Quantum solution of the relationship between the 19-vertex model and the Jones polynomial

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Abstract.
The challenge is to create an efficient quantum algorithm for the bosonic model capable of calculating the Jones polynomials for a knot resulting from interweaving or interlacing n - vertices. This weave is the construction of braid group representations from nineteen-vertex model. We present eigenbases and eigenvalues for lattice generators and their usefulness for the direct computation of Jones polynomials. The calculation shows that the Temperley-Lieb operators can be used for any braid word. Therefore, we propose a quantum sequence using these singular operators as quantum gates operating on the state of n qubits. We show that quantum calculations give the Jones polynomial for achiral knots and links.

1. Introduction

This short article presents the quantization of an exactly solvable nineteen-vertex model, which is a model of classical statistical mechanics. The term "decidable" means that the Boltzmann model weights or R-matrix $R(u)$ with spectral parameter k satisfy the Yang-Baxter equation (IBE) or star-delta relation. When the IBE is satisfied, the model transfer matrices in the family change: $[T(x), T(x)] = 0$.This means that the existence of such commuting operators is a sufficient condition for the solvability of the model [1],[2]. Indeed, in many cases there are methods for obtaining an exact expression for the free energy: for example, we can use Ansatz Bethe, quantum backscattering [1] or some other methods [2].

The Hamiltonian is determined by the logarithmic derivative $T(x)$. The set of mutually commuting operators $(T(x)$ $n = 0, 1, 2, ..$ includes the spin Hamiltonian. There are so many commuting operators that it is impossible to integrate. First, we define vertex models and their associated quantum spin chains in general features, and then we will define the nineteen-vertex models and associated spin chains that will be discussed in this article.

The challenge is to build an efficient quantum algorithm that can compute the Jones polynomial for any knot or link obtained by weaving or closing a $3n$ - braid words.Let us consider the construction of representations of braid groups from vertex models knot. We present eigenstates and eigenvalues for constructing braid generators and their usefulness for computing the Jones polynomial directly. Calculations show that sets of unit operators can be attached to any word in a line. Therefore, we propose a quantum algorithm that uses these unit operators as quantum gates that operate on the states of $3n$ qubits. We show that quantum computation yields a Jones polynomial for knots and achiral links [3],[4],[5],[6]. In the 1970s, physicist Stephen Wiesner mentioned the idea of combining quantum computing and knot theory [7].

Quantum algorithms have proven to be more effective than classical algorithms in solving many problems [8]. Grover's search algorithm in a quantum mechanical system can be in a state of superposition and simultaneously check many names. By properly tuning the phases of different operations, successful calculations reinforce each other while others randomly interfere. Compared to Shor's quantum algorithm, Grover's search algorithm does not change the complexity class, but can be used to speed up a wide range of algorithms. Thus, the capabilities of quantum algorithms are explored in the context of knot theory and vertex models of statistical mechanics. The classification of knots and links in three-dimensional space is one of the open problems. Jones presented a recursive procedure to determine the polynomial relationship of these knots and links. Jones polynomials classify some knots and links [9]. Using quantum walks to build a new quantum algorithm for bosonic models and determine the distinctness of elements and their generalization. To determine element distinctness (the problem of finding two equal elements among N given elements), we obtain the quantum query algorithm $O(N^{2/3})$ [10]. This improves the previous quantum algorithm $O(N^{3/4})$ of Berman et al [11] and matches the lower bound of Aronson and Shi [12]. There are other generalized polynomials that improve classification, but none have achieved perfect classification [13], [14]. The classical calculation of the Jones polynomial P is known to be a difficult problem [15].Therefore, it is interesting to study the polynomial computation of knots and links using quantum algorithms. In physics, there are different ways to obtain polynomials for knots and links. According to Alexander's theory, each knot can be considered as a trailing or closing n thread. Therefore, polynomials for knots and links can be determined by studying the theoretical representation of B_n-entanglement. The general part of this approach is to find another representation of the braid group B_n. We now present a summary of some of these approaches:

1) N-state vertex models, which are two-dimensional statistical mechanical models in which the square lattice terms carry $SU(3)$ spin n representations. The number of possible 1 spin states is denoted by $N = n + 1$. The properties of these models are described by the so-called R-matrix, which is an $N^2 \times N^2$ matrix. The number of nonzero elements in the R-matrix for N-states vertex models.

