tag{20}$$

$$\Rightarrow S^{\mu\nu} = \frac{1}{\mathcal{M}}\epsilon^{\mu\nu\kappa\lambda}P_\kappa s_\lambda\,, \tag{21}$$

where the second equality applies on the $S^{\mu\nu}P_\nu = 0$ hypersurface. This vector partially reduces the number of the many dependent components of the tensor $S^{\mu\nu}$, but not fully, since it also holds that $s^\mu P_\mu = 0$.

Now one would like to transform to canonical coordinates so that the bracket (19) reduces to a simple form. One soon realizes that one of the biggest obstacles is the *zeroth-order problem*. That is, even in flat space-time $R_{\mu\nu\kappa\lambda} = 0$ and Minkowski coordinates, the bracket is quite non-trivial. Specifically, it reduces in the pure Minkowski case to

$$[x^\mu, P_\nu] = \delta^\mu_\nu\,,\ [P_\mu, P_\nu] = 0\,,\ [P_\mu, s^\lambda] = 0\,, \tag{22}$$

$$[x^\mu, x^\nu] = \frac{1}{\mathcal{M}^3}\epsilon^{\mu\nu\kappa\lambda}P_\kappa s_\lambda\,, \tag{23}$$

$$[x^\mu, s^\nu] = \frac{P^\mu s^\nu - s^\mu P^\nu}{\mathcal{M}}\,, \tag{24}$$

$$[s^\mu, s^\nu] = -\frac{1}{\mathcal{M}}\epsilon^{\mu\nu\kappa\lambda}P_\kappa s_\lambda\,. \tag{25}$$

There is no obvious and/or elegant way to cover this Dirac algebra with canonical coordinates.

2.4. *Is Newton-Wigner the only way?*

The difficulty with Poisson brackets of spinning systems in flat space-time was already the concern of the works of Newton and Wigner,[15] even though there the motivation was the quantum analogue of this problem for the commutation relations of operators. Jordan[16] proved that the Newton-Wigner basis of positions and spins is the only basis such that the position "transforms as a position should". (For a review and a classical rederivation, see the recent paper by Schwartz and Giulini.[17])

The results on the Newton-Wigner basis can be rephrased in the context of this work as follows. Consider any physical Dirac-constraint procedure that 1) reduces the unphysical spin degrees of freedom, 2) forces the time parametrization to be coordinate time $t = x^0$ (by using the $P^\mu P_\mu = -\mathcal{M}^2$ on-shell constraint[13, 18]), and 3) parametrizes the phase-space in terms of variables $X^i, P_j, \tilde{S}^k$ such that P_j are spatial components of P_μ and such that the final bracket is

$$[X^i, X^j]' = 0\,, \tag{26}$$

$$[X^i, P_j]' = \delta^i_j\,, \tag{27}$$

$$[X^i, \tilde{S}^j]' = 0\,, \tag{28}$$

$$[\tilde{S}^j, P_i]' = 0\,, \tag{29}$$

$$[\tilde{S}^i, \tilde{S}^j]' = \epsilon^{ijk}\tilde{S}^k\,, \tag{30}$$

where ϵ^{ijk} is the permutation symbol. Then (up to singular systems), the variables $X^i, P_j, \tilde{S}^k$ are *necessarily* the Newton-Wigner variables with $\xi^\mu = \delta^\mu_0$.

One conclusion of this theorem is that if one was to cover the algebra (22)-(25) with coordinates X^μ canonically conjugate to P_μ and two other canonically conjugate coordinates on the spin sector of the phase space, then these would necessarily be Newton-Wigner variables when reduced to spatial sections. Even this result would be interesting, since it would "covariantize" the Newton-Wigner prescription. Of course, it is also possible to construct canonical coordinates that do not have $\sim P_\mu$ as canonical variables, but these seem to almost always violate manifest rotational and/or translational invariance of the dynamics.

3. Summary and Outlooks

I have shown that capturing the spin-curvature coupling of a test body on a curved background in a covariant Hamiltonian formalism – while eliminating redundant degrees of freedom – is a non-trivial matter. This matter is also far from closed. In the future, I hope to find a set of "covariant Newton-Wigner variables" for the brackets (22)-(25) and formulate a (possibly implicit) iteration problem to canonicalize the full curved-spacetime brackets (19). These should provide a first, truly general basis for the canonical treatment of spinning test particles in general relativity (but see also the work of Steinhoff in the ADM formalism[19]).

