Ultraviolet Fixed Point Structure of Renormalizable Four-Fermion Theory in Less Than Four Dimensions *

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Abstract

We study the renormalization properties of the four-fermion theory in less than four dimensions ($D < 4$) in 1/N expansion scheme. It is shown that β function of the bare coupling has a nontrivial ultraviolet fixed point with a large anomalous dimension ($\gamma_{\bar{\psi}\psi} = D - 2$) in a similar manner to QED and gauged Nambu-Jona-Lasinio (NJL) model in ladder approximation. The anomalous dimension has no discontinuity across the fixed point in sharp contrast to gauged NJL model. The operator product expansion of the fermion mass function is also given.

Introduction

Recently the possibility that QED may have a nontrivial ultraviolet(UV) fixed point has been paid much attention from the viewpoints of "zero charge" problem in QED and raising condensate in technicolor model. Actually such a possibility was pointed out in ladder approximation in which the cutoff Schwinger-Dyson equation for the fermion self-energy possesses a spontaneous-chiral-symmetry-breaking solution for the bare coupling larger than a non-zero value ($\alpha_0 \equiv e_0^2/4\pi > \pi/3 \equiv \alpha_c$). We can make this solution finite by letting α_0 have a cutoff dependence in such a way that $\alpha_0(\Lambda) \to \alpha_c + 0$ ($\Lambda \to \infty$), α_c being identified as the critical point with scaling behavior of essential-singularity type. At the critical point, fermion mass operator $\bar{\psi}\psi$ has a large anomalous dimension $\gamma_{\bar{\psi}\psi} = 1$, which is indeed crucial to the technicolor.[2]

This problem was further analyzed in ladder approximation in the two-coupling space of the gauged Nambu-Jona-Lasinio (NJL) model, i.e., QED plus a (possibly "induced") four-fermion interaction whose physical dimension becomes 4(=6-2$\gamma_{\bar{\psi}\psi}$) at the critical point due to a large $\gamma_{\bar{\psi}\psi}$ (=1).[3] Quite recently a critical line of this model was discovered in the whole prameter space of two couplings ($\alpha_0(\Lambda), g_0(\Lambda)$), with $g_0(\Lambda)$ being the dimensionless bare four-fermion coupling.[4][5] The most striking feature of the model is the appearance of an even larger anomalous dimension $\gamma_{\bar{\psi}\psi} = 1 + \sqrt{1 - \alpha_0/\alpha_c}\,(\geq 1)$ at the critical line, which in fact suggests the four-fermion interaction may become a relevant operator and renormalizable, in sharp contrast to the symmetric phase where one obtains a smaller $\gamma_{\bar{\psi}\psi} = 1 - \sqrt{1 - \alpha_0/\alpha_c}\,(< 1)$ and accordingly the four-fermion interaction is irrelevant.[6]

An important application of this dynamical symmetry breaking with a very large $\gamma_{\bar{\psi}\psi}(\simeq 2$ for $\alpha_0 \simeq 0)$ is a "top-mode standard model" in which a top quark condensate is responsible for the electroweak symmetry breaking.[7]

However the existence of a critical point for the bare coupling $\alpha_0(\Lambda)$ does not necessarily imply the UV fixed point for the renormalized one $\alpha_{(\mu)}$ in the continuum theory. In fact the β

* This talk is based on the work done in collaboration with K. Yamawaki.[1]

function was argued to be non-negative, $\beta(\alpha_{(\mu)}) \geq 0$, based on the spectral representation.[8] In ladder approximation, there is no simple way to compute $\beta(\alpha_{(\mu)})$ and/or $\beta(g(\mu))$ through the calculation of vertex Green functions and hence no direct comparison with $\hat{\beta}(\alpha_0(\Lambda))$ and/or $\hat{\beta}(g_0(\Lambda))$ obtained through the gap equation (ladder Schwinger-Dyson equation) for the fermion propagator. Also the above discontinuity of $\gamma_{\bar{\psi}\psi}$,$1\pm\sqrt{1-\alpha_0/\alpha_c}$, across the critical line seems to be rather paradoxical (an artifact of ladder approximation ?), though not obviously in contradiction to the operator product expansion (OPE).[9]