In the literature, vertex models are called to either as N-states or n-vertex models; both are equivalent. For example, vertex models with three states have spin 1 at vertex lattice and are equivalently called nineteen-vertex models, where "nineteen" denotes the number of non-zero elements of the R-matrix. In [16], [17] representations of braid groups and vertex polynomials were obtained from R-matrices of vertex models with $N = 2, 3, 4$ states.

2) Chern-Simons theory is a topological field theory that provides a natural basis for the study of knots and connections [18]. Vertex polynomials are obtained by averaging the observed Wilson loops. In particular, the Jones polynomial corresponds to the representation of the Wilson loop with spin 3/2 in the Chern-Simons $SU(3)$ theory. It is obvious that arbitrary representations of any compact calibration group G can lead to generalized polynomials [19]. Polynomies are related to the variable q, which is a function of the connection constant to the rank of the measurement group. Field-theoretic polynomials were obtained by combining the Chern-Simons theory on a WZW 3-manifold with a boundary and the corresponding conformal Wess-Zumino-Witten (WZW) field theory on the boundary. Polynomies depended significantly on different representations of monodromy or matrix entanglement in WZW models. Recently, Fridman et al.[20] attempted to model topological field theories using quantum computers. Topological quantum calculations proposed in [20],[21], [22] are at the mathematically abstract level. He uses the connection between the partial quantum Hall states and the Chern-Simons theory with the corresponding integer connection k.

3) Sum of states method for obtaining polynomials in parentheses [23], [24]. The source

[24] shows the construction of a unified representation of the three-strand braid B_n. Moreover, with this approach it was shown that a quantum computer cannot estimate the knot polynomial. However, the number of connections can be determined by a specific choice of polynomial variable [24]. Our goal is to determine the Jones polynomial for any knot or link obtained from braids using a quantum algorithm. To do this, we need to define a matrix representation for the braid generators. Let us repeat the construction of a representation of a group of braids from a model with nineteen vertices. It is important to emphasize that quantum computing in this paper fundamentally depends on the mapping of any entangled word into a product of unitary operators. Polynomials for knots and connections can be calculated directly by choosing the appropriate eigenbasis of the braid matrices.

We reproduce the hierarchy of connectivity invariants associated with a set of vertex N-state models in a method different from the original construction of Akutsu, Deguchi and Wadati. An alternative method replaces the "cross-symmetry" property exhibited by the Boltzmann weights of vertex models with a similar property that encodes the same information for the purpose of constructing coupling invariants, but requires only a constraint on the Boltzmann weights as the spectral parameter varies to infinity.

For braid matrices obtained from a nineteen vertex model, the top matrix element gives the squared modulus of the Jones polynomial (up to a common constant). This article is organized as follows. In Section II, we present a general method for finding braid group representations using N-state knot models. We discuss in detail the nineteen vertex model, the intertwining of eigenvalues and eigenstates. Using these eigenstates, we estimate the Jones polynomial. In Section III, we present a method for estimating the squared modulus of a Jones polynomial as a quantum computation, treating each vertex or network as a collection of cups, a series of interconnected and closing actions. In the last part, we summarize the results obtained and discuss the importance of the quantum algorithm.