This result should then have obvious pay-offs for the relativistic two-body problem and gravitational-wave modelling. For example, the procedure of Vines *et al.*[7] suggests that the minimally coupled spinning particle (that is, with no additional multipoles) under the Tulczyjew-Dixon condition represents a "test Kerr black hole" on the curved background, since its effective quadrupole in the Newton-Wigner variables matches the Hansen multipole of a Kerr black hole.[20] Does this apply at all orders? A more general and robust formalism is needed for an answer. The question of the minimally coupled particle is all the more interesting due to the fact that the motion of a spining test particle under the Tulzcyjew-Dixon condition is integrable at $O(S)$ in Kerr space-time,[10, 21, 22] but apparently non-integrable at $O(S^2)$.[23] This then naturally ties-in to the important question of the integrability and smoothness of gravitational-wave inspirals and our ability to efficiently generate predictions for them.[24, 25]

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References

1. M. Mathisson, Neue mechanik materieller systemes, *Acta Phys. Polon.* **6**, 163 (1937).
2. A. Papapetrou, Spinning test-particles in general relativity. I, *Proc. Roy. Soc. Lond.* **A209**, 248 (1951).
3. W. Dixon, A covariant multipole formalism for extended test bodies in general relativity, *Il Nuovo Cimento (1955-1965)* **34**, 317 (1964).
4. L. F. O. Costa and J. Natário, Center of mass, spin supplementary conditions, and the momentum of spinning particles, in *Equations of Motion in Relativistic Gravity*, (Springer, 2015) pp. 215–258.
5. L. F. O. Costa, G. Lukes-Gerakopoulos and O. Semerák, On spinning particles in general relativity: Momentum-velocity relation for the Mathisson-Pirani spin condition, *Phys. Rev.* **D97**, p. 084023 (2018).
6. K. Kyrian and O. Semerák, Spinning test particles in a kerr field–ii, *Mon. Not. Roy. Astron. Soc.* **382**, p. 1922 (2007).
7. J. Vines, D. Kunst, J. Steinhoff and T. Hinderer, Canonical Hamiltonian for an extended test body in curved spacetime: To quadratic order in spin, *Phys. Rev.* **D93**, p. 103008 (2016).
8. E. Barausse, E. Racine and A. Buonanno, Hamiltonian of a spinning test particle in curved spacetime, *Phys. Rev.* **D80**, p. 104025 (2009), [Erratum: Phys. Rev.D85,069904(2012)].
9. V. Witzany, J. Steinhoff and G. Lukes-Gerakopoulos, Hamiltonians and canonical coordinates for spinning particles in curved space-time, *Classical and Quantum Gravity* **36**, p. 075003 (2019).
10. V. Witzany, Hamilton-Jacobi equation for spinning particles near black holes, *Physical Review D* **100**, p. 104030 (2019).
11. E. Barausse and A. Buonanno, Improved effective-one-body Hamiltonian for spinning black-hole binaries, *Phys. Rev. D* **81**, p. 084024 (April 2010).
12. O. Semerák, Spinning test particles in a Kerr field—I, *Monthly Notices of the Royal Astronomical Society* **308**, 863 (1999).
13. P. A. M. Dirac, *Lectures on quantum field theory* (Yeshiva Univ., 1966).

14. A. Hanson, T. Regge and C. Teitelboim, *Constrained Hamiltonian Systems* (Accademia Nazionale dei Lincei, 1976).
15. T. D. Newton and E. P. Wigner, Localized States for Elementary Systems, *Rev. Mod. Phys.* **21**, 400 (1949).
16. T. F. Jordan, Simple derivation of the Newton-Wigner position operator, *Journal of Mathematical Physics* **21**, 2028 (1980).
17. P. K. Schwartz and D. Giulini, Classical perspectives on the Newton–Wigner position observable, *International Journal of Geometric Methods in Modern Physics* **17**, p. 2050176 (2020).
18. A. J. Hanson and T. Regge, The Relativistic Spherical Top, *Annals Phys.* **87**, p. 498 (1974).
19. J. Steinhoff, Canonical formulation of spin in general relativity, *Annalen der Physik* **523**, 296 (2011).
20. R. O. Hansen, Multipole moments of stationary space-times, *J. Math. Phys.* **15**, 46 (1974).
21. R. Rüdiger, Conserved quantities of spinning test particles in general relativity. I, *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences* **375**, 185 (1981).
22. R. Rüdiger, Conserved quantities of spinning test particles in general relativity. II, *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences* **385**, 229 (1983).
23. O. Zelenka, G. Lukes-Gerakopoulos, V. Witzany and O. Kopáček, Growth of resonances and chaos for a spinning test particle in the schwarzschild background, *Physical Review D* **101**, p. 024037 (2020).
24. E. E. Flanagan and T. Hinderer, Transient resonances in the inspirals of point particles into black holes, *Physical review letters* **109**, p. 071102 (2012).
25. G. Lukes-Gerakopoulos and V. Witzany, Non-linear effects in EMRI dynamics and their imprints on gravitational waves, *arXiv:2103.06724, to appear in "Handbook of Gravitational Wave Astronomy"* (Eds. C. Bambi, S. Katsanevas, and K. Kokkotas; Springer, Singapore) (2021).