In this talk, we wish to clarify these issues by explicitly calculating $\beta(g)$, $\gamma_{\bar{\psi}\psi}(g)$ and the corresponding "bare" quantities $\hat{\beta}(g_0)$, $\hat{\gamma}_{\bar{\psi}\psi}(g_0)$ of the four-fermion theory in less than four dimensions ($2 < D < 4$) in 1/N expansion; the theory in fact was shown to be renormalizable and was also demonstrated to have a nontrivial UV fixed point for the renormalized coupling, $g_{(\mu)} = g^* \neq 0$, and a large anomalous dimension $\gamma_{\bar{\psi}\psi}(g^*) = D-2$ at the fixed point.[10] We shall show $\hat{\beta}(g_0)$ and $\hat{\gamma}_{\bar{\psi}\psi}(g_0)$ are very similar to $\beta(g)$ and $\gamma_{\bar{\psi}\psi}(g)$, respectively; $\hat{\beta}(g_0)$ possesses a UV fixed point $g_0(\Lambda) = g_c$ in much the same way as the ladder QED and the gauged NJL model, while $\hat{\gamma}_{\bar{\psi}\psi}(g_0)$ becomes large, $\hat{\gamma}_{\bar{\psi}\psi}(g_c) = D-2$, although having no discontinuity across the fixed point in contrast to the gauged NJL model. The discontinuity of $\hat{\gamma}_{\bar{\psi}\psi}(g_0)$ may be traced to the fact that usually in ladder approximation $g_0(\Lambda)$ is not renormalized in the symmetric phase : Taking account of the renormalization of $g_0(\Lambda)$ in our model indeed fill in the gap of $\hat{\gamma}_{\bar{\psi}\psi}$. The large anomalous dimension without discontinuity will be shown to be consistent with the operator product expansion of the fermion mass function, which actually holds in a quite nontrivial fashion.[1]

1/N expansion and Renormalized theory

Let us start with the following four-fermion theory,

$$\mathcal{L}_{4F}(x) = \bar{\psi}^a i\partial\!\!\!/ \psi^a + G_0(\bar{\psi}^a\psi^a)^2/2N,$$

where $\psi^a(x)$ is a four-component Dirac fermion and the suffix runs from one to N. The space-time dimension is less than four. This system has a symmetry under discrete chiral transformation; $\psi^a(x) \to \gamma_5\psi^a(x)$. By introducing an auxiliary field $\sigma(x)$, we rewrite the Lagrangian into

$$\mathcal{L}_\sigma(x) = \bar{\psi}^a i\partial\!\!\!/ \psi^a - M\bar{\psi}^a\psi^a - (N/2G_0)\tilde{\sigma}^2 - \tilde{\sigma}\bar{\psi}^a\psi^a - (NM/G_0)\tilde{\sigma},$$

where the σ field has been shifted to $\tilde{\sigma}$ by a vacuum expectation value $\langle\sigma\rangle = M$ determined through a self-consistent equation, the gap equation, which is derived from the condition that the new variable has no vacuum expectation value.

We now perform 1/N expansion to evaluate Green functions. The fermion propagator and the vertex are of order $O(N^0)$, while the boson propagator and the tadpole are of order $O(1/N)$ and $O(N)$, respectively. The boson propagator, the gap equation and the scalar vertex in 1/N leading order are given by,

$$D_\sigma(p)^{-1} = iN\Big[\frac{1}{G_0} - i\int^\Lambda \frac{d^Dk}{(2\pi)^D}\mathrm{Tr}(\frac{1}{p\!\!\!/ + k\!\!\!/ - M}\frac{1}{k\!\!\!/ - M})\Big],$$

$$\langle\tilde{\sigma}\rangle \propto M(1 - iG_0\int^\Lambda \frac{d^Dk}{(2\pi)^D}\frac{4}{k^2-M^2}) = 0, \qquad (1)$$

$$\Gamma_0^{\bar{\psi}\psi}(p,q) = iND_\sigma(p-q)/G_0.$$

The gap equation has two solutions, a symmetric solution $M = 0$ and a spontaneously broken one $M \neq 0$.