2. N-STATE VERTEX MODEL

In this section, we will discuss the construction of braid group representations from N-state vertex models [16]. To compare the eigenstates of the wreath operator with in the qubit states, the six-point model (spin 1 on the links of the square lattice) is relevant. Therefore, we present the explicit form of the R-matrix and the braid matrix for nineteen point model. As mentioned in the introduction, vertex models are two-dimensional statistical mechanical models. models with n spins lying on the links of the square lattice. The properties of these models are described by the elements of the R-matrix between the edge states $(m1, m2)$ and (n_1, n_2): $R^{n_1 n_2}_{m_1 m_2}(u)$, where u is the spectral parameter. Here m_1, m_2, n_1, n_2 have the values $n/2, \dot{n}/2$. The condition for the integrability of these models requires that the following equations must be fulfilled:

$$R^{n_1,n_2}_{m_1,m_2}(u) = \sum_{l=0,1,2} \begin{bmatrix} 1 & 1 & l \\ m_2 & m_1 & M \end{bmatrix} \lambda_l(u) \begin{bmatrix} 1 & 1 & l \\ n_1 & n_2 & M \end{bmatrix}, \tag{1}$$

where m_1, m_2, n_1, n_2 are called Yang-Baxter elements and the relation (1) is called Yang-Baxter equation. The terms in parenthesis are the quantum Clebsch-Gordan coefficients (q-CG) [25] which are nonzero if and only if m takes a value in the range $m = l, l+1, \ldots + l$ and satisfies the condition $m_1 + m_2 = m = n_1 + n_2$. Here $\lambda_l(u)$ is given by

$$\lambda_l(u) = \prod_{s=l+1}^{j_1+j_2} \sinh(\mu - u) \prod_{l=|j_1-j_2|+1}^{J} \sinh(\mu + u) \prod_{s=l-1}^{j_1-j_2} \sinh(2\mu - u) \prod_{l=|j_1+j_2|-1}^{J} \sinh(2\mu + u). \tag{2}$$

In other words, braid group generators $b_i \in Bn$ (spectral parameter independent operators) are obtained from $R_i(u)$ by the formula taking the spectral parameter $u \to \propto$:

$$\lim_{u \to \propto} \lambda_l(u) = b_i. \tag{3}$$

Let us define a quantity $\sigma^{n_1 n_2}_{m_1 m_2}$ as follows:

$$\lim_{u \to \propto} R^{n_1 n_2}_{m_1 m_2} = \sigma^{n_1 n_2}_{m_1 m_2}. \tag{4}$$

The explicit form of the $\sigma^{n_1 n_2}_{m_1 m_2}$ m_1, m_2-matrix elements in this limit turns out to be:

$$\frac{(R^{1,1})^{n_1,n_2}_{m_1,m_2}(u=0)}{(R^{1,1})^{\uparrow,\uparrow}_{\uparrow,\uparrow}(u=0)} = \left(\begin{array}{c|ccccccccc} m_1, m_2/n_1, n_2 & 1,1 & 1,0 & 1,-1 & 0,1 & 0,0 & 0,-1 & -1,1 & -1,0 & -1,-1 \\ \hline 1,1 & \chi_1(u) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1,0 & 0 & \chi_3(u) & 0 & \chi_2(u) & 0 & 0 & 0 & 0 & 0 \\ 1,-1 & 0 & 0 & \chi_6(u) & 0 & \chi_5(u) & 0 & 0 & 0 & 0 \\ 0,1 & 0 & \chi_2(u) & 0 & \chi_3'(u) & 0 & 0 & 0 & 0 & 0 \\ 0,0 & 0 & 0 & \chi_5(u) & 0 & \chi_7(u) & 0 & \chi_5'(u) & 0 & 0 \\ 0,-1 & 0 & 0 & 0 & 0 & 0 & \chi_3(u) & 0 & \chi_2(u) & 0 \\ -1,1 & 0 & 0 & \chi_4(u) & 0 & \chi_5'(u) & 0 & \chi_6'(u) & 0 & 0 \\ -1,0 & 0 & 0 & 0 & 0 & 0 & \chi_2(u) & 0 & \chi_3'(u) & 0 \\ -1,-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \chi_1(u) \end{array}\right). \tag{5}$$