This model can be simultaneously renormalized for both solutions as follows. We define the renormalized coupling $G_{(\mu)}(\equiv g(\mu)\mu^{2-D})$;

$$\frac{1}{G(\mu)} \equiv \frac{Z_G}{G_0} \equiv \frac{1}{G_0} - i\int^{\Lambda} \frac{d^Dk}{(2\pi)^D}\mathrm{Tr}(\frac{1}{\not p + \not k}\frac{1}{\not k})\Big|_{p^2=-\mu^2},$$

$$Z_G = 1 - G_0(\frac{\Lambda^{D-2}}{g_c} - \frac{\mu^{D-2}}{g^*}), \quad (2)$$

where

$$g^* = \frac{(4\pi)^{D/2}(D-2)}{[8(D-1)\Gamma(2-D/2)B(D/2,D/2)]}, \quad g_c = (4\pi)^{D/2}\frac{(D-2)}{8}\Gamma(D/2),$$

with $\Gamma(B)$ being the gamma (beta) function. Notice that the renormalization constant Z_G can be defined to be mass-independent (Zero Mass Renormalization Procedure)[11] even for the broken solution $M \neq 0$, and also that $Z_\psi = Z_\sigma = 1$ in 1/N leading order.

The β function of the renormalized coupling, $\beta(g) = \mu\frac{\partial g(\mu)}{\partial\mu}$, is calculated from (2);

$$\beta(g) = (D-2)\frac{g}{g^*}(g^* - g),$$

which is valid in both phases (solid line in Fig.1). It is now evident that g^* is the UV fixed point which separates the symmetric and broken phases of the symmetry. In D=2 we have $g^* = 0$ ($(D-2)/g^* \to 2/\pi$), which is just the asymptotic freedom of Gross-Neveu model.

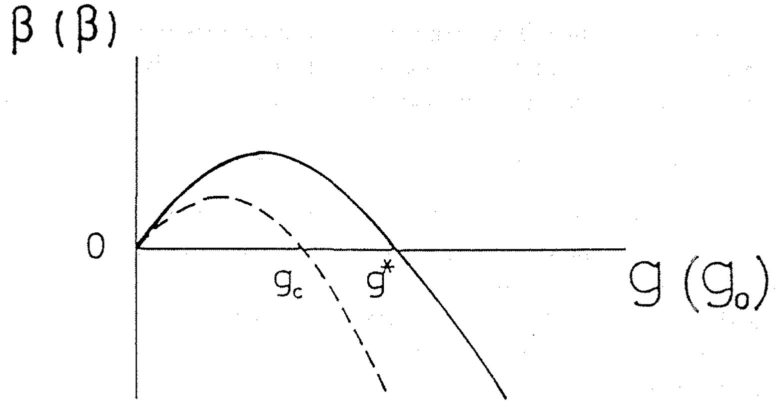

Fig. 1. β functions of renormalized coupling (solid line), and of bare coupling (dashed line).

For the scalar vertex, we can take $Z_{\bar\psi\psi} = Z_G$. Thus the anomalous dimension of the mass operator $\bar\psi^a\psi^a$, $\gamma_{\bar\psi\psi} = \mu\frac{\partial lnZ_{\bar\psi\psi}}{\partial\mu}$, is also obtained from (2),

$$\gamma_{\bar\psi\psi}(g) = (D-2)\frac{g}{g^*}.$$

(solid line in Fig.2). There is no discontinuity at $g(\mu) = g^*$.