Therefore, the above (2) matrix elements indicate that we can choose the basis for the n-strand braid mi denotes the one-qubit basis and the above basis state is the n-qubit basis state. Even though we have explicitly diagonalized the matrix, we must remember that everything the generators of the braid group bi cannot be simultaneously diagonalized. The spectral parameter independent form of (2) are defining relations of the braid group B_n, which implies that we can simultaneously diagonalize either b_{2i} or $b'_{2i+1}s$.

Consider any knot or link shown, which is technically called a plating or braid closure. The type of knot is determined by the area of the braid, indicated by the shaded area. It involves a sequence of braiding operations, which is usually written as the word braid with the appropriate orientation. Taking the initial and final state as $|\phi_{(0,0)} >$, next matrix element

$$< \phi_{(0,0)}|B|\phi_{(0,0)} > \tag{6}$$

gives the Jones polynomial (up to general normalization).

Diagonal element matrices are functions of the braid eigenvalues. The relative orientations and number of intersections between corresponding braids are taken into account when writing the functional form. If q is a root of one, these diagonal matrices are unitary. Thus, the braid word $B_a \in B_9$ can be equivalently represented as a product of unitary matrices of size $3^9 \otimes 3^9$

$$B_b = (b_2^{-1} b_4)(\hat{b}_1^{-2} \hat{b}_3^{-3} \hat{b}_6^{-3}) b_1 \hat{b}_2 \in B_6. \tag{7}$$

Similarly, we can find a unitary representation of U_B for any braid word $B \in B_{3n}$ through products of unitary matrices of size $3^{3n} \otimes 3^{3n}$. These unitary representations play the role of quantum gates in the quantum computation of the Jones polynomial. For a subclass of knots (links), called achiral knots (links), D_K not changed as $q \to q^{-1}$. In other words, the elements of the matrix $< 0|S_L|0 >$ will be real. For these achiral knots and links, the quantum algorithm directly produces a Jones polynomial (up to general normalization).

3. QUANTUM SOLUTION

In this section, we attempt to compute Jones polynomials for the knots and links resulting from the sheathing or closure of a $2n$-strand braid using a quantum algorithm. In the previous section we already explained that we can associate a S_L (product of unitary matrices) with each word of the braid $B \in B_{2n}$ The quantum algorithm includes the following steps:

Point 1: Let the initial state of $2n$-qubit be $|0>$ ($|mug>$).

Point 2: We perform the sequence of unitary operations S_L corresponding to the braid word $B \in B_{2n}$. The unitarily transformed state will be

$$|\varphi >= S_L|0 > \tag{8}$$

Point 3: Finally, we determine the probability of the unitary state $|\varphi >$ in a certain final state $|f >$ as

$$| < f|S_{|L|}0 > |^3. \tag{9}$$

Taking the final state $|mug >= |0 >$, we obtain the squared modulus of the Jones' polynomial (up to general normalization) D_K (19).

4. CONCLUSION

In this article, we show a matrix representation of braid matrices for the nineteen-vertex model. From the obtained elements in this model, the eigenbases and eigenvalues of the braid generator are found, obtained from the nineteen-vertex model of the braid representation theory. An exact estimate of the Jones polynomial for any given braid is given by the knot. Consequently, the estimate shows that any braid word can be associated with a set of operators. This is a pretty seminal result of the paper that led to quantum computing. We show a quantum algorithm that combines these unitary operators and can determine the quadratic form of the Jones polynomial for any knot or link using the Clebsch-Gordan coefficients. The quantum sequence gives the Jones polynomial for achiral knots and links.

Quantum calculations essentially determine the probability of transition from the initial state of the unit to the final state. In this case, the number of operators per unit depends on the braid word and cannot exceed double the length of the braid word.

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