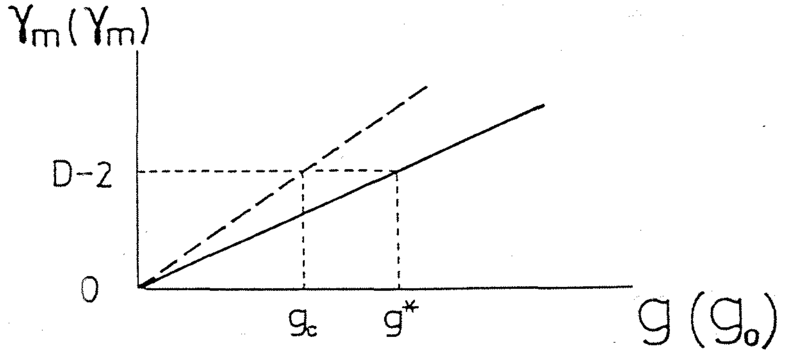

Fig. 2. Anomalous dimensions of mass operator, renormalized one (solid line) and bare one (dashed line).

Bare quantities

Let us now turn to the cutoff dependence of the bare coupling in the present model. From (2) we obtain, for $\Lambda \gg \mu$,

$$\hat{\beta}(g_0) = \Lambda \frac{\partial}{\partial \Lambda} g_0 \Big|_{\mu, G_{(\mu)}-fixed} \simeq (D-2)\frac{g_0}{g_c}(g_c - g_0), \tag{3}$$

where $g_0 \equiv G_0 \Lambda^{D-2}$ (dashed line in Fig.1). Since the gap equation (1) leads to a relation,

$$\frac{\Lambda^{D-2}}{g_c}\frac{g_c - g_0(\Lambda)}{g_0(\Lambda)} = \frac{\mu^{D-2}}{g^*}\frac{g^* - g(\mu)}{g(\mu)},$$

g_c turns out to be the critical point which divides the two phases, corresponding to g^*. On the other hand, from the gap equation (1), we obtain (for $g_0 > g_c$)

$$\hat{\beta}(g_0) = \Lambda \frac{\partial}{\partial \Lambda} g_0 \Big|_{M-fixed} \simeq (D-2)\frac{g_0}{g_c}(g_c - \frac{g_0}{(1 + M^2/\Lambda^2)}),$$

which is actually the β function widely discussed in the ladder QED and the gauged NJL model. In the limit $M/\Lambda \ll 1$, this reduces to (3). This reflects the fact that we can renormalize simultaneously the gap equation and the boson propagator by the renormalization of the coupling.

The anomalous dimension $\hat{\gamma}_{\bar{\psi}\psi}(g_0(\Lambda)) = -\Lambda \frac{\partial ln Z_{\bar{\psi}\psi}}{\partial \Lambda}$ is also calculated from (2),

$$\hat{\gamma}_{\bar{\psi}\psi}(g_0) \simeq (D-2)\frac{g_0}{g_c},$$

(dashed line in Fig.2). This does not have a discontinuity at $g_0(\Lambda) = g_c$ in contrast to the gauged NJL model.

Note that (3) is valid both in the symmetric and the broken phases. This is contrasted with the ladder QED in which the renormalization of α_0 is performed only through the gap equation for the fermion propagator, which is trivial in the symmetric phase ($\hat{\beta}(\alpha_0) = 0$), but not through that of other Green functions such as fermion-photon vertex and fermion four-point function. The lack of renormalization of the bare coupling in the symmetric phase is also shared by the ladder gauged NJL model ($\hat{\beta}(g_0) = \hat{\beta}(\alpha_0) = 0$ below the critical line).

It may be this non-renormalization of the bare coupling in the symmetric phase that caused the discontinuity of the anomalous dimension, $\gamma_{\bar{\psi}\psi} = 1 \pm \sqrt{1 - \alpha_0/\alpha_c}$, across the critical line in the gauged NJL model. In order to clarify this point in our model, we incorporate a fermion bare mass m_0 into the gap equation (1);

$$M = m_0 + iG_0 \int^{\Lambda} \frac{d^D k}{(2\pi)^D} \frac{4}{k^2 - M^2} = m_0 - G_0 \frac{\langle \bar{\psi}^a \psi^a \rangle_0}{N}, \tag{4}$$

where M is defined as $S_F^{-1}(p) = \not{p} - M$. m_0 is renormalized as $m_0 = Z_{\bar{\psi}\psi} m_R$. In the symmetric phase, (4) itself does not require the renormalization of g_0. Were it not for any renormalization of g_0 through other Green functions than the fermion propagator, one would conclude that $m_0 \sim M(1 - \frac{g_0}{\Lambda^{D-2}} \frac{\Lambda^{D-2}}{g_c}) \sim \Lambda^0$. This is indeed what happened to the ladder gauged NJL model. However, in the case at hand, g_0 is actually renormalized through the boson propagator renormalization (2) in such a way that $1 - g_0(\Lambda)/g_c \sim \Lambda^{2-D}$; namely $m_0 \sim 1/\Lambda^{D-2}$.

Mass function and OPE

We define mass function of fermions, the effective coupling and the effective mass as follows,

$$\Sigma(q; g_R, m_R, \mu) \equiv B(q; g_R, m_R, \mu)/A(q; g_R, m_R, \mu),$$

$$S_R^{-1}(q; g_R, m_R, \mu) = A(q; g_R, m_R, \mu)\not{q} - B(q; g_R, m_R, \mu),$$

and

$$Q\frac{dm(Q)}{dQ} = -\gamma_{\bar{\psi}\psi} m(Q)\,; \quad m(\mu) = m_R,$$

$$Q\frac{dg(Q)}{dQ} = \beta(g)\,; \quad g(\mu) = g_R,$$

$$m(Q) = m_R \ \exp\Big\{-\int_\mu^Q \gamma_{\bar{\psi}\psi} \frac{dQ'}{Q'}\Big\}.$$

Based on OPE and renormalization group equation, the general formula of the asymptotic behavior of the mass function is;

$$\begin{aligned}\Sigma(q; g_R, m_R, \mu) =& m(Q)\ \mathrm{Tr}\{\Gamma_R^{\bar{\psi}\psi}(q, q; g(Q), 0, Q)\}/4 \\ &+ iQ^2 C_{\bar{\psi}\psi}(q; g(Q), Q)\ \exp\Big\{+\int_\mu^Q \gamma_{\bar{\psi}\psi} \frac{dQ'}{Q'}\Big\}\ \langle 0|\,\bar{\psi}\psi\,|0\rangle_{(\mu)},\end{aligned} \tag{5}$$

where $-q^2 \equiv Q^2 \gg \mu^2, m_R^2$. The second term in R.H.S. is a dynamical mass associated with the spontaneous chiral symmetry breaking, which was given by Politzer.[12] The first term, a current mass with the explicit chiral symmetry breaking, can be obtained as follows. In the scheme of ZMRP, we straightforwardly have

$$\Gamma_R^{\bar{\psi}\psi}(q; g_R, m_R, \mu) = -\frac{\partial}{\partial m_R} S_R^{-1}(q; g_R, m_R, \mu),$$

from the corresponding formula in terms of bare quantities. There is no singularity in the limit that the renormalized mass goes to zero, so that, we next expand each term in both sides with respect to the renormalized mass m_R. Then,

$$\Gamma_R^{\bar{\psi}\psi}(q, q; g_R, 0, \mu) = B'(q; g_R, 0, \mu) - A'(q; g_R, 0, \mu)\not{q}.$$

The first term in the expansion of B function is a dynamical mass in broken phase, or is equal to zero in symmetric phase because of chiral symmetry. Here we take the explicit breaking term as perturbation to broken phase, so that,

$$
\begin{aligned}
\Sigma(q;g_R,m_R,\mu) &\simeq m_R\frac{B'(q;g_R,0,\mu)}{A(q;g_R,0,\mu)}+O(m_R{}^2)+(Dynamical\ Mass),\\
&\simeq \frac{m_R}{A(q;g_R,0,\mu)}\mathrm{Tr}\{\Gamma_R^{\bar\psi\psi}(q,q;g_R,0,\mu)\}/4+(Dynamical\ Mass).
\end{aligned}
\tag{6}
$$

Renormalization group equations for $A(q;g_R,0,\mu)$ and $\Gamma_R^{\bar\psi\psi}(q,q;g_R,0,\mu)$ can be solved as follows.

$$
A(q;g_R,0,\mu)=\exp\Big\{-\int_\mu^Q 2\gamma_\psi\frac{dQ'}{Q'}\Big\},
\tag{7}
$$

$$
\Gamma_R^{\bar\psi\psi}(q,q;g_R,0,\mu)=\exp\Big\{-\int_\mu^Q(\gamma_{\bar\psi\psi}+2\gamma_\psi)\frac{dQ'}{Q'}\Big\}\Gamma_R^{\bar\psi\psi}(q,q;g(Q),0,Q),
\tag{8}
$$

where $-q^2=Q^2$. What we want follows from eq.(6),(7),and (8).

In the case of the four-fermion theory considered, the scalar vertex and the Wilson coefficient function in the zero mass limit are

$$
\Gamma_R^{\bar\psi\psi}(q,q;g(Q),0,Q)=iND_\sigma(0;g(Q),0,Q)/G(Q)=\frac{g^*}{g^*-g(Q)},
$$

$$
C_{\bar\psi\psi}(q;g(Q),0,Q)=\frac{iG(Q)}{NQ^2}.
$$

Thus we obtain the asymptotic form of the fermion mass,

$$
\begin{aligned}
M &\simeq m(Q)\,\frac{g^*}{g^*-g(Q)}\;-\frac{G(Q)}{N}\;\exp\Big\{+\int_\mu^Q\gamma_{\bar\psi\psi}\frac{dQ'}{Q'}\Big\}\;\langle 0|\,\bar\psi\psi\,|0\rangle_{(\mu)},\\
&\simeq m_R\,\frac{g^*}{g^*-g(\mu)}-\frac{G(\mu)}{N}\;\langle 0|\,\bar\psi\psi\,|0\rangle_{(\mu)}.
\end{aligned}
\tag{9}
$$

In the second equality, we use the relation

$$
\frac{G(\mu)}{G(Q)}=\frac{g^*-g(\mu)}{g^*-g(Q)}=\exp\Big\{+\int_\mu^Q\gamma_{\bar\psi\psi}\frac{dQ'}{Q'}\Big\},
$$

which follows from $Z_G=Z_{\bar\psi\psi}$. Eq.(9) in fact agrees with the gap equation (1), if it is expanded in the renormalized mass, and the subtraction of the operator $\bar\psi\psi$ is considered. The subtraction procedure is as follows,[13]

$$
:(\bar\psi\psi):\equiv(\bar\psi\psi)_0-\frac{Nm_0}{G_0}+\frac{Nm_0|\Gamma_0^{\bar\psi\psi}(q,q;g_R,0,\mu)|}{G_0}\equiv Z_{\bar\psi\psi}^{-1}(\bar\psi\psi)_R
$$

by means of which $\bar\psi\psi$ can be renormalized by $Z_{\bar\psi\psi}$ defined through the scalar vertex and does not mix with the operator **1**. Note that the effective mass $m_{(Q)}$ is multiplied by the nontrivial factor $g^*/(g^*-g(Q))$, which precisely compensates rapid damping of $m_{(Q)}\sim Q^{-\gamma_{\bar\psi\psi}(g^*)}$ to yield the first term of (9), a constant mass. This is a remarkable difference from that in QCD-like theories.

Summary

The four-fermion theory in $D < 4$ dimensions is an explicit example of the model possessing a nontrivial UV fixed point in both the bare and the renormalized couplings. It shares many interesting features with QED and gauged NJL model in ladder approximation in four dimensions. The gap in the anomalous dimension across the fixed point in the gauged NJL model may be caused by the non-renormalization of the coupling in the symmetric limit. The large anomalous dimension is consistent with the OPE formula of mass function including the nontrivial coefficient.

Acknowledgement

I would like to express my sincere thanks to K. Yamawaki for collaboration and many enlightening discussions.